

NONLINEAR RESPONSES OF A HINGED CIRCULAR PLATE  
WITH A CONCENTRIC RIGID MASS

by

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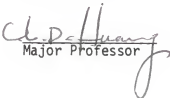
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## NOMENCLATURE

|                                     |   |
|-------------------------------------|---|
| $r, \theta, z$                      | cylindrical coordinates   |
| $h$                                 | thickness of the plate  |
| $a$                                 | outer radius of the plate   |
| $b$                                 | inner radius of the plate and radius of the concentric rigid core                                 |
| $M_C$                               | mass of the rigid core  |
| $t$                                 | time variable   |
| $u, w$                              | radial and transverse displacements of the middle plane, respectively                             |
| $\epsilon_r, \epsilon_\theta$       | radial and circumferential strain components, respectively  |
| $\sigma_r, \sigma_\theta, \sigma_z$ | radial, circumferential and normal stress components, respectively                                |
| $D, E, \nu$                         | flexural rigidity, elastic modulus and Poisson's ratio, respectively                              |
| $N_r, N_\theta$                     | radial and circumferential membrane forces per unit length, respectively                          |
| $M_r, M_\theta$                     | radial and circumferential bending moments per unit length, respectively                          |
| $T_1, T_2$                          | Kinetic energy of the plate and of the concentric rigid mass, respectively                        |
| $V_1, V_2$                          | strain energy due to bending of the plate and due to stretching of the middle plane, respectively |
| $\epsilon_r^0, \epsilon_\theta^0$   | radial and circumferential strain components acting on the middle plane, respectively             |
| $\rho$                              | density of the plate material   |
| $I$                                 | action integral of the vibratory system   |
| $\delta I$                          | first variation of $I$  |
| $n, \zeta$                          | arbitrary functions in the admissible variations of $w$ and $u$ , respectively                    |
| $\epsilon$                          | arbitrary infinitesimal constant  |

|                             |   |
|-----------------------------|---|
| $X$                         | nondimensional transverse displacement                |
| $\xi, \tau$                 | nondimensional space and time variables, respectively |
| $R$                         | radius ratio  |
| $\gamma$                    | mass ratio  |
| $\omega$                    | nondimensional angular frequency                      |
| $\psi, \phi$                | stress functions                                      |
| $A, \alpha$                 | amplitude parameters                                  |
| $g(\xi), f(\xi)$            | shape functions of vibrations                         |
| $\lambda$                   | nondimensional nonlinear eigenvalue                   |
| $\delta X$                  | first variation of $X$                                |
| $\delta W$                  | virtual work of all transverse forces                 |
| $\bar{Y}, \bar{Z}, \bar{H}$ | (6 x 1) vector functions                              |
| (M), (N)                    | coefficient matrices                                  |
| $\bar{n}$                   | adjustable data in the related initial value problem  |
| (J <sub>1</sub> )           | Jacobian matrix                                       |
| $W_R/h$                     | relative amplitude at $\xi = R$                       |
| $\omega_2$                  | linear angular frequency                              |

## INTRODUCTION

The transverse oscillations of a thin circular plate carrying a concentric rigid mass are important in many engineering applications ranging from telephone receiver diaphragm oscillations to structures supporting vibratory machinery and to modeling printed circuit boards. Since vibrations may be disastrous, reliable predictions of their nature is of great importance. The amplitude of vibrations may be of sufficient magnitude to result in malfunction of delicate components. In addition, undesirable noise associated with vibrations may result in human discomfort as well.

When the amplitude of vibration is of the same order of magnitude as the thickness of the plate, classical linear plate theory must be extended to include the effects of middle plane deformation. Considering this membrane effect results in a set of two equations, well known as the dynamical von Karman's equations, which are non-linear and coupled. Due to the complexity of the governing equations, the only present means of solution is by approximate methods.

Employing various approximate numerical methods, several authors have examined the vibratory characteristics of circular plates of constant thickness [3,7,11]. However, limited attention has been given to the large amplitude vibration of simply supported plates with a rigidly attached concentric rigid mass.

Handelman and Cohen [10] studied the effects of adding a concentric rigid mass to a clamped circular plate. Small amplitude vibration response curves were obtained for various mass and radii ratios by employing a minimum principle. Extending the problem, Laura and Gutierrez [13] solved the problem of variable thickness circular



plates carrying a concentric rigid mass by means of the Ritz method. Defining a thickness ratio and a flexibility parameter, frequencies were obtained for various conditions.

Approximate solutions are commonly obtained by separating the variables and using function space methods for the purpose of eliminating the space function from the governing equations. An alternate solution was proposed by Huang and Sandman [14] to assume the existence of harmonic vibrations and eliminate the time variable by a Kantorovich time-averaging method.

The present investigation is concerned with the axisymmetric vibrations of a thin isotropic circular plate carrying a concentric rigid mass. Hamilton's principle is utilized to derive the von Karman form of the governing differential equations and the associated natural boundary conditions. Harmonic vibrations are assumed and the time variable is eliminated by a Kantorovich averaging method. Thus, the governing equations of motion are reduced to a pair of ordinary differential equations, which form a non-linear eigenvalue problem. Numerical solutions are obtainable by introducing the related initial value problem. Free vibrations of the plate-mass system are investigated for various mass and radius ratios, and the fundamental angular frequencies and its corresponding results are presented. Agreement with prior work is obtained for a flat circular plate when the radius of the rigid mass tends to zero and a mass ratio of unity is prescribed.

## BASIC DIFFERENTIAL EQUATIONS

### Formulation of Governing Equations

Consider a flat circular plate having an outer radius  $a$ , constant thickness  $h$ , and an attached concentric rigid mass,  $M_c$ . The radius of the rigid mass is  $b$  and equals the inner radius of the plate. Let the origin of polar coordinates  $(r, \theta, z)$  be located at the center of the middle plane of the plate as shown in Figure 1. The plate material is assumed to be elastic, homogeneous, and isotropic. Formulation of the governing equations of motion are based on the following assumptions:

1. Planes normal to the middle plane of the undeformed plate remain normal to the middle plane in the deformed state.
2. Normal stresses to the middle plane,  $\sigma_z$ , are small compared to other stress components and may be neglected from the stress-strain relationships.
3. Deflections of the plate are symmetrical with respect to the  $z$ -axis.
4. Effects of stretching of the middle plane are not negligible.

In accordance with these assumptions, the non-zero radial and circumferential strain-displacement relations are found to be [1,4]

$$\epsilon_r = u_r + \frac{1}{2}(w_r)^2 - zw_{rr} \quad (1a)$$

$$\epsilon_\theta = \frac{u}{r} - \frac{z}{r}w_r \quad (1b)$$

where  $\epsilon_r$  and  $\epsilon_\theta$  denote the radial and circumferential strain components respectively. Also,  $u(r,t)$  and  $w(r,t)$  denote the radial and transverse displacements of the middle plane of the plate. The subscripts of the displacement components represent partial derivatives, e.g.  $w_{rr} = \frac{\partial^2 w}{\partial r^2}$ .

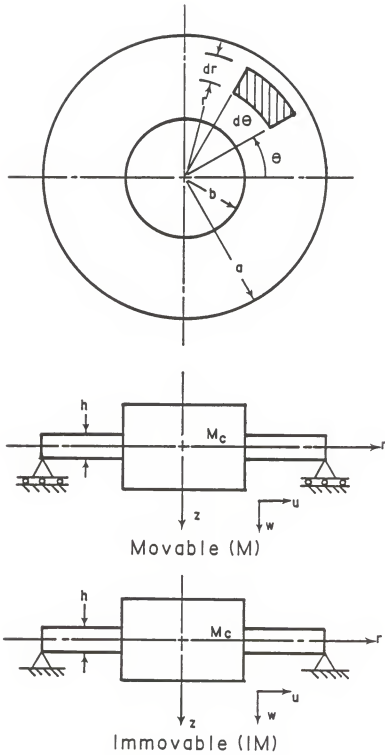


Fig. 1 Circular Plate and Polar Coordinate System.

From Hooke's Law the stress-strain relationships are derived as

$$\sigma_r = \frac{E}{1-\nu^2} (\epsilon_r + \nu\epsilon_\theta) \quad (2a)$$

$$\sigma_\theta = \frac{E}{1-\nu^2} (\epsilon_\theta + \nu\epsilon_r) \quad (2b)$$

Expressions for the radial and circumferential forces per unit length,  $N_r$  and  $N_\theta$ , are obtained by integrating the corresponding stresses over the thickness of the plate

$$N_r = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_r dz = \frac{12D}{h^2} (u_r + \frac{1}{2}w_r^2 + \frac{\nu}{r}u) \quad (3a)$$

$$N_\theta = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_\theta dz = \frac{12D}{h^2} (\frac{u}{r} + \nu u_r + \frac{\nu}{2}w_r^2) \quad (3b)$$

Also, expressions for the radial and circumferential moments per unit length,  $M_r$  and  $M_\theta$ , are obtained by integrating the moments of the forces about the middle plane

$$M_r = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_r z dz = -D(w_{rr} + \frac{\nu}{r}w_r) \quad (4a)$$

$$M_\theta = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_\theta z dz = -D(\frac{1}{r}w_r + \nu w_{rr}) \quad (4b)$$

Hamilton's Principle, [5,6], is used in the derivation of the governing equations of motion and related boundary conditions. It states that between any two instants,  $t_1$  and  $t_2$ , the first variation of the action integral is equal to zero, e.g.  $\delta I = 0$

where 
$$I = \int_{t_1}^{t_2} (T_1 + T_2 - V_1 - V_2) dt \quad (5)$$

and  $T_1$  = Kinetic energy of the circular plate

$T_2$  = Kinetic energy of the rigid mass

$V_1$  = strain energy due to bending

$V_2$  = strain energy due to stretching of middle plane

The kinetic energy of the circular plate, considering only the transverse velocity component, is given as

$$T_1 = \frac{\rho h}{2} \int_0^{2\pi} \int_b^a w_t^2 r dr d\theta$$

where  $\rho$  denotes the mass per unit volume of the plate. Integrating yields

$$T_1 = \pi \rho h \int_b^a w_t^2 r dr \quad (6)$$

The kinetic energy of the rigid mass is given as

$$T_2 = \frac{1}{2} M_c w_t^2 \Big|_{r=b} \quad (7)$$

The strain energy due to bending is given by [1,4]

$$V_1 = - \int_0^{2\pi} \int_b^a \left\{ \frac{M_r w_{rr}}{2} + \frac{M_\theta w_r}{2r} \right\} r dr d\theta$$

substituting  $M_r$  and  $M_\theta$ , then integrating yields

$$V_1 = \pi D \int_b^a \left\{ w_{rr}^2 + 2 \frac{w_r}{r} w_{rr} + \frac{1}{r^2} w_r^2 \right\} r dr \quad (8)$$

The strain energy due to stretching of the middle plane is given by [1,4]

$$V_2 = \int_0^{2\pi} \int_b^a \left( -\frac{N_r \epsilon_r^0}{2} + \frac{N_\theta \epsilon_\theta^0}{2} \right) r dr d\theta$$

where  $\epsilon_r^0 = u_r + \frac{1}{2}w_r^2$

and  $\epsilon_\theta^0 = \frac{u}{r}$

are the strain components acting on the middle plane. Substituting  $N_r$  and  $N_\theta$ , then integrating yields

$$V_2 = \frac{12\pi D}{h^2} \int_b^a \left\{ u_r^2 + u_r w_r^2 + \frac{\nu}{r} u (2u_r + w_r^2) + \frac{w_r^4}{4} + \frac{u^2}{r^2} \right\} r dr \quad (9)$$

Introducing equations (6), (7), (8), and (9) into (5) results in the following expression

$$I = \int_{t_1}^{t_2} \int_b^a F^*(r, t; w, w_r, w_{rr}, w_t, u, u_r) dr - G^*(r=b, t; w_t) dt \quad (10)$$

where 
$$F^* = \rho h r w_t^2 - D [r w_{rr}^2 + 2\nu w_r w_{rr} + \frac{1}{r} w_r^2] - \frac{12D}{h^2} [r u_r^2 + r u_r w_r^2 + \nu u w_r^2 + \frac{1}{r} u^2 + 2\nu u u_r + \frac{1}{4} r w_r^2]$$

and 
$$G^* = \frac{1}{2} M_c w_t^2 \Big|_{r=b}$$

The displacement functions  $w(r, t)$  and  $u(r, t)$  are given an admissible variation  $\epsilon \eta(r, t)$  and  $\epsilon \zeta(r, t)$  respectively, where  $\eta(r, t)$  and  $\zeta(r, t)$  are continuously differentiable functions which satisfy the constraints of the plate, and  $\epsilon$  is an arbitrary infinitesimal constant. The change in the action integral  $I$  is given in the following expression.

