DEFORMATIONS OF DIFFERENTIAL OPERATORS

by

BRYAN E. BISCHOF

B.S., Westminster College, 2008

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

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Abstract

The Weyl algebra is the algebra of differential operators on a commutative ring of polynomials in finitely many variables. In [8], Hayashi defines an algebra which he refers to as the quantized n-th Weyl algebra given by a deformation of the classical Weyl algebra. In [21], Lunts and Rosenberg define β and quantum differential operators for localization of quantum groups by deforming the relations that algebras of differential operators satisfy. In [13], Iyer and Mccune compute the quantum differential operators on the polynomial algebra with n variables. One naturally wonders "What is the relationship between the quantized Weyl algebra and the quantum differential operators on the polynomial algebra with n variables?" In this thesis we answer this question by comparing the natural representations of $U_q(\mathfrak{sl}_2)$ emerging from each algebra. Additionally, we connect the differential operators on the big cell of the flag variety of $U_q(\mathfrak{sl}_n)$ with our deformed algebras. We also show the relationship between these algebras of differential operators and those appearing in the quantum Beilinson-Bernstein equivalence. Next we discuss analagous results in the case of β-differential operators, as introduced in [21]. We consider both deformations on the underlying coordinate rings, and of the algebra of differential operators. We relate these results to the gluing problem for differential operators on noncommutative coordinate rings. We collect some of the different deformations of the usual Weyl algebra, and compare them based on a common bicharacter β. Finally, we show a geometric result need in order to be able to glue deformed spaces and have their algebras of deformed differential operators cohere.

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Abstract

The Weyl algebra is the algebra of differential operators on a commutative ring of polynomials in finitely many variables. In [8], Hayashi defines an algebra which he refers to as the quantized n-th Weyl algebra given by a deformation of the classical Weyl algebra. In [21], Lunts and Rosenberg define β and quantum differential operators for localization of quantum groups by deforming the relations that algebras of differential operators satisfy. In [13], Iyer and Mccune compute the quantum differential operators on the polynomial algebra with n variables. One naturally wonders "What is the relationship between the quantized Weyl algebra and the quantum differential operators on the polynomial algebra with n variables?" In this thesis we answer this question by comparing the natural representations of $U_q(\mathfrak{sl}_2)$ emerging from each algebra. Additionally, we connect the differential operators on the big cell of the flag variety of $U_q(\mathfrak{sl}_n)$ with our deformed algebras. We also show the relationship between these algebras of differential operators and those appearing in the quantum Beilinson-Bernstein equivalence. Next we discuss analagous results in the case of β-differential operators, as introduced in [21]. We consider both deformations on the underlying coordinate rings, and of the algebra of differential operators. We relate these results to the gluing problem for differential operators on noncommutative coordinate rings. We collect some of the different deformations of the usual Weyl algebra, and compare them based on a common bicharacter β. Finally, we show a geometric result need in order to be able to glue deformed spaces and have their algebras of deformed differential operators cohere.

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Acknowledgments

I will leave thanks to those who wont read this to my mind, my heart, and the wind; instead focusing on those who might.

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Dedication

This work is dedicated to the memory of my advisor Alexander Rosenberg; "Sasha", as many knew him (and I was sternly informed not to refer to him as professionally). Alex passed away during my last year of work towards this thesis. The last conversation between us, over the phone, through intense physical pain on his end, was ninety-nine percent mathematics. His only concern at that time was whether the quantum Schubert cells had hyperbolic algebras as their differential operator algebras. They do, although in a different category than **Vect**.

This thesis, and my entire mathematical perspective, could not exist but for Alex. The memories I have of him, at the board, in white khakis, a black shirt, and chalk on his nose, are unforgettable.

I once made the mistake of referring to someone's "former advisor". Alex was quick to correct me: "there is no 'former advisor', just like there is no 'former father"'. I can't possibly express it any better; as I have two advisors, I have two fathers.

You've always been one of each.

