# PHYSICS EDUCATION RESEARCH SECTION

# Distinguishing between "change" and "amount" infinitesimals in first-semester calculus-based physics

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From the perspective of an introductory calculus course, an integral is simply a Riemann sum: a particular limit of a sum of small quantities. However, students connect those mathematical quantities to physical representations in different ways. For example, integrals that add up mass and integrals that add up displacement use infinitesimals differently. Students who are not cognizant of these differences may not understand what they are doing when they integrate. Further, they may not understand how to set up an integral. We propose a framework for scaffolding students' knowledge of integrals using a distinction between "change" and "amount" infinitesimals. In support of the framework, we present results from two qualitative studies about student understanding of integration. © *2014 American Association of Physics Teachers.*
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## I. INTRODUCTION

The integral is an essential mathematical tool in calculus-based undergraduate physics, playing a role in both the introductory and upper-division curriculum. The American Institute of Physics reports[1] that in the 2007–2008 academic year, 179,000 undergraduates were enrolled in a calculus-based introductory physics course—as compared to only 153,000 students who participated in algebra-based courses. Calculus-based physics is usually required not only by an institution's physics program, but also by the engineering, chemistry, and some pre-med programs.[2]

We believe that it is not enough for students to be able to compute integrals; they should also be able to connect (or "map") integrals, including infinitesimal quantities, to multiple representations. Starting from this assumption, we will describe a framework for instruction about integrals in physics courses. We will focus on a distinction between two types of infinitesimals, *changes* and *amounts*. However, changes and amounts are not the only types of infinitesimals. A third type of infinitesimal, the product infinitesimal, will also play a role in our discussion in helping to make a connection with expert knowledge.

We chose to study first-semester physics even though electricity and magnetism place more emphasis on integrals. Our second-semester course introduces integrals so abruptly that it can be challenging to assist students with the complexities they face in the first few weeks of the semester. Therefore, we propose to familiarize students with integration starting with the previous course—the first semester of calculus-based physics—when the physics is less abstract and more grounded in students' concrete experiences.

Our framework will not consist solely of a description of students' prior knowledge, misconceptions, or difficulties. Good instruction requires not only knowing about students' prior knowledge but also developing techniques, such as bridging analogies,[3] that connect students' prior knowledge to the desired learning outcomes. In addition to bridging analogies, researchers may invent other types of conceptual tools, ideas, or procedures that scaffold their students' understanding.[4] Therefore, our framework will describe conceptual tools for scaffolding students' understanding of integration, especially regarding the use of infinitesimals. In this paper, such conceptual tools will be called *scaffolding knowledge*.

More specifically, we define scaffolding knowledge to be any knowledge that is not in general used by experts (except in pedagogical circumstances), but which can potentially be helpful to novices. A bridging analogy is an example of scaffolding knowledge. Experts seldom use bridging analogies to communicate a concept to other experts, but they may use bridging analogies to help less knowledgeable students to understand a concept. As an example, high school algebra students are often taught to use a "FOIL" mnemonic (First, Outside, Inside, Last.) Experts do not necessarily use this mnemonic, but it is helpful to students. As another example of scaffolding knowledge, we point out that physics education researchers commonly recommend problem solving frameworks[5] that help students to go through problems step-by-step, setting up equations and making use of multiple representations. While experts do set up equations and make use of multiple representations, they probably do not apply memorized frameworks with enumerated sequences of steps in an algorithmic manner. Therefore, the problem solving

frameworks are a type of scaffolding knowledge rather than expert knowledge.

Since scaffolding knowledge is not expert knowledge, experts are not necessarily aware of it, in that they may have forgotten the scaffolds or never learned them in the first place. Furthermore, instructors or researchers can invent new scaffolding knowledge. What should new scaffolding knowledge look like? We posit the following four conditions pertaining to scaffolding knowledge. Scaffolding knowledge should…

1. Be connected to students' prior knowledge, because as per constructivist theory all new knowledge is built upon existing knowledge.
2. Be within students' Zone of Proximal Development (ZPD)—the set of tasks that students can perform with assistance.[6]
3. Address the major difficulties that are likely to confuse students.
4. Help students build toward an expert understanding.

This paper presents a framework for scaffolding knowledge about integration that addresses the four conditions above. Because a connection to students' prior knowledge and ZPD are required (conditions 1 and 2), the development of scaffolding knowledge must be based on empirical research about student thinking. In the research program presented in this paper, we have asked several students to talk and reason about integrals in order to better understand how they think and what confuses them about integration.

To address student difficulties (condition 3), we make use of resource theory, which holds that students have ideas or intuitions about the physical world, called resources.[7] Resources are neither true nor false in and of themselves; however, they may be applied correctly or incorrectly, depending upon the context. Resource theorists can address a student difficulty by helping a student to see in what contexts their intuition is valid and in what contexts it is invalid. For example, Hammer and Elby[8] describe how to help students to understand Newton's third law by refining their intuition that a car "reacts more strongly" than a truck when the two collide. This intuition is correct if "reaction" means a change in velocity, but not if it means a change in momentum.

In an analogous manner, our scaffolding framework can be used to refine students' intuition that infinitesimals are small changes in a quantity. This intuition is correct in the context of change infinitesimals, such as $dv$, but not in the context of amount infinitesimals, such as $dM$.

Finally, we require that the framework connect to expert understanding (condition 4). We accomplish this using Sherin's idea of symbolic forms.[9] Although experts do not always distinguish between types of infinitesimals, we note that in thermodynamics, infinitesimals can be classified as exact or path-independent (such as $dV$) and path-dependent (such as $\delta W$. We connect this expert distinction to our students' understanding and to the proposed scaffolding knowledge through symbolic forms.

Thus, we can argue on theoretical grounds, using constructivism, resource theory, and symbolic forms, that we have developed a useful framework for scaffolding knowledge about integration. Our development process is shown in Fig. 1.

As shown in the figure, our initial inspiration comes from differential geometry, a subject that we discuss in the appendix. As the a priori framework evolved, it was incorporated
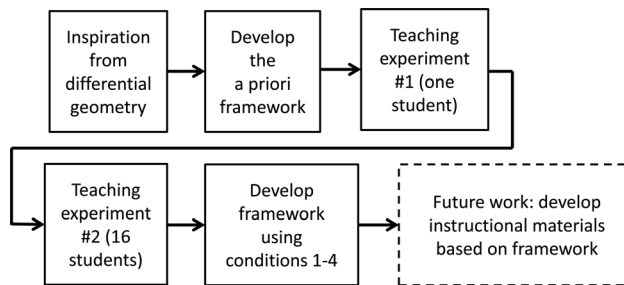


Fig. 1. Our process for developing the scaffolding framework.

into the instructional materials—worksheets, tutorials, and labs—for our two teaching experiments. The final framework is justified using conditions 1–4, with evidence from our observations in the two experiments. Future research could help find the optimal pedagogical strategies to facilitate students to use this knowledge or to determine which aspects of the framework ought to be taught to students.