$$\Delta I = \int_{t_1}^{t_2} \left\{ \pi \int_b^a [F^*(r, t; w + \epsilon \eta, w_r + \epsilon \eta_r, w_{rr} + \epsilon \eta_{rr}, u + \epsilon \zeta, u_r + \epsilon \zeta_r, w_t + \epsilon \eta_t) - F^*] dr - [G^*(r=b, t; w_t + \epsilon \eta_t) - G^*] \right\} dt \quad (11)$$

Properly expanding the integrand, integrating by parts, and retaining only the linear terms of  $\epsilon$  contained in  $\Delta I$ , produces the first variation,  $\delta I$ . The resulting expression is listed in Appendix A. Extremization of the resulting integrand is accomplished by noting the necessary condition that the double and single integrals must vanish separately. Therefore, combining the double integrals with like terms of the admissible functions and equating to zero yield the governing equations of motion,

$$D[w_{rrrr} + \frac{2}{r}w_{rrr} - \frac{1}{r^2}w_{rr} + \frac{1}{r^3}w_r] + \rho h w_{tt} - \frac{12D}{h^2} \left[ \frac{u_r w_r}{r} \right] \quad (12a)$$

$$+ u_{rr} w_r + u_r w_{rr} + \frac{v}{r} u_r w_r + \frac{v}{r} u w_{rr} + \frac{w_r^3}{2r} + \frac{3}{2} w_r^2 w_{rr} = 0$$

$$\frac{u_r}{r} + u_{rr} + \frac{1}{2r} w_r^2 + w_r w_{rr} - \frac{v}{2r} w_r^2 - \frac{u}{r^2} = 0 \quad (12b)$$

while combining the single integrals with like terms of the admissible functions and equating to zero yield the natural boundary conditions,

$$2\pi D r (w_{rr} + \frac{v}{r} w_r) \eta_r \Big|_b^a = 0 \quad (13a)$$

$$\frac{24\pi D}{h^2} r (u_r + \frac{1}{2} w_r^2 + \frac{v}{r} u) \zeta \Big|_b^a = 0 \quad (13b)$$

$$2\pi D r (w_{rrr} + \frac{w_{rr}}{r} - \frac{w_r}{r^2}) \eta \Big|_b^a - \frac{24\pi D}{h^2} r w_r (u_r + \frac{v}{r} u + \frac{w_r^2}{2}) \zeta \Big|_b^a + M_c w_{tt} \eta \Big|_b = 0 \quad (13c)$$

Introducing a stress function  $\psi(r,t)$ , which satisfies the equilibrium equation of the plate, given by

$$\psi = rN_r \quad \frac{\partial \psi}{\partial r} = N_\theta$$

where

$$N_r = \frac{12D}{h^2} \left( u_r + \frac{w_r^2}{2} + \frac{v}{r} u \right) \quad \text{(from (3))}$$

and

$$N_\theta = \frac{12D}{h^2} \left( \frac{u}{r} + v u_r + \frac{v}{2} w_r^2 \right)$$

transforms the governing equations of motion, (12a) and (12b), into

$$\frac{D}{r} \left[ r \left( \frac{1}{r} (r w_r)_r \right)_r \right] + \rho h w_{tt} - \frac{1}{r} (\psi w_r)_r = 0 \quad (14a)$$

$$\psi_{rr} + \frac{1}{r} \psi_r - \frac{1}{r^2} \psi + \frac{Eh}{2r} (w_r)^2 = 0 \quad (14b)$$

Equations (3) may be used to derive a useful expression of the radial displacement,  $u(r,t)$ , needed later in the definition of the boundary conditions

$$u(r,t) = \frac{r}{Eh} (N_\theta - v N_r) . \quad (15)$$

Using the following dimensionless variables

$$\begin{aligned} X &= \frac{w}{a} & \xi &= \frac{r}{a} \\ R &= \frac{b}{a} & \gamma &= \frac{M_c}{\pi b^2 \rho h} \\ \phi &= \frac{\psi}{Eha} & \tau &= t \left( \frac{D}{4} \right)^{\frac{1}{2}} \end{aligned} ,$$

transforms the governing equations of motions into the following forms:



$$X_{\xi\xi\xi\xi} + \frac{2}{\xi}X_{\xi\xi\xi} - \frac{1}{\xi^2}X_{\xi\xi} + \frac{1}{\xi^3}X_{\xi} + X_{\tau\tau} - \frac{12(1-\nu^2)}{\xi} \left(\frac{a}{h}\right)^2 (\phi X_{\xi})_{\xi} = 0 \quad (16a)$$

$$\phi_{\xi\xi} + \frac{1}{\xi}\phi_{\xi} - \frac{1}{\xi^2}\phi + \frac{1}{2\xi}(X_{\xi})^2 = 0 \quad (16b)$$

### Boundary Conditions

Depending on the type of support on the outer edge and the physical constraint on the inner edge of the circular plate, the geometric boundary conditions are supplemented by the natural boundary conditions of equations (13a), (13b), and (13c) providing a complete set of boundary conditions to satisfy.

An edge is termed immovable if it is rigidly held so as to eliminate any radial displacement. The geometric boundary conditions for a hinged-immovable plate with an attached concentric rigid mass are as follows:

$$\begin{aligned} (w)_{r=a} &= 0 & (w_r)_{r=b} &= 0 \\ (u)_{r=a} &= 0 & (u)_{r=b} &= 0 \end{aligned}$$

An edge is termed movable if it is allowed radial displacement, thus eliminating radial forces at the boundary. The geometric boundary conditions for a hinged-movable plate with an attached concentric rigid mass are given as:

$$(w)_{r=a} = 0 \quad (w_r)_{r=b} = 0 \quad (u)_{r=b} = 0$$

Utilizing the geometric and natural boundary conditions, the prescribed dimensionless variables and equation (15), Table 1 represents appropriate

non-dimensional boundary conditions for simply supported plate-mass systems.

Table 1. Non-dimensional Boundary Conditions

| Type of Edge        | Boundary Condition   |   |
|---------------------|--|---|
|                     | $\xi = 1$  | $\xi = R$   |
| HINGED<br>IMMOVABLE | $X = 0$ $X_{\xi\xi} + \frac{\nu}{\xi}X_{\xi} = 0$ $\phi_{\xi} - \frac{\nu}{\xi}\phi = 0$ | $X_{\xi} = 0$ $X_{\xi\xi\xi} + \frac{1}{\xi}X_{\xi\xi} + \frac{\xi}{2}\gamma X_{\tau\tau} = 0$ $\phi_{\xi} - \frac{\nu}{\xi}\phi = 0$ |
| HINGED<br>MOVABLE   | $X = 0$ $X_{\xi\xi} + \frac{\nu}{\xi}X_{\xi} = 0$ $\frac{\phi}{\xi} = 0$                 | $X_{\xi} = 0$ $X_{\xi\xi\xi} + \frac{1}{\xi}X_{\xi\xi} + \frac{\xi}{2}\gamma X_{\tau\tau} = 0$ $\phi_{\xi} - \frac{\nu}{\xi}\phi = 0$ |

## APPROXIMATE ANALYSIS

An exact solution to the problem defined by the non-linear differential equations of (16a) and (16b) which must satisfy the associated boundary conditions of Table (1) is at present unknown. Thus, the analysis and solution of the problem is accomplished by approximate methods. Approximate solutions to large amplitude vibration problems are commonly obtained by using function space methods to eliminate the space coordinate with an assumed mode shape function [7]. Thus, the non-linear partial differential equations are reduced to a set of non-linear ordinary differential equations with time,  $t$ , as an independent variable. In this investigation a time function is assumed, then a Kantorovich averaging method is employed to reduce the non-linear partial differential equations to a set of ordinary differential equations.

### Kantorovich Averaging Method

The Kantorovich method is used to find an assumed time mode solution to the equations (16) which satisfies the boundary conditions of Table (1) [9]. Assuming a harmonic solution for the system of equations (16), the behavior of the plate is given by:

$$X(\xi, \tau) = Ag(\xi) \sin \omega \tau \quad (17a)$$

$$\phi(\xi, \tau) = A^2 f(\xi) \sin^2 \omega \tau \quad (17b)$$

where  $A$  is an amplitude parameter,  $\omega$  is the angular frequency, and  $g(\xi)$  and  $f(\xi)$  are shape functions of vibration to be determined.

Substituting equations (17) into the differential equation (16b) yields the expression

$$f'' + \frac{1}{\xi} f' - \frac{1}{2\xi} f - \frac{1}{2\xi} (g')^2 = 0 \quad (18)$$

where the superscript denotes total differentiation, i.e.  $f' = \frac{df}{d\xi}$ .

Noting that equations (17) cannot satisfy the differential eq. (16a)

for all  $\tau$ , the residual may be found and minimized by the Ritz-Kantorovich method. For any instant in time  $\tau$ , the virtual work of the transverse forces moving through a virtual displacement  $\delta X = \delta G \sin \omega \tau$  is given by the following integral.

$$\delta W = \int_R \left[ X_{\xi\xi\xi\xi} + \frac{2}{\xi} X_{\xi\xi\xi} - \frac{1}{\xi^2} X_{\xi\xi} + \frac{1}{\xi^3} X_{\xi} + X_{\tau\tau} \right. \\ \left. - \frac{12(1-\nu^2)}{\xi} \left( \frac{a}{h} \right)^2 (\phi X_{\xi})_{\xi} \right] \delta X_{\xi} d\xi \quad (19)$$

Substituting eqs. (17) and  $\delta X$  into the above integral and equating the average virtual work over one complete period of oscillation to zero, that is

$$\int_0^{2\pi/\omega} \delta W d\tau = 0 \quad (20)$$

yields, upon integration with respect to  $\tau$ , the following expression:

$$g'''' + \frac{2}{\xi} g'' - \frac{1}{\xi^2} g'' + \frac{1}{\xi^3} g' - \lambda g - \frac{9(1-\nu^2)}{\xi} \alpha (g'f)' = 0 \quad (21)$$

where  $\lambda = \omega^2$  and  $\alpha = \left( \frac{Aa}{h} \right)^2$  are additional dimensionless parameters associated with the angular frequency and amplitude respectively. Thus, the time

variable is eliminated from the governing differential equations with an average minimum error over one complete cycle of assumed motion. Therefore, the motion of the plate-mass system becomes governed by the pair on non-linear ordinary differential eqs. (18) and (21).

Similarly, by substitution of equations (17), the boundary conditions of Table (1) are transformed to the final form given in Table (2). In addition to the boundary conditions of Table (2), a unique relationship between  $\alpha$  and  $\lambda$  is assured by imposing the following condition:

$$g|_{\xi=R} = 1. \quad (22)$$

Table 2. Final Non-dimensional Boundary Conditions

| Type of Edge     | Boundary Condition  |   |
|------------------|---|---|
|                  | $\xi = 1$   | $\xi = R$   |
| HINGED IMMOVABLE | $g = 0$<br>$g'' + \frac{\nu}{\xi}g' = 0$<br>$f' - \nu f = 0$    | $g = 1$<br>$g' = 0$<br>$f' - \nu f = 0$<br>$g'''' + \frac{1}{\xi}g''' - \frac{\xi}{2}\gamma\lambda g = 0$ |
| HINGED MOVABLE   | $g = 0$<br>$g'' + \frac{\nu}{\xi}g' = 0$<br>$\frac{f}{\xi} = 0$ | $g = 1$<br>$g' = 0$<br>$f' - \nu f = 0$<br>$g'''' + \frac{1}{\xi}g''' - \frac{\xi}{2}\gamma\lambda g = 0$ |

## NUMERICAL ANALYSIS

The equations (18) and (21) along with the boundary conditions of Table (2) comprise a nonlinear two-point boundary value problem describing the harmonic motion of a circular plate carrying a concentric rigid mass. Although solutions to boundary value problems are complicated, they may be solved by numerical integration to the associated initial value problem.

### Matrix Formulation

In order to solve the problem numerically, equations (18) and (21) are rewritten as six first order differential equation as follows

$$\frac{d\bar{Y}}{d\xi} = \bar{H}(\xi, \bar{Y}; \alpha, \lambda, \gamma) \quad R < \xi < 1 \quad (23a)$$

$$\text{where } \bar{Y}(\xi) = \begin{bmatrix} g \\ g' \\ g'' \\ g''' \\ f \\ f' \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix}$$

$$\text{and } \bar{H} = \begin{bmatrix} y_2 \\ y_3 \\ y_4 \\ -\frac{2}{\xi}y_4 + \frac{1}{\xi^2}y_3 - \frac{1}{\xi^3}y_2 + \lambda y_1 + \frac{9(1-\nu^2)}{\xi} \alpha (y_2y_6 + y_3y_5) \\ y_6 \\ -\frac{1}{\xi}y_6 + \frac{1}{\xi^2}y_5 - \frac{1}{2\xi}y_2^2 \end{bmatrix}$$

The boundary and normalization conditions of Table (2) may be written in the generalized form:

$$\overline{MY}(R) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (23b)$$

and

$$\overline{NY}(1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (23c)$$

where M and N are 4 x 6 and 3 x 6 coefficient matrices respectively, shown in Table (3) for the two different sets of boundary conditions considered.

