K-THEORY IN CATEGORICAL GEOMETRY

by

ERIC BUNCH

B.S., Baylor University, 2009

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

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Abstract

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In this thesis, we are concerned with geometries extending beyond traditional algebraic geometry coming from the algebraic structure of rings. We consider monoids in a monoidal category as the appropriate generalization of rings–rings being monoids in the monoidal category of Abelian groups. Drawing inspiration from the definition of the spectrum of an Abelian category in [13], and the exploration of it in [15], we define the spectrum of a monoidal category, which we will call the *monoidal spectrum*. We prove a descent condition which is the mathematical formalization of the statment "$R-mod$ is the category of quasi-coherent sheaves on the monoidal spectrum of $R-mod$". In addition, we prove a functoriality condidition for the spectrum, and show that for a commutative Noetherian ring, the monoidal spectrum of $R-mod$ is homeomorphic to the prime spectrum of the ring R.

In [1], Paul Balmer defined the *prime tensor ideal spectrum* of a tensor triangulated category; this can be thought of as the beginning of "tensor triangulated categorical geometry". This definition is very transparent and digestible, and is the inspiration for the definition in this thesis of the *prime tensor ideal spectrum* of an *monoidal Abelian* category. It it shown that for a polynomial identity ring R such that the catgory $R-mod$ is monoidal Abelian, the prime tensor ideal spectrum is homeomorphic to the prime ideal spectrum.

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Acknowledgments

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Dedication

This thesis is dedicated to the memory of Alexander Rosenberg. The memories that I have of Alex relating an anecdote during a pause in teaching mathematics, and laughing so hard at his own jokes you thought he might faint, will stay with me for a long time. But what will stay with me even longer are the things that I took away from observing Alex during the time I knew him. My mathematical philosophy was developed during the years that I spent learning from Alex, and it is one of the most valuable things I will take away from my graduate school experience. Alex was a great example of somebody who approached mathematics and life with calm confidence and good humor. As I look back on my time in graduate school, I will remember to move out of the light of the streetlamp.

Preface

In classical algebraic geometry, we study schemes; these are locally ringed spaces $(X, \mathcal{O}_X)$ that have an open cover $\{U_i\}$ such that $(U_i, \mathcal{O}_{X|U_i}) \simeq (Spec(R_i), \mathcal{O}_{R_i})$ for some commutative unital rings R_i. Although the notion of a scheme was created in order to study pre-existing geometric objects of interest, we will in this thesis take schemes as the basic object of interest, and proceed with the intent to generalize the notions of scheme theory.

One of the first employments of categorical geometry was by Alexander Rosenberg [13, 14] in laying the foundations of noncommutative algebraic geometry. Here, one of the first things one desires is the spectrum of a noncommutative ring which is analagous to the prime spectrum of a commutative ring. It is the *left spectrum*, denoted $Spec_l(R)$, defined in [13] that fills this role. The left spectrum of a ring R, whose points are certain left ideals of R, has the property that it is *Morita equivalence invariant*. That is, if two rings R and S are Morita equivalent (i.e. there is a category equivalence between $R-mod$ and $S-mod$), then $Spec_l(R)$ is homeomorphic to $Spec_l(S)$. This leads naturally to the question *Can a space that is homeomorphic to $Spec_l(R)$ be defined just from the category $R-mod$?* This was answered in the affirmative in [13], by giving the definition of the spectrum of an *Abelian category* $\mathcal{A}$, which we denote by $\mathbf{Spec}^0(\mathcal{A})$, such that $\mathbf{Spec}^0(R-mod)$ is homeomorphic to $Spec_l(R)$.

Recall that if R is a commutative ring, then in addition to the prime spectrum $Spec(R)$, we have the *structure sheaf*

$$\mathcal{O}_R : (Open(Spec(R)))^{op} \to CommRing$$

defined by $U(S) \mapsto S^{-1}R$, where $S \subseteq R$ is a saturated multiplicative system defining an open set in $Spec(R)$, and $S^{-1}R$ is the *localization of R at S*. That is, $S^{-1}R$ is the ring wich formally adds to R the inverses of elements in S. The pair $(Spec(R), \mathcal{O}_R)$ form the *affine scheme* associated to R. The fact that $\mathcal{O}_R$ is a sheaf gives validity to the statement that R *is the ring of functions on the space* $Spec(R)$. For every open set U in $\mathbf{Spec}^0(R-mod)$ (or $Spec_l(R)$), we do not have associated a localization of the ring R, but rather a localization of the category $R-mod$, which we will denote by $R-mod_U$. In general, it is *not* the case that $R-mod_U$ is equivalent to a category of modules over a ring; however, it is a full subcategory of $R-mod$. When R is commutative, $\mathbf{Spec}^0(R-mod)$ is homeomorphic to $Spec(R)$, and the topology on $\mathbf{Spec}^0(R-mod)$ coincides with the Zariski topology on $Spec(R)$. In general, we have a presheaf

$$\mathcal{O}_{\mathcal{A}} : (Open(\mathbf{Spec}^0(\mathcal{A})))^{op} \to Cat$$
$$U \mapsto \mathcal{A}_U$$

where $\mathcal{A}_U$ is a certain localization of the category $\mathcal{A}$ defined by the open set U, and Cat is the category of small categories. Since this functor takes values in categories, we do not speak of a strict sheaf condition. However, in Chapter 1 Section 1.4 we discuss the following: given a finite open cover $\mathcal{U}$ of $\mathbf{Spec}^0(\mathcal{A})$, there is a monad associated to $\mathcal{O}_{\mathcal{A}}$ and $\mathcal{U}$, which we will denote $\mathbf{T}$. This monad satisfies a condition called *effective descent*, which we will see specializes to give the sheaf condition on $\mathcal{O}_{\mathcal{A}}$ when $\mathcal{A} = R-mod$ for a commutative ring R.

The definition of $\mathbf{Spec}^0(\mathcal{A})$ for an Abelian category $\mathcal{A}$ can be considered the first step in "categorical geometry". In this thesis, we wish to do categorical geometry when the category $\mathcal{A}$ at hand is not necessarily an Abelian category, but is instead a monoidal category.

In Chapter 1, we recall the definition of the prime spectrum of a commutative ring in a way that is amenable to generalization. Then we introduce the left spectrum of a ring defined in [13], as well as the spectrum of an Abelian category, as defined in [13], and developed in [14]. We then discuss the descent condition, and how it generalizes the sheaf condition.

In Chapter 2, we define the spectrum of a monoidal category $\mathcal{A}$, denoted $\mathbf{Spec}^{\otimes}(\mathcal{A})$. We then go on to prove a functoriality condition, as well as a descent condition. It is shown that when R is a commutative Noetherian ring, $\mathbf{Spec}^{\otimes}(\mathcal{A})$ is homeomorphic to $Spec(R)$. The example of the monoidal category $A-mod$ where A is a commutative monoid is considered.

In Chapter 3, we consider Abelian categories $\mathcal{A}$ that also have a closed, monoidal structure. In [1], Paul Balmer defines the *prime tensor ideal spectrum* of a tensor triangulated category. This definition is much more digestible transparent than that given in Chapter 2. A definition in terms of $\otimes$-ideals is not always available, since we do not always have an additive structure or a zero object. In Chapter 3, we use the definition in [1] as inspiration, and define the *prime $\otimes$-ideal spectrum* of an Abelian closed monoidal category A, denoted $\mathbf{Prime}(\mathcal{A})$. We show that if R is a Noetherian PI ring such that $R-mod$ is Abelian closed monoidal, then we have homeomorphisms

$$Spec(R) \simeq Spec_l(R) \simeq \mathbf{Spec}^0(R-mod) \simeq \mathbf{Prime}(R-mod^{fg}).$$

It is also shown that if R is left Artinian such that $R-mod$ is closed braided monoidal, then $\mathbf{Spec}^1(R-mod) \simeq \mathbf{Prime}(R-mod^{fg})$.

Chapter 1

Noncommutative Algebraic Geometry

In Section 1.1 of this chapter, we first recall the definition of the prime spectrum $Spec(R)$ of a commutative ring R. We then describe $Spec(R)$ in terms of filters of ideals in R, which is more amenable to generalization than the usual description. We describe the Zariski topology on $Spec(R)$, and the structure sheaf on $Spec(R)$. Then we define, in Section 1.2, the left spectrum $Spec_l(R)$ of a (possibly noncommutative) ring R, as defined in [13]. We then describe the Zariski topology on $Spec_l(R)$, as well as the "structure presheaf" on $Spec_l(R)$, both as defined in [13]. Next, in Section 1.3 we define the spectrum of an Abelian category, which is defined in [13]. We then give, in Section 1.4, a brief expository about descent theory and why it is important in the theory of categorical geometry.

1.1 The Prime Spectrum of a Commutative Ring

Let R be a unital, commutative, associative ring. Denote by $I(R)$ the collection of ideals of R. Recall that an ideal $\mathbf{p} \in I(R)$ is said to be *prime* if for $a, b \in R$, if $ab \in \mathbf{p}$, then either $a \in \mathbf{p}$ or $b \in \mathbf{p}$.

Definition 1. *The* prime spectrum *of R is $Spec(R) := \{\mathbf{p} \in I(R) \mid \mathbf{p}$ is prime$\}$.*

The *Zariski topology* on $Spec(R)$ is defined as follows. For every ideal $\mathbf{m} \in I(R)$, define

$$U(\mathbf{m}) := \{\mathbf{p} \in Spec(R) \mid \mathbf{m} \not\subseteq \mathbf{p}\}.$$

These form the open sets of the Zariski topology on $Spec(R)$ [4]. This topology has a basis given by the open sets

$$U(f) := U(\langle f \rangle) = \{\mathbf{p} \in Spec(R) \mid f \notin \mathbf{p}\},$$

where $\langle f \rangle$ is the ideal generated by f. In general, we will call the elements of a basis of a topological space *basic open sets*, or *basic opens* for the given topology. This basis is closed under finite intersection; in fact, for $f, g \in R$

$$U(f) \cap U(g) = U(fg).$$

Before we define the structure sheaf on $Spec(R)$, we recall some sheaf theory. Let X is a topological space, and β be a basis of open sets for the topology τ on X. Assume also that β is closed under finite intersections (this is not completely necessary, but is more convenient). Then Proposition I-12 in Chapter 1 of [4] says that if

$$F : \beta^{op} \to AbGroups$$

is a presheaf of abelian groups that satisfies the sheaf condition with respect to β, then F uniquely extends to a presheaf

$$F' : \tau^{op} \to AbGroups$$

that satisfies the sheaf condition with respect to τ. For this reason, we will typically define presheaves on basic open sets of a topological space, and only show that they satisfy the sheaf condition with respect to some basis that is closed under finite intersection.

The *structure sheaf* $\mathcal{O}_R$ is the presheaf

$$\mathcal{O}_R : (Open(Spec(R)))^{op} \to CommRing$$

that is defined on basic open sets by $\mathcal{O}_R(U(f)) := R_f$, where R_f is the localization of R at the element f. The presheaf $\mathcal{O}_R$ is in fact a sheaf, which means that for an open cover by basic open sets $\{U(f_j)\}_{j\in J}$ the following diagram is an equalizer diagram:

$$R \to \prod_{j\in J} \mathcal{O}_R(U(f_j)) \rightrightarrows \prod_{i,j\in J} \mathcal{O}_R(U(f_i) \cap U(f_j)). \tag{1.1}$$

The diagram (1.1) is isomorphic to the diagram

$$R \to \prod_{j\in J} R_{f_j} \rightrightarrows \prod_{i,j\in J} R_{f_i f_j}.$$

We now wish to describe $Spec(R)$ is a different fashion.

Definition 2. *Let $F \subseteq I(R)$. We say F is a* filter *if F satisfies the following conditions*

1. $R \in F$,
2. *if* $\mathbf{m} \in F$, *and* $\mathbf{n} \in I(R)$, *and* $\mathbf{m} \subseteq \mathbf{n}$, *then* $\mathbf{n} \in F$,
3. *if* $\mathbf{m}, \mathbf{n} \in F$, *then* $\mathbf{m} \cap \mathbf{n} \in F$.

Denote by $Filt(I(R))$ the collection of all filters in $I(R)$. Note that $Filt(I(R))$ is partially ordered by inclusion. Here we make a short digression that will hopefully make the relevance of the above definition clear. For a subset $F \subseteq I(R)$, denote by $\mathcal{T}_F$ the full subcategory of $R-mod$ whose objects are

$$\{M \in ObR-mod \mid \text{ for each } x \in M, \text{ there exists } \mathbf{n} \in F \text{ such that } \mathbf{n}x = 0\}$$
$$= \{M \in ObR-mod \mid \text{ for each } x \in M, \text{ there exists } \mathbf{n} \in F \text{ such that } \mathbf{n} \subseteq ann(x)\}.$$

The subcategory was originally defined in [6], and is discussed in [10]. We now explore some properties of $\mathcal{T}_F$. First, let $M \in Ob\mathcal{T}_F$, and let $f : N \to M$ be a monomorphism. We can see that for any $x \in N$ that $ann(x) \subseteq ann(f(x))$. Since f is a monomorphism, we have that $ann(x) = ann(f(x))$. Since $M \in Ob\mathcal{T}_F$, there exists an element $\mathbf{n} \in F$ such that $\mathbf{n} \subseteq ann(f(x)) = ann(x)$. Hence $N \in Ob\mathcal{T}_F$, and so $\mathcal{T}_F$ is closed under taking subobjects.

Now suppose that $F \in Filt(I(R))$. Condition 1 says that $0 \in \mathcal{T}_F$. If $M \in Ob\mathcal{T}_F$, and $x \in M$, then there is an $\mathbf{n} \in F$ such that $\mathbf{n} \subseteq ann(x)$. Condition 2 then gives that in fact $ann(x) \in F$. Thus, if $F \in Filt(I(R))$, we can write

$$Ob\mathcal{T}_F = \{M \in ObR - mod \mid ann(x) \in F \text{ for each } x \in M\}. \tag{1.2}$$

Now let $M \in Ob\mathcal{T}_F$, and $g : M \to N$ be an epimorphism. If $x \in N$, then there exists an element $y \in M$ such that $g(y) = x$. Then we have that $ann(y) \subseteq ann(x)$. Since $M \in Ob\mathcal{T}_F$, we have that $ann(y) \in F$. Condition 2 then gives that $ann(x) \in F$, and we conclude that $N \in Ob\mathcal{T}_F$. Hence $\mathcal{T}_F$ is closed under taking quotients.

Finally, let $\{M_\lambda\}_{\lambda\in\Lambda} \in Ob\mathcal{T}_F$, and let $x \in \coprod_{\lambda\in\Lambda} M_\lambda$. Because only finitely many of the components of x are nonzero, we can write $x = \sum_{j\in J} x_j$ for some finite subset $J \subseteq \Lambda$. Then $ann(x) = ann(\sum_{j\in J} x_j) = \cap_{j\in J} ann(x_j)$. Since $ann(x_j) \in F$ for all x_j, Condition 3 gives that $ann(x) \in F$. Thus $\coprod_{\lambda\in\Lambda} M_\lambda \in Ob\mathcal{T}_F$. Hence $\mathcal{T}_F$ is closed under arbitrary coproducts. By Proposition 1.2.2 in Chapter 1 of [15], this is equivalent to the inclusion functor $\mathcal{T}_F \to R-mod$ having a right adjoint. We have just shown that $\mathcal{T}_F$ is what is known as a *reflective topologizing subcategory* of $R - mod$. We will define this formally in Section 1.2.1.

For $\mathbf{n} \in I(R)$ and $w \subseteq R$, recall that

$$(\mathbf{n} : w) := \{x \in R \mid xw \subseteq \mathbf{n}\}.$$

Note that $(\mathbf{n} : w) \in I(R)$.

Definition 3. *We say that a filter $F \in Filt(I(R))$ is a* radical filter *if the following condition holds:*

- *Let $\mathbf{m} \in F$ and $\mathbf{n} \in I(R)$. If $(\mathbf{n} : r) \in F$ for all $r \in \mathbf{m}$, then $\mathbf{n} \in F$.*

Denote by $RadF(I(R))$ the collection of all radical filters in $I(R)$. For $\mathbf{m} \in I(R)$, define

$$[\mathbf{m}] := \{\mathbf{n} \in I(R) \mid \mathbf{m} \subseteq \mathbf{n}\}.$$

It is clear that $[\mathbf{m}] \in Filt(I(R))$, and that it is the smallest filter containing $\mathbf{m}$. That is, if $F \in Filt(I(R))$, and $\mathbf{m} \in F$, then $[\mathbf{m}] \subseteq F$. We also define

$$\langle \mathbf{m} \rangle := \{\mathbf{n} \in I(R) \mid \mathbf{n} \nsubseteq \mathbf{m}\}.$$

It is not always true that $\langle \mathbf{m} \rangle$ is in $Filt(I(R))$. However, $\langle \mathbf{m} \rangle$ always satisfies the conditions 1 and 2 in the definition of a filter. Whenever $\langle \mathbf{m} \rangle$ is an element of $Filt(I(R))$, we can see that it is the largest filter not containing $\mathbf{m}$. Before the next proposition, we take the lead of [13] and make the

Definition 4. *Let $\mathbf{m} \in I(R)$ be a proper ideal. Set*

$$\mathbf{m}^{\vee} := \{r \in R \mid (\mathbf{m} : r) \in \langle \mathbf{m} \rangle\}.$$

The set $\mathbf{m}^{\vee}$ is not necessarily closed under addition, but note that for any proper ideal $\mathbf{m} \in I(R)$, we have that:

1. $\mathbf{m} \subseteq \mathbf{m}^{\vee} \neq R$,

2. We have for any $x \in R$ and $r \in \mathbf{m}^{\vee}$, that $(\mathbf{m} : r) \subseteq ((\mathbf{m} : r) : x) = (\mathbf{m} : xr)$. Since $\langle \mathbf{m} \rangle$ satisfies 2 from the definition of a filter, and $(\mathbf{m} : r) \nsubseteq \mathbf{m}$ by definition of $\mathbf{m}^{\vee}$, we have that $(\mathbf{m} : xr) \nsubseteq \mathbf{m}$. Hence $xr \in \mathbf{m}^{\vee}$. That is, $\mathbf{m}^{\vee}$ satisfies the ideal condition.

The following proposition is a specialization of Proposition 1.9.1 in Chapter 1 of [13].

Proposition 1. *For a proper ideal* $\mathbf{p} \in I(R)$*, the following are equivalent*

1. $\mathbf{p}$ *is a prime ideal*

2. $\mathbf{p} = \mathbf{p}^\vee$

3. $\langle \mathbf{p} \rangle$ *is a radical filter.*

Proof. $(1 \Rightarrow 2)$ Let $x \in \mathbf{p}^\vee$. Then $(\mathbf{p} : x) \nsubseteq \mathbf{p}$, which means there exists an element $y \in R \backslash \mathbf{p}$ such that $yx \in \mathbf{p}$. Since $\mathbf{p}$ is prime, we have $x \in \mathbf{p}$, and thus $\mathbf{p}^\vee \subseteq \mathbf{p}$.

$(2 \Rightarrow 1)$ Suppose $\mathbf{p} = \mathbf{p}^\vee$. Let $y \in R \setminus \mathbf{p}$ and $x \in R$ such that $yx \in \mathbf{p}$. Then $y \in (\mathbf{p} : x)$, and thus $(\mathbf{p} : x) \nsubseteq \mathbf{p}$. Hence $x \in \mathbf{p}^\vee = \mathbf{p}$, and $\mathbf{p}$ is prime.

$(1 \Rightarrow 3)$ Let $\mathbf{p} \subseteq R$ be a proper prime ideal. We have already noted that $\langle \mathbf{p} \rangle$ satisfies conditions 1 and 2 to be a filter. To show that $\langle \mathbf{p} \rangle$ also satisfies condition 3, suppose $\mathbf{m}, \mathbf{n} \in \langle \mathbf{p} \rangle$ and $\mathbf{m} \cap \mathbf{n} \notin \langle \mathbf{p} \rangle$. Then $\mathbf{mn} \subseteq \mathbf{m} \cap \mathbf{n} \subseteq \mathbf{p}$. Since $\mathbf{p}$ is prime, then either $\mathbf{m} \subseteq \mathbf{p}$ or $\mathbf{n} \subseteq \mathbf{p}$, which is a contradiction. Thus $\mathbf{m} \cap \mathbf{n} \nsubseteq \mathbf{p}$; hence $\mathbf{m} \cap \mathbf{n} \in \langle \mathbf{p} \rangle$.

