COMPACTNESS IN TOPOLOGICAL SPACES

by

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INTRODUCTION

The property of compactness is an abstraction of an important property of certain sets of real numbers [10,58].[1] This property is characterized by the Heine-Borel theorem which states that if X is a closed and bounded subset of the real line R, then any collection of open subsets of the real line whose union contains X has a finite subcollection whose union also contains X [12,110].

The sphere and the torus, geometric figures of Euclidean three space, are closed surfaces contained in a finite portion of the space. In contrast, a surface such as the paraboloid is unbounded. The bounded and closed properties form the basis for the topological notion of compactness. However, a general definition is required; that is, one which does not depend upon a surrounding set as do the above examples [10,58].

Specifically, consider the closed and bounded subset in the Heine-Borel theorem as forming a topological space in its own right. The theorem then becomes the basis for a general

[1] The reference [10,58] refers to page 58 of reference number 10 of the literature cited.

definition of compactness in arbitrary topological spaces. "As is often the case with crucial theorems in analysis, the conclusion of the Heine-Borel theorem is converted into a definition in topology [12,110]."

The importance of the Heine-Borel theorem is manifest by its use in guaranteeing that continuous functions defined on closed and bounded sets of real numbers will themselves be bounded and uniformly continuous. In contrast, consider the function f defined on the open interval of real numbers $\{x : 0 < x < 1\}$ by $f(x) = 1/x$. Although the function is continuous on this bounded set, f is neither bounded nor uniformly continuous.

The importance of compactness to topology extends beyond its relationship to continuous functions. As Solomon Lefschetz writes in his book Algebraic Topology, "It is not too much to say that all the spaces of chief interest in general topology, and even more so in algebraic topology, are compact spaces or their subsets [7,17]."

In the following it is assumed that the reader is familiar with some of the basic definitions and theorems of elementary topology.

CHARACTERIZATIONS OF COMPACTNESS

Compactness may be characterized in a number of ways. A common choice for defining compactness follows from the theorem of Heine-Borel. The following terms defined in the topological space $(X, \mathcal{J})$ simplify the discussion.

A family $\mathcal{A}$ of subsets of the topological space $(X, \mathcal{J})$ is said to be a _cover_ for or _to cover_ set $B \subset X$ if B is contained in $\bigcup \mathcal{A}$. A cover of B is called an _open cover_ if each member of the cover is an open set. A _subcover_ for B of a family $\mathcal{A}$ is a subfamily of $\mathcal{A}$ which is also a cover of B. If a cover for a set B contains a finite number of sets, then it is called a _finite cover_.

Definition 1. A topological space is said to be _compact_ if each open cover of the space contains a finite subcover.

Compactness can be characterized in terms of neighborhoods of the points of a topological space.

Theorem 1. A topological space is compact if and only if for every cover of the space by neighborhoods of the points of the space, there is a finite subcover.

Proof: Let a topological space X be compact.[2] Assume β is a cover of X by neighborhoods of points in X. Then for each x in X, there is a neighborhood N_x of x in β. By definition, for each x, there is an open set A_x such that $x \in A_x \subset N_x$. The collection $\mathcal{A}$ of all A_x is an open cover for X. The compactness of X implies there is a finite subcover of $\mathcal{A}$. Let $\{A_{x_i}: i=1,2,\ldots,n\}$ be such a subcover. For each $i \leq n$, $A_{x_i} \subset N_{x_i}$ and hence

$$X \subset \bigcup_{i=1}^{n} A_{x_i} \subset \bigcup_{i=1}^{n} N_{x_i}.$$

Therefore, the collection $\{N_{x_i}: i=1,2,\ldots,n\}$ is also a finite subcover of X.

Conversely, let $\mathcal{A}$ be an open cover of X. Each x belongs to some member of $\mathcal{A}$. Thus $\mathcal{A}$ itself is a collection of neighborhoods which covers X. By hypothesis, there is a finite subcover of X by members of $\mathcal{A}$.

The following two theorems characterize compactness in terms

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[2]Where no confusion is likely to result, a topological space $(X, \mathcal{J})$ will be referred to simply as the space X or the topological space X.

of closed subsets of the space.

Theorem 2. A topological space is compact if and only if the intersection of a family of closed sets is empty implies there exists a finite subfamily whose intersection is also empty.

Proof: Assume that a topological space X is compact and that a family $\mathcal{F}$ of closed sets in X has the property $\bigcap \mathcal{F} = \emptyset$. If $A \in \mathcal{F}$ then C(A) is open and $\mathcal{U} = \{C(A) : A \in \mathcal{F}\}$ is a family of open sets. The set $C(\bigcap \mathcal{F}) = C(\emptyset)$ implies $\bigcup_{A \in \mathcal{F}} C(A) = X$. Therefore, since $\bigcup_{U \in \mathcal{U}} U = \bigcup_{A \in \mathcal{F}} C(A) = X$, $\mathcal{U}$ is an open cover of X. The compactness of X implies there is a finite subcover $\{U_i : i=1,2,\ldots,n\}$ of $\mathcal{U}$ where each $U_i = C(A_i)$ for some $A_i \in \mathcal{F}$. Therefore, we have $C(\bigcup_{i=1}^{n} U_i) = C(X) = \emptyset$. This can be rewritten $\bigcap_{i=1}^{n} C(U_i) = \emptyset$ and hence it is true that $\mathcal{F}$ has a finite subfamily $\{A_i : i=1,2,\ldots,n\}$ such that $\bigcap_{i=1}^{n} A_i = \emptyset$.

Conversely, let $\mathcal{U}$ be an open cover of X. Then $\{C(U) : U \in \mathcal{U}\}$ is a family of closed sets such that $\bigcap_{U \in \mathcal{U}} C(U) = \emptyset$. By hypothesis, there is a finite subfamily of these closed sets whose intersection is also empty. Let $\mathcal{V}$ be such a finite subfamily of $\mathcal{U}$. Then $C(\bigcap_{V \in \mathcal{V}} C(V)) = \bigcup_{V \in \mathcal{V}} V = X$ and X is compact.

A family of subsets of a topological space is said to have the <u>finite intersection property</u> if the intersection of each finite subfamily is nonempty. The following theorem follows immediately from the preceding one.

<u>Theorem 3</u>. A topological space is compact if and only if each family of closed sets which has the finite intersection property has a nonempty intersection.

If a topological space is compact, then by definition, every open cover has a finite subcover, even if the sets are basic open sets. Since a subbasic cover defines an open cover, the subbasic cover of a compact space also has a finite subcover. That the converse of these two implications is true is demonstrated in the following theorems.

<u>Theorem 4</u>. If β is an arbitrary base for the open sets of a topological space X and if for each open cover of X by members of β there is a finite subcover, then X is compact.