Table 3. Coefficient Matrices (M) and (N) of Boundary Conditions

| Type of Edge        | (M)   | (N)  |
|---------------------|---|--|
| Hinged<br>Immovable | $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{R}{2}\gamma\lambda & 0 & \frac{1}{R} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{v}{R} & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & v & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -v & 1 \end{bmatrix}$ |
| Hinged<br>Movable   | $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{R}{2}\gamma\lambda & 0 & \frac{1}{R} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{v}{R} & 1 \end{bmatrix}$ | $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & v & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$  |

Initial Value Problem

To conveniently study the system of equations (23), a related initial value problem may be expressed as

$$\frac{d\bar{Z}}{d\xi} = \bar{H}(\xi, \bar{Z}; \alpha, \lambda, \gamma) \quad (24a)$$

$$\text{where } \bar{Z}(R) = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{bmatrix}_{\xi=R} = \begin{bmatrix} 1 \\ 0 \\ \eta_1 \\ -\frac{\eta_1}{R} + \frac{R}{2}\gamma\lambda \\ \eta_2 \\ \frac{\nu}{R}\eta_2 \end{bmatrix} \quad (24b)$$

Equation (24b) represents the initial value vector constructed from the boundary and normalization conditions at  $\xi = R$  and  $\eta_1$ ,  $\eta_2$  and  $\lambda$  are unknown adjustable initial value parameters. Substituting equations (24b) into (23b) for  $Y(R)$  yields the following system of equations

$$M\bar{Z}(R) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (25)$$

that must be satisfied at the inner boundary  $\xi = R$ .

A solution to the initial value problem of (24) may be symbolically written as:

$$\bar{Z}(\xi) = \bar{Z}(R) + \int_R^\xi \bar{H}(\xi, \bar{Z}; \bar{\eta}, \alpha, \gamma) d\xi \quad (26)$$

where  $\bar{\eta} = [\eta_1 \ \eta_2 \ \lambda]^T$



Given the parameters  $\alpha$  and  $\gamma$ , the values of  $\eta_1$ ,  $\eta_2$  and  $\lambda$  are searched for such that a solution to equations (25) also satisfies the boundary conditions of (23c),

$$N\bar{Z}(1; \bar{\eta}, \alpha, \gamma) = \bar{0} \quad (27)$$

A solution to the boundary value problem defined by equations (18) and (21) is found when a continuous set of solutions satisfying equations (23) are obtained for the related initial value problem (24). Therefore, for given values of  $\alpha$  and  $\gamma$ ,

$$\bar{Y}(\xi) = \bar{Z}(\xi; \bar{\eta}^*, \alpha, \gamma)$$

represents a solution to the boundary value problem where  $\bar{\eta}^*$  is a root of equation (27).

By direct application of Newtons method [8], a root,  $\bar{\eta}^*$ , of equation (27) may be found. Starting with an initial guess,  $\bar{\eta}_1$ , and fixed parameters  $\alpha$  and  $\gamma$ , the iterative sequence

$$\bar{\eta}_{k+1} = \bar{\eta}_k + \Delta\bar{\eta}_k ; k = 1, 2, 3, \dots \quad (28)$$

is generated. Retaining only first order terms of the Taylor series expansion of (27) about  $\bar{\eta}_k$  provides the linear correction,

$$\Delta\bar{\eta}_k = - [N(J_1)_k]^{-1} N\bar{Z}(1; \bar{\eta}_k, \alpha, \gamma) , \quad (29)$$

at the k-th step of iteration. The Jacobian matrix  $(J_1)$  is defined as

$$(J_1) = \left( \frac{\partial \bar{Z}}{\partial \bar{\eta}} \right)_{\xi=1} = \left( \frac{\partial Z_i}{\partial \eta_j} \right)_{\xi=1} ; \begin{matrix} i=1, \dots, 6 \\ j=1, 2, 3 \end{matrix}$$

and represents the change of final values with respect to a change in the initial  $\bar{n}$ . The term  $N\bar{Z}(1; \bar{n}, \alpha, \gamma)$  represents the error vector at the  $k$ -th iteration. Given constant values of  $\alpha$  and  $\gamma$ , the following vectors provide the linear correction of the initial values  $n_1$ ,  $n_2$  and  $\lambda$  for hinged immovable and hinged movable edge constraints respectively.

$$\begin{bmatrix} \Delta n_1 \\ \Delta n_2 \\ \Delta \lambda \end{bmatrix} = - \begin{bmatrix} \frac{\partial z_1}{\partial n_1} & \frac{\partial z_1}{\partial n_2} & \frac{\partial z_1}{\partial \lambda} \\ \nu \frac{\partial z_2}{\partial n_1} + \frac{\partial z_3}{\partial n_1} & \nu \frac{\partial z_2}{\partial n_2} + \frac{\partial z_3}{\partial n_2} & \nu \frac{\partial z_2}{\partial \lambda} + \frac{\partial z_3}{\partial \lambda} \\ -\nu \frac{\partial z_5}{\partial n_1} + \frac{\partial z_6}{\partial n_1} & -\nu \frac{\partial z_5}{\partial n_2} + \frac{\partial z_6}{\partial n_2} & -\nu \frac{\partial z_5}{\partial \lambda} + \frac{\partial z_6}{\partial \lambda} \end{bmatrix}^{-1} \begin{bmatrix} z_1 \\ \nu z_2 + z_3 \\ -\nu z_5 + z_6 \end{bmatrix} \quad (31)$$

$$\begin{bmatrix} \Delta n_1 \\ \Delta n_2 \\ \Delta \lambda \end{bmatrix} = - \begin{bmatrix} \frac{\partial z_1}{\partial n_1} & \frac{\partial z_1}{\partial n_2} & \frac{\partial z_1}{\partial \lambda} \\ \nu \frac{\partial z_2}{\partial n_1} + \frac{\partial z_3}{\partial n_1} & \nu \frac{\partial z_2}{\partial n_2} + \frac{\partial z_3}{\partial n_2} & \nu \frac{\partial z_2}{\partial \lambda} + \frac{\partial z_3}{\partial \lambda} \\ \frac{\partial z_5}{\partial n_1} & \frac{\partial z_5}{\partial n_2} & \frac{\partial z_5}{\partial \lambda} \end{bmatrix}^{-1} \begin{bmatrix} z_1 \\ \nu z_2 + z_3 \\ z_5 \end{bmatrix} \quad (32)$$

If the initial values of  $\bar{n}$  are chosen in a sufficiently small proximity of the root  $\bar{n}^*$ , the sequence (28) will converge to the root  $\bar{n}^*$  of (27).

Since an explicit solution to the initial value problem (24) cannot be readily obtained, due to the nonlinearity of the vector function  $\bar{H}$ , an expression for the Jacobian matrix cannot be determined directly. Therefore, a technique for constructing the Jacobian matrix at any iteration step is necessary. Differentiating the initial value problem (24) with respect to  $\bar{n}$  yields the following variational equations.

$$\frac{d}{d\xi} \left( \frac{\partial \bar{Z}}{\partial n_1} \right) = \left( \frac{\partial \bar{H}}{\partial Z} \right) \left( \frac{\partial \bar{Z}}{\partial n_1} \right) \quad (33a)$$

$$\left( \frac{\partial \bar{Z}}{\partial n_1} \right) \Big|_{\xi=R} = \left[ 0 \ 0 \ 1 \ -\frac{1}{R} \ 0 \ 0 \right]^T \quad (33b)$$

$$\frac{d}{d\xi} \left( \frac{\partial \bar{Z}}{\partial n_2} \right) = \left( \frac{\partial \bar{H}}{\partial Z} \right) \left( \frac{\partial \bar{Z}}{\partial n} \right) \quad (33c)$$

$$\left( \frac{\partial \bar{Z}}{\partial n_2} \right) \Big|_{\xi=R} = \left[ 0 \ 0 \ 0 \ 0 \ 1 \ \frac{\nu}{R} \right]^T \quad (33d)$$

$$\frac{d}{d\xi} \left( \frac{\partial \bar{Z}}{\partial \lambda} \right) = \left( \frac{\partial \bar{H}}{\partial Z} \right) \left( \frac{\partial \bar{Z}}{\partial \lambda} \right) + \left( \frac{\partial \bar{H}}{\partial \lambda} \right) \quad (33e)$$

$$\left( \frac{\partial \bar{Z}}{\partial \lambda} \right) \Big|_{\xi=R} = \left[ 0 \ 0 \ 0 \ \frac{R}{2\gamma} \ 0 \ 0 \right]^T \quad (33f)$$

Performing the operations indicated by equations (33a), (33c) and (33e) results in eighteen first order equations which are presented in Appendix B. The initial conditions for the associated variational problem are given by equations (33b), (33d) and (33f). For given values of  $\alpha$  and  $\gamma$ , and a vector  $\bar{n}$ , the initial value problem (24) may be integrated numerically by a Runge-Kutta method. Evaluation of the resulting solution to the variational problem at  $\xi = 1$  provides the Jacobian matrix ( $J_1$ ) corresponding to the given values of  $\alpha$ ,  $\gamma$  and  $\bar{n}$ . Therefore, by setting  $\bar{n} = \bar{n}^1$  and integrating equations (24) and (33) simultaneously from  $\xi = R$  to  $\xi = 1$ , the first corrected vector,  $\bar{n}^2$ , can be calculated from equations (31) or (32). Repeating the same operations with  $\bar{n} = \bar{n}^2$ , one obtains the second corrected vector,  $\bar{n}^3$ . Repetition of this procedure yields the desired solution  $\bar{n}^*$ , a root of equation (27).

Once a root  $\bar{n}^j$  has been obtained for a given  $\alpha = \alpha^j$  and  $\gamma$ , the value of  $\alpha$  can be perturbed,

$$\alpha = \alpha^j + \Delta\alpha^j = \alpha^{(j+1)} \quad j = 1, 2, \dots, m$$

For this value of  $\alpha$ , the iteration is restarted from the previous root  $\bar{n}^j$ . If  $\Delta\alpha^j$  is chosen sufficiently small,  $\bar{n}^j$  will be within the new contraction domain of Newton's method, and iteration will converge to a new root  $\bar{n}^{(j+1)}$  corresponding to  $\alpha = \alpha^{(j+1)}$ . The range of  $\alpha$  is limited to finite amplitudes for which equations (18) and (21) were derived.

In determining the vibration characteristics of a plate-mass system, the outline of the numerical computation procedure used is as follows.

By first considering the linear vibration case, with  $\alpha$  set to zero, initial estimates may be obtained for  $\bar{n}^{-1}$  [11]. Next, the initial value problems (24) and (33) are integrated numerically over the interval  $[R, I]$  with a fourth order Runge-Kutta-Gill method having a step size  $\Delta u = 1/40$ . Successive corrections and integrations are carried out until the final values,  $\bar{Z}(1)$ , satisfy the inequality,

$$\max_{1 \leq i \leq 3} \left| \sum_{j=1}^6 n_{ij} z_j(1) \right| \leq 0.1 \times 10^{-5}$$

where  $n_{ij} = (N)$  and the prescribed error is consistent with the Runge-Kutta-Gill method employed. Having obtained an approximate solution for the linear vibration of the system, the corresponding values  $n_1$ ,  $n_2$  and  $\lambda$  are stored. Successively incrementing  $\alpha$  and starting iteration from the previously obtained root,  $\bar{n}^*$ , provides the resonance curves of the plate-mass system. Three or four iterations were required for

convergence to a solution when the increments,  $\Delta\alpha^i$ , in the value of  $\alpha$  satisfied the constraint  $0.0 < |\Delta\alpha^i| \leq 0.1$ . Two Fortran computer programs are listed in Appendix C for the purpose of illustrating specific steps in the numerical solution to hinged immovable and hinged movable plate-mass systems.

## RESULTS OF NUMERICAL COMPUTATION

To support the method of solution in the present investigation, the plate-mass system may be reduced for comparison to the characteristics of a flat circular plate with a hinged immovable outer boundary. Since the mass ratio,  $\gamma$ , is defined as the ratio of the rigid mass to the mass of the plate it replaces, setting  $\gamma = 1$  describes a plate-mass system having the same mass as a flat circular plate. As the radii ratio tends to zero, the conditions of zero slope and radial deflection are approached. Appropriately,  $R = 0.025$  was selected since it falls on a radius defined by a convenient step size in the numerical integration technique. The response curve  $(W_R/h - \omega)$  obtained by setting  $\gamma = 1$  and  $R = 0.025$  is presented in Figure 2, where  $W_R/h$  is the relative amplitude at the inner boundary and  $\omega$  is the dimensionless angular frequency. Huang and Al-Khattat [12] also employed the von Karman equations to study the characteristics of a flat circular plate. Values taken from their investigation are presented by circles in Figure 2. The close agreement supports the correct method of solution used in the present investigation.

Although the mass of the system under consideration is the same as that of a flat circular plate, the present investigation examines a flat plate with a small finite rigid region, while Huang and Al-Khattat had examined a complete elastic flat circular plate. Therefore, the response of the plate-mass system is slightly higher than that in [12], due to an increase in the stiffness of the plate.