Chapter 1

Introduction

1.1 An overview

For commutative algebras, one has the usual definition of the algebra of differential operators. This algebra is a filtered, noncommutative algebra, containing the original commutative algebra in the zero-eth degree. If we wish to consider a noncommutative analog of these differential operators there are two main trajectories:

1. one can consider algebras which are noncommutative, and find their differential operators

2. one can consider differential operators which satisfy specific noncommutativity rules.

We will consider both. The former, we refer to as noncommutative spaces, as we think of the noncommutative algebras as function rings on some "spaces". The latter will be deformations of differential operators, which will be used in relation to quantum groups. We will also combine these two into a coherent picture.

These two considerations lead to three problems which I address in this thesis:

1. The relationship between the differential operators of Lunts and Rosenberg, which is known to be necessary for a quantum analog of the Beilinson-Bernstein localization

theorem, and a classical deformation of the Weyl algebra, and why they have similar actions on quantum spaces. Henceforth, we refer to this as *comparing Weyl algebras*.

2. To what degree deforming algebras of differential operators and deforming the spaces on which they act are related. This problem will be called *twisting and untwisting* in the sequel.

3. How gluing pieces of these spaces together behaves with respect to differential operators, and how we might glue algebras of differential operators together to get sheaves of quantum differential operator algebras on commutative spaces. We will identify this as a *gluing problem* for the rest of this thesis.

1.2 Outline

Exploring the connections between a semi-simple Lie algebra $\mathfrak{g}$ and Weyl algebras is a classical story. In the particular geometric setting of the flag variety G/B of $\mathfrak{g}$, there are (at least) two connections between the Weyl algebra and $\mathfrak{g}$:

(I). there is a morphism $\phi : U(\mathfrak{g})/\operatorname{ann}(M_\lambda) \twoheadrightarrow A_n$ for $n = |\Lambda^+|$, the number of positive roots of $\mathfrak{g}$, M_λ a Verma module, and $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. This is the so-called *Conze embedding*.

(II). there is a morphism $\psi : U(\mathfrak{g})/\operatorname{ann}(M_\lambda) \to D_\lambda(G/B)$, arising from the Beilinson-Bernstein localization theorem, with $D_\lambda(G/B)$ the twisted differential operators on the flag variety.

In an unpublished note by Hodges and Smith [10], a morphism between A_n and $D_\lambda(G/B)$ which commutes with these two maps is established. In particular, A_n is isomorphic to the twisted differential operators on the Schubert cell corresponding to the longest element of the Weyl group of G, a subalgebra of $D_\lambda(G/B)$.

We begin this thesis by establishing an analagous morphism in the quantum case; i.e. between $A_q(n)$, the quantum Weyl algebra, and $D_q(\mathbb{C}_q[n])$, the quantum differential operators on the big cell of the quantum flag variety for $U_q(\mathfrak{sl}_n)$.

We consider $U_q(\mathfrak{g})$, the quantum group at q, for q not a root of unity. In [8], Hayashi defined a deformation of A_n which we denote by $A_q(n)$. In [8] and [24] it is shown that this algebra, a deformation of the Weyl algebra, has a useful ring theoretic property known as the *generalized Weyl algebra structure*. Further, for it exists an analog of (I), i.e., a mapping $U_q(\mathfrak{sl}_n) \to A_q(n)$.

In [21] a related deformation of the Weyl algebra, denoted D_q, is given by deformation at the level of endomorphisms, on a ring using the so-called Grothendieck definition of differential operators. In [21], [22], and [26], it is shown that the latter deformation preserves much of the geometric structure and an analog of (II) for D_q holds.

We compare these two deformations from a purely algebraic point of view. In the classical case one can use morphisms (I) and (II) to make statements about representation theory, so we consider these statements in the quantum case. In particular, we see that representations of $U_q(\mathfrak{g})$ induced by the quantum differential operators of [8] are reflected in the quantum flag variety and those quantized differential operators of [21]. Note that this reconciles the result of [9] and [17] with the results of [21] which were different approaches to finding quantum analogs of (II).