Proof: Let $\mathcal{A}$ be an arbitrary open cover of X and β an arbitrary base for the open sets of X. Then the collection $\{B_i : i \in J\}$ of members of β which are contained in some member of $\mathcal{A}$ is also a cover for X. By hypothesis $\{B_i : i \in J\}$ has a finite subfamily $\{B_i : i=1,2,\ldots,n\}$ which also covers X. Since each B_i, $i=1,2,\ldots,n$ is contained in some member A_i of $\mathcal{A}$,

$$X \subset \bigcup_{i=1}^{n} B_i \subset \bigcup_{i=1}^{n} A_i .$$

Thus $\mathcal{A}$ has a finite subcover.

The proof that the existence of a finite subcover for any cover of a topological space by subbasic sets implies compactness is less immediate than the analogous theorem for basic open sets. The following preparations will simplify the proof. The development is taken from Theral O. Moore's Elementary General Topology [9,125].

A binary relation $\leq$ on a set X is a partial ordering in X (or partially orders X) if $\leq$ is transitive and $x \leq y$ and $y \leq x$ implies $x = y$ for $x,y \in X$. A subset Y of X is simply ordered by $\leq$ if $x \leq y$ or $y \leq x$ for each $x,y \in Y$. Notice that given any family of subsets $\mathcal{C}$ of a set Z, the inclusion relation partially orders $\mathcal{C}$ since $\subset$ is transitive and if $A,B \in \mathcal{C}$ then $A \subset B$ and $B \subset A$ implies $A = B$. A simply ordered set Y contained in a partially ordered set X is a maximal simply ordered set if no simply ordered subset of X properly contains Y.

Axiom (Hausdorff Maximal Principle). If A is a simply ordered subset of a partially ordered set X, then there is a maximal simply ordered set Y such that $A \subset Y \subset X$ [9,126].

Notice that if $a \in X$, where X is a partially ordered set, then there is a maximal simply ordered set $Y \subset X$ such that $a \in Y$. This is true since $\{a\}$ is a simply ordered set. Hence, by the Hausdorff Maximal Principle, there is a maximal simply ordered set Y such that $\{a\} \subset Y$ or $a \in Y$. A family $\mathcal{C}$ of subsets of X is inadequate if $\mathcal{C}$ does not cover X and finitely inadequate if no finite subfamily of $\mathcal{C}$ covers X. A finitely inadequate family $\mathcal{C}$ is a maximal finitely inadequate open family of a space X if no finitely inadequate family of open subsets of X properly contains $\mathcal{C}$.

Lemma 1. Let X be a topological space and β be a maximal finitely inadequate open family of subsets of X. If some member of β contains the set $G_1 \cap G_2 \cap \ldots \cap G_n$ where each G_i, $i=1,2,\ldots,n$ is open, then G_k belongs to β for some $k \leq n$, $n \in N$.[3]

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[3]The letter N will be reserved for the set of natural numbers.

Proof: Assume $G_k \notin \beta$ for all $k \leq n$. Since β is a maximal finitely inadequate open family, and $G_k \notin \beta$, then the union of each G_i, $i \leq n$ with some finite subfamily of β, call it $\mathcal{A}_i$, must fail to be finitely inadequate and hence cover X. Thus

$G_i \cup [\cup \mathcal{A}_i] = X$ and certainly $\bigcap_{i=1}^{n} \left[G_i \cup [\cup \mathcal{A}_i]\right] = X$ also. Consider the set

$$C = \left[\bigcap_{i=1}^{n} G_i\right] \cup \left[\bigcup_{i=1}^{n} [\cup \mathcal{A}_i]\right].$$

If $x \in X$ and $x \notin \bigcap_{i=1}^{n} G_i$, then $x \notin G_j$ for some $j \leq n$ and hence $x \in [\cup \mathcal{A}_j]$. Thus, every x in X belongs to C and C = X. Therefore, $\bigcap_{i=1}^{n} G_i$ cannot belong to a member of β since β is maximally finitely inadequate.

<u>Lemma 2</u>. Let $\mathcal{D}$ be a finitely inadequate subfamily of $\mathcal{J}$, where $(X, \mathcal{J})$ is a topological space. Then there is a maximal finitely inadequate subfamily β of $\mathcal{J}$ such that $\mathcal{D} \subset \beta$.

Proof: Let $\mathcal{C}$ be the collection of all finitely inadequate subfamilies of $\mathcal{J}$ and let $\mathcal{C}$ be ordered by set inclusion. Now $\mathcal{D} \in \mathcal{C}$ and $\mathcal{C}$ is partially ordered. Let $\mathcal{A}$ be a maximal simply ordered subcollection of $\mathcal{C}$ such that $\mathcal{D} \in \mathcal{A}$. Let $\beta = \cup \mathcal{A}$.

(Observe that $\mathcal{B}$ so composed is a family of open sets since each member of $\mathcal{A}$ was such a family.) To prove that $\mathcal{B}$ is finitely inadequate, let $B_1, B_2, \ldots, B_k$ be subsets of $\mathcal{B}$. Then for each $i \leq k$, there is some $\mathcal{C}_i \in \mathcal{A}$ $(\mathcal{C}_i \in \mathcal{C})$ such that $B_i \in \mathcal{C}_i$. Since $\mathcal{A}$ is simply ordered by set inclusion, one of these $\mathcal{C}$'s, say $\mathcal{C}_j$, contains each of the other $\mathcal{C}$'s. Thus, $B_i \in \mathcal{C}_j$ for each $i \leq k$. Therefore, $B_1 \cup B_2 \cup \ldots \cup B_k \neq X$ since $\mathcal{C}_j$ is finitely inadequate. This implies that $\mathcal{B}$ is finitely inadequate.

The proof that $\mathcal{B}$ is also a maximal finitely inadequate subfamily of $\mathcal{J}$ is indirect. Assume that $\mathcal{B}$ is not maximal; that is, there is some open set $G \notin \mathcal{B}$ such that $\mathcal{B} \cup \{G\}$ is still finitely inadequate. Since $\mathcal{B}$ contains the elements of each member of $\mathcal{A}$, $\mathcal{A} \cup \{\mathcal{B} \cup \{G\}\}$ would be simply ordered. Thus $\mathcal{A} \subset \mathcal{A} \cup \{\mathcal{B} \cup \{G\}\}$ since $G \notin \mathcal{B}$ and hence $G \in \bigcup \mathcal{A}$. This contradicts the maximality of $\mathcal{A}$ which means that $\mathcal{B}$ is the required maximal finitely inadequate family such that $\mathcal{D} \subset \mathcal{B}$.