To show that $\langle \mathbf{p} \rangle$ satisfies the condition of Definition 3, of a radical filter, suppose $\mathbf{m} \in \langle \mathbf{p} \rangle$, and $\mathbf{n} \in I(R)$ such that $(\mathbf{n} : r) \in \langle \mathbf{p} \rangle$ for all $r \in \mathbf{m}$. Rephrasing, this means that there exists an element $a \in \mathbf{m} \setminus \mathbf{p}$ and that for all $r \in \mathbf{m}$, there exists an element $x_r \in (\mathbf{n} : r) \setminus \mathbf{p}$ such that $x_r r \in \mathbf{n}$. In particular, for $a \in \mathbf{m} \setminus \mathbf{p}$, there exists x_a such that $x_a a \in \mathbf{n}$. Now suppose that $\mathbf{n} \notin \langle \mathbf{p} \rangle$; i.e. $\mathbf{n} \subseteq \mathbf{p}$. Then $a \notin \mathbf{p}$, and $x_a \notin \mathbf{p}$, but $x_a a \in \mathbf{n} \subseteq \mathbf{p}$, which contradicts the assumption that $\mathbf{p}$ is prime. Thus we have $\mathbf{n} \nsubseteq \mathbf{p}$; i.e. $\mathbf{n} \in \langle \mathbf{p} \rangle$. Hence $\langle \mathbf{p} \rangle$ is a radical filter.

$(3 \Rightarrow 2)$ Suppose that $\langle \mathbf{p} \rangle$ is a radical filter. First, we will show that $\mathbf{p}^\vee$ is closed under addition. Let $x, y \in \mathbf{p}^\vee$, which means that both $(\mathbf{p} : x)$ and $(\mathbf{p} : y)$ are in $\langle \mathbf{p} \rangle$. Then by condition 3 of a filter, we know that $(\mathbf{p} : x) \cap (\mathbf{p} : y) \in \langle \mathbf{p} \rangle$. Since $(\mathbf{p} : x) \cap (\mathbf{p} : y) \subseteq (\mathbf{p} : x + y)$, we have that also $(\mathbf{p} : x + y) \in \langle \mathbf{p} \rangle$. Thus $x + y \in \mathbf{p}^\vee$. Thus $\mathbf{p}^\vee$ is a proper ideal.

To show that $\mathbf{p} = \mathbf{p}^\vee$, suppose that $\mathbf{p}^\vee \in \langle \mathbf{p} \rangle$. We have $(\mathbf{p} : x) \nsubseteq \mathbf{p}$ for all $x \in \mathbf{p}^\vee$ by definition of $\mathbf{p}^\vee$. Then because $\langle \mathbf{p} \rangle$ is a radical filter, this implies that $\mathbf{p} \in \langle \mathbf{p} \rangle$, which is a contradiction. Thus $\mathbf{p}^\vee \notin \langle \mathbf{p} \rangle$, which means that $\mathbf{p}^\vee \subseteq \mathbf{p}$, and hence $\mathbf{p} = \mathbf{p}^\vee$. □

This allows us to describe the prime spectrum as follows:

$$Spec(R) = \{\mathbf{p} \in I(R) \mid \langle \mathbf{p} \rangle \in RadF(I(R))\}. \tag{1.3}$$

1.2 The Left Spectrum of a Ring

In this section we generalize the definition of the prime spectrum of a commutative ring as described in (1.3). Let R be an associative, unital ring (possibly noncommutative!). Denote by $I_l R$ the collection of left ideals of R. We will write $I(R)$ to denote the collection of all two-sided ideals of R. It is straightforward to show that if $\mathbf{m} \in I_l R$, then $(\mathbf{m} : w) \in I_l R$. Define the relation $\leq$ on $I_l R$ as follows

$$\mathbf{m} \leq \mathbf{n} \text{ if } (\mathbf{m} : w) \subseteq \mathbf{n} \text{ for some finite set } w \subseteq R.$$

Lemma 1.1 in Chapter 1 of [13] gives that the relation $\leq$ is a preorder. Note that if $\mathbf{m} \in I_l R$ is in fact a two-sided ideal, then $\mathbf{m} \leq \mathbf{n}$ if and only if $\mathbf{m} \subseteq \mathbf{n}$, for any $\mathbf{n} \in I_l R$. We can see that if $\mathbf{m} \subseteq \mathbf{n}$, then since $\mathbf{m} = (\mathbf{m} : 1)$, we have thus $\mathbf{m} \leq \mathbf{n}$. Now if w is any subset of R, then we see that $\mathbf{m} \subseteq (\mathbf{m} : w)$, (again, for $\mathbf{m}$ two-sided). For if $x \in \mathbf{m}$ and $y \in w$, then $xy \in \mathbf{m}$, and thus $x \in (\mathbf{m} : w)$. So now if $\mathbf{m}$ is two-sided, and $\mathbf{m} \leq \mathbf{n}$, then there is some finite subset $w \subseteq R$ such that $\mathbf{m} \subseteq (\mathbf{m} : w) \subseteq \mathbf{n}$. Thus for a commutative ring the preorder $\leq$ on ideals is the same as the preorder $\subseteq$.

Definition 5. *Let $F \subseteq I_l R$. We say F is a* topologizing filter *if the following conditions hold*

1. *$R \in F$,*
2. *if $\mathbf{m} \in F$ and $\mathbf{n} \in I_l R$ with $\mathbf{m} \subseteq \mathbf{n}$, then $\mathbf{n} \in F$,*
3. *$\mathbf{m}, \mathbf{n} \in F$ implies that $\mathbf{m} \cap \mathbf{n} \in F$*

4. $\mathbf{m} \in F$*,* $r \in R$ *implies that* $(\mathbf{m} : r) \in F$*.*

Denote by $TopFilt(I_l R)$ the collection of all uniform filters in $I_l R$. We can see that $TopFilt(I_l R)$ is partially ordered by inclusion, and becomes a lattice when the infimum is taken to be intersection. If R is commutative, we have that $\mathbf{m} \subseteq (\mathbf{m} : w)$ for any $\mathbf{m} \in I(R)$ and $w \subseteq R$. Thus any filter of ideals in $I(R)$ for a commutative ring is a topologizing filter.

Definition 6. *We say that a topologizing filter* $F \in TopFilt(I_l R)$ *is a* radical filter *if the following additional condition holds*

- *If* $\mathbf{m} \in F$*, and* $\mathbf{n} \in I_l R$*, and* $(\mathbf{n} : r) \in F$ *for all* $r \in \mathbf{m}$*, then* $\mathbf{n} \in F$*.*

Denote by $RadF(I_l R)$ the collection of all radical filters in $I_l R$. We have seen that if R is commutative, then $TopFilt(I_l R) = Filt(I(R))$, and (abusing notation) $RadF(I_l R) = RadF(I(R))$. Let $\mathbf{m} \in I_l R$. As in [13], Chapter 1, Example 1.3, define

$$[\mathbf{m}] := \{\mathbf{n} \in I_l R \mid \mathbf{m} \leq \mathbf{n}\}.$$

We can see that conditions 1, 2, & 4 of a topologizing filter clearly hold for $[\mathbf{m}]$. To see that 3 holds as well, note that if $(\mathbf{m} : x) \subseteq \mathbf{n}$ and $(\mathbf{m} : y) \subseteq \mathbf{n}'$ for some finite subsets $x, y \subseteq R$, then $(\mathbf{m} : x + y) = (\mathbf{m} : x) \cap (\mathbf{m} : y) \subseteq \mathbf{n} \cap \mathbf{n}'$. Thus $[\mathbf{m}] \in TopFilt(I_l R)$, and in fact $[\mathbf{m}]$ is the smallest topologizing filter containing $\mathbf{m}$. As in [13], Chapter 1, Example 1.4, define

$$\langle \mathbf{m} \rangle := \{\mathbf{n} \in I_l R \mid \mathbf{n} \not\leq \mathbf{m}\}.$$

It is not always true that $\langle \mathbf{m} \rangle$ is in $TopFilt(I_l R)$. However, $\langle \mathbf{m} \rangle$ always satisfies the conditions 1, 2, & 4 in the definition of a topologizing filter. Whenever $\langle \mathbf{m} \rangle$ is an element of $TopFilt(I_l R)$, we can see that it is the largest topologizing filter not containing $\mathbf{m}$.

In [13], the *left spectrum* of a ring is defined. We do not present it in its original form, but rather invoking Proposition 1.9.1 in [13], we present the left spectrum as the following.

Definition 7. *Let R be an assiciative unital ring. We define the* left spectrum of R *as*

$$Spec_l(R) := \{\mathbf{p} \in I_l R \mid \langle \mathbf{p} \rangle \in RadF(I_l R)\}.$$

We can see that if R is commutative, then the left spectrum is exactly the prime spectrum.

We now describe a topology on the left spectrum of a ring. Given a two-sided ideal $\mathbf{m} \subseteq R$, define

$$U(\mathbf{m}) := \{\mathbf{p} \in Spec_l(R) \mid \mathbf{m} \nsubseteq \mathbf{p}\}.$$

It is shown in Chapter 1, Lemma 1.10.2.1 of [13] that the subsets $U(\mathbf{m}) \subseteq Spec_l(R)$ for $\mathbf{m} \in I(R)$ (the collection of two-sided ideals of R), form a topology on $Spec_l(R)$, which we will call the *Zariski topology*. We can see that if R is commutative, the $U(\mathbf{m})$ as defined above coincide with the classical Zariski topology.

Now, one of the departing points of noncommutative algebraic geometry from commutative algebraic geometry is the notion of a structure sheaf. In the noncommutative setting, localization of the ring is more subtle; our solution is to use localization of the category $R - mod$ instead.

Let $F \subseteq I_l R$, and denote by $R - mod_F$ the full subcategory of $R - mod$ whose objects are $M \in R - mod$ such that the canonical morphism

$$M \to Hom_R(\mathbf{m}, M), \quad z \mapsto (r \mapsto r \cdot z) \text{ for all } r \in \mathbf{m}, z \in M$$

is a bijection for all $\mathbf{m} \in F$. The inclusion functor

$$\iota_F : R - mod_F \to R - mod$$

preserves limits, and thus has a left adjoint, $\mathcal{L}_F$. Since ι_F is fully faithful, $\mathcal{L}_F$ is a localization functor. The R-module $R_F := \iota_F \mathcal{L}_F(R)$ has a structure of a ring uniquely determined by the fact that the adjunction morphism $\eta_F(R) : R \to R_F$ is a ring morphism. There is a canonical natural transformation

$$\tau_F : R_F \otimes_R - \to \iota_F \mathcal{L}_F.$$

However, τ_F is not always a natural isomorphism. This means that the category $R{-}mod_F$ is not always equivalent to the category $R_F - mod$.

In general, the localization $\mathcal{L}_F$ does not preserve finite limits. Denote by $\overline{F}$ the set of all left ideals $\mathbf{m} \in I_l R$ such that the canonical morphism $M \to Hom_R(\mathbf{m}, M)$ is a bijection for all $M \in ObR - mod_F$. We can see that $R - mod_F = R - mod_{\overline{F}}$, and thus $\mathcal{L}_F = \mathcal{L}_{\overline{F}}$. It follows from Theorem 1 that the localization $\mathcal{L}_F = \mathcal{L}_{\overline{F}}$ preserves finite limits if and only if $\overline{F}$ is a radical filter.

Now for an open set $U(\mathbf{m}) \subseteq Spec_l(R)$, define the filter

$$\langle U(\mathbf{m}) \rangle := \bigcap_{\mathbf{p} \in U(\mathbf{m})} \langle \mathbf{p} \rangle.$$

Each $\langle \mathbf{p} \rangle$ is a radical filter, and the intersection of a collection of radical filters is a radical filter; hence $\langle U(\mathbf{m}) \rangle$ is a radical filter. Thus the localization

$$\mathcal{L}_{\langle U(\mathbf{m}) \rangle} : R - mod \to R - mod_{\langle U(\mathbf{m}) \rangle}$$

is flat, i.e. preserves finite limits, and has a right adjoint. Define the presheaf

$$\mathcal{O}_R : (Open(Spec_l(R)))^{op} \to AbCat$$
$$U(\mathbf{m}) \mapsto R - mod_{\langle U(\mathbf{m}) \rangle}.$$

We call $\mathcal{O}_R$ the *structure presheaf on $Spec_l(R)$*. Since $\mathcal{O}_R$ takes values in categories, we do not speak of a sheaf condition. However, in Section 1.4, we discuss a condition to replace the sheaf condition, such that when R is commutative, specializes to give that $\mathcal{O}_R$ is a sheaf.

If R is commutative, then $\langle U(\mathbf{m})\rangle = F_S$, where $S \in Mult(R)$ is the largest saturated multiplicative set in R such that $\mathbf{m} \cap S = \emptyset$, and $R - mod_{\langle U(\mathbf{m})\rangle} = S^{-1}R - mod$.

There is a feature of the left spectrum that will be apparent in what follows; that feature is that if R and S are two Morita equivalent rings, then $Spec_l(R)$ is homeomorphic to $Spec_l(S)$. In addition to making this fact apparent, the next section will give us the tools to generalize the left spectrum of a ring to the spectrum of an Abelian category.

1.2.1 Invoking a Theorem of Gabriel

Denote by $R - mod$ the category of left modules over R, and let $\mathcal{T}$ be a full subcategory of $R - mod$. For now, the following definitions will be restricted to the special case of the category of modules over R. In the next chapter, we will generalize them to arbitrary Abelian categories.

Definition 8. *We say that $\mathcal{T}$ is a* topologizing subcategory of $R - mod$ *(or just* topologizing*) if $\mathcal{T}$ is closed under taking subobjects, quotients, and finite direct sums.*

Denote by $T(R - mod)$ the collection of all topologizing subcategories of $R - mod$. Since every element $\mathcal{T} \in T(R - mod)$ is a full subcategory, we can define a partial order $\subseteq$ on $T(R - mod)$ in the following way. For $\mathcal{T}, \mathcal{T}' \in T(R - mod)$, we say $\mathcal{T} \subseteq \mathcal{T}'$ if $\mathcal{T}$ is a full subcategory of $\mathcal{T}'$; equivalently, if $Ob\mathcal{T} \subseteq Ob\mathcal{T}'$. Furthermore, we can endow $T(R - mod)$ with a lattice structure by defining the infimum of a collection $\{\mathcal{T}_\lambda \subseteq T(R - mod)\}$ to be the full subcategory whose set of objects is $\bigcap_\lambda Ob\mathcal{T}_\lambda$. We will denote this infimum $\bigcap_\lambda \mathcal{T}_\lambda$. The supremum of a collection $\{\mathcal{T}_\lambda\} \subseteq T(R - mod)$ is then simply the infimum of all $\mathcal{T}' \in T(R - mod)$ that contain every $\mathcal{T}_\lambda$.

Definition 9. *We say that $\mathcal{T}$ is* (co)reflective *if the inclusion functor $\mathcal{T} \hookrightarrow R - mod$ has a left (right) adjoint.*

Denote by $T^r(R-mod)$ the collection of all reflective topologizing subcategories of $R-mod$, and denote by $T_c(R-mod)$ the collection of all coreflective topologizing subcategories of $R-mod$. Both $T^r(R-mod)$ and $T_c(R-mod)$ are subsets of $T(R-mod)$, and it is straightforward to check that each is closed under arbitrary infima (taken in $T(R-mod)$). Denote by $T_c^r(R-mod)$ the collection of topologizing subcategories of $R-mod$ which are both reflective and coreflective.

Definition 10. *We say that $\mathcal{T}$ is* thick *if $\mathcal{T}$ is topologizing, and is closed under extensions.*

Denote by $Th(R-mod)$ the collection of all thick subcategories of $R-mod$. Note that $Th(R-mod) \subseteq T(R-mod)$, and $Th(R-mod)$ is closed under arbitrary infima taken in $T(R-mod)$. Denote by $Th_c(R-mod)$ the collection of thick, coreflective subcategories of $R-mod$. Note that $T_c(R-mod)$ is closed under arbitrary infima taken in $T(R-mod)$.

The following is a result of Gabriel [6].

Theorem 1. *Let R be a (noncommutative) ring. Let $F \subseteq I_lR$, and define $\mathcal{T}_F$ to be the full subcategory of $R-mod$ defined in equation (1.2). If $F \in TopFilt(I_lR)$, then $\mathcal{T}_F \in T_c(R-mod)$, and the map*

$$TopFilt(I_lR) \to T_c(R-mod)$$

$$F \longmapsto \mathcal{T}_F$$

is an isomorphism of lattices. Furthermore, this map induces a bijection between $RadF(I_lR)$ and $Th_c(R-mod)$.

In addition to the above theorem, we have a theorem of Rosenberg from [13] (Chapter 3, Proposition 6.4.1), that is reworded below.

Theorem 2. *The map in Theorem 1, induces a bijection between the topologizing filters of the form $[\mathbf{m}]$, where $\mathbf{m} \in I(R)$ (a two-sided ideal), and the set $T_c^r(R-mod)$. Explicitly, the* *topologizing filter* $[\mathbf{m}]$ *is sent to the subcategory* $R/\mathbf{m} - mod \subseteq R - mod$. *Further, for any collection* Λ *of two-sided ideals of* R, *the filter* $[sup\Lambda]$ *is sent to the topologizing subcategory* $\bigcap_{\mathbf{m}\in\Lambda} R/\mathbf{m} - mod$.

We now introduce some lattice theory terminology. Let L be a lattice with arbitrary suprema and infima. In Chapter 3 of [12], an element $x \in L$ is said to be *compact* if whenever $x \leq \bigvee A$, for some $A \subseteq L$, then there exists a finite subset $F \subseteq A$ such that $x \leq \bigvee F$. Denote the collection of all compact elements of L by $cpt(L)$.

Definition 11. *Let* $x \in L$. *Define the* support of x in L *to be*

$$Supp(x) := \{y \in L \mid x \nleq y\}.$$

Definition 12. *Call an element* $x \in L$ strongly compact *if* $Supp(x)$ *has a greatest element. Denote the subset of all strongly compact elements of* L *by* $Cpt(L)$.

Equivalently, $x \in Cpt(L)$ if and only if $\bigvee Supp(x) \in Supp(x)$. Because of this, we make the notation of

$$\widehat{x} := \bigvee Supp(x).$$

Finally, with this notation we can see that

$$Cpt(L) = \{x \in L \mid x \nleq \widehat{x}\}.$$

Using our previous notation, we can see that if $\mathbf{m} \in I_l R$ is a left ideal such that $\langle \mathbf{m} \rangle \in TopFilt(I_l R)$, then in fact $\langle \mathbf{m} \rangle = \widehat{[\mathbf{m}]}$.

Lemma 1. *Let* L *be a lattice with arbitrary suprema and infima. Then* $Cpt(L) \subseteq cpt(L)$.

Proof. Let $A \subseteq L$. Suppose $x \in Cpt(L)$, and $x \leq \bigvee A$, but $x \nleq y$ for all $y \in A$. Then by definition of $\widehat{x}$, we have that $y \leq \widehat{x}$ for all $y \in A$. Then we have $\bigvee A \leq \widehat{x}$, which means

$x \leq \bigvee A \leq \widehat{x}$. By hypothesis, $x \in Cpt(L)$, which means $x \nleq \widehat{x}$, a contradiction. Thus our assumption that $x \nleq y$ for all $y \in A$ was not permissible; hence there must be at least one $y \in A$ with $x \leq y$, meaning $x \in cpt(L)$. □

By the dual of Theorem 3.2 in [12], we know that the compact elements of $Filt(L)$, the lattice of filters in L are filters of the form $1/x := \{y \in L \mid x \leq y\}$, for $x \in L$. Specializing this, we see that the compact elements of $TopFilt(I_l R)$ are the filters $[\mathbf{m}]$ for $\mathbf{m} \in I_l R$. Thus we have the following rewriting of the left spectrum of a ring

$$\begin{aligned} Spec_l(R) &= \{\mathbf{p} \in I_l R \mid \langle \mathbf{p} \rangle \in RadF(I_l R)\} \\ &\simeq \{[\mathbf{p}] \in TopFilt(I_l R) \mid \widehat{[\mathbf{p}]} \in RadF(I_l R)\} \\ &\simeq \{F \in cpt(TopFilt(I_l R)) \mid \widehat{F} \in RadF(I_l R)\} \\ &\simeq \{\mathcal{T} \in cpt(T_c(R - mod)) \mid \widehat{\mathcal{T}} \in Th_c(R - mod)\}. \end{aligned}$$

Where the last bijection is obtained by invoking Theorem 1. It should be noted that $\widehat{F}$ is obtained by viewing F as an element of the lattice $TopFilt(I_l R)$ (as opposed to $RadF(I_l R)$), and $\widehat{\mathcal{T}}$ is obtained by viewing $\mathcal{T}$ as an element of the lattice $T_c(R - mod)$.