When a rigid mass occupying a finite area is concentrically added to a thin circular plate, the effect upon the response is not immediately

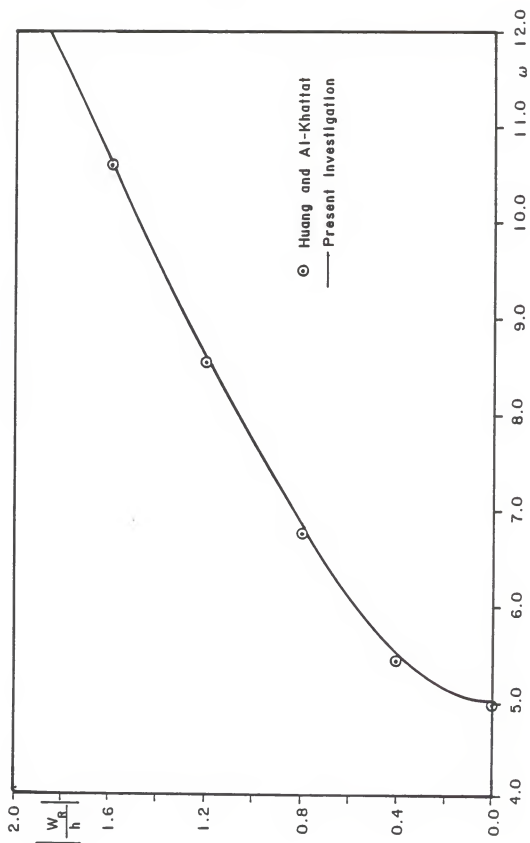


Fig. 2 Harmonic Response of Hinged Immovable Circular Plate .

obvious. Insertion of a rigid mass produces a change in kinetic energy which must compete with a change in potential energy due to the additional stiffening effect at the inner boundary. Thus, the behavior of the plate-mass system is dependent upon the specified mass and radii ratios given to the system under consideration.

An examination of the linear vibration characteristics is found to facilitate the study of the nonlinear vibration of plate-mass systems. Accordingly, the nonlinear problem is reduced by setting the amplitude parameter to zero, ie.  $\alpha = 0$ . The linear problem may then be solved for various mass and radii ratios. The dimensionless linear frequencies presented as a function of radii ratio are shown in Figure 3 and are valid for both immovable and movable edge constraints. The linear problem has been studied by other authors [10,13] where similar curves were produced.

For mass ratios of less than 2.0, the stiffening produced by insertion of a rigid core of any radius will increase the natural frequency of the system. Increasing the radii ratio simultaneously increases the stiffness and the frequency of the plate-mass system. As the mass ratio is assigned to values greater than 2.0, the frequencies must be examined more carefully. To gain further insight, a refined view of Figure 3 is presented in Figure 4. For mass ratios greater than 2.0, there exists a critical radii ratio for which the frequency is a minimum. As the radii ratio is increased, the frequency decreases for radii ratios less than the critical value. Hence, the effect of additional mass is seen to have more influence than the increase in stiffness on the frequency. The frequency increases as the radii ratio



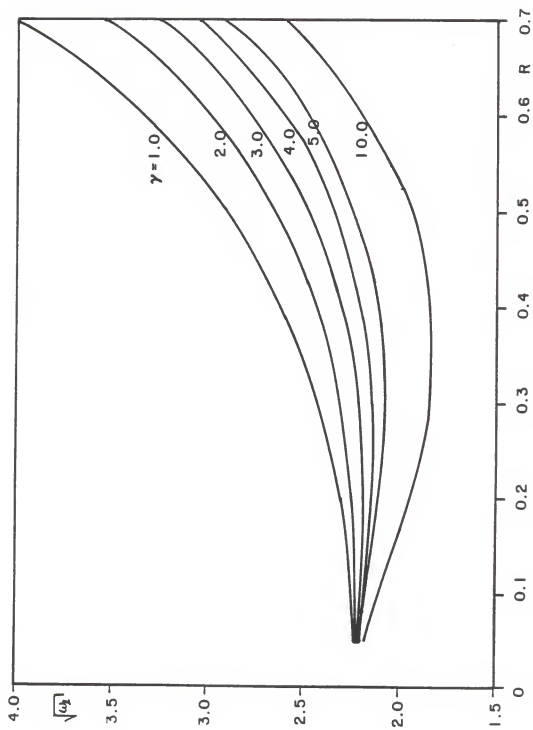


Fig. 3 Linear Resonant Frequencies of Hinged Immovable and Movable Plate-Mass Systems.

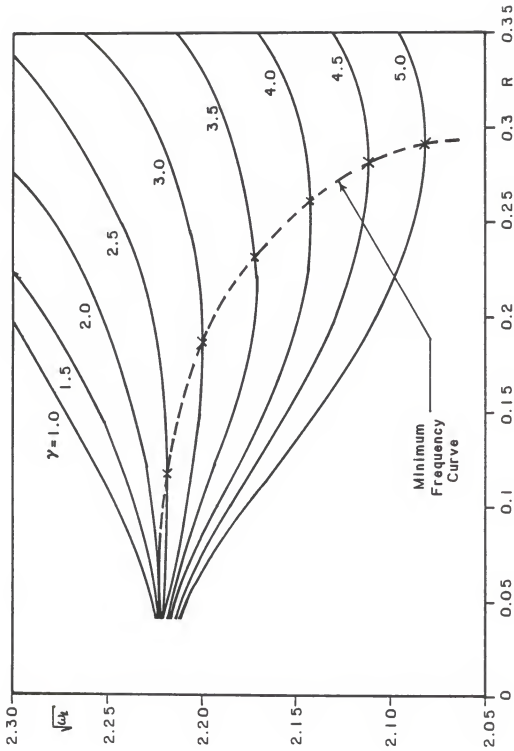


Fig. 4 Linear Resonant Frequencies of Hinged Immovable and Movable Plate-Mass Systems.

is increased beyond the critical value. Here, the influence of the stiffening effect becomes dominant over any addition of mass. Referring to Figure 4, the dashed curve represents the critical radii ratios where the transition occurs.

Since the primary intention of the present investigation is to determine the response of nonlinear vibratory plate-mass systems, determining the exact critical radii ratios of Figure 4 was not attempted. It will later be apparent how the critical radii ratios are related to the response of plate-mass systems at finite amplitudes.

Advancing the problem to nonlinear considerations, Figure 5 represents how the effects of mass and radii ratios and edge constraints influence the response of the plate-mass system. Similarities are observed between the linear case,  $\alpha = 0$ , and the curves in Figure 5,  $\alpha = 1.0$ . Obviously, the movable edge condition produces lower frequencies than the immovable edge due to the greater flexibility.

Nonlinear response curves obtained from numerical results are given in Figures 6-21. The figures are presented with dimensionless relative amplitudes,  $W_R/h$ , plotted as a function of dimensionless angular frequency,  $\omega$ . Response curves are given for constant radii ratios in Figures 6-11. For both immovable and movable edge constraints, the addition of mass decreases the frequency of the system. The response curves of Figures 12-17 are presented with constant mass ratios, providing an alternate view to the nonlinear behavior of plate-mass systems. Recalling the observations found for linear vibratory systems, the effects of varying the radii ratio for mass ratios greater than 2.0 are evident. The frequency decreases for radii ratios less than the

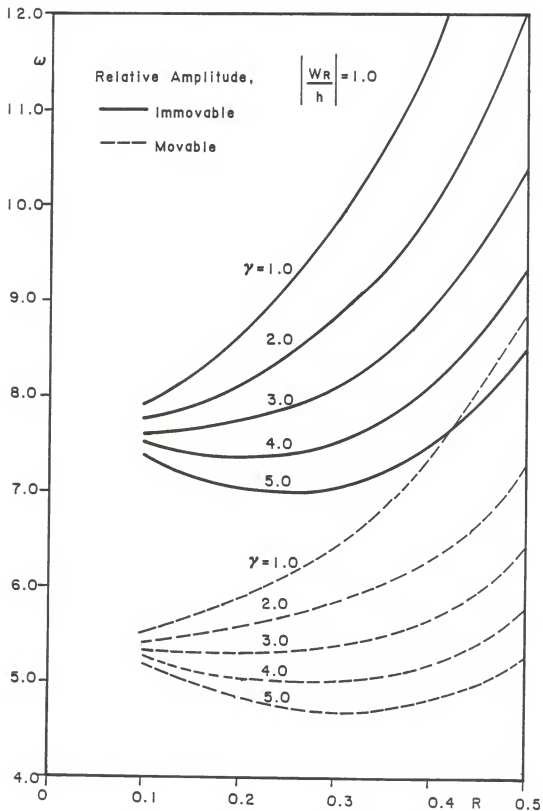
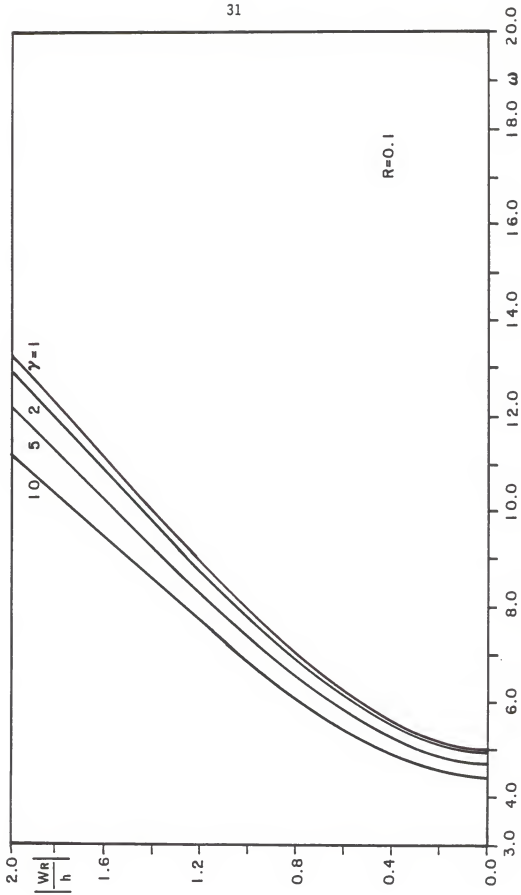
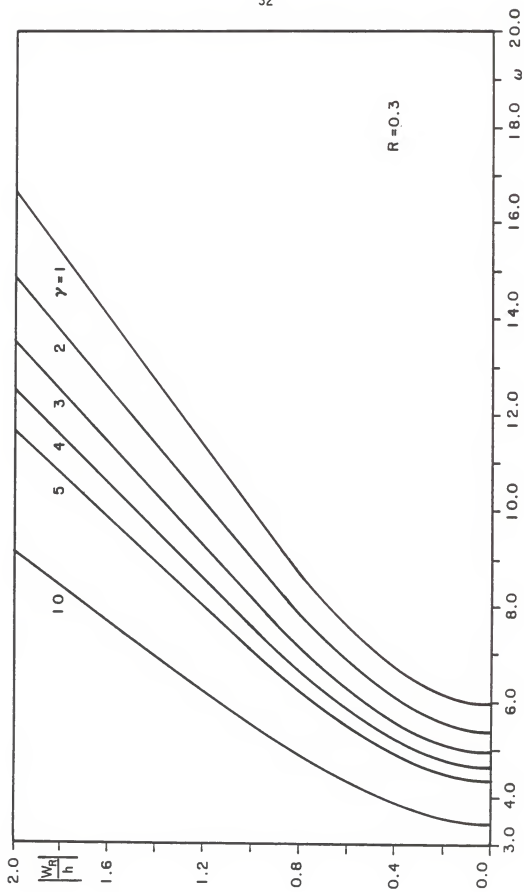


Fig. 5 Nondimensional Angular Frequencies of Hinged Immovable and Movable Plate-Mass Systems.

critical value while increasing for radii ratios greater than the critical value. For design considerations, Figures 18-21 present the nonlinear response envelopes of immovable and movable plate-mass systems.

Fig. 6 Harmonic Response of Hinged Immovable Plate-Mass System with  $R = 0.1$ .

Fig. 7. Harmonic Response of Hinged Immovable Plate-Mass System with  $R = 0.3$ .

