Calculation of Electromagnetic Wave Scattering by a Small Impedance Particle of an Arbitrary Shape

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Abstract

Scattering of electromagnetic (EM) waves by small ($ka \ll 1$) impedance particle D of an arbitrary shape, embedded in a homogeneous medium, is studied. Analytic, closed form, formula for the scattered field is derived. The scattered field is of the order $O(a^{2-\kappa})$, where $\kappa \in [0,1)$ is a number. This field is much larger than in the case of Rayleigh-type scattering. The numerical results demonstrate a wide range of applicability of the analytic formula for the scattered field. Comparison with Mie-type solution is carried out for various boundary impedances and radii of the particle.

Key words: electromagnetic waves; wave scattering by small body; boundary impedance; computational testing.

1 Introduction

In this paper a theory of electromagnetic (EM) wave scattering by a small impedance particle (body) D of an arbitrary shape is developed. The particle is embedded in a homogeneous medium which is described by the constant permittivity $\epsilon_0 > 0$ and permeability $\mu_0 > 0$. The smallness of a particle means that $ka \ll 1$, where $a = 0.5 diamD$ is the characteristic dimension of a particle, $k = \omega(\epsilon_0\mu_0)^{1/2}$ is the wave number in the medium exterior to the particle. The approach can be easily extended to the case of many particles. Although scattering of EM waves by small bodies has a long history, going back to Rayleigh (1871), see [2], [17], the results of this

paper are new and useful in applications because light scattering by colloidal particles in a solution, and light scattering by small dust particles in the air are examples of the problems to which our theory is applicable. The Mie theory deals with the EM wave scattering by a sphere, not necessarily small, and gives the solution to the scattering problem in terms of the series in spherical harmonics. If the sphere is small, $ka \ll 1$, then the first term in the Mie series yields the main part of the solution. Our theory is applicable only to small particles. They can be of an arbitrary shape. The solution to the scattering problem for one small particle of an arbitrary shape is given analytically, in closed form. Wave scattering problem in the case of one body can be studied theoretically only in the limiting cases of a small body, $ka \ll 1$, or a large body, $ka \gg 1$. In the latter case the geometrical optics is applicable. This paper deals with the case $ka \ll 1$. Rayleigh (1871) understood that the scattering by a small body is given mainly by the dipole radiation. For a small body of an arbitrary shape this dipole radiation is determined by the polarization moment, which is defined by the polarizability tensor. For homogeneous bodies of arbitrary shape analytic formulas, which allow one to calculate this tensor with any desired accuracy, were derived in [17],[18]. These bodies were assumed dielectric or conducting in [17]. Under the Rayleigh assumption the scattered field is proportional to a^3, that is, to the volume of a small body. The physically novel feature of our theory is a conclusion that for small *impedance* particle with the impedance $\zeta = ha^{-\kappa}$, where h is a given constant, Re$h \geq 0$, and $\kappa \in [0,1)$ is a given number, the scattered field is proportional to $a^{2-\kappa}$. Therefore, *the scattered field in this case is much larger than in the Rayleigh case*, since $a^3 \ll a^{2-\kappa}$ if $a \ll 1$ and $\kappa \geq 0$. This physically interesting conclusion may be of practical importance.

In this paper wave scattering by one small impedance particle is studied. Besides high intrinsic interest in this problem, the theory we develop allows one to generalize it to the case of many particles and to obtain some physically interesting conclusions about the changes of the material properties of the medium in which many small particles are embedded. These results will be presented in another paper and will be used for developing a method for creating materials with a desired refraction coefficient by embedding many small impedance particles into a given material, similarly to the results in a series of papers [4]-[15] for scalar wave scattering. The novel physical idea in this paper is to reduce solving the scattering problem to finding some constant pseudovector Q (see formula (18)), rather than a function J (see formula (10)) on the surface of the scatterer. The quantity Q is a pseudovector which somewhat analogous to the total charge on the surface of

the perfect conductor with the shape of D, while the function J is somewhat analogous to the surface charge density in electrostatics. We assume for simplicity that the impedance ζ (see formula (5)) is a constant given in (16). A similar assumption appeared in paper [5], where scalar wave scattering theory was developed. The results of the theory in [5] was a recipe for creating materials with a desired refraction coefficient in acoustics (see [13], [14], [15]). The boundary impedance (16) grows to infinity as $a \to 0$. The impedance boundary condition is widely applicable in physics and does not require that the body be small or large. One can pass to the limit in the equation for the effective (self-consistent) field in the medium, obtained by embedding many small impedance particles into a given medium. Such a theory is briefly summarized in paper [14] for the scalar wave scattering, where the equation for the limiting field in the medium is derived. The aim of this paper is to develop a similar theory for the EM wave scattering by a small impedance particle embedded in a given material and to carry out the numerical calculations showing the possibility to use the developed theory in the wide range of the parameters, such as the radius a of the particle, its boundary impedance ζ, the distance d between neighboring particles, and the wavelength.

For EM wave scattering by one small body D of an arbitrary shape with an impedance boundary condition the following analytic formula for the electromagnetic field in the region $r := |x| \gg a$, is derived:

$$E(x) = E_0(x) + [\nabla \frac{e^{ikr}}{4\pi r}, Q], \qquad r \gg a, \tag{1}$$

where $g(x, y) := \frac{e^{ikr}}{4\pi r}, \quad r = |x - y|$, E_0 is the incident field, which satisfies Maxwell's equations in the absence of the scatterer D, $[A, B] = A \times B$ is the cross product of two vectors, $(Q, e_j) = Q \cdot e_j$ is the dot product, $\{e_j\}_{j=1}^3$ is an orthonormal basis in $\mathbb{R}^3$,

$$Q_j := (Q, e_j) = -\frac{\zeta |S|}{i\omega\mu_0} \Xi_{jp} (\nabla \times E_0(O))_p, \quad \Xi := (I + \alpha)\tau, \tag{2}$$

over the repeated index p summation is understood from 1 to 3, ζ is the boundary impedance, $|S|$ is the surface area of the particle, the matrix Ξ_{jp} is defined by the formula

$$\Xi_{jp} := (I + \alpha)\Big(\delta_{jp} - |S|^{-1} \int_S N_j(s)N_p(s)ds\Big) := (I + \alpha)\tau_{jp}, \tag{3}$$

where $N_j(s)$ is the $j-$th component of the unit normal $N(s)$ to the surface

4

S at a point $s \in S$, pointing *out of D*, $\tau := I - b$, b is a matrix:

$$b_{jp} := |S|^{-1} \int_S N_j(s) N_p(s) ds,$$

$k = \omega(\epsilon_0 \mu_0)^{1/2}$ is the wave number, $O \in D$ is the origin, I is the identity matrix, and α is a matrix, $I + \alpha := (I + \beta)^{-1}$, where the matrix β is defined in (29).

