MATRIX ANALYSIS OF ELASTIC GRIDWORKS

by

445

KUO-KUANG HU

Diploma, Taipei Institute of Technology, 1956

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Applied Mechanics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1966

ABSTRACT

In this report, matrix analysis is used to derive formulas for the force and displacement analysis of simply connected and rigidly connected rectangular and circular gridworks, supported on either elastic or rigid foundations.

The regularity which these complicated structures have permits the basic equations to be written in the form of matrix direct products, and allows the greatest possible simplification in the computations required. This simplification occurs essentially because the regularity properties of the structures permit a certain factoring which simplifies the solution of the basic equation.

Although the methods used are general, the factoring property disappears as the gridworks become less regular, and the equations become progressively more complicated until for arbitrary gridworks they finally become a completely general set which allows no essential simplification.

MATRIX ANALYSIS OF ELASTIC GRIDWORKS

by

KUO-KUANG HU

Diploma, Taipei Institute of Technology, 1956

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Applied Mechanics

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1966

Approved by:

TABLE OF CONTENTS

Page

INTRODUCTION 1

STIFFNESS MATRICES OF ELASTIC MEMBERS. 3

Stiffness Matrices of Laterally-Constrained Beams 3

Stiffness Matrices of Laterally-Constrained Circular Rings. 5

Stiffness Matrices of Straight Member 7

DEFORMATION OF SIMPLY CONNECTED GRIDWORKS ON ELASTIC FOUNDATIONS . . . 21

Rectangular Gridwork. 21

Circular Gridwork 30

DEFORMATION OF GRIDWORKS CONSIDERING TORSION 37

Rectangular Gridwork. 37

Polar Gridwork. 43

NUMERICAL EXAMPLES 50

Simply Connected Gridwork on An Elastic Foundation. 50

Gridwork Bridge Without Torsion 55

Rectangular Gridwork Considering Torsion. 58

CONCLUDING REMARKS 70

ACKNOWLEDGEMENT. 72

APPENDICES 73

INTRODUCTION

Gridworks of beams are structures which are commonly used in footings, bridges, ribbed shells, etc.

Because the mutual interactions between beams distributes the loads among them, beam gridworks can be designed to be efficient and inexpensive structures.

In this report, formulas are derived for the force and displacement analysis of rectangular and circular gridworks of two classes, simply connected and rigidly connected gridworks, supported on elastic or rigid foundations.

Simply connected gridworks are those in which only tensile or compressive forces are transmitted from one beam to another at a connection; rigidly connected gridworks are those in which couples can be transmitted at connections as well.

The analysis of the gridworks was made by the use of matrix methods. It yielded formulas which are presented for forces, deflections, and stresses.

In establishing the matrix equations for various gridworks, the following assumptions were made:

(a) The beams intersect orthogonally at the joints except at points of singularity.

(b) The gridwork is in a plane, and is loaded by forces and couples only at the joints.

(c) The deformations of the gridwork are infinitesimal and the stresses are within the elastic limit. Thus, the deformations and stresses are expressible as linear functions of the external loads and vice versa.

The stiffness matrices of straight members, circular members and rings, which are included in both simply connected and rigidly connected gridworks, have been derived in the first part of the report. These form the basis of the displacement method for the matrix analysis. The second part of the report contains the development of the matrix equations for solving simply-connected gridworks. Rigidly-connected gridworks are discussed in the third part.

Three numerical examples are given to show applications of the results which appear in the first three parts of the report. The analysis and calculation of deformation and stresses are shown in detail. Additional properties of the matrix equations and some further applications are given in the Discussion section.

STIFFNESS MATRICES OF ELASTIC MEMBERS

Stiffness Matrices of Laterally-Constrained Beams

A laterally-constrained beam, as considered here, is a continuous beam with m-constraint points, shown in Fig. 1, such that the beam connot deflect freely in z-direction at these points. The loads and deformations are in the xz-plane only.

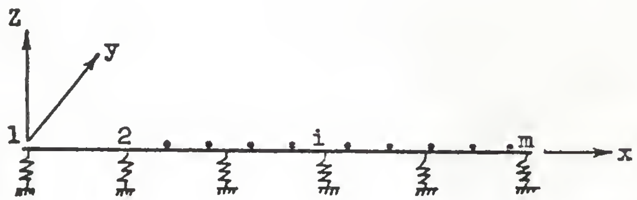


Fig. 1. Laterally-constrained beam.

The stiffness matrix ϕ is the matrix which transforms the displacement vector $\{Z\}$ into the corresponding load vector $\{F\}$. The components of vectors $\{Z\}$ and $\{F\}$ are shown in Fig. 2.

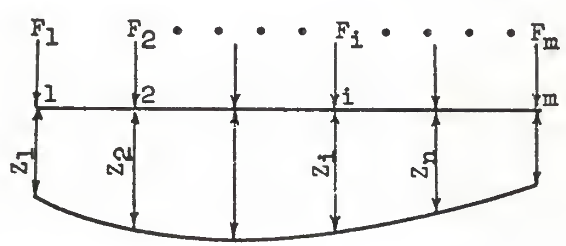


Fig. 2. Load vector $\{F\}$ and displacement vector $\{Z\}$.

The force-displacement relationships for the beam shown in Fig. 2 are characterized by the matrix equation

$$\begin{Bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ F_m \end{Bmatrix} = \begin{bmatrix} (\overline{m}, m) \\ \phi \end{bmatrix} \begin{Bmatrix} Z_1 \\ Z_2 \\ \cdot \\ \cdot \\ \cdot \\ Z_m \end{Bmatrix} \tag{a}$$

For compactness, Eq. (a) can be rewritten as

$$F = \phi Z. \tag{1}$$

The stiffness matrix ϕ can be obtained by using the principle of superposition, as indicated in Fig. 3.

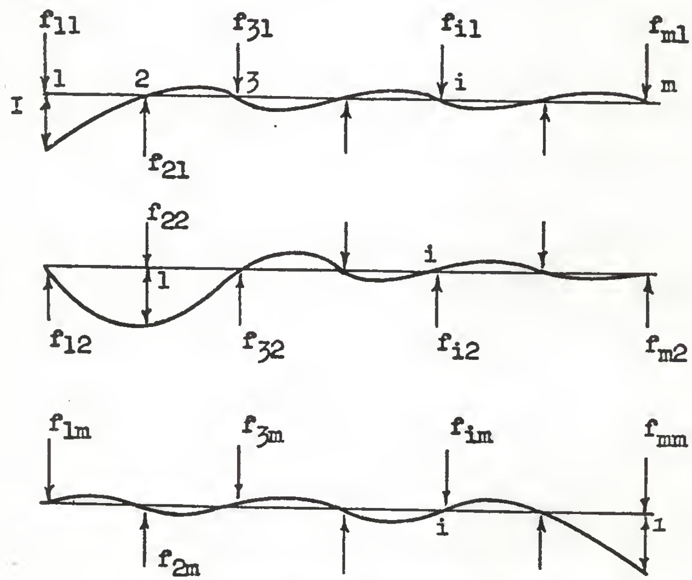


Fig. 3. Relations between loads and deformations.

This leads to the equations

$$\begin{Bmatrix} f_{11} \\ f_{21} \\ \cdot \\ \cdot \\ \cdot \\ f_{m1} \end{Bmatrix} = [\phi] \begin{Bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix},$$

$$\begin{Bmatrix} f_{12} \\ f_{22} \\ \cdot \\ \cdot \\ \cdot \\ f_{m2} \end{Bmatrix} = [\phi] \begin{Bmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} ,$$

and $\qquad \vdots \qquad \vdots$

$$\begin{Bmatrix} f_{1m} \\ f_{2m} \\ \cdot \\ \cdot \\ \cdot \\ f_{mm} \end{Bmatrix} = [\phi] \begin{Bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{Bmatrix} .$$

The above relations yield the stiffness matrix ϕ as

$$\phi = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1m} \\ f_{21} & f_{22} & & f_{2m} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ f_{m1} & f_{m2} & & f_{mm} \end{bmatrix} . \tag{2}$$

Fig. 3 indicates that the problem of finding the stiffness matrix ϕ is that of finding the reactions of a continuous beam when one of the supports undergoes a unit displacement. Therefore, any technique for solving continuous beams can be used for finding the elements of the matrix ϕ.

The matrix ϕ for a uniform beam with equally-spaced constraints is given in Appendix I.

Stiffness Matrices for Laterally-Constrained Circular Rings

Consider a circular ring with m-points of lateral constraint, as shown in Fig. 4, such that at these points the ring cannot be freely displaced in the z-direction. The possible loads and deformations considered here are perpendicular to the ring.

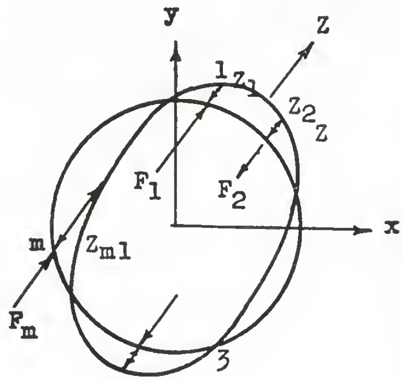


Fig. 4. Force and displacement vectors.

The stiffness matrix ϕ of a circular ring has the same notation as used in section 1 for straight beams.

Let $\{Z\}$ = displacement vector which denotes the displacements at the constraint points of the ring, and

$\{F\}$ = force vector which denotes the corresponding loads.

The components of $\{Z\}$ and $\{F\}$ are shown in Fig. 4. The equations which describe the ring can be written as

$$\{F\} = (\phi)\{Z\},$$

where ϕ is the stiffness matrix.

In order to satisfy static equilibrium conditions, there must be at least three constraint points of a ring. Thus, the number of external redundants for a ring of m-constraint points is equal to m-3, which means that not all constraint forces are independent of each other. The stiffness matrix ϕ can be obtained by the following procedures:

(a) Select three constraint points as ground points and express these forces at these points as linear functions of the other external constraint forces. The latter ones are called external redundants.

(b) Cut the ring at the i-th constraint point and select its sectional bending moment M_o and torsional moment T_o as internal redundants.

(c) Express the bending moment and torsional moment of every section in terms of redundant forces F_j's, M_o and T_o.

(d) Calculate the strain energy U of the system.

(e) Using Castigliano's principle, the principle of least work, establish the elastic equations as

$$\begin{cases} \dfrac{\partial U}{\partial F_j} = \begin{bmatrix} 1 & j = i \\ 0 & j \neq i \end{bmatrix} \quad j = 1^*, 2^*, \ldots\ldots, (m-3)^* \\[2ex] \dfrac{\partial U}{\partial M_o} = 0 \\[2ex] \dfrac{\partial U}{\partial T_o} = 0 \,. \end{cases}$$

(f) Solve these equations for redundant forces and calculate the reactions at the selected ground points.

The calculated external force vector $\{F\}_i$ gives the i-th column of the stiffness matrix ϕ.

Figure 5 is an example showing the first three steps of the above processes. A more detailed treatment is given in Appendix II.

Stiffness Matrices of Straight Member

In the three dimensional case, the deformation of a certain section A of a member can be described completely by a 6-component row vector as $\lfloor D_{a1}, D_{a2}, D_{a3}; R_{a1}, R_{a2}, R_{a3} \rfloor$, where $\lfloor D_{a1}, D_{a2}, D_{a3} \rfloor$ denote the linear

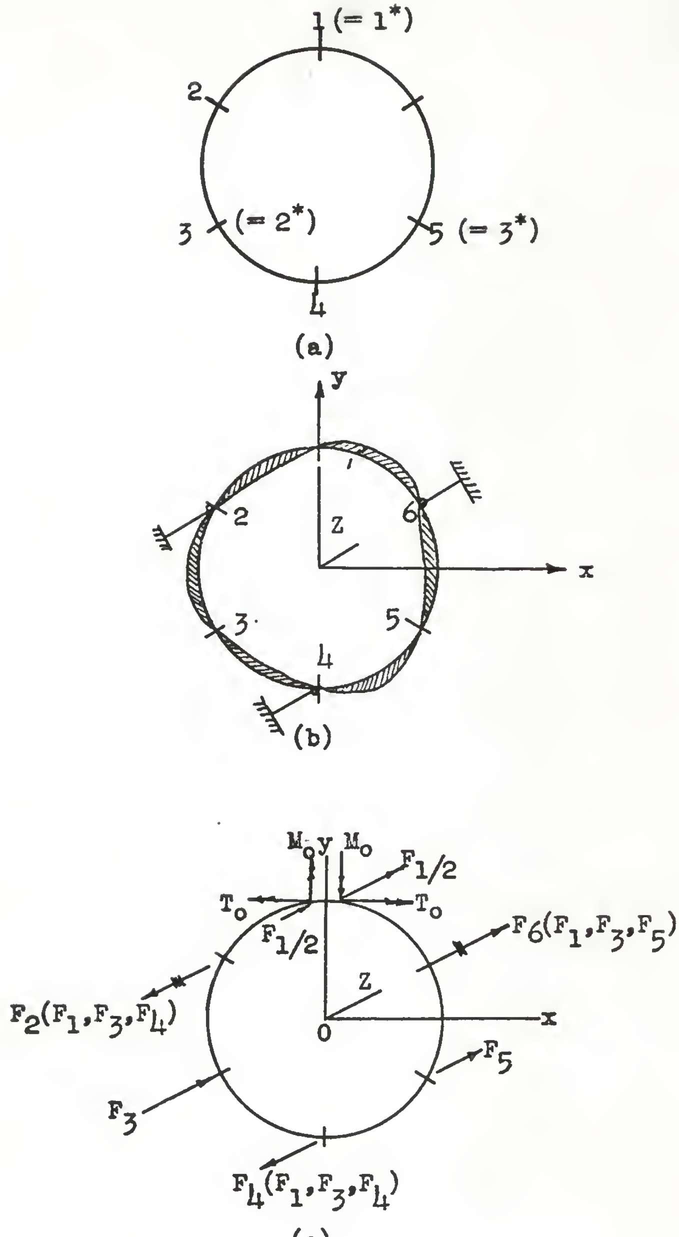


Fig. 5. (a) A ring of six constraints.
(b) Deformation of the ring when point 1 undergoes a unit displacement with points 2, 4 and 6 considered as ground points.
(c) Forces applied to the ring when point 1 is the cut point.

displacements and $\lfloor R_{a1}, R_{a2}, R_{a3} \rfloor$ denote the rotations of the section with respect to X_1, X_2 and X_3, respectively. Similarly, the forces and couples which act on section A can also be described by a 6-component vector $\lfloor F_{a1}, F_{a2}, F_{a3}, C_{a1}, C_{a2}, C_{a3} \rfloor$, where $\lfloor F_{a1}, F_{a2}, F_{a3} \rfloor$ denote the forces and $\lfloor C_{a1}, C_{a2}, C_{a3} \rfloor$ denote the couples in the X_1, X_2 and X_3 directions, respectively. The signs of rotations and couples follow the right-hand rule.

If a member has m-constraint points, then the deformation of the member (or the forces applied on the member) can be described by a 6m-component vector.

As a matter of convenience, the stiffness matrix of a two-end constrained member is derived first. This matrix is then used to construct the stiffness matrix of the member when it is multi-constrained. A further treatment is given in Appendix III.

Stiffness Matrix of a Uniform, End-Constrained, Member.

Let

$$\{F_a\} = \begin{Bmatrix} F_{a1} \\ F_{a2} \\ F_{a3} \\ C_{a1} \\ C_{a2} \\ C_{a3} \end{Bmatrix}, \quad \{D_a\} = \begin{Bmatrix} D_{a1} \\ D_{a2} \\ D_{a3} \\ R_{a1} \\ R_{a2} \\ R_{a3} \end{Bmatrix},$$

be the end forces and displacements of the end A; and

$$\{F_b\} = \begin{Bmatrix} F_{b1} \\ F_{b2} \\ F_{b3} \\ C_{b1} \\ C_{b2} \\ C_{b3} \end{Bmatrix}, \quad \{D_b\} = \begin{Bmatrix} D_{b1} \\ D_{b2} \\ D_{b3} \\ R_{b1} \\ R_{b2} \\ R_{b3} \end{Bmatrix},$$

be the end forces and displacements of the end B, as shown in Fig. 6.

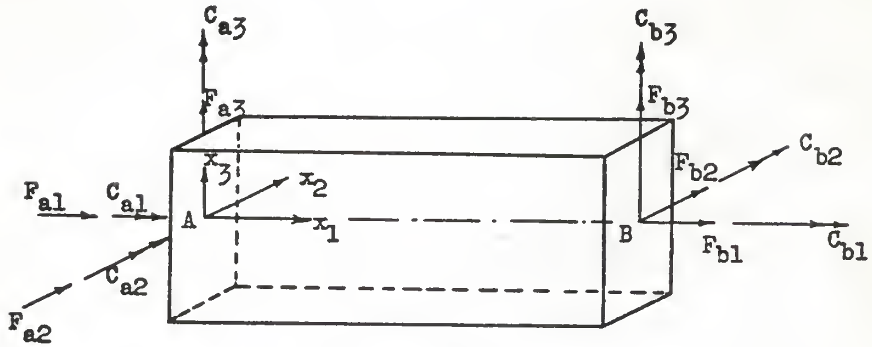


Fig. 6. End-constrained member.