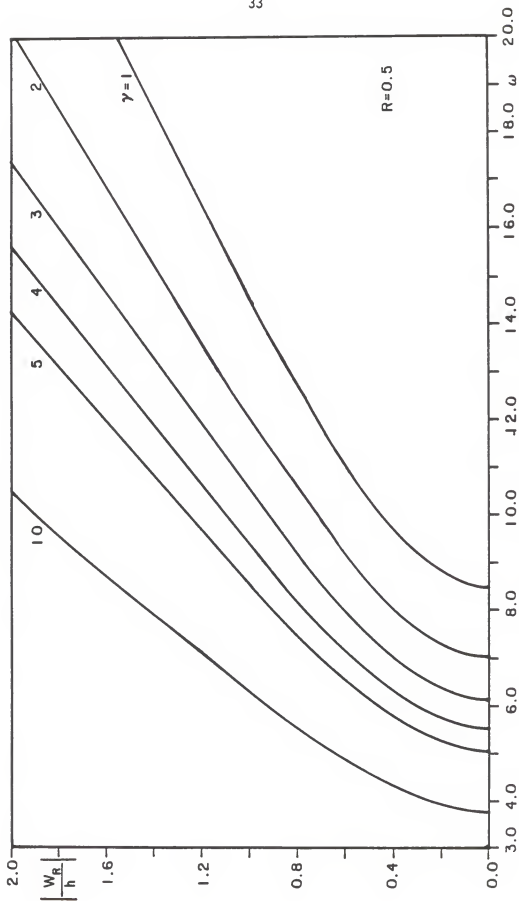
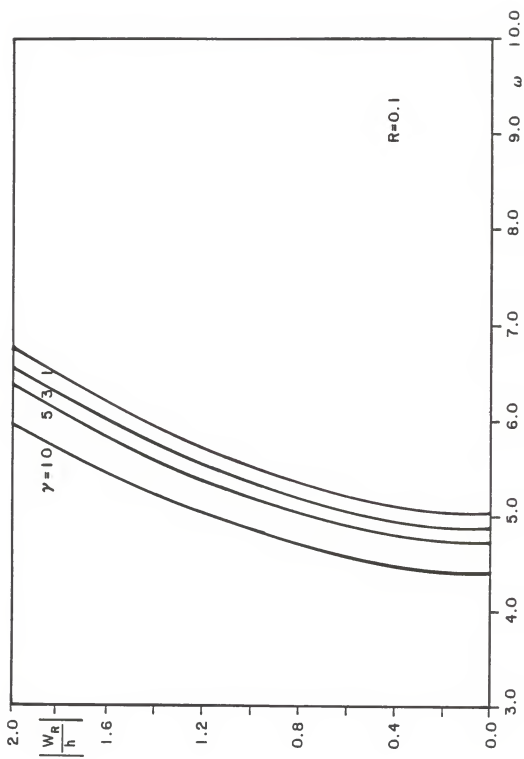
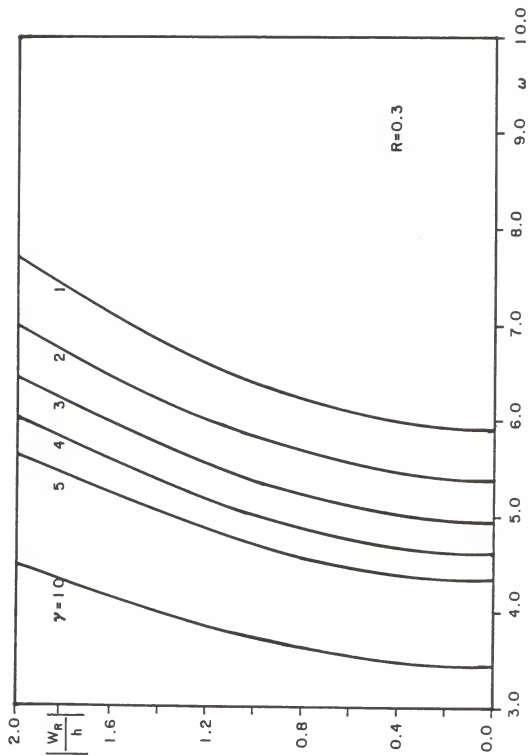


Fig. 8 Harmonic Response of Hinged Immovable Plate-Mass System with  $R = 0.5$ .



Fig. 9 Harmonic Response of Hinged Movable Plate-Mass System with  $R = 0.1$ .

Fig. 10 Harmonic Response of Hinged Movable Plate-Mass System with  $R = 0.3$ .

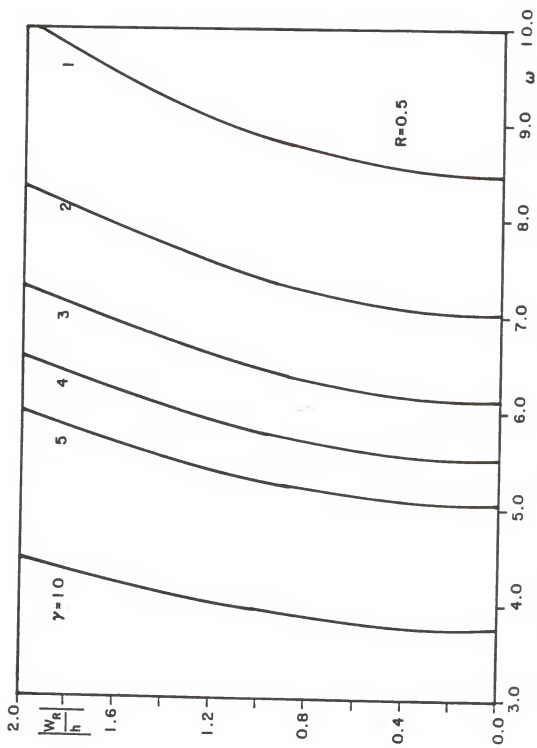
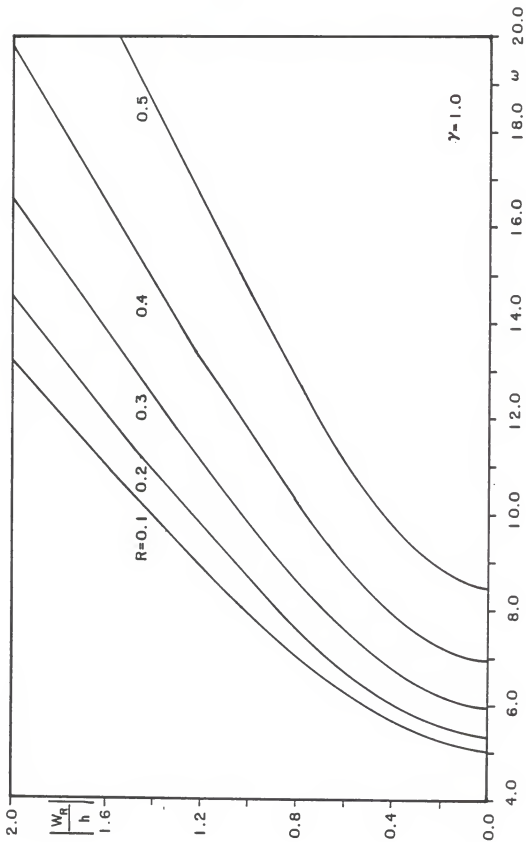
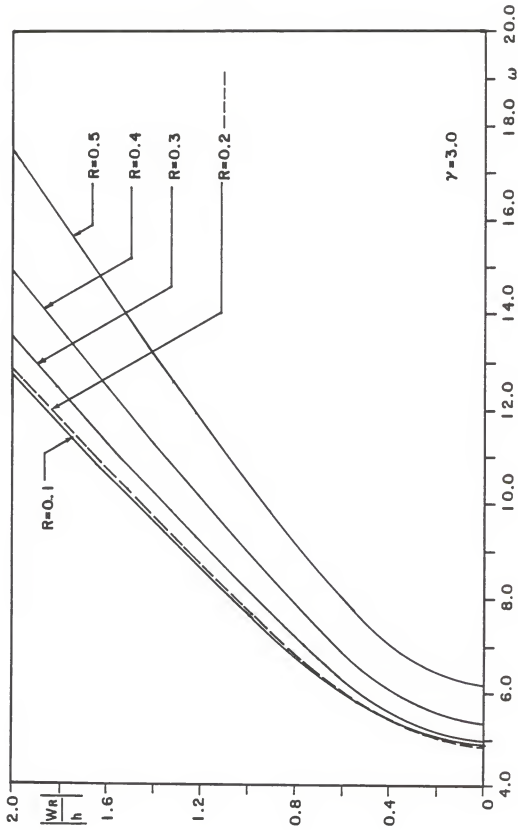


Fig. 11 Harmonic Response of Hinged Movable Plate-Mass System with  $R = 0.5$ .

Fig. 12 Harmonic Response of Hinged Immovable Plate-Mass System with  $\gamma = 1.0$ .

Fig. 13 Harmonic Response of Hinged Immovable Plate-Mass System with  $\gamma = 3.0$ .

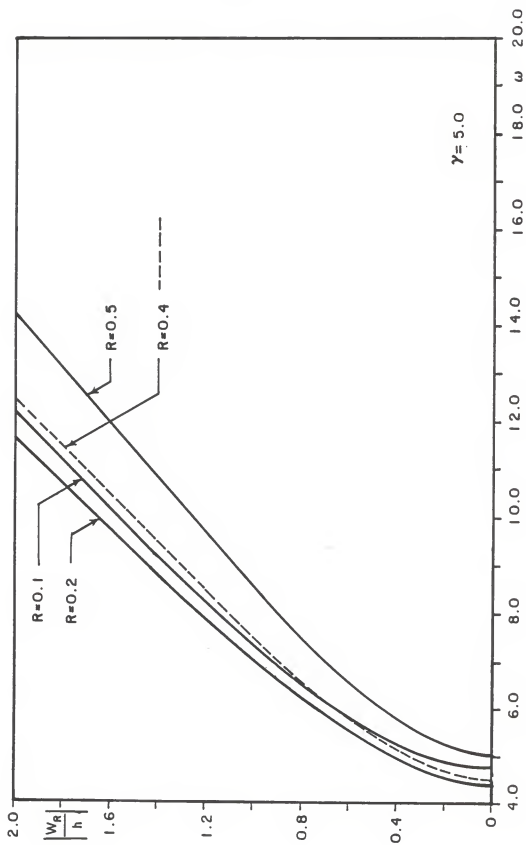
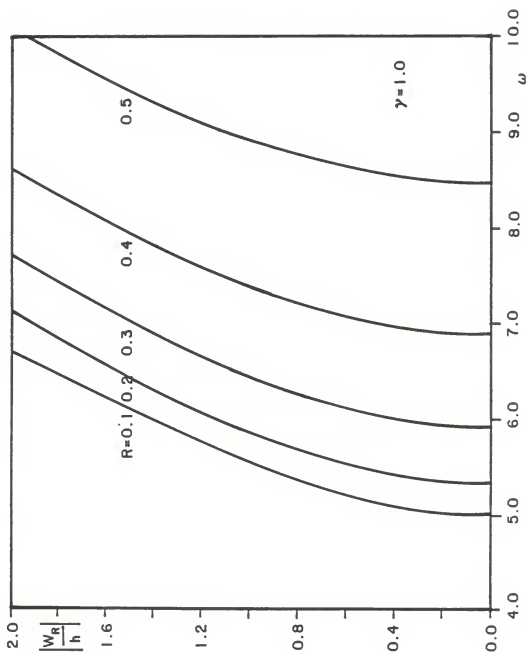


Fig. 14 Harmonic Response of Hinged Immovable Plate-Mass System with  $\gamma = 5.0$ .

Fig. 15 Harmonic Response of Hinged Movable Plate-Mass System with  $\gamma = 1.0$ .

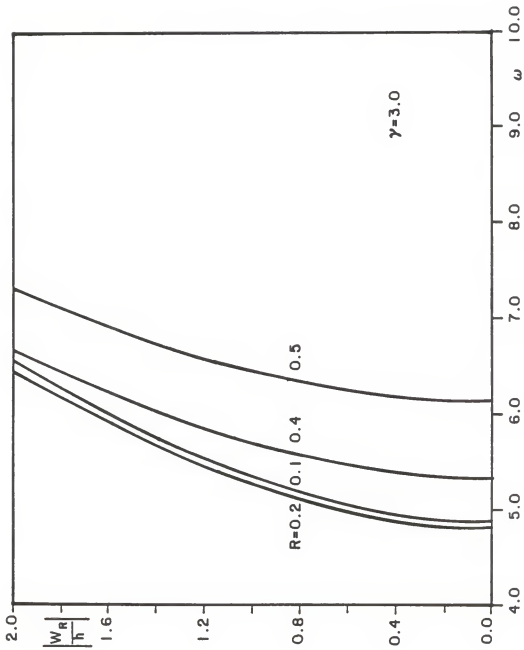


Fig. 16 Harmonic Response of Hinged Movable Plate-Mass System with  $\gamma = 3.0$ .