Chapter 2 concerns itself with (I) and (II) and is organized as follows: it begins by recalling the definitions of Hayashi's quantum Weyl algebra as a deformation of the usual Weyl algebras (Section 2.1), and the construction of quantum differential operators by Lunts-Rosenberg, and the explicit construction by Iyer-McCune (Section 2.2); next we construct explicitly the connection between Hayashi's quantum Weyl algebra in the 1-dimensional case, and the quantum differential operators on the affine line (Section 2.3); further, we extend this relationship to the 2-dimensional quantum Weyl algebra, and the quantum affine line. Since both of these geometries lead to $U_q(\mathfrak{sl}_2)$-modules, we compare their representations, and prove that they are the same (Section 2.3.4). We conclude by extending these results to $U_q(\mathfrak{sl}_n)$ by constructing representations of $U_q(\mathfrak{sl}_n)$ on quantum differential operators on quantum affine spaces (Section 2.4).

Now equipped with q-differential operators on general rings, we are primed to consider other deformations of algebras of differential operators. In [21], in the process of defining the quantum differential operators, Lunts and Rosenberg define β-differential operators on a graded noncommutative ring. If the ring is graded by an abelian group Γ, then β is taken to be a bicharacter $\Gamma \times \Gamma \to \mathbb{K}$ for some ground field $\mathbb{K}$. What we discuss in Chapter 2, is when β is specialized to be q^{ab} for a and b are graded degrees of ring elements. This special case is what the authors of [21] call q-differential operators and play the role in the quantum case that differential operators serve in the classical case.

Recall that given a graded algebra over $\mathbb{K}$, there is a natural deformation of that ring by a 2-cocycle of the grading group. In particular, for a Γ-graded algebra R, and γ a 2-cocycle in the group cohomology for Γ, we write R^γ for the noncommutative algebra with a new multiplication

$$r \star s = \gamma(|r|\,, |s|)rs,$$

where concatenation is usual algebra multiplication, and $|\bullet|$ denotes graded degree.

A natural question when observing the definition of the ring of β-differential operators is: "when will β-differential operators on a ring coincide with 'untwisted' differential operators on a deformation of the ring?" A more precise formulation is: given a Γ-graded ring and γ some 2-cocycle for Γ, is there a relationship between β, a bicharacter, and γ for $D_\beta(R^\gamma)$. Chapter 3 is dedicated to proving the following result,

Theorem 1.2.1. $(D_{\beta^\gamma}(R))^\gamma \simeq D_\beta(R^\gamma)$

where β^γ is another bicharacter.

In Chapter 2, the relationship between the quantum Weyl algebra constructed in [8], and the algebra of quantum differential operators on affine space and quantum affine space ([13],

[14]) is indentified. Keeping with this theme, we define a similar β-Weyl algebra (modeled after $\mathcal{A}_q(n)$) and show that the morphisms existing in Chapter 2, also exist in this setting. Further, we discuss the relationship between $D_q(\mathbb{C}_q[x_1, \ldots, x_n])$ and $D_\beta(\mathbb{C}[x_1, \ldots, x_n])$. Finally, we show some "functoriality" properties of $D_q(-)$ similar to those in [12], which are useful for computation purposes.

For the last task of this thesis, we return to the geometry. We explore the gluing conditions necessary for these deformed algebras. When considering algebraic spaces, localization is the recipe for building sheaves of algebras from our algebras associated to rings of functions. In our Chapter 2 discussion of sheaves of D-modules, we used the fact that localization by a multiplicative subset, S^{-1}, of a ring R, commutes with the functor D. D is the functor which sends a quasi-coherent sheaf to a sheaf of D-modules. In light of Chapters 2 and 3, we need to check three new cases: β-differential operators for commutative rings, differential operators for noncommutative rings, and β-differential operators for noncommutative rings. In particular, if R is commutative, S_w a multiplicative subset, then for what pairs of bicharacters (β, β') is

$$S_w^{-1} D_\beta(R) \subset D_{\beta'}(S_w^{-1} R)?$$

And similarly for R noncommutative with S_w an Ore subset. We answer this question in Chapter 4.

