Existence and uniqueness of the global solution to the Navier–Stokes equations

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A proof is given of the global existence and uniqueness of a weak solution to Navier–Stokes equations in unbounded exterior domains.

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1. Introduction

Let \( D \subset \mathbb{R}^3 \) be a bounded domain with a connected \( C^2 \)-smooth boundary \( S \), and \( D' := \mathbb{R}^3 \setminus D \) be the unbounded exterior domain.

Consider the Navier–Stokes equations:

\[
\begin{align*}
    u_t + (u, \nabla)u &= -\nabla p + \nu \Delta u + f, & x \in D', \ t \geq 0, \\
    \nabla \cdot u &= 0, \quad (1) \\
    u|_S &= 0, \quad u|_{t=0} = u_0(x). \quad (2)
\end{align*}
\]

Here \( f \) is a given vector-function, \( p \) is the pressure, \( u = u(x,t) \) is the velocity vector-function, \( \nu = \text{const} > 0 \) is the viscosity coefficient, \( u_0 \) is the given initial velocity, \( u_t := \partial_t u, \ (u, \nabla)u := u_a \partial_a u, \partial_a u := \frac{\partial u}{\partial x_a} := u_{a;1}, \) and \( \nabla \cdot u_0 := u_{a;1} = 0. \) Over the repeated indices \( a \) and \( b \) summation is understood, \( 1 \leq a, b \leq 3. \) All functions are assumed real-valued.

We assume that \( u \in W, \)

\[
W := \{ u \in L^2(0,T;H^1_0(D')) \cap L^\infty(0,T;L^2(D')) \cap u_t \in L^2(D' \times [0,T]); \nabla \cdot u = 0 \},
\]

where \( T > 0 \) is arbitrary.

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Let \((u, v) := \int_D u_a v_a dx\) denote the inner product in \(L^2(D')\), \(\|u\| := (u, u)^{1/2}\). By \(u_{ja}\) the \(a\)-th component of the vector-function \(u_j\) is denoted, and \(u_{ja;b}\) is the derivative \(\frac{\partial u_{ja}}{\partial x_b}\). Eq. (2) can be written as \(u_{a;\alpha} = 0\) in these notations. We denote \(\frac{\partial u^2}{\partial x_a} := (u^2)_a\), \(u^2 := u_b u_b\). By \(c > 0\) various estimation constants are denoted.

Let us define a weak solution to problem (1)–(3) as an element of \(W\) which satisfies the identity:

\[ (u_t + \nu \nabla u, \nabla v) = (f, v), \quad \forall v \in W. \]  

(4)

Here we took into account that \(- (\Delta u, v) = (\nabla u, \nabla v)\) and \((\nabla p, v) = -(p, v_{a;\alpha}) = 0\) if \(v \in \mathcal{H}^1_0(D')\) and \(\nabla \cdot v = 0\). Eq. (4) is equivalent to the integrated equation:

\[ \int_0^t [(u_s, v) + (u_{a;\alpha}, v_b)] + \nu (\nabla u, \nabla v) ds = \int_0^t (f, v) ds, \quad \forall v \in W. \]  

(*)

Eq. (4) implies Eq. (*), and differentiating Eq. (*) with respect to \(t\) one gets Eq. (4) for almost all \(t \geq 0\).

The aim of this paper is to prove the global existence and uniqueness of the weak solution to the Navier–Stokes boundary problem, that is, solution in \(W\) existing for all \(t \geq 0\). Let us assume that

\[ \sup_{t \geq 0} \int_0^t \|f\|ds \leq c, \quad (u_0, u_0) \leq c. \]  

(A)

**Theorem 1.** If assumptions (A) hold and \(u_0 \in \mathcal{H}^1_0(D)\) satisfies Eq. (2), then there exists for all \(t > 0\) a solution \(u \in W\) to (4) and this solution is unique in \(W\) provided that \(\|\nabla u\|^4 \in L^1_{loc}(0, \infty)\).