Taking the effect of shearing forces to be second order and using the elementary theory of structures, this member can be described as

$$\left\{\begin{matrix} F_a \\ \cdots \\ F_b \end{matrix}\right\} = (\phi)\left\{\begin{matrix} D_a \\ \cdots \\ D_b \end{matrix}\right\} = \left(\begin{array}{c:c} \phi^a_{ab} & \phi^b_{ab} \\ \hdashline \phi^a_{ba} & \phi^b_{ba} \end{array}\right)\left\{\begin{matrix} D_a \\ \cdots \\ D_b \end{matrix}\right\} \tag{3}$$

where

$$(\phi^a_{ab}) = \left[\begin{array}{c:c:c|c:c:c} \frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & \frac{12EI_3}{L^3} & 0 & 0 & 0 & \frac{+6EI_3}{L^2} \\ \hdashline 0 & 0 & \frac{12EI_2}{L^3} & 0 & \frac{-6EI_2}{L^2} & 0 \\ \hline 0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\ \hdashline 0 & 0 & \frac{-6EI_2}{L^2} & 0 & \frac{4EI_2}{L} & 0 \\ \hdashline 0 & \frac{+6EI_3}{L^2} & 0 & 0 & 0 & \frac{4EI_3}{L} \end{array}\right] \tag{4}$$

$$
(\phi^{b}_{ab}) = \left[\begin{array}{ccc|ccc}
\frac{-AE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-12EI_3}{L^3} & 0 & 0 & 0 & \frac{+6EI_3}{L^2} \\
0 & 0 & \frac{-12EI_2}{L^3} & 0 & \frac{-6EI_2}{L^2} & 0 \\
\hline
0 & 0 & 0 & \frac{-GJ}{L} & 0 & 0 \\
0 & 0 & \frac{+6EI_2}{L^2} & 0 & \frac{2EI_2}{L} & 0 \\
0 & \frac{-6EI_3}{L^2} & 0 & 0 & 0 & \frac{2EI_3}{L}
\end{array}\right] \tag{5}
$$

$$
(\phi^{b}_{ba}) = \left[\begin{array}{ccc|ccc}
\frac{AE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{12EI_3}{L^3} & 0 & 0 & 0 & \frac{-6EI_3}{L^2} \\
0 & 0 & \frac{12EI_2}{L^3} & 0 & \frac{6EI_2}{L^2} & 0 \\
\hline
0 & 0 & 0 & \frac{GJ}{L} & 0 & 0 \\
0 & 0 & \frac{6EI_2}{L^2} & 0 & \frac{4EI_2}{L} & 0 \\
0 & \frac{-6EI_3}{L^2} & \frac{-6EI_3}{L^2} & 0 & 0 & \frac{4EI_3}{L}
\end{array}\right] \tag{6}
$$

and

$$
(\phi^{a}_{ba}) = \left[\begin{array}{ccc|ccc}
\frac{-AE}{L} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{-12EI_3}{L^3} & 0 & 0 & 0 & \frac{-6EI_3}{L^2} \\
0 & 0 & \frac{-12EI_2}{L^3} & 0 & \frac{6EI_2}{L^2} & 0 \\
\hline
0 & 0 & 0 & \frac{-GJ}{L} & 0 & 0 \\
0 & 0 & \frac{-6EI_2}{L^2} & 0 & \frac{2EI_2}{L} & 0 \\
0 & \frac{6EI_3}{L^2} & 0 & 0 & 0 & \frac{2EI_3}{L}
\end{array}\right] \tag{7}
$$

Stiffness Matrix of a Multiply-Constrained Straight Member.

Figure 7 indicates a straight member which is constrained at sections 1, 2, ... and m.

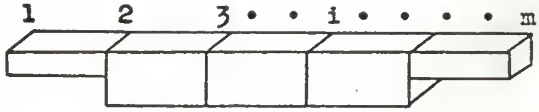


Fig. 7. Multiply-constrained member.

The stiffness matrix of the straight member with m terminal points can be represented as follows:

$$(\Phi) = \begin{bmatrix} \phi^{1}_{12} & \phi^{2}_{12} & & & & \\ \phi^{1}_{21} & \phi^{2}_{21} + \phi^{2}_{23} & \phi^{3}_{23} & & & \\ & \phi^{2}_{32} & \phi^{3}_{32} + \phi^{3}_{34} & \phi^{4}_{34} & & \\ & & \cdot & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & & \phi^{m-1}_{m,m-1} & \phi^{m}_{m,m-1} \end{bmatrix} \tag{8}$$

Stiffness Matrix of Circular Members

Consider the general case of a circular ring, i.e., the ring is subjected to vector forces and couples, and is constrained in all directions at the constraint points. The displacements and forces at any point can be described by 6-component vectors. For simplicity, the forces, couples, linear displacements and rotations can be classified into two independent cases:

(a) Plane deformations: all deformations are in the plane of the ring only.

(b) Lateral deformations: all deformations are perpendicular to the ring only.

Suppose that the points of constraint on the ring are equally spaced. Then every segment between any two such points has the same stiffness. As the stiffness matrix of one segment has been derived, the stiffness matrix of the whole ring can immediately be deduced from it.

Stiffness Matrix of an Arc-Segment in the Case of Plane-Deformation.

The end-constrained circular segment can be described by the partitioned matrix equation as

$$\begin{Bmatrix} F_a \\ \cdots \\ F_b \end{Bmatrix} = \left[\begin{array}{c|c} \phi^a_{ab} & \phi^b_{ab} \\ \hline \phi^a_{ba} & \phi^b_{ba} \end{array}\right] \begin{Bmatrix} D_a \\ \cdots \\ D_b \end{Bmatrix} , \tag{9}$$

where

$$F_a = \begin{Bmatrix} C_{za} \\ F_{ra} \\ F_{ta} \end{Bmatrix} , \quad D_a = \begin{Bmatrix} R_{za} \\ D_{ra} \\ D_{ta} \end{Bmatrix} ;$$

and

$$F_b = \begin{Bmatrix} C_{zb} \\ F_{rb} \\ F_{tb} \end{Bmatrix} , \quad D_b = \begin{Bmatrix} R_{zb} \\ D_{rb} \\ D_{tb} \end{Bmatrix} .$$

The end forces and displacements of segment $\widehat{AB}$ are shown in Fig. 8. The matrices ϕ^a_{ab}, ϕ^b_{ab}, ϕ^a_{ba}, ϕ^b_{ba} in the stiffness matrix ϕ can be obtained in the following way:

Consider a circular segment AB with end B fixed and free end A subjected to loads F_{ra}, F_{ta} and C_{za}, as shown in Fig. 8. The deformations of the end A form the elements of ϕ.

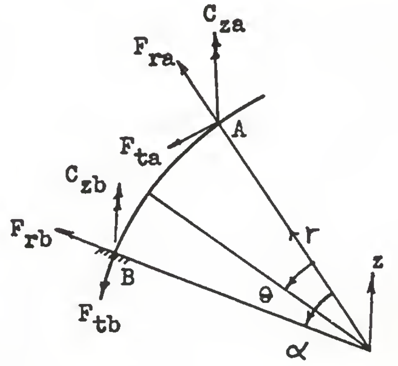


Fig. 8. Circular of plane deformation.

The forces of section θ of the segment can be described by

$$\begin{cases} M_z(\theta) = C_{za} + F_{ra} r\sin\theta + F_{ta} r(1 - \cos\theta) \\ N_t(\theta) = F_{ta}\cos\theta - F_{ra}\sin\theta \\ V_r(\theta) = F_{ra}\cos\theta + F_{ta}\sin\theta \quad . \end{cases} \tag{10}$$

The strain energy can be calculated as

$$U = \int_0^{\alpha} \left[\frac{M_z^2(\theta)}{2EI_z} + \frac{N_t^2(\theta)}{2AE} + \frac{V_r^2(\theta)}{2GA} \right] r d\theta \quad . \tag{11}$$

The deformations of end A can be obtained by using Castigliano's theorem, and can be expressed as

$$\begin{Bmatrix} \dfrac{\partial U}{\partial C_{za}} \\ \dfrac{\partial U}{\partial F_{ra}} \\ \dfrac{\partial U}{\partial F_{ta}} \end{Bmatrix} = (A) \begin{Bmatrix} C_{za} \\ F_{ra} \\ F_{ta} \end{Bmatrix} = \begin{Bmatrix} R_{za} \\ D_{ra} \\ D_{ta} \end{Bmatrix} , \tag{12}$$

where

$$A = \frac{r}{EI_z} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} ,$$

$$a_{11} = \alpha ,$$

$$a_{12} = a_{21} = r(1 - \cos\alpha) ,$$

$$a_{22} = \frac{r^2 + t_1 + t_2}{2}\alpha - \frac{\sin 2\alpha}{4}(r^2 + t_1 - t_2) ,$$

$$a_{23} = a_{32} = r^2(1 - \cos\alpha - \frac{\sin^2\alpha}{2}) - \frac{t_1 \sin^2\alpha}{2} + \frac{t_2(2\alpha - \sin 2\alpha)}{4} ,$$

$$a_{33} = \frac{\alpha}{2}(3r^2 + t_1 + t_2) - 2r^2\sin\alpha + \frac{\sin 2\alpha}{4}(r^2 + t_1 - t_2) ,$$

$$t_1 = \frac{I_z}{A} ,$$

and

$$t_2 = \frac{EI_z}{GJ} ,$$

or as

$$\begin{Bmatrix} C_{za} \\ F_{ra} \\ F_{ta} \end{Bmatrix} = (A)^{-1} \begin{Bmatrix} R_{za} \\ D_{ra} \\ D_{ta} \end{Bmatrix} .$$

By the definition of stiffness matrix,

$$(\phi^a_{ab}) = (A)^{-1} \tag{13}$$

From the equilibrium conditions, the reactions at end B are

$$\begin{Bmatrix} C_{zb} \\ F_{rb} \\ F_{tb} \end{Bmatrix} = \begin{bmatrix} -1 & -r\sin\alpha & -r(1 - \cos\alpha) \\ 0 & -\cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix} \begin{Bmatrix} C_{za} \\ F_{ra} \\ F_{ta} \end{Bmatrix} .$$

Thus

$$(\phi^a_{ba}) = \begin{bmatrix} -1 & -r\sin & -r(1 - \cos\alpha) \\ 0 & -\cos & -\sin\alpha \\ 0 & \sin & -\cos\alpha \end{bmatrix} (A)^{-1} . \tag{14}$$

The matrices ϕ^b_{ab} and ϕ^b_{ba} can be obtained in the same fashion.

Stiffness Matrix of an Arc-Segment in Case of Lateral Deformation.

In this case, the properties of segment AB can be described by the partioned matrix equation,

$$\begin{Bmatrix} F_a \\ \cdots \\ F_b \end{Bmatrix} = \left[\begin{array}{c|c} \phi^a_{ab} & \phi^b_{ab} \\ \hline \phi^a_{ba} & \phi^b_{ba} \end{array}\right] \begin{Bmatrix} D_a \\ \cdots \\ D_b \end{Bmatrix} , \tag{15}$$

where

$$F_a = \begin{Bmatrix} F_{za} \\ C_{ra} \\ C_{ta} \end{Bmatrix}, \qquad D_a = \begin{Bmatrix} D_{za} \\ R_{ra} \\ R_{ta} \end{Bmatrix} ;$$

$$F_b = \begin{Bmatrix} F_{zb} \\ C_{rb} \\ C_{tb} \end{Bmatrix}, \quad \text{and } D_b = \begin{Bmatrix} D_{zb} \\ R_{rb} \\ R_{tb} \end{Bmatrix} .$$

Then the ϕ_s' in the stiffness matrix can be obtained as follows:

Consider the circular segment shown in Fig. 9. With end B considered as fixed, the forces on section θ can be expressed as

$$\begin{cases} M_r(\theta) = F_{za} r\sin\theta + C_{ra}\cos\theta - C_{ta}\sin\theta \\ M_t(\theta) = F_{za} r(1 - \cos\theta) + C_{ra}\sin\theta + C_{ta}\cos\theta \\ V_z(\theta) = F_{za} \\ N(\theta) = 0 \end{cases} \tag{16}$$

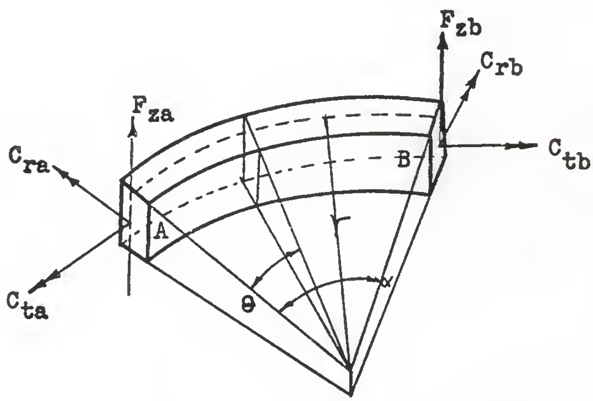


Fig. 9. Circular segment of lateral deformation.

The total strain energy stored in the body can be calculated as

$$U = \int_0^\alpha \left[\frac{M_r^2(\theta)}{2EI_r} + \frac{M_t^2(\theta)}{2GJ} + \frac{K_3 V^2(\theta)}{2GA} \right] r d\theta$$

$$= \frac{r}{EI} \int_0^\alpha \left[\frac{M_r^2(\theta)}{2} + \frac{K_1}{2} M_t^2(\theta) + \frac{K_2}{2} V^2(\theta) \right] d\theta \ , \tag{17}$$

where

$$K_1 = \frac{EI_r}{GJ} \ ; \text{ and } K_2 = \frac{K_z EI_r}{GA} \ . \tag{18}$$

EI_r, GJ and K/GA are the flexile rigidity, torsional rigidity, and shearing rigidity of the ring, respectively. The deformation of the free end can be calculated by using Castigliano's theorem as

$$\begin{Bmatrix} D_{za} \\ R_{ra} \\ R_{ta} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial U}{\partial F_{za}} \\ \frac{\partial U}{\partial C_{ra}} \\ \frac{\partial U}{\partial C_{ta}} \end{Bmatrix} = \begin{pmatrix} \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} \begin{Bmatrix} F_{za} \\ C_{ra} \\ C_{ta} \end{Bmatrix} \ , \tag{19}$$

or

$$\begin{Bmatrix} D_{za} \\ R_{ra} \\ R_{ta} \end{Bmatrix} = (\eta_2) \begin{Bmatrix} F_{za} \\ C_{ra} \\ C_{ta} \end{Bmatrix} \ , \tag{20}$$

where

$$
\left\{
\begin{aligned}
\eta_{11} &= \frac{r^3}{4EI}\left[\, 2\alpha - \sin 2\alpha + K_1(6\alpha + \sin 2\alpha + 4\sin^2\alpha\,) + \frac{K_2\alpha}{r^2}\right] , \\
\eta_{12} &= \frac{r^3}{4EI}\left[\frac{2\sin^2\alpha}{r} - \frac{2K_1}{r}(2 - 2\cos\alpha + \sin^2\alpha\,)\right] \quad (=\eta_{21}) , \\
\eta_{13} &= \frac{r^3}{4EI}\left[-\frac{2\alpha + \sin 2\alpha}{r} + \frac{K_1}{r}(4\sin\alpha - 2\alpha - \sin 2\alpha\,)\right] \quad (=\eta_{31}) , \\
\eta_{22} &= \frac{r^3}{4EI}\left[\frac{2\alpha + \sin 2\alpha}{r} + \frac{K_1}{r}(2\alpha - \sin 2\alpha\,)\right] , \\
\eta_{23} &= \frac{r^3}{4EI}\left[\frac{2(K_1 - 1)}{r}\sin^2\alpha\right] \quad (=\eta_{32}) , \\
\eta_{33} &= \frac{r^3}{4EI}\left[\frac{2\alpha - \sin 2\alpha}{r} + \frac{K_1}{r}(2\alpha + \sin 2\alpha\,)\right]
\end{aligned}
\right.
\tag{21}
$$

therefore,

$$
\phi^a_{ab} = (\eta_2)^{-1} . \tag{22}
$$

The reactions of end B can be calculated by substituting $\begin{Bmatrix} F_{za} \\ C_{ra} \\ C_{ta} \end{Bmatrix}$ into the stress function. Thus

$$
\phi^a_{ba} = -\begin{pmatrix} 1 & 0 & 0 \\ r\sin\alpha & \cos\alpha & -\sin\alpha \\ r(1 - \cos\alpha\,) & \sin\alpha & \cos\alpha \end{pmatrix} (\eta_2)^{-1} \tag{23}
$$

Stiffness Matrix of a Multiply Constrained Circular Member.

Figure 10 indicates a non-closed circular member with section 1, 2, ..., and m as constraint points.

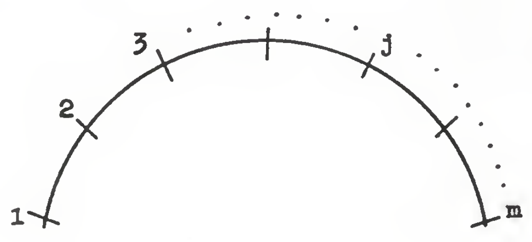


Fig. 10. Multiply constrained circular member.

The stiffness matrix of the circular member can be obtained by combining the stiffness matrices of the elements in the following fashion:

$$\Phi = \begin{bmatrix} \phi^1_{12} & \phi^2_{12} & \cdot & \cdot & \cdot & \cdot & \cdot \\ \phi^1_{21} & \phi^2_{21} + \phi^2_{23} & \phi^3_{23} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \phi^2_{32} & \phi^3_{32} + \phi^3_{34} & \phi^4_{34} & \cdot & \cdot & \cdot \\ \cdot & & \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \phi^{m-2}_{m-1,m-2} & \phi^{m-1}_{m-1,m-2} + \phi^{m-1}_{m-1,m} & \phi^m_{m-1,m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \phi^{m-1}_{m,m-1} & \phi^m_{m,m-1} \end{bmatrix} \qquad (24)$$

The matrix elements ϕ's in the stiffness matrix can represent either plane deformation or lateral deformation.

Stiffness Matrix of a Circular Ring.

Figure 11 indicates a closed circular ring with section 1, 2, ..., m as constraint points.

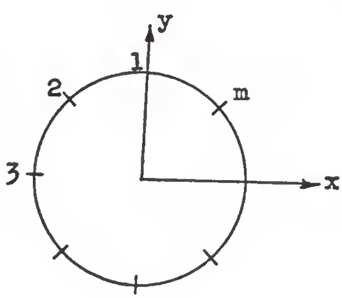


Fig. 11. Multiply constrained circular ring.

The stiffness matrix of a circular ring is different from the stiffness matrix of a circular member because a ring has the property that its initial point is also its terminal point.

$$\Phi = \begin{bmatrix} \phi^1_{12} & \phi^2_{12} & \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & & & \phi^m_{1,m} \\ \phi^1_{21} & \phi^2_{21} + \phi^2_{23} & \phi^3_{23} & \cdot \quad \cdot \quad \cdot & & \\ \vdots & \phi^2_{32} & \phi^3_{32} + \phi^3_{34} & \phi^4_{34} & \cdot \quad \cdot \quad \cdot & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & \\ \phi^1_{m1} & \cdot \quad \cdot \quad \cdot \quad \cdot & & \phi^{m-1}_{m,m-1} & \phi^m_{m,m-1} + \phi^m_{m1} & \end{bmatrix} \tag{25}$$

DEFORMATION OF SIMPLY CONNECTED GRIDWORKS ON ELASTIC FOUNDATIONS

Rectangular Gridwork

Consider a gridwork which consists of longitudinal beams and transverse beams as shown in Fig. 12. The longitudinal beams are supported by an elastic foundation at the grid points and the transverse beams are continuous across the longitudinal beams and intersect them at right angles. The connection between the two types of beams is assumed to transmit only tension or compression, and it is assumed that the gridwork is loaded by forces at the joints only.