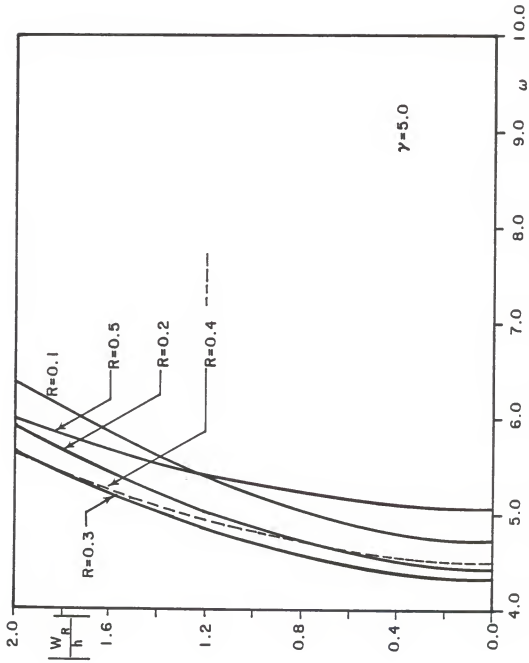
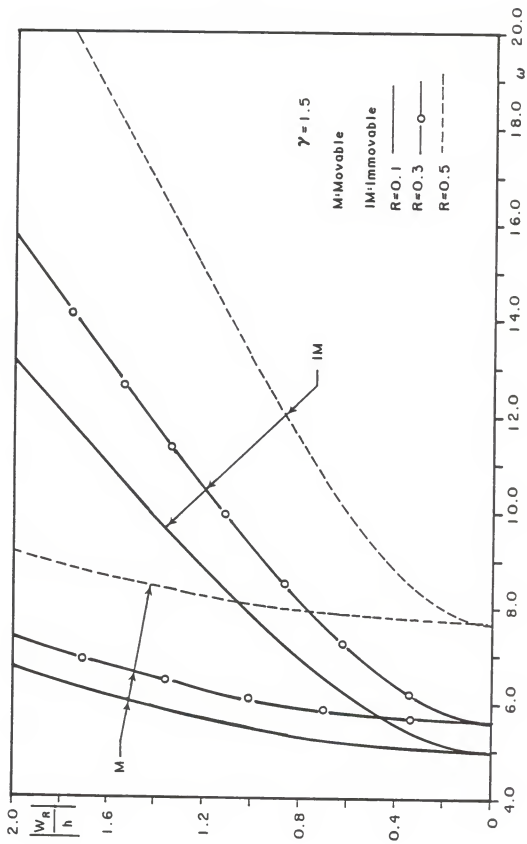


Fig. 17 Harmonic Response of Hinged Movable Plate-Mass System with  $\gamma = 5.0$ .

Fig. 18 Harmonic Response Envelope of Hinged Plate-Mass Systems with  $\gamma = 1.5$ .

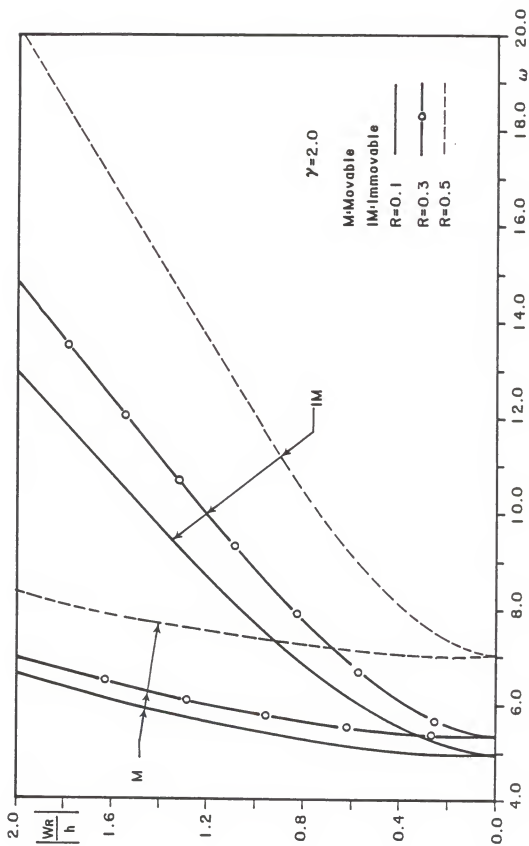
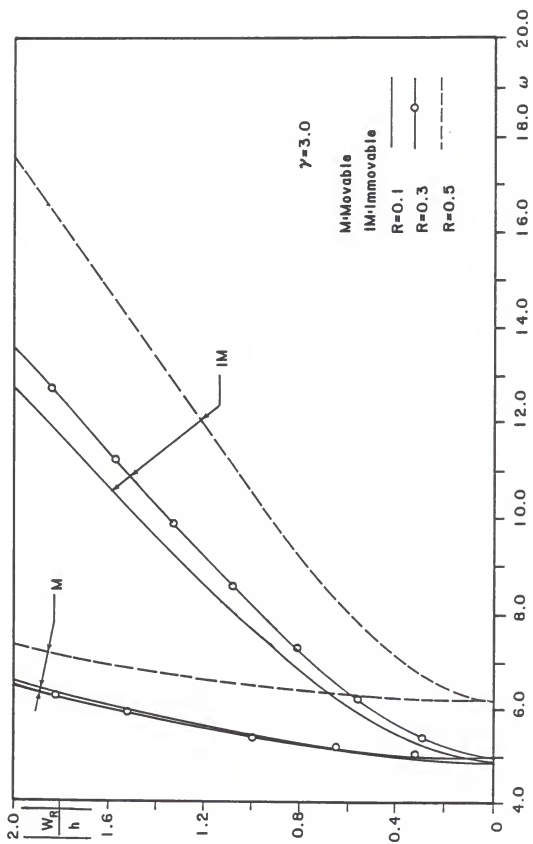
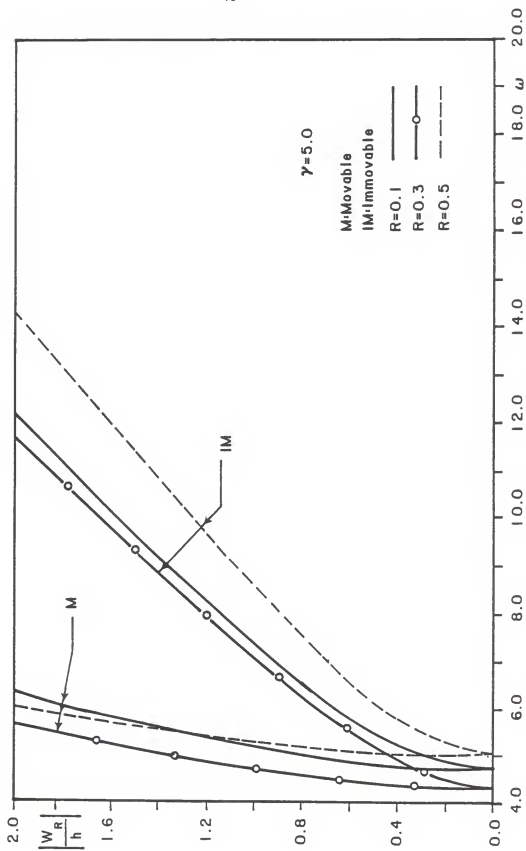


Fig. 19 Harmonic Response Envelope of Hinged Plate-Mass Systems with  $\gamma = 2.0$ .

Fig. 20 Harmonic Response Envelope of Hinged Plate-Mass Systems with  $\gamma = 3.0$ .

Fig. 21 Harmonic Response Envelope of Hinged Plate-Mass Systems with  $\gamma = 5.0$ .

## CONCLUSIONS

Utilizing the method of variational calculus facilitated the derivation of the governing differential equations of motion, geometric and the natural boundary conditions. By eliminating the time variable, the governing equations for the transverse and in-plane displacements were reduced to a pair of ordinary differential equations. Results for the responses of a plate-mass system are obtained using numerical integration. This method can easily be extended to investigate forced vibrations and stress distributions of a plate-mass system.

As noted in references [10,13], the linear behavior of the plate is dependent upon the specific mass and radii ratios imposed on the system under consideration. The non-linear behavior of the plate-mass system under consideration was also found to be dependent upon the specified mass and radii ratios. Excellent agreement was obtained with previous results [12] for the case of a hinged-immovable edge condition when the present problem was reduced to approximate a flat circular plate. With both edge constraints considered, characteristics were exhibited by the response of the plate as similar to that of a hard spring.

Varying the mass and radii ratios of the plate-mass system had interesting effects on the behavior of the system. The frequency of the plate decreased with the addition of mass for both hinged immovable and hinged movable boundaries with a constant radii ratio. In addition, the dependency of the frequency on amplitude weakens for large mass ratios with a constant radii ratio.

For both kinds of edge constraints, increasing the radii ratio simultaneously increased the frequency for mass ratios less than 2.0. For mass ratios greater than 2.0, there exists a critical radii ratio for which the frequency is a minimum. As the radii ratio is increased, the frequency decreases for radii ratios less than the critical point while the frequency increases for radii ratios greater than the critical point.

The findings of this investigation are intended to yield some essential information of the behavior of plate-mass systems. Figures 6-17 are useful in prediction or alteration of the frequency response of a plate-mass system. With the edge constraints analyzed being mathematical idealizations, Figures 18-21 are of valuable importance to the designer in defining a range envelope for the frequency response of simply supported thin circular plates carrying a concentric rigid mass, and provide the upper and lower bounds for the plates.

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APPENDIX A

The first variation of the action integral I is defined to be

$$\begin{aligned}
 \delta I = & -2\pi b h \epsilon \int_{t_1}^{t_2} \int_b^a r w_{tt} \eta \, dr \, dt + \epsilon \int_{t_1}^{t_2} M w_{tt} \eta |_{r=b} \, dt \\
 & -2\pi D \epsilon \int_{t_1}^{t_2} r (w_{rr} + \frac{v}{r} w_r) \eta_r \int_b^a \, dt \\
 & +2\pi D \epsilon \int_{t_1}^{t_2} r (w_{rr} + \frac{1}{r} w_{rr} - \frac{1}{r^2} w_r) \eta \int_b^a \, dt \\
 & -2\pi D \epsilon \int_{t_1}^{t_2} \int_b^a r (w_{rrr} + \frac{2}{r} w_{rrr} - \frac{1}{r^2} w_{rr} + \frac{1}{r^3} w_r) \eta \, dr \, dt \\
 & - \frac{24\pi D \epsilon}{h^2} \int_{t_1}^{t_2} r (u_r + \frac{1}{2} w_r^2 + \frac{v}{r} u) \zeta \int_b^a \, dt \\
 & - \frac{24\pi D \epsilon}{h^2} \int_{t_1}^{t_2} r (u_r w_r + \frac{v}{r} u w_r + \frac{1}{2} w_r^3) \eta \int_b^a \, dt \\
 & + \frac{24\pi D \epsilon}{h^2} \int_{t_1}^{t_2} \int_b^a r (\frac{1}{r} u_r + u_{rr} + \frac{1}{2} w_r^2 + w_r w_{rr} + \frac{v}{r} u_r) \zeta \, dr \, dt \\
 & + \frac{24\pi D \epsilon}{h^2} \int_{t_1}^{t_2} \int_b^a r (\frac{u_r w_r}{r} + u_{rr} w_r + u_r w_{rr} + \frac{v}{r} u_r w_r + \frac{v}{r} u w_{rr} \\
 & \qquad \qquad \qquad + \frac{w_r^3}{2r} + \frac{3}{2} w_r^2 w_{rr}) \eta \, dr \, dt \\
 & - \frac{24\pi D \epsilon}{h^2} \int_{t_1}^{t_2} \int_b^a r (\frac{v}{2r} w_r^2 + \frac{u}{r^2} w_r + \frac{v}{r} u_r) \zeta \, dr \, dt
 \end{aligned}$$