We can see that for $\mathbf{p} \in Spec_l(R)$, we have $\langle \mathbf{p} \rangle \in TopFilt(I_l R)$ by definition, and so $\langle \mathbf{p} \rangle = \widehat{[\mathbf{p}]}$. This implies that in fact $[\mathbf{p}] \in Cpt(TopFilt(I_l R))$. Thus we have the final description of the left spectrum

$$Spec_l(R) = \{\mathcal{T} \in Cpt(T_c(R - mod)) \mid \widehat{\mathcal{T}} \in Th_c(R - mod)\}. \tag{1.4}$$

At this point, we can see that the following proposition is now apparent

Proposition 2. *If R and S are two Morita equivalent rings, then $Spec_l(R)$ is homeomorphic to $Spec_l(S)$.*

1.3 The Spectrum of an Abelian Category

At this point, it is not much more work to generalize the definition of the left spectrum of a ring to the spectrum of an Abelian category. We will do so in this section. Let $\mathcal{A}$ be an Abelian category, and $\mathcal{T}$ a full subcategory of $\mathcal{A}$.

Definition 13. *We say that $\mathcal{T}$ is a* topologizing subcategory of $\mathcal{A}$ *(or just* topologizing*) if $\mathcal{T}$ is closed under taking subobjects, quotients, and finite direct sums.*

Denote by $T(\mathcal{A})$ the collection of all topologizing subcategories of $\mathcal{A}$. Note that $T(\mathcal{A})$ is partially ordered by inclusion of subcategories, and becomes a lattice with intersection as infimum.

Definition 14. *We say that a topologizing $\mathcal{T}$ is* (co)reflective *if the inclusion functor $\mathcal{T} \hookrightarrow \mathcal{A}$ has a left (right) adjoint.*

Denote by $T^r(\mathcal{A})$ the collection of all reflective topologizing subcategories of $\mathcal{A}$, and denote by $T_c(\mathcal{A})$ the collection of all coreflective topologizing subcategories of $\mathcal{A}$. Denote by $T_c^r(\mathcal{A})$ the collection of topologizing subcategories of $\mathcal{A}$ wich are both reflective and coreflective.

Definition 15. *We say that $\mathcal{T}$ is* thick *if $\mathcal{T}$ is topologizing, and is closed under extensions. That is, if $M, L \in Ob\mathcal{T}$, and $N \in Ob\mathcal{A}$ such that there is an exact sequence*

$$0 \to M \to N \to L \to 0$$

then N must be in $Ob\mathcal{T}$ as well.

Denote by $Th(\mathcal{A})$ the collection of all thick subcategories of $\mathcal{A}$. Note that $Th(\mathcal{A}) \subseteq T(\mathcal{A})$, and $Th(\mathcal{A})$ is closed under arbitrary infima taken in $T(\mathcal{A})$. Denote by $Th_c(\mathcal{A})$ the collection of thick, coreflective subcategories of $\mathcal{A}$. Note that $T_c(\mathcal{A})$ is closed under arbitrary infima taken in $T(\mathcal{A})$.

Definition 16. *Let $\mathcal{A}$ be an abelian category. Define the* 0-Spectrum *to be*

$$\mathbf{Spec}^0(\mathcal{A}) := \{\mathcal{T} \in Cpt(T_c(\mathcal{A})) \mid \widehat{\mathcal{T}} \in Th_c(\mathcal{A})\}.$$

For $\mathcal{T} \in T_c^r(\mathcal{A})$, define

$$U(\mathcal{T}) := \{\mathcal{T}' \in \mathbf{Spec}^0(\mathcal{A}) \mid \mathcal{T}' \nsubseteq \mathcal{T}\} = \{\mathcal{T}' \in \mathbf{Spec}^0(\mathcal{A}) \mid \mathcal{T} \subseteq \widehat{\mathcal{T}'}\}.$$

The subsets $U(\mathcal{T})$ for $\mathcal{T} \in T_c^r(\mathcal{A})$ form the open sets in a topology on $\mathbf{Spec}^0(\mathcal{A})$, which we will call the *Zariski topology*. Using Thoerem 2, we can see that if $\mathcal{A} = R - mod$, then for any $\mathcal{T} \in T^r(R - mod)$, there is a two-sided ideal $\mathbf{m} \subseteq R$ such that $\mathcal{T} = R/\mathbf{m} - mod$. The subset $U(R/\mathbf{m} - mod)$ of $\mathbf{Spec}^0(R - mod)$ corresponds to the subset $U(\mathbf{m})$ of $Spec_l(R)$. The discussion in Section C1.5 and C1.6 in Chapter 2 of [15] show that the sets $U(\mathcal{T})$ for $\mathcal{T} \in T_c^r(\mathcal{A})$ form a topology on $\mathbf{Spec}^0(\mathcal{A})$.

Proposition 3. *Let R be a ring. Then $\mathbf{Spec}^0(R - mod)$ is homeomorphic to $Spec_l(R)$.*

Proof. This follows from the description of the left spectrum in equation (1.4), and the discussion above. Explicity, the map that induces this homeomorphism is

$$Spec_l(R) \to \mathbf{Spec}^0(R - mod)$$
$$\mathbf{p} \mapsto [R/\mathbf{p}],$$

where $[R/\mathbf{p}]$ is the smallest topologizing subcategory of $R - mod$ containing the R-module $R/\mathbf{p}$; i.e. is is the intersection of all the topologizing subcategories containing $R/\mathbf{p}$. □

For each $\mathcal{T} \in Th(\mathcal{A})$, we have the localization functor $\mathcal{L}_{\mathcal{T}} : \mathcal{A} \to \mathcal{A}/\mathcal{T}$. Define the presheaf with values in small Abelian categories

$$\mathcal{O}_{\mathcal{A}} : (Open(\mathbf{Spec}^0(\mathcal{A})))^{op} \to AbCat$$
$$U(\mathcal{T}) \mapsto \mathcal{A}/\left(\bigcap_{\mathcal{T}' \in U(\mathcal{T})} \widehat{\mathcal{T}'}\right).$$

Since for every $\mathcal{T} \in \mathbf{Spec}^0(\mathcal{A})$, the subcategory $\widehat{\mathcal{T}}$ is an element of $Th_c(\mathcal{A})$, the intersection $\bigcap_{\mathcal{T} \in V} \widehat{\mathcal{T}}$ for any subset $V \subseteq \mathbf{Spec}^0(\mathcal{A})$ is also an element of $Th_c(\mathcal{A})$.

We will call $\mathcal{O}_{\mathcal{A}}$ the *structure presheaf* on $\mathbf{Spec}^0(\mathcal{A})$. This will take the place of the structure sheaf in commutative algebraic geometry. Exactly how this does so will be explained in Section 1.4.

We now make a purely lattice-theoretical definition. Let L be a lattice with arbitrary infima, and $x \in L$. Define

$$x^* := \bigwedge \{y \in L : x \lneq y\} \tag{1.5}$$

Note that we always have that either x^* does not exist (in the case there are no elements y strictly greater than x), or $x \leq x^*$. We will be interested in x such that x^* exists, and $x < x^*$ with $x \neq x^*$, i.e., when $x \lneq x^*$.

Lemma 2. *Let L be a lattice with arbitrary suprema and infima, and let $x \in Cpt(L)$. Then we have $\widehat{x} \lneq (\widehat{x})^*$. In fact, $(\widehat{x})^* = x \vee \widehat{x}$.*

Proof. Let $x \in Cpt(L)$, and $y \in L$ with $\widehat{x} \leq y$. If $y \in Supp(x)$, then $y \leq \widehat{x}$, and $\widehat{x} = y$. If $y \notin Supp(x)$, then $x \leq y$, and thus $x \vee \widehat{x} \leq y$. Since y is arbitrary, this gives that $(\widehat{x})^* = x \vee \widehat{x}$. □

Definition 17. *Let $\mathcal{A}$ be an abelian category. Define the* 1-Spectrum *to be*

$$\mathbf{Spec}^1(\mathcal{A}) := \{\mathcal{T} \in Th_c(\mathcal{A}) \mid \mathcal{T} \subsetneq \mathcal{T}^*\}.$$

The main reason for introducing a second spectrum of an Abelian category is that $\mathbf{Spec}^1(\mathcal{A})$ will be easier to generalize later to categories that are not necessarily Abelian or even additive.

For $\mathcal{T} \in T_c^r(\mathcal{A})$, define

$$U^1(\mathcal{T}) := \{\mathcal{T}' \in \mathbf{Spec}^1(\mathcal{A}) \mid \mathcal{T} \subseteq \mathcal{T}'\}.$$

The subsets $U^1(\mathcal{T}) \subseteq \mathbf{Spec}^1(\mathcal{A})$ where $\mathcal{T} \in T_c^r(\mathcal{A})$ form the open sets in a topology on $\mathbf{Spec}^1(\mathcal{A})$, which we will call the *Zariski topology*. Define the presheaf

$$\mathcal{O}_{\mathcal{A}} : (Open(\mathbf{Spec}^1(\mathcal{A})))^{op} \to AbCat$$
$$U^1(\mathcal{T}) \mapsto \mathcal{A}/\left(\bigcap_{\mathcal{T}' \in U(\mathcal{T})} \mathcal{T}'\right).$$

We will call $\mathcal{O}_{\mathcal{A}}$ the *structure presheaf* on $\mathbf{Spec}^1(\mathcal{A})$. Again, this will take the place of the structure sheaf from commutative algebraic geometry, and we will discuss the importance of $\mathcal{O}_{\mathcal{A}}$ in Section 1.4.

1.3.1 Reconciling Definitions

Let $\mathcal{A}$ be an Abelian category. If $M \in Ob\mathcal{A}$, denote by $[M]$ the smallest topologizing subcategory containing M. That is, $[M]$ is the intersection of all topologizing subcategories of $\mathcal{A}$ that contain M. Explicitly, $[M]$ is the full subcategory of $\mathcal{A}$ whose objects are all $N \in Ob\mathcal{A}$ such that there exists $n \in \mathbb{N}$ and $L \in Ob\mathcal{A}$ and a diagram

$$\bigoplus_{j=1}^{n} M \xleftarrow{f} L \xrightarrow{g} N,$$

with f a monomorphism and g an epimorphism. That is, N is the subquotient of a finite direct sum of copies of M. Also define ${}^{\perp}M := \{N \in Ob\mathcal{A} \mid Hom(N,M) = 0\} = Ker(Hom(-,M))$.

There are natural maps

$$\begin{aligned} \mathbf{Spec}^0(\mathcal{A}) &\to \mathbf{Spec}^1(\mathcal{A}) \\ T &\mapsto \widehat{T}, \end{aligned} \tag{1.6}$$

(this coming from Lemma 2) and for a ring R,

$$\begin{aligned} Spec_l(R) &\to \mathbf{Spec}^0(R-mod) \\ \mathbf{p} &\mapsto [R/\mathbf{p}]. \end{aligned} \tag{1.7}$$

The map 1.7 is a homeomorphism, but in general, the map 1.6 is not a homeomorphism. However, combining Proposition 8.7.2 and Corollary 8.7.3 from [14], we have the following

Proposition 4. *Let R be a Noetherian polynomial identity ring. Then the maps*

$$\begin{array}{ccccc} Spec_l(R) & \longrightarrow & \mathbf{Spec}^0(R-mod) & \longrightarrow & \mathbf{Spec}^1(R-mod) \\ \mathbf{p} & \longmapsto & [R/\mathbf{p}] & \longmapsto & \widehat{[R/\mathbf{p}]}. \end{array}$$

are homeomorphisms.

If R is a Noetherian PI ring, then it is known that

$$\widehat{[R/\mathbf{p}]} = {}^{\perp}E_R(R/\mathbf{p}),$$

where $E_R(R/\mathbf{p})$ is the injective hull of $R/\mathbf{p}$. In the case where R is commutative, we have that $\widehat{[R/\mathbf{p}]} = \ker(- \otimes_R R_{\mathbf{p}})$ where $R_{\mathbf{p}}$ is the localization of R at the multiplicative set $S = R \setminus \mathbf{p}$.

1.4 Descent

In this section we describe the notion of *descent*, and describe the importance of this property. Much of the expository information and definitions that follow are taken from Section 2 of [8].

Definition 18. *Let $\mathcal{A}$ be a category. A* monad *on $\mathcal{A}$ is a triple $\mathbf{T} = (T, \mu, \eta)$, where $T : \mathcal{A} \to \mathcal{A}$ is a functor, $\mu : T^2 \to T$, and $\eta : Id_{\mathcal{A}} \to T$ are natural transformations such that the following hold*

- $\mu \circ T\mu = \mu \circ \mu T$
- $\mu \circ T\eta = \mu \circ \eta T = 1_T$

where 1_T is the identity natural transformation of T.

Dually, a comonad *on $\mathcal{A}$ is a triple $\mathbf{K} = (K, \Delta, \epsilon)$, where $K : \mathcal{A} \to \mathcal{A}$ is a functor, $\Delta : K \to K^2$, and $\epsilon : K \to Id_{\mathcal{A}}$ are natural transformations such that the following hold*

- $\Delta \circ K\Delta = \Delta K \circ \Delta$
- $\epsilon K \circ \Delta = K\epsilon \circ \Delta = 1_K$,

where 1_K is the identity natural transformation of K.

Monads and comonads arise naturally from pairs of adjoint functors in the following way. Let $F : \mathcal{A} \to \mathcal{B}$ have a right adjoint $G : \mathcal{B} \to \mathcal{A}$, and let η and ϵ be the unit and counit respectively for this adjunction. Then the triple $(GF, G\epsilon F, \eta)$ defines a monad on $\mathcal{A}$, and the triple $(FG, F\eta G, \epsilon)$ forms a comonad on $\mathcal{A}$. We will call a (co)monad arising in such a way from an adjunction the *(co)monad associated to the adjunction* (F, G).

Definition 19. *Let* $\mathbf{T} = (T, \mu, \eta)$ *be a monad on a category* $\mathcal{A}$. *The* Eilenberg-Moore category of $\mathbf{T}$-modules, *denoted* $\mathcal{A}^{\mathbf{T}}$, *is defined as follows. Objects of* $\mathcal{A}^{\mathbf{T}}$ *are pairs* (M, h), *where* $M \in Ob\mathcal{A}$ *and* $h \in Hom(TM, M)$ *such that*

$$h \circ Th = h \circ \mu_M \quad and \quad h \circ \eta_M = id_M.$$

A morphism in $\mathcal{A}^{\mathbf{T}}$ *from* (M, h) *to* (M', h') *is a morphism* $f : M \to M'$ *in* $\mathcal{A}$ *such that* $h' \circ Tf = f \circ h$.

We have a forgetful functor

$$U^{\mathbf{T}} : \mathcal{A}^{\mathbf{T}} \to \mathcal{A} \quad (M, h) \mapsto M.$$

We have that $U^{\mathbf{T}}$ admits a left adjoint, called the *free* $\mathbf{T}$*-module functor*,

$$F^{\mathbf{T}} : \mathcal{A} \to \mathcal{A}^{\mathbf{T}},$$

which is defined on an object M and morphism f by

$$F^{\mathbf{T}}(M) = (TM, \mu_M), \qquad F^{\mathbf{T}}(f) = Tf.$$

We can see that $\mathbf{T}$ is the monad associated to the adjunction $(F^{\mathbf{T}}, U^{\mathbf{T}})$. The comonad associated to the adjunction $(F^{\mathbf{T}}, U^{\mathbf{T}})$ is

$$\mathbf{K}^{\mathbf{T}} := (F^{\mathbf{T}}U^{\mathbf{T}}, F^{\mathbf{T}}\eta U^{\mathbf{T}}, \epsilon^{\mathbf{T}}),$$

where $\epsilon^{\mathbf{T}}$ is the counit of the adjunction $(F^{\mathbf{T}}, U^{\mathbf{T}})$, and is given by

$$(\epsilon^{\mathbf{T}})_{(M,h)} = m : (TM, \mu_M) \to (M, h).$$

Definition 20. *Let* $F : \mathcal{C} \leftrightarrows \mathcal{D} : U$ *be a pair of adjoint functors, with unit* $\eta : Id_{\mathcal{C}} \to UF$

and counit $\epsilon : FU \to Id_{\mathcal{D}}$. *Let* $\mathbf{T}$ *denote the associated monad. The* canonical $\mathbf{T}$-module functor

$$Can^{\mathbf{T}} : \mathcal{D} \to \mathcal{C}^{\mathbf{T}}$$

is defined on an object D *and morphism* f *by*

$$Can^{\mathbf{T}}(D) := (UD, U_{\epsilon_D}), \qquad Can^{\mathbf{T}}(f) := Uf.$$

We say the functor U is *monadic* if $Can^{\mathbf{T}}$ is an equivalence of categories. In order to define descent, we need to describe the dual situation for comonads, which we do now.

Definition 21. *Let* $\mathbf{K} = (K, \Delta, \epsilon)$ *be a comonad on* $\mathcal{D}$*. The* Eilenberg-Moore category of K-comodules*, denoted by* $\mathcal{D}_{\mathbf{K}}$*, is defined as follows. The objects of* $\mathcal{D}_{\mathbf{K}}$ *are pairs* (D, δ)*, where* $D \in Ob\mathcal{D}$*, and* $\delta \in Hom_{\mathcal{D}}(D, \mathbf{K}D)$ *such that*

$$K\delta \circ \delta = \Delta_D \circ \delta \quad and \quad \epsilon_D \circ \delta = id_D.$$

A morphism in $\mathcal{D}_{\mathbf{K}}$ *from* (D, δ) *to* (D', δ') *is a morphism* $f : D \to D'$ *in* $\mathcal{D}$ *such that* $Kf \circ \delta = \delta' \circ f$.

Let $\mathbf{K} = (K, \Delta, \epsilon)$ be a comonad on $\mathcal{D}$. The forgetful functor $U_{\mathbf{K}} : \mathcal{D}_{\mathbf{K}} \to \mathcal{D}$ admits a right adjoint

$$F_{\mathbf{K}} : \mathcal{D} \to \mathcal{D}_{\mathbf{K}},$$

called the *cofree* $\mathbf{K}$*-comodule functor*, which is defined on an object X and morphism f by

$$F_{\mathbf{K}}(X) = (KX, \Delta_X) \qquad F_{\mathbf{K}}(f) = Kf.$$

Note that $\mathbf{K}$ is the comonad associated to the adjunction $(U_{\mathbf{K}}, F_{\mathbf{K}})$; however, the monad associated to this adjunction is

$$\mathbf{T_K} := \{F_\mathbf{K} U_\mathbf{K}, F_\mathbf{K} \epsilon U_\mathbf{K}, \eta_\mathbf{K}\},$$

where $\eta_\mathbf{K}$ is the unit of the adjunction $(U_\mathbf{K}, F_\mathbf{K})$, given by

$$(\eta_\mathbf{K})_{(D,\delta)} = \delta : (D, \delta) \to (KD, \Delta_D).$$

Definition 22. *Let $F : \mathcal{C} \leftrightarrows \mathcal{D} : U$ be a pair of adjoint functors, with unit $\eta : Id_\mathcal{C} \to UF$ and counit $\epsilon : FU \to Id_\mathcal{D}$. Let $\mathbf{K}$ denote the associated comonad. The* canonical $\mathbf{K}$-comodule fucntor

$$Can_\mathbf{K} : \mathcal{C} \to \mathcal{D}_\mathbf{K}$$

is defined on an object X and morphism f by

$$Can_\mathbf{K}(C) = (FC, F\eta_C) \qquad Can_\mathbf{K}(f) = Ff.$$

We say the functor F is *comonadic* if $Can_\mathbf{K}$ is an equivalence of categories.