Theorem 5 (Alexander). Let $\mathcal{S}$ be a subbase for the topology of the topological space $(X, \mathcal{J})$. If each cover of X by members of $\mathcal{S}$ has a finite subcover, then X is compact.

Proof: Let $\mathcal{F}$ be any finitely inadequate subfamily of $\mathcal{J}$. By lemma 2, let $\mathcal{D}$ be a maximal finitely inadequate subfamily of

$\mathcal{J}$ which contains $\mathcal{F}$. The subfamily $\mathcal{B} \cap \mathcal{S}$ of $\mathcal{B}$ (the collection of all sets which are intersections of a member of $\mathcal{B}$ with a member of $\mathcal{S}$) is finitely inadequate since $\mathcal{B}$ is finitely inadequate. Since every finitely inadequate family of $\mathcal{S}$ is inadequate, it follows that $\mathcal{B} \cap \mathcal{S}$ is inadequate. Now suppose $p \in \bigcup \mathcal{B}$. Then $p \in V$ for some $V \in \mathcal{B}$. Since V is open and $\mathcal{S}$ is a subbase, $p \in G_1 \cap G_2 \cap \ldots \cap G_k \subset V$ for some G_i's in $\mathcal{S}$. By lemma 1, $G_j \in \mathcal{B}$ for some $j \leq k$. Thus $p \in G_j$, G_j an element of $(\mathcal{B} \cap \mathcal{S})$ and $p \in \bigcup (\mathcal{B} \cap \mathcal{S})$. This implies $\bigcup \mathcal{B} \subset \bigcup(\mathcal{B} \cap \mathcal{S})$ and hence $\mathcal{B}$ is also inadequate. Clearly $\mathcal{F} \subset \mathcal{B}$ is also inadequate and X therefore compact.

The terminology of the previous section offers another characterization of compactness. A topological space $(X, \mathcal{J})$ is compact if each finitely inadequate subfamily of $\mathcal{J}$ is inadequate.

By generalizing the concept of a sequence, E. H. Moore and H. L. Smith were able to completely describe the topology of a space (and hence also compactness) in terms of convergence [6,62; 9,155]. A few definitions are required to make this generalization.

A binary relation $\geq$ on a nonempty set A is said to <u>direct</u> A if a) $\geq$ is transitive and reflexive and b) $m,n \in A$ implies there is a $p \in A$ such that $p \geq m$ and $p \geq n$. The pair $(A, \geq)$ is called a directed set. If a directed set $(A, \geq)$ and a mapping f of A into a set X is given, f is called a net in X and is denoted by $(f,X,A,\geq)$. Given a net $(f,X,A,\geq)$ and a set $Y \subset X$, f is a) eventually in Y if there is an m in A such that $f(n) \in Y$ for all $n \geq m$ in A and b) frequently in Y if for each m in A there is a $p \geq m$ in A such that $f(p) \in Y$. A net $(f,X,A,\geq)$ in a topological space X is said to converge to $p \in X$ if f is eventually in each neighborhood of p. A net $(f,X,A,\geq)$ is said to have a cluster point $x \in X$ if f is frequently in each neighborhood of x. If $(f,X,A,\geq)$ and $(g,X,B,\geq')$ are nets in X, g is said to be a subnet of f if there is a map h of B into A such that a) $g = f \circ h$ and b) for each $a \in A$, there is a $k \in B$ such that $h(b) \geq a$ for $b \geq' k$.

Lemma 3. If x is a cluster point of a net $(f,X,A,\geq)$ which is eventually in a closed set B, then $x \in B$.

Proof: Suppose $x \notin B$. Then, since B is closed, there is a neighborhood M of x such that $M \cap B = \emptyset$. Since f is eventually in B, then for some $m \in A$, $f(a) \in B$ for every $a \geq m$. But since $M \cap B = \emptyset$, then no $f(a)$ for $a \geq m$ belongs to M, contradicting the fact that x is a cluster point of the net.

Lemma 4. Let $(f,X,A,\geq)$ be a net in a topological space X. For each a in A, let $Y_a = \{f(b): b \geq a, \text{ b in A}\}$. If $x \in \bigcap \{\overline{Y}_a : \text{a in A}\}$, then x is a cluster point of f.[4]

Proof: Suppose x is not a cluster point of f. Then there is some neighborhood M of x and some a in A such that $M \cap Y_a = \emptyset$. Thus $x \notin \overline{Y}_a$ for this a and hence $x \notin \bigcap \{\overline{Y}_a : \text{a in A}\}$.

Theorem 6. A topological space X is compact if and only if each net in X has a cluster point.

Proof: Suppose each net in X has a cluster point and let $\mathcal{F}$ be a family of closed sets with the finite intersection property. Let $\mathcal{C}$ be the collection of all sets which are intersections over finite collections of $\mathcal{F}$. Since the intersection over each finite subfamily of $\mathcal{C}$ is a member of $\mathcal{C}$, it follows that $\mathcal{C}$ is directed by set inclusion. For each $C \in \mathcal{C}$, let $f(C) \in C$. By hypothesis, the net $(f, X, \mathcal{C}, \subset)$ has a cluster point, say x. Let B be any member of $\mathcal{C}$. For each $C \in \mathcal{C}$ such that $C \subset B$, $f(C) \in C \subset B$. So f is eventually in every closed set B. By lemma 3, $x \in B$ and therefore $x \in \bigcap \mathcal{C}$. Since $\mathcal{F} \subset \mathcal{C}$, then $x \in \bigcap \mathcal{F}$ and hence $\bigcap \mathcal{F} \neq \emptyset$ and X is compact.

Conversely, suppose X is compact and let $(f, X, A, \geq)$ be a net in X. For a in A, let $Y_a = \{f(b): b \geq a, \text{ b in A}\}$. Then

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[4]The symbol $\overline{Y}$ denotes the closure of the set Y.

$\{\overline{Y}_a$:a in A$\}$ has the finite intersection property since for any two elements m and n in A, there exists an element p in A such that $p \geq m$ and $p \geq n$ and hence $\overline{Y}_m \cap \overline{Y}_n \supset \overline{Y}_p$. Since X is compact, there exists some point y in $\bigcap\{\overline{Y}_a$:a in A$\}$. Therefore, y must be a cluster point by lemma 4.

Lemma 5. Let $(f,X,A,\geq)$ be a net and x a point in a topological space X. Then x is a cluster point of f if and only if some subnet of f converges to x.

Proof: Let some subnet g of f converge to x. Then g is eventually in each neighborhood of x. From the definition of a subnet it follows that f is frequently in each neighborhood of x and hence x is a cluster point of f.