This paper will develop scaffolding knowledge about infinitesimal quantities in integrals in a physics context. We intend that this knowledge will help students to "map" symbolic expressions to physical representations and verbal descriptions. We begin by describing the types of maps we are looking for involving finite and infinitesimal quantities (Sec. II) and reviewing prior work on this topic (Sec. III). We will then describe two studies we have completed to investigate integration in introductory physics (Sec. IV). Next, we will outline our proposed framework for scaffolding knowledge (Sec. V). Expert physicists as well as novices use representations such as diagrams, pictures, words or language, and symbolic equations ($dx = vdt$) to reason and communicate about integrals.[10,11] Section V uses examples from our two studies to illustrate students' ideas about integrals and representations of integrals. This section supports our scaffolding framework using student ideas. In addition, we connect our framework to an expert understanding of integrals using Sherin's idea of symbolic forms.[9] Lastly, we provide a discussion and summary of our work (Sec. VI) and then finish by suggesting implications of our work for instruction and assessment (Sec. VII).

## II. MAPPINGS

A physics course might have at least two goals for students' understanding of mathematics; we will call them *processing* and *mapping*. First, students might be asked to process equations. According to this goal, students should be able to start with a given set of equations, apply standard rules of inference, and derive a new set of equations. Processing often requires no knowledge about the meaning of the equations. For instance, it follows from $a = b^2$ that $\sqrt{a} = \pm b$. This relation is true regardless of whether the equations refer to the area and side of a square or a statistical variance and standard deviation.

Second, students could be required to map physical meanings to equations. This means that students should be able to consider a problem statement in ordinary language and transform it into a precise mathematical formula. For example, students could look at a problem about a falling ball and recognize that $mv^2/2 + mgh$ is the formula for the energy of the system at any given moment. Another type of mapping is

converting a mathematical formula back into a physical scenario; if the student uses the formula above and finds that $v = 0$, then she should conclude that this means the ball is not moving. This paper will focus on mapping rather than processing.

Redish describes four activities[12]—modeling, processing, interpreting, and evaluating—that are common to many uses of mathematics in physics. Redish's paper includes a diagram of student thinking in math and physics, which we reproduce in Fig. 2. In this paper, we will use the word *mapping* to refer to both "modeling" and "interpreting," since both of these activities involve a connection between physics and mathematics.

In the examples above, a student who recognizes that $v = 0$ refers to motionlessness is performing a *simple* map. That is, only a single physical idea is being mapped. On the other hand, the student who maps a ball's height, mass, and velocity to find its kinetic energy is performing a *composite* map; multiple physical quantities are involved in the map.

This paper is concerned with students' facility with composite maps in physical integration problems—their ability to interpret multiple quantities related to the same integral. In particular, we are interested in maps that require both finite and infinitesimal quantities. For instance, consider the problem in Fig. 3, which can be found in a popular introductory physics textbook.[13] Notice the number of distinct quantities that students are asked to comprehend.

The solution to the problem specified in Fig. 3 is given in Fig. 4. In this problem, the student must use a composite map involving not only finite quantities such as $\varphi$ and $\ell$, but also infinitesimal quantities such as $d\varphi$ and $dW_{\text{total}}$. To achieve this, we imagine that a student might draw a diagram showing a small change of angle $d\varphi$ along with a macroscopic angle $\varphi$. Such a diagram, which is displayed in Fig. 4, would help the student to map the mathematics to the physics, so that he or she can visualize the situation in order to recognize that trigonometry is necessary and choose between $dW_{\text{total}} = mg\ell d\varphi \cos \varphi$ and $dW_{\text{total}} = mg\ell d\varphi \sin \varphi$.

Derivations of integral formulas also require composite maps. For instance, if a student were asked to explain the meaning of $W = \int F(x)dx$, they might explain that $dx$ refers to an infinitesimal distance, $F(x)$ to a force at a particular point in a trajectory, and $dW = F(x) dx$ to an infinitesimal work over that distance. By adding up these infinitesimal works, we obtain a macroscopic work, $W$. This explanation gives an account of two infinitesimal quantities $(dx, dW)$ as well as two finite quantities $(F, W)$.

Another example of the importance of composite maps is the result of Pollock *et al.*,[14] showing that many students are
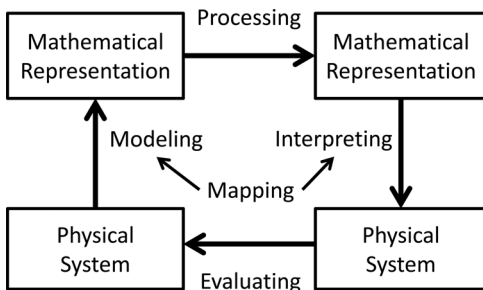
unable to interpret an integral such as $\int z dy$. We suggest that if students were able to map $z$ to the height of the curve $z(y)$ and $z\, dy$ to the area of a small rectangle, they would be more adept with this kind of integral.

## III. PRIOR WORK

Prior work on integration has been carried out by both mathematics education researchers and physics education researchers. However, these researchers have not focused on classifying infinitesimal quantities.

Some mathematicians[15,16] have described the sum and product structure of the integral using a "layered" framework. They have pointed out that the integral $\int f(x)dx$ is found by first setting locations $x_n = x_0 + nL/N$ at equal intervals along a length $L = x_N - x_0$, then taking the product of $f(x_n)$ and $L/N$, then summing over many values of $n$, and finally taking the limit as $N \to \infty$. Mathematically, this can be expressed as

$$\int_{x_0}^{x_0+L} f(x)\ dx = \lim_{N \to \infty} \sum f\left(x_0 + \frac{n}{N}L\right)\frac{L}{N}. \quad (1)$$

Thus, students must understand a product, sum, and limit layer in order to comprehend the integral. This framework is applicable to a physics context; however, we emphasize that in physics, representing the finite and infinitesimal quantities



Fig. 3. A problem from a popular introductory physics textbook illustrating a composite map. The student must map both finite and infinitesimal quantities.



Fig. 2. Redish's diagram of student thinking in math and physics annotated with the *mapping* and *processing* phases.