Definition 23. *The* descent category *of a monad $\mathbf{T}$ on a category $\mathcal{C}$, denoted $D(\mathbf{T})$, is the category $(\mathcal{C}^\mathbf{T})_{\mathbf{K}^\mathbf{T}}$ of $\mathbf{K}^\mathbf{T}$-comodules in the category of $\mathbf{T}$-modules. The objects of $D(\mathbf{T})$ are called* descent data*. The monad $\mathbf{T}$ satisfies* descent *if*

$$Can_{\mathbf{K}^\mathbf{T}} : \mathcal{C} \to D(\mathbf{T})$$

is fully faithful. If $Can_{\mathbf{K}^\mathbf{T}}$ is an equivalence of categories, i.e., if the free $\mathbf{T}$-module functor $F^\mathbf{T} : \mathcal{C} \to \mathcal{C}^\mathbf{T}$ is comonadic, then $\mathbf{T}$ satisfies effective descent*.*

Before we continue, we need a very useful theorem, and it's dual. And before we can state that theorem, we need some terminology. Let $\mathcal{A}$ be a category. A *reflexive pair* is a pair of morphisms $s, t \in Hom_{\mathcal{A}}(X, Y)$ for some $X, Y \in Ob\mathcal{A}$ such that there exists a morphism $i \in Hom_{\mathcal{A}}(Y, X)$ with $si = 1_Y = ti$. We say that a functor $G : \mathcal{A} \to \mathcal{C}$ *reflects isomorphisms* if, for each morphism f of $\mathcal{A}$, t is an isomorphism whenever $G(t)$ is. The following theorem is a slightly modified version (in order match language and notation defined here) of Beck's theorem can be found as Theorem 2 in Chapter IV, Section 4 of [11].

Theorem 3. *Let $G : \mathcal{C} \to \mathcal{A}$ be a functor with a left adjoint, and $\mathbf{T}$ the associated monad on $\mathcal{A}$. We have the canonical $\mathbf{T}-$module functor $Can^{\mathbf{T}} : \mathcal{C} \to \mathcal{A}^{\mathbf{T}}$.*

1. *If $\mathcal{C}$ has coequalizers of all reflexive pairs, then $Can^{\mathbf{T}}$ has a left adjoint, which we will denote by $Q^{\mathbf{T}}$, often called the "indecomposables" functor.*

2. *If, in addition, G preserves coequalizers of reflexive pairs, the unit of this adjunction is an isomorphism; that is, $Id_{\mathcal{A}^{\mathbf{T}}} \simeq Can^{\mathbf{T}} \circ Q^{\mathbf{T}}$.*

3. *If, in addition to 1 and 2, G reflects isomorphisms, then the counit of this adjunction is also an isomorphism; that is, $Q^{\mathbf{T}} \circ Can^{\mathbf{T}} \simeq Id_{\mathcal{C}}$. Consequently, G is monadic in this case.*

We will also need the theorem dual to Beck's theorem. Before we give the statement, we give some terminology. A *coreflexive pair* of morphisms in a category $\mathcal{A}$ is a pair $s, t \in Hom_{\mathcal{A}}(X, Y)$ such that there exists a morphism $i \in Hom_{\mathcal{A}}(Y, X)$ such that $is = 1_X = it$. The following is the dual to Beck's theorem

Theorem 4. *Let $F : \mathcal{A} \to \mathcal{C}$ be a functor with a right adjoint, and $\mathbf{K}$ the associated comonad on $\mathcal{C}$. We have the canonical $\mathbf{K}-$comodule functor $Can_{\mathbf{K}} : \mathcal{A} \to \mathcal{A}_{\mathbf{K}}$.*

1. *If $\mathcal{A}$ has equalizers of all coreflective pairs, then $Can_{\mathbf{K}}$ has a right adjoint, which we will denote by $Prim_{\mathbf{K}}$, often called the "primitives" functor.*

2. *If, in addition, F preserves equalizers of coreflexive pairs, then the adjunction morphism is an isomorphism; that is, $Can_{\mathbf{K}} \circ Prim_{\mathbf{K}} \simeq Id_{\mathcal{A}_{\mathbf{K}}}$.*

3. *If, in addition to 1 and 2, F reflects isomorphisms, then the counit morphism is also an isomorphism; that is, $Id_{\mathcal{C}} \simeq Prim_{\mathbf{K}} \circ Can_{\mathbf{K}}$. Consequently, F is comonadic in this case.*

The following is an important application of Beck's theorem (Theorem 3).

Lemma 3. *Let $\mathcal{A}$ be an Abelian category with injective hulls and the property* (sup) *(defined later in Chapter 2, Section 2.1.1). Let $\mathcal{T} \in Th_c(\mathcal{A})$, and $\mathcal{L}_{\mathcal{T}} : \mathcal{A} \to \mathcal{A}/\mathcal{T}$ the corresponding localization functor. Then $\mathcal{L}_{\mathcal{T}}$ has a right adjoint $\iota_{\mathcal{T}}$, and $\iota_{\mathcal{T}}$ is monadic.*

Proof. Lemma 2.4.4 and Corollary 2.4.8.2 of Chapter 3 in [13] together show that for $\mathcal{A}$ as in the hypothesis, and $\mathcal{T} \in Th_c(\mathcal{A})$, then $\mathcal{L}_{\mathcal{T}}$ has a right adjoint $\iota_{\mathcal{T}}$. Proposition 1.3 of Chapter 1 in [7] gives that since $\mathcal{L}_{\mathcal{T}}$ is a localization functor, $\iota_{\mathcal{T}}$ is fully faithful. Because of this, we can and will assume that $\mathcal{A}/\mathcal{T}$ is a full subcategory of $\mathcal{A}$, and that $\iota_{\mathcal{T}}$ is the inclusion functor. Further, we can (and will) assume that for all $M \in Ob\mathcal{A}/\mathcal{T}$, we have that $\mathcal{L}_{\mathcal{T}}(M) \simeq M$. That is, $\mathcal{L}_{\mathcal{T}}$ is isomorphic the identity functor when restricted to the full subcategory $\mathcal{A}/\mathcal{T}$.

Corollary 3.2 of Chapter 1 in [7] gives that $\mathcal{A}/\mathcal{T}$ has all finite colimits. This shows that condition 1 of Beck's theorem is satisfied.

To show condition 2, we show that in fact $\iota_{\mathcal{T}}$ preserves all finite colimits. First, let $\mathfrak{D}$ be a finite diagram in $\mathcal{A}/\mathcal{T}$. Denote by $colim_{\mathcal{A}/\mathcal{T}}\mathfrak{D}$ the colimit of $\mathfrak{D}$ in $\mathcal{A}/\mathcal{T}$, and by $colim_{\mathcal{A}}\mathfrak{D}$ the colimit of $\mathfrak{D}$ in $\mathcal{A}$. Because $\mathcal{L}_{\mathcal{T}}$ has a right adjoint, it preserves arbitrary colimits, so $\mathcal{L}_{\mathcal{T}}(colim_{\mathcal{A}}\mathfrak{D}) = colim_{\mathcal{A}/\mathcal{T}}(\mathcal{L}_{\mathcal{T}}(\mathfrak{D}))$. Because the diagram $\mathfrak{D}$ is contained in $\mathcal{A}/\mathcal{T}$, we have that $\mathcal{L}_{\mathcal{T}}(\mathfrak{D}) = \mathfrak{D}$. Hence, $colim_{\mathcal{A}/\mathcal{T}}\mathfrak{D} = colim_{\mathcal{A}}\mathfrak{D}$. This shows that the inclusion functor $\iota_{\mathcal{T}}$ preserves the colimit of the diagram $\mathfrak{D}$. Hence condition 2 of Beck's theorem is satisfied.

Since $\iota_{\mathcal{T}}$ is an inclusion of a full subcategory, it reflects isomorphisms. Thus condition 3 of Beck's theorem is satisfied. □

Let $\mathcal{A}$ and $\mathcal{T}$ be as in Lemma 3. The conclusion states that $\iota_{\mathcal{T}}$ is monadic. If we let $\mathbf{T}$ denote the monad associated the adjunction $(\mathcal{L}_{\mathcal{T}}, \iota_{\mathcal{T}})$, then this means that $\mathcal{A}/\mathcal{T} \simeq \mathcal{A}^{\mathbf{T}}$.

Proposition 5. *Let $\mathcal{A}$ be an Abelian category with injective hulls and the property* (sup) *(defined later in Chapter 2, Section 2.1.1). Let $\{U(\mathcal{T}_j)\}_{j\in J}$ be a finite Zariski open cover of* $\mathbf{Spec}^0(\mathcal{A})$*, with $\mathcal{T}_j \in T_c^r(\mathcal{A})$. Denote by $\mathcal{L}_j$ the localization functors*

$$\mathcal{L}_j : \mathcal{A} \to \mathcal{O}_{\mathcal{A}}(U(\mathcal{T}_j)).$$

Denote by $\mathcal{L}$ the product of all the $\mathcal{L}_j$:

$$\mathcal{L} : \mathcal{A} \to \prod_{j\in J} \mathcal{O}_{\mathcal{A}}(U(\mathcal{T}_j)).$$

Then $\mathcal{L}$ has a right adjoint ι. Denote by $\mathbf{T}$ the monad associated to the adjoint pair $(\mathcal{L}, \iota)$. Then $\mathbf{T}$ satisfies effective descent.

Proof. Lemma 2.4.4 and Corollary 2.4.8.2 of Chapter 3 in [13] together show that for $\mathcal{A}$ as in the hypothesis, then $\mathcal{L}_j$ has a right adjoint ι_j. Because $\mathcal{L}_j$ is a localization functor, we can take ι_j to be an inclusion functor of a full subcategory. For each j, let $\mathbf{T}_j$ be the monad associated to the adjunction $(\mathcal{L}_j, \iota_j)$. Then we can apply Lemma 3 and see that $\mathcal{O}_{\mathcal{A}}(U(\mathcal{T}_j)) \simeq \mathcal{A}^{\mathbf{T}_j}$. The functor

$$\mathcal{L} : \mathcal{A} \to \prod_{j\in J} \mathcal{O}_{\mathcal{A}}(U(\mathcal{T}_j)), \qquad M \mapsto (\mathcal{L}_j(M))_{j\in J}$$

has right adjoint

$$\iota : \prod_{j\in J} \mathcal{O}_{\mathcal{A}}(U(\mathcal{T}_j)) \to \mathcal{A}, \qquad (M_j)_{j\in J} \mapsto \prod_{j\in J} \iota_j M_j.$$

We can see that the product of all the $\mathcal{O}_{\mathcal{A}}(U(\mathcal{T}_j)) \simeq \mathcal{A}^{\mathbf{T}_j}$ is equivalent to the category of modules over the product of the $\mathbf{T}_j$. Alternatively, we can see this by applying Beck's theorem directly to ι. Letting $\mathbf{T}$ denote the monad associated to the adjunction $(\mathcal{L}, \iota)$. We have seen that $\prod_{j\in J} \mathcal{O}_{\mathcal{A}}(U(\mathcal{T}_j)) \simeq \mathcal{A}^{\mathbf{T}}$.

Since ι is monadic, we can see that $\mathcal{L}$ is equivalent to the free $\mathbf{T}$-monad functor. Thus the comonad $\mathbf{K^T}$ is simply the comonad associated to the adjunction $(\mathcal{L}, \iota)$. We will then apply the dual to Beck's theorem to $\mathcal{L}$. Since $\mathcal{A}$ is an Abelian category, condition 1 of the Theorem 4 is satisfied. Since each $\mathcal{L}_j$ preserves finite limits, we can see that $\mathcal{L}$ also preserves finite limits. Thus condition 2 of Theorem 4 is satisfied.

The requirement that the collection $\{U(\mathcal{T}_j)\}_{j\in J}$ be an open cover is equivalent to the condition $\bigcap_{j\in J} \mathcal{T}_j = 0$, the zero subcategory. This implies that $Ker(\mathcal{L}) = 0$, which means that $\mathcal{L}$ reflects isomorphisms. Thus condition 3 of Theorem 4 is satisfied. Hence $\mathcal{L}$ is comonadic. This means that $\mathcal{A} \simeq (\prod_{j\in J} \mathcal{O}_{\mathcal{A}}(U(\mathcal{T}_j)))_{\mathbf{K^T}} \simeq (\mathcal{A}^{\mathbf{T}})_{\mathbf{K^T}} = D(\mathbf{T})$; that is, $\mathbf{T}$ satisfies effective descent. $\square$

Let $\mathcal{A}$, $\{U(\mathcal{T}_j)\}_{j\in J}$, $\mathcal{L}_j$, $\mathbf{T}_j$, $\mathbf{T}$ and $\mathbf{K^T}$ as in Proposition 5 (and its proof). We will now modify a discussion from [15] that details how descent relates to sheaves on $\mathbf{Spec}^0(\mathcal{A})$. A $\mathbf{K^T}$-comodule is a pair (M', δ') where $M' = (M_j)_{j\in J} \in Ob \prod_{j\in J} \mathcal{O}_{\mathcal{A}}(U(\mathcal{T}_j))$, and $\delta \in Hom_{\mathcal{A}_{\mathcal{U}}}(L', \mathbf{K^T}M')$ such that $\epsilon_{M'} \circ \delta = id_{M'}$ and $\mathbf{K^T} \circ \delta = \Delta_{M'} \circ \delta$. This means that $\delta = (\delta_j)_{j\in J} \in \prod_{j\in J} Mor\mathcal{O}_{\mathcal{A}}(U(\mathcal{T}_j)) = \prod_{j\in J} \mathcal{O}_{\mathcal{A}}(U(\mathcal{T}_j))$. We have that δ_j is a morphism

$$\delta_j : M_j \to \mathcal{L}_j \iota_{\mathcal{U}}(M') = \mathcal{L}_j \left(\prod_{i\in J} \iota_i(M_i) \right) = \mathcal{L}_j \left(\prod_{i\in J} M_i \right).$$

The last equality comes from suppressing the ι_i, since they are inclusion functors. Furthermore, δ_j is a morphism that must equalize the pair of morphisms

$$\mathcal{L}_j \left(\prod_{i\in J} M_i\right) \underset{\mathcal{L}_j(\iota_i)\delta_i}{\overset{\mathcal{L}_j \eta_{\iota_{\mathcal{U}} M'}}{\rightrightarrows}} \mathcal{L}_j \Big(\prod_{m\in J} \mathcal{L}_m \big(\prod_{i\in J} M_i \big) \Big),$$

and such that $\epsilon_j(M') \circ \delta_j = id_{M_j}$ for all $j \in J$. The exactness of the diagram

$$M' \xrightarrow{\delta} \mathbf{K^T}(M') \underset{\mathbf{K^T}\delta}{\overset{\Delta_{M'}}{\rightrightarrows}} (\mathbf{K^T})^2(M') \tag{1.8}$$

is equivalent to the exactness of the diagram

$$M_j \xrightarrow{\delta_j} \mathcal{L}_j\left(\prod_{i\in J} M_i\right) \underset{\mathcal{L}_j(\delta_i)}{\overset{\mathcal{L}_j\eta_{\iota_{\mathcal{U}}M'}}{\rightrightarrows}} \mathcal{L}_j\Big(\prod_{m\in J}\mathcal{L}_m\big(\prod_{i\in J} M_i\big)\Big) \tag{1.9}$$

for every $j \in J$. Since the functors $\mathcal{L}_j$ are flat localizations, they preserve finite products. Since J is finite, the diagram (1.9) is isomorphic to the diagram

$$M_j \xrightarrow{\delta_j} \prod_{i\in J}\mathcal{L}_j M_i \underset{\mathcal{L}_j(\delta_i)}{\overset{\mathcal{L}_j\eta_{\iota_{\mathcal{U}}M'}}{\rightrightarrows}} \prod_{m,i\in J}\mathcal{L}_j\mathcal{L}_m M_i \tag{1.10}$$

The diagram (1.10) can be viewed as a sheaf-like property. The object M_j can be regarded as the section of the object M over the open set $U(\mathcal{T}_j)$. Likewise, the objects $\mathcal{L}_j M_i$ and $\mathcal{L}_j\mathcal{L}_m M_i$ can be viewed as the section of M_j over $U(\mathcal{T}_i)\cap U(\mathcal{T}_j)$, and the section of M_j over $U(\mathcal{T}_i)\cap U(\mathcal{T}_j)\cap U(\mathcal{T}_m)$ respectively.

From now on, for us a descent condition will take the place of a sheaf condition on a "structure presheaf". The examples below explore concrete situations where the descent condition comes even closer to the sheaf condition.

1.4.1 Examples

Let R be a commutative ring. An open cover $\{U(\mathcal{T}_i)\}_{i\in I}$ of $\mathbf{Spec}^0(R-mod)$ corresponds to an open cover $\{U(\mathbf{m}_i)\}$ of $Spec(R)$ for some collection $\mathbf{m}_i \in I(R)$. Furthermore, $\mathcal{O}_{\mathcal{A}}(U(\mathcal{T}_j)) \simeq R-mod/\langle U(\mathbf{m}_i)\rangle \simeq \mathcal{L}_i(R)-mod$, where $\mathcal{L}_i$ is the localization functor $\mathcal{L}_i : R-mod \to R-mod/\langle U(\mathbf{m}_i)\rangle$.

Let us take the open cover to be a finite cover of basic open sets; that is, let $\mathcal{T}_i = R/\langle f_i\rangle - mod$ for some finite collection of elements $f_i \in R$, ($\langle f_i\rangle$ is the ideal generated by f_i). Then $\mathcal{L}_i(R) = R_{f_i}$, and $\mathcal{L}_i \simeq R_{f_i}\otimes -$ for all $i \in I$. Using the notation of equation (1.9), let $M' = R$. Then (1.8) becomes (suppressing the names of the morphisms)

$$R \to \prod_{i\in I} R_{f_i} \rightrightarrows \prod_{i,m\in I} R_{f_m f_i}$$

This is exactly means that the functor

$$\mathcal{O}_R : (Open(Spec(R)))^{op} \to CommRing$$
$$U(f) \mapsto R_f$$

is a sheaf on $Spec(R)$.

We have noted that for every open set $U(\mathcal{T}) \subseteq \mathbf{Spec}^0(R-mod)$, it is not always the case that $R-mod/\left(\bigcap_{\mathcal{T}' \in U(\mathcal{T})} \widehat{\mathcal{T}'}\right) \simeq \mathcal{L}_{U(\mathcal{T})}(R)-mod$, where

$$\mathcal{L}_{U(\mathcal{T})} : R-mod \to R-mod/\left(\bigcap_{\mathcal{T}' \in U(\mathcal{T})} \widehat{\mathcal{T}'}\right)$$

is the flat localization functor associated to $U(\mathcal{T})$. If this were the case, then all of the information contained in the structure presheaf

$$\mathcal{O}_{R-mod} : (Open(\mathbf{Spec}^0(R-mod)))^{op} \to AbCat$$
$$U(\mathcal{T}) \mapsto R-mod/\left(\bigcap_{\mathcal{T}' \in U(\mathcal{T})} \widehat{\mathcal{T}'}\right).$$

would be contained in the presheaf

$$\mathcal{O}_R : (Open(\mathbf{Spec}^0(R-mod)))^{op} \to Ring$$
$$U(\mathcal{T}) \mapsto \mathcal{L}_{U(\mathcal{T})}(R).$$

In such a situation, the descent condition is equivalent to a sheaf-like condition on the above presheaf $\mathcal{O}_R$. The following is a corollary to Lemma 4.4 in [13].