Conversely, if x is a cluster point of f, let $\mathcal{B}$ be the family of all neighborhoods of x. Let B =
$\{(a,U)$:a in A, U in $\mathcal{B}$, f(a) in U$\}$. For $(a,U), (b,V) \in B$, let $(a,U) \geq' (b,V)$ if and only if $a \geq b$ and $U \subset V$. The relation $\geq'$ is clearly transitive and reflexive. Also, since $(A,\geq)$ is given as a directed set, $a,b \in A$ implies there exists a $c \in A$ such that $c \geq a$ and $c \geq b$. Similarly, U and V are neighborhoods of x and therefore $U \cap V$ is also a neighborhood of x. Set theory assures that $U \cap V \subset U$ as well as $U \cap V \subset V$. Thus B is a directed set. Let h be the map of B into A such that h(a,U) = a for each

(a,U) in B. Then $f \circ h$ is a subnet of f. Now if $U \in \mathscr{U}$, a is an arbitrary element of A such that $f(a) \in U$, and (b,W) is a member of B such that $(b,W) \geq' (a,U)$, then $f \circ h(b,W) = f(b) \in W \subset U$. Therefore, $f \circ h$ is eventually in U and converges to x.

The following theorem follows easily from lemma 5 and theorem 6.

Theorem 7. A space X is compact if and only if each net in X has a subnet which converges to some point in X.

SOME PROPERTIES OF COMPACTNESS

Compact topological spaces possess many important and interesting properties. A number of elementary properties are discussed in the theorems and examples which follow.

The property of being a compact subset and a closed subset of a topological space are related. They are not equivalent, however, in a general topological space. The following theorems and examples relate the two properties.

Theorem 8. A closed subset of a compact topological space is compact.

Proof: Let Y be a closed subset of the compact topological space X. Let $\mathcal{A}$ be an open cover for Y. Then $\beta = \{C(Y)\} \cup \mathcal{A}$ is an open cover for X. Since X is compact, there is a finite subcover of β which also covers X. If $\{B_i : i=1,2,\ldots,n\}$ represents this subcover, then $\{B_i : i=1,2,\ldots,n\} - \{C(Y)\}$ is a finite subcover for Y, and Y is therefore compact.

The converse of theorem 8 is not true in a general topological space. That a compact subset need not be closed is illustrated in the example which follows.

Example 1. Let S be the set of real numbers of the closed interval [0,1] and define the topology on S by letting a subset F of S be closed if it is finite or S itself. The empty set is open as the complement of S and closed since it is finite. The entire set is open as the complement of the empty set. Since the finite union of finite sets is finite, the finite intersection of open sets is open. If $\mathcal{F}$ is a family of closed sets, then $\bigcap \mathcal{F}$ is closed since $\bigcap \mathcal{F} \subset F$ for each $F \in \mathcal{F}$ or $\bigcap \mathcal{F}$ is S itself. Thus the arbitrary union of open sets is open. Let A be a subset of S and $\mathcal{U}$ an open cover of A. Let $U \in \mathcal{U}$ be such that $U \neq \emptyset$. The set $A - U = F$ is a relatively closed set and hence finite. Each element of F belongs to some member of $\mathcal{U}$ and, therefore, some finite collection of members of $\mathcal{U}$ covers F. The union of this collection with $\{U\}$ forms a finite subcover of $\mathcal{U}$ for A and hence A is compact. Thus every subset of S is compact and since S is not finite, some of these sets are not closed.

If a topological space satisfies the Hausdorff separation property, being a compact subset does imply the subset is closed.[5]

Theorem 9. If a subset F of a Hausdorff space X is compact, then it is closed.

Proof: Let F be a compact subset of a Hausdorff space. Let z be a point of A = C(F). Since X is a Hausdorff space, for each x in F, there exists open sets U_x and V_x such that $z \in U_x$, $x \in V_x$, and $U_x \cap V_x = \emptyset$. The family $\{V_x : x \text{ in } F\}$ is an open cover for F which by hypothesis has a finite subcover $\{V_{x_i} : i=1,2,\ldots,n\}$. The intersection $U = \bigcap_{i=1}^{n} U_{x_i}$ is a neighborhood of z which does not intersect F since it does not intersect any of the elements of $\{V_{x_i} : i=1,2,\ldots,n\}$. Thus $U \subset A$, from

[5] A topological space satisfies the Hausdorff separation property if for any two points x,y in the space, there are two neighborhoods U and V of x and y respectively, such that $U \cap V = \emptyset$. Such a space is called a Hausdorff space.

which it follows that A is a neighborhood of each of its points and hence open. Therefore, F = C(A) is closed.

The following theorem is a direct consequence of theorems 8 and 9.

Theorem 10. A subset of a compact Hausdorff space is compact if and only if it is closed.

Although example 1 clearly illustrates that a subset of a space need not be closed to be compact, it is natural to inquire whether the closure of a subset of a space which is compact must itself be compact. This is trivally true in a Hausdorff space. Example 2 illustrates it is not generally true.

Example 2. Let X be the set of real numbers and $\mathcal{J}$ be the family consisting of $\emptyset$, X, and all open right rays in X; that is, all sets of the form $\{x:x > a\}$ for some a in X. The collection $\mathcal{J}$ is a topology for X. Now consider a finite set $\{p\}$, p in X. Since every neighborhood of any point q, $q < p$ contains a right open ray containing p, each neighborhood also contains p. Thus every

$q < p$ is a limit point of $\{p\}$.[6] On the other hand, every point $r > p$ is contained in a neighborhood of the form $\{x:x > \frac{r+p}{2}\}$ which does not intersect $\{p\}$. The closure of $\{p\}$ is therefore $\{x:x \leq p\}$. The set $\{p\}$ is compact because it is finite, but $\overline{\{p\}}$ is not compact. Let $\mathcal{A}$ be an open cover for $\overline{\{p\}}$ such that $X \notin \mathcal{A}$. Consider any finite subcollection $\{A_i:i=1,2,\ldots,n\}$ of $\mathcal{A}$. Each A_i is of the form $A_i = \{x:x > a_i\}$ where $a_i \in X$. The collection $\{a_i:i=1,2,\ldots,n\}$ has a least element, call it a_m. For any $x \leq a_m$, $x \notin \bigcup_{i=1}^{n} A_i$. Hence $\overline{\{p\}}$ fails to be compact.

Not only may the closure of a compact set fail to be compact, the intersection of two compact sets may fail to be compact. The space cannot be a Hausdorff space, however, since in a Hausdorff space the intersection of two compact (closed) sets is compact (closed). An example is given at a later point. The following theorem states that the finite union of compact sets is compact.

<u>Theorem 11</u>. Let X be a topological space and $\mathcal{A}$ a family of compact subsets of X. Then the finite union $\bigcup_{i=1}^{n} A_i$ of members of $\mathcal{A}$

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[6] A point x in X is a <u>limit point</u> of a set $A \subset X$ if every neighborhood of x contains points of $A - \{x\}$.

is a compact subset of X.