For small particles, $a \to 0$, one may set matrix $\alpha = 0$ and get $\Xi = \tau$.

By S^2 the unit sphere in $\mathbb{R}^3$ is denoted. The boundary S of the small body D is assumed smooth. It is sufficient to assume that in the local coordinates the equation of S is $x_3 = \phi(x_1, x_2)$, where the function ϕ has first derivatives satisfying a Hölder condition.

The main analytical results of this paper are formulas (1)-(3) and equations (32) and (33).

The scattering problem by one small body is formulated and studied in Section 2. The numerical results demonstrating the applicability of the proposed theory to computation of the electromagnetic field in a wide range of problem's parameters are given in Section 3. In Section 4 the conclusions are formulated.

In this paper we do not solve the boundary integral equation to which the scattering problem can be reduced in a standard approach, but find an asymptotically exact closed form expression for the pseudovector Q, which defines the behavior of the scattered field at the distances $d \gg a$.

These distances d can be very small if a is sufficiently small, and d can be much less than the wavelength $\lambda = \frac{2\pi}{k}$.

2 EM wave scattering by one small impedance particle

Let us use in this Section the following notations: D is a small body, $D' := \mathbb{R}^3 \setminus D$, $k > 0$ is a wave number, $ka \ll 1$, $a = 0.5 diamD$, $k = \frac{2\pi}{\lambda}$, λ is the wavelength of the incident EM wave, $k^2 = \omega^2 \epsilon_0 \mu_0$, where ω is the frequency and ϵ_0, μ_0 are constant permittivity and permeability of the medium. Our arguments remain valid if one assumes that the medium has a constant conductivity $\sigma_0 \geq 0$. In this case ϵ_0 is replaced by $\epsilon_0 + i\frac{\sigma_0}{\omega}$. Denote by S the boundary of D, by $|S|$ its surface area, by V the volume of D, by $[E, H] = E \times H$ the cross product of two vectors, and by $(E, H) = E \cdot H$ the dot product of two vectors, N is the unit normal to S *pointing out of D*, ζ is the boundary impedance of the particle.

Let D be embedded in a homogeneous medium with constant parameters ϵ_0, μ_0. Electromagnetic (EM) wave scattering problem consists of finding vectors E and H satisfying Maxwell's equations:

$$\nabla \times E = i\omega\mu_0 H, \nabla \times H = -i\omega\epsilon_0 E \text{ in } D' := \mathbb{R}^3 \setminus D, \qquad (4)$$

the impedance boundary condition:

$$[N, [E, N]] = \zeta[N, H] \quad \text{on } S \qquad (5)$$

and the radiation condition:

$$E = E_0 + v_E, \quad H = H_0 + v_H, \qquad (6)$$

where E_0, H_0 are the incident fields satisfying equations (4) in all of $\mathbb{R}^3$, $v_E := v$ and v_H are the scattered fields. In the literature, for example in [2], the impedance boundary condition is written sometimes as $E^t = \zeta[H^t, N]$, where N is the unit normal on S pointed *into* D. Since our N is pointed *out of* D, our impedance boundary condition (5) is the same as in [2].

One often assumes that the incident wave is a plane wave, i.e., $E_0 = \mathcal{E}e^{ik\alpha\cdot x}$, $\mathcal{E}$ is a constant vector, $\alpha \in S^2$ is a unit vector, S^2 is the unit sphere in $\mathbb{R}^3$, $\alpha \cdot \mathcal{E} = 0$, v_E and v_H satisfy the Sommerfeld radiation condition: $r(\frac{\partial v}{\partial r} - ikv) = o(1)$ as $r := |x| \to \infty$, and, consequently, $[r^0, E] = H + o(r^{-1})$ and $[r^0, H] = -E + o(r^{-1})$ as $r := |x| \to \infty$, where $r^0 := x/r$.

It is assumed in this paper that the impedance ζ is a constant, $Re\,\zeta \geq 0$.

This assumption guarantees the uniqueness of the solution to Maxwell's equations satisfying the radiation condition. For completeness a proof of the uniqueness result is given in Lemma 1. The tangential component of E on S, E^t, is defined as:

$$E^t = E - N(E, N) = [N, [E, N]]. \qquad (7)$$

This definition differs from the one used often in the literature, namely, from the definition $E^t = [N, E]$. Our definition (7) corresponds to the geometrical meaning of the tangential component of E and, therefore, should be used. The impedance boundary condition is written in [2] as $E^t = \zeta[H^t, N_i]$, where ζ is the boundary impedance and N_i is the unit normal to S pointing *into* D. In our paper N is the unit normal pointing *out* of D. Therefore, the impedance boundary condition in our paper is written as in equation (5). If one uses definition (7), then this condition reduces to (5), because $[[N, [H, N]], N] = [H, N]$. The assumption $Re\zeta \geq 0$ is physically justified by

the fact that this assumption guarantees the uniqueness of the solution to the boundary problem (4)-(6).

Lemma 1. *Problem* (4)-(6) *has at most one solution.*

For a proof of Lemma 1, we refer the reader to [4]. Let us note that problem (4)-(6) is equivalent to the problem

$$\nabla \times \nabla \times E = k^2 E \text{ in } D', \quad H = \frac{\nabla \times E}{i\omega\mu_0}, \tag{8}$$

$$[N, [E, N]] = \frac{\zeta}{i\omega\mu_0}[N, \nabla \times E] \text{ on } S, \tag{9}$$

together with the radiation condition (6). Thus, we have reduced the scattering problem to finding one vector $E(x)$. If $E(x)$ is found, then $H = \frac{\nabla \times E}{i\omega\mu_0}$, and the pair E and H solves Maxwell's equations, satisfies the impedance boundary condition and the radiation condition (6).