Proposition 6. *Let R be a ring such that for every $\mathbf{p} \in Spec_l(R)$, $\mathbf{p}$ satisfies the equivalent conditions of Lemma 4.4 in [13]. Let $\{U(\mathcal{T}_j)\}_{j\in J}$ be a finite cover of $\mathbf{Spec}^0(R-mod)$ of open sets, and let $\mathcal{L}_j$ be as in Theorem 5. Then we have exact diagrams*

$$R \to \prod_{i\in J} \mathcal{L}_i(R) \rightrightarrows \prod_{i,m\in J} \mathcal{L}_i \circ \mathcal{L}_m(R)$$

and

$$R \to \prod_{i\in J} \mathcal{L}_i(R) \rightrightarrows \prod_{i,m\in J} \mathcal{L}_m \circ \mathcal{L}_i(R).$$

Proof. This is a direct consequence of Lemma 4.4 in [13], and the discussion in Section 1.4. □

In the above diagrams, the term $\mathcal{L}_i(R) \otimes \mathcal{L}_m(R)$ can be thought of as the section of $\mathcal{L}_m(R)$ over the open set $U(\mathcal{T}_i)$, and the term $\mathcal{L}_m(R) \otimes \mathcal{L}_i(R)$ can be thought of as the section of $\mathcal{L}_i(R)$ over the open set $U(\mathcal{T}_m)$. Because $R-mod$ is braided monoidal, there is an isomorphism $\mathcal{L}_i(R) \otimes \mathcal{L}_m(R) \simeq \mathcal{L}_m(R) \otimes \mathcal{L}_i(R)$, but this isomorphism does not have to be the identity. Thus when restricting R, the "ring of functions on $\mathbf{Spec}^0(R-mod)$", to the intersection of two open sets, the noncommutivity of R makes it important to keep track of which open set we are restricting to first.

Chapter 2

Geometry of Monoidal Categories

2.1 Flat Localizations of Closed Braided Monoidal Categories of Modules

In this chapter, we turn our attention to categories $\mathcal{A}$ that have a monoidal structure, but not necessarily an Abelian or additive structure. We define the spectrum of a monoidal category and investigate some properties of this spectrum.

2.1.1 Saturated Multiplicative Systems of an Abelian Category

Before we turn our attention to monoidal categories, there is an important observation to be made about the case when $\mathcal{A}$ is an Abelian category. First, we say a collection of morphisms $\Sigma \subseteq Mor\mathcal{A}$ a *multiplicative system* if the following conditions hold (Definition 10.3.4 in Chapter 10 of [18])

1. Σ is closed under composition.

2. (Ore condition) If $t : Z \to Y$ is in Σ, then for every $g : X \to Y$ in $\mathcal{A}$ there is a commutative diagram in $\mathcal{A}$ with $s \in \Sigma$

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "s"'] & Z \arrow[d, "t"] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

Moreover, the symmetric statement is also valid.

3. (Cancellation) If $f, g : X \to Y$ are parallel morphisms in $\mathcal{A}$, then the following two conditions are equivalent:

   (a) $sf = sg$ for some $s \in \Sigma$ with source Y.

   (b) $ft = gt$ for some $t \in \Sigma$ with target X.

Assume that $\mathcal{A}$ has finite limits and colimits. As described in [7], the canonical localization functor that inverts the morphisms of Σ (defined in [7], Chapter 1 Section 1.1),

$$\mathcal{L}_\Sigma : \mathcal{A} \to \Sigma^{-1}\mathcal{A}$$

preserves finite limits and colimits if and only if Σ is a multiplicative system ([7], Chapter 1, Section 3 Proposition 3.1). For a functor between two categories $F : \mathcal{A} \to \mathcal{B}$, define the collection of morphisms of $\mathcal{A}$

$$\Sigma_F := \{g \in Mor\mathcal{A} \mid F(g) \in Mor\mathcal{B} \text{ is invertible}\}.$$

We say that a multiplicative system $\Sigma \subseteq Mor\mathcal{A}$ is *saturated* if $\Sigma_{\mathcal{L}_\Sigma} = \Sigma$. Note that we always have that $\Sigma \subseteq \Sigma_{\mathcal{L}_\Sigma}$. Denote the collection of all saturated multiplicative systems of a category $\mathcal{A}$ by $\mathcal{M}(\mathcal{A})$. Since the intersection of an arbitrary collection $\{\Sigma_\lambda\}_{\lambda \in \Lambda} \subseteq \mathcal{M}(\mathcal{A})$ is again a saturated multiplicative system, the set $\mathcal{M}(\mathcal{A})$ becomes a lattice with arbitrary infima and suprema, where infimum taken to be intersection. Denote by $\mathcal{M}_f(\mathcal{A})$ the subset of all $\Sigma \in \mathcal{M}(\mathcal{A})$ such that $\mathcal{L}_\Sigma$ has a right adjoint. We call saturated multiplicative systems that are elements of $\mathcal{M}_f(\mathcal{A})$ *flat*. Before we proceed, we make some definitions needed for the next lemma.

Definition 24. *Let $\mathcal{A}$ be an Abelian category. We say $\mathcal{A}$* has the property (*sup*) *if it has the following property*

(sup) *Fory any ascending chain Ω of subobjects of an object $M \in Ob\mathcal{A}$, the supremum of Ω exists; and for any subobject L of M, the natural morphism*

$$sup\{X \cap L \mid X \in \Omega\} \to (sup\Omega) \cap L$$

is an isomorphism.

Also recall that an Abelian category $\mathcal{A}$ is said to *have injective hulls* if for every object $M \in Ob\mathcal{A}$, there is a smallest injective object $E_M \in Ob\mathcal{A}$ such that M is a subobject of E_M. Note that for a ring R, the category $R-mod$ has both the property *(sup)* and injective hulls.

We also make the following notation. If $\mathcal{B}$ if a full subcategory of $\mathcal{A}$, define

$$\Sigma_{\mathcal{B}} := \{g \in Mor\mathcal{A} \mid ker(g) \in \mathcal{B} \text{ and } coker(g) \in \mathcal{B}\}.$$

The following lemma is an amalgamation of various facts that we present here in a coherent form.

Lemma 4. *Let $\mathcal{A}$ be an Abelian category. For $\mathcal{T} \in Th(\mathcal{A})$, the set $\Sigma_{\mathcal{T}}$ is in $\mathcal{M}(\mathcal{A})$. In fact, the map*

$$Th(\mathcal{A}) \to \mathcal{M}(\mathcal{A}) \tag{2.1}$$

$$\mathcal{T} \to \Sigma_{\mathcal{T}}$$

is a bijection of lattices. Moreover, if $\mathcal{A}$ has the property (sup), *and injective hulls then the map* (2.1) *restricts to a bijection of lattices*

$$Th_c(\mathcal{A}) \to \mathcal{M}_f(\mathcal{A}). \tag{2.2}$$

Proof. The fact that (2.1) is bijective follows from the discussion in [7], Section 3 p.19. The inverse map is given by $\Sigma \mapsto \mathcal{T}_\Sigma$, where

$$\mathcal{T}_\Sigma := \{M \in Ob\mathcal{A} \mid \mathcal{L}_\Sigma(M) \simeq 0\}. \tag{2.3}$$

The bijection (2.2) is obtained by combining Lemma 2.4.4 and Corollary 2.4.8.2 in Chapter 1 of [13]. $\square$

2.1.2 Monoidal Multiplicative Systems

Let $(\mathcal{A}, \otimes)$ be a monoidal (now not necessarily additive!) category. For any subset $\Sigma \subseteq Mor\mathcal{A}$, define the *monoidal interior* as in [2] (p.5) of Σ as

$$\Sigma^o := \{s \in \Sigma \mid s \otimes 1_M \in \Sigma \text{ and } 1_M \otimes s \in \Sigma \text{ for all } M \in Ob\mathcal{A}\}.$$

Note that we always have $\Sigma^o \subseteq \Sigma$. If $\Sigma = \Sigma^o$, we say that Σ is *monoidal.* For any $\Sigma \subseteq Mor\mathcal{A}$, denote by

$$\mathcal{L}_\Sigma : \mathcal{A} \to \Sigma^{-1}\mathcal{A}$$

the canonical localization functor that formally inverts the morphisms of Σ, as defined in [7], Chapter 1 Section 1.1.

It is clear to see that if $\Sigma \subseteq Mor\mathcal{A}$ is such that $\Sigma^{-1}\mathcal{A}$ is a monoidal category, and $\mathcal{L}_\Sigma$ is a strong monoidal functor, then Σ is monoidal. Corollary 1.4 in [2] shows that if Σ is monoidal, then $\Sigma^{-1}\mathcal{A}$ is a monoidal category, and $\mathcal{L}_\Sigma$ is a strong monoidal functor. Define

$$\mathcal{M}^\otimes(\mathcal{A}) := \{\Sigma \in \mathcal{M}(\mathcal{A}) \mid \Sigma = \Sigma^o\},$$

and

$$\mathcal{M}_f^{\otimes}(\mathcal{A}) := \{\Sigma \in \mathcal{M}^{\otimes}(\mathcal{A}) \mid \mathcal{L}_\Sigma \text{ has a right adjoint}\}.$$

We call the set $\mathcal{M}^{\otimes}(\mathcal{A})$ *monoidal multiplicative systems of* $\mathcal{A}$, and the set $\mathcal{M}_f^{\otimes}(\mathcal{A})$ *flat monoidal multiplicative systems of* $\mathcal{A}$. Both $\mathcal{M}^{\otimes}(\mathcal{A})$ and $\mathcal{M}_f^{\otimes}(\mathcal{A})$ are lattices with arbitrary infima and suprema when the infimum of taken to be intersection.

2.2 Spectrum of a Monoidal Category

We make the following definition of the spectrum of a monoidal category.

Definition 25. *Let* $\mathcal{A}$ *be a monoidal category. We define the* monoidal spectrum of $\mathcal{A}$ *to be*

$$\mathbf{Spec}^{\otimes}(\mathcal{A}) := \{\Sigma \in \mathcal{M}_f^{\otimes}(\mathcal{A}) \mid \Sigma \subsetneq \Sigma^*\}.$$

Here, the $(-)^*$ operator considers Σ as an element of the lattice $\mathcal{M}^{\otimes}(\mathcal{A})$, and not just $\mathcal{M}_f^{\otimes}(\mathcal{A})$. Our definition $\mathbf{Spec}^{\otimes}(\mathcal{A})$ is analagous to $\mathbf{Spec}^1(\mathcal{A})$ for an Abelian category defined in Chapter 1. We also put a topology on $\mathbf{Spec}^{\otimes}(\mathcal{A})$ as follows. For $\Sigma \in \mathbf{Spec}^{\otimes}(\mathcal{A})$, set

$$U_1^{\otimes}(\Sigma) := \{\Sigma' \in \mathbf{Spec}^{\otimes}(\mathcal{A}) \mid \Sigma' \nsubseteq \Sigma\}.$$

Set $\mathbb{U} := \{U_1^{\otimes}(\Sigma) \mid \Sigma \in \mathbf{Spec}^{\otimes}(\mathcal{A})\}$. Let τ_1 be the smallest topology on $\mathbf{Spec}^{\otimes}(\mathcal{A})$ containing $\mathbb{U}$; that is $\mathbb{U}$ is a subbasis for τ_1. Explicitly, the collection $B_{\mathbb{U}}$, consisting of the set $\mathbf{Spec}^{\otimes}(\mathcal{A})$ as well as all finite intersections of elements of $\mathbb{U}$, forms a basis for the topology τ_1. Clearly $B_{\mathbb{U}}$ is closed under finite intersections. We call τ_1 the *Zariski topology* (we will justify this name later).

In the case when $\mathcal{A}$ is a braided monoidal category, we can put another topology on $\mathbf{Spec}^{\otimes}(\mathcal{A})$ as follows. Let $g \in Hom_{\mathcal{A}}(I, I)$, where $I \in Ob\mathcal{A}$ is the identity object with respect to the monoidal product. Then define

$$U_2^{\otimes}(g) := \{\Sigma \in \mathbf{Spec}^{\otimes}(\mathcal{A}) \mid g \in \Sigma\}.$$

We can see that sets of this form cover $\mathbf{Spec}^{\otimes}(\mathcal{A})$. Now let $g, h \in Hom_{\mathcal{A}}(I, I)$. Then

$$U_2^{\otimes}(g) \cap U_2^{\otimes}(h) = \{\Sigma \in \mathbf{Spec}^{\otimes}(A) \mid g \in \Sigma \text{ and } h \in \Sigma\}.$$

Since by definition every $\Sigma \in \mathbf{Spec}^{\otimes}(\mathcal{A})$ is a multiplicative system, we have that $g, h \in \Sigma$ implies that $gh \in \Sigma$ and $hg \in \Sigma$. Since $\mathcal{A}$ is braided monoidal, we have that $hg = h \otimes g \simeq g \otimes h = gh$. Thus $gh \in \Sigma$ if and only if $hg \in \Sigma$. This gives that

$$U_2^{\otimes}(g) \cap U_2^{\otimes}(h) \subseteq U_2^{\otimes}(gh) = \{\Sigma \in \mathbf{Spec}^{\otimes}(\mathcal{A}) \mid gh \in \Sigma\}.$$

The discussion in Section 1.1.3 in Chapter 5 of [15] gives that because Σ is saturated for every $\Sigma \in \mathbf{Spec}^{\otimes}(\mathcal{A})$, we have that $hg \simeq gh \in \Sigma$ implies that g and h are in Σ. This gives the reverse inclusion $U_2^{\otimes}(gh) \subseteq U_2^{\otimes}(g) \cap U_2^{\otimes}(h)$, and hence

$$U_2^{\otimes}(g) \cap U_2^{\otimes}(h) = U_2^{\otimes}(gh).$$

Thus the sets $U_2^{\otimes}(g)$ for $g \in Hom_{\mathcal{A}}(I, I)$ are closed under finite intersection; hence they form the base of a topology on $\mathbf{Spec}^{\otimes}(\mathcal{A})$. Denote this topology by τ_2.

We define the "structure presheaf" functor for each of the topologies above. In what follows, $MonCat$ will denote the category of small monoidal categories and strong monoidal functors. For the first topology, we will define the functor $\mathcal{O}_{\mathcal{A}}$ only on subbasic open sets

$$\begin{aligned} \mathcal{O}_{\mathcal{A}} : Open(\mathbf{Spec}^{\otimes}(\mathcal{A}))^{op} &\to MonCat \\ U_1^{\otimes}(\Sigma) &\mapsto \left(\bigcap_{\Sigma' \in U_1^{\otimes}(\Sigma)} \Sigma' \right)^{-1} \mathcal{A}. \end{aligned}$$

For the second topology, we define $\mathcal{O}_{\mathcal{A}}$ in a similar way. Here we define $\mathcal{O}_{\mathcal{A}}$ on basic open sets

$$\mathcal{O}_{\mathcal{A}} : Open(\mathbf{Spec}^{\otimes}(\mathcal{A}))^{op} \to MonCat$$
$$U_2^{\otimes}(g) \mapsto \left(\bigcap_{\Sigma \in U_2^{\otimes}(g)} \Sigma \right)^{-1} \mathcal{A}$$

As before, we call $\mathcal{O}_{\mathcal{A}}$ the *structure presheaf* on $\mathbf{Spec}^{\otimes}(\mathcal{A})$. Also as before, we have a condition involving descent theory that replaces the sheaf condition of our structure presheaf, which we discuss in Section 2.4.

2.2.1 The Spectrum with respect to Flat Monoidal Multiplicative Systems

Before we move on to examples, we formulate another spectra that will be simpler to work with in the examples of a commutative Noetherian ring and of a commutative monoid. First, if $\mathcal{A}$ is a monoidal category, and $\Sigma \in \mathcal{M}_f^{\otimes}(\mathcal{A})$, set

$$\Sigma^{\circledast} := \bigcap \{\Sigma' \in \mathcal{M}_f^{\otimes}(\mathcal{A}) \mid \Sigma \subsetneq \Sigma'\}.$$

The element $\Sigma^{\circledast} \in \mathcal{M}_f^{\otimes}(\mathcal{A})$ is simply the operator $(-)^*$ of equation (1.5), where Σ is considered only an element of $\mathcal{M}_f^{\otimes}(\mathcal{A})$, and not of $\mathcal{M}^{\otimes}(\mathcal{A})$ as was the case in the definition of $\mathbf{Spec}^{\otimes}(\mathcal{A})$; Definition 25. If $\Sigma \in \mathcal{M}_f^{\otimes}(\mathcal{A})$, we have that $\Sigma^* \subseteq \Sigma^{\circledast}$ (here the operator $(-)^*$ views Σ as an element of $\mathcal{M}^{\otimes}(\mathcal{A})$).

Definition 26. *For a monoidal category $\mathcal{A}$, set*

$$\mathbf{Spec}_f^{\otimes}(\mathcal{A}) = \{\Sigma \in \mathcal{M}_f^{\otimes}(\mathcal{A}) \mid \Sigma \subsetneq \Sigma^{\circledast}\}.$$

Because of the fact that for any $\Sigma \in \mathcal{M}_f^{\otimes}(\mathcal{A})$, we have $\Sigma^* \subseteq \Sigma^{\circledast}$, we can see that if $\Sigma \in \mathcal{M}_f^{\otimes}(\mathcal{A})$ is such that $\Sigma \subsetneq \Sigma^*$, we automatically have that $\Sigma \subsetneq \Sigma^{\circledast}$. This gives the inclusion $\mathbf{Spec}^{\otimes}(\mathcal{A}) \subseteq \mathbf{Spec}_f^{\otimes}(\mathcal{A})$ for any monoidal category $\mathcal{A}$.

Question 1. *For what monoidal categories $\mathcal{A}$ do we have $\mathbf{Spec}^{\otimes}(\mathcal{A}) \simeq \mathbf{Spec}_f^{\otimes}(\mathcal{A})$?*

2.2.2 The Spectrum of a Commutative Noetherian Ring

The Gabriel-Krull Dimension of an Abelian Category

Let $\mathcal{A}$ be an Abelian category with the property *(sup)*. The *Gabriel filtraion of* $\mathcal{A}$ assigns to every ordinal α a thick coreflective subcategory $\mathcal{A}_\alpha$ of $\mathcal{A}$ which is constructed as follows.

Set $\mathcal{A}_0 = 0$, the zero subcategory.

If α is not a limit ordinal, then $\mathcal{A}_\alpha$ is the smallest thick coreflective subcategory of $\mathcal{A}$ containing all objects M such that the object $\mathcal{L}_{\alpha-1}(M)$ has finite length. The functor $\mathcal{L}_{\alpha-1}$ is the localization

$$\mathcal{L}_{\alpha-1} : \mathcal{A} \to \mathcal{A}/\mathcal{A}_{\alpha-1}.$$

If β is a limit ordinal, then $\mathcal{A}_\beta$ is the smallest thick coreflective subcategory containing all the subcategories $\mathcal{A}_\alpha$ for $\alpha < \beta$.

Let $\mathcal{A}_\omega$ denote the smallest thick coreflective subategory containing all the subcategories $\mathcal{A}_\alpha$. The category $\mathcal{A}/\mathcal{A}_\omega$ has no simple objects.

An object $M \in Ob\mathcal{A}$ is said to have *Gabriel-Krull dimension* β if β is the smallest ordinal such that M belongs to $\mathcal{A}_\beta$. The category $\mathcal{A}$ is said to have *Gabriel-Krull dimension* if $\mathcal{A} = \mathcal{A}_\omega$. Note that $R-mod$ for R a left Noetherian ring has Gabriel-Krull dimension.

Lemma 5. *Let R be a commutative Noetherian ring, where the monoidal product in $R-mod$ is given by tensor product of modules. Then*

$$\mathbf{Spec}^1(R-mod) \simeq \mathbf{Spec}_f^{\otimes}(R-mod).$$

Proof. This follows almost immediately from Proposition 4.11.1 in Chapter 7 of [15]. Before we can apply it directly, we need to make a few observations. Because of the bijection in equations (2.1) and (2.2), $\mathbf{Spec}^1(R-mod)$ can be described as follows

$$\begin{aligned}\mathbf{Spec}^1(R-mod) &= \{\mathcal{T} \in Th_c(R-mod) \mid \mathcal{T} \subsetneq \mathcal{T}^*\} \\ &\simeq \{\Sigma \in \mathcal{M}_f(R-mod) \mid \Sigma \subsetneq \Sigma^*\}.\end{aligned}$$

Using Lemma 7, and the fact that for $\Sigma \in \mathcal{M}_f^{\otimes}(R-mod)$, we have $\Sigma^* \subseteq \Sigma^{\circledast}$, which gives that

$$\mathbf{Spec}^1(R-mod) \subseteq \mathbf{Spec}_f^{\otimes}(R-mod).$$

In order to apply Proposition 4.11.1 of Chapter 7 in [15], we must make sure the hypotheses are met. We need that $R-mod$ is an Abelian category with the property *(sup)*, injective hulls, and has Gabriel-Krull dimension. For R a commutative Noetherian ring, these are true. Proposition 4.11.1 of Chapter 7 in [15] then gives the reverse inclusion

$$\mathbf{Spec}_f^{\otimes}(R-mod) \subseteq \mathbf{Spec}^1(R-mod). \tag{2.4}$$

□

The following theorem shows that for a commutative Noetherian ring, all the various spectra we have defined agree when applied to the category $R-mod$.