Another task of Chapter 4 is to compute our "quantum quotient rules". Similar to the quotient rule of differential calculus, these give computational rules for how to evaluate the images of differential operators under localizations on functions involving rational functions in the commutative and noncommutative settings.

However, the main utility of these computations is for gluing. This quantum analogy of transition maps from differential geometry requires a cohesion of deformation parameters; this thesis culminates in proving the existence and conditions for this cohesion. We tease with a final section on an example of the kind of quantum space that might be built with all these deformed geometries.

So, to recapitulate, we show what deformations of differential operators have been in the past, collimate them, generalize the notion, prove how they behave with deformed spaces, and provide the tools to create some new spaces out of them.

1.3 Essential preliminaries on representation theory

Representations of a given algebraic object are structure-preserving self-maps of a vector space which satisfy the algebraic relations of that object. One of our most prototypical examples is a group action on a vector space as a group homomorphism:

$$G \to \mathrm{End}(V)$$

for a group G. Here $\mathrm{End}(V)$ is the ring of all linear transformations on a vector space V.

We will take G to be a connected simple reductive algebraic group, with $\mathfrak{g}$ its corresponding semisimple Lie algebra. For B a Borel subgroup of G, define:

Definition 1.3.1. The *flag variety* of G is the space of Borel subalgebras of $\mathfrak{g}$ as identified with the quotient variety, G/B. This is a projective variety with transitive G action, for G a reductive algebraic group over $\mathbb{C}$.

Remark 1.3.2. If we recall the notion of a flag manifold as the space of full flags in $\mathbb{C}^n$, we can see that the stabilizer under group action of the standard flag in $\mathbb{C}^n$ is a Borel subgroup.

Example 1.3.3. If $G = \mathrm{SL}_n$, then B is the upper triangular matrices in G.

Definition 1.3.4. The *Gel'fand Model*, $\mathcal{R}$, associated to a Lie algebra, $\mathfrak{g}$, is defined as a $\mathcal{P}^+$-graded algebra $\mathcal{R} := \oplus_{\lambda \in \mathcal{P}^+} R_\lambda$ for $\mathcal{P}^+$ the dominant integral weights in $\mathfrak{h}^*$, with B corresponding to the negative roots, and R_λ the highest-weight representation, with weight λ. Further, we observe that this ring has its multiplication $\otimes$ given the by tensor product of irreducible representations composed with projection onto its highest weight component, i.e. $R_\lambda \otimes R_\mu \to R_{\lambda+\mu}$, so $\mathcal{R}$ is a $\mathcal{P}^+$-graded algebra.

Lemma 1.3.5. *$G/B \simeq \mathbf{Proj}(\mathcal{R})$ with $\mathbb{N}$-gradation by weights.*

Returning to the flag description of the flag variety, we can define a geometric decomposition. Fix a maximal torus T in G.

Definition 1.3.6. Let W be the Weyl group of G. For $w \in W$, define F_ω for ω as a T-fixed point of G/B corresponding to ω, i.e., $F_\omega := \omega F$ for F the standard coordinate flag, and we define $C_\omega := BF_\omega = UF_\omega$. Call these orbits of B the *Schubert Cells*.

Definition 1.3.7. The Zariski closures $\overline{C_\omega}$ are called *Schubert Varieties*, as they are closed subvarieties of G/B.

The main observation about the Schubert cells decomposition of G/B from the Bruhat decomposition, $G = \dot{\coprod}_{w\in W} B\omega B$, is:

Proposition 1.3.8. $G/B = \dot{\coprod}_{\omega\in W} C_\omega$.