Let us look for E of the form

$$E = E_0 + \nabla \times \int_S g(x,t)J(t)dt, \quad g(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \tag{10}$$

where E_0 is the incident field, which satisfies Maxwell's equations in the absence of the scatterer D, t is a point on the surface S, $t \in S$, dt is an element of the area of S, and $J(t)$ is an unknown pseudovector-function on S, which is tangential to S, i.e., $N(t) \cdot J(t) = 0$, where $N(t)$ is the unit normal to S at the point $t \in S$. That $J = J(t)$ is a pseudovector follows from the fact that E is a vector and $\nabla \times (gJ)$ is a vector only if J is a pseudovector, because g is a scalar.

It is assumed that J is a smooth function on S, for example, $J \in C^2(S)$.

The right-hand side of (10) solves equation (8) in D for any continuous $J(t)$, because E_0 solves (8) and

$$\nabla \times \nabla \times \nabla \times \int_S g(x,t)J(t)dt =$$

$$graddiv\nabla \times \int_S g(x,t)J(t)dt-$$

$$\nabla^2\nabla \times \int_S g(x,t)J(t)dt = \tag{11}$$

$$k^2\nabla \times \int_S g(x,t)J(t)dt, \quad x \in D'.$$

Here we have used the known identity $divcurlE = 0$, valid for any smooth vector field E, and the known formula

$$-\nabla^2 g(x,y) = k^2 g(x,y) + \delta(x-y). \tag{12}$$

The integral $\int_S g(x,t)J(t)dt$ satisfies the radiation condition. Thus, formula (10) solves problem (8), (9), (6), if $J(t)$ is chosen so that boundary condition (9) is satisfied.

Let $O \in D$ be a point inside D. The following known formula (see, for example, [3]) is useful:

$$[N, \nabla \times \int_S g(x,t)J(t)dt]_{\mp} = \int_S [N_s, [\nabla_x g(x,t)|_{x=s},$$
$$J(t)]]dt \pm \frac{J(s)}{2}, \tag{13}$$

where the $\pm$ signs denote the limiting values of the left-hand side of (13) as $x \to s$ along the normal to S at the point s from D, respectively, from D'. To derive an integral equation for $J = J(t)$, substitute $E(x)$ from (10) into impedance boundary condition (9), and get

$$0 = f + [N, [\nabla \times \int_S g(s,t)J(t)dt, N]] -$$
$$\frac{\zeta}{i\omega\mu_0}[N, \nabla \times \nabla \times \int_S g(s,t)J(t)dt], \tag{14}$$

where

$$f := [N, [E_0(s), N]] - \frac{\zeta}{i\omega\mu_0}[N, \nabla \times E_0]. \tag{15}$$

We assume that

$$\zeta = \frac{h}{a^\kappa}, \tag{16}$$

where Re $h \geq 0$. and $\kappa \in [0,1)$ is a constant.

The requirement $Re\zeta \geq 0$ is the only physical requirement that is imposed on the boundary impedance. Our assumption (16) satisfies this requirement. This assumption allows us to derive the formulas (1) and (2) for the scattered field. Formulas (1)-(2) are new and of physical importance because they yield the value for the scattered field that is $O(a^{2-\kappa})$, which is *much larger* than the usual Rayleigh-type estimate $O(a^3)$ for the field scattered by a small body of characteristic size a.

Let us write (10) as

$$E(x) = E_0(x) + [\nabla_x g(x,O), Q] + \nabla \times \int_S \Big(g(x,t) -$$
$$g(x,O) \Big) J(t)dt, \tag{17}$$

where
$$Q := \int_S J(t)dt. \tag{18}$$

The central physical idea of the theory, developed in this paper, is: the third term in the right-hand side of (17) is negligible compared with the second term if $ka \ll 1$. Consequently, the scattering problem is solved if Q is found. The traditional approach requires finding an unknown function $J(t)$, which is usually found numerically by the boundary integral equations (BIE) method. The reason for the third term in the right-hand side of (17) to be negligible compared with the second one, is explained below. In the estimates, used in this explanation, the smallness of the body is used essentially: even if one is in the far zone, that is, $\frac{a}{d} \ll 1$, one cannot conclude that estimate (21) (see below) holds unless one assumes that $ka \ll 1$. Thus, *the third term in* (17) *cannot be neglected in the far zone if the condition $ka \ll 1$ does not hold.*

We prove below that
$$Q = O(a^{2-\kappa}).$$

To prove that the third term in the right-hand side of (17) is negligible compared with the second one, let us establish several estimates valid if $a \to 0$ and $d := |x - O| \gg a$. Under these assumptions one has

$$j_1 := |[\nabla_x g(x, O), Q]| \leq O\left(\max\left\{\frac{1}{d^2}, \frac{k}{d}\right\}\right) O(a^{2-\kappa}), \tag{19}$$

$$\begin{aligned} j_2 := |\nabla \times \int_S (g(x, t) - g(x, O))\sigma(t)dt| \\ \leq aO\left(\max\left\{\frac{1}{d^3}, \frac{k^2}{d}\right\}\right) O(a^{2-\kappa}), \end{aligned} \tag{20}$$

and
$$\left|\frac{j_2}{j_1}\right| = O\left(\max\left\{\frac{a}{d}, ka\right\}\right) \to 0, \quad \frac{a}{d} = o(1), a \to 0. \tag{21}$$

These estimates show that one may neglect the third term in (17)*, and write*

$$E(x) = E_0(x) + [\nabla_x g(x, O), Q]. \tag{22}$$

The error of this formula tends to zero as $a \to 0$ under our assumptions.

Note that the inequality $|x| \gg ka^2$ is satisfied even for $|x| = O(a)$ if $ka \ll 1$. Thus, formula (22) is applicable in a wide region.

Let us estimate Q asymptotically, as $a \to 0$.

Take the cross product of (14) with N and integrate the resulting equation over S to get
$$I_0 + I_1 + I_2 = 0,$$
where I_0 is defined in formula (23) (see below), I_1 is defined in (26), and
$$I_2 := -\int_S \frac{\zeta}{i\omega\mu_0}[N, [N, \nabla \times \nabla \times \int_S g(s,t)J(t)dt]]ds.$$

Let us estimate the order of I_0 as $a \to 0$. One has
$$I_0 = \int_S \left([N, E_0] - \frac{\zeta}{i\omega\mu_0}[N, [N, \nabla \times E_0]]\right)ds. \tag{23}$$

In what follows we keep only the main terms as $a \to 0$, and denote by the sign $\simeq$ the terms equivalent up to the terms of higher order of smallness as $a \to 0$.