APPENDIX B

Differentiation of the initial value problem with respect to  $\eta_1$ ,  $\eta_2$ , and  $\lambda$  yields the variational equations defined by (33a), (33c) and (33e) as follows:

$$\frac{d}{d\xi}\left(\frac{\partial z_1}{\partial \eta_1}\right) = \frac{\partial z_2}{\partial \eta_1}$$

$$\frac{d}{d\xi}\left(\frac{\partial z_2}{\partial \eta_1}\right) = \frac{\partial z_3}{\partial \eta_1}$$

$$\frac{d}{d\xi}\left(\frac{\partial z_3}{\partial \eta_1}\right) = \frac{\partial z_4}{\partial \eta_1}$$

$$\begin{aligned} \frac{d}{d\xi}\left(\frac{\partial z_4}{\partial \eta_1}\right) = & -\frac{2}{\xi}\left(\frac{\partial z_4}{\partial \eta_1}\right) + \frac{1}{\xi^2}\left(\frac{\partial z_3}{\partial \eta_1}\right) - \frac{1}{\xi^3}\left(\frac{\partial z_2}{\partial \eta_1}\right) + \lambda\left(\frac{\partial z_1}{\partial \eta_1}\right) \\ & + \frac{9(1-\nu^2)}{\xi} \alpha \left[ z_3\left(\frac{\partial z_5}{\partial \eta_1}\right) + z_5\left(\frac{\partial z_3}{\partial \eta_1}\right) + z_6\left(\frac{\partial z_2}{\partial \eta_1}\right) + z_2\left(\frac{\partial z_6}{\partial \eta_1}\right) \right] \end{aligned}$$

$$\frac{d}{d\xi}\left(\frac{\partial z_5}{\partial \eta_1}\right) = \frac{\partial z_6}{\partial \eta_1}$$

$$\frac{d}{d\xi}\left(\frac{\partial z_6}{\partial \eta_1}\right) = -\frac{1}{\xi}\left(\frac{\partial z_6}{\partial \eta_1}\right) + \frac{1}{\xi^2}\left(\frac{\partial z_5}{\partial \eta_1}\right) - \frac{1}{\xi} z_2\left(\frac{\partial z_2}{\partial \eta_1}\right)$$

$$\frac{d}{d\xi}\left(\frac{\partial z_1}{\partial \eta_2}\right) = \frac{\partial z_2}{\partial \eta_2}$$

$$\frac{d}{d\xi}\left(\frac{\partial z_2}{\partial \eta_2}\right) = \frac{\partial z_3}{\partial \eta_2}$$

$$\frac{d}{d\xi}\left(\frac{\partial z_3}{\partial \eta_2}\right) = \frac{\partial z_4}{\partial \eta_2}$$

$$\begin{aligned} \frac{d}{d\xi} \left( \frac{\partial z_4}{\partial n_2} \right) &= -\frac{2}{\xi} \left( \frac{\partial z_4}{\partial n_2} \right) + \frac{1}{\xi^2} \left( \frac{\partial z_3}{\partial n_2} \right) - \frac{1}{\xi^3} \left( \frac{\partial z_2}{\partial n_2} \right) + \lambda \left( \frac{\partial z_1}{\partial n_2} \right) \\ &\quad + \frac{g(1-v^2)}{\xi} \alpha \left[ z_3 \left( \frac{\partial z_5}{\partial n_2} \right) + z_5 \left( \frac{\partial z_3}{\partial n_2} \right) + z_6 \left( \frac{\partial z_2}{\partial n_2} \right) + z_2 \left( \frac{\partial z_6}{\partial n_2} \right) \right] \end{aligned}$$

$$\frac{d}{d\xi} \left( \frac{\partial z_5}{\partial n_2} \right) = \frac{\partial z_6}{\partial n_2}$$

$$\frac{d}{d\xi} \left( \frac{\partial z_6}{\partial n_2} \right) = \frac{1}{\xi} \left( \frac{\partial z_6}{\partial n_2} \right) + \frac{1}{\xi^2} \left( \frac{\partial z_5}{\partial n_2} \right) + \frac{1}{\xi} z_2 \left( \frac{\partial z_2}{\partial n_2} \right)$$

$$\frac{d}{d\xi} \left( \frac{\partial z_1}{\partial \lambda} \right) = \frac{\partial z_2}{\partial \lambda}$$

$$\frac{d}{d\xi} \left( \frac{\partial z_2}{\partial \lambda} \right) = \frac{\partial z_3}{\partial \lambda}$$

$$\frac{d}{d\xi} \left( \frac{\partial z_3}{\partial \lambda} \right) = \frac{\partial z_4}{\partial \lambda}$$

$$\begin{aligned} \frac{d}{d\xi} \left( \frac{\partial z_4}{\partial \lambda} \right) &= -\frac{1}{\xi} \left( \frac{\partial z_4}{\partial \lambda} \right) + \frac{1}{\xi^2} \left( \frac{\partial z_3}{\partial \lambda} \right) - \frac{1}{\xi^3} \left( \frac{\partial z_2}{\partial \lambda} \right) + z_1 + \lambda \left( \frac{\partial z_1}{\partial \lambda} \right) \\ &\quad + \frac{g(1-v^2)}{\xi} \alpha \left[ z_5 \left( \frac{\partial z_3}{\partial \lambda} \right) + z_3 \left( \frac{\partial z_5}{\partial \lambda} \right) + z_6 \left( \frac{\partial z_2}{\partial \lambda} \right) - z_2 \left( \frac{\partial z_6}{\partial \lambda} \right) \right] \end{aligned}$$

$$\frac{d}{d\xi} \left( \frac{\partial z_5}{\partial \lambda} \right) = \frac{\partial z_6}{\partial \lambda}$$

$$\frac{d}{d\xi} \left( \frac{\partial z_6}{\partial \lambda} \right) = -\frac{1}{\xi} \left( \frac{\partial z_6}{\partial \lambda} \right) + \frac{1}{\xi^2} \left( \frac{\partial z_5}{\partial \lambda} \right) - \frac{1}{\xi} z_2 \left( \frac{\partial z_2}{\partial \lambda} \right)$$

## APPENDIX C

### Fortran Computer Programs for Free Vibrations

The relationship between the equations given in (23a) and Appendix B to the following programs is described as,

$$Y(I) = y_I$$

$$Y(I+6) = \frac{\partial z_I}{\partial \eta_1}$$

$$Y(I+12) = \frac{\partial z_I}{\partial \eta_2} \quad I = 1, \dots, 6$$

$$Y(I+18) = \frac{\partial z_I}{\partial \lambda}$$

```

C *****
C HINGED CIRCULAR PLATE WITH ATTACHED CONCENTRIC RIGID MASS
C INITIAL-VALUE METHOD FOR THE FREE VIBRATION OF A SIMPLY-
C SUPPORTED CIRCULAR PLATE WITH CONCENTRIC RIGID MASS
C RIGID MASS INSIDE AND HINGED-IMMOVABLE OUTSIDE BOUNDARY
C POISSON'S RATIO = 0.3
C *****
C
C      IMPLICIT REAL*8(A-H,O-Z),INTEGER(I-N)
C      DIMENSION Y(24),Q(24),TP(3,4),D(6,41),C(3),ER(3)
112  FORMAT(5X,'AMP=',D22.14,3X,'FREQ=',D22.14)
C 113  FORMAT(9X,'W',19X,'DW',18X,'DDW',17X,'DDDW')
C 114  FORMAT(4D22.14)
C 115  FORMAT(//9X,'F',19X,'DF')
C 116  FORMAT(2D22.14)
C 117  FORMAT(1H )
C 118  FORMAT(1S)
C 119  FORMAT(15X,'RADIUS RATIO=',D22.14)
C 120  FORMAT(10X,'MASS RATIO=',D22.14)
C *****
C      A=AMPLITUDE PARAMETER
C      R=RATIO OF INNER RADIUS TO OUTER RADIUS B/A
C      GAMMA=RATIO OF RIGID MASS TO MASS OF PLATE IT REPLACES
C      DA=INCREMENT IN AMPLITUDE
C      DR=INCREMENT IN RADIUS RATIO
C      DG=INCREMENT IN MASS RATIO
C      H=STEP-SIZE FOR NUMERICAL INTEGRATION
C      VV=POISSON'S RATIO
C      IK=COUNTER FOR AMPLITUDE INCREASE
C      IR=COUNTER FOR RADIUS RATIO INCREASE
C      IG=COUNTER FOR MASS RATIO INCREASE
C      LL=STEPS REQUIRED FOR RADIUS RATIO TO REACH UNITY (R=1.0)
C          IN RUNGE-KUTTA-GILL INTEGRATION (LL=((1.0+R)/H)+1.0)
C *****
C      DA=0.1D-0
C      DR=0.1D-0
C      DG=1.0D-0
C      VV=0.3D-0
C      R=0.0D-0
C      H=2.5D-2
C      LL=41
C ***  LOOPS10 INCREMENTS THE RADIUS RATIO
C      DO 510 IR=1,5
C      LL=LL-4
C      R=R+DR
C      GAMMA=0.5D-0
C      WRITE(6,117)
C      WRITE(6,119) R

```

```

C *** LOOP 520 INCREMENTS THE MASS RATIO
  DO 520 I6=1,1
    IK=1
    P=4.975D-0
    A=0.0D-0
    GAMMA=GAMMA+DG
    WRITE(6,117)
    WRITE(6,120) GAMMA
C *** CONSTRUCT INITIAL VALUES
500 DO 10 I=1,24
  10 Y(I)=0.0D-0
    Y(1)=1.0D-0
    Y(3)=-4.67D-0
    Y(4)=-Y(3)/R+0.5D-0*R*GAMMA*P
    Y(5)=0.82D-0
    Y(6)=(VV*Y(5))/R
    Y(9)=1.0D-0
    Y(10)=-1.0D-0/R
    Y(17)=1.0D-0
    Y(18)=VV/R
    Y(22)=0.5D-0*R*GAMMA
    IF(IK.EQ.1) GO TO 600
    DO 15 I=1,6
  15 Y(I)=D(I,1)
C *** X=INDEPENDENT VARIABLE
600 X=R
  DO 20 I=1,24
  20 Q(I)=0.0D-0
    DO 21 I=1,6
  21 D(I,1)=Y(I)
C *** PERFORM RUNGE-KUTTA-GILL INTEGRATION
  DO 25 I=2,LL
    CALL RKGPL(X,H,Y,Q,P,A)
    DO 30 J=1,6
  30 D(J,I)=Y(J)
  25 CONTINUE
C *** ER(I)=ERROR VECTOR FOR BOUNDARY CONDITIONS AT X=1.0
  ER(1)=D(1,LL)
  ER(2)=D(2,LL)*VV+D(3,LL)
  ER(3)=D(6,LL)-VV*D(5,LL)
  DO 35 I=1,3
  DER=DABS(ER(I))
  IF(DER.GT.0.1D-5) GO TO 36
  35 CONTINUE
  GO TO 900
  36 CONTINUE

```

```

C *** TP(I,J) IS THE JACOBIAN OF THE MAPPING OF INITIAL VALUES
C *** TO FINAL VALUES
      TP(1,1)=Y(7)
      TP(2,1)=Y(8)*VV+Y(9)
      TP(3,1)=Y(12)-VV*Y(11)
      TP(1,2)=Y(13)
      TP(2,2)=Y(14)*VV+Y(15)
      TP(3,2)=Y(18)-VV*Y(17)
      TP(1,3)=Y(19)
      TP(2,3)=Y(20)*VV+Y(21)
      TP(3,3)=Y(24)-VV*Y(23)
      DO 40 I=1,3
40    TP(I,4)=ER(I)
      CALL GAUSSX(TP,C,3,4)
C *** C(I)=CORRECTION VECTOR
      DO 76 I=1,6
76    Y(I)=D(I,1)
      Y(3)=Y(3)-C(1)
      Y(5)=Y(5)-C(2)
      P=P-C(3)
      DO 80 I=7,24
80    Y(I)=0.0D-0
      Y(4)=-Y(3)/R+0.5D-0*R*GAMMA*P
      Y(6)=(VV*Y(5))/R
      Y(9)=1.0D-0
      Y(10)=-1.0D-0/R
      Y(17)=1.0D-0
      Y(18)=VV/R
      Y(22)=0.5D-0*R*GAMMA
      GOTO 600
900   SRA=DSQRT(A)
      SP=DSQRT(P)
      WRITE(6,117)
      WRITE(6,112) SRA,SP
C     WRITE(6,117)
C     WRITE(6,113)
C     DO 910 J=1,LL
C 910  WRITE(6,114) (D(I,J),I=1,4)
C     WRITE(6,115)
C     DO 920 J=1,LL
C 920  WRITE(6,116) (D(L,J),L=5,6)
C     WRITE(6,117)
      A=A+DA
      IK=IK+1
      IF(IK.BT.40) GO TO 520
      GO TO 500
520  CONTINUE
510  CONTINUE
550  STOP
      END
C     *****

```





```

SUBROUTINE DERIVL(X,H,Y,DY,P,AP)
C *** THIS ROUTINE EVALUATES THE DERIVATIVES OF THE RELATED INITIAL-
C *** VALUE PROBLEM AND THE ASSOCIATED VARIATIONAL EQUATIONS
      IMPLICIT REAL*8(A-H,D-Z),INTEGER(I-N)
      DIMENSION Y(24),DY(24)
      DO 10 I=1,3
10  DY(I)=Y(I+1)
      DY(5)=Y(6)
      DO 15 I=7,9
15  DY(I)=Y(I+1)
      DY(11)=Y(12)
      DO 20 I=13,15
20  DY(I)=Y(I+1)
      DY(17)=Y(18)
      DO 25 I=19,21
25  DY(I)=Y(I+1)
      DY(23)=Y(24)
50  DY(4)=-2.0D-0*(Y(4)/X)+Y(3)/(X*X)-Y(2)/(X*X*X)+P*Y(1)
      DY(4)=DY(4)+B.19D-0*AP*(Y(3)*Y(5)+Y(2)*Y(6))/X
      DY(6)=-Y(6)/X+Y(5)/(X*X)-(Y(2)*Y(2))/(2.0D-0*X)
      DY(10)=-2.0D-0*(Y(10)/X)+Y(9)/(X*X)-Y(8)/(X*X*X)+P*Y(7)
      &+B.19D-0*AP*(Y(5)*Y(9)+Y(3)*Y(11)+Y(2)*Y(12)+Y(6)*Y(8))/X
      DY(12)=-Y(12)/X+Y(11)/(X*X)-(Y(2)*Y(8))/X
      DY(16)=-2.0D-0*(Y(16)/X)+Y(15)/(X*X)-Y(14)/(X*X*X)+P*Y(13)
      &+B.19D-0*AP*(Y(3)*Y(17)+Y(5)*Y(15)+Y(2)*Y(18)+Y(6)*Y(14))/X
      DY(18)=-Y(18)/X+Y(17)/(X*X)-(Y(2)*Y(14))/X
      DY(22)=-2.0D-0*(Y(22)/X)+Y(21)/(X*X)-Y(20)/(X*X*X)+P*Y(19)+Y(1)
      &+B.19D-0*AP*(Y(3)*Y(23)+Y(5)*Y(21)+Y(2)*Y(24)+Y(6)*Y(20))/X
      DY(24)=-Y(24)/X+Y(23)/(X*X)-(Y(2)*Y(20))/X
70  RETURN
      END
C *****