In Section 2 we prove Theorem 1. There is a large literature on Navier–Stokes equations, of which we mention only [1,2]. The global existence and uniqueness of the solution to Navier–Stokes boundary problems has not yet been proved without additional assumptions. Our additional assumption is \(\|\nabla u\|^4 \in L^1_{loc}(0, \infty)\). The history of this problem see, for example, in [1]. In [2] the uniqueness of the global solution to Navier–Stokes equations is established under the assumption \(\|u\|_{L^4(D')}^8 \in L^1_{loc}(0, \infty)\).

2. Proof of Theorem 1

**Proof of Theorem 1.** The steps of the proof are: (a) derivation of a priori estimates; (b) proof of the existence of the solution in \(W\); (c) proof of the uniqueness of the solution in \(W\).

(a) **Derivation of a priori estimates**

Take \(v = u\) in (4). Then

\[ (u_{a;\alpha}, u_b) = -(u_{a;\alpha}, u_b) = -\frac{1}{2} (u_a, (u^2)_\alpha) = \frac{1}{2} (u_{a;\alpha}, u^2) = 0, \]

where the equation \(u_{a;\alpha} = 0\) was used. Thus, Eq. (4) with \(v = u\) implies

\[ \frac{1}{2} \partial_t (u, u) + \nu (\nabla u, \nabla u) = (f, u) \leq \|f\| \|u\|. \]  

(5)

We will use the known inequality \(\|u\| \|f\| \leq \epsilon \|u\|^2 + \frac{1}{4\epsilon} \|f\|^2\) with a small \(\epsilon > 0\), and denote by \(c > 0\) various estimation constants.

One gets from (5) the following estimate:

\[ (u(t), u(t)) + 2\nu \int_0^t (\nabla u, \nabla u) ds \leq (u_0, u_0) + 2 \int_0^t \|f\| ds \sup_{s \in [0, t]} \|u(s)\| \leq c + c \sup_{s \in [0, t]} \|u(s)\|. \]  

(6)

Recall that assumptions (A) hold. Denote \(\sup_{s \in [0, t]} \|u(s)\| := b(t)\). Then inequality (6) implies

\[ b^2(t) \leq c + cb(t), \quad c = const > 0. \]  

(7)
Since $b(t) \geq 0$, inequality (7) implies
\[
\sup_{t \geq 0} b(t) \leq c. \tag{8}
\]
Remember that $c > 0$ denotes various constants, and the constant in Eq. (8) differs from the constant in Eq. (7). From (6) and (8) one obtains
\[
\sup_{t \geq 0} [(u(t), u(t)) + \nu \int_0^t (\nabla u, \nabla u) ds] \leq c. \tag{9}
\]
A priori estimate (9) implies for every $T \in [0, \infty)$ the inclusions
\[
u \in L^\infty(0, T; L^2(D')), \quad u \in L^2(0, T; H^1_0(D')).
\]
This and Eq. (4) imply that $u_t \in L^2(D' \times [0, T])$ because Eq. (4) shows that $(u_t, v)$ is bounded for every $v \in W$. Note that $L^\infty(0, T; L^2(D')) \subset L^2(0, T; L^2(D'))$, and that bounded sets in a Hilbert space are weakly compact. Weak convergence is denoted by the sign $\rightharpoonup$.

(b) **Proof of the existence of the solution $u \in W$ to (4) and (**) 

The idea of the proof is to reduce the problem to the existence of the solution to a Cauchy problem for ordinary differential equations (ODE) of finite order, and then to use a priori estimates to establish convergence of these solutions of ODE to a solution of Eqs. (4) and (**). This idea is used, for example, in [1]. Our argument differs from the arguments in the literature in treating the limit of the term $\int_0^t (u^n, v) ds$.