One has
$$I_{00} := \int_S [N, E_0]ds = \int_D \nabla \times E_0 dx \simeq \nabla \times E_0(O)V = O(a^3),$$

where dx is the element of the volume, $O \in D$ is a point chosen as the origin, and V is the volume of D.

Denoting by $|S|$ the surface area of S, one obtains
$$I_{01} := -\int_S \frac{\zeta}{i\omega\mu_0}[N, [N, \nabla \times E_0]]ds = \frac{\zeta|S|}{i\omega\mu_0}\tau\nabla \times E_0, \tag{24}$$

where
$$\tau := I - b, \quad b_{mj} := \frac{1}{|S|}\int_S N_m(t)N_j(t)dt, \tag{25}$$

$b = (b_{mj})$ is a matrix which depends on the shape of S, and $I := \delta_{mj}$ is the unit matrix. Since $\zeta = O(a^{-\kappa})$ and $|S| = O(a^2)$, one concludes that $I_{01} = O(a^{2-\kappa})$. Consequently, $|I_{00}| \ll |I_{01}|$, because $I_{00} = O(a^3)$ as $a \to 0$. Thus,
$$I_0 \simeq I_{01} = O(a^{2-\kappa}), \quad a \to 0.$$

Let us consider I_1:
$$I_1 = \int_S \left(\int_S [N, [\nabla g(s,t), J(t)]]dt + \frac{J(s)}{2}\right)ds := I_{11} + \frac{Q}{2}. \tag{26}$$

One has
$$I_{11} = \int_S ds \int_S dt \left(\nabla g(s,t)N(s) \cdot J(t) - J(t)\frac{\partial g(s,t)}{\partial N(s)}\right).$$

10

It is well known that $\int_S \frac{\partial g_0(s,t)}{\partial N(s)} ds = -\frac{1}{2}$, where $g_0 := \frac{1}{4\pi|s-t|}$. Since

$$g(s,t) - g_0(s,t) = \frac{ik}{4\pi} + O(|s-t|), \quad |s-t| \to 0,$$

and $|s - t| = O(a)$, one concludes that if $a \to 0$ then

$$\int_S \frac{\partial g(s,t)}{\partial N(s)} ds \simeq -\frac{1}{2}, \tag{27}$$

and

$$\int_S dt \int_S ds \nabla g(s,t) N(s) \cdot J(t) := O(a^3), \tag{28}$$

because $|N(s) \cdot J(t)| = O(|s-t|)$ as $|s-t| \to 0$, $\nabla g = O(\frac{1}{|s-t|^2})$, and

$$\int_S dt \int_S ds |s-t|^{-1} = O(a^3). \tag{29}$$

Therefore,

$$I_1 \simeq Q. \tag{30}$$

Matrix b for spheres is calculated at the end of Section 2. The estimate $N(s) \cdot J(t) = O(|s-t|)$ as $|s-t| \to 0$ holds because J is tangential to S and this implies the estimate $\cos \theta(s,t) = O(|s-t|)$, where $\theta(s,t)$ is the angle between $N(s)$ and $J(t)$.

Let us show that $I_2 = O(a^{4-\kappa})$ and therefore I_2 is negligible compared with I_0 as $a \to 0$. If this is done, then the equation for Q is

$$Q = -\frac{\zeta|S|}{i\omega\mu_0} \tau \nabla \times E_0. \tag{31}$$

From (31) it follows that

$$E(x) = E_0(x) - \frac{\zeta|S|}{i\omega\mu_0}[\nabla_x g(x,O), \tau \nabla \times E_0(O)]. \tag{32}$$

This equation is our first main result which gives an analytic formula for the solution of the EM wave scattering problem by a small body of an arbitrary shape, on the boundary of which an impedance boundary condition holds.

In the far zone $r := |x| \to \infty$ one has $\nabla_x g(x,O) = ikg(x,O)x^0 + O(r^{-2})$, where $x^0 := x/r$ is a unit vector in the direction of x. Consequently, for $r \to \infty$ one can rewrite formula (32) as

$$E(x) = E_0(x) - \zeta|S|\left(\frac{\epsilon_0}{\mu_0}\right)^{1/2}\frac{e^{ikr}}{r}[x^0, \tau \nabla \times E_0(O)]. \tag{33}$$

11

This field is orthogonal to the radius-vector x in the far zone as it should be.

Let us show that the term I_2 is negligible as $a \to 0$. Remember that $curlcurl = graddiv - \nabla^2$ and $-\nabla^2 g(x,t) = k^2 g(x,t) + \delta(x-y)$. Therefore,

$$-\nabla^2 \int_S g(x,t)J(t)dt = k^2 \int_S g(x,t)J(t)dt.$$

Consequently,

$$-i\omega\mu_0 I_2 \simeq \zeta \int_S ds[N,[N, graddiv \int_S g(x,t)J(t)dt]]|_{x\to s} := I_{21}.$$

Since the function $J(t)$ is assumed smooth, one has

$$div \int_S g(x,t)J(t)dt = \int_S g(x,t)div J(t)dt, \quad div J = \frac{\partial J_m}{\partial t_m},$$

summation is understood here (and below) over the repeated indices.

Furthermore,

$$grad \int_S g(x,t)\frac{\partial J_m}{\partial t_m}dt = e_p \int_S g(x,t)\frac{\partial^2 J_m}{\partial t_p \partial t_m}dt,$$

where the relation $\frac{\partial g(x,t)}{\partial x_p} = -\frac{\partial g(x,t)}{\partial t_p}$ was used, and an integration by parts with respect to t_p has been done over the closed surface S. Therefore

$$I_2 \leq c|\zeta| \int_S ds \int_S |g(s,t)| = O(a^{4-\kappa}) \ll I_0.$$

The constant $c > 0$ here is a bound on the second derivatives of J on S.

Example of calculation of matrix b. Let us calculate b for a sphere of radius a centered at the origin. One has:

$$b_{jm} = \frac{1}{|S|} \int_S N_j(t)N_m(t)dt = \frac{1}{3}\delta_{jm}.$$

Note that $|S| = 4\pi a^2$, $dt = a^2 \sin\theta d\theta d\phi$, and $N_m(t)$ is proportional to the spherical harmonic $Y_{1,m}$, so the above formula for $j \neq m$ follows from the orthogonality properties of the spherical harmonics, and for $m = j$ this formula is a consequence of the normalization $|N| = 1$.

We do not discuss calculation of β because $\lim_{a\to 0}\beta = 0$.

Let us formulate the result of this Section as a theorem.