Fig. 4. The solution to the Tarzan problem. The diagram on the right illustrates both $d\varphi$ and $\varphi$ for a swinging Tarzan. This diagram could help a student find the formula for infinitesimal work.

must come before understanding the product. That is, the student needs to know *what* they are taking the product of. In this sense, our work fits into Sealey's "orienting" or sense-making layer,[15] a preliminary layer in which students understand the connection between the physical situation and framework of mathematics.

Some physics researchers have focused on the conditions under which students know that an integral must be performed. For instance, Meredith and Marrongelle[17] discuss students' use of recall, dependence (a quantity that varies with another quantity), and parts-of-a-whole cues. They found that students could sometimes misuse these cues, writing $kqdr/r^2$ in place of $kdq/r^2$ when integrating the electric field. Students were "simply sticking in a $dr$ next to the formula for a point particle" because they knew the integral must be with respect to $r$. We suggest that distinguishing between these two expressions requires mapping "$dr$" to physical representations such as diagrams. The students would have to start with the finite expression $kq/r^2$, draw a diagram to visualize *what* is being added up, and then replace the $q$ by $dq$ to indicate a small amount of charge (see Fig. 5). Although the diagram in Fig. 5 alone is not sufficient to understand the problem, it is very helpful if properly understood. Such a diagram shows that $dr$ is associated with the charge producing the electric field, and that this charge is infinitesimal. Thus, the expression $kqdr/r^2$ cannot be correct because it counts the charge as a finite quantity and it considers the charge to be independent of $dr$. We do not claim that simply showing this diagram to students would lead to a better understanding; rather, such a diagram could be one component in an effective lesson.

Our work is closest to that of Nguyen and Rebello,[18] who discussed students' ability to set up integrals and to find the correct infinitesimal form. They pointed out that students sometimes do not write down *any* infinitesimal when writing an integral. Nguyen and Rebello also found that students often attempted to find a resistance by integrating the incorrect formula $dR = \rho L dx/A$ rather than the correct formula $dR = \rho dL/A$. We suggest that students might have been more successful had they mapped $L$, $dL$, and $dx$ to a diagrammatic representation.

Dray and Manogue[19] point out that differentials can mean many things in the context of mathematics, including: "arbitrarily small changes in given quantities; a shorthand notation for limits; differential forms; or the infinitesimals of the hyperreal numbers." One source of inspiration for our a priori framework was the mathematical field of differential geometry. In differential geometry, $dx$ refers to a construct called a differential form; and in this context the "$d$" is called an exterior derivative. This paper will not require any knowled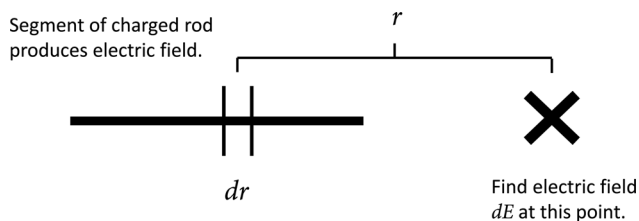ge of differential geometry, but further information can be found in the appendix. The appendix will contrast several meanings of $d$. The interested reader can also consult Frankel's book.[20] However, in this paper we will use $d$ to mean simply "something infinitesimal; vanishingly small," without giving a mathematical definition of what it means to be infinitesimal or vanishingly small.

## IV. METHODOLOGY

We will develop our scaffolding framework by examining students' prior knowledge and their response to instruction, especially students' presentations about integrals in multiple representations, including words, diagrams, and symbols. The student work presented here comes from two separate studies. In both studies, we recruited undergraduate student participants from an introductory first semester calculus-based physics course. Von Korff and Rebello[21] describe the first study, including all assigned problems. Problems assigned in the second study are available as supplementary material.[22] (The first lesson's worksheet was the same for both experiments.) In this section, we will briefly discuss the methodology for each study.

The first project was a case study, involving a single student participant, "Amber."[21] Amber took the introductory calculus-based mechanics course over the summer, with just nine other students. This course compressed a semester's worth of material into two months. In addition to her coursework, Amber participated in our sequence of seven two-hour lessons on the topic of integration in introductory mechanics. Amber was paid $10 per hour to participate, and did not receive course credit. We offered this opportunity to all 10 students in Amber's course. Three students had accepted the offer for the first lesson. However, two students dropped out after that, one of them citing a commitment to a summer job. In this paper, only Amber's work will be presented.

Amber had taken AP Physics B in high school, making her physics preparation above average for students in her course. Her mathematics preparation was typical; she was concurrently taking a second semester of calculus. Her physics course contained little material about integration, so our lessons were her primary introduction to integration in a physics context.

In the sequence of lessons, Amber solved problems, set up integrals, took part in Socratic dialogues[23] with the instructor, wrote essays, examined physical objects related to integrals, and solved debate problems. In a debate problem,[18,24] students consider a set of hypothetical statements made by fictitious students, and analyze or judge these statements. Amber also gave oral presentations at the blackboard.

In most lessons, Amber and the instructor were seated at the same table, located in the same room as Amber's labs for her mechanics course. Most commonly, Amber and the instructor were engaged in Socratic dialogue about a worksheet or the lab equipment. In the current paper, we will show work from Amber's oral presentations about integrals. We will highlight Amber's use of multiple representations in her presentations.

Our second project involved 16 students recruited from the same introductory physics class taught by a different instructor in the following (fall) semester. The fall course enrolled 277 students. The professor who taught this course lectured slightly more about integrals than the professor who taught Amber's physics course. In addition, he assigned more activities related to integrals. These activities included



Fig. 5. A diagram that could be used to show the meaning of $dr$ and $r$ for an electric field integral.

three short (20 to 60 min) labs or problem-solving lessons about integration. In addition to their coursework, the 16 students in our study attended a sequence of six 1.5-h lessons, which were modeled after our work with Amber. The 16 students were paid $100 each to participate in our study, but they did not receive course credit. During these lessons, students worked at tables in groups of three or four, sharing their ideas with one another by drawing on whiteboards. They also gave group oral presentations to one another using the same whiteboards.

Finally, we gave the students Livescribe "Smartpens," a type of pen containing an optical sensor and an audio recorder. The Smartpen is able to record everything the student says and writes, since its optical sensor can detect the pen's position as it writes on special paper. Using these Smartpens, we were able to collect students' written and spoken representations of integrals, just as we could collect Amber's written and spoken representations by videotaping her oral presentations.

In the six fall lessons, our 16 students primarily considered two types of tasks: debate problems and explanatory tasks. Debate problems are defined above, in our discussion of the first experiment. In an explanatory task, students were asked to summarize the mathematical and physical derivation of a particular integral equation. For instance, they might be instructed to explain why displacement is equal to the time integral of velocity, starting from the algebraic equation $x = vt$.