Let us look for a solution to Eq. (4) of the form $u^n := \sum_{j=1}^n c^n_j(t) \phi_j(x)$, where $\{\phi_j\}_{j=1}^\infty$ is an orthonormal basis of the space $L^2(D')$ of divergence-free vector functions belonging to $H^1_0(D')$ and in the expression $u^n$ the upper index $n$ is not a power. If one substitutes $u^n$ into Eq. (4), takes $v = \phi_m$, and uses the orthonormality of the system $\{\phi_j\}_{j=1}^\infty$ and the relation $(\nabla \phi_j, \nabla \phi_m) = \lambda_m \delta_{jm}$, where $\lambda_m$ are the eigenvalues of the vector Dirichlet Laplacian in $D$ on the divergence-free vector fields, then one gets a system of ODE for the unknown coefficients $c^n_m$:
\[
\partial_t c^n_m + \nu \lambda_m c^n_m + \sum_{i,j=1}^n (\phi_{ia} \phi_{jb;m}, \phi_{mb}) c^n_i c^n_j = f_m, \quad c^n_m(0) = (u_0, \phi_m). \tag{10}
\]
Problem (10) has a unique global solution because of the a priori estimate that follows from (9) and from Parseval’s relations:
\[
\sup_{t \geq 0} (u^n(t), u^n(t)) = \sup_{t \geq 0} \sum_{j=1}^n |c^n_j(t)|^2 \leq c. \tag{11}
\]
Consider the set $\{u^n = u^n(t)\}_{n=1}^\infty$. Inequalities (9) and (11) for $u = u^n$ imply the existence of the weak limits $u^n \rightharpoonup u$ in $L^2(0, T; H^1_0(D'))$ and in $L^\infty(0, T; L^2(D'))$. This allows one to pass to the limit in Eq. (**) in all the terms except the first, namely, in the term $\int_0^t (u^n_a, v(s)) ds$. The weak limit of the term $(u^n_a u^n_b, v_b)$ exists and is equal to $(u_a u_b, v_b)$ because
\[
(u^n_a u^n_b, v_b) = -(u^n_a u^n_b, v_b) \rightarrow -(u_a u_b, v_b) = (u_a u_b, v_b).
\]
Note that $v_b \in L^2(D')$ and $u^n_a u^n_b \in L^4(D')$. The relation $(u^n_a u^n_b, v_b) = -(u^n_a u^n_b, v_b)$ follows from an integration by parts and from the equation $u^n_{aca} = 0$.

The following inequality is essentially known:
\[
\|u\|_{L^4(D')} \leq 2^{1/2} \|u\|^{1/4} \|\nabla u\|^{3/4}, \quad \|u\| := \|u\|_{L^2(D')}, \quad u \in H^1_0(D'). \tag{12}
\]
In [1] this inequality is proved for \( D' = \mathbb{R}^3 \), but a function \( u \in H^1_0(D') \) can be extended by zero to \( D = \mathbb{R}^3 \setminus D' \) and becomes an element of \( H^1(\mathbb{R}^3) \) to which inequality (12) is applicable.

It follows from (12) and Young’s inequality \((ab \leq \frac{a^p}{p} + \frac{b^q}{q}, p^{-1} + q^{-1} = 1)\) that
\[
\|u\|_{L^4(D')}^2 \leq \epsilon \|\nabla u\|^2 + \frac{27}{16\epsilon^2} \|u\|^2, \quad u \in H^1_0(D'),
\]
where \( \epsilon > 0 \) is an arbitrary small number, \( p = \frac{4}{3} \) and \( q = 4 \). One has \( u_n^a u_n^b \rightharpoonup u_a u_b \) in \( L^2(D') \) as \( n \to \infty \), because bounded sets in a reflexive Banach space \( L^4(D') \) are weakly compact. Consequently, \( (u_n^a u_n^b, v_b) \to (u_a u_b, v_b) \) when \( n \to \infty \), as claimed. Therefore, \( \int_0^t (u_n^a u_n^b, v_b) ds \to \int_0^t (u_a u_b, v_b) ds \). The weak limit of the term \( \nu \int_0^t (\nabla u^n, \nabla v) ds \) exists because of the a priori estimate (9) and the weak compactness of the bounded sets in a Hilbert space. Since Eq. (\( \ast \)) holds, and the limits of all its terms, except \( \int_0^t (u_n^a, v) ds \), do exist, then there exists the limit \( \int_0^t (u_n^a, v(s)) ds \to \int_0^t (u_a, v(s)) ds \) for all \( v \in W \). By passing to the limit \( n \to \infty \) one proves that the limit \( u \) satisfies Eq. (\( \ast \)). Differentiating Eq. (\( \ast \)) with respect to \( t \) yields Eq. (4) almost everywhere.