Proof: Let $\mathcal{U}$ be an open cover of $A = \bigcup_{i=1}^{n} A_i$. Then $\mathcal{U}$ is a cover of each A_i for i=1,2,...,n. Since A_i is compact there exists a finite subcover $\{B_{i_j} \in \mathcal{U} : j=1,2,\ldots,m_i\}$ of A_i. Therefore $\{B_{i_j} : j=1,2,\ldots,m_i;\ i=1,2,\ldots,n\}$ is a finite cover of A itself.

Although theorem 11 guarantees that any finite union of compact sets is compact, an arbitrary union of compact sets may fail to be compact. An example to illustrate this fact is presented at a later point. The important fact that the product of an arbitrary collection of compact spaces is compact follows.

Theorem 12 (Tychonoff). The Cartesian product of a collection of compact topological spaces is compact relative to the product topology.

Proof: Let $\mathcal{C} = \prod\{X_a : a \text{ in } A\}$ where each X_a is a compact topological space and $\mathcal{C}$ has the product topology. Let $\mathcal{S}$ be the subbase for the product topology consisting of all sets of the form $P_a^{-1}(U)$ where P_a is the projection into the a-th coordinate and U is open in X_a. By theorem 5, $\mathcal{C}$ will be compact if an arbitrary subfamily $\mathcal{A}$ of $\mathcal{S}$ which is finitely inadequate is inadequate. For each index a, let β_a be the family of all open sets U in X_a such that $P_a^{-1}(U) \in \mathcal{A}$. Then no finite subfamily of β_a covers X_a since $\mathcal{A}$ is finitely inadequate. By the compactness of X_a there is a point x_a such that x_a is contained in $X_a - U$ for each U in β_a. The point x whose a-th coordinate is x_a for all $a \in A$ then belongs to no member of $\mathcal{A}$ and $\mathcal{A}$ is inadequate [6,143].

A property of a topological space is said to be a <u>topological invariant</u> if each topological space homeomorphic to the given space also possesses this property. One important and useful property of compactness is that it is a topological invariant. The following theorems show this fact.

<u>Theorem 13</u>. If $f:X \rightarrow Y$ is a continuous mapping from the topological space X into the topological space Y and $A \subset X$ is compact, then f(A) is a compact subset of Y.

Proof: Let $\mathcal{U}$ be an open cover for f(A). Thus $f(A) \subset \bigcup_{U \in \mathcal{U}} U$ and consequently $A \subset \bigcup_{U \in \mathcal{U}} f^{-1}(U)$ so that $\{f^{-1}(U):U \in \mathcal{U}\}$ is a cover for A. Since f is continuous, $\{f^{-1}(U):U \in \mathcal{U}\}$ is an open cover for A. By the compactness of A, there is a finite subcover $\{f^{-1}(U_i):i=1,2,\ldots,n,\ U_i \in \mathcal{U}\}$ for A.

This implies that $f(A) \subset \bigcup \{U_i : i=1,2,\ldots,n,\ U_i \in \mathcal{U}\}$ and therefore that f(A) is compact.

Theorem 14. Let X and Y be homeomorphic topological spaces. Then X is compact if and only if Y is compact.

Proof: The homeomorphism implies that there exist functions f and g, $f:X \rightarrow Y$ and $g:Y \rightarrow X$ such that f and g are continuous and onto and g is the inverse of f. Therefore, applying theorem 13, X is compact if and only if Y is compact.

Theorem 15. If $f:X \rightarrow Y$ is a continuous mapping from a topological space X into a topological Hausdorff space Y, then A compact in X implies that f(A) is closed in Y.

Proof: Theorem 13 guarantees f(A) is compact and by theorem 9 f(A) is closed.

In metric spaces, there is a relationship between compactness and uniform continuity. The following lemmas and theorem establish this relationship.

Lemma 6. If X is a compact topological space, then every infinite subset of X has at least one limit point in X.

Proof: Let A be an arbitrary infinite subset of X that has no limit points. Since A is infinite, an infinite sequence of distinct points $E = \{x_i : i \in N\}$, N the set of natural numbers, may be chosen from A.[7] The set E has no limit points in E since it is a subset of A and hence each point x_n, n in N, is not a limit point of E. This means that for every n in N, there exists an open set G_n containing x_n such that $E \cap (G_n - \{x_n\}) = \emptyset$. Then $E \cap G_n = \{x_n\}$ for every n in N. Since E has no limit points it is closed and C(E) is therefore open. The collection $\{C(E)\} \cup \{G_n : n \in N\}$ is an open cover for X which has no finite subcover since for each $x_n \in E$, $x_n \notin G_j$ when $j \neq n$ and $n \in N$. This contradicts the assumption.

<u>Lemma 7</u>. Let (X,d) be a compact metric space and let $\mathcal{C}$ be an open cover for X. Then there is a $\delta > 0$ (called a <u>Lebesgue number</u> of $\mathcal{C}$) such that each subset of X of diameter less than δ is contained in some member of $\mathcal{C}$.

Proof: Suppose that the conclusion of the lemma is false. Then for each n in N, N the set of natural numbers, there is a subset $A_n \subset X$ such that A_n has diameter less than 1/n and A_n is contained in no member of $\mathcal{C}$. For each n in N, let p_n belong to

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[7] A sequence S in a space X is the collection of elements $\{S(n) : n \in N\}$ defined by the net $(S, X, N, \geq)$ where N is the set of natural numbers.

A_n. The infinite set $\{p_n : n \in N\}$ has a limit point p by lemma 6. Let U be a neighborhood of p. Then $U - \{p\} \cap \{p_n : n \in N\} \neq \emptyset$. Let G in $\mathcal{C}$ be such that $p \in G$. Let $r > 0$ such that $S(p;r) \subset G$.[8] Let $k \in N$ such that $\frac{1}{k} < \frac{r}{2}$ and such that $p_k \in S(p;\frac{r}{2})$. If $y \in A_k$, then

$$d(p,y) \leq d(p,p_k) + d(p_k,y) < \frac{r}{2} + \frac{1}{k} < \frac{r}{2} + \frac{r}{2} = r.$$

Hence $A_k \subset S(p;r) \subset G$, a contradiction of the hypothesis.

Theorem 16. Let $f:(X,d) \rightarrow (Y,d')$ be a mapping from a compact metric space X into a metric space Y. Then f is continuous if and only if f is uniformly continuous [8,185].