Our initial, a priori scaffolding framework evolved from our attempt to reconcile the standard physical notation in which a single symbol $d$ can be used to represent infinitesimals ranging from $dx$ to $\delta W$ to $dM$, with our observation that our students were not proficient with certain infinitesimals. In order to develop an a priori framework that could describe diverse physical integrals, we initially turned to differential geometry (see Appendix). We refined and tested this a priori framework by observing students working with integrals in the context of lessons we developed, as discussed above. We connected our framework to students' prior knowledge (condition 1) and ZPD (condition 2) by paying special attention to students' use of representations after instruction. We also paid attention to student difficulties (condition 3). If students correctly represented two integrals similarly, we took this as evidence that these integrals could be classified similarly. For example, our students depicted work and kinematic trajectories using similar diagrams (see Sec. V D). The students' ability to pick up on this similarity shows that the classification is within the students' ZPD. This connection

with students' ZPD provides evidence (according to condition 2) that work and kinematics should be linked in our framework. On the other hand, if students made a physics error due to treating one integral like another, we classified the two integrals differently. For example, students selected the wrong equation for a moment of inertia integral because they treated moment of inertia like a change infinitesimal (see Sec. V B). As discussed below, we occasionally attempted to teach parts of the taxonomy—a form of member checking—to the students.

# V. SCAFFOLDING FRAMEWORK FOR INTEGRATION

As students work with composite maps in integration (Sec. II), they explore the physical meanings of integrals, infinitesimals, and finite quantities. It may be tempting to assume that students can "understand infinitesimals" in a general sense. We might guess that a student who understands infinitesimals should be able to manipulate $dv$, $dM$, $\delta W$ and similar quantities with equal facility. However, we will argue that from a physical perspective, $dv$ and $dM$ are simply not the same kind of physical quantity. Students use different conventions for drawing and reasoning about these two quantities, therefore we should not assume that students would possess equal facility with both quantities. Thus, in order to most easily connect our framework to student understanding, it will be productive to classify quantities based on the way they are commonly represented, especially in diagrammatic, verbal, and symbolic representations.

In the following sections, we will discuss two interrelated classification systems. The first system involves infinitesimal quantities, which are the variables under the integral sign that have a "$d$." The second classification system will categorize integrals according to their use of space, time, and matter. This second system will be closely related to the way students use diagrams when talking about integrals. We will call the categories in this second system "integral types." Throughout these sections, we will give examples of student representations pertaining to each classification system.

Table I summarizes our scaffolding framework. We will not present complete information about the entries in the third column (our framework) immediately (we will do so throughout the remainder of Sec. V, starting in Sec. V A); however, we can provide a brief summary. In the table, change infinitesimals are infinitesimal differences, while amount infinitesimals are infinitesimal parts of a whole. Products come from multiplying a finite quantity by an

Table I. From the mathematical (first-year calculus) point of view, all integrals are Riemann sums. From the physical point of view, there are many integrals, each of which must be understood on its own terms. Our framework is an intermediate step between these two ways of understanding and will be described in detail later in the article.

| | Mathematical | Our framework | Physical |
|---|---|---|---|
| Infinitesimal quantity | $\frac{L}{N}$ | Amount | $dM$, $dA$, $dI$, $dF$ |
| | | Change | $da$, $dt$, $dv$, $dU$ |
| | | Product | $\delta W$, $\delta Q$ (heat) |
| Integral | Riemann sum: $\lim\limits_{N\to\infty}\sum\limits_{n=0}^{N-1} f\left(x_0 + L\frac{n}{N}\right)\frac{L}{N}$ | Static object | $\int \rho_1 \, dx$, $\int \rho_3 \, dV$, $\int dM \, R^2$ |
| | | Trajectory | $\int v \, dt$, $\int a \, dt$, $\int F \, dx$ |
| | | Thermodynamic | $\int P \, dV$, $\int T \, dS$ |

infinitesimal. Static object integrals involve the body of an object, while trajectory integrals involve a moving point particle, and thermodynamic integrals take place in thermodynamic state space. The reader is advised to refer back to Table I throughout this section. Table I also depicts mathematical and physical aspects of integrals and suggests that our taxonomy occupies a space between the two.

## A. Expert knowledge about infinitesimals in thermodynamics

According to condition 4 (helping students build toward expert understanding), our scaffolding framework must connect to experts' knowledge about infinitesimals. Physicists view thermodynamic quantities such as work and heat as path dependent, sometimes putting a line through the $d$ in their differentials $\delta W$ and $\delta Q$. Other quantities, such as $V$ and $T$, are path independent, and their differentials are "exact." The differential $dV$ refers to a change in volume, and as such, can be imagined as a difference $V_f - V_i$ over an infinitesimal period of time. In contrast, $\delta W$ should not be imagined as a difference $W_f - W_i$, because work is a property of a process rather than a property of the initial or final state of the system. Instead, the infinitesimal work can be thought of as a product $PdV = P(V_f - V_i)$. Thus, we can reinterpret the expert distinction between path dependence and path independence as a distinction between types of infinitesimals: some infinitesimals are changes and some are products. In Sec. V C, we will connect our scaffolding framework to this expert distinction (condition 4) and to a major student difficulty (condition 3) by applying Sherin's idea of symbolic forms to both.[9]

In order to extend our framework beyond thermodynamics, we must consider the general nature of infinitesimal and finite quantities in physics. Infinitesimal quantities, which are always indicated with a $d$, are small quantities that can be added up. Thus, a trajectory is conceptualized as a sum of $dx$'s, and the mass of a solid object is conceptualized as a sum of $dM$'s. Such quantities often have a linear relationship with infinitesimal amounts of space or time. For instance, the infinitesimal equation $dW = F\,dx$ says that work increases linearly with displacement for small displacements $dx$. Likewise, $dv = a\,dt$ says that the change in velocity increases linearly with time for small changes in time $dt$.

Finite quantities, such as velocity, acceleration, density, pressure, and force, are defined by a different pattern: they are quantities that take on values at points (a "point quantity"). Depending on the type of integral, finite quantities may take values at points in space, time, phase space, or several of these.

## B. Student difficulties with infinitesimal types

In this section, we seek examples beyond thermodynamics. We also argue that our framework should distinguish between "infinitesimal changes" and "infinitesimal amounts." Infinitesimal changes can be written as differences, as with the example $dV = V_f - V_i$ above, while infinitesimal amounts refer to small elements of a substance, as with a volume element $dV$. Our Smartpen study addressed both infinitesimal changes and infinitesimal amounts. The first three lessons in the sequence were about kinematics and work integrals such as $\int vdt$ and $\int Fdx$. In each case, the integrand involved an infinitesimal change, either $dt$ or $dx$,

A rod moving in a circle, around an axis parallel to itself, is shown in the picture:
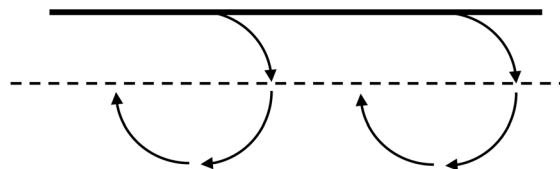


For moment of inertia, the *finite* equation is $I = MR^2$. Which of these people is correct about the *infinitesimal* equation, and explain why?