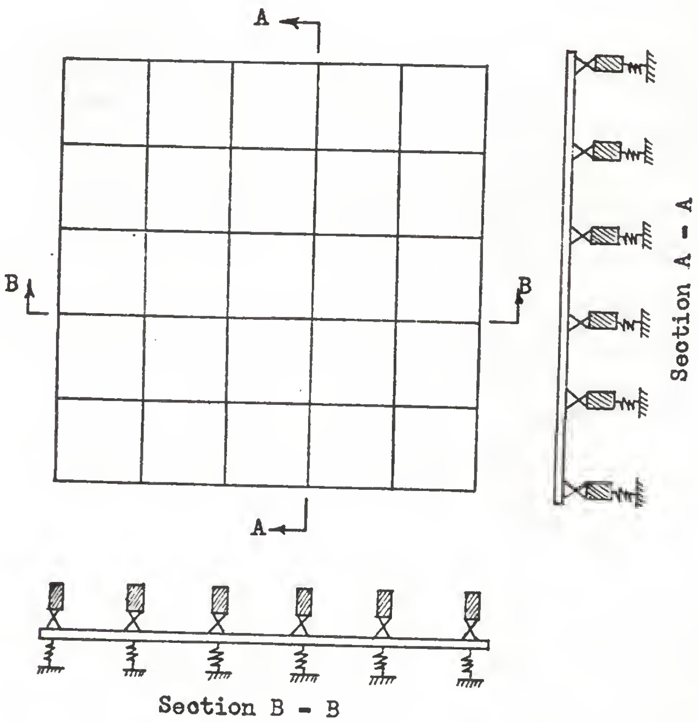


Fig. 12. Rectangular gridwork.

Transverse Beams.

$\{F_i\}$, $\{f_i\}$, $\{Z_i\}$, $\{\Phi_i\}$ are defined as:

$\{F_i\} = \lfloor F_{1i}, F_{2i}, \ldots\ldots, F_{mi} \rfloor^T$ denotes the external load vector which is applied to the i-th transverse beam (in x-direction).

$\{f_i\} = \lfloor f_{1i}, f_{2i}, \ldots\ldots, f_{mi} \rfloor^T$ denotes the vector reaction between the i-th transverse beam and the longitudinal beams.

$\{Z_i\} = \lfloor Z_{1i}, Z_{2i}, \ldots\ldots, Z_{mi} \rfloor^T$ denotes the displacement vector of the i-th transverse beam.

(Φ_i) = The stiffness matrix of the i-th transverse beam and is given by Eq. (2) in a previous chapter.

The components of $\{F_i\}$, $\{f_i\}$ and $\{Z_i\}$ are shown in Fig. 13.

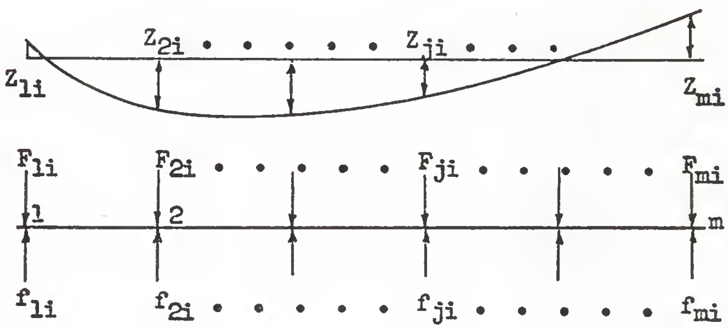


Fig. 13. The i-th transverse beam.

The i-th transverse beam can be characterized by the matrix equation

$$\{F_i\} - \{f_i\} = (\Phi_i)\{Z_i\} \quad . \tag{26}$$

Thus, the set of transverse beams can be described by the following partitioned matrix equation:

$$\begin{Bmatrix} F_1 \\ \hdashline F_2 \\ \hdashline \cdot \\ \cdot \\ \cdot \\ \hdashline F_n \end{Bmatrix} - \begin{Bmatrix} f_1 \\ \hdashline f_2 \\ \hdashline \cdot \\ \cdot \\ \cdot \\ \hdashline f_n \end{Bmatrix} = \begin{bmatrix} \Phi_1 & & & \\ & \Phi_2 & & \\ & & \ddots & \\ & & & \Phi_n \end{bmatrix} \begin{Bmatrix} Z_1 \\ \hdashline Z_2 \\ \hdashline \cdot \\ \cdot \\ \cdot \\ \hdashline Z_n \end{Bmatrix} \tag{27}$$

If all transverse beams are the same, i.e.,

$$\Phi_i \equiv \Phi$$

for $i = 1, 2, \ldots, n$. Then Eq. (27) can be rewritten as

$$(F_1 \,\vdots\, F_2 \,\vdots\, \cdots \,\vdots\, F_n) - (f_1 \,\vdots\, f_2 \,\vdots\, \cdots \,\vdots\, f_n) = (\Phi)(Z_1 \,\vdots\, Z_2 \,\vdots\, \cdots \,\vdots\, Z_n) \; . \tag{28'}$$

The subscripts in Eqs. (26) to (28) refer to the transverse beam numbers. For compactness, Eq. (28') can be rewritten as

$$(F) - (f) = (\Phi)(Z), \tag{28}$$

where

$$(F) = (F_1 \,\vdots\, F_2 \,\vdots\, \cdots \,\vdots\, F_n) \; ,$$

$$(f) = (f_1 \,\vdots\, f_2 \,\vdots\, \cdots \,\vdots\, f_n) \; ,$$

and

$$(Z) = (Z_1 \,\vdots\, Z_2 \,\vdots\, \cdots \,\vdots\, Z_n) \; .$$

Longitudinal Beams.

$\{f'_j\}$, $\{z_j\}$, $\{\phi_j\}$, $\{c_j\}$, $\{g_j\}$, are defined as:

$\{f'_j\} = \lfloor f'_{1j}, f'_{2j}, \ldots\ldots, f'_{nj} \rfloor^T$ denotes the vector reaction between the j-th longitudinal beam and the transverse beams.

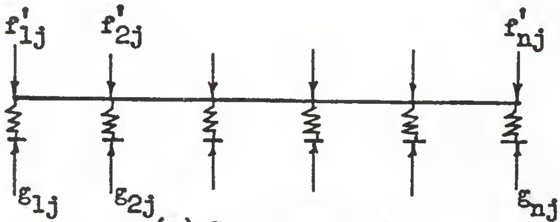
$\{z_j\} = \lfloor z_{1j}, z_{2j}, \ldots, z_{nj} \rfloor^T$ denotes the displacement vector of the j-th longitudinal beam.

(ϕ_j) = The stiffness matrix of the j-th longitudinal beam. It is given by Eq. (2).

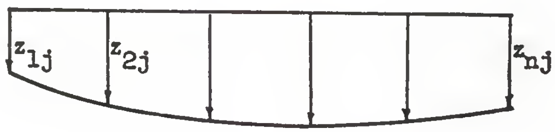
$(c_j) = \begin{pmatrix} c_{1j} & 0 & \cdots & 0 \\ 0 & c_{2j} & \ddots & \\ 0 & \cdots & \cdots & c_{nj} \end{pmatrix}$ is the stiffness matrix for the elastic foundations under the j-th longitudinal beam. It relates forces and deformations by

$\{g_j\} = (c_j)\{z_j\} = \lfloor g_{1j}, g_{2j}, \ldots, g_{nj} \rfloor^T$, the vector reaction of the elastic foundations.

The components of vectors such as $\{f'_j\}$, $\{z_j\}$, and $\{g_j\}$ are shown in Fig. 14.



(a) Force vectors



(b) Displacement vector

Fig. 14. The j-th longitudinal beam.

Similarly, the j-th longitudinal beam can be described by

$$\{f'_j\} - \{g_j\} = (\phi_j)\{z_j\} \ ,$$

or

$$\{f'_j\} = (c_j + \phi_j)\{z_j\} \, , \tag{29}$$

since

$$\{g_j\} = (c_j)\{z_j\} \, .$$

From Eq. (29), the set of longitudinal beams can be described as the partitioned matrix equation

$$\begin{Bmatrix} f'_1 \\ f'_2 \\ \cdot \\ \cdot \\ \cdot \\ f'_m \end{Bmatrix} = \left[\begin{array}{c|c|c} c_1 + \phi_1 & \cdot \quad \cdot \quad \cdot \quad \cdot & \cdot \\ \hline \cdot \quad \cdot & c_2 + \phi_2 & \cdot \quad \cdot \quad \cdot \\ \hline \cdot \quad \cdot & \ddots & \cdot \\ \cdot \quad \cdot & & \cdot \\ \cdot \quad \cdot & & \cdot \\ \hline \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & & c_m + \phi_m \end{array}\right] \begin{Bmatrix} z_1 \\ z_2 \\ \cdot \\ \cdot \\ \cdot \\ z_m \end{Bmatrix} , \tag{30}$$

where the subscripts refer to the longitudinal beam numbers.

If all the longitudinal beams have the same properties and all the elastic foundations are equal in strength, then Eq. (30) can be rewritten as

$$(f'_1 \mid f'_2 \mid \cdots \mid f'_m) = (CI + \phi)(z_1 \mid z_2 \mid \cdots \mid z_m) \, , \tag{31'}$$

since $C_j = CI$, and $\phi_j = \phi$ for $j = 1, 2, \ldots, m$. For compactness, Eq. (31') can be rewritten as

$$(f') = (CI + \phi)(z) \, , \tag{31}$$

where

$$(f') = (f'_1 \mid f'_2 \mid \cdots \mid f'_m) \, ,$$

$$(z) = (z_1 \mid z_2 \mid \cdots \mid z_m) \, ,$$

and

$$(I) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = (\overline{n} \times n) \, .$$

Matrix Equation of the Gridwork.

The most general case of the gridwork considered here is that

(a) The stiffnesses of the transverse beams are different;

(b) The stiffnesses of the longitudinal beams are different; and

(c) The strengths of the elastic foundations are different.

For convenience, let

$$\{F\} = \begin{Bmatrix} F_1 \\ \hdashline F_2 \\ \hdashline \cdot \\ \cdot \\ \cdot \\ \hdashline F_n \end{Bmatrix} = (\overline{n.m} \times 1) \quad , \qquad \{Z\} = \begin{Bmatrix} Z_1 \\ \hdashline Z_2 \\ \hdashline \cdot \\ \cdot \\ \cdot \\ \hdashline Z_n \end{Bmatrix} = (\overline{n.m} \times 1) \quad ,$$

$$\{f\} = \begin{Bmatrix} f_1 \\ \hdashline f_2 \\ \hdashline \cdot \\ \cdot \\ \cdot \\ \hdashline f_n \end{Bmatrix} = (\overline{n.m} \times 1) \quad ; \qquad \{f'\} = \begin{Bmatrix} f'_1 \\ \hdashline f'_2 \\ \hdashline \cdot \\ \cdot \\ \cdot \\ \hdashline f'_m \end{Bmatrix} = (\overline{n.m} \times 1) \quad ,$$

$$\{z\} = \begin{Bmatrix} z_1 \\ \hdashline z_2 \\ \hdashline \cdot \\ \cdot \\ \cdot \\ \hdashline z_m \end{Bmatrix} \quad ,$$

$$[\Phi] = \begin{pmatrix} \Phi_1 & & & & \\ & \Phi_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \Phi_n \end{pmatrix} = (\overline{m.n} \times m.n) \quad ,$$

$$[\phi] = \begin{pmatrix} \phi_1 & & & \\ & \phi_2 & & \\ & & \ddots & \\ & & & \phi_m \end{pmatrix} = (\overline{n \cdot m} \times n \cdot m) ,$$

and

$$[C] = \begin{pmatrix} C_1 & & & \\ & C_2 & & \\ & & \ddots & \\ & & & C_m \end{pmatrix} = (\overline{n \cdot m} \times n \cdot m) .$$

Equations (27) and (30) can be written in the compact forms as

$$\{F\} - \{f\} = [\Phi]\{Z\} \tag{27'}$$

and

$$\{f'\} = ([\phi] + [C])\{Z\} \tag{30'}$$

Since the transverse and the longitudinal beams are joined at the grid points, the deformation is the same at these points.

The rearrangement matrix (R) may be defined as

$$(R) = \sum_{k=1}^{n} \sum_{i=1}^{m} (E_{ki,\ ((i-1)n+k)}) \tag{32}$$

where

$$(E_{i,j}) = \begin{array}{cl} \begin{pmatrix} 0 & 0 \cdots 0 & 0 \cdots 0 \cdots 0 \\ 0 & 0 \cdots 0 & 0 \cdots 0 \cdots 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 \cdots 0 & 1 \cdots 0 \cdots 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 \quad 0 & 0 \quad 0 \quad 0 \end{pmatrix} & \leftarrow i^{th} \text{ row} \\ \uparrow j^{th} \text{ column} & \end{array} \tag{33}$$

Then $\{z\}$ and $\{f'\}$ can be rearranged as

$$(R)\{z\} = \{Z\} \tag{34}$$

and $$(R)\{f'\} = \{f\} \tag{35}$$

Substituting Eqs. (34) and (35) into Eqs. (27') and (30') and eliminating $\{f\}$ yields

$$\{F\} = [(R)([\phi] + [C]) + [\Phi](R)]\{z\} \quad . \tag{36}$$

with $$(\eta) = (R)([\phi] + [C]) + [\Phi](R) \tag{37}$$

The deformation of the gridwork can now be expressed as

$$\{Z\} = (R)(\eta)^{-1}\{F\} \tag{38}$$

In the special case where all transvorse beams of the gridwork are alike, all longitudinal beams are alike, and where the elastic foundations are equal in strength, the transverse beams and longitudinal beams can be described by Eqs. (28) and (31), respectively.

$$(F) - (f) = (\Phi)(Z) \tag{28}$$

$$(f') = (CI + \phi)(z) \tag{31}$$

From Fig. 12 it is seen that the station number of the transverse beam is the same as the longitudinal beam number.

Thus

$$(f')^T = (f) \tag{39}$$

$$(z)^T = (Z) \tag{40}$$

Transposing Eq. (31) yields

$$(f) = (Z)(CI + \phi)^T \tag{41}$$

Substitution of this equation into Eq. (28) yields

$$(F) = (\Phi)(Z) + (Z)(CI + \phi)^T \tag{42}$$

The matrix equation can be solved[1] as follows: Since the stiffness matrices Φ and ϕ are positive definite matrices, they can always be reduced to diagonal form by

$$(\Phi) = (\xi)(\Lambda_1)(\xi)^{-1} = (\xi)(\Lambda_1)(\xi)^T \tag{43}$$

and

$$(CI + \phi)^T = (\eta)(\Lambda_2)(\eta)^{-1} = (\eta)(\Lambda_2)(\eta)^T \ , \tag{44}$$

where (ξ) and (Λ_1) are the eigenvector and eigenvalue matrices for Φ, and (η) and (Λ_2) are the eigenvector and eigenvalue matrices for $(CI + \phi)^T$, respectively. Substituting Eqs. (43) and (44) into Eq. (42) yields

$$(F) = (\xi)(\Lambda_1)(\xi)^T(Z) + (Z)(\eta)(\Lambda_2)(\eta)^T \ . \tag{45}$$

Pre-multiplication by $(\xi)^T$ and post-multiplication by (η) converts Eq. (45) to

$$(\xi)^T(F)(\eta) = (\Lambda_1)(\xi)^T(Z)(\eta) + (\xi)^T(Z)(\eta)(\Lambda_2) \ . \tag{46}$$

Let

$$(X) = (\xi)^T(Z)(\eta) \ .$$

Then Eq. (46) becomes

$$(\Lambda_1)(X) + (X)(\Lambda_2) = (\xi)^T(F)(\eta) \ . \tag{47}$$

Element by element, this equation is

$$\lambda_{1i}X_{ij} + X_{ij}\lambda_{2j} = (\xi^T F \eta)_{ij} \ , \tag{48}$$

from which

$$(X) = \frac{(\xi^T F \eta)_{ij}}{\lambda_{1i} + \lambda_{2j}} \ . \tag{49}$$

Thus,

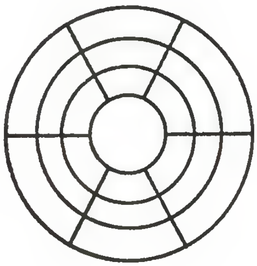
$$(Z) = (\xi) \ \frac{(\xi^T F \eta)_{ij}}{\lambda_{1i} + \lambda_{2j}} (\eta)^T \ . \tag{50}$$

---

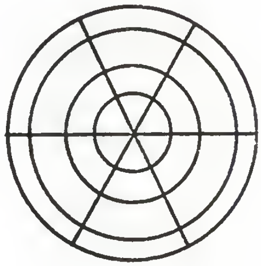
[1]The method given here is a digest from pp. 28-2 in "Notes on a Course of Lectures on Applied Linear Analysis," given by Philip G. Kirmser.

Circular Gridwork

Consider the gridwork as consisting of concentric circular rings and radial beams on an elastic foundation, as shown in Fig. 15. It is assumed that the connections between the radial and circular beams can transmit only tension or compression, and that the gridwork is loaded by vertical forces at the joints only.



(a) Circular gridwork without singular point.



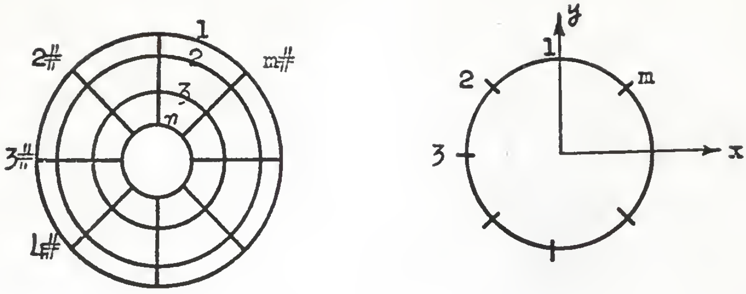
(b) Circular gridwork including singular point.

Fig. 15.

The gridwork shown in Fig. 15(a) has no point of singularity, i.e., the members of the gridwork intersect at right angles. The gridwork shown in Fig. 15(b) has a singular point at the center where all of the radial beams intersect. These two gridworks are different and must be treated separately.

Gridwork Without Singular Point.

Figure 16 indicates a circular gridwork without singluar point and thus orthogonal everywhere. Figure 16(b) is the i-th ring.



(A) Orthogonal circular gridwork (b) i-th ring

Fig. 16. Gridwork without singular point.