Theorem 5. *For R a commutative Noetherian ring, we have that*

$$\mathbf{Spec}^{\otimes}(R-mod) \simeq \mathbf{Spec}_f^{\otimes}(R-mod) \simeq \mathbf{Spec}^1(R-mod) \simeq \mathbf{Spec}^0(R-mod) \simeq Spec(R),$$

where the monoidal structure on $R-mod$ is tensor product of modules.

Proof. Corollary 8.7.3 in Chapter 2 of [14] gives us the last two equivalences. For every $\Sigma \in \mathcal{M}_f(R - mod)$, we have an inclusion of sets

$$\{\Sigma' \in \mathcal{M}(R - mod) \mid \Sigma \subsetneq \Sigma'\} \subseteq \{\Sigma' \in \mathcal{M}^{\otimes}(R - mod) \mid \Sigma \subsetneq \Sigma'\}.$$

Then by this and Lemma 7, we can see that we have the inclusion

$$\mathbf{Spec}^1(R - mod) \subseteq \mathbf{Spec}^{\otimes}(R - mod).$$

We have noted that we always have the inclusion

$$\mathbf{Spec}^{\otimes}(R - mod) \subseteq \mathbf{Spec}^{\otimes}_f(R - mod),$$

and equation (2.4) gives that

$$\mathbf{Spec}^1(R - mod) \subseteq \mathbf{Spec}^{\otimes}(R - mod) \subseteq \mathbf{Spec}^{\otimes}_f(R - mod) \subseteq \mathbf{Spec}^1(R - mod).$$

This gives us the desired result. $\square$

Let R be a commutative ring, and denote by $Mult(R)$ the collection of saturated multiplicative systems of R. Note that the intersection of an arbitrary collection $\{S_j\} \subseteq Mult(R)$ is again an element of $Mult(R)$. In this way, $Mult(R)$ becomes a lattice with arbitrary infima and suprema where infimum is taken to be intersection.

Lemma 6. *Let R be a commutative ring, and $S \in Mult(R)$. Let Σ be the collection of all morphisms $M \to N$ in $R - mod$ such that $S^{-1}R \otimes M \to S^{-1}R \otimes N$ is an isomorphism. Then $\Sigma \in \mathcal{M}^{\otimes}_f(R - mod)$, and*

$$\Sigma^{-1}R - mod \simeq S^{-1}R - mod.$$

Proof. Since for every $M \in Ob R-mod$, we have that the natural map $M \to S^{-1}R \otimes M$ is in Σ, we have that the subcategory $S^{-1}R-mod$ of $R-mod$ is localizing (as in [18]). Since $\Sigma \cap S^{-1}R-mod$ consists only of isomorphisms, we have by Corollary 10.3.14 in [18], that $S^{-1}R-mod \simeq \Sigma^{-1}R-mod$.

The localization $-\otimes S^{-1}R : R-mod \to S^{-1}R-mod$ preserves finite limits. Thus we have $\Sigma \in \mathcal{M}_f^{\otimes}(R-mod)$. □

Lemma 7. *Let R be a commutative ring. Then the map*

$$\mathcal{M}_f(R-mod) \to Mult(R)$$

$$\Sigma \mapsto S_\Sigma := \{x \in R \mid (r \mapsto rx) \in \Sigma\}$$

is an isomorphism of lattices. Further, $\mathcal{M}_f(R-mod) = \mathcal{M}_f^{\otimes}(R-mod)$.

Proof. Let $\Sigma \in \mathcal{M}_f^{\otimes}(R-mod)$. We have the adjuction

$$R-mod \underset{\iota_\Sigma}{\overset{\mathcal{L}_\Sigma}{\rightleftarrows}} \Sigma^{-1}R-mod,$$

where we will take ι_Σ to be an inclusion functor. For every $M \in Ob\Sigma^{-1}R-mod$, we have isomorphisms $Hom_{\Sigma^{-1}R-mod}(\mathcal{L}_\Sigma(R), M) \simeq Hom_{R-mod}(R, \iota_\Sigma(M)) \simeq \iota_\Sigma(M)$. This shows that ι_Σ is defined by the object $\mathcal{L}_\Sigma(R)$. Since $\mathcal{L}_\Sigma$ is left adjoint to ι_Σ, we have then that $\mathcal{L}_\Sigma$ is defined uniquely up to isomorphism by the object $\mathcal{L}_\Sigma(R)$. We have that

$$\mathcal{L}_\Sigma(R) \simeq Hom_R(R, \mathcal{L}_\Sigma(R)) \simeq Hom_{\Sigma^{-1}R-mod}(R, \mathcal{L}_\Sigma(R)) \simeq Hom_{\Sigma^{-1}R-mod}(R, \iota_\Sigma \circ \mathcal{L}_\Sigma(R))$$
$$\simeq Hom_R(\mathcal{L}_\Sigma(R), \mathcal{L}_\Sigma(R)) \simeq Hom_{\Sigma^{-1}R-mod}(\mathcal{L}_\Sigma(R), \mathcal{L}_\Sigma(R)).$$

Since $Hom_{\Sigma^{-1}R-mod}(\mathcal{L}_\Sigma(R), \mathcal{L}_\Sigma(R))$ is the ring obtained by inverting exactly the morphisms corresponding to S_Σ in $Hom_R(R, R) \simeq R$, we can see that $\mathcal{L}_\Sigma(R) \simeq S_\Sigma^{-1}R$.

Hence by Lemma 6, we have that

$$- \otimes S_{\Sigma}^{-1}R : R - mod \to S_{\Sigma}^{-1}R - mod$$

is a flat localization of the category $R - mod$, and we can see that $R \otimes S_{\Sigma}^{-1}R \simeq S_{\Sigma}^{-1}R$. Thus this determines, up an isomorphism, the same localizaiton as Σ; that is, $\mathcal{L}_{\Sigma} \simeq - \otimes S_{\Sigma}^{-1}R$. $\square$

A consequence of the proof of the above lemma is that for $\Sigma \in \mathcal{M}_f^{\otimes}(R - mod)$, we have that

$$\Sigma^{-1}R - mod \simeq \mathcal{L}_{\Sigma}(R) - mod \simeq S_{\Sigma}^{-1}R - mod.$$

Proposition 7. *Let R be a commutative Noetherian ring. Then the topology τ_1 coincides with the usual Zariski topology on $Spec(R)$.*

Proof. By Theorem 5, we have that $\mathbf{Spec}^{\otimes}(R - mod)$ is in bijection with $Spec(R)$. Using the bijection

$$\begin{gathered} \mathcal{M}_f^{\otimes}(R - mod) \to Mult(R) \\ \Sigma \mapsto S_{\Sigma} \end{gathered}$$

from Lemma 7, we can write the previous bijection as

$$\begin{gathered} \mathbf{Spec}^{\otimes}(R - mod) \to Spec(R) \\ \Sigma \mapsto R \setminus S_{\Sigma}. \end{gathered} \tag{2.5}$$

Let $\Sigma \in \mathbf{Spec}^{\otimes}(R - mod)$, and let $\mathbf{p} \in Spec(R)$ such that $\mathbf{p} = R \setminus S_{\Sigma}$. Let us describe the set $U_1^{\otimes}(\Sigma)$ for $\Sigma \in \mathbf{Spec}^{\otimes}(R - mod)$ under this bijection:

$$U_1^{\otimes}(\Sigma) = \{\Sigma' \in \mathbf{Spec}^{\otimes}(R-mod) \mid \Sigma' \nsubseteq \Sigma\}$$
$$= \{\mathbf{p}' \in Spec(R) \mid \mathbf{p} \nsubseteq \mathbf{p}'\}.$$

The discussion in Section 1.10 in Chapter 1 of [13] (especially Section 1.10.3) yield that if R is a commutative Noetherian ring, then the sets $\{\mathbf{p}' \in Spec(R) \mid \mathbf{p} \nsubseteq \mathbf{p}'\}$ for $\mathbf{p} \in Spec(R)$ form a subbasis for the Zariski topology on $Spec(R)$. □

Proposition 8. *Let R be a commutative Noetherian ring. The topology τ_2 coincides with the usual Zariski topology on $Spec(R)$.*

Proof. Let $g \in Hom_R(R, R) \simeq R$. Employing again the bijection in equation (2.5), we describe $U_2^{\otimes}(g)$:

$$U_2^{\otimes}(g) = \{\Sigma \in \mathbf{Spec}^{\otimes}(R-mod) \mid g \in \Sigma\}$$
$$= \{\mathbf{p} \in Spec(R) \mid g \notin \mathbf{p}\}.$$

□

Question 2. *Given a commutative ring R, the full subcategory of finitely generated projective modules, $Proj_R$, is a closed symmetric monoidal category Note that $Proj_R$ is not an Abelian category, but an exact category.* What is $\mathbf{Spec}^{\otimes}(Proj_R)$? *Is it in bijection with* $\mathbf{Spec}^{\otimes}(R-$*mod*$)$*?*

2.2.3 Application to $\mathbb{F}_1$ Geometry: The Spectrum of a Monoid

The catgory of commutative monoids in Set, $CMon(Set)$, is viewed as dual to the category $Aff_{\mathbb{F}_1}$ (as in [3] and [16]) of affine spaces over $\mathbb{F}_1$, the field with one element.

Before the following theorem, let A be a commutative monoid (in $Sets$). We say that a subset $\mathbf{m} \subseteq A$ is an *ideal* of A if $a \in A$ and $x \in \mathbf{m}$ implies that $ax \in \mathbf{m}$. We say that an ideal $\mathbf{p} \subseteq A$ is *prime* if $xy \in \mathbf{p}$ implies that either $x \in \mathbf{p}$ or $y \in \mathbf{p}$. Denote the collection of prime ideals of A by $Prime(A)$.

Definition 27. *We say that a subset $S \subseteq A$ is a* saturated multiplicative set *if the following conditions hold*

1. $1 \in S$
2. *if $x, y \in S$, then $xy \in S$*
3. *if $xy \in S$, then $x \in S$ and $y \in S$.*

Denote the collection of saturated multiplicative sets in A by $Mult(A)$. The intersection of an arbitrary collection $\{S_j\} \subseteq Mult(A)$ is again an element of $Mult(A)$. In this way, the collection $Mult(A)$ becomes a lattice with arbitrary infima and suprema, where infimum is taken to be intersection. As is in [3], we can define the *localization* of a monoid A at a saturated multiplicative set S. Denote by $S^{-1}A$ the set $A \times S$ modulo the equivalence

$$(a, s) \sim (a', s') \text{ if and only if } \exists s'' \in S \text{ such that } s''s'a = s''sa'.$$

multiplication in $S^{-1}A$ is given by $(a, s)(a', s') = (aa', ss')$.

The saturated multiplicative sets of a commutative monoid are in bijection with the collection of prime ideals of the monoid A, through taking complements:

$$\begin{aligned} Mult(A) &\longrightarrow Prime(A) \\ S &\longmapsto A \setminus S \\ A \setminus \mathbf{p} &\longleftarrow\!\shortmid\ \mathbf{p}. \end{aligned}$$

By $A-mod$, we denote the category whoe objects are sets equipped with an A-action, and morphisms are set maps that respect the A-action. For $X, Y \in ObA-mod$, the cartesian product of X and Y will be the set $X \times Y$ equipped with the diagonal action. We have the following

Lemma 8. *Let A be a commutative monoid, and $S \in Mult(A)$. Let Σ be the collection of all morphisms $M \to N$ in $A-mod$ such that $S^{-1}A \times M \to S^{-1}A \times N$ is an isomorphism. Then $\Sigma \in \mathcal{M}_f^{\otimes}(A-mod)$, and*

$$\Sigma^{-1}A - mod \simeq S^{-1}A - mod.$$

Proof. Since for every $M \in ObA-mod$, we have that the natural map $M \to S^{-1}A \times M$ is in Σ, we have that the subcategory $S^{-1}A-mod$ of $A-mod$ is localizing (as in [18]). Since $\Sigma \cap S^{-1}A-mod$ consists only of isomorphisms, we have by Corollary 10.3.14 in [18], that $S^{-1}A-mod \simeq \Sigma^{-1}A-mod$.

By Proposition 2.3.8 of [17], we have that the localization $-\times S^{-1}A : A-mod \to S^{-1}A-mod$ preserves finite limits. Thus we have $\Sigma \in \mathcal{M}_f^{\otimes}(A-mod)$. □

Lemma 9. *Let A be a commutative monoid. Then the map*

$$\begin{aligned} \mathcal{M}_f^{\otimes}(A-mod) &\to Mult(A) \\ \Sigma &\mapsto S_\Sigma := \{x \in A \mid (a \mapsto ax) \in \Sigma\} \end{aligned}$$

is an isomorphism of lattices.

Proof. Let $\Sigma \in \mathcal{M}_f^{\otimes}(A-mod)$. We have the adjuction

$$A-mod \underset{\iota_\Sigma}{\overset{\mathcal{L}_\Sigma}{\rightleftarrows}} \Sigma^{-1}A-mod,$$

where we will take ι_Σ to be an inclusion functor. For every $M \in \Sigma^{-1}A - mod$, we have isomorphisms $Hom_{\Sigma^{-1}A-mod}(\mathcal{L}_\Sigma(A), M) \simeq Hom_{A-mod}(A, \iota_\Sigma(M)) \simeq \iota_\Sigma(M)$. This shows that ι_Σ is defined by the object $\mathcal{L}_\Sigma(A)$. Since $\mathcal{L}_\Sigma$ is left adjoint to ι_Σ, we have then that $\mathcal{L}_\Sigma$ is defined uniquely up to isomorphism by the object $\mathcal{L}_\Sigma(A)$. We have that

$$\begin{aligned}\mathcal{L}_\Sigma(A) &\simeq Hom_A(A, \mathcal{L}_\Sigma(A)) \simeq Hom_{\Sigma^{-1}A-mod}(A, \mathcal{L}_\Sigma(A)) \simeq Hom_{\Sigma^{-1}A-mod}(A, \iota_\Sigma \circ \mathcal{L}_\Sigma(A)) \\ &\simeq Hom_A(\mathcal{L}_\Sigma(A), \mathcal{L}_\Sigma(A)) \simeq Hom_{\Sigma^{-1}A-mod}(\mathcal{L}_\Sigma(A), \mathcal{L}_\Sigma(A)).\end{aligned}$$

Since $Hom_{\Sigma^{-1}A-mod}(\mathcal{L}_\Sigma(A), \mathcal{L}_\Sigma(A))$ is the monoid by inverting exactly the morphisms corresponding to S_Σ in $Hom_A(A, A) \simeq A$, we can see that $\mathcal{L}_\Sigma(A) \simeq S_\Sigma^{-1}A$.

Hence by Lemma 8, we have that

$$- \times S_\Sigma^{-1}A : A - mod \to S_\Sigma^{-1}A - mod$$

is a flat localization of the category $A - mod$, and we can see that $A \times S_\Sigma^{-1}A \simeq S_\Sigma^{-1}A$. Thus this determines, up an isomorphism, the same localizaiton as Σ; that is, $\mathcal{L}_\Sigma \simeq - \times S_\Sigma^{-1}A$. $\square$

A consequence of the proof of the above lemma is that for $\Sigma \in \mathcal{M}_f^{\otimes}(A - mod)$, we have that

$$\Sigma^{-1}A - mod \simeq \mathcal{L}_\Sigma(A) - mod \simeq S_\Sigma^{-1}A - mod.$$

Theorem 6. *Let A be a commutative monoid. For a prime ideal $\mathbf{p} \subseteq A$, set*

$$\mathbf{p}_{\circledast} := \bigcup \{\mathbf{p}' \in Prime(A) \mid \mathbf{p}' \subsetneq \mathbf{p}\}.$$

Then $\mathbf{Spec}_f^{\otimes}(A - mod)$ is homeomorphic to the collection of prime ideals $\mathbf{p}$ of A such that

$$\mathbf{p}_{\circledast} \subsetneq \mathbf{p}.$$

Proof. By Lemma 9, we have that $\mathcal{M}_f^{\otimes}(A-mod) \simeq Mult(S)$. Thus we have

$$\begin{aligned}\mathbf{Spec}_f^{\otimes}(A-mod) &\simeq \{\Sigma \in \mathcal{M}_f^{\otimes}(A-mod) \mid \Sigma \subsetneq \Sigma^{\circledast}\} \simeq \{S \in Mult(A) \mid S \subsetneq S^{\circledast}\} \\ &\simeq \{\mathbf{p} \in Prime(A) \mid \mathbf{p}_{\circledast} \subsetneq \mathbf{p}\},\end{aligned}$$

□

Corollary 1. *Lets Sets be the category of sets with cartesian product as the monoidal structure. Then* $\mathbf{Spec}_f^{\otimes}(Sets)$ *is a space with a single point.*

Proof. The category $Sets$ is isomorphic to the category $\mathbb{F}_1 - mod$, where $\mathbb{F}_1$ is the monoid with one element. There is only one ideal (let alone prime ideal) of $\mathbb{F}_1$, and thus by Theorem 6, we have that $\mathbf{Spec}_f^{\otimes}(Sets) \simeq \mathbf{Spec}_f^{\otimes}(\mathbb{F}_1 - mod) = \{*\}$. □

Theorem 7. *Lets Sets be the category of sets with cartesian product as the monoidal structure. Then* $\mathbf{Spec}^{\otimes}(Sets)$ *is a space with a single point.*

Proof. Denote by 1 the one-emelent set. Let $\Sigma \in \mathcal{M}^{\otimes}(Sets)$. Let $\mathcal{L}_{\Sigma}^{-1}(1)$ be the full subcategory of $Sets$ whose objects are all $X \in ObSets$ such that $\mathcal{L}_{\Sigma}(X) \simeq 1$ in $\Sigma^{-1}Sets$. Note that $X \in \mathcal{L}_{\Sigma}^{-1}(1)$ if and only if the unique morphism $X \to 1$ is in Σ. Let $X \in \mathcal{L}_{\Sigma}^{-1}(1)$. Then because Σ is monoidal, we have that the morphism $X \times X \to 1 \times 1 \simeq 1$ is in Σ. Thus $X \times X \in \mathcal{L}_{\Sigma}^{-1}(1)$.

A morphism $h : A \to B$ in any category is an epimorphism if and only if the diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "h"'] & B \arrow[d, "id"] \\ B \arrow[r, "id"] & B \end{tikzcd}$$

is a pushout diagram. The dual statement also holds true for monomorphisms. Since Σ is multiplicative, $\mathcal{L}_{\Sigma}$ preserves finite limits and colimits, and thus preserves epimorphisms and monomorphisms. Let $X \in \mathcal{L}_\Sigma^{-1}(1)$, and $X \to Y$ an epimorphism. Then we have $1 \simeq \mathcal{L}_\Sigma(X) \to \mathcal{L}_\Sigma(Y)$ is an epimorphism as well, and thus $Y \simeq 1$. Since every nonempty set Y with cardinality less than X has an epimorphism $X \to Y$, we have that if $X \in \mathcal{L}_\Sigma^{-1}(1)$, then so is every nonempty set Y with cardinality less than or equal to X. In fact, by the above, we can see that if $X \in \mathcal{L}_\Sigma^{-1}(1)$, then so is every nonempty set Y with cardinaliy less than or equal to any finite product $X \times X \times \cdots \times X$. Hence if there is any set $X \in \mathcal{L}_\Sigma^{-1}(1)$ such that the cardinality of X is larger than 1, we have that $\Sigma = MorSets$. If there is no such X in $\mathcal{L}_\Sigma^{-1}(1)$, then we must have that $\Sigma = Iso(Sets)$. Thus we can see that $\mathcal{M}^\otimes(Sets)$ has only two elements; $\mathcal{M}^\otimes(Sets) = \{Iso(Sets), MorSets\}$.