Ron: The infinitesimal equation is $dI = M\,(dR)^2$.

Sam: The infinitesimal equation is $dI = MR\,dR$.

Tara: The infinitesimal equation is $dI = dM\,R^2$.

Fig. 6. A rod revolving around an axis parallel to itself. Students were asked what infinitesimal equation is appropriate in this case. This was a particularly thin debate problem, in that Ron, Sam, and Tara do not provide rationales for their suggestions; in other debate problems, we provided more justification for each choice.

respectively. Students frequently used the language "change in time" or "change in $x$" to describe the differentials.

The fourth lesson addressed mass and moment of inertia integrals and required students to think about infinitesimal equations such as $dM = \rho dx$ and $dI = dMR^2$. The Smartpen problem (Fig. 6) required students to consider a rod revolving around an axis parallel to itself. They were asked to explain whether $dI = dMR^2$, $dI = MRdR$, or $dI = M(dR)^2$ is the most appropriate equation. Students were given some hints about a version of the scaffolding framework as it existed at the time of the experiment; in particular, the instructor lectured for roughly five minutes about amount and change infinitesimals as well as time orientation (see Sec. V E) and gave some examples of amounts and changes, including an infinitesimal amount of mass. The instructor also gave the students a hint by sectioning off a small segment of the rod in Fig. 6 using a pair of short vertical lines.

Students tended to interpret the differentials in these integrals according to conventions that were most familiar to them. In some cases, in their Smartpen oral presentations, students referred to the $d$ as a change. This happened in spite of the mini lecture elucidating the meaning of $d$. For instance, one student said, "Mass is not changing its value. It is always the same mass. Therefore, $dM$, the change in mass, is not going to give the right equation." This student discounted the correct answer (Tara's answer) because he imagined that $dM$ refers to a mass that changes over time. In other words, the student viewed $dM$ as referring to a change $M_2 - M_1$ over an infinitesimal period of time.

Several students asserted that the correct equation should contain $dR$ because, they stated, $R$ is changing. For instance: "Tara is wrong 'cause she says the change in mass times $R$ squared. Since mass isn't changing and $R$ is, that means Tara is wrong." In fact, the radius $R$ is constant, although the radius vector $\vec{R}$ is changing as the rod revolves around the axis. Apparently, students were so determined to find a changing quantity that they were willing to consider a change in direction to be a change in $R$.

Some students used the word "amount" rather than "change." For instance, one student said "You're taking a

Table II. Examples of our three infinitesimal types.

| Infinitesimal type | Symbolic form | Example of form | Examples of quantity |
|---|---|---|---|
| Change | $\square - \square = \Delta$ | $dv = v_2 - v_1$ | $dt$, $dv$, $dx$ (moving object), $dM$ (sand on a cart), $dV$ (thermo) |
| Amount | $\Delta$ | $dM$ | $dM$, $dA$, $dV$ (density integrals) |
| Product | $\square \cdot \Delta$ | $P \cdot dV$, $T \cdot dS$ | $\delta W$, $\delta Q$ (thermo) |

small amount of the rod as the $dM$, it's rotating around the radius so $R$ stays the same, you have a small amount of inertia that the piece of rod will have." Thus, we can discern from students' language whether they classified mass as an infinitesimal amount or an infinitesimal change. This student used the word "amount" to describe $dM$ rather than focusing on a change of the rod's position or mass.

Of the 15 students who attempted this lesson, nine selected the right answer (Tara's answer). Those who selected this answer used the following words to describe $dM$: five called it an "amount," one called it a "section," one a "mass of a small rod," one a "piece," and one a "change." Of the five who called $dM$ an "amount," three had other names for it as well. One also called it a "mass of a small segment," one a "little mass," and one both a "piece" and a "change." Two explicitly stated that $dM$ is *not* a change in mass, of whom one used the word "amount" and one did not. Of the six who selected a wrong answer, two called $dM$ a "change," while the other four did not use any descriptive name for $dM$.

We can use these data to build evidence that students who picked up the vocabulary from our lesson were able to select the right answer. None of the five students who used the word "amount" in lesson four had used that word when describing a work integral in lesson three; this suggests that these five students had picked up on the appropriate language in our mini lecture. All five students selected the right answer in lesson four. In addition to these five, a sixth student did not use the word "amount" but stated that "$dM$ does not mean in this sense a change in mass," indicating that he may have been influenced by our lecture. This student also selected the right answer. On the other hand, we should consider the possibility that some students listened to the mini lecture and understood it, but gave the wrong answer because they believed that $dM$ is a change rather than an amount in this context. Two students used the word "change" to describe $dM$ in lesson four and selected the wrong answer. However, both of these students had also used the word "change" when describing "$dx$" in the work integral in lesson three. This suggests that the students who used the word "change" were doing so out of familiarity with this usage rather than by picking it up from our mini lecture. Thus, the data are consistent with the possibility that our scaffolding framework was helpful for as many as six students and was not harmful to any students.

In the setting of a calculus-based physics course, one solution to these difficulties would be to expunge infinitesimal amounts from the course vocabulary. Thus, rather than writing $dM$ for a small amount of mass, we could write the infinitesimal product $\rho_1\, dx$, where $dx$ can be interpreted as an infinitesimal change along an imaginary path and $\rho_1$ is a linear mass density. However, the use of amounts is difficult to avoid in multiple dimensions, as the elements $dA$ and $dV$ are amounts of space. Furthermore, the notation $dM$ is standard and convenient. We propose that a scaffolding framework

that distinguishes between amount and change infinitesimals could help students to think before interpreting $dM$ and to ask themselves whether it is an amount or a change.

## C. Symbolic forms

Sherin[9] has found that students possess conceptual understandings of equations in a physics context, which he calls "symbolic forms." These symbolic forms guide students' intuitions about the relationships between physical quantities, including (we suggest) finite and infinitesimal quantities. For example, Sherin's symbolic form "Base ± Change," symbolized by "$[\square \pm \Delta]$", could be used to explain the equation $x_1 + dx = x_2$, a base displacement plus an increment equals a new displacement. We will adopt a similar nomenclature throughout this paper, using $\square$ to denote finite quantities, and $\Delta$ to denote infinitesimals.