Let

$\{Z_i\} = \lfloor Z_{1i}, Z_{2i}, \ldots\ldots, Z_{mi} \rfloor^T$ denote the displacement vector of the i-th ring;

$\{F_i\} = \lfloor F_{1i}, F_{2i}, \ldots\ldots, F_{mi} \rfloor^T$ denote the external load vector of the i-th ring;

$\{f_i\} = \lfloor f_{1i}, f_{2i}, \ldots\ldots, f_{mi} \rfloor^T$ denote the reaction vector between the i-th ring and the radial beams;

$\{z_j\} = \lfloor z_{1j}, z_{2j}, \ldots\ldots, z_{nj} \rfloor$ denote the displacement vector of the j-th radial beam;

$\{f'_j\} = \lfloor f'_{1j}, f'_{2j}, \ldots\ldots, f'_{nj} \rfloor$ denote the vector reaction between the j-th radial beam and the rings:

$$(C_j) = \begin{pmatrix} C_{1j} & & & \\ & C_{2j} & & \\ & & \ddots & \\ & & & C_{nj} \end{pmatrix}$$ denote the stiffness matrix which characterizes the elastic supports under the j-th radial beam; and

$\{g_i\} = (C_j)\{z_j\}$ denote the vector reaction of the elastic supports under the j-th radial beam.

The i-th ring is characterized by

$$\{F_i\} - \{f_i\} = (\Phi_i)\{Z_i\} \quad , \tag{51}$$

where Φ_i is the stiffness matrix of the i-th ring as treated in Section 1 of the previous chapter. The j-th radial beam is characterized by

$$\{f'_j\} - \{g_j\} = (\phi_j)\{z_j\} \quad , \tag{52}$$

or

$$\{f'_j\} = (C_j + \phi_j)\{z_j\} \quad . \tag{53}$$

Figure 15 and Eqs. (51) and (53) show that

(a) The station number of the ring is the same as the radial beam number and the station number of the radial beam is the same as the ring number.

(b) The form of Eqs. (51) and (53) is the same as that of Eqs. (26) and (29), respectively. This means that the rings and their radial beams are analogous to the transverse and longitudinal beams of the rectangular gridwork.

Therefore, the rings can be characterized by

$$\{F\} - \{f\} = [\Phi]\{Z\} \quad , \tag{54}$$

and the radial beams by

$$\{f'\} = ([\phi] + [C])\{z\} \quad . \tag{55}$$

Equations (54) and (55) are analogous to Eqs.(27') and (30'), which have been defined previously.

The deformation of the circular gridwork can be represented in form similar to that of Eq. (38), i.e.,

$$\{Z\} = (R)(\eta)^{-1}\{F\} \quad , \tag{56}$$

where

$$(\eta) = (R)([\phi] + [C]) + [\Phi](R) \ ,$$

and

$$(R) = \sum_{k=1}^{n} \sum_{i=1}^{m} (E_{ki,((i-1)n+k)})$$

are defined by Eqs. (37) and (32), respectively.

Gridwork With Singular Point.

If the radial beams of the gridwork are alike and hinge connected at the origin o, the two kinds of beams can be represented separately as rings and radial beams.

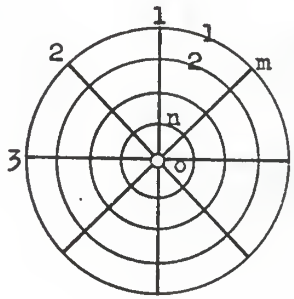


Fig. 17. Gridwork with Singular point at origin.

For the separate cases of:

(a) Rings:

Let

$(F) = (F_1 \vdots F_2 \vdots \cdots \vdots F_n)$ be the external load matrix;

$(f) = (f_1 \vdots f_2 \vdots \cdots \vdots f_n)$ be the inter-reaction matrix;

$(H) = (F) - (f)$ be the load matrix which is carried by the ring;

$(Z) = (Z_1 \vdots Z_2 \vdots \cdots \vdots Z_n)$ be the displacement matrix; and

Φ_i be the stiffness matrix of the i-th ring as discussed in section (2). Where the subscripts $i = 1, 2, \ldots, n$ denote the ring number. Then the system of rings can be described as

$$(H) = (F) - (f) = \sum_{i=1}^{n} \Phi_i (Z)(E_{ii}) , \qquad (57')$$

where E is defined by Eq. (33).

(b) Radial Beams:

Let

$(f') = (f'_1 \mid f'_2 \mid \cdots \mid f'_m)$ be the reaction matrix between radial beams and the ring;

$(z) = (z_1 \mid z_2 \mid \cdots \mid z_n)$ be the displacement matrix of the radial beams;

(g') be the reaction matrix corresponding to radial beams; and

$(h) = (f') - (g')$ be the matrix which represents the loads carried by the radial beams.

Then from Eq. (31), the radial beams can be described by

$$(h) = (f') - (g') = (\phi)(z) \quad . \tag{58}$$

After partitioning, Eq. (58) becomes

$$\begin{pmatrix} [h_1] \\ \cdots \\ \lfloor h_o \rfloor \end{pmatrix} = \left(\begin{array}{c|c} \phi_{\overline{n},n} & \phi_{\overline{n}1} \\ \hline \phi_{\overline{1},n} & \phi_{\overline{1}1} \end{array}\right) \begin{pmatrix} [z_1] \\ \cdots \\ \lfloor z_o \rfloor \end{pmatrix} \quad ,$$

or

$$[h_1] = (\phi_{\overline{n}n})[z_1] + (\phi_{\overline{n}1})\lfloor z_o \rfloor \tag{59}$$

$$\lfloor h_o \rfloor = \lfloor \phi_{\overline{1}n} \rfloor [z_1] + (\phi_{\overline{1}1})\lfloor z_o \rfloor \tag{60}$$

Let

c_o be the stiffness matrix of the elastic foundation of the origin,

d_o be the displacement of the origin,

g_o be the reaction of the elastic foundation at origin; and

F_o be the external load applied at the origin.

Since all radial beams are connected at the origin

$$\lfloor z_o \rfloor = d \lfloor 1, 1, \ldots, 1 \rfloor \quad .$$

Application of the equilibrium conditions at the origin yields

$$F_o = (\phi_{\overline{1n}})\lceil z_1 \rceil \begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix} + (m\phi_{\overline{11}} + C)d_o \ .$$

or

$$F_o = (\phi_{\overline{1n}})\lceil z_1 \rceil \begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix} + k_o d_o \ , \tag{61}$$

where

$$K_o = m\phi_{\overline{11}} + C \ .$$

Thus,

$$d_o = \frac{F_o}{k_o} - \frac{1}{K_o}(\phi_{\overline{1n}})\lceil z_1 \rceil \begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix} \ , \tag{62}$$

and

$$\lfloor z_o \rfloor = \frac{F_o}{K_o}\lfloor 1, 1, \ldots, 1 \rfloor - \frac{1}{K_o}(\phi_{\overline{1n}})\lceil z_1 \rceil \ . \tag{63}$$

Substituting Eq. (63) into Eq. (59) yields

$$\lceil h_1 \rceil = (\phi_{\overline{nn}}) - \frac{1}{K_o}(\phi_{\overline{n1}})(\phi_{\overline{1n}})\lceil z_1 \rceil + \frac{F_o}{K_o}(\phi_{\overline{n1}})\lfloor 1, 1, \ldots, 1 \rfloor \tag{64}$$

Now

$$\lceil z_1 \rceil^T = (Z) \ , \tag{a}$$

$$\lceil h_1 \rceil^T = (f) - C(Z) \ , \tag{b}$$

and

$$(H) = (F) - (f) \ , \tag{c}$$

which yields

$$(F) = (H) + [h_1]^T + c(Z) \quad . \tag{d}$$

By using the results of Eqs. (57) and (64), the matrix equation for the gridwork becomes

$$\overset{(\bar{m},n)}{(F)} - \frac{F_o}{K_o} \overset{(\bar{m},1)}{\begin{Bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{Bmatrix}} \overset{(\bar{1},n)}{(\phi_{\bar{n},1})^T} = \sum_{i=1}^{n} \overset{(\bar{m},m)}{(\Phi_i)} \overset{(\bar{m},n)}{(\ Z\ )} \overset{(\bar{n},n)}{(E_{ii})} + \overset{(\bar{m},n)}{(\ Z\ )} \overset{(\bar{n},n)}{(\phi_{\overline{nn}})^T} - \frac{1}{K_o} \overset{(\bar{n},1)}{(\phi_{\bar{1},n})^T} \overset{(\bar{1},n)}{(\phi_{\bar{n},1})^T} \quad . \tag{65}$$

DEFORMATION OF GRIDWORKS CONSIDERING TORSION

Rectangular Gridwork

In a gridwork as shown in Fig. 18, the beams are interconnected by rigid joints which can transmit torsional moments and which are loaded at joints only.

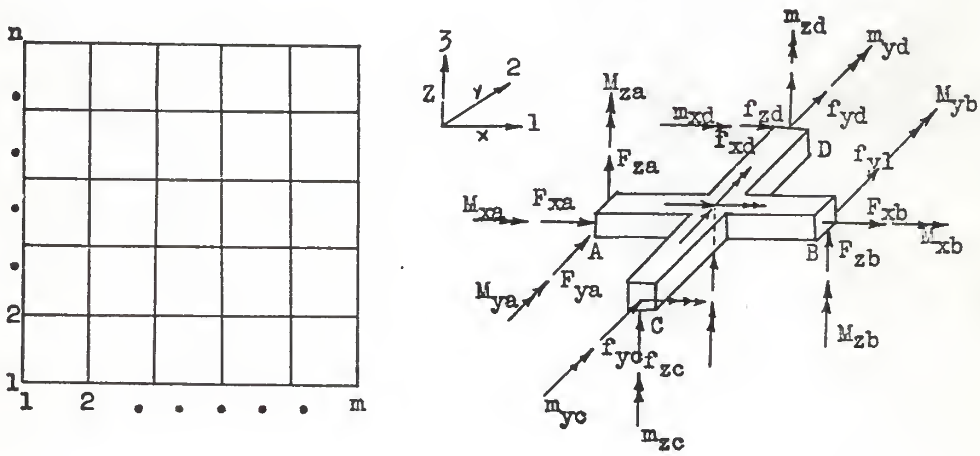


Fig. 18. Rigidly-connected gridwork.

The notation used in the analyses of such gridworks is given below.

$\{F_{ij}\} = \lfloor F_{ij}^{(1)}, F_{ij}^{(2)}, F_{ij}^{(3)} \rfloor^T$ denotes the force vector carried by the j-th transverse beam at the i-th station. The superscripts (1), (2) and (3) stand for the directions of the vector components.

$\{C_{ij}\} = \lfloor C_{ij}^{(1)}, C_{ij}^{(2)}, C_{ij}^{(3)} \rfloor^T$ denotes the couple vector carried by the j-th transverse beam at the i-th station. The superscripts (1), (2) and (3) stand for the directions of the vector components. The sign of a couple is determined by the right-hand rule.

$\{D_{ij}\} = \lfloor D_{ij}^{(1)}, D_{ij}^{(2)}, D_{ij}^{(3)} \rfloor^T$ is the linear displacement of the i-th station of the j-th transverse beam.

$\{R_{ij}\} = \lfloor D_{ij}^{(1)}, D_{ij}^{(2)}, D_{ij}^{(3)} \rfloor^T$ is the angular displacement of the i-th station of the j-th transverse beam.

$$\{F_j\} = \begin{Bmatrix} F_{1j} \\ \hdashline C_{1j} \\ \hdashline F_{2j} \\ \hdashline C_{2j} \\ \hdashline \vdots \\ \hdashline F_{mj} \\ \hdashline C_{mj} \end{Bmatrix} \quad ; \quad \{D_j\} = \begin{Bmatrix} D_{1j} \\ \hdashline R_{1j} \\ \hdashline D_{2j} \\ \hdashline R_{2j} \\ \hdashline \vdots \\ \hdashline D_{mj} \\ \hdashline R_{mj} \end{Bmatrix}$$

$\{F_j\}$ is the force vector whose components F_{ij} and C_{ij} are the forces and couples, respectively, which are carried by the j-th transverse beam.

$\{D_j\}$ is the corresponding displacement vector whose elements D_{ij} and R_{ij} are the displacements and rotations which occur in the j-th transverse beam.

$(F) = (F_1 \vdots F_2 \vdots \ldots\ldots \vdots F_n)$ is the total force matrix whose columns are the force matrices for the individual transverse beams.

$(D) = (D_1 \vdots D_2 \vdots \ldots\ldots \vdots D_n)$ is the total displacement matrix whose columns are the displacement matrices for the individual transverse beams.

The longitudinal beams are described similarly, except that the vector forces, couples, displacements and rotations are denoted by f, c, d and r in place of the corresponding F, C, D and R. The forces and couples that are carried by the i-th longitudinal beam, and the displacement of this beam, are described as $\{f_i\}$ and $\{d_i\}$, respectively. Similarly, the total forces and couples that are carried by the longitudinal beams of the gridwork and the corresponding displacements are expressed in matrix form as (f) and (d), respectively.

Gridwork of Uniform Transverse Beams and Uniform Longitudinal Beams.

In this case, each beam is characterized separately by the matrix equation

$$\{F_i\} = (\Phi)\{D_i\} \tag{66}$$

for the i-th transverse beam and by

$$\{f_j\} = \phi\{d_j\} \tag{67}$$

for the j-th longitudinal beam. Here Φ and ϕ are the stiffness matrix of the individual beams as given by Eq. (8).

If all transverse beams are alike and all longitudinal beams are alike, the individual systems of these beams are characterized by the two matrix equations

$$(F) = (\Phi)(D) \tag{68}$$

and

$$(f) = (\phi)(d) \ . \tag{69}$$

Since the longitudinal beams are rigidly connected to the transverse beams at the grid points or joints, they undergo common displacements at a joint, i.e., if the transverse and longitudinal beams are joined, (d) and (D) are different representations of the same deflections. (d) can be rearranged to form (D) by the equation

$$\sum_{j=1}^{n} \sum_{i=1}^{m} \overset{(\overline{6m},6n)}{(H_{ij})} \overset{(\overline{6n},m)}{(d)} \overset{(\overline{m},n)}{(E_{ij})} = \overset{(\overline{6m},n)}{(D)} \ , \tag{70}$$

where n is the total number of transverse beams, m is the total number of longitudinal beams, and

$$
\begin{array}{c} (6m,6n) \\ (H_{ij}) \end{array} =
\begin{array}{cl}
\begin{array}{cccccc} & & & j^{\text{th}}\text{ column (matrix element)} & & \end{array} & \\
\left[\begin{array}{cccccc}
0 & 0 & \cdots & \cdot & \cdots & 0\\
0 & 0 & \cdots & \cdot & \cdots & 0\\
\vdots & \vdots & & \vdots & & \vdots\\
\vdots & \vdots & \cdots & \boxed{I} & \cdots & \vdots\\
\vdots & \vdots & & \vdots & & \vdots\\
0 & 0 & \cdots & \cdot & \cdots & 0
\end{array}\right] & \begin{array}{l} \\ \\ \\ \leftarrow i^{\text{th}}\text{ row (matrix element)} \\ \\ \\ \end{array}
\end{array}
\quad . \qquad (71)
$$

Equation (71) is a given matrix of matrix elements which are themselves $(6,6)$ matrices, and

$$
(E_{ij}) =
\begin{array}{cl}
\begin{array}{cccccc} & & & j^{\text{th}}\text{ column} & & \end{array} & \\
\left[\begin{array}{cccccc}
0 & 0 & \cdots & 0 & \cdots & 0\\
0 & 0 & \cdots & 0 & \cdots & 0\\
\vdots & \vdots & & \vdots & & \vdots\\
0 & 0 & \cdots & 1 & \cdots & 0\\
\vdots & \vdots & & \vdots & & \vdots\\
0 & 0 & \cdots & 0 & \cdots & 0
\end{array}\right] & \begin{array}{l} \\ \\ \\ \leftarrow i^{\text{th}}\text{ row} \\ \\ \\ \end{array}
\end{array}
\qquad (72)
$$

Let (P) be the external load matrix which has the same arrangement as the matrix (F) as defined at the beginning of this section. The static equilibrium equations are

$$(P) = (F) + \sum_{j=1}^{n} \sum_{i=1}^{m} (H_{ij})(f)(E_{ij}) \quad .$$

Substitution of Eq. (69) yields

$$(P) = (\Phi)(D) + \sum_{j=1}^{n} \sum_{i=1}^{m} (H_{ij})(\phi)(d)(E_{ij}) \quad , \qquad (73)$$

which, by using Eq. (70) becomes

$$(P) = (\Phi)\sum_{j=1}^{n}\sum_{i=1}^{m}\left[(H_{ij})(d)(E_{ij})\right] + \sum_{j=1}^{n}\sum_{i=1}^{m}\left[(H_{ij})(\phi)(d)(E_{ij})\right] \quad . \tag{74}$$

The effects of boundary conditions for the gridwork are included in the stiffness matrices Φ and ϕ.

Uniform Gridwork on Elastic Foundation

When the gridwork on an elastic foundation is of uniform strength, Eq. (73) can be modified to

$$(P) = (F) + \sum_{j=1}^{n}\sum_{i=1}^{m}\left[(H_{ij})(f)(E_{ij})\right] + C(D) \ ,$$

where C is the stiffness matrix for the elastic support. Thus,

$$(P) = (\Phi + CI)\sum_{j=1}^{n}\sum_{i=1}^{m}\left[(H_{ij})(d)(E_{ij})\right] + \sum_{j=1}^{n}\sum_{i=1}^{m}\left[(H_{ij})(\phi)(d)(E_{ij})\right] \tag{75}$$

where I is the $(\overline{6m}, 6m)$ unit matrix.

The General Gridwork.

For the general case where the transverse beams and longitudinal beams are not like themselves or each other, Eqs. (68) and (69) become

$$(F) = \sum_{i=1}^{n}(\Phi_i)(D)(E_{ii}) \tag{76}$$

and

$$(f) = \sum_{j=1}^{m}(\phi_j)(d)(E_{jj}) \quad , \tag{77}$$

where n is the number of transverse beams, m is the number of longitudinal beams, and E_{jj} is the matrix given by Eq. (72) .

Let (g) be the reaction matrix for the elastic supports arranged in the same order as for (P). Then

$$(g) = \sum_{i=1}^{n} (C_i)(D)(E_{ii}) \ . \tag{78}$$

From the static equilibrium conditions, it is seen that

$$(P) = (F) + (g) + \sum_{j=1}^{n} \sum_{i=1}^{m} (H_{ij})(f)(E_{ij}) \ .$$

Substituting Eqs. (76), (77), (78), and (70) yields

$$(P) = \sum_{k=1}^{n} (\Phi_k + C_k) \left\{ \sum_{j=1}^{n} \sum_{i=1}^{m} (H_{ij})(d)(E_{ij} \right\} (E_{kk})$$

$$+ \sum_{j=1}^{n} \sum_{i=1}^{m} (H_{ij}) \left[\sum_{l=1}^{m} (\phi_l)(d)(E_{ll}) \right] (E_{ij}) \tag{79}$$

as the basic equation relating forces and deformations.