```

```

SUBROUTINE GAUSSX(A,X,N,N1)
C *** THIS ROUTINE PERFORMS A GAUSSIAN ELIMINATION
      IMPLICIT REAL*8(A-H,O-Z),INTEGER(I-N)
      DIMENSION A(N,N1),X(N)
      DO 200 J=1,N
C      * * * * *
      J1=J+1
      IF(J1.GT.N) GO TO 980
      BIG=DABS(A(J,J))
      M=J
      DO 900 L=J1,N
      IF(DABS(A(L,J)).LE.BIG) GO TO 900
      M=L
      BIG=DABS(A(L,J))
900  CONTINUE
      DO 990 JJ=J,N1
      DUMMY=A(M,JJ)
      A(M,JJ)=A(J,JJ)
990  A(J,JJ)=DUMMY
980  CONTINUE
C      * * * * *
      S=1.0D-0/A(J,J)
      DO 201 K=J,N1
201  A(J,K)=A(J,K)*S
      DO 202 I=1,
      IF(I-J) 203,202,203
203  AIJ=-A(I,J)
      DO 204 K=J,N1
204  A(I,K)=A(I,K)+AIJ*A(J,K)
202  CONTINUE
200  CONTINUE
      DO 300 I=1,N
300  X(I)=A(I,N1)
      RETURN
      END

```

```

C *****
C HINGED CIRCULAR PLATE WITH ATTACHED CONCENTRIC RIGID MASS
C INITIAL-VALUE METHOD FOR THE FREE VIBRATION OF A SIMPLY-
C SUPPORTED CIRCULAR PLATE WITH CONCENTRIC MASS
C RIGID MASS INSIDE AND HINGED-MOVABLE OUTSIDE
C POISSON'S RATIO = 0.3
C *****
C
      IMPLICIT REAL*8(A-H,O-Z),INTEGER(I-N)
      DIMENSION Y(24),Q(24),TP(3,4),D(6,41),C(3),ER(3)
112  FORMAT(5X,'AMP=',D22.14,3X,'FREQ=',D22.14)
C 113  FORMAT(9X,'W',19X,'DW',18X,'DDW',17X,'DDDW')
C 114  FORMAT(4D22.14)
C 115  FORMAT(/9X,'F',19X,'DF')
C 116  FORMAT(2D22.14)
      117  FORMAT(1H )
C 118  FORMAT(1S)
119  FORMAT(15X,'RADIUS RATIO=',D22.14)
120  FORMAT(10X,'MASS RATIO=',D22.14)
C *****
C A=AMPLITUDE PARAMETER
C R=RATIO OF INNER RADIUS TO OUTER RADIUS B/A
C GAMMA=RATIO OF RIGID MASS TO MASS OF PLATE IT REPLACES
C DA=INCREMENT IN AMPLITUDE
C DR=INCREMENT IN RADIUS RATIO
C DG=INCREMENT IN MASS RATIO
C H=STEP-SIZE FOR NUMERICAL INTEGRATION
C VV=POISSON'S RATIO
C IK=COUNTER FOR AMPLITUDE INCREASE
C IR=COUNTER FOR RADIUS RATIO INCREASE
C IG=COUNTER FOR MASS RATIO INCREASE
C LL=STEPS REQUIRED FOR RADIUS RATIO TO REACH UNITY (R=1.0)
C      IN RUNGE-KUTTA-GILL INTEGRATION (LL=((1.0+R)/H)+1.0)
C *****
      DA=0.1D-0
      DR=0.1D-0
      DG=1.0D-0
      VV=0.3D-0
      R=0.0D-0
      H=2.5D-2
      LL=41
C *** LOOP 10 INCREMENTS THE RADIUS RATIO
      DO 510 IR=1,5
      LL=LL-4
      R=R+DR
      GAMMA=0.5D-0
      WRITE(6,117)
      WRITE(6,119) R

```

```

C *** LOOP 520 INCREMENTS THE MASS RATIO
  DD 520 IG=1,1
  IK=1
  P=4.975D-0
  A=0.0D-0
  GAMMA=GAMMA+DB
  WRITE(6,117)
  WRITE(6,120) GAMMA
C   CONSTRUCT INITIAL VALUES
500 DD 10 I=1,24
  10 Y(I)=0.0D-0
  Y(1)=1.0D-0
  Y(3)=-4.67D-0
  Y(4)=-Y(3)/R+0.5D-0*R*GAMMA*P
  Y(5)=0.82D-0
  Y(6)=(VV*Y(5))/R
  Y(9)=1.0D-0
  Y(10)=-1.0D-0/R
  Y(17)=1.0D-0
  Y(18)=VV/R
  Y(22)=0.5D-0*R*GAMMA
  IF(IK.EQ.1) GO TO 600
  DD 15 I=1,6
  15 Y(I)=D(I,1)
C *** X=INDEPENDENT VARIABLE
600 X=R
  DD 20 I=1,24
  20 Q(I)=0.0D-0
  DD 21 I=1,6
  21 D(I,1)=Y(I)
C *** PERFORM RUNGE-KUTTA-GILL INTEGRATION
  DD 25 I=2,LL
  CALL RKBPL(X,H,Y,Q,P,A)
  DD 30 J=1,6
  30 D(J,I)=Y(J)
  25 CONTINUE
C *** ER(I)=ERROR VECTOR FOR BOUNDARY CONDITIONS AT X=1.0
  ER(1)=D(1,LL)
  ER(2)=D(2,LL)*VV+D(3,LL)
  ER(3)=D(5,LL)
  DD 35 I=1,3
  DER=DABS(ER(I))
  IF(DER.GT.0.1D-5) GO TO 36
  35 CONTINUE
  GO TO 900
  36 CONTINUE

```

```

C *** TP(I,J) IS THE JACOBIAN OF THE MAPPING OF INITIAL VALUES
C *** TO FINAL VALUES
      TP(1,1)=Y(7)
      TP(2,1)=Y(8)*VV+Y(9)
      TP(3,1)=Y(11)
      TP(1,2)=Y(13)
      TP(2,2)=Y(14)*VV+Y(15)
      TP(3,2)=Y(17)
      TP(1,3)=Y(19)
      TP(2,3)=Y(20)*VV+Y(21)
      TP(3,3)=Y(23)
      DO 40 I=1,3
40    TP(I,4)=ER(I)
      CALL GAUSSX(TP,C,3,4)
C *** C(I)=CORRECTION VECTOR
      DO 76 I=1,6
76    Y(I)=D(I,1)
      Y(3)=Y(3)-C(1)
      Y(5)=Y(5)-C(2)
      P=P-C(3)
      DO 80 I=7,24
80    Y(I)=0.0D-0
      Y(4)=-Y(3)/R+0.5D-0*R*GAMMA*P
      Y(6)=(VV*Y(5))/R
      Y(9)=1.0D-0
      Y(10)=-1.0D-0/R
      Y(17)=1.0D-0
      Y(18)=VV/R
      Y(22)=0.5D-0*R*GAMMA
      GO TO 600
900  RA=DSQRT(A)
      SP=DSQRT(P)
      WRITE(6,117)
      WRITE(6,112) SRA,SP
C     WRITE(6,117)
C     WRITE(6,113)
C     DO 910 J=1,LL
C 910  WRITE(6,114) (D(I,J),I=1,4)
C     WRITE(6,115)
C     DO 920 J=1,LL
C 920  WRITE(6,116) (D(L,J),L=5,6)
C     WRITE(6,117)
      A=A+DA
      IK=IK+1
      IF(IK.GT.40) GO TO 520
      GO TO 500
520  CONTINUE
510  CONTINUE
550  STOP
      END
C *****

```



```

SUBROUTINE DERIVL(X,H,Y,DY,P,AP)
C *** THIS ROUTINE EVALUATES THE DERIVATIVES OF THE RELATED INITIAL-
C *** VALUE PROBLEM AND THE ASSOCIATED VARIATIONAL EQUATIONS
      IMPLICIT REAL*8(A-H,O-Z),INTEGER(I-N)
      DIMENSION Y(24),DY(24)
      DO 10 I=1,3
10  DY(I)=Y(I+1)
      DY(5)=Y(6)
      DO 15 I=7,9
15  DY(I)=Y(I+1)
      DY(11)=Y(12)
      DO 20 I=13,15
20  DY(I)=Y(I+1)
      DY(17)=Y(18)
      DO 25 I=19,21
25  DY(I)=Y(I+1)
      DY(23)=Y(24)
50  DY(4)=-2.0D-0*(Y(4)/X)+Y(3)/(X*X)-Y(2)/(X*X*X)+P*Y(1)
      DY(4)=DY(4)+8.19D-0*AP*(Y(3)*Y(5)+Y(2)*Y(6))/X
      DY(6)=-Y(6)/X+Y(5)/(X*X)-(Y(2)*Y(2))/(2.0D-0*X)
      DY(10)=-2.0D-0*(Y(10)/X)+Y(9)/(X*X)-Y(8)/(X*X*X)+P*Y(7)
      &+8.19D-0*AP*(Y(5)*Y(9)+Y(3)*Y(11)+Y(2)*Y(12)+Y(6)*Y(8))/X
      DY(12)=-Y(12)/X+Y(11)/(X*X)-(Y(2)*Y(8))/X
      DY(16)=-2.0D-0*(Y(16)/X)+Y(15)/(X*X)-Y(14)/(X*X*X)+P*Y(13)
      &+8.19D-0*AP*(Y(3)*Y(17)+Y(5)*Y(15)+Y(2)*Y(18)+Y(6)*Y(14))/X
      DY(18)=-Y(18)/X+Y(17)/(X*X)-(Y(2)*Y(14))/X
      DY(22)=-2.0D-0*(Y(22)/X)+Y(21)/(X*X)-Y(20)/(X*X*X)+P*Y(19)+Y(1)
      &+8.19D-0*AP*(Y(3)*Y(23)+Y(5)*Y(21)+Y(2)*Y(24)+Y(6)*Y(20))/X
      DY(24)=-Y(24)/X+Y(23)/(X*X)-(Y(2)*Y(20))/X
70  RETURN
      END
C *****

```

```

SUBROUTINE GAUSSX(AX,N,N1)
C *** THIS ROUTINE PERFORMS A GAUSSIAN ELIMINATION
      IMPLICIT REAL*8(A-H,O-Z),INTEGER(I-N)
      DIMENSION A(N,N1),X(N)
      DO 200 J=1,N
C      * * * * *
      J1=J+1
      IF(J1.GT.N) GO TO 980
      BIG=DABS(A(J,J))
      M=J
      DO 900 L=J1,N
      IF(DABS(A(L,J)).LE.BIG) GO TO 900
      M=L
      BIG=DABS(A(L,J))
900 CONTINUE
      DO 990 JJ=J,N1
      DUMMY=A(M,JJ)
      A(M,JJ)=A(J,JJ)
990 A(J,JJ)=DUMMY
980 CONTINUE
C      * * * * *
      S=1.0D-0/A(J,J)
      DO 201 K=J,N1
201 A(J,K)=A(J,K)*S
      DO 202 I=1,N
      IF(I-J) 203,202,203
203 AIJ=-A(I,J)
      DO 204 K=J,N1
204 A(I,K)=A(I,K)+AIJ*A(J,K)
202 CONTINUE
200 CONTINUE
      DO 300 I=1,N
300 X(I)=A(I,N1)
      RETURN
      END

```



NONLINEAR RESPONSES OF A HINGED CIRCULAR PLATE  
WITH A CONCENTRIC RIGID MASS

by

HUGH S. WALKER, JR.

B.S., Kansas State University, 1985

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## ABSTRACT

Large amplitude axisymmetric vibrations of a thin elastic circular plate with an attached concentric rigid mass are investigated. The problem is formulated and results in a set of von Karman's dynamic equations by employing Hamilton's Principle and the method of Calculus of Variations. Harmonic oscillations are assumed and the time variable is eliminated by the Kantorovich averaging method. The resulting differential equations of motion form an eigenvalue problem. Successive corrections of the unknown initial values by Newton's method and perturbations of the amplitude parameter provide approximate solutions to the eigenvalue problem. The effects of adding a mass to the plate are studied for very small amplitudes. The behavior of both hinged immovable and movable plates are examined when a mass is added. Various mass and radii ratios are investigated to provide complete non-linear characteristics.