(c) Proof of the uniqueness of the solution \( u \in W \)

Suppose there are two solutions to Eq. (4), \( u \) and \( w \), \( u, w \in W \), and let \( z := u - w \). Then
\[
(z_t, v) + \nu(\nabla z, \nabla v) + (u_a u_{b:a} - w_a w_{b:a}, v_b) = 0.
\]
Since \( z \in W \), one may set \( v = z \) in (14) and get
\[
(z_t, z) + \nu(\nabla z, \nabla z) + (u_a u_{b:a} - w_a w_{b:a}, z_b) = 0, \quad z = u - w.
\]
Note that \((u_a u_{b:a} - w_a w_{b:a}, z_b) = (z_a u_{b:a}, z_b) + (w_a z_{b:a}, z_b)\), and \((w_a z_{b:a}, z_b) = 0\) due to the equation \( w_{a:a} = 0 \). Thus, Eq. (15) implies
\[
\partial_t (z, z) + 2\nu(\nabla z, \nabla z) \leq 2|\langle z_a u_{b:a}, z_b \rangle|.
\]
Since \( |z_a u_{b:a} z_b| \leq |z|^2 \|\nabla u\| \), one has the following estimate:
\[
|\langle z_a u_{b:a}, z_b \rangle| \leq \int_{D'} |z|^2 |\nabla u| dx \leq \|z\|^2 \|\nabla u\|_{L^4(D')} \|\nabla u\| \leq \|\nabla u\| \left( \epsilon \|\nabla z\|^2 + \frac{27}{16\epsilon^2} \|z\|^2 \right).
\]
Denote \( \phi := (z, z) \), take into account that \( \|\nabla u\|^4 \in L^1_{loc}(0, \infty) \), choose \( \epsilon = \|\nabla u\| \) in the inequality (13), in which \( u \) is replaced by \( z \), use inequality (17) and get
\[
\partial_t \phi + \nu(\nabla z, \nabla z) \leq \frac{27}{16\epsilon^2} \|\nabla u\|^4 \phi, \quad \phi|_{t=0} = 0.
\]
In the derivation of inequality (18) the idea is to compensate the term \( \nu \|\nabla z\|^2 \) on the left side of inequality (16) by the term \( \epsilon \|\nabla u\| \|\nabla z\|^2 \) on the right side of inequality (17). To do this, choose \( \|\nabla u\| = \nu \) and obtain inequality (18). It follows from inequality (18) that
\[
\partial_t \phi \leq \frac{27}{16\nu^3} \|\nabla u\|^4 \phi, \quad \phi|_{t=0} = 0.
\]
Since we have assumed that \( \|\nabla u\|^4 \in L^1_{loc}(0, \infty) \) this implies that \( \phi = 0 \) for all \( t \geq 0 \).

Theorem 1 is proved. \( \Box \)

Remark 1. One has (summation is understood over the repeated indices):
\[
2|\langle z_a u_{b:a}, z_b \rangle| = 2|\langle z_a u_b, z_{b:a} \rangle| \leq 18 \|z\| \|z\| \|u\| \leq \nu \|\nabla z\|^2 + \frac{81}{\nu} \|z\| \|u\|^2.
\]
Thus,
\[
\partial_t \phi + \nu(\nabla z, \nabla z) \leq \frac{81}{\nu} \|z\| \|u\|^2.
\]
If one assumes that $|u(\cdot, t)| \leq c(T)$ for every $t \in [0, T]$, then $\partial_t \phi \leq c \phi, \phi(0) = 0$, on any interval $[0, T]$, $c = c(T, \nu) > 0$ is a constant. This implies $\phi = 0$ for all $t \geq 0$. The same conclusion holds under a weaker assumption $\|u(\cdot, t)\|_{L^4(\Omega)} \leq c(T)$ for every $t \in [0, T]$, or under even weaker assumption $\|u(\cdot, t)\|_{L^4(\Omega)}^8 \in L^1_{loc}(0, \infty)$.

In [1] it is shown that the smoothness properties of the solution $u$ are improved when the smoothness properties of $f$, $u_0$ and $S$ are improved.

References