Notes Concerning Gridworks.

From consideration of the basic formula for general gridworks, it is apparent that

(a) If a gridwork is loaded by forces which consist of P_z, M_x and M_y only, the deformation of the gridwork is perpendicular to the plane of the structure;

(b) If a gridwork is loaded by forces which consist of M_z, P_x and P_y only, the deformation of the gridwork is in the plane of the structure. Thus the vector loads $\lfloor P_z, M_x, M_y \rfloor$ and $\lfloor M_z, P_x, P_y \rfloor$ are independent, as shown in Fig. 19. From this it follows that

(c) An arbitrary vector load acting on a gridwork can always be decomposed into two kinds of loads such that for each load the deformation vector at every joint of the gridwork has only three components. The matrix equations in this case can be derived easily.

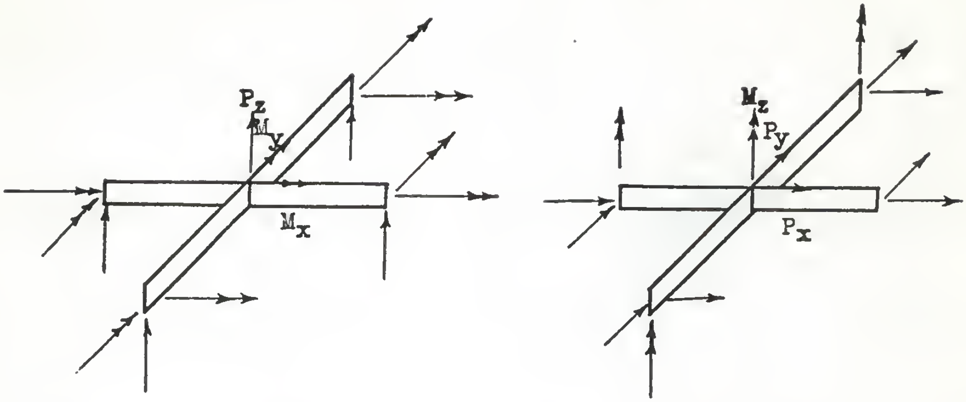
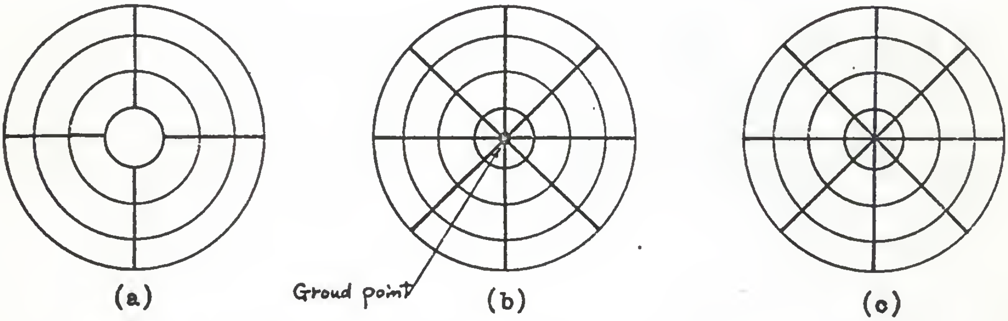


Fig. 19. Independent force systems in plane-gridwork.

Polar Gridworks

There are two kinds of polar gridworks as shown in Fig. 20. The first contains no singular point; the second contains one point of singularity, the origin of the polar coordinates.



(a) (b) (c)

Fig. 20. Circular gridworks.
(a) and (b) without singular point.
(c) with singular point.

In the first, since the gridwork does not contain the origin O, every joint of the gridwork is an orthogonal intersection. The analysis in this case is almost the same as for rectangular gridworks.

In the second case, all of the radial beams meet and are joined at the origin. The two sets of beams do not intersect orthogonally at this point, and special treatment is required.

Polar Gridworks Without Singular Point.

Let the rings and radial beams be treated as the transverse and longitudinal beams of the rectangular gridworks, respectively, and use the corresponding notations. Then, separately, the i-th ring and the j-th radial beam can be characterized by

$$\{F_i\} = (\Phi_i)\{D_i\} \tag{80}$$

and

$$\{f_j\} = \phi\{d_j\} \quad , \tag{81}$$

respectively, where the stiffness matrices Φ_i's are given by Eq. (25), and ϕ is given by Eq. (8). The force vectors $\{F_i\}$, $\{f_j\}$ and the displacement vectors $\{D_i\}$ and $\{d_j\}$ are indexed in the sense of cylindrical coordinates. Similarly, the rings and the radial beams of the gridwork can be characterized together by the equations

$$(F) = \sum_{i=1}^{n} (\Phi_i)(D)(E_{ii}) \tag{82}$$

and

$$(f) = (\phi)(d) \ . \tag{83}$$

Equations (82) and (83) are written directly as the analogs of Eqs. (76) and (69), respectively, with the radial beams assumed to be alike. Then, following the procedures used in the derivation of Eq. (79), the matrix equation for the polar gridwork becomes

$$(P) = \sum_{j=1}^{n} \sum_{i=1}^{m} (H_{ij})(\phi)(d)(E_{ii}) + \sum_{i=1}^{n} (\Phi_i)\left[\sum_{k=1}^{n} \sum_{j=1}^{m} (H_{jk})(d)(E_{jk})\right] (E_{ii}) \tag{84}$$

If the gridwork is supported elastically, this equation becomes

$$(P) = \sum_{j=1}^{n} \sum_{i=1}^{m} (H_{ij})(\phi)(d)(E_{ii})$$

$$+ \sum_{i=1}^{n} (\Phi_i + c_i) \left[\sum_{k=1}^{n} \sum_{j=1}^{m} (H_{jk})(d)(E_{jk}) \right] (E_{ii}) \ . \qquad (85)$$

Polar Gridworks With Singular Points.

The set of rings, shown in Fig. 21, is characterized by the matrix equation

$$(F) = \sum_{i=1}^{n} (\Phi_i)(D)(E_{ii}) \qquad (86)$$

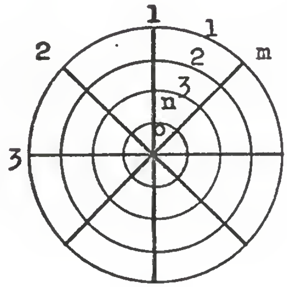


Fig. 21. Gridwork with a singular point.

Equation (86) is the same as Eq. (82). The set of radial beams is characterized by

$$(f) = (\phi)(d) \qquad (87)$$

The forces and deflections at the singular point require special treatment.

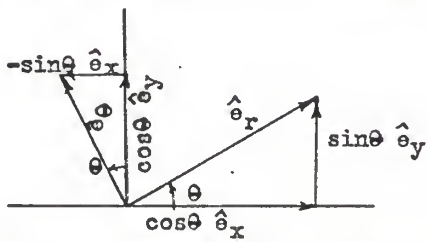


Fig. 22. Transformation of coordinates.

Let θ_j denote the direction angle of the j-th radial beam. Then the matrix

$$(\Xi)_j = \begin{pmatrix} \cos\theta_j & -\sin\theta_j & 0 \\ \sin\theta_j & \cos\theta_j & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{88}$$

transforms the displacement vector of the j-th radial beam at origin $\begin{Bmatrix} d_{oj} \\ r_{oj} \end{Bmatrix}$ to

$$\begin{Bmatrix} d_{oj} \\ \hline r_{oj} \end{Bmatrix}^{*} = \left(\begin{array}{c:c} (\Xi)_j & \\ \hdashline & (\Xi)_j \end{array}\right) \begin{Bmatrix} d_{oj} \\ \hline r_{oj} \end{Bmatrix} = \begin{Bmatrix} d_o \\ \hline r_o \end{Bmatrix}$$

The displacement vectors for all radial beams must yield the same Cartesian projection because they have a common displacement at the origin.

This equation is equivalent to

$$\begin{Bmatrix} d_{oj} \\ \hline r_{oj} \end{Bmatrix} = \left(\begin{array}{c:c} (\Xi)_j^T & 0 \\ \hdashline 0 & (\Xi)_j^T \end{array}\right) \begin{Bmatrix} d_o \\ \hline r_o \end{Bmatrix} .$$

Let

$$(T_j) = \left(\begin{array}{c:c} (\Xi)_j^T & \\ \hdashline & (\Xi)_j^T \end{array}\right) . \tag{89}$$

Then

$$\begin{Bmatrix} d_{oj} \\ \cdots \\ r_{oj} \end{Bmatrix} = (T_j) \begin{Bmatrix} d_o \\ \cdots \\ r_o \end{Bmatrix} . \tag{90}$$

Equation (87) can be rewritten in partitioned form as

$$\begin{pmatrix} f_{\overline{6n},m} \\ \hdashline f_{\overline{6},m} \end{pmatrix} = \left(\begin{array}{c:c} \phi_{\overline{6n},6n} & \phi_{\overline{6n},6} \\ \hdashline \phi_{\overline{6},6n} & \phi_{\overline{6},6} \end{array}\right) \begin{pmatrix} d_{\overline{6n},m} \\ \hdashline d_{\overline{6},m} \end{pmatrix} , \tag{91}$$

which, for compactness, is rewritten as

$$\begin{pmatrix} f_1 \\ \hdashline f_o \end{pmatrix} = \left(\begin{array}{c:c} A & B \\ \hdashline K & L \end{array}\right) \begin{pmatrix} d_1 \\ \hdashline d_o \end{pmatrix} \tag{91'}$$

Equation (91) implies that

$$(f_1) = (A)(d_1) + (B)(d_o) \tag{92}$$

and

$$(f_o) = (K)(d_1) + (L)(d_o) \quad , \tag{93}$$

where (f_o) is the matrix of forces carried by the radial beams at the origin 0; (d_o) is the matrix of displacements at the same point; and both (f_o) and (d_o) have components in sense of cylindrical coordinates. Let the transformations of (f_o) and (d_o) into Cartesian coordinates be denoted as $(f_o)^*$ and $(d_o)^*$, respectively. Then

$$(f_o)^* = \sum_{j=1}^{m} (T_j)(f_o)(E_{jj}) \tag{94}$$

Let $\{P_o\}^*$ be the external load which is applied at point 0, and which is represented in the sense of Cartesian coordinates. Then application of the static equilibrium condition at 0 gives

$$\{P_o\}^* = (f_o)^* \begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix} \quad ,$$

which, on substitution of Eqs. (93) and (94), becomes

$$\{P_o\}^* = \left[\sum_{j=1}^{m} (T_j)(K)(d_1)(E_{ii}) \right] \begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix} + \left[\sum_{j=1}^{m} (T_j)(L)(d_o)(E_{jj}) \right] \begin{Bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{Bmatrix} \quad . \tag{95}$$

By analogy to Eqs. (90) and (94), it is found that (d_o) can be represented as

$$(d_o) = \sum_{j=1}^{m} (T_j)^T \{d_o\}^* \lfloor 1, 1, \ldots, 1 \rfloor (E_{jj}) \quad , \tag{96}$$

where $\{d_o\}^*$ is the displacement vector at the origin 0 in the sense of rectangular coordinates. Thus,

$$\{P_o\}^* = \left[\sum_{j=1}^{M} (T_j)(K)(d_1)(E_{jj}) \right] \begin{Bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ 1 \end{Bmatrix}$$

$$+ \left[\sum_{j=1}^{m} (T_j)(L)(T_j)^T \right] \{d_o\}^* \lfloor 1, 1, \ldots, 1 \rfloor (E_{jj}) \begin{Bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{Bmatrix} \quad . \tag{97}$$

Let (P) be the external load matrix which is applied at all regular points other than origin, and which is arranged in the same order as (F). Then the condition of static equilibrium at each regular joint yields

$$(P) = (F) + \sum_{j=1}^{n} \sum_{i=1}^{m} (H_{ij})(f_1)(E_{ii}) \quad .$$

On substitution of Eqs. (70), (86), (92) and (96), this equation becomes

$$(P) = \sum_{i=1}^{n} (\Phi_i) \left[\sum_{k=1}^{n} \sum_{j=1}^{m} (H_{jk})(d_1)(E_{jj}) \right] (E_{ii})$$

$$+ \sum_{k=1}^{n} \sum_{j=1}^{m} (H_{jk})(A)(d_1)(E_{jj})$$

$$+ \sum_{k=1}^{n} \sum_{j=1}^{m} (H_{jk})(B)(T_j)^T \{d_o\}^* \lfloor 1, 1, \ldots, 1 \rfloor (E_{jj}) \tag{98}$$

To find the deformations of the gridwork, Eqs. (97) and (98) must be solved simultaneously. If the gridwork is supported by elastic foundations which are defined by the stiffness matrices (C_o), the stiffness matrix of the elastic foundation at the origin, and (C_i), the stiffness matrix of the elastic foundation under the i-th ring, (C_o) is $(6, 6)$ in size, and (C_i) is $(6m, 6m)$ in size, where m is the number of radial beams.

Matrix Eqs. (97) and (98) can be modified to yield

$$\{P_o\}^* = \left[\sum_{j=1}^{m}(T_j)(K)(d_1)(E_{ii})\right]\lfloor 1, 1, \ldots, 1\rfloor^T + (C_o)\{d_o\}^*$$

$$+ \left[\sum_{j=1}^{m}(T_j)(L)(T_j)^T\right]\{d_o\}^* \lfloor 1, 1, \ldots, 1\rfloor (E_{jj}) \lfloor 1, 1, \ldots, 1\rfloor^T \tag{99}$$

and

$$[P] = \sum_{i=1}^{n}(\Phi_i + C_i)\left[\sum_{k=1}^{n}\sum_{j=1}^{m}(H_{jk})(d_1)(E_{jj})\right](E_{ii})$$

$$+ \sum_{k=1}^{n}\sum_{j=1}^{m}(H_{jk})(B)(T_j)^T\{d_o\}^* \lfloor 1, 1, \ldots, 1\rfloor (E_{jj})$$

$$+ \sum_{k=1}^{n}\sum_{j=1}^{m}(H_{jk})(A)(d_1)(E_{jj}) \quad . \tag{100}$$

NUMERICAL EXAMPLES

Simply Connected Gridwork on an Elastic Foundation

Given Conditions.

The gridwork shown in Fig. 23 is a rectangular gridwork which is simply connected and supported by an elastic foundation at the grid points. It is loaded by a unit vertical load at point (2, 2).

In Fig. 23, the beams in x-direction are numbered 1-x, 2-x, 3-x, and 4-x; and the beams in y-direction are numbered 1-y, 2-y, ..., 6-y. The properties of the x-beams and y-beams and the stiffness of the supports are given in Table 1.

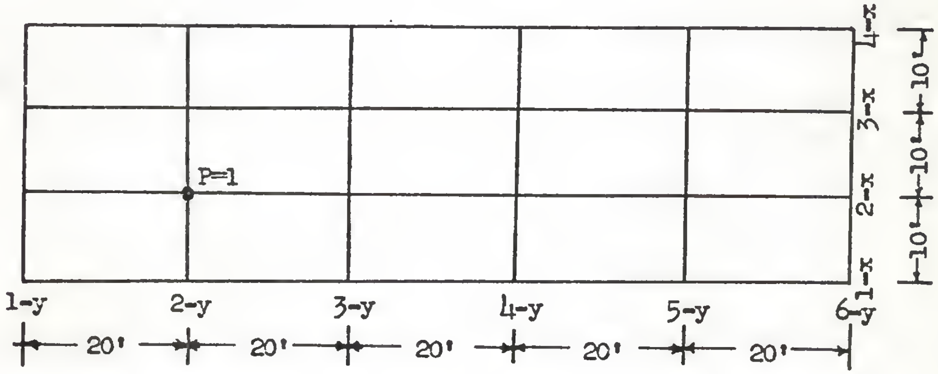


Fig. 23. Gridwork on elastic foundation.

Table 1. Physical constants for gridwork.

X-beam	Y-beam	Foundation
$GJ_x = 2EI$	$EI_x = 2EI$	
$EI_y = 8EI$	$GJ_y = \frac{1}{2}EI$	$C = \frac{EI}{100}$
$EI_z = EI$	$EI_z = \frac{1}{2}EI$	
$\lambda_x = 20$	$\lambda_y = 10$	

Stiffness Matrix.

On substituting the above physical constants into the derived results in Appendix I the stiffness of the x-beams and y-beams are as follows:

(a) X-beams.

On substitution of $EI_y = 8EI$ and $\lambda_x = 20$ into (A-4), Φ_x is obtained as

$$\Phi_x = \frac{EI}{1463}\begin{bmatrix} 2.352 & -5.334 & 3.780 & -1.008 & 0.252 & -0.042 \\ -5.334 & 14.448 & -13.902 & 6.048 & -1.512 & 0.252 \\ 3.780 & -13.902 & 20.496 & -15.414 & 6.048 & -1.008 \\ -1.008 & 6.408 & -15.414 & 20.496 & -13.902 & 3.780 \\ 0.252 & -1.512 & 6.048 & -13.902 & 14.448 & -5.334 \\ -0.042 & 0.252 & -1.008 & 3.780 & -5.334 & 2.352 \end{bmatrix} .$$

(b) Y-beams.

On substitution of $EI_x = 2EI$ and $\lambda_y = 10$ into (A-2), ϕ_y is obtained as

$$\phi_y = \frac{EI}{1000}\begin{bmatrix} 3.2 & -7.2 & 4.8 & -0.8 \\ -7.2 & 19.2 & -16.8 & 4.8 \\ 4.8 & -16.8 & 19.2 & -7.2 \\ -0.8 & 4.8 & -7.2 & 3.2 \end{bmatrix} .$$

Loading Matrix.

Since the gridwork is loaded by a unit vertical force at point (2, 2), the matrix of loads can be represented as

$$(F) = \begin{bmatrix} (0) & (0) & (0) & (0) \\ (0) & (P) & (0) & (0) \\ (0) & (0) & (0) & (0) \\ (0) & (0) & (0) & (0) \\ (0) & (0) & (0) & (0) \\ (0) & (0) & (0) & (0) \end{bmatrix} ,$$

where

$$(0) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} , \text{ and}$$

$$(P) = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} .$$

Matrix Equation.