We can relate symbolic forms to students' language about infinitesimals in the previous section. For example, the students related the symbol $d$ to the word change; in particular, changes in mass or changes in $R$. However, in the kinematics and work problems the students had experienced earlier in our course, such changes had always referred to mathematical differences. For example, in kinematics, the infinitesimal $dx$ represents a change in position $x_2 - x_1$ or, symbolically, $\square - \square = \Delta$. We conceptualize $dx$ by imagining two positions at nearby points on a trajectory and taking the difference between them. Contrast this with the quantity $dM$. This quantity can be described as a small amount of mass of a small segment of rod. To explain $dM$ to a student, we would not imagine two masses and take the difference $M_2 - M_1$. Rather, we imagine a small amount of matter, and $dM$ is a property of this amount. Therefore, the symbolic form for $dM$ will be represented by the symbol $\Delta$ alone to indicate that there is no arithmetical relationship. In this paper, we will describe three types of infinitesimal quantities, of which two will emerge from arithmetical relationships and one will not. Table II depicts the three types.

Note that in this paper, we do not use the term "product" for a quantity that is also an amount or a change. Thus, $dv = a\,dt$ can be written as a product but is better thought of as a change, and $dM = \rho\,dV$ can be written as a product but is better thought of as an amount. However, $dW = P\,dV$ is neither an amount nor a change, so we classify it as a product. Product infinitesimals are usually defined for students in terms of products; that is, students are told that the work is *by definition* equal to $P\,dV$ or $F\,dx$. The same is not true of $dM$ or $dv$.

Sometimes, there may be an ambiguity about which symbolic form should apply to a given situation. The most serious ambiguity in our framework is whether $dx$ in a length or one-dimensional density integral should be considered an "amount" of length or a "change" in position

along an imaginary path. We consider both conventions to be acceptable, depending upon the context of the problem. In other cases the issue is more clear-cut. In that same density integral, $dM$ refers to an infinitesimal amount of mass of an object. In this case, thinking about $dM$ as a change of mass, $\square - \square$, seems awkward and unusual in one dimension,[25] and impossible in more than one. It is possible for $dM$ to be a change quantity in the case that an object has a changing mass, for instance if sand is being poured onto a moving cart; but this is an entirely different usage of $dM$.

## D. Integral types and diagrammatic representations

Having defined three types of infinitesimals, we will now extend our framework by asking what types of integrals are associated with each infinitesimal type. We ground this development in students' representations of integrals. Because our students were participating in a sequence of lessons that we designed, we are examining their Zone of Proximal Development—their ability to talk about integrals with our assistance—and not only their prior knowledge.

We will first consider integrals involving an object's motion, which we call *trajectory integrals*. For such integrals, our students' diagrammatic representations were characterized by curly braces indicating spatial or temporal amounts, sequences of images to show that an object is moving (freeze frames), and arrows to indicate the influence of forces or the direction of velocity.

Figures 7–10 show a few examples of student work from the Smartpen study that demonstrate several possible variations. Notice that segments of a trajectory can be characterized as time intervals, displacements, or both in the same diagram. Students could indicate the motion using a sequence of intervals or using a single interval. As pointed out in the figure captions, students used similar techniques to depict work and displacement trajectories. Lynn (Figs. 9 and 10) was especially successful at connecting the mathematical idea of a sum to the two diagrammatic representations. This lends support to our grouping work and displacement integrals together as a single type.

However, many students' presentations about the trajectory domain omitted the diagram entirely, relying more on graphs. From these examples, we see that trajectory integrals can be associated with changes ($dt$, $dx$, $da$, $dv$) or products ($dW = F\,dx$) but not amounts.

Our second type of integral, the *static object integral*, involves an integral over the body of an object at a single instant in time. For this type of integral, students relied more strongly on the diagram during their presentation, using it at various points throughout their argument. Curly brackets were less common, possibly because the diagram depicts all points in the body of the object, rather than just the start and end points as in the trajectory integral. Amber's work from the case study is shown in Fig. 11. The picture is from her oral presentation on a blackboard, which a researcher copied by hand. Notice Amber's use of diagrams to represent both the stationary and rotating trough, as well as a small volume element "$dv = D\,y\,dr$."

It seems that static object integrals are usually associated with amounts such as $dV$ and $dM$, but are often associated with $dx$ as well, which may be viewed either as an amount of length or as a change $x_2 - x_1$.
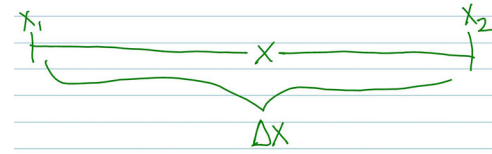


Fig. 7. Beatrice's depiction of $\int v\,dt$. This figure, as well as Fig. 8, illustrates Beatrice's representations of trajectory integrals. Whether she is drawing a displacement or work integral, Beatrice uses a similar diagram.
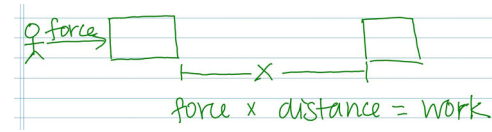


Fig. 8. Beatrice's depiction of $\int F\,dx$. For both displacement and work integrals, Beatrice shows the initial and final position of an object and draws an $x$ in the ground between the two.
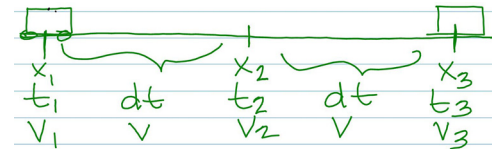


Fig. 9. Lynn's depiction of $\int v\,dt$. This figure, as well as Fig. 10, illustrates Lynn's representations of trajectory integrals. Whether she is drawing a displacement or work integral, Lynn uses a similar diagram.
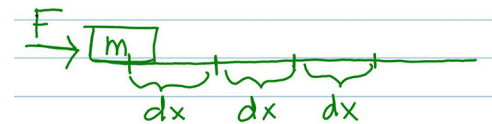


Fig. 10. Lynn's depiction of $\int F\,dx$. For both displacement and work integrals, Lynn shows multiple intervals of motion of an object. She also uses curly brackets to indicate infinitesimals, whether these infinitesimals are labeled $dt$ or $dx$.
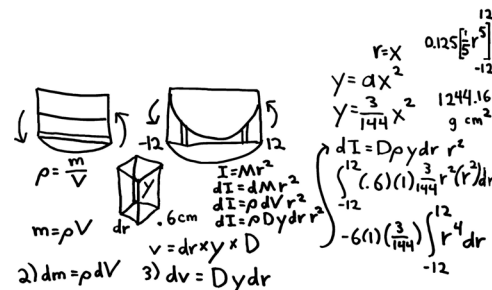


Fig. 11. Amber's depiction of the moment of inertia integral for a rotating trough of water.