Additionally, we see that the Schubert varieties are built from subordinate cells:

Proposition 1.3.9. $\overline{C_\omega} = \bigcup_{\nu\leq\omega} C_\nu$.

Definition 1.3.10. The big cell U_{ω_0}, for ω_0 the longest element of W, is the Schubert cell associated to ω_0. We will also be particularly interested in its translates. An important aspect of big cells is that they are affine spaces isomorphic to $\mathbb{A}^N$ for N the number of positive roots of $\mathfrak{g}$.

Can one describe the Schubert cells of a flag variety G/B in terms of ideals of $\mathcal{R}$?

Consider $\mathcal{O}_X$ the structure sheaf of a scheme X. We recall some facts about intersections of subschemes:

Definition 1.3.11. Consider X a scheme with cover $\{U_i \mid i \in I\}$ and $M \in \mathcal{O}_X - mod$. We have maps

$$u_i : \Gamma(X, M) \to \Gamma(U_i, M),$$

for all i. In the case when $\Gamma(U, M) = M(U)$ then this is given by localization of rings. For U_{ij} the intersection of U_i and U_j,

$$\Gamma(U_i, M) \to \Gamma(U_{ij}, M) \leftarrow \Gamma(U_j, M)$$

we consider these *transition maps* from the embedded U_{ij} in U_i to U_j.

We will also need to introduce classical D-modules; we recall from [6]. For a $\mathbb{K}$-algebra, R.

Example 1.3.12. We can see that the Leibniz rule is the condition for the degree 1 differential operators. Assume that R consists of commutative ring of functions, and ∂ is a degree one differential operator, if $\partial \in \mathrm{End}_{\mathbb{K}}(R)$

$$\partial(fg) = \partial(f)g + f\partial(g),$$

for all $f, g \in R$, regarding $f \in R$ and $R \hookrightarrow \mathrm{End}_{\mathbb{K}}(R)$ by left multiplication. We see that in $\mathrm{End}_{\mathbb{K}}(R)$:

$$[[\partial, f], g] = [\partial, f]g - g[\partial, f] = \partial \circ f \circ g - f \circ \partial \circ g - g \circ \partial \circ f + g \circ f \circ \partial = 0.$$

Definition 1.3.13. The *algebra of differential operators* on a ring R, $D(R)$, is the set of linear endomorphisms ∂, of R with some $n \in \mathbb{N}$, such that $[\dots[[\partial, r_0], r_1], \dots, r_n] = 0$ for any sequence $\{r_i\}_{1 \leq i \leq n}$ of $r_i \in R$. This set has an algebra structure we we denote $D(R)$.

It is easy to see a filtration on this algebra by $\mathbb{N}$:

Definition 1.3.14. For $\partial \in D(R)$ and n such that $[\dots[[\partial, r_0], r_1], \dots, r_n] = 0$ for any sequence $\{r_i\}_{0 \leq i \leq n}$ of $r_i \in A$, then we say ∂ is a differential operator of degree at most n and collect these into a set D^n. This filtration is called the *D-filtration*.

This is the correct definition to generalize to the noncommutative ring case.

Definition 1.3.15. For X a affine algebraic variety over $\mathbb{C}$, *differential operators on X of degree at most k*, written $\mathcal{D}_X^k(X)$, are differential operators of degree k for the ring $\mathcal{O}_X(X)$ the ring of global sections.

We can also think of these as sheaves of algebras if we sheafify the presheaf of differential operator algebras on the images of the open sets under the global sections functor. Denote $\mathcal{T}_X$ the *tangent sheaf* defined as the sheaf of algebras of derivations on the algebras of functions.

Remark 1.3.16. We see that $\mathcal{O}_X \hookrightarrow (\mathcal{D}_X)^0 \subset \mathcal{D}_X$ as sheaves. In fact, one can show that $\mathcal{O}_X$ and $\mathcal{T}_X$ generate $\mathcal{D}_X$. Thus $\mathcal{D}_X^k \in \mathcal{O}_X - \mathbf{Mod}$.