Substituting the above data into Eq. (42), and with $(P) = \frac{1463}{EI}(F)$, yields the matrix equation

$$(P) = \begin{bmatrix} 2.352 & -5.334 & 3.780 & -1.008 & 0.252 & -0.042 \\ -5.334 & 14.448 & -13.902 & 6.048 & -1.512 & 0.252 \\ 3.780 & -13.902 & 20.496 & -15.414 & 6.048 & -1.008 \\ -1.008 & 6.408 & -15.414 & 20.496 & -13.902 & 3.780 \\ 0.252 & -1.512 & 6.048 & -13.902 & 14.448 & -5.334 \\ -0.042 & 0.252 & -1.008 & 3.780 & -5.334 & 2.352 \end{bmatrix} (Z)$$

$$+ 14.63 \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} (Z) + 1.463\,(Z) \begin{bmatrix} 3.2 & -7.2 & 4.8 & -0.8 \\ -7.2 & 19.2 & -16.8 & 4.8 \\ 4.8 & -16.8 & 19.2 & -7.2 \\ -0.8 & 4.8 & -7.2 & 3.2 \end{bmatrix} .$$

Deformation Matrix.

The solution of this matrix equation yields the deformation matrix Z as

$$\begin{array}{c} \begin{array}{cccc} \quad x\text{-}1 \quad & \quad x\text{-}2 \quad & \quad x\text{-}3 \quad & \quad x\text{-}4 \quad \end{array} \\ \begin{bmatrix} .002567759 & .004803425 & .003114347 & -.000273293 \\ .007763838 & .026656725 & .011664789 & -.001950476 \\ .002300447 & .008100884 & .004999434 & -.002098453 \\ .001555673 & -.000740060 & -.000230730 & .001642161 \\ -.000427034 & -.000652101 & -.000508289 & .000040628 \\ -.000016253 & .000034741 & .000025311 & -.000020781 \end{bmatrix} \end{array}$$

Effective Load Matrix.

The matrix of the effective load forces on the x and y-beams are found to be:

(a) Loads applied on x-beams.

$$
\begin{array}{cccc}
\text{x-1} & \text{x-2} & \text{x-3} & \text{x-4} \\
\end{array}
$$
$$
\left[\begin{array}{rrrr}
-.021180262 & -.099687797 & -.035893763 & .007356748 \\
.046661482 & .243415280 & .081798300 & -.017568394 \\
-.023175992 & -.178960800 & -.047466476 & .011161286 \\
-.006465764 & .025539586 & -.007218990 & -.000607322 \\
.004994692 & .010581076 & .009113060 & -.002059184 \\
.000477768 & -.000887345 & -.000332115 & .000401942
\end{array}\right] .
$$

(b) Loads applied on y-beams.

$$
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array}
$$
$$
\left[\begin{array}{rrrr}
-.011186727 & .013815637 & .005928907 & -.008557817 \\
-.117997470 & .239849810 & -.125707190 & .003854855 \\
-.031738148 & .060444850 & -.025675252 & -.003031450 \\
.006276786 & -.014712472 & .010594586 & -.002158900 \\
-.040996163 & .125706110 & -.128423730 & .043713783 \\
-.005439335 & .015977113 & -.015636221 & .005098443
\end{array}\right] .
$$

Stresses.

The shearing forces and bending moments in the x-beams and y-beams are given by the following matrices:

(a) Shearing Forces in the x-beams.

$$
\begin{array}{cccc}
\text{x-1} & \text{x-2} & \text{x-3} & \text{x-4} \\
\end{array}
$$
$$
\left[\begin{array}{rrrr}
-.021180262 & -.099687797 & -.035893763 & .007356748 \\
.025481220 & .143727490 & .045904537 & -.010211646 \\
.002305228 & -.035233310 & -.001561939 & .000949640 \\
-.004160536 & -.009693724 & -.008780929 & .000342318 \\
.000834156 & .000887352 & .000332131 & -.001716866
\end{array}\right] .
$$

(b) Bending Moments in the x-beams.

$$\begin{array}{c}
\begin{array}{cccc} \text{x-1} & \text{x-2} & \text{x-3} & \text{x-4} \end{array} \\
\begin{bmatrix}
.000000000 & 0.000000000 & 0.000000000 & 0.000000000 \\
-.423605240 & -1.993755900 & -.717875260 & .147134960 \\
.086019160 & .880793900 & .200215480 & -.057097960 \\
.132123720 & .176127700 & .168976700 & -.038105160 \\
.048913000 & -.017746780 & -.006641880 & -.031258800 \\
.065596120 & .000000260 & .000000740 & -.065596120
\end{bmatrix}
\end{array}$$

(c) Shearing forces in the y-beams.

$$\begin{array}{c}
\begin{array}{ccc} \text{1-2} & \text{2-3} & \text{3-4} \end{array} \\
\begin{bmatrix}
-.011186727 & .002628910 & .008557817 \\
-.117997470 & .121852340 & -.003854850 \\
-.031738148 & .028706702 & .003031450 \\
.006276786 & -.008435686 & .002158900 \\
-.040996163 & .084709950 & -.043713780 \\
-.005439335 & .010537778 & -.005098443
\end{bmatrix}
\end{array}$$

(d) Bending Moment in the y-beams.

$$\begin{array}{c}
\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\
\begin{bmatrix}
.000000000 & -.111867270 & -.085578170 & 0.000000000 \\
.000000000 & -1.179974700 & .038548700 & .000000200 \\
.000000000 & -.317381480 & -.030314460 & .000000040 \\
.000000000 & .062767860 & -.021589000 & 0.000000000 \\
.000000000 & -.409961630 & .437137870 & .000000070 \\
.000000000 & -.054393350 & .050984430 & 0.000000000
\end{bmatrix}
\end{array}$$

Gridwork Bridge Without Torsion

Given Data.

The size of the gridwork bridge is the same as the gridwork given in Example 1. It is taken to be simply-supported at the end of the x-beams, thus

$$\{z_1\} = \{z_6\} = 0$$

Stiffness Matrix.

By using the boundary conditions, it is seen that the end reaction of an x-beam depends on the deformations at its interior grid-points. Thus, the first and last columns, and the first and last rows of the stiffness matrix $\bar{\Phi}_x$ can be omitted because they have no influence on the deformation at the interior points.

$$\Phi_x = \frac{EI}{1463}\begin{bmatrix} 14.448 & -13.902 & 6.048 & -1.512 \\ -13.902 & 20.496 & -15.414 & 6.048 \\ 6.048 & -15.414 & 20.496 & -13.902 \\ -1.512 & 6.048 & -13.902 & 14.448 \end{bmatrix} .$$

The stiffness of y-beams is the same as that previously obtained, and

$$\phi_y = \frac{EI}{100}\begin{bmatrix} 3.2 & -7.2 & 4.8 & -0.8 \\ -7.2 & 19.2 & -16.8 & 4.8 \\ 4.8 & -16.8 & 19.2 & -7.2 \\ -0.8 & 4.8 & -7.2 & 3.2 \end{bmatrix} .$$

Matrix Equation.

The final equation becomes

$$\frac{1463}{EI}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 14.448 & -13.902 & 6.048 & -1.512 \\ -13.902 & 20.496 & -15.414 & 6.048 \\ 6.048 & -15.414 & 20.496 & -13.902 \\ -1.512 & 6.048 & -13.902 & 14.448 \end{bmatrix}(Z)$$

$$+\,1.463(Z)\begin{bmatrix} 3.2 & -7.2 & 4.8 & -0.8 \\ -7.2 & 19.2 & -16.8 & 4.8 \\ 4.8 & -16.8 & 19.2 & -7.2 \\ -0.8 & 4.8 & -7.2 & 3.2 \end{bmatrix}.$$

Calculated Values.

The following matrices indicate the calculated deformations, shearing forces and bending moments in the x-beams and y-beams.

(a) Deformation matrix (Z).
Beam number

$$\begin{array}{c} \begin{array}{cccc} \text{x-1} & \text{x-2} & \text{x-3} & \text{x-4} \end{array} \\ \begin{bmatrix} .000000000 & 0.000000000 & 0.000000000 & 0.000000000 \\ .271979400 & .242649240 & .156951500 & .057514600 \\ .400064900 & .318350630 & .213680750 & .093193100 \\ .363419700 & .274349050 & .183777290 & .089822220 \\ .210323240 & .156495260 & .104111920 & .053106280 \\ .000000000 & 0.000000000 & 0.000000000 & 0.000000000 \end{bmatrix} \end{array}$$

(b) Effective force matrix (F_x).
Beam number

$$\begin{array}{cccc} x\text{-}1 & x\text{-}2 & x\text{-}3 & x\text{-}4 \end{array}$$

$$\begin{bmatrix} .251818470 & .328032610 & .188477370 & .031670955 \\ -.247809920 & -.502727540 & -.251113180 & .001651478 \\ -.088956050 & .130727740 & .005413465 & -.047185000 \\ -.003067700 & -.007946735 & .025096835 & -.014082450 \\ .005150978 & -.005541912 & -.004367408 & .004759072 \\ .082864280 & .057456122 & .036493218 & .023185898 \end{bmatrix}$$

(c) Shearing force matrix (V_x).
Beam number

$$\begin{array}{cccc} x\text{-}1 & x\text{-}2 & x\text{-}3 & x\text{-}4 \end{array}$$

$$\begin{bmatrix} .251818470 & .328032610 & .188477370 & .031670955 \\ .004008550 & -.174694930 & -.062635810 & .033322433 \\ -.084947500 & -.043967190 & -.057222345 & -.013862567 \\ -.088015200 & -.051913925 & -.032125510 & -.027945017 \\ -.082864222 & -.057455837 & -.036492918 & -.023185945 \\ .000000058 & .000000285 & .000000300 & -.000000047 \end{bmatrix}$$

(d) Bending moment matrix (M_x).
Beam number

$$\begin{array}{cccc} x\text{-}1 & x\text{-}2 & x\text{-}3 & x\text{-}4 \end{array}$$

$$\begin{bmatrix} .000000000 & 0.000000000 & 0.000000000 & 0.000000000 \\ 5.036369400 & 6.560652200 & 3.769547400 & .633419100 \\ 5.116540400 & 3.066753600 & 2.516831200 & 1.299867800 \\ 3.417590400 & 2.187409800 & 1.372384300 & 1.022616500 \\ 1.657286400 & 1.149131300 & .729874100 & .463716200 \\ .000002000 & .000014600 & .000015740 & -\ .000002700 \end{bmatrix}$$

(e) Shearing force matrix (V_y).
Station number

$$
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\left[\begin{array}{r}
.000000000 \\ .247810190 \\ .088956186 \\ .003067764 \\ -.005150743 \\ .000000000
\end{array}\right. &
\begin{array}{r}
0.000000000 \\ -.219461640 \\ -.041771320 \\ .011014565 \\ .000391685 \\ 0.000000000
\end{array} &
\begin{array}{r}
0.000000000 \\ .001651490 \\ -.047184995 \\ -.014082460 \\ .004759073 \\ 0.000000000
\end{array} &
\left.\begin{array}{r}
0.000000000 \\ .000000005 \\ -.000000005 \\ 0.000000000 \\ 0.000000000 \\ 0.000000000
\end{array}\right]
\end{array}
$$

(f) Bending moment matrix (M_y).
Station number

$$
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\left[\begin{array}{r}
.000000000 \\ .000000000 \\ .000000000 \\ .000000000 \\ .000000000 \\ .000000000
\end{array}\right. &
\begin{array}{r}
0.000000000 \\ 2.478101900 \\ .889561860 \\ .030677640 \\ -.051507430 \\ 0.000000000
\end{array} &
\begin{array}{r}
0.000000000 \\ -.016514500 \\ .471848660 \\ .140823290 \\ -.047590580 \\ 0.000000000
\end{array} &
\left.\begin{array}{r}
0.000000000 \\ .000000400 \\ -.000001290 \\ -.000001310 \\ .000000150 \\ 0.000000000
\end{array}\right]
\end{array}
$$

Rectangular Gridworks Considering Torsion

Given Conditions.

The gridwork analyzed here takes the form of a beam-grid bridge. The main dimensions are shown in Fig. 24.

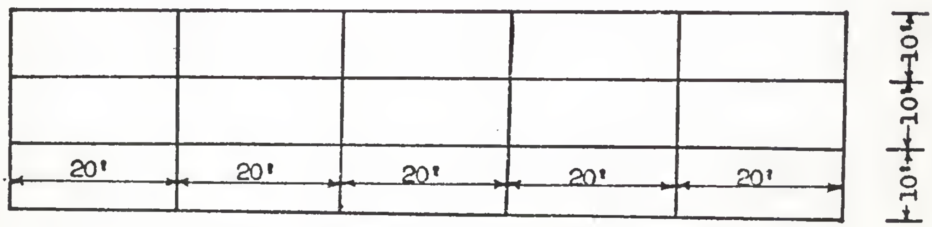


Fig. 24. Rigidly-connected rectangular gridwork.

The beams of the gridwork are the same as those used in Example 1.

An external unit load is applied at (2,2).

The X-beams are simply supported at their ends by rollers such that the vertical deflections D_z and the rotations R_x at these points are zero, i.e.,

$$\left\{\begin{array}{l} D_{1zi} = 0 \\ D_{6zi} = 0 \\ R_{1xi} = 0 \\ R_{6xi} = 0 \end{array}\right. \qquad \text{for } i = 1,\ 2,\ 3,\ 4.$$

Equivalent Force Systems.

To reduce the number of unknowns, the unit load which is applied at (2,2) can be replaced by the superposition of the following force systems:

$$P = \left[P_{x}s_{y}s + P_{x}s_{y}a + P_{x}a_{y}s + P_{x}a_{y}a\right]/4$$

where

$$P_{x}s_{y}s = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \qquad P_{x}s_{y}a = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

$$P_{x}a_{y}s = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ and } \quad P_{x}a_{y}a = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Only the unknowns in the first quadrant need to be determined, since the others may be found by use of symmetry or anti-symmetry.

Stiffness Matrices.

The following matrices apply for:

(a) X-beams in case of symmetry.

$$\Phi_x^s = \frac{EI}{100}\begin{bmatrix} 160 & 12.0 & 0 & 80.0 & 0 & 0 & 0 \\ -12.0 & 2.4 & 0 & 0 & -1.2 & 0 & -12.0 \\ 0 & 0 & 20.0 & 0 & 0 & -10.0 & 0 \\ 80.0 & 0 & 0 & 320.0 & 12.0 & 0 & 80.0 \\ 0 & -1.2 & 0 & 12.0 & 1.2 & 0 & 12.0 \\ 0 & 0 & -10.0 & 0 & 0 & 10.0 & 0 \\ 0 & -12.0 & 0 & 80.0 & 12.0 & 0 & 24 \end{bmatrix}$$

(b) X-beam in case of anti-symmetry.

$$\Phi_x^a = \frac{EI}{100}\begin{bmatrix} 160 & 12.0 & 0 & 80.0 & 0 & 0 & 0 \\ -12.0 & 2.4 & 0 & 0 & -1.2 & 0 & -12.0 \\ 0 & 0 & 20.0 & 0 & 0 & -10.0 & 0 \\ 80.0 & 0 & 0 & 320 & 12.0 & 0 & 80.0 \\ 0 & -1.2 & 0 & 12.0 & 3.6 & 0 & -12.0 \\ 0 & 0 & -10.0 & 0 & 0 & 30.0 & 0 \\ 0 & -12.0 & 0 & 80.0 & -12.0 & 0 & 400.0 \end{bmatrix}$$

such that

$$\lfloor M_{1y};\ P_{2z},\ M_{2x},\ M_{2y};\ P_{3z},\ M_{3x},\ M_{3y} \rfloor$$

$$= \lfloor R_{1y};\ D_{2z},\ R_{2x},\ R_{2y};\ D_{3z},\ R_{3x},\ R_{3y} \rfloor\ (\Phi_x)^T$$

(c) Y-beam stiffness in the case of symmetry.

$$\phi_y^s = \begin{bmatrix} 2.4 & 12.0 & 0 & 2.4 & 12.0 & 0 \\ 12.0 & 80.0 & 0 & -12.0 & 40.0 & 0 \\ 0 & 0 & 5.0 & 0 & 0 & -5.0 \\ -2.4 & -12.0 & 0 & 2.4 & -12.0 & 0 \\ 12.0 & 40.0 & 0 & -12.0 & 120.0 & 0 \\ 0 & 0 & -5.0 & 0 & 0 & 5.0 \end{bmatrix}$$

(d) Y-beam in case of anti-symmetry.

$$\phi_y^a = \begin{bmatrix} 2.4 & 12.0 & 0 & -2.4 & 12.0 & 0 \\ 12.0 & 80.0 & 0 & -12.0 & 40.0 & 0 \\ 0 & 0 & 5.0 & 0 & 0 & -5.0 \\ -2.4 & -12.0 & 0 & 7.2 & 12.0 & 0 \\ 12.0 & 40.0 & 0 & 12.0 & 200 & 0 \\ 0 & 0 & -5.0 & 0 & 0 & 15.0 \end{bmatrix}$$

such that

$$\lfloor f_{z1}, M_{x1}, M_{y1}; f_{z2}, M_{x2}, M_{y2} \rfloor$$
$$= \lfloor d_{z1}, r_{x1}, r_{y1}; d_{z2}, r_{x2}, r_{y2} \rfloor \; (\phi_y)^T$$

(e) Exterior beam stiffness in the case of symmetry,

$$(\phi_{y1}^s) = \frac{EI}{100}\begin{pmatrix} 5.0 & -5.0 \\ -5.0 & 5.0 \end{pmatrix} \quad ;$$

in the case of anti-symmetry,

$$(\phi_{y1}^a) = \frac{EI}{100}\begin{pmatrix} 5.0 & -5.0 \\ -5.0 & 15.0 \end{pmatrix}$$

such that

$$\lfloor M_{y1}, M_{y2} \rfloor_1 = \lfloor r_{y1}, r_{y2} \rfloor_1 (\phi_{y1})^T .$$

Columnar Matrix Equation.