Now if $Iso(Sets) \subsetneq \Sigma \subsetneq MorSets$, then there exsists an $X \in \mathcal{L}_\Sigma^{-1}(1)$ different from both 1 and $\emptyset$. We can now see that for our choice of Σ, we must have that $\mathcal{L}_\Sigma^{-1}(1)$ contains all finite sets. By definition of $Iso(Sets)^*$, this implies that $\mathcal{L}_{Iso(Sets)^*}^{-1}(1)$ must also contain all finite sets, and thus $Iso(Sets) \subsetneq Iso(Sets)^*$. Thus $Iso(Sets) \in \mathbf{Spec}^\otimes(Sets)$.

We have shown that the only elements of $\mathcal{M}^\otimes(Sets)$ are $Iso(Sets)$ and $MorSets$. In fact both of these are elements of $\mathcal{M}_f^\otimes(Sets)$, which in turn implies that $\mathcal{M}_f^\otimes(Sets) = \{Iso(Sets), MorSets\}$. We cannot have that $MorSets \in \mathbf{Spec}^\otimes(Sets)$ because there are no elements of $\mathcal{M}^\otimes(Sets)$ properly containing $MorSets$. Hence $\mathbf{Spec}^\otimes(Sets) = \{Iso(Sets)\}$. □

Example

Proposition 9. *Let $\mathbb{N}$ denote the commutative monoid of non-negative integers with binary operation given by multiplication. Then $\mathbf{Spec}_f^\otimes(\mathbb{N} - mod)$ is in bijection with the prime numbers.*

Proof. If $m \in \mathbb{N}$, we denote $\langle m \rangle := \{nm \mid n \in \mathbb{N}\} \subseteq \mathbb{N}$. We can see that for every $m \in \mathbb{N}$, the set $\langle m \rangle$ is an ideal of the monoid $\mathbb{N}$. We can also see that if $p \in \mathbb{N}$ is a prime number, that $\langle p \rangle$ is a prime ideal. If $m, n \in \mathbb{N}$, we denote by $\langle m, n \rangle := \langle m \rangle \cup \langle n \rangle$, the set-theoretic union of $\langle m \rangle$ and $\langle n \rangle$. We can see that for any $m, n \in \mathbb{N}$ the set $\langle m, n \rangle$ is an ideal, and if

$p_1, p_2 \in \mathbb{N}$ are prime numbers that $\langle p_1, p_2 \rangle$ is a prime ideal.

We claim that every prime ideal is of the form $\langle p_1, p_2, \ldots, p_j, \ldots \rangle_{j \in J}$ where $\{p_j\}_{j \in J}$ is some collection of prime numbers. Indeed, let $\mathbf{p} \subseteq \mathbb{N}$ be a prime ideal of $\mathbb{N}$, and let m be the smallest element of $\mathbf{p}$. If m is not prime, then there exist $a, b \in \mathbb{N}$ not equal to 1 such that $ab = m$. Necessarily, $a, b < m$. Thus $a, b \notin \mathbf{p}$, which contradicts the assumption that $\mathbf{p}$ is prime. So $m = p_1$ is a prime number. Set $\mathbf{p}_1 := \mathbf{p} \setminus \langle p_1 \rangle$. Suppose $ab \in \mathbf{p}_1$. Because $\mathbf{p}$ is prime, we know either a or b is in $\mathbf{p}$. Without loss of generality, assume $a \in \mathbf{p}$. If $a \in \langle p_1 \rangle$, then because $\langle p_1 \rangle$ is an ideal, we have that $ab \in \langle p_1 \rangle$, which contradicts our assumption that $ab \in \mathbf{p}_1 = \mathbf{p} \setminus \langle p_1 \rangle$. Thus $a \in \mathbf{p}_1$, meaning $\mathbf{p}_1$ satisfies the prime condition (although it is *not* an ideal of $\mathbb{N}$). Now let p_2 be the smallest element of $\mathbf{p}_1$. We can again seet that p_2 is a prime number, and we define $\mathbf{p}_2 := \mathbf{p}_1 \setminus \langle p_2 \rangle$. Again, $\mathbf{p}_2$ satisfies the prime condition, but is not an ideal of $\mathbb{N}$. We can recursively continue this process, which will yeild a (possibly infinite) sequence $\{p_j\}_{j \in J}$ of prime numbers. It is clear that $\mathbf{p} = \bigcup_{j \in J} \langle p_j \rangle = \langle p_1, \ldots, p_j, \ldots \rangle_{j \in J}$.

From here, it is clear to see that the prime ideals $\mathbf{p}$ of $\mathbb{N}$ that satisfy the condition $\mathbf{p}_{\circledast} \subsetneq \mathbf{p}$ are the $\mathbf{p}$ generated by a single prime number; that is, prime ideals of the form $\mathbf{p} = \langle p \rangle$ for some prime number $p \in \mathbb{N}$. $\square$

Question 3. *If A is a noncommutative monoid, can we describe $\mathbf{Spec}^{\otimes}(A - mod)$, or $\mathbf{Spec}^{\otimes}_f(A - mod)$? The monoid $Hom_A(A, A) \simeq A^{op}$ is not isomorphic to A, which was a key part in the proof of Lemma 9.*

Question 4. *For what monoids A do we have $\mathbf{Spec}^{\otimes}(A - mod) \simeq \mathbf{Spec}^{\otimes}_f(A - mod)$?*

2.3 Functoriality of the Spectrum

Theorem 8. *Let $\mathcal{A}$ and $\mathcal{B}$ be two monoidal categories. Let $F : \mathcal{A} \to \mathcal{B}$ be a flat strong monoidal localization. Then F defines a continuous function*

$$\mathbf{Spec}^{\otimes}(F) : \mathbf{Spec}^{\otimes}(\mathcal{B}) \to \mathbf{Spec}^{\otimes}(\mathcal{A}).$$

Proof. The map $\mathbf{Spec}^{\otimes}(F)$ is defined by

$$\begin{aligned} \mathbf{Spec}^{\otimes}(\mathcal{B}) &\to \mathbf{Spec}^{\otimes}(\mathcal{A}) \\ \Sigma' &\mapsto F^{-1}(\Sigma'), \end{aligned}$$

where $F^{-1}(\Sigma') = \{f \in Mor\mathcal{A} \mid F(f) \in \Sigma'\}$. Since F is flat, F has a right adjoint ι, which we can and will assume is an inclusion functor. Thus $\mathcal{B}$ is a full subcategory of $\mathcal{A}$. Since F is a localization functor, we have that $\mathcal{B}$ is equivalent to the category $\Sigma_F^{-1}\mathcal{A}$, and that $\Sigma_F \in \mathcal{M}_f(\mathcal{A})$. Let us set $\Sigma_F = \Sigma$. Let us make the notation

$$\Sigma\backslash 1 := \{\Sigma' \in \mathcal{M}(\mathcal{A}) \mid \Sigma \subseteq \Sigma'\}.$$

From Lemma 2.2.1 in Chapter 5 of [15], we have that the map

$$\begin{aligned} \mathcal{M}(\mathcal{B}) &\to \Sigma\backslash 1 \\ \Sigma' &\mapsto F(\Sigma') \end{aligned} \tag{2.6}$$

induces an isomorphism of lattices $\mathcal{M}(\mathcal{B}) \simeq \Sigma\backslash 1$. The inverse map to (2.6) is given by

$$\begin{aligned} \Sigma\backslash 1 &\to \mathcal{M}(\mathcal{B}) \\ \Sigma' &\mapsto \Sigma' \cap \mathcal{B}, \end{aligned} \tag{2.7}$$

where we are here using the fact that we have taken $\mathcal{B}$ to be a full subcategory of $\mathcal{A}$.

Since F is strong monoidal, it is straightforward to see that the map $\Sigma' \mapsto F^{-1}(\Sigma')$ induces an isomorphism $\mathcal{M}^{\otimes}(\mathcal{B}) \simeq \Sigma\backslash 1_{\otimes}$, where

$$\Sigma\backslash 1_{\otimes} := \{\Sigma' \in \mathcal{M}^{\otimes}(\mathcal{A}) \mid \Sigma \subseteq \Sigma'\}.$$

The bijection (2.6) restricts to give a bijection

$$\mathcal{M}_f(\mathcal{B}) \simeq (\Sigma\backslash 1)_f := \{\Sigma' \in \mathcal{M}_f(\mathcal{A}) \mid \Sigma \subseteq \Sigma'\}.$$

Since F is strong monoidal, this once again induces to bijection

$$\mathcal{M}_f^{\otimes}(\mathcal{B}) \simeq (\Sigma\backslash 1_{\otimes})_f := \{\Sigma' \in \mathcal{M}_f^{\otimes}(\mathcal{A}) \mid \Sigma \subseteq \Sigma'\}.$$

With this, we can now describe $\mathbf{Spec}^{\otimes}(\mathcal{B})$ as follows

$$\begin{aligned}
\mathbf{Spec}^{\otimes}(\mathcal{B}) &= \{\Sigma' \in (\Sigma\backslash 1_{\otimes})_f \mid \Sigma \subsetneq \Sigma^*\} \\
&= \{\Sigma' \in \mathcal{M}_f^{\otimes}(\mathcal{A}) \mid \Sigma \subseteq \Sigma' \text{ and } \Sigma \subsetneq \Sigma^*\} \\
&= \{\Sigma' \in \mathbf{Spec}^{\otimes}(\mathcal{A}) \mid \Sigma \subseteq \Sigma'\}.
\end{aligned}$$

With this description, the map $\mathbf{Spec}^{\otimes}(F)$ is an inclusion of sets.

Now we wish to show that $\mathbf{Spec}^{\otimes}(F)$ is continuous. Let $U_1^{\otimes}(\Sigma') \subseteq \mathbf{Spec}^{\otimes}(\mathcal{A})$ be a subbasic open subset, where $\Sigma' \in \mathbf{Spec}^{\otimes}(\mathcal{A})$. Because $\mathbf{Spec}^{\otimes}(F)$ is inclusion, we must show that the intersection of $U_1^{\otimes}(\Sigma')$ with the subset $\mathbf{Spec}^{\otimes}(\mathcal{B})$ is an open set in $\mathbf{Spec}^{\otimes}(\mathcal{B})$. If $\Sigma \nsubseteq \Sigma'$, then for every $\Sigma'' \in \Sigma\backslash 1$ we have that $\Sigma'' \nsubseteq \Sigma'$. Thus $U_1^{\otimes}(\Sigma') \cap \mathbf{Spec}^{\otimes}(\mathcal{B}) = \mathbf{Spec}^{\otimes}(\mathcal{B})$.

Now suppose that $\Sigma \subseteq \Sigma'$. Then we have

$$U_1^{\otimes}(\Sigma') \cap \mathbf{Spec}^{\otimes}(\mathcal{B}) = \{\Sigma'' \in \mathbf{Spec}^{\otimes}(\mathcal{B}) \mid \Sigma'' \nsubseteq \Sigma'\} \tag{2.8}$$

Since $\Sigma' \in \mathbf{Spec}^{\otimes}(\mathcal{A})$ with $\Sigma \subseteq \Sigma'$, we have that $\Sigma \in \mathbf{Spec}^{\otimes}(\mathcal{B})$ as well. This fact shows that the subset in (2.8) is a subbasic open set in $\mathbf{Spec}^{\otimes}(\mathcal{B})$. $\square$

2.4 Descent

Let $\mathcal{A}$ be a monoidal category. Recall that a basic open set for the topology τ_1 is a finite intersection of sets of the form

$$U_1^{\otimes}(\Sigma) = \{\Sigma' \in \mathbf{Spec}^{\otimes}(\mathcal{A}) \mid \Sigma' \nsubseteq \Sigma\},$$

where $\Sigma \in \mathbf{Spec}^{\otimes}(\mathcal{A})$.

Theorem 9. *Let $\mathcal{A}$ be a monoidal category with finite limits and colimits, and let $\mathcal{U} = \{U_j\}_{j\in J}$ be a finite cover of basic open sets of $\mathbf{Spec}^{\otimes}(\mathcal{A})$. For each U_j, let $\mathcal{L}_j$ denote the localization functor*

$$\mathcal{L}_j : \mathcal{A} \to \left(\bigcap_{\Sigma\in U_j} \Sigma\right)^{-1} \mathcal{A} = \mathcal{O}_{\mathcal{A}}(U_j),$$

and denote by ι_j the right adjoint to $\mathcal{L}_j$. Set $\mathcal{L} = \prod_j \mathcal{L}_j$. Then $\mathcal{L}$ has a right adjoint ι, and the monad $\mathbf{T} := \iota \circ \mathcal{L}$ satisfies effective descent. That is, $\mathcal{A}$ is equivalent to the descent category $D(\mathbf{T})$.

Proof. Denote by $\mathbf{T_j}$ the monad associated to the adjuntion $(\mathcal{L}_j, \iota_j)$. By applying Beck's theorem (Theorem 3) to the functor ι_j, we have that $(\mathcal{A})^{\mathbf{T_j}} \simeq \mathcal{O}_{\mathcal{A}}(U_j)$. The functor

$$\mathcal{L} : \mathcal{A} \to \prod_{j\in J} \mathcal{O}_{\mathcal{A}}(U_j), \qquad M \mapsto (\mathcal{L}_j(M))_{j\in J}$$

has right adjoint

$$\iota : \prod_{j\in J} \mathcal{O}_{\mathcal{A}}(U_j) \to \mathcal{A}, \qquad (M_j)_{j\in J} \mapsto \prod_{j\in J} \iota_j(M_j).$$

Now we can see that $\prod_{j \in J} \mathcal{A}^{\mathbf{T}_j} \simeq \mathcal{A}^{\mathbf{T}}$. Alternatively, we could apply Beck's theorem directly to the functor ι.

Because $\mathcal{L}$ is a flat localization, we can clearly see that it satisfies conditions 1 and 2 of Theorem 4. The fact that $\mathcal{U}$ is a cover of $\mathbf{Spec}^{\otimes}(\mathcal{A})$ implies that $ker(\mathcal{L}) = 0$. Thus $\mathcal{L}$ satisfies condition 3 of Theorem 4. Hence $\mathcal{L}$ is comonadic; that is, $\mathcal{A} \simeq (\prod_{j \in J} \mathcal{O}_{\mathcal{A}}(U_j))_{\mathbf{K}^{\mathbf{T}}} \simeq (\mathcal{A}^{\mathbf{T}})_{\mathbf{K}^{\mathbf{T}}} = D(\mathbf{T})$. $\square$

As detailed in Chapter 1, Section 1.4, we will think of the above theorem as a replacement for the sheaf condition on the presheaf $\mathcal{O}_{\mathcal{A}}$.

Chapter 3

Connection with Prime $\otimes$-ideals

In this chapter, we focus on categories that have both an Abelian structure and a closed monoidal structure such that they are compatible in an appropriate way. In [1], triangulated categories with a monoidal sctructure are studied, and the *prime $\otimes$-ideal spectrum* is defined as the thick triangulated prime $\otimes$-ideals. This is a very transparent and digestible definition, and we use it as inspiration in this chapter for the definition of the prime $\otimes$-ideal spectrum of an Abelian category with a compatible closed monoidal structure. We make the

Definition 28. *We say that a category $\mathcal{A}$ is a* monoidal Abelian category *if $\mathcal{A}$ is at once a monoidal category as well as an Abelian category such that the monoidal product is additive in each variable. This implies that for each $A, B, C, D \in Ob\mathcal{A}$, we have a homomorphism of Abelian groups*

$$Hom_{\mathcal{A}}(A, B) \otimes_{\mathbb{Z}} Hom_{\mathcal{A}}(C, D) \to Hom_{\mathcal{A}}(A \otimes C, B \otimes D).$$

If $\mathcal{A}$ is a monoidal Abelian category such that the monoidal product is symmetric (resp. braided), we will call $\mathcal{A}$ a *symmetric monoidal Abelian category* (resp. *braided monoidal Abelian category*).

3.1 Prime $\otimes$-ideal Spectrum

Definition 29. *Let $\mathcal{A}$ be a monoidal Abelian category. We say that a full subcategory $\mathcal{T}$ of $\mathcal{A}$ is a* $\otimes$-ideal *if for all $X \in Ob\mathcal{A}$ and $Y \in Ob\mathcal{T}$, we have $X \otimes Y \in Ob\mathcal{T}$. Moreover, we say that $\mathcal{T}$ is a* prime $\otimes$-ideal *if $X \otimes Y \in Ob\mathcal{T}$ implies either $X \in Ob\mathcal{T}$ or $Y \in Ob\mathcal{T}$.*

Definition 30. *We define the* prime $\otimes$-ideal spectrum *of a monoidal Abelian category $\mathcal{A}$ to be*

$$\mathbf{Prime}(\mathcal{A}) := \{\mathcal{T} \in Th_c(\mathcal{A}) \mid \mathcal{T} \textit{ is a prime } \otimes -\textit{ideal}\}.$$

3.2 Commutative Noetherian Rings

Denote by $R - mod^{fg}$ the category of finitely generated R-modules. In this section we will show that if R is commutative and Noetherian, then $\mathbf{Prime}(R - mod^{fg}) \simeq Spec(R)$. We wil use the fact that $\mathbf{Spec}^1(R - mod) \simeq Spec(R)$, as shown in Proposition 4.

Lemma 10. *Let R be a commutative Noetherian ring. Then $Spec(R) \subseteq \mathbf{Prime}(R - mod^{fg})$.*

Proof. First, let $\mathbf{p} \in Spec(R)$. We have a localization functor $- \otimes_R R_{\mathbf{p}} : R - mod \to R_{\mathbf{p}} - mod$, which restrics to a functor between the subcategories of finitely generated modules, which by abuse of notation we will write also as $- \otimes_R R_{\mathbf{p}} : R - mod^{fg} \to R_{\mathbf{p}} - mod^{fg}$, and for distinction we will write $ker'(- \otimes_R R_{\mathbf{p}}) := ker(- \otimes_R R_{\mathbf{p}}) \cap R - mod^{fg}$. The kernel $ker'(- \otimes_R R_{\mathbf{p}}) \subset R - mod^{fg}$ is in fact a thick coreflective subcategory $R - mod^{fg}$. It is easy to see that $ker(- \otimes_R R_{\mathbf{p}})$ is closed under quotients and extensions. Since $ker(- \otimes_R R_{\mathbf{p}})$ is closed under taking subobjects, and R is Noetherian, subobjects of finitely generated R-modules are finitely generated, thus $ker'(- \otimes_R R_{\mathbf{p}})$ is closed under subobjects. The right adjoint to the inclusion $ker(- \otimes_R R_{\mathbf{p}}) \hookrightarrow R - mod$ is given by $M \mapsto ker\eta_M$ where $\eta : Id_{R-mod} \to i_{\mathbf{p}} \circ - \otimes_R R_{\mathbf{p}}$, and $i_{\mathbf{p}} : R_{\mathbf{p}} - mod^{fg} \to R - mod^{fg}$ is the inclusion functor. So $ker'(- \otimes_R R_{\mathbf{p}})$ is a thick coreflective subcategory of $R - mod^{fg}$.