Theorem 1. *If $ka \ll 1$, then the solution to the scattering problem (4)-(6) is given by formula (32).*

In Section 3 numerical calculations of the EM waves, scattered by a small body (particle), are carried out for a variety of the problem's parameters.

3   Numerical results

In paper [1] some numerical results were obtained on the comparison of the asymptotic solution, found in [19], and the Mie-type solution given in [1]. The asymptotic solution from [1] was compared with the Mie-type solution in far zone. In Section 2 analytic formula (32) for the EM field everywhere outside an immediate neighborhood of the small particle was given, and formula (33) for this field in the far zone was derived.

It is of practical interest to evaluate the accuracy of formula (33) and compare it with the accuracy of formula (32). It is also of practical interest to find out for which values of the problem parameters these two formulas differ from the Mie-type solution by not more than 5% -10%. To compare the asymptotic solution (33) with the Mie-type solution one should establish the range of applicability of the asymptotic formula (33) and compare the results, obtained by this formula, with the results obtained by formula (32) for a variety of the values of the parameters a, r, and ζ, where r is distance to the far zone, and ζ is the boundary impedance. Let us rewrite formulas (32) and (33) in the form, suitable for numerical calculations. To do this, let us use the following formulas for $\nabla_x g(x, O)$ and $\nabla_x E_0(O)$:

$$\nabla_x g(x, O) = \nabla_x \frac{e^{ik|x|}}{4\pi|x|} = \frac{e^{ik|x|}}{4\pi|x|^2}(ik - \frac{1}{|x|})x \tag{34}$$

and

$$\nabla \times E_0(O) = \nabla \times \begin{pmatrix} e^{ikx_3} \\ 0 \\ 0 \end{pmatrix}_{|x=0} = \begin{pmatrix} 0 \\ ik \\ 0 \end{pmatrix}. \tag{35}$$

Parameter $\tau = 2/3$ because of $b_{jm} = 1/3\delta_{jm}$. Using these formulas and formulas (34) and (35), one gets the following representation for the EM field:

$$E(x) = e^{ikx_3} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{8a^2\pi k\zeta}{3\omega\mu_0} \frac{e^{ik|x|}}{4\pi|x|^2}(ik - \frac{1}{|x|}) \begin{pmatrix} -x_3 \\ 0 \\ x_1 \end{pmatrix}, \tag{36}$$

and the corresponding formula valid in the far zone,

$$E(r) = e^{ikx_3} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{8a^2\pi\zeta}{3}\sqrt{\frac{\varepsilon_0}{\mu_0}} \frac{e^{ikr}}{r} ik \begin{pmatrix} -x_3^0 \\ 0 \\ x_1^0 \end{pmatrix}, \tag{37}$$

where $x_1^0 = x_1/r$, $x_3^0 = x_3/r$ are the unit vectors in the directions x_1 and x_3.
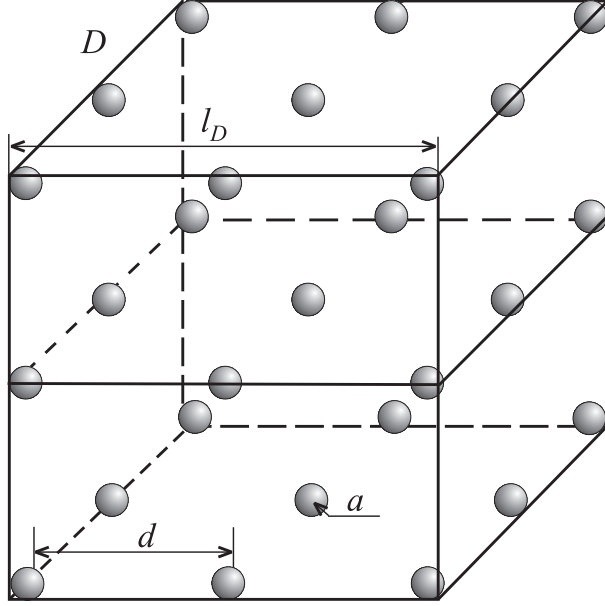
13

Figure 1: Relative error for E_1 component $k = 0.001$

3.1 Relative error of formula (37) for far zone

The dependence of relative error of the asymptotic solution (37) on the parameter r/a is investigated here. In Figs. 1-3 the numerical results demonstrate the relative error of asymptotic formula (37) for the E_1, E_2, and E_3 components for various values of the boundary impedance ζ, when $k = 0.001$ and $a = 1.0$. The relative error is calculated by formula

$$RE = \frac{|S_{36} - S_{37}|}{|S_{36}|}, \tag{38}$$

where S_{36} is the exact solution by formula (36), and S_{37} is the asymptotic solution by formula (37). One can see that the relative error for the E_1 component is maximal for $\zeta = 100$ and is equal to 3.5% when $r/a = 5.0$. The error for $\zeta = 100$ decays fast as r/a grows and is equal to 0.3% at $r/a = 20.0$. The relative error decays slower for larger values of ζ. Nevertheless, the relative error for $\zeta = 500$ remains small: for all considered values of r/a this error changes from 0.5% when $r/a = 5.0$ to 0.3% when $r/a = 20.0$.

The relative error for component E_2 practically does not depend on the r/a, except for $\zeta = 500$. The maximal value of error is attained at $\zeta = 20$, it is equal to 15.3% and it diminishes if ζ decreases. The minimal value of
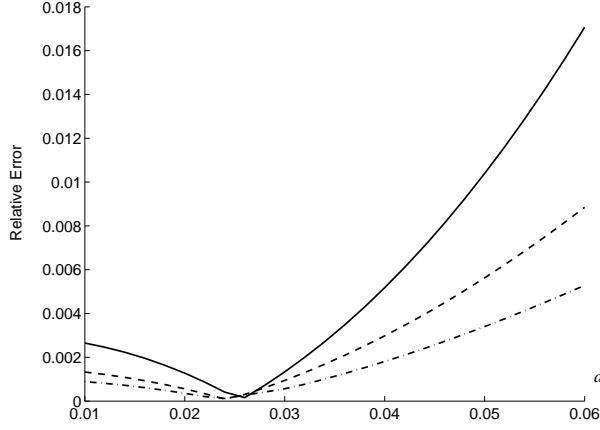
Figure 2: Relative error for E_2 component, $k = 0.001$

error for this component is attained for $\zeta = 500$ at $r/a = 20$ and it is equal to 0.6%.