Let $P_{11} = C_{1y}$, $\{P_{21}\} = \begin{Bmatrix} P_{2z} \\ C_{2x} \\ C_{2y} \end{Bmatrix}$, $\{P_{31}\} = \begin{Bmatrix} P_{3z} \\ C_{3x} \\ C_{3y} \end{Bmatrix}$, $P_{12} = C_{4y}$, $\{P_{22}\} = \begin{Bmatrix} P_{5z} \\ C_{5x} \\ C_{5y} \end{Bmatrix}$

and $\{P_{32}\} = \begin{Bmatrix} P_{6z} \\ C_{6x} \\ C_{6y} \end{Bmatrix}$ denote the external loads at the corresponsing points, with the first subscript denoting the station number and the second subscript the X-beam number. Let $F_{11} = M_{1y}$, $F_{12} = M_{4y}$,

$F_{21} = \begin{Bmatrix} F_{2z} \\ M_{2x} \\ M_{2y} \end{Bmatrix}$, $F_{22} = \begin{Bmatrix} F_{5z} \\ M_{5x} \\ M_{5y} \end{Bmatrix}$, $F_{31} = \begin{Bmatrix} F_{3z} \\ M_{3x} \\ M_{3y} \end{Bmatrix}$, and $F_{23} = \begin{Bmatrix} F_{6z} \\ M_{6x} \\ M_{6y} \end{Bmatrix}$ denote the forces that are carried by the X-beams; $f_{11} = m_{1y}$, $f_{21} = m_{2y}$, $f_{12} = \begin{Bmatrix} f_{3z} \\ m_{3x} \\ m_{3y} \end{Bmatrix}$, $f_{22} = \begin{Bmatrix} f_{4z} \\ m_{4x} \\ m_{4y} \end{Bmatrix}$,

$f_{13} = \begin{Bmatrix} f_{5z} \\ m_{5x} \\ m_{5y} \end{Bmatrix}$, and $f_{23} = \begin{Bmatrix} f_{6z} \\ m_{6x} \\ m_{6y} \end{Bmatrix}$ denote the forces that are carried by Y-beams.

Then

$$\begin{Bmatrix} F_{11} \\ F_{21} \\ F_{31} \\ F_{12} \\ F_{22} \\ F_{32} \\ f_{11} \\ f_{21} \\ f_{12} \\ f_{22} \\ f_{13} \\ f_{23} \end{Bmatrix} = \begin{bmatrix} \Phi_x & & & & \\ & \Phi_x & & & \\ & & \phi_{y1} & & \\ & & & \phi_y & \\ & & & & \phi_y \end{bmatrix} \cdot \begin{bmatrix} 1 & & & & & \\ & (I) & & & & \\ & & (I) & & & \\ & & & 1 & & \\ & & & & (I) & \\ & & & & & (I) \\ 1 & & & & & \\ & & & 1 & & \\ & (I) & & & & \\ & & & & (I) & \\ & & (I) & & & \\ & & & & & (I) \end{bmatrix} \begin{Bmatrix} D_{11} \\ D_{21} \\ D_{31} \\ D_{12} \\ D_{22} \\ D_{32} \end{Bmatrix},$$

where

$$(I) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } (1) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

This can be written as

$$\{F\} = [\Phi][C]\{D\} \quad ,$$

with

$$\{P\}^T = \lfloor P_{11}, P_{21}, P_{31}; P_{12}, P_{22}, P_{32} \rfloor \quad ,$$

which yields the columnar matrix equation as

$$\{P\} = [C]^T [\Phi] [C]\{D\} \quad .$$

There are four sets of matrix equations as follows:

$$\tfrac{1}{4}\{P\}_{x^s y^s} = [C]^T [\Phi]_{x^s y^s} [C]\{D\}_{x^s y^s} \quad ,$$

$$\tfrac{1}{4}\{P\}_{x^s y^a} = [C]^T [\Phi]_{x^s y^a} [C]\{D\}_{x^s y^a} \quad ,$$

$$\tfrac{1}{4}\{P\}_{x^a y^s} = [C]^T [\Phi]_{x^a y^s} [C]\{D\}_{x^a y^s} \quad ,$$

and

$$\tfrac{1}{4}\{P\}_{x^a y^a} = [C]^T [\Phi]_{x^a y^a} [C]\{D\}_{x^a y^a} \quad .$$

Each of the above equations can be solved as

$$\{D\} = [C]^T [\Phi]^{-1} [C]\{P\}$$

and

$$\{F\} = [\Phi][C]\{D\}$$

for the deformations and forces of the first quadrant.

Rearrangement of the Deformation Matrix.

The deformation matrix of the gridwork can be obtained by the following successive rearranging operations:

Let $(I) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $(I') = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $(E) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $(C_1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$

$$(C_2) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix} \quad , (C_3) = \begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 1 & 1 \end{pmatrix} \quad , \text{ and } (C_4) = \begin{pmatrix} -1 & -1 \\ +1 & +1 \\ +1 & +1 \end{pmatrix} \quad .$$

(a) Then the first quadrant of the deformation matrix can be constructed as

$$(D_1) = \left[\begin{array}{ccc:ccc} (E) & & & & & \\ & (I) & & & & \\ & & (I) & & & \\ \hdashline & & & & & \\ & & & & & \\ & & & & & \end{array}\right] \{D\} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \left[\begin{array}{ccc:ccc} & & & (E) & & \\ & & & & (I) & \\ & & & & & (I) \\ \hdashline & & & & & \\ & & & & & \\ & & & & & \end{array}\right] \{D\} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad .$$

(b) The magnitude of the elements of the deformation matrix can be obtained from (D_1) as

$$(D_2) = \left[\begin{array}{ccc:ccc} (I) & & & & & \\ & (I) & & & & \\ & & (I) & & & \\ \hdashline & & (I') & (I) & & \\ & (I') & & & (I) & \\ (I') & & & & & (I) \end{array}\right] \quad (D_1) \quad \left[\begin{array}{cc:cc} 1 & & & 1 \\ & 1 & 1 & \\ \hdashline & & 1 & \\ & & & 1 \end{array}\right] \quad .$$

(c) For correction of signs, let

$$[C_i] = \begin{bmatrix} (C_i) \\ (C_i) \\ (C_i) \end{bmatrix} \quad , i = 1, 2, 3, 4.$$

Then

$$(D_{x^s y^s}) = (D_2^{x^s y^s}) * \begin{pmatrix} [C_1] & [C_3] \\ [C_2] & -[C_4] \end{pmatrix} \quad ,$$

$$(D_{x^a y^s}) = (D_2^{x^a y^s}) * \begin{pmatrix} [C_1] & -[C_3] \\ [C_2] & [C_4] \end{pmatrix} \quad ,$$

$$(D_{x^s y^a}) = (D_2^{x^s y^a}) * \begin{pmatrix} [C_1] & [C_3] \\ -[C_2] & [C_4] \end{pmatrix} \quad ,$$

and

$$(D_{x^a y^a}) = (D_2^{x^a y^a}) * \begin{pmatrix} [C_1] & -[C_3] \\ -[C_2] & -[C_4] \end{pmatrix} \quad .$$

(d) Then the deformation matrix is

$$(D) = (D_{x_s y_s} + D_{x_a y_s} + D_{x_s y_a} + D_{x_a y_a}) \quad .$$

Calculated Values.

The calculated values follow:

(a) Deformation matrix.
Beam number.

x-1	x-2	x-3	x-4
.000000000	0.000000000	0.000000000	0.000000000
.000000000	0.000000000	0.000000000	0.000000000
-.009080726	- .020648467	- .014732295	- .005380284
.138699030	.329974610	.147200060	.006071108
.016266322	- .000434481	- .019948211	- .018098622
-.005933934	- .006661680	- .005064918	- .001414836
.210344210	.255397000	.194126000	.045049286
.009959712	- .001323410	- .013089954	- .012859176
.000219348	.007266244	.001748690	- .001309046
.137945840	.146978040	.091231960	.042897466
.001199970	- .001528650	- .005066670	- .006734766
.005609952	.006125582	.004754444	.001318178
.035990048	.033291580	.025532390	.011893198
.007860616	- .000519805	- .001184439	.006020108
.003526194	.003293988	.002528834	.001224508
.000000000	0.000000000	0.000000000	0.000000000
.000000000	0.000000000	0.000000000	0.000000000
.001962632	.000847195	.000645985	.001299922

(b) Load matrix for x-beams.

x-1	x-2	x-3	x-4
.158033490	.468764870	.180989150	-.008129724
.241228330	-.004344810	-.238572310	-.089244480
.103242310	-.123000890	.017184450	.043074547
-.174801130	-.631132000	-.217640960	.023574349
-.382859530	-.004544485	.268064680	.076250470
.003475560	-.011622856	-.010266936	.018413000
-.034325674	.068085401	-.017199583	-.016559828
.054033789	.006836897	.011650253	.087414775
-.035234500	.062822230	-.012299250	-.015288820
.004213274	-.009235973	.009243031	-.004220272
.075638749	.012140847	-.017532107	.026774655
-.002577840	.009434010	.010325810	-.017181080
.000287891	-.002116780	-.001687140	.003516071
.011917635	-.004890395	.120716340	-.174573130
.001324080	.002664776	.002695976	-.006684160
.046592155	.105634470	.046295514	.001819403
.000041033	-.005198054	-.144326850	.073377710
-.004236007	.000424054	.000719506	-.003229933

(c) Load matrix for y-beams.

0.000000000	.174801130	.034325674	- .004213274
- .000287891	0.000000000	0.000000000	.382859530
- .054033789	- .075638749	- .011917635	0.000000000
- .103242310	- .003475560	.035234500	.002577840
- .001324080	.004236007	0.000000000	- .368868000
- .068085401	.009235973	.002116780	0.000000000
0.000000000	.004544485	- .006836897	- .012140847
.004890395	0.000000000	.123000890	.011622856
- .062822230	- .009434010	- .002664776	- .000424054
0.000000000	.217640960	.017199583	- .009243031
.001687140	0.000000000	0.000000000	- .268064680
- .011650253	.017532107	- .120716340	0.000000000
- .017184450	.010266936	.012299250	- .010325810
- .002695976	- .000719506	0.000000000	- .023574349
.016559828	.004220272	- .003516071	0.000000000
0.000000000	- .076250470	- .087414775	- .026774655
.174573130	0.000000000	- .043074547	- .018413000
.015288820	.017181080	.006684160	.003229933

(d) Stresses in x-beams.

VX Matrix (Shearing Forces)

$$
\begin{array}{cccc}
x\text{-}1 & x\text{-}2 & x\text{-}3 & x\text{-}4 \\
\end{array}
$$

$$
\begin{bmatrix}
.158033490 & .468764870 & .180989150 & -.008129724 \\
-.016767640 & -.162367130 & -.036651810 & .015444625 \\
-.051093314 & -.094281730 & -.053851393 & -.001115203 \\
-.046880040 & -.103517700 & -.044608362 & -.005335475 \\
-.046592149 & -.105634480 & -.046295502 & -.001819404
\end{bmatrix}
$$

TMX Matrix (Torsional Moments)

x-1	x-2	x-3	x-4
.241228330	-.004344810	-.238572310	-.089244480
-.141631200	-.008889295	.029492370	-.012994010
-.087597420	-.002052398	.041142623	.074420765
-.011958671	.010088449	.023610516	.101195420
-.000041036	.005198054	.144326850	-.073377710

BMX Matrix (Bending Moments)

x-1	x-2	x-3	x-4
.103242310	- .123000890	.017184450	.043074547
3.263912100	9.252296600	3.636967400	- .119519940
3.267387600	9.240673800	3.626700500	- .101106940
2.932034800	5.993332200	2.893664300	.207785560
2.896800300	6.056154400	2.881365100	.192496740
1.874934100	4.170519400	1.804337300	.170192680
1.872356300	4.179953400	1.814663100	.153011600
.934754640	2.109600000	.922494980	.046302140
.938656560	2.102830700	.914865140	.056799060
.004236620	- .000425204	- .000719024	.003229980

(e) Stresses in Y-beams.

VY Matrix (Shearing Forces)

Y-1	Y-2	Y-3	Y-4	Y-5	Y-6
0.000000	.174801	.034326	-.004213	-.000288	0.000000
0.000000	-.194067	-.033760	.005023	.001829	0.000000
0.000000	.023574	-.016560	-.004220	.003516	0.000000

BMY Matrix (Bending Moments)

$$
\begin{array}{cccccc}
Y\text{-}1 & Y\text{-}2 & Y\text{-}3 & Y\text{-}4 & Y\text{-}5 & Y\text{-}6 \\
\end{array}
$$
$$
\begin{bmatrix}
0.000000 & .382860 & -.054034 & -.075639 & -.011918 & 0.000000 \\
0.000000 & -1.365152 & -.397291 & -.033506 & -.009039 & 0.000000 \\
0.000000 & -1.360607 & -.404127 & -.045647 & -.004148 & 0.000000 \\
0.000000 & .580061 & -.066530 & -.095874 & -.022437 & 0.000000 \\
0.000000 & .311997 & -.078180 & -.078342 & -.143154 & 0.000000 \\
0.000000 & .076256 & .087421 & -.036138 & -.178314 & 0.000000
\end{bmatrix}
$$

TMY Matrix (Torsional Moments)

$$
\begin{array}{cccccc}
Y\text{-}1 & Y\text{-}2 & Y\text{-}3 & Y\text{-}4 & Y\text{-}5 & Y\text{-}6 \\
\end{array}
$$
$$
\begin{bmatrix}
-.103242 & -.003476 & .035235 & .002578 & -.001324 & .004236 \\
.019759 & .008147 & -.027588 & -.006856 & -.003989 & .003812 \\
.002574 & .018414 & -.015288 & -.017182 & -.006685 & .003092 \\
-.040500 & .000001 & .000000 & -.000000 & -.000000 & .006322
\end{bmatrix}
$$

CONCLUDING REMARKS

In this report, matrix analysis has been used to analyze various types of beam gridworks which have certain regularity properties.

The analysis can be extended in various directions:

(1) Although it has been used only for gridworks having beams which intersect orthogonally, non-orthogonal gridworks can be treated without difficulty. In fact, the angles of intersection of the two sets of beams play no role whatever in simply connected gridworks, and introduce only minor complications if the beams of the gridwork are rigidly connected.

(2) Other structures, such as ribbed cylinders, anisotropic plates, etc., that can be approximated by linear partial difference equations which lead to the same matrix equations used here, can be treated directly.

(3) The matrix equation for rigidly connected gridworks supported on an elastic foundation can be extended to include gridworks supported by columns or other structures merely by replacing the foundation property matrix C by the stiffness matrix of the supporting structure.

(4) Although the equations for arbitrary gridworks (i.e. those in which the beams in a given direction have stiffness and spacings different from each other) can be derived without difficulty by methods used here, their solutions have no particular advantage over the usual methods.

The simplification brought about by the matrix treatment of gridworks presented here occurs because of the regularity which they have. This regularity allows the same stiffness matrix to describe every beam of a given set, and this permits a factoring which simplifies the solution of the basic matrix equation. As the gridworks become less regular, and thus more general, this factoring property disappears, and the equations become progressively more complicated until for arbitrary gridworks they finally become a completely general linear set which allows no essential simplification.

ACKNOWLEDGEMENT

The author wishes to express the deep acknowledgement to Dr. Philip G. Kirmser for his valuable suggestions, direction, correction, and encouragement during the preparation of this report, and also for his outstanding lectures on Applied Linear Analysis. These equipped the author with the most useful tools for engineering analysis, and enabled him to treat such complicated problems as the general beam gridworks discussed here.

APPENDICES

APPENDIX I

STIFFNESS OF UNIFORM LATERALLY CONSTRAINED BEAMS

The stiffness matrix in Figs. A-1 through A-4 was prepared for a uniform beam with uniformly spaced points of constraint not exceeding six in number.

The beams and the corresponding stiffness matrices are given in the figures and equations which follow.

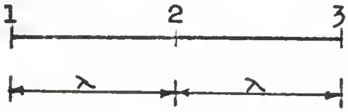


Fig. A-1.

$$\Phi = \frac{EI}{10\lambda^3}\begin{pmatrix} 15 & -30 & 15 \\ -30 & 60 & -30 \\ 15 & -30 & 15 \end{pmatrix} \qquad \text{(A-1)}$$

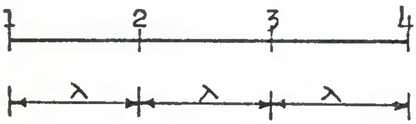


Fig. A-2.

$$\Phi = \frac{EI}{10\lambda^3}\begin{pmatrix} 16 & -36 & 24 & -4 \\ -36 & 96 & -84 & 24 \\ 24 & -84 & 96 & -36 \\ -4 & 24 & -36 & 16 \end{pmatrix} \qquad \text{(A-2)}$$

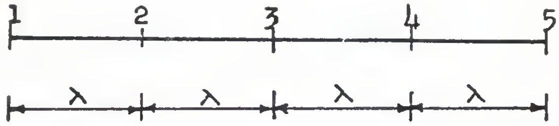


Fig. A-3.

$$\Phi = \frac{EI}{28\lambda^3}\begin{bmatrix} 45 & -102 & 72 & -18 & 3 \\ -102 & 276 & -264 & 108 & -18 \\ 72 & -264 & 384 & -264 & 72 \\ -18 & 108 & 264 & 276 & -102 \\ 3 & -18 & 72 & -102 & 45 \end{bmatrix} \qquad \text{(A-3)}$$

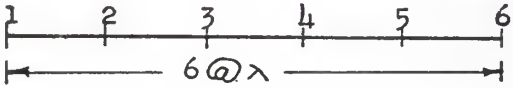


Fig. A-4.

$$\Phi = \frac{EI}{1463\lambda^3}\begin{bmatrix} 2{,}352 & -5{,}334 & 3{,}780 & -1{,}008 & 252 & -42 \\ -5{,}334 & 14{,}448 & -13{,}902 & 6{,}048 & -1{,}512 & 252 \\ 3{,}780 & -13{,}902 & 20{,}496 & -15{,}414 & 6{,}048 & -1{,}008 \\ -1{,}008 & 6{,}048 & -15{,}414 & 20{,}496 & -13{,}902 & 3{,}780 \\ 252 & -1{,}512 & 6{,}048 & -13{,}902 & 14{,}448 & -5{,}334 \\ -42 & 252 & -1{,}008 & 3{,}780 & -5{,}334 & 2{,}352 \end{bmatrix}, \qquad \text{(A-4)}$$

where EI = flexile rigidity of the beam, and λ = spacing between the constraint points.

APPENDIX II

STIFFNESS MATRICES OF LATERALLY CONSTRAINED CIRCULAR RINGS

The cases where $m = 4$ and for all $i = 1, 2, \ldots, m$, (i.e., the ring has at least four points of constraint of equal spacing) are treated here. The treatments are based on the principle of least work.

Ring of 4-Constraints

Internal Stresses (F_1 and M_o are selected as redundants).

Bending moments $M(\theta)$ and torsional moments $T(\theta)$ of section θ are given by Eq. (A-5).

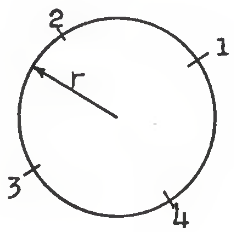


Fig. A-5. Ring of 4-constraints.