Proof: Let f be continuous. Then given an $\varepsilon > 0$, for each $x \in X$ there is a $\delta_x > 0$ such that if y belongs to the spherical neighborhood $S(x;\delta_x)$, then $f(y) \in S(f(y);\frac{\varepsilon}{2})$. The collection $\{S(x;\delta_x) : x \in X\}$ is an open cover of X. Since X is compact, this open cover has a Lebesgue number. Let δ be a positive number less than this Lebesgue number. Then if $z,z' \in X$, and $d(z,z') < \delta$ so that z and z' are in a sphere of radius less than δ, we have $z,z' \in S(x;\delta_x)$ for some $x \in X$ and hence $f(z)$, $f(z') \in S(f(x);\frac{\varepsilon}{2})$. Therefore,

[8] The set $S(p;r)$ is the set $\{x \in X : d(p,x) < r\}$.

$$d'(f(z),f(z')) \leq d'(f(z),f(x)) + d'(f(x),f(z')) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and f is uniformly continuous.

Conversely, if f is uniformly continuous, then it is necessarily continuous.

As illustrated earlier, the property of being closed and that of being compact are related. In the space of real numbers, boundedness is also related to compactness.

Theorem 17. A subset A of the reals R with the usual topology is closed and bounded if it is compact.

Proof: Since the space of reals is a Hausdorff space, A compact implies A is closed. Certainly $R = \bigcup_{n \in N} (-n,n)$ where $(-n,n)$ is an open interval and N is the set of natural numbers. Since $A \subset R$, $\bigcup_{n \in N} (-n,n)$ is also a cover for A. The compactness of A implies there is a finite subcover $\bigcup_{n \in K} (-n,n)$, K some finite subset of N. Since K has some largest element, call it k, $A \subset \bigcup_{n \in K} (-n,n)$ implies $A \subset (-k,k)$ and hence that A is bounded.

The converse is also true and can be shown by using the fact that any closed and finite interval is compact.

Lemma 8. The closed interval [0,1] is compact.

Proof: Let $\mathcal{A}$ be an open cover of [0,1] that contains no finite subcover. Divide [0,1] into two segments, [0,1/2] and [1/2,1]. At least one of these segments cannot be covered by a finite subcover of $\mathcal{A}$. Denote this set by $[a_1,b_1]$. Then at least one of the two segments $\left[a_1, \frac{a_1+b_1}{2}\right], \left[\frac{a_1+b_1}{2}, b_1\right]$ cannot be covered by a finite subcover of $\mathcal{A}$. Denote this set by $[a_2,b_2]$. From this process the sequence $S_a = \{a_1, a_2, \ldots, a_n, \ldots\}$ can be formed which is bounded above by 1 and is non-decreasing. Therefore, S_a has a least upper bound, call it a_o. The sequence $S_b = \{b_1, b_2, \ldots, b_n, \ldots\}$ is bounded below and is non-increasing and hence has a greatest lower bound, call it b_o. Furthermore,

$$a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots \leq a_o \leq \ldots \leq b_o \leq \ldots \leq b_n \leq b_{n-1} \leq \ldots \leq b_1.$$

Since $|b_n - a_n|$ approaches zero, $a_o = b_o$. Let G_k be a member of such that G_k contains a_o. Then for n sufficiently large, $[a_n,b_n]$ is contained in G_k which contradicts the hypothesis about $[a_n,b_n]$. Therefore, the hypothesis that there is no finite subcover is false and [0,1] is compact [2,66; 8,172].

Since the closed interval [0,1] is homeomorphic to each closed interval [a,b], a and b finite, and compactness is a topological invariant, it is true that [a,b] is compact.

Theorem 18. If A is a closed and bounded subset of the reals R, then A is compact [12,114].

Proof: Since A is bounded, it is a subset of some closed interval $[-k,k]$ for k sufficiently large. Since A is a closed subset of a compact Hausdorff space, it is compact.

The following is a direct consequence of theorems 17 and 18.

Theorem 19. A subset of the space of real numbers is compact if and only if the subset is closed and bounded.

The following examples utilize the previous theorems to illustrate that neither the finite intersection nor the arbitrary union of compact sets must be compact.

Example 3. Let $X = \{a,b\}$ be a space which is not discrete and Y be the space of the real numbers with the usual topology. Let M be a subset of $X \times Y$ such that

$$M = \{(a,y):1 \leq y < 6\} \cup \{(b,y):5 \leq y \leq 7\}$$

and let S be a subset of $X \times Y$ such that

$$S = \{(a,y):2 < y \leq 7\} \cup \{(b,y):1 \leq y \leq 3\}.$$

Let $\mathcal{C}$ be an arbitrary open cover for M. Since X is not discrete, the members of $\mathcal{C}$ must be of the form $X \times V$ where V is open in Y. Let P_y be the projection into Y. Then $P_y(X \times V) = V$ for each $X \times V \in \mathcal{C}$. Since $\mathcal{C}$ covers M, each y such that

$$y \in \{1 \le y < 6\} \cup \{5 \le y \le 7\} = [1,7]$$

is contained in some open set V. Hence the collection $\mathcal{V}$ of all V for which $X \times V \in \mathcal{C}$, covers [1,7]. By the compactness of [1,7], there is a finite subcollection $\{V_i : i=1,2,\ldots,n\}$ of $\mathcal{V}$ which also covers [1,7]. Hence, $X \times V_i$, $i=1,2,\ldots,n$, is a finite subcollection of $\mathcal{C}$ and M is compact. A similar argument also leads to the compactness of S. The set $M \cap S$ is the set $\{(a,y) : 2 < y < 6\}$ which is homeomorphic to the open set $\{y : 2 < y < 6\} = (2,6)$ by the projection P_y. Thus $M \cap S$ fails to be compact [6,161].

Example 4. Let $\mathcal{a}$ be the set of all open intervals of the form $(-k,k)$ where each k belongs to the set of natural numbers. Then the reals, R, are contained in $\bigcup \mathcal{a}$. Assume $\mathcal{a}$ has a finite subcover β such that each k belongs to some finite subset M of N, the natural numbers. Now M has a largest element m and $(-m,m)$ contains all the other members of β. But $p > m$ does not belong to any member of β. Therefore R is not compact. On the other hand, $R = \bigcup_{k \in N} [-k,k]$ is a union of compact sets.

When the terminology of Euclidean n-space (R^n) is applied to the real line, the unit cube is the closed interval $[0,1] = I$. Generally, $I^2 = I \times I$ and $I^n = I \times I \times \ldots \times I$ for n factors. Define $C_k = [-k,k]$, $C_k^2 = C_k \times C_k$ and $C_k^n = C_k \times C_k \times \ldots \times C_k$ for n factors. Since each $C_k = [-k,k]$ is compact, C_k^n is compact by theorem 12. With this notation, a generalization of theorem 19 can be given.

<u>Theorem</u> <u>20</u> (Heine-Borel theorem). A subset of Euclidean n-space is compact if and only if it is closed and bounded [4,75].