The relative error for the field component E_3 depends significantly on ζ. The maximal value of this error is attained for $\zeta = 500$ when $r/a = 5.0$ and is equal to 2.3%. The minimal value of the error is equal to 0.1% and is attained for $\zeta = 100$ when $r/a = 20.0$. Summarizing the above results, one concludes that the relative error of the asymptotical solution (37) when $r/a \geq 20.0$ does not exceed 1% for E_1 and E_3 components for all considered values of ζ. The relative error for E_2 component when $r/a = 20.0$ depends on ζ and varies from 15.3% to 1.2%.

3.2   Comparison with the Mie-type solution

In the previous subsection, we have estimated the accuracy of formula (37) for the EM field in the far zone by comparing it with the exact formula (36). From the practical point of view it is of interest to compare the accuracy of the solutions (36) and (37) with the accuracy of the Mie-type solution obtained in paper [1]. In the far zone, this solution can be calculated by the following formula:

$$
\begin{aligned}
E(p) = {}& e^{ip}\cos(\theta)e_1 + 2ia_1\frac{e^{ip}}{p^2}\sin(\theta)\cos(\varphi)\widehat{e}_r + \\
& + (-b_1\frac{e^{ip}}{p}\cos(\varphi) + ia_1\frac{e^{ip}(1+ip)}{p2}\cos(\theta)\cos(\varphi))\widehat{e}_\theta + \\
& + (b_1\frac{e^{ip}}{p}\cos(\theta)\sin(\varphi) - ia_1\frac{e^{ip}(1+ip)}{p2}\sin(\varphi))\widehat{e}_\varphi,
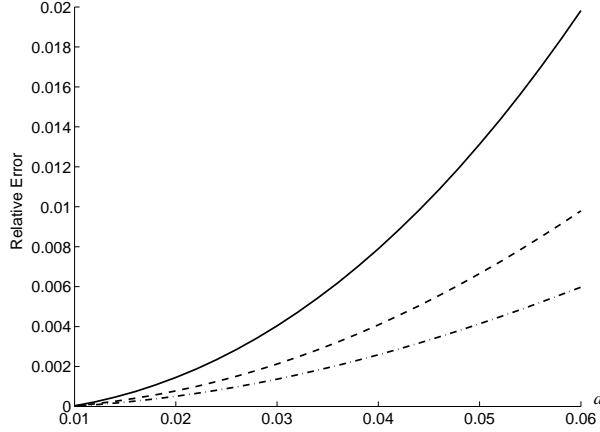\end{aligned}
\tag{39}
$$

15

Figure 3: Relative error for E_3 component, $k = 0.001$

where $p = kr$ is distance to far zone. We will use formula (39) for comparison with solutions (36) and (37), because it was established in [1] that the relative error of formula (39) does not exceed 0.1% for all values of parameters k, a, r, and ζ used below. This was established by comparing the numerical results, obtained by formula (39) with the exact solution, given in [1], see formulas (43)-(44) in [1]. Since the solution (39) is given in the spherical coordinates, we use the formulas for the electric field also in the spherical coordinates:

$$
\begin{aligned}
E_r &= E_1 \sin(\theta) \cos(\varphi) + E_2 \sin(\theta) \sin(\varphi) + E_3 \cos(\theta), \\
E_\theta &= E_1 \cos(\theta) \cos(\varphi) + E_2 \cos(\theta) \sin(\varphi) - E_3 \sin(\theta), \quad (40) \\
E_\varphi &= -E_1 \sin(\varphi) + E_2 \cos(\varphi),
\end{aligned}
$$

In Fig. 4, the results of the comparison of the solution, obtained by formulas (36) and (39) are given for the E_r component. The dependence of the relative error on the value of the parameter r/a is shown. Solution (39) is considered here as the benchmark (exact) solution. The relative error depends on r/a and ζ. For example, when $r/a = 2.0$ and $\zeta = 20$, then the relative error is equal to 7.1%, if r/a grows this error decays and does not exceed 0.9% when $r/a = 20.0$ for all considered values of ζ. The relative error for E_φ component is 7.1% when $r/a = 2.0$ and is not more than 0.2% at $r/a = 20.0$ for all considered values of ζ. The relative error of formula (37) in the far zone is about the same as the relative error of formula (36). The comparison is made using formula (39) in the far zone.

In Figs. 5-7, the relative error of the solution, calculated by formula (37) is given. The Mie-type solution (39) is considered as the exact solution.
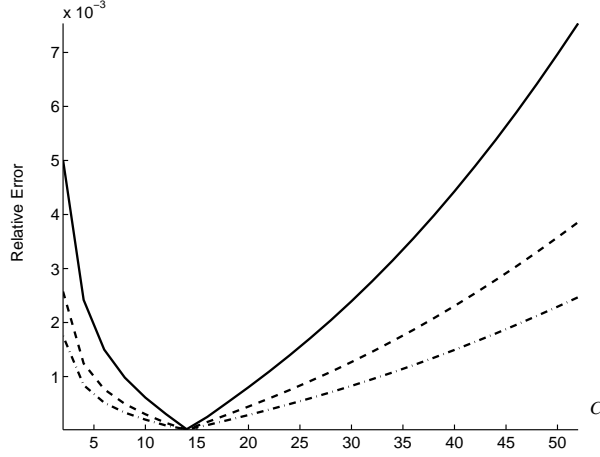
16

Figure 4: Relative error of (36) for E_r component, $k = 0.001$

The results are shown for the components E_r, E_θ, and E_φ, respectively. The relative error for the E_r component is maximal when $\zeta = 500$ and $r/a = 2.0$: this error is equal to 0.2%. The maximal relative error when $r/a = 20.0$ and $\zeta = 500$ is equal to 0.02%.

The relative error for the E_θ component (see Fig. 6) practically does not depend on r/a and ζ and is equal to 1.23%. The relative error for the E_φ component (see Fig. 7) does not exceed 0.01% for all considered values of r/a and ζ. The relative error depends on the value of k for the considered range of r/a. In Fig. 8, the relative error for the E_r component is shown for $k = 0.5$. This error is about 40% when $r/a = 2.0$ and $\zeta = 500$, it decays fast when r/a grows for all considered values of ζ and it it is less than 3.6% when $r/a = 20.0$ and $\zeta = 500$. The minimal value of the error for $r/a = 20.0$ is attained for $\zeta = 20$ and is equal to 0.1%. The relative error for the E_θ component has the same order as the E_r component, and the relative error for the E_φ component changes from 9.3% to 0.9% for the considered values of r/a and ζ. The above numerical results show that the accuracy of solution (32) increases as ka decreases.