Table III. Some integral types occurring in introductory mechanics and thermodynamics. The "Thermodynamic" type did not appear in our studies, but it is relevant to our framework due to its connection with expert knowledge about infinitesimals.

| Type | | Description | Orientation | Examples |
|---|---|---|---|---|
| Static object | | A sum over pieces of an object. | None | $M = \int \rho_1 dx$, $M = \int \rho_3 dV$, $I = \int dM r^2$ |
| Source-Path-Goal | Trajectory | An integral involving a moving object. | Time | $\Delta x = \int v\,dt$, $\Delta v = \int a\,dt$, $W = \int F\,dx$ |
| | Thermodynamic | Can be visualized as a path in state space or as the heating, expansion, and contraction of a gas. | Time | $W = \int P\,dV$, $Q = \int T\,dS$ |

### E. Integral types: Space, time, and matter

As we have seen, amounts do not arise in trajectory integrals, while they do appear in static object integrals. We postulate that an awareness of these integral types (trajectories and static object integrals) may be useful for students, since this would help students to predict whether infinitesimals should be interpreted as changes or amounts. Knowledge of integral types is also valuable for instructors who wish to apply our scaffolding framework, since it is helpful to be aware that different integral and infinitesimal types are likely to appear when particular subjects are taught.

Table III describes three types of integrals that occur in introductory mechanics and thermodynamics and Table IV illustrates their relationship with space, time, and matter. Each domain is also either "time-oriented" or "not oriented." Each domain's orientation has to do with the order in which the path of integration should be followed in one's imagination. Time-oriented domains have orders that are pre-defined by the arrow of time, since they represent a system's time evolution. An integral can be time-oriented even if time is not in the integrand; for instance, thermodynamic state space does not have a time dimension, but nevertheless the state space integral $\int P\,dV$ refers to a gas that is expanding or contracting over a period of time. One could also imagine an oriented space integral, such as a line integral of the electric field to find $\Delta V$. Integrals with no orientation have no intrinsic direction, and can be integrated in either direction with identical results. (Two or three dimensional integrals never have any orientation).[26]

Lakoff and Nuñez[27] describe the "Source-Path-Goal schema" as a fundamental metaphor in human cognition, whereby a moving entity (the "trajector") travels along a path. This schema fits with our trajectory and thermodynamic types, so we have classified both as "source-path-goal" integrals in Table III. In both cases, the source is defined as being the initial time and the goal the final time. A static object integral, on the other hand, is not oriented; it has no specific source or goal. Although it may be parameterized using a coordinate (such as $x$) that varies from $x_i$ to $x_f$,

the value of the integral is the same regardless of which end of an object is designated initial and which is designated final. Although thermodynamic and trajectory integrals are both classified as source-path-goal-integrals, we predict that students could represent the two integrals differently. An ideal gas can be visualized in a diagram either as a state space trajectory *or* as a depiction of a container full of the gas, but the latter representation does not apply to the trajectory integral. This would imply that our three integral types differ according to the ways in which they can be represented.

### F. Finite types

Finite quantities, like infinitesimal quantities, can be categorized using symbolic forms. Finite quantities such as force, density, and acceleration may be nontrivial for students to understand even outside of the context of the integral. However, we have not studied any student difficulties with finite quantities; therefore, we will present only a brief and tentative theory of finite types as a suggestion for future work.

Many finite quantities take the form of ratios; for example, $a = dv/dt$ and $\rho = dM/dV$. Acceleration is really a derived quantity that comes from a ratio of two infinitesimal changes, and density is derived from a ratio of two infinitesimal amounts. On the other hand, when force appears in the expression $dW = F\,dx$, it is not a derived quantity. The definition of a force given in a first-semester physics course is not $F = dW/dx$, but rather something like "a push or a pull." Thus, we could say that force is a "primitive" quantity and not a ratio. Coordinates, such as $r$, $x$, $y$, or $z$, are a special type of primitive quantity, so a theory of finite types in integrals might include three types: ratios, primitives, and coordinates.

### VI. DISCUSSION

Two attributes have been described in the present work: infinitesimal type and integral type. In separating these attributes, we do not mean to imply that they are independent; on the contrary, we include the integral type primarily to

Table IV. This table describes qualitatively how space, time, and matter play a role (or do not play a role) for each of our three integral types.

| Type | Space | Time | Matter |
|---|---|---|---|
| Static object | Amount of space | No time | Amount of an object |
| Trajectory | Motion through space | Motion takes time | Moving object |
| Thermodynamic | "State space" only | Time evolution is implied, even though $t$ does not appear in equations. | Transformations of an object (ideal gas) |

show how infinitesimal types are used. Infinitesimal amounts do not occur in trajectory integrals, and infinitesimal changes are less common in static object integrals. Nevertheless, we differentiate between the two attributes because they are concerned with different aspects of the integral. The integral type has to do with an integral's use of space, time, and matter, and whether it is conceptualized as an oriented path; an infinitesimal type has to do with the meaning of infinitesimal quantities, such as $dt$.

Because we group multiple physics integrals into a single category, our work is more abstract than a typical physics treatment of integration. On the other hand, our scaffolding framework is far less abstract than the mathematical treatment, which handles all integrals in the same way.[28] Therefore, we can construct a hierarchy of integral frameworks, with the mathematical understanding at the most abstract. Table I illustrates this hierarchy.

## VII. CONCLUSIONS

In this paper, we have presented a framework for scaffolding integration in the context of introductory calculus-based mechanics. This taxonomy includes three infinitesimal types and three integral types. We have emphasized student knowledge, student difficulties, and expert knowledge as tools for developing these categories, and we have presented student work in the form of spoken words, diagrams, and equations.

We proposed using Sherin's[9] concept of a "symbolic form" to classify infinitesimal and finite types. Our categorization posits that although physical quantities can be expressed mathematically in various ways, certain symbolic forms are more likely to connect with student understanding and expert usage. For instance, we can write $dx = x_2 - x_1$, but not $dM = M_2 - M_1$, if $dM$ is to represent a small amount of mass in an object. The infinitesimal type is especially relevant to instruction, as students' errors may arise from a confusion about which infinitesimal type is meant in a particular expression. This confusion may lead them to have difficulties understanding which integral type is being represented, because particular infinitesimal types are associated with specific integral types. In particular, some students described in Sec. V B assumed that $d$ referred to a trajectory integral involving the change of a physical quantity over time, so they attended to the motion of the rod rather than its spatial extent.