Proof: Let $F \subset R^n$ be compact and let $A_k^n = A_k \times A_k \times \ldots \times A_k$ where A_k is the open interval $(-k,k)$. The collection $\{A_k^n : k \in N\}$, N the set of natural numbers, is an open cover for R^n and hence also for F. By the compactness of F, there is a subset $\{A_j^n : j \in B\}$, B a finite subset of N, which also covers F. Since $A_j \subset C_j$ for each $j \in B$, F is also covered by $\{C_j^n : j \in B\}$. Set B has a greatest element p and hence $F \subset C_p^n$ and is bounded. The set F, as a compact subset of a Hausdorff space, is also closed.

Conversely, if F is closed and bounded, F is contained in some set C_k^n for k sufficiently large. Hence F is a closed subset of a compact space and is itself compact.

The following two properties are easy consequences of the preceding work.

<u>Theorem</u> <u>21</u>. If $f:X \rightarrow R^n$ is a continuous mapping from a compact space X into R^n, then f(X) is bounded.

Proof: By theorem 13, f(X) is compact and since $f(X) \subset R^n$, by theorem 20, it is also bounded.

<u>Theorem</u> <u>22</u>. If $f:X \rightarrow R$ is a continuous mapping from a compact space X into the reals R, then f assumes a maximum and minimum in X.

Proof: Let $f:X \rightarrow R$ satisfy the conditions of the theorem. Then f(X) is compact and as a subset of R is closed and bounded. Since f(X) is bounded, f(X) has a least upper bound m and a greatest lower bound n which are limit points of f(X). Since f(X) is closed, m and n belong to f(X). Therefore, there exist some u and v in X such that f(u) = m and f(v) = n and hence f attains a maximum and minimum in X.

THREE OTHER TYPES OF COMPACTNESS

Compactness was defined and characterized in a general topological space. Related concepts can be defined which are equivalent to compactness under special conditions. In this section three other types of compactness which are generally found in elementary topology are considered.

Definition 2. A topological space is said to be countably compact if every countable open cover has a finite subcover.

This definition, given by Kelly and Simmons, differs from a definition given by Pervin, Moore, and Bushaw [4,83; 6,162; 9,67; 11,58; 12,114]. Pervin states that a subset A of a topological space is said to be countably compact if every infinite subset of A has at least one limit point in A. The following two theorems and example establish the relationship between definition 2 and that given by Pervin.

Theorem 23. If every infinite subset A of a T_1-space X has a limit point in A, then X is countably compact.[9]

Proof: Let $\{G_n : n \in N\}$, N the set of natural numbers, be a countable, finitely inadequate family of open sets. Then

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[9] A T_1-space is one which satisfies the following conditions: If x and y are two distinct points of X, then there exist two open sets, one containing x but not y and the other containing y but not x.

$\bigcup_{i=1}^{n} G_i \neq X$ for any $n \in N$. The set $F_n = C(\bigcup_{i=1}^{n} G_i)$ is a nonempty closed set and $F_{n+1} \subset F_n$. From F_n choose a point x_n for each $n \in N$ and let $A = \bigcup_{n \in N}\{x_n\}$. If A is finite, then there is some point in an infinitely many F_n and hence all of the sets F_n and $\{G_n : n \in N\}$ is inadequate. If A is infinite, it contains a limit point x. Any open set about x contains a point of $A-\{x\}$ and thus x is a limit point of each of the sets $A_n = \bigcup_{i>n}\{x_i\}$. For each n, however, A_n is contained in the closed set F_n, and hence x must belong to F_n for every $n \in N$. Thus $\bigcap_{n \in N} F_n \neq \emptyset$ implies $C(\bigcap_{n \in N} F_n) \neq X$ and since $F_n = C(\bigcup_{i=1}^{n} G_n)$,

$$C(\bigcap_{n \in N} F_n) = \bigcup_{n \in N} C(F_n) = \bigcup_{n \in N} (\bigcup_{i=1}^{n} G_n) = \bigcup_{n \in N} G_n \neq X$$

and $\{G_n : n \in N\}$ is inadequate.

Theorem 24. If a topological space X is countably compact, then every infinite subset of X has at least one limit point in X.

Proof: Since, from lemma 6, the collection $\{C(E)\} \cup \{G_n : n \in N\}$ is a countable open cover, the proof of lemma 6 also serves as a proof of this theorem.

The following example illustrates that the two definitions given for a countably compact space are not equivalent.

Example 5. Let X be the set of natural numbers. Let $B_i = \{2i, 2i-1\}$ and let $\beta = \{B_i : i \in N\}$ be a basis for a topology for X where $\emptyset$ and X are open. Certainly if $\mathcal{U}$ is a collection of open sets, $\cup\,\mathcal{U}$ is also open since $\cup\,\mathcal{U} = \cup_{U \in \mathcal{U}} \cup_{B \subset U} B$. Also, if U and V are two open sets such that $U \cap V \neq \emptyset$, then

$$U \cap V = (\cup_{B \subset U} B) \cap (\cup_{B \subset V} B) = (\cup_{B \subset U \cap V} B) = W$$

is an open set. The set X, with this topology, satisfies the condition that every infinite subset A of X has a limit point in X but fails to satisfy definition 2. Let A be any infinite subset of X and x some point of A. If x is of the form 2i-1, then 2i is a limit point of A. If x is of the form 2i, then 2i-1 is a limit point of A. Therefore A has a limit point in X. On the other hand, let β be an open cover for X. No finite subcover of β covers X since a finite subcover contains at most a finite number of elements of X.

Since a countable open cover is an open cover, the following theorem holds.

Theorem 25. A compact topological space is countably compact.

<u>Theorem 26</u>. A second axiom countably compact topological space is compact.[10]

Proof: Let X be a topological space which satisfies the second axiom of countability. Let $\mathcal{U}$ be an open cover of X and β a countable base for the open sets of X. Choose all elements B_i, $i \in I$ from β whose unions form members of $\mathcal{U}$. This collection $\{B_i : i \in I\}$ is also a cover for X, and since β is countable a finite collection, $\{B_i : i=1,2,\ldots,m\} \subset \beta$ also covers X. Since each B_i is contained in some member of $\mathcal{U}$, call it U_i, every x in $\bigcup_{i=1}^{m} B_i$ also belongs to $\bigcup_{i=1}^{m} U_i$ and hence $\{U_i : i=1,2,\ldots,m\}$ is also a cover for X.

A property related to the characterization of compactness given in theorem 7 is called sequential compactness.

<u>Definition 3</u>. A topological space is said to be <u>sequentially compact</u> if every sequence in the space has a convergent subsequence.