The numerical results related to investigation of the dependence of relative error for the E_r component in formula (37) on ζ at fixed r are shown in Fig. 9. Solution (39) is considered as the benchmark solution. The relative error grows as ζ increases. When $\zeta = 20$ the maximal relative error is attained at $r = 5.0$ and it is equal to 0.01%. When $\zeta = 500$ the maximal relative error is attained at $r = 5.0$ and it is equal to 3.4%. The minimal
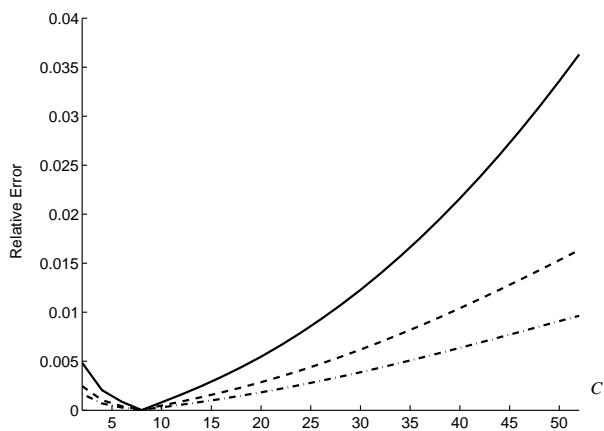
17

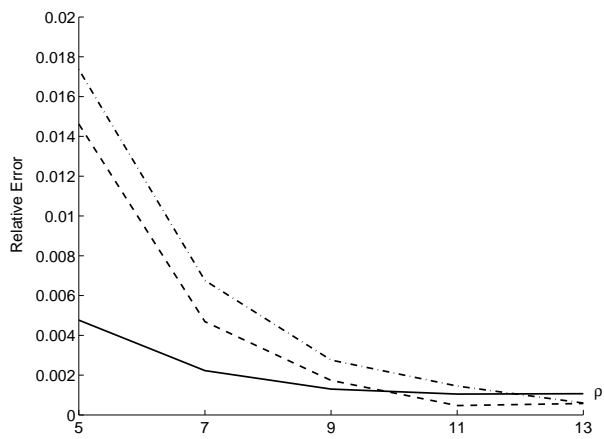Figure 5: Relative error of (37) for E_r component, $k = 0.001$

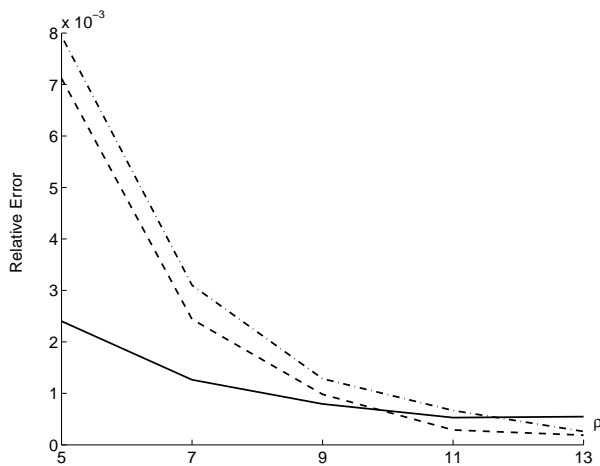
Figure 6: Relative error of (37) for E_θ component, $k = 0.001$

18

Figure 7: Relative error of (37) for E_φ component, $k = 0.001$
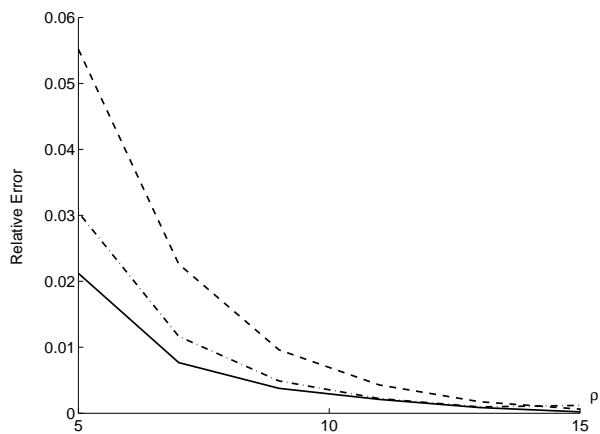

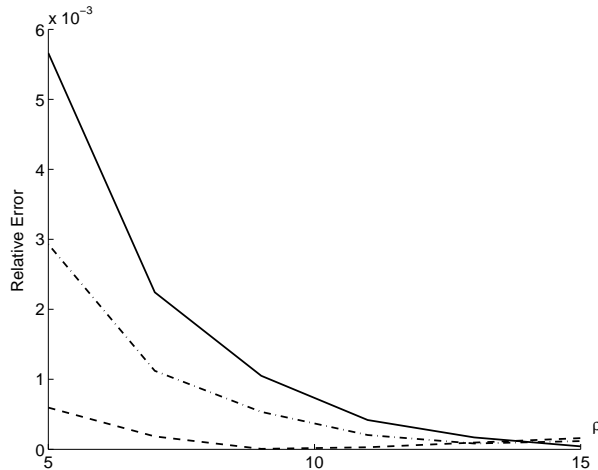Figure 8: Relative error of (37) for E_r component, $k = 0.5$

19

Figure 9: Relative error of (37) for E_r component, $k = 0.1$

error for $\zeta = 500$ is attained when $r = 20.0$ and it is equal to 0.2%. The calculations show that error diminishes when k decreases. When $k = 0.001$ the maximal error for $r = 5.0$ and $\zeta = 500$ is equal to 0.3%. This error for $r = 20.0$ is equal to 0.05%. These results show that the EM field for the far zone given by formula (37) are very close to the benchmark (exact) results obtained by using the Mie-type solution (39) in the far zone. The maximal relative error does not exceed 3.5% for the considered range of the parameters k, a, r, and ζ.

3.3   Comparative results on the accuracy of formula (36)

In paper [1], the numerical experiments related to investigation of the accuracy of formula (53) were given. Formula (53) from [1] looks as follows:

$$E(x) = E_0(x) - \frac{8a^2 \pi i \zeta}{3\omega \mu_0} [\nabla_x g(x, O), \nabla \times E_0(O)]. \qquad (41)$$

Formula (36) is more accurate than (41) because it is based on a theoretically exact argument and, because of this, it contains the additional term τ, which was neglected in [1]. We give below the relative error of both formulas for various ζ in order to explain the advantages of the new formula (36) over formula (41).