To show $ker'(- \otimes_R R_{\mathbf{p}})$ is a prime $\otimes$-ideal, let $M, N \in Ob R - mod^{fg}$, and suppose $M \otimes N \in ker'(- \otimes_R R_{\mathbf{p}})$. Recall that since M and N are finitely generated, $Supp(M \otimes N) = Supp(M) \cap Supp(N)$. Thus $M \otimes N \in ker'(- \otimes_R R_{\mathbf{p}})$ if and only if $Spec(R_{\mathbf{p}}) \cap Supp(M) \cap Supp(N) = \emptyset$, where we view $Spec(R_{\mathbf{p}})$ as a subset of $Spec(R)$. This happens if and only if for all primes $\mathbf{p}' \subseteq \mathbf{p}$, we have that $\mathbf{p}' \notin Supp(M) \cap Supp(N)$. Suppose $M, N \notin ker'(- \otimes_R R_{\mathbf{p}})$. Then there exist $\mathbf{p}', \mathbf{p}'' \in Spec(R)$ such that $M_{\mathbf{p}'} \neq 0$ and $N_{\mathbf{p}''} \neq 0$. This implies that $M_{\mathbf{p}} \neq 0$ and $N_{\mathbf{p}} \neq 0$. Thus $\mathbf{p} \in Spec(R_{\mathbf{p}}) \cap Supp(M) \cap Supp(N)$, which implies that $M \otimes N \notin ker'(- \otimes_R R_{\mathbf{p}})$. This gives a contradiction. Hence $ker'(- \otimes_R R_{\mathbf{p}})$ is a prime $\otimes$-ideal, and we have shown that $Spec(R) \subseteq \mathbf{Prime}(R - mod^{fg})$. □

If $x \in R$ (for the moment, let R be an arbitrary ring, not necessarily commutative), then we will denote by $ann(x) \in I_l R$ the annihilator of x in R, and if $M \in Ob R - mod$, we will denote by $Ann(M) \in I(R)$ the annihilator of M in R. Explicitly,

$$ann(x) = \{r \in R \mid rx = 0\}$$

and

$$Ann(M) = \{r \in R \mid rx = 0 \text{ for all } x \in M\}.$$

We can see that $ann(x) \in I_l R$, $Ann(M) \in I(R)$, and that $Ann(M) = \bigcap_{x \in M} ann(x)$. The following is a modification of Theorem 1 in Chapter 1.

Lemma 11. *Let R be a commutative ring. Then the map*

$$\begin{aligned} (TopFiltI(R))^{op} &\to T(R - mod^{fg}) \\ F &\mapsto \mathcal{T}_F := \langle M \in Ob R - mod^{fg} \mid Ann(M) \in F \rangle \end{aligned} \tag{3.1}$$

is an isomorphism of lattices. Moreover, restriction of this map to

$(RadFI(R))^{op}$ *induces an isomorphism* $(RadFI(R))^{op} \to Th^c(R - mod^{fg})$.

Proof. A straightforward modification of the proof of Lemma 16.2 in [5] will yield that the map

$$(TopFiltI(R))^{op} \to T(R - mod^{fg}),$$
$$F \mapsto \mathcal{T}_F := \langle M \in ObR - mod^{fg} \mid ann(x) \in F \text{ for all } x \in M\rangle$$

is an isomorphism of lattices. But we can see that if $x_1, ..., x_n \in M$ generate M, then for $y \in M$, we have $y = \sum_{i=1}^n r_i x_i$ for some $r_i \in R$. If $r \in \bigcap_{i=1}^n ann(x_i)$, then

$$ry = r\sum_{i=1}^n r_i x_i = \sum_{i=1}^n r_i r x_i = 0.$$

This is for any $y \in M$, so $r \in Ann(M)$, and thus $\bigcap_{i=1}^n ann(x_i) \subseteq Ann(M)$. Since $Ann(M) \subseteq ann(x)$ for all $x \in M$, we have $Ann(M) \subseteq \bigcap_{i=1}^n ann(x_i)$, thus $Ann(M) = \bigcap_{i=1}^n ann(x_i)$. Since $ann(x_i) \in F$ for all $1 \le i \le n$, and F is closed under finite intersections, we have that $Ann(M) \in F$. Thus $\mathcal{T}_F = \langle M \in ObR - mod^{fg} \mid Ann(M) \in F\rangle$. □

Reminder: $T(R - mod^{fg})$ is the collection of topologizing (not necessarily coreflective) subcategories of $R - mod^{fg}$.

Since Lemma 11 gives an isomorphism between $(TopFiltI(R))^{op}$ and $T(R - mod^{fg})$, we actually have an isomorphism between $T_c(R - mod)$ and $T(R - mod^{fg})$, that restrics to give an isomorphism between $Th_c(R - mod)$ and $Th_c(R - mod^{fg})$. Thus we have the

Corollary 2. *Let R be a commutative ring. Then*

$$\begin{aligned}\mathbf{Spec}^1(R - mod) &:= \{\mathcal{T} \in Th_c(R - mod) \mid \mathcal{T} \subsetneq \mathcal{T}^*\} \\ &\simeq \{\mathcal{T} \in Th_c(R - mod^{fg}) \mid \mathcal{T} \subsetneq \mathcal{T}^*\}.\end{aligned} \tag{3.2}$$

Note that in the first line of equation (2), the operator $(-)^*$ applied to $\mathcal{T} \in Th_c(R-mod)$ considers $\mathcal{T}$ as an element of the lattice $T_c(R-mod)$, while in the second line the operator $(-)^*$ considers $\mathcal{T} \in Th_c(R-mod^{fg})$ as an element of $T(R-mod^{fg})$. Invoking Chapter 1, Proposition 4, we now have that

$$Spec(R) \simeq \{\mathcal{T} \in Th_c(R-mod^{fg}) \mid \mathcal{T} \subsetneq \mathcal{T}^*\}.$$

Lemma 12. *For R a Noetherian commutative ring, we have*

$$\mathbf{Prime}(R-mod^{fg}) \subseteq \mathbf{Spec}^1(R-mod).$$

Proof. Recall that if $M \in ObR-mod^{fg}$, $[M]$ is the smallest topologizing subcategory of $R-mod^{fg}$ containing M. It is not difficult to see that by Lemma 11,

$$[M] = \big\langle L \in ObR-mod^{fg} \mid Ann(M) \subseteq Ann(L)\big\rangle.$$

From this we can see that $[M] = [R/Ann(M)]$. For $\mathbf{m}, \mathbf{n} \in I(R)$, we have by Theorem 2 that $[R/\mathbf{m}] \cap [R/\mathbf{n}] = [R/(\mathbf{m}+\mathbf{n})]$. By Exercise 9 in Chapter 4 Section 5 of [9], we have that $R/(\mathbf{m}+\mathbf{n}) \simeq R/\mathbf{m} \otimes R/\mathbf{n}$. Thus $[R/\mathbf{m}] \cap [R/\mathbf{n}] = [R/\mathbf{m} \otimes R/\mathbf{n}]$.

For $\mathcal{T} \in T(R-mod^{fg})$, denote by $F_{\mathcal{T}}$ the topologizing filter that corresponds to $\mathcal{T}$ under the isomorphism in Lemma 3.1. Then $\mathbf{m} \in F_{\mathcal{T}}$ if and only if $R/\mathbf{m} \in \mathcal{T}$, if and only if $[R/\mathbf{m}] \subseteq \mathcal{T}$.

Recall that for $\mathcal{T}, \mathcal{S} \in T(R-mod^{fg})$ the Gabriel product of $\mathcal{T}$ and $\mathcal{S}$ is

$$\begin{aligned}\mathcal{T} \bullet \mathcal{S} = &\big\langle M \in ObR-mod^{fg} \mid \exists N \in Ob\mathcal{T} \text{ and } L \in Ob\mathcal{S} \\ &\text{and and exact sequence } 0 \to N \to M \to L \to 0\big\rangle.\end{aligned}$$

We can see that since both $\mathcal{S}$ and $\mathcal{T}$ are topologizing, $\mathcal{T} \bullet \mathcal{S}$ contains both $\mathcal{S}$ and $\mathcal{T}$.

Let $\mathcal{T}, \mathcal{T}' \in T(R - mod^{fg})$ with $\mathcal{T} \subsetneq \mathcal{T}'$. Then there exists $M \in Ob\mathcal{T}' \setminus Ob\mathcal{T}$, which means that $[M] \nsubseteq \mathcal{T}$. Thus $\mathcal{T} \subsetneq [M] \bullet \mathcal{T} \subseteq \mathcal{T}'$. Thus

$$\begin{aligned}\mathcal{T} &\subseteq \bigcap \left\{[M] \bullet \mathcal{T} \mid M \in ObR - mod^{fg} \setminus Ob\mathcal{T}\right\} \\ &\subseteq \bigcap \left\{\mathcal{T}' \mid \mathcal{T} \subsetneq \mathcal{T}' \in T(R - mod^{fg})\right\} = \mathcal{T}^*.\end{aligned}$$

This implies that $\mathcal{T}^* = \bigcap \left\{[M] \bullet \mathcal{T} \mid M \in R - mod^{fg} \setminus \mathcal{T}\right\}$. Since $M \notin Ob\mathcal{T}$ if and only if $Ann(M) \notin F_{\mathcal{T}}$, we can see that $\mathcal{T}^*$ can be written as

$$\mathcal{T}^* = \bigcap_{\mathbf{m} \notin F_{\mathcal{T}}} ([R/\mathbf{m}] \bullet \mathcal{T}).$$

By Lemma C1.2.1 in the Complementary Facts to Chapter 2 of [15],

$$\bigcap_{\mathbf{m} \notin F_{\mathcal{T}}} ([R/\mathbf{m}] \bullet \mathcal{T}) = \left(\bigcap_{\mathbf{m} \notin F_{\mathcal{T}}} [R/\mathbf{m}]\right) \bullet \mathcal{T}.$$

By Theorem 2, we have that for any collection $\Lambda \subseteq I(R)$, we have that

$$\bigcap_{\mathbf{m} \in \Lambda} [R/\mathbf{m}] = [R/sup(\mathbf{m} \mid \mathbf{m} \in \Lambda)].$$

Since R is Noetherian, there are $\mathbf{m}_1, ..., \mathbf{m}_n \in F_{\mathcal{T}}$ such that $\mathbf{m}_1 + \cdots \mathbf{m}_n = sup(\mathbf{m} \mid \mathbf{m} \notin F_{\mathcal{T}})$. By Exercise 9 in Chapter 4 Section 5 of [9], we have that $R/(\mathbf{m}_1 + \cdots + \mathbf{m}_n) \simeq R/\mathbf{m}_1 \otimes \cdots \otimes R/\mathbf{m}_n$. Thus

$$\mathcal{T}^* = \bigcap_{i=1}^{n} ([R/\mathbf{m}_i] \bullet \mathcal{T}) = [R/\mathbf{m}_1 + \cdots \mathbf{m}_n] \bullet \mathcal{T} = [R/\mathbf{m}_1 \otimes \cdots \otimes R/\mathbf{m}_n] \bullet \mathcal{T}.$$

Now let $\mathcal{T} \in \mathbf{Prime}(R - mod^{fg})$. Let $\mathbf{m}_1, ..., \mathbf{m}_n \notin F_{\mathcal{T}}$ be as above such that $\mathcal{T}^* =$

$[R/\mathbf{m}_1 \otimes \cdots \otimes R/\mathbf{m}_n] \bullet \mathcal{T}$. Since $\mathbf{m}_i \notin F_{\mathcal{T}}$, we have that $R/\mathbf{m}_i \notin \mathcal{T}$ for all $1 \leq i \leq n$. Since $\mathcal{T}$ is a prime $\otimes$-ideal, $R/\mathbf{m}_1 \otimes \cdots \otimes R/\mathbf{m}_n \notin \mathcal{T}$. Thus $\mathcal{T} \subsetneq \mathcal{T}^*$. Then appealing to Corollary 2, we conclude that $\mathbf{Prime}(R - mod^{fg}) \subseteq \mathbf{Spec}^1(R - mod)$. □

Corollary 3. *For R a Noetherian commutative ring, we have bijections*

$$Spec(R) \simeq \mathbf{Spec}^0(R - mod) \simeq \mathbf{Spec}^1(R - mod) \simeq \mathbf{Prime}(R - mod^{fg}).$$

Proof. We have seen that the first bijection follows from the definition of $\mathbf{Spec}^0(R - mod)$, and the second bijection follows from Proposition 8.7.2 in [14]. The last bijection follows from Lemma 12 and Lemma 10. □

3.3 Left Noetherian Polynomial Identity Rings

In this section, we will consider a left Noetherian polynomial identity ring R.

Lemma 13. *Let R be a left Noetherian PI ring. Then the map*

$$\begin{aligned} &(TopFiltI_lR)^{op} \to T(R - mod^{fg}) \\ &F \mapsto \mathcal{T}_F := \langle M \in ObR - mod^{fg} \mid Ann(M) \in F \rangle \end{aligned}$$

is an isomorphism of lattices. Moreover, restriction of this map to $(RadFI_lR)^{op}$ induces an isomorphism $(RadFI_lR)^{op} \to Th_c(R - mod^{fg})$.

Proof. A straightforward modification of the proof of Lemma 16.2 in [5] will yield that the map

$$\begin{aligned} &(TopFiltR)^{op} \to T(R - mod^{fg}) \\ &F \mapsto \mathcal{T}_F := \langle M \in ObR - mod^{fg} \mid ann(x) \in F \text{ for all } x \in M \rangle \end{aligned}$$

is an isomorphism of lattices, and that the restriction

$$(RadFI_lR)^{op} \to Th_c(R-mod^{fg})$$

is an isomorphism as well.

Let $F \in TopFiltI_lR$, and $M \in Ob\mathcal{T}_F$. Let $x_1, ..., x_n$ generate M. Since R is a PI ring, for every $\mathbf{m} \in I_lR$, there exists a finite subset $X \subseteq R$ such that $(\mathbf{m} : R) = (\mathbf{m} : X)$. Thus for the ideals $ann(y_i)$, there exist finite sets $X_i \subseteq R$ such that $(ann(y_i) : R) = (ann(y_i) : X_i)$. Then

$$Ann(M) = \bigcap_{i=1}^{n}(ann(y_i) : X_i) = \bigcap_{i=1}^{n}\bigcap_{j=1}^{k_i}(ann(y_i) : x_j)$$

where k_i is the size of the finite set X_i. Since $ann(y_i) \in F$ for all $1 \leq i \leq n$, we have that $(ann(y_i) : x_j) \in F$. Further, since F is closed under finite intersection, we have that $Ann(M) \in F$. Since for every $z \in R$, we have that $Ann(M) \subseteq ann(z)$, we can see that in fact $\mathcal{T}_F = \langle M \in ObR - mod \mid Ann(M) \in F\rangle$. □

Corollary 4. *Let R be a left Noetherian PI ring. Then*

$$\begin{aligned}\mathbf{Spec}^1(R-mod) &:= \{\mathcal{T} \in Th_c(R-mod) \mid \mathcal{T} \subsetneq \mathcal{T}^*\} \\ &\simeq \{\mathcal{T} \in Th_c(R-mod^{fg}) \mid \mathcal{T} \subsetneq \mathcal{T}^*\}.\end{aligned}$$

Theorem 10. *Let R be a left Noetherian PI ring such that $R-mod^{fg}$ is a monoidal Abelian category. Then*

$$Spec(R) \simeq Spec_l(R) \simeq \mathbf{Spec}^1(R-mod) \simeq \mathbf{Prime}(R-mod^{fg}).$$

Proof. The first two equivalences have been shown in Theorem 4. For the final equality, we will actually show $\mathbf{Prime}(R-mod^{fg}) \simeq \{\mathcal{T} \in Th_c(R-mod^{fg}) \mid \mathcal{T} \subsetneq \mathcal{T}^*\}$, and then we will invoke 4 to conclude that $\mathbf{Prime}(R-mod^{fg}) \simeq \mathbf{Spec}^1(R-mod)$. Let $\mathcal{T} \in Th_c(R-mod^{fg})$ such that $\mathcal{T} \subsetneq \mathcal{T}^*$, and $M, N \in ObR-mod^{fg} \setminus Ob\mathcal{T}$, but $M \otimes N \in Ob\mathcal{T}$. Note that by Lemma 13, $[M] = [R/Ann(M)]$, $[N] = [R/Ann(N)]$, and

$$\begin{aligned}[M] \cap [N] &= [R/Ann(M)] \cap [R/Ann(N)] = [R/(Ann(M) + Ann(N))] \\ &= [R/Ann(M) \otimes R/Ann(N)].\end{aligned}$$

Notice that we must have that $R/Ann(M), R/Ann(N) \in ObR-mod^{fg} \setminus Ob\mathcal{T}$, and that $R/Ann(M) \otimes R/Ann(N) \in \mathcal{T}$. Indeed, if we had $R/Ann(M) \in Ob\mathcal{T}$, then because $\mathcal{T}$ is a topologizing subcategory of $R-mod^{fg}$, and $[R/Ann(M)] = [M]$ is the smallest subcategory containing $R/Ann(M)$, as well as the smallest topologizing subcategory containing M, we must have $[R/Ann(M)] = [M] \subseteq \mathcal{T}$. Thus $M \in Ob\mathcal{T}$, which would be a contradiction. Similarly we can see $R/Ann(N) \in ObR-mod^{fg} \setminus Ob\mathcal{T}$, and $R/Ann(M) \otimes R/Ann(N) \in Ob\mathcal{T}$.

Then $\mathcal{T}$ is properly contained in both $[R/Ann(M)] \bullet \mathcal{T}$ and $[R/Ann(N)] \bullet \mathcal{T}$. Then by Lemma C1.2.1 in the Complementary Facts to Chapter 2 of [15],

$$\big([R/Ann(M)] \bullet \mathcal{T}\big) \bigcap \big([R/Ann(N)] \bullet \mathcal{T}\big) = \big([R/Ann(M)] \cap [R/Ann(N)]\big) \bullet \mathcal{T}.$$

We have that $\mathcal{T} \subsetneq \mathcal{T}^* \subsetneq \big([R/Ann(M)] \cap [R/Ann(N)]\big) \bullet \mathcal{T}$. On the other hand, since $M \otimes N \in \mathcal{T}$, we also have that $\big([R/Ann(M)] \cap [R/Ann(N)]\big) \bullet \mathcal{T} = [R/Ann(M) \otimes R/Ann(N)] \bullet \mathcal{T} = \mathcal{T}$, since we have already seen that we must have $R/Ann(M) \otimes R/Ann(N) \in Ob\mathcal{T}$. This contradicts our assumption that $\mathcal{T} \subsetneq \mathcal{T}^*$. Thus we must have that $M \otimes N \in ObR-mod^{fg} \setminus Ob\mathcal{T}$, and hence $\mathcal{T} \in \mathbf{Prime}(R-mod^{fg})$.

The reverse inclusion folows from an argument almost identical to the proof of Lemma 12, and the conclusion follows. $\square$

3.4 Left Artinian Rings

In this section, let R be a left Artinian ring, possibly noncommutative, and that $R-mod$ is closed braided monoidal. We look back to Lemma 11. We have the following

Lemma 14. *Let R be a left Artinian ring. The map*

$$(TopFiltI_lR)^{op} \to T(R-mod^{fg}),$$
$$F \mapsto \mathcal{T}_F := \langle M \in ObR-mod^{fg} \mid Ann(M) \in F \rangle$$

is an isomorphism of lattices. Moreover, restriction of this map to $(RadFI_lR)^{op}$ induces an isomorphism $(RadFI_lR)^{op} \to Th^c(R-mod^{fg})$.

Proof. Again, a straightforward modification of the proof of Lemma 16.2 in [5] will yield

$$(TopFiltI_lR)^{op} \to T(R-mod^{fg}),$$
$$F \mapsto \mathcal{T}_F := \langle M \in ObR-mod^{fg} \mid ann(x) \in F \text{ for all } x \in M \rangle$$

is an isomorphism of lattices. We know $Ann(M) = \bigcap_{x\in M} ann(x)$. Order the elements of M by some ordinal α. Then we have a descending chain of left ideals

$$ann(x_0) \supseteq ann(x_0) \cap ann(x_1) \supseteq ... \supseteq \bigcap_{\lambda\in\beta<\alpha} ann(x_\lambda) \supseteq ... = Ann(M).$$

Since R is left Artinian, we know that this chain stabilizes in finitely many steps, so that there exist $x_1, ..., x_n \in M$ such that $Ann(M) = \bigcap_{i=1}^{n} ann(x_i)$. Thus,

$$\mathcal{T}_F = \langle M \in ObR-mod^{fg} \mid Ann(M) \in F \rangle.$$

$\square$

With this lemma in hand, we can prove an analog of Theorem 10.

Theorem 11. *Let R be a left Artinian ring such that $R - mod^{fg}$ is a monoidal Abelian category. Then*

$$\mathbf{Spec}^1(R - mod) \simeq \mathbf{Prime}(R - mod^{fg}).$$

Proof. The proof is almost identical to Theorem 10. $\square$

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