The following four theorems relate compactness, countable

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[10] A topological space is said to satisfy the second axiom of countability if there exists a countable base for the open sets.

compactness, and sequential compactness.

Theorem 27. A first axiom compact T_1-space X is sequentially compact.[11]

Proof: Let $\{x_n : n \in N\}$ be an arbitrary sequence of points in X. If $x_n = x$ for infinitely many n, $\{x_n : n \in N\}$ is convergent and X is sequentially compact. If no element is repeated infinitely often, then a subsequence may be chosen by removing all the points of $\{x_n : n \in N\}$ which are identical to elements which have appeared before. Denote the subsequence by $\{x_m : m \in N\}$. By theorems 24 and 25, $\{x_m : m \in N\}$ must have a limit point. Let p be such a limit point. Every open set G_p which contains p also contains infinitely many of the $\{x_m : m \in N\}$. At p there is a countable family of open sets $\{G_i : i \in N\}$ which is a basis for the neighborhoods of p and directed by set inclusion. Let $\{x_{m_i} : i \in N\}$ be a subsequence of $\{x_m : m \in N\}$ chosen such that x_{m_1} is some element of G_1; x_{m_2} is some element of $G_1 \cap G_2 - \{x_{m_1}\}$; and, in general,

$$x_{m_i} \in \left(\bigcap_{k=1}^{i} G_k - \{x_{m_j} : j=1,2,\ldots,i-1\}\right).$$

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[11] A topological space is said to be a first axiom space if at each point $x \in X$ there is a countable system of neighborhoods for x.

For each G_i, $j > i$ implies $x_j \in G_i$, then $\{x_{m_i} : i \in N\}$ is convergent and X is sequentially compact [3,22].

Theorem 28. A second axiom countably compact T_1-space is sequentially compact.

Proof: The countable basis for the open sets determined by a second axiom space also determines a countable base for the complete system of neighborhoods of each point. Therefore, theorem 28 follows from theorems 26 and 27.

Theorem 29. A sequentially compact T_1-space X is countably compact.

Proof: Let E be an infinite subset of X and let $\{x_n : n \in N\}$, N the set of natural numbers, be a sequence of distinct points in E. By the sequential compactness of X, there is a subsequence $\{x_{n_i} : i \in N\}$ of $\{x_n : n \in N\}$ which converges to a point $p \in E$. Therefore, for each open set G which contains p, there is an associated integer $q \in N$ such that whenever $r > q$, $x_{n_r} \in G$. Since the elements of $\{x_{n_i} : i \in N\}$ are distinct, each open set which contains p contains also other points of E and thus p is a limit point of E. Therefore, since X is a T_1-space, X is countably compact by theorem 23.

Theorem 30. A second axiom sequentially compact T_1-space X is compact.

Proof: It follows from theorem 29 that a second axiom sequentially compact T_1-space is countably compact and by theorem 26 that X is compact.

Many spaces which fail to be compact still possess many properties related to compactness. By "localizing" the concept of compactness a useful property results.

Definition 4. A topological space S is said to be locally compact if each point of X is contained in a compact neighborhood of the point.

Pervin points out that another definition is adopted by some authors. He states that a topological space S is said to be locally compact if each point of x is contained in a neighborhood whose closure is compact [11,59]. Clearly the alternative definition implies definition 4. If the topological space X is a Hausdorff space, then the compact neighborhoods of definition 4 are closed and the two definitions are equivalent.

Since an entire topological space is a neighborhood of each of its points, it follows that every compact space is locally compact. On the other hand, spaces do exist which are locally compact but fail to be compact. The set of reals is not compact under the usual topology, but every point is contained in some closed and bounded set and therefore a compact neighborhood.

The property of being locally compact is such a general one that most of the spaces in elementary topology are locally compact. There do exist spaces which fail to be locally compact, however, as the following example illustrates.

Example 6. Let X be the space of real numbers with the usual topology and I be the set of integers. Let $\mathcal{D}$ be a family of subsets of X whose members are I and all sets $\{x\}$ for $x \in X - I$. Then $\mathcal{D}$ with the quotient topology is an example of a space which is not locally compact [6,165].

Among the properties of local compactness are several related to the properties of compactness. One of particular interest is described by Baum as "a very pretty topological analog of the process of stereographic projection [1,92]." It is possible to consider a locally compact Hausdorff space as being a subspace of a compact space containing one additional point. The one-point compactification is presented in the following theorem.

<u>Theorem</u> <u>31</u>. Let X be a topological space and p a point which does not belong to X. Assume a base for the system of neighborhoods of p which consists of the complements in $X \cup \{p\}$ of compact subsets of X. Then, if X is a locally compact Hausdorff space, $X \cup \{p\}$ is a compact Hausdorff space.

Proof: By the local compactness of X, each point $x \in X$ is contained in some closed compact neighborhood $F_x \subset X$. The complement of F_x in $Y = X \cup \{p\}$ contains p and is disjoint from F_x. Therefore, Y is Hausdorff. To show that Y is also compact, let $\mathcal{U}$ be an open cover of Y. Then p belongs to some member $U \in \mathcal{U}$. Since U is a neighborhood of p, it contains a member M of a base $\mathcal{N}$ for the system of neighborhoods of p which are complements of compact subsets of X. The complement of M is therefore compact. Being compact, it can be covered by a finite number of members of $\mathcal{U}$ which cover Y. This finite collection together with the set $U \supset M$ is a finite subcover of Y [6,150].

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COMPACTNESS IN TOPOLOGICAL SPACES

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COMPACTNESS IN TOPOLOGICAL SPACES

Compactness is an abstraction of the property of a closed and bounded subset of the space of real numbers. Specifically, if one considers a closed and bounded subset of the real numbers as a topological space in its own right, this space is compact. Compactness is of particular interest since continuous mappings from compact metric spaces into metric spaces are necessarily uniformly continuous.

A topological space is defined to be compact if every collection of open sets whose union contains the space has a finite subcollection whose union also contains the space. Equivalent characterizations are made in terms of neighborhoods, closed sets, basic and subbasic collections of open sets, and Moore-Smith convergence.

Compactness is related to many other topological properties. In a compact Hausdorff space, the collection of closed subsets and the collection of compact subsets are identical. In a general topological space the closure of a compact subspace need not be compact. Finite unions of compact sets are themselves compact, but arbitrary unions may fail to be compact. Finite intersections of compact sets may also fail to be compact. Under the product topology, the product of arbitrary collections of compact spaces is also compact. Compactness is shown to be a topological invariant. In Euclidean n-space, the compact subsets are the closed and bounded subsets.

There are many properties closely related to compactness which are special types of compactness. Three of these--countable compactness, sequential compactness, and local compactness--are defined. The conditions are given which relate countable compactness, sequential compactness, and compactness.