The Mie-type solution (39) is considered as the benchmark solution. The results are shown in Figs. 10-12 for the components E_r, E_θ, and E_φ, respectively, $k = 0.5$, $a = 0.1$. The thin lines correspond to the solution
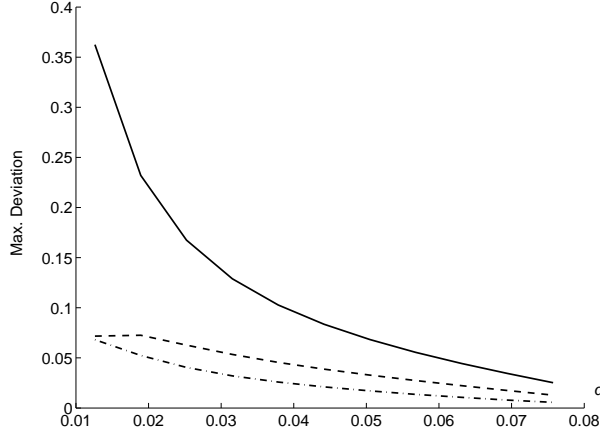
20

Figure 10: Relative errors of (36) and (41) for E_r component

(41) from [1], and the thick ones correspond to the solution (36). One can see from Fig. 10, that the maximal decrease of the relative error for E_r component is observed at $r/a = 20.0$ for $\zeta = 500$, the error decreases from 7.3% to 1.2%. Considerable decrease of the error is observed for $\zeta = 200$: the error decreases from 10.9% to 2.6% at $r/a = 5.0$ and it decreases from 2.9% to 1.4% when $r/a = 20.0$. For $\zeta = 50$ and $\zeta = 100$ the relative error practically does not depend on r/a and decreases not as much. The comparison of the relative errors for the E_θ component shows that the maximal decrease of this error is observed for $\zeta = 500$: the relative error decreases from 40.2% to 8.1% when $r/a = 5.0$, and it decreases from 3.7% to 1.1% when $r/a = 20.0$. For $\zeta = 200$ the relative error decreases from 19.4% to 1.5% when $r/a = 5.0$, and from 1.8% to 1.2% when $r/a = 20.0$. For $\zeta = 50$ and $\zeta = 100$, the relative error for the E_θ component in solution (41) depends on relation r/a slightly and it decreases from 7.2% up to 6.9% and from 9.7% up to 3.5%, respectively. For solution (36), this decrease amounts from 5.9% up to 5.6% and from 3.1% up to 2.8%, respectively. The relative errors for the E_φ component behave similarly. The maximal decrease of the relative error is observed for $\zeta = 500$: this error decreases from 13.5% to 2.1% when $r/a = 5.0$ and it decreases from 0.7% to 0.01% when $r/a = 20.0$. For $\zeta = 200$, the error decreases from 5.4% to 0.3% when $r/a = 5.0$ and from 0.3% to 0.08% when $r/a = 20.0$. For $\zeta = 50$ and $\zeta = 100$ the relative error of the new solution (32) does not exceed 0.34%. These results show that the new solution (36) is much more accurate than the solution (41) from [1].
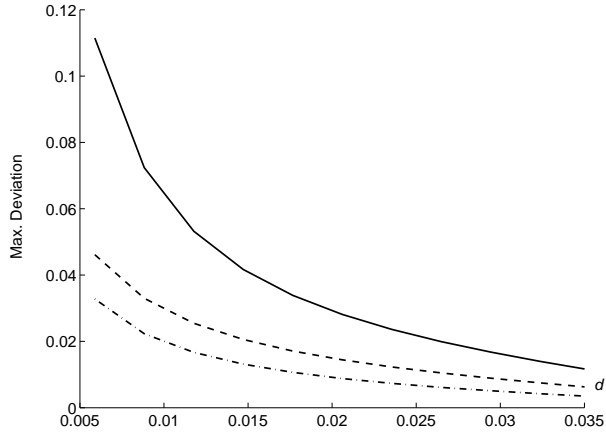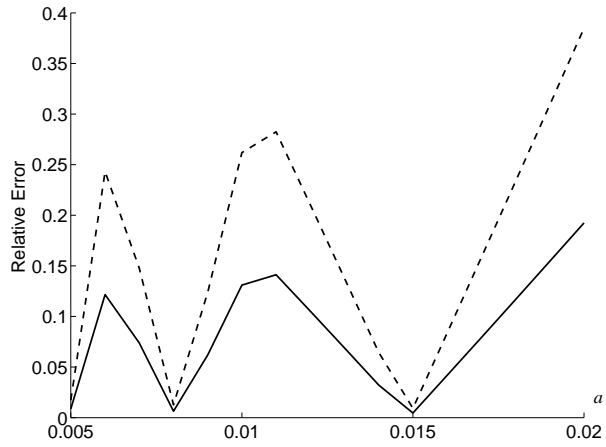
Figure 11: Relative errors of (36) and (41) for E_θ component

Figure 12: Relative errors of (36) and (41) for E_φ component

22

4   Conclusions

The explicit analytical formulas (32) and (33) for the EM field components are derived and tested numerically. To establish the limits of the applicability of the new formulas they are compared numerically with the Mie-type solution, obtained in [1] for a spherical particle and the impedance boundary condition. This solution is used as the benchmark (exact) solution for comparison with the new formulas (32) and (33), and with their versions (36) and (37), used in computations.

Comparison of formula (36) for the EM field components and its representation (37) for the far zone demonstrates that formula (36) yields the relative error not exceeding 2.5% when $r/a \geq 15.0$ for all the values of ζ in the range from 20 to 100. This error is larger at the smaller values of r/a as ζ increases and it grows slightly at greater r/a if ζ increases. Formulas (37) and (39) representing the new solution and the Mie-type solution in the far zone yield very similar numerical results. The relative error of formula (37) does not exceed 0.012% for the considered range of parameters k, a, r, and ζ. Therefore, formula (37) is applicable in a wide range of parameters. The numerical results show that formula (36) is much more accurate than formula (41). This is a consequence of an exact theoretical derivation, which took into account the additional parameter τ, which was neglected in [1].

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