We suggest that our framework could be useful for instructors in a number of ways. First, instructors may wish to consider these issues when deciding whether to teach about multiple infinitesimal types in a particular course. If students are to learn about both $dv$ and $dM$, instructors should be aware that these two infinitesimals may seem very different to students. Teaching these ideas will give students a richer vocabulary for expressing their knowledge about infinitesimals, but will also require more support from the instructor. We suggest that instructors could scaffold students' understanding by using different notations for the change $dM$ and the amount $dM$. For instance, instructors could write $\delta M$, $\hat{d}M$, or *some* invented notation to indicate that we are talking about an amount and not a change. Such a change of notation would not be a quick, easy fix, but would have to be part of a carefully designed sequence of lessons. This would be an appropriate topic for future work.

Second, we found that most of the integrals in the first half of our introductory mechanics course were trajectory integrals, and most in the second half were static object integrals. Instructors who teach introductory mechanics in this order must take care

to provide guidance during the transition between these integral types because this transition may not be intuitive from the students' point of view. Third, we feel that any assessment of students' understanding of integration must take into account the diversity of infinitesimal and integral types. It would be a mistake to test students' understanding of trajectory integrals and then claim that they understand static object integrals.

Our observations of students have focused on a few integral types, namely trajectories and static object integrals. However, we have tentatively proposed another type—thermodynamic integrals—and we suggest that future work could analyze student representations of this type. We restrict ourselves to these three types of integrals partly because of the context of the course—first-semester calculus-based physics—in which we have carried out our investigations. Electricity and magnetism would present new variations, but would not necessarily require an entirely new framework. For instance, in the expression $dQ/dt$ (the rate of change of charge on a capacitor per unit time), $dQ$ is a change infinitesimal; whereas in $dQ = \lambda dx$, $dQ$ is an amount infinitesimal. These examples illustrate the use of change and amount infinitesimals in the context of electricity and magnetism. One new development is that we can perform a line integral to compute the difference in electric potential between two points. This integral is a source-path-goal integral, although it is unlike our trajectory integral in that its trajector—the object that moves along the trajectory—is a hypothetical test charge and not a physical object. In addition, $d\vec{E}$ is fundamentally an amount—an electric field that is generated by an amount of charge $dq$—but might correspond to a new infinitesimal subtype. To find the electric field at a point, we perform a static object integral over the body of the charged object, giving us the electric field at a point in space due to that object. The process differs from a basic static object integral in that as we sum the expression involving $dq$, we are imagining a superposition of tiny electric field vectors $d\vec{E}$. We predict that our scaffolding framework is general enough that it could be easily extended to integration problems in electricity and magnetism and even to differential equations. As a final thought, we note that all oriented integrals mentioned in this paper have been one-dimensional. We expect that an extended framework would be required to discuss topics that involve oriented two-dimensional integrals, such as flux integrals encountered in upper-division physics courses.

In summary, we propose that our scaffolding framework for infinitesimal and integral types may be useful to instructors of calculus-based mechanics courses and to developers of assessments about integration. Our framework provides a bridge between a mathematical view of integration, in which all integrals can be viewed as Riemann sums, and a physical view, in which every integral is different from every other.

## ACKNOWLEDGMENTS

## APPENDIX: DIFFERENTIAL GEOMETRY

Physicists and mathematicians do not generally agree on the meaning of the symbol $d$. As mentioned previously, $d$ has at least four possible mathematical interpretations, one of which comes from differential geometry.

We will present an overview of the relevant idea in differential geometry, keeping in mind that the subject is extraordinarily rich and complex, and it would be impossible for us to give a rigorous or complete picture in this small amount of space.

In some interpretations of the differential, $dx$ is not an independent object; rather, the $d$ is a shorthand for a limit[19]—a notation that is not meaningful unless it is part of the derivative operator $d/dx$ or the integral operator $\int dx$. However, in differential geometry, the situation is different: the $d$ refers to an exterior derivative, and it operates on objects called differential forms. Differential forms allow us to explain what $dx$ means, although we will run into trouble when we try to define $dM$ and $\delta W$. For the simplest example of an exterior derivative, consider a coordinate such as $r$. This coordinate can be viewed as a function that assigns a value to each point $p$ in space. The assigned value $r(p)$ is the distance from the origin to that point. Many quantities in physics can be viewed as functions of a point in space or time, including coordinates $x$, $y$, $z$, density $\rho$, time $t$, velocity $v$, and so forth.

Now the exterior derivative $d$ changes $r$ into a different function, called $dr$. Whereas $r$ assigns values to points in space $p$, $dr$ effectively assigns values to infinitesimal segments of paths by describing the change in $r$ per each unit of the path.

To be more accurate, we should say that $dr$ assigns values to parameterized paths in space that start from a given point $p$. If a path $P$ is parameterized by a variable s, then $dr$ of the path, which we can denote $dr(P)$, is equal to the derivative $dr/ds$. Due to this definition, the only part of the path that matters is the part very close to the point p. Then $dr/ds$ is effectively a ratio between infinitesimal changes in $r$ and infinitesimal changes in $s$.

Now, one problem with $dM$ is that there is no such thing as $M(p)$. Given a particular point on a three-dimensional object, we should not ask "what is the $M$ of this point?" The same problem occurs with work in state space. Given that the state variables are $P$ and $V$, we cannot ask for the "work at a point" $W(P, V)$. Work is a property of state space trajectories, not of points. Therefore, differential geometry does not permit us to talk about "$dW$" in state space. Another way of saying this is that work is path-dependent and is not a state variable. Some physicists will write $\delta W$ instead of $dW$ to remind themselves of this, but as far as we know, physicists do not write $\delta M$. (Mathematicians do not abide perfectly by this notation either; they commonly write $dV$ to indicate an infinitesimal volume, although they are aware that there is no differential form called $V(p)$ to take the $d$ of.)

One idea in differential geometry that proved important in our framework is the idea of *exact* differential forms. Exact forms are those that can be written as the exterior derivative of another form, as above. Thus, the infinitesimal $dM$ does not correspond to an exact differential form. Changes correspond to exact forms, while amounts and products do not. Another aspect of differential geometry that we considered incorporating into our framework was the idea of the *manifold*—a mathematical space that could represent physical space, time, thermodynamic state space, and many other "spaces." However, the framework presented here differentiates between trajectory and static object integral because the abstraction of the manifold is farther from students' experiences.

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[28] The definition of the Riemann sum might vary slightly between polar and rectangular coordinates or between one- and two-dimensional integrals, but our taxonomy has to do with types of physical quantities and not with coordinatization or dimensionality.