EI_r = flexile rigidity. GJ = torsional rigidity.

$$\left\{\begin{array}{ll} \left\{\begin{array}{l} M_1(\theta) = M_o\cos\theta + \dfrac{F_1 r}{2}\sin\theta\,, \\ T_1(\theta) = M_o\sin\theta - \dfrac{F_1 r}{2}(1 - \cos\theta)\,, \end{array}\right. & \text{for } (0 \le \theta \le \dfrac{\pi}{2})\;; \\ \left\{\begin{array}{l} M_2(\theta) = M_o\cos\theta + \dfrac{F_1 r}{2}\sin\theta - F_1 r\sin(\theta - \dfrac{\pi}{2})\,, \\ T_2(\theta) = M_o\sin\theta - \dfrac{F_1 r}{2}(1 - \cos\theta) + F_1 r(1 - \cos(\theta - \dfrac{\pi}{2}))\,, \end{array}\right. & \text{for } (\dfrac{\pi}{2} \le \theta \le \pi)\,. \end{array}\right. \tag{A-5}$$

Strain Energy.

The strain energy of the ring is given by Eq. (A-6).

$$U = 2\int_0^{\frac{\pi}{2}} \left(\frac{M_1^2(\theta)}{2EI_r} + \frac{T_1^2(\theta)}{2GJ}\right) r\,d\theta + 2\int_{\frac{\pi}{2}}^{\pi} \left(\frac{M_2^2(\theta)}{2EI_r} + \frac{T_2^2(\theta)}{2GJ}\right) r\,d\theta \tag{A-6}$$

Stiffness Matrix.

By the method of least work one obtains

$$\begin{cases} \dfrac{\partial U}{\partial M_o} = 0 \\[2ex] \dfrac{\partial U}{\partial F_1} = 1 \end{cases} \tag{A-7}$$

Solving Eq. (A-7) for M_o and F_1 gives the stiffness matrix as

$$\Phi = \begin{bmatrix} F & -F & F & -F \\ -F & F & -F & F \\ F & -F & F & -F \\ -F & F & -F & F \end{bmatrix} \tag{A-8}$$

Ring of 6-Constraints

External Forces (F_1, F_4, F_6 are selected as external redundants).

By using static equilibrium conditions the constraint forces can be represented as functions of f_1, f_4, and F_6 as given in Eq. (A-9).

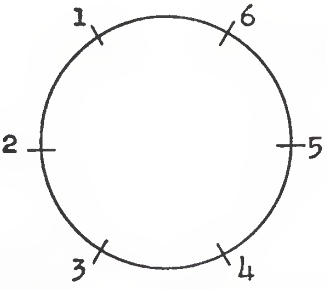


Fig. A-6. Ring of 6-constraints.

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{3}{2} & \frac{1}{2} & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_4 \\ F_6 \end{Bmatrix} \qquad \text{(A-9)}$$

Internal Stress.

$$\left\{\begin{array}{l} \left\{\begin{array}{l} M_1(\theta) = M_o\cos\theta + \frac{F_1}{2}r\sin\theta \\ T_1(\theta) = -M_o\sin\theta - \frac{F_1}{2}r(1-\cos\theta) \end{array}\right. \qquad 0 \leq \theta \leq 60^\circ \\ \left\{\begin{array}{l} M_2(\theta) = M_1(\theta) + F_2 r\sin(\theta - \frac{\pi}{6}) \\ T_2(\theta) = T_1(\theta) - F_2 r(1 - \cos(\theta - \frac{\pi}{3}) \end{array}\right. \\ \left\{\begin{array}{l} M_3(\theta) = M_2(\theta) + F_3 r\sin(\theta - \frac{2\pi}{3}) \\ T_3(\theta) = T_2(\theta) - F_3 r(1 - \cos(\theta - \frac{2\pi}{3})\ ) \end{array}\right. \end{array}\right. \qquad \text{(A-10)}$$

Strain Energy.

$$U = 2\sum_{j=1}^{3}\int_{\theta_{j-1}}^{\theta_j}\left[\frac{M_j{}^2(\theta)}{2EI} + \frac{T_j{}^2(\theta)}{2GJ}\right] r d\theta\ , \ (\theta_j = \frac{j\pi}{3}) \qquad \text{(A-11)}$$

Stiffness Matrix.

By the method of least work one obtains

$$\begin{Bmatrix} \dfrac{\partial U}{\partial M_o} \\ \dfrac{\partial U}{\partial F_1} \\ \dfrac{\partial U}{\partial F_4} \\ \dfrac{\partial U}{\partial F_6} \end{Bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} . \qquad \text{(A-12)}$$

After solving for M_o, F_1, F_4 and F_6 and calculating $\{F\}$, the stiffness matrix can be deduced from $\{F\}$ by use of the property of shift symmetry of $\{F\}$.

Ring of 8-Constraints

External Forces.

Select F_1, F_2, F_4 as redundants then the external forces can be represented as

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -\dfrac{2+\sqrt{2}}{2} & -\dfrac{2-\sqrt{2}}{2} \\ 0 & 0 & 1 \\ 1 & \sqrt{2} & -\sqrt{2} \\ 0 & 0 & 1 \\ -1 & -\dfrac{2+\sqrt{2}}{2} & -\dfrac{2-\sqrt{2}}{2} \\ 0 & 1 & 0 \end{pmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_4 \end{Bmatrix} . \qquad \text{(A-13)}$$

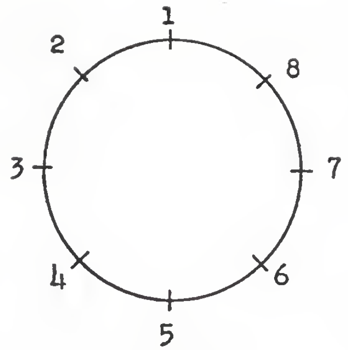


Fig. A-7.

Internal Stresses.

$$\left.\begin{aligned} M_1(\theta) &= M_o\cos\theta + \frac{F_1}{2}r\sin\theta \ , \\ T_1(\theta) &= M_o\sin\theta - \frac{F_1}{2}r(1 - \cos\theta) \end{aligned}\right\} \text{ for } 0 \le \theta \le \frac{\pi}{4} \ ;$$

$$\left.\begin{aligned} M_i(\theta) &= M_{i-1}(\theta) + F_i r\sin(\theta - \theta_{i-1}) \ , \\ T_i(\theta) &= T_{i-1}(\theta) - F_i r(1 - \cos(\theta - \theta_{i-1})) \end{aligned}\right\} \text{ for } \theta_{i-1} \le \theta \le \theta_{i+1} \ ; \qquad \text{(A-14)}$$

where

$$\theta_{i-1} = \frac{\pi}{4}(i - 1) \qquad i = 1, 2, 3, 4.$$

Strain Energy.

$$U = 2 \sum_{i=1}^{4} \int_{\theta_{i-1}}^{\theta_i} r \left[\frac{M_i^2(\theta)}{2EI} + \frac{T_i^2(\theta)}{2GJ} \right] d\theta \qquad \text{(A-15)}$$

Stiffness Matrix.

By method of least work,

$$\left\lfloor \frac{\partial U}{\partial M_o}, \frac{\partial U}{\partial F_1}, \frac{\partial U}{\partial F_2}, \frac{\partial U}{\partial F_4} \right\rfloor = \lfloor 0, 1, 0, 0 \rfloor \qquad \text{(A-16)}$$

is obtained.

After solving for the F's by the property of shift symmetry of the vector $\{F\}$, the stiffness matrix Φ can be deduced immediately.

APPENDIX III

STIFFNESS OF STRAIGHT MEMBERS

Stiffness Matrix of Decomposed Force System

Stiffness Matrices for the Association of $\lfloor F_z, M_x, M_y \rfloor$ and $\lfloor D_z, R_x, R_y \rfloor$.

From Fig. A-8 it can be seen that the stiffnesses $\Phi_x^{(1)}$ and $\phi_y^{(1)}$ are given by Eqs. (A-17) and (A-18).

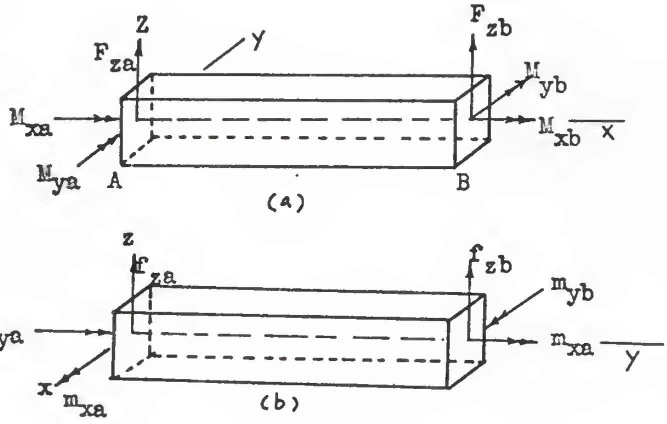


Fig. A-8.

$$\Phi_x^{(I)} = \left[\begin{array}{ccc:ccc} \frac{12EI_y}{L^3} & 0 & \frac{-6EI_y}{L^2} & \frac{-12EI_y}{L^3} & 0 & \frac{-6EI_y}{L^2} \\ 0 & \frac{GJ}{L} & 0 & 0 & \frac{-GJ}{L} & 0 \\ \frac{-6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & \frac{6EI_y}{L^2} & 0 & \frac{2EI_y}{L} \\ \hdashline \frac{-12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & \frac{12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} \\ 0 & \frac{-GJ}{L} & 0 & 0 & \frac{GJ}{L} & 0 \\ \frac{-6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} \end{array}\right] \tag{A-17}$$

$$\phi_y^{(I)} = \left[\begin{array}{ccc|ccc} \frac{12EI_x}{l^3} & \frac{6EI_x}{l^2} & 0 & \frac{-12EI_x}{l^3} & \frac{6EI_x}{l^2} & 0 \\ \frac{6EI_x}{l^2} & \frac{4EI_x}{l} & 0 & \frac{-6EI_x}{l^2} & \frac{2EI_x}{l} & 0 \\ 0 & 0 & \frac{GJ}{l} & 0 & 0 & \frac{-GJ}{l} \\ \hline \frac{-12EI_x}{l^3} & \frac{-6EI_x}{l^2} & 0 & \frac{12EI_x}{l^3} & \frac{-6EI_x}{l^2} & 0 \\ \frac{6EI_x}{l^2} & \frac{2EI_x}{l} & 0 & \frac{-6EI_x}{l^2} & \frac{4EI_x}{l} & 0 \\ 0 & 0 & \frac{-GJ}{l} & 0 & 0 & \frac{GJ}{l} \end{array}\right] \tag{A-18}$$

such that

$$\lfloor F_{za}, M_{xa}, M_{ya}; F_{zb}, M_{xb}, M_{yb} \rfloor = \lfloor D_{za}, R_{xa}, R_{ya}; D_{zb}, R_{xb}, R_{yb} \rfloor \, (\Phi_x^{(1)})^T$$

and

$$\lfloor f_{za}, m_{xa}, m_{ya}; f_{zb}, m_{xb}, M_{yb} \rfloor = \lfloor d_{za}, r_{xa}, r_{ya}; d_{zb}, r_{xb}, r_{yb} \rfloor \, (\phi_y^{(1)})^T$$

Stiffness Matrices for the Association of $\lfloor F_x, F_y, M_z \rfloor$ and $\lfloor D_x, D_y, R_z \rfloor$.

From Fig. A-9, it can be seen that the stiffness matrices $\Phi_x^{(II)}$ and $\phi_y^{(II)}$ are given by Eqs. (A-19) and (A-20).

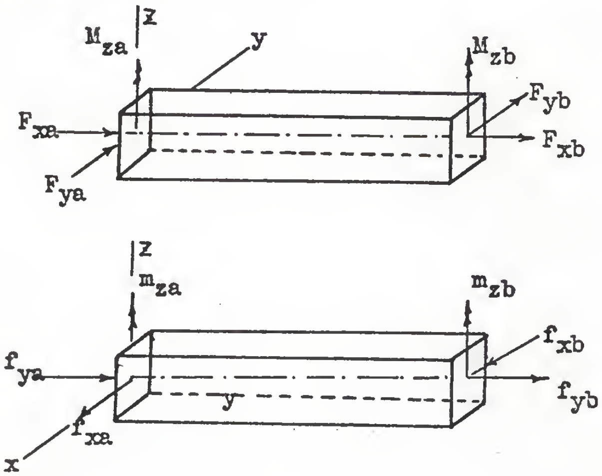


Fig. A-9.

$$\Phi_x^{(II)} = \left[\begin{array}{ccc:ccc} \frac{AE}{L} & 0 & 0 & \frac{-AE}{L} & 0 & 0 \\ 0 & \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & 0 & \frac{-12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & 0 & \frac{-6EI_z}{L^2} & \frac{2EI_z}{L} \\ \hdashline \frac{-AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\ 0 & \frac{-12EI_z}{L^3} & \frac{-6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & \frac{-6EI_z}{L^2} \\ 0 & \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & 0 & \frac{-6EI_z}{L^2} & \frac{4EI_z}{L} \end{array}\right] \qquad \text{(A-19)}$$

$$\phi_y^{(II)} = \left[\begin{array}{ccc:ccc} \frac{12EI_z}{l^3} & 0 & \frac{-6EI_z}{l^2} & \frac{-12EI_z}{l^3} & 0 & \frac{-6EI_z}{l^2} \\ 0 & \frac{AE}{l} & 0 & 0 & \frac{-AE}{l} & 0 \\ \frac{-6EI_z}{l^2} & 0 & \frac{4EI_z}{l} & \frac{6EI_z}{l^2} & 0 & \frac{2EI_z}{l} \\ \hdashline \frac{-12EI_z}{l^3} & 0 & \frac{6EI_z}{l^2} & \frac{12EI_z}{l^3} & 0 & \frac{6EI_z}{l^2} \\ 0 & \frac{-AE}{l} & 0 & 0 & \frac{AE}{l} & 0 \\ \frac{-6EI_z}{l^2} & 0 & \frac{2EI_z}{l} & \frac{6EI_z}{l^2} & 0 & \frac{4EI_z}{l} \end{array}\right] \qquad \text{(A-20)}$$

such that

$$\lfloor F_{xa}, F_{ya}, M_{za}; F_{xb}, F_{yb}, M_{zb} \rfloor = \lfloor D_{xa}, D_{ya}, R_{za}; D_{xb}, D_{yb}, R_{zb} \rfloor \, (\Phi_x^{(II)})^T$$

and

$$\lfloor f_{xa}, f_{ya}, m_{za}; f_{xb}, f_{yb}, m_{zb} \rfloor = \lfloor d_{xa}, d_{ya}, r_{za}; d_{xb}, d_{yb}, r_{zb} \rfloor \, (\phi_y^{(II)})^T$$

(A-21)

Special Stiffness Matrices for the $\lfloor F_z, M_x, M_y \rfloor$ Force System.

(a) Deformation is symmetrical with respect to Y-axis. The deformations are shown in Fig. A-10.

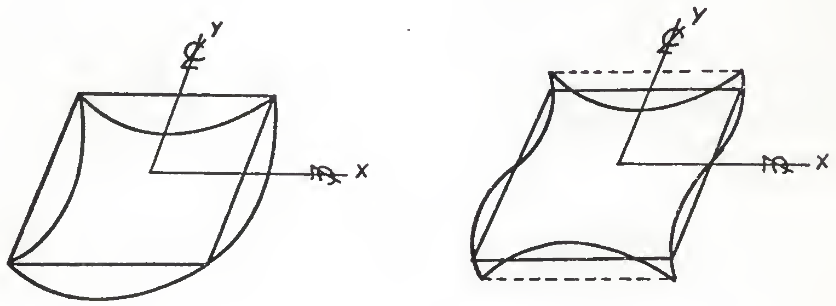


Fig. A-10.

From Fig. A-10 it is seen that: $\Phi_{xa}^{y^s x^s} = \Phi_{xa}^{y^s x^a} = \Phi_x^{y^s}$, and $D_{bz} = D_{az}$, $R_{bx} = R_{ax}$, $R_{by} = -R_{ay}$.

Substituting into Φ_{ab} gives

$$\Phi_x^{y^s} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2EI_y/h \end{bmatrix} \tag{A-22}$$

(b) Deformation anti-symmetrical with respect to X-axis. The deformations are shown in Fig. A-11.

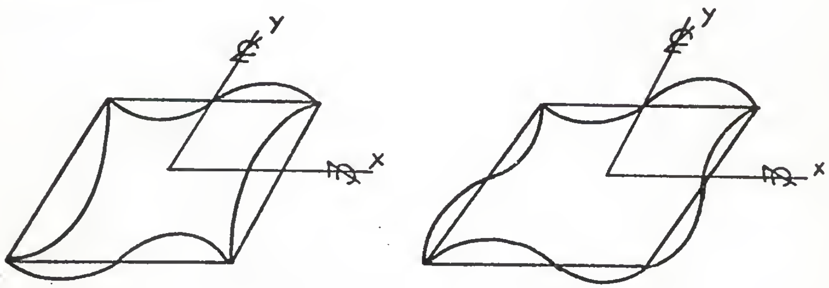


Fig. A-11.

From Fig. A-11 it is seen that $\overline{\Phi}_{xa}^{y^a x^s} = \overline{\Phi}_{xa}^{y^a x^a} = \overline{\Phi}_{xa}^{y^a}$, (A-23)

and $D_{bz} = -D_{az}$, $R_{bx} = -R_{ax}$, $R_{by} = R_{ay}$.

Thus

$$\overline{\Phi}_{xa}^{y^a} = \begin{bmatrix} \frac{24EI_y}{\lambda^3} & 0 & \frac{-12EI_y}{\lambda^3} \\ 0 & \frac{2GJ}{\lambda} & 0 \\ \frac{-12EI_y}{\lambda^2} & 0 & \frac{6EI_y}{\lambda} \end{bmatrix} \tag{A-24}$$

such that

$$\lfloor F_{za}, M_{xa}, M_{ya} \rfloor = \lfloor D_{za}, R_{xa}, R_{ya} \rfloor \; (\overline{\Phi}_{xa})^T \tag{A-25}$$

Similarly, the stiffness matrix $\phi_{ya}^{x^s}$ and $\phi_{ya}^{x^a}$ are obtained as

(c)

$$\phi_{ya}^{x^s} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{2EI_x}{\lambda} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{A-26}$$

and

(d)

$$\phi_{ya}^{x^a} = \begin{bmatrix} \frac{24EI_x}{\lambda^3} & \frac{12EI_x}{\lambda^2} & 0 \\ \frac{12EI_x}{\lambda^2} & \frac{6EI_x}{\lambda} & 0 \\ 0 & 0 & \frac{2GJ}{\lambda} \end{bmatrix} , \tag{A-27}$$

such that

$$\lfloor f_{za}, m_{xa}, m_{ya} \rfloor = \lfloor d_{za}, r_{xa}, r_{ya} \rfloor \; (\phi_{ya})^T . \tag{A-28}$$