Definition 1.3.17. For $f \in \mathcal{O}_X(X)$, and X_f the principle open set of X defined by f,

$$\mathcal{D}_X(X_f)^k = \mathcal{D}_X(X)^k \otimes_{\mathcal{O}_X(X)} \mathcal{O}_X(X_f),$$

i.e. localization of differential operators is localization as $\mathcal{O}_X$-modules.

We will omit the subscript X if it understood.

Remark 1.3.18. $\mathcal{D}$ is quasi-coherent $\mathcal{O}_X - \mathbf{Mod}$.

Definition 1.3.19. A *$\mathcal{D}$-module* is a sheaf of $\mathcal{D}$-modules, which by our previous remark is an $\mathcal{O}_X$-module with $\mathcal{T}_X$ actions. Localization is by extension of scalars. We denote the category of left (resp. right) $\mathcal{D}$-modules of X by $\mathcal{D}_X - \mathbf{Mod}^l$, (resp. $\mathcal{D}_X - \mathbf{Mod}^r$).

Definition 1.3.20. For an affine scheme $X = \operatorname{Spec} R$, R a commutative ring, and M an R module, we define the *localization functor* as the map which sends $R - \mathbf{Mod} \to \mathcal{O}_X - \mathbf{Mod}$ by extension of scalars:

$$M \mapsto \mathcal{O}_X \otimes_R M$$

which we see to be a sheaf of modules.

Definition 1.3.21. A variety is *$\mathcal{D}$-affine* if each D-module in $\mathcal{D}_X - \mathbf{Mod}$ is generated by global sections, or equivalently

$$\Gamma(U, \mathcal{D}_X) := \mathcal{D}_X(U) \simeq D(\mathcal{O}_X(U))$$

Remark 1.3.22. This can be rephrased as $H^i(X, \mathcal{D}_X) = 0$ for $i \geq 1$.

This is an analogue of Serre's affine global sections theorem, that affine schemes are those whose quasi-coherent sheaves are built by localizations of modules over the ring of global sections.

Theorem 1.3.23. *(Beilinson-Bernstein) G/B is $\mathcal{D}$-affine, and if X is $\mathcal{D}$-affine, then $\Gamma : \mathcal{D}_X - \mathbf{Mod} \to \mathcal{D}(X) - \mathbf{Mod}$ and its adjoint, $\mathcal{D}_X \otimes_{\mathcal{D}_X} - : \mathcal{D}(X) - \mathbf{Mod} \to \mathcal{D}_X - \mathbf{Mod}$, is an equivalence of categories.*

Remark 1.3.24. One sees that those modules are naturally $U(\mathfrak{g})$-modules by the fact that G acts on G/B, and thus the localization functor above constructs a $U(\mathfrak{g})$-module from a $\mathcal{D}$-module. This theorem is also true for sheaves which are twisted by weights λ and modules of the form $U(\mathfrak{g})/\operatorname{ann}(M_\lambda)$. These twists are constructed as line bundles associated to the weights and tensored with our sheaf.

Consequently, the theorem of Beilinson and Bernstein establishes our motivational connection between algebras of differential operators and representations of Lie algebras.

1.4 Quantum algebras

We wish to consider noncommutative spaces, and above we sketched the connection to representation theory. This thesis' raison d'étre is to transition from classical representation theory of Lie algebras, to the representation theory of quantum groups. It is precisely the quantum geometries analogous to the flag variety which carry the noncommutative nature, and necessitate the deformation theory we explore in this thesis. Where once were algebraic schemes corresponding to lie algebras, now come algebraic stacks corresponding to a deformation of the Lie algebras.

Quantum groups arise as a particular kind of deformation of the Hopf algebras associated to the classical Lie algebras. For the noble ideal of self-containedness, we will now outline an appropriately focused introduction to quantum groups. Additionally, we introduce the essential geometric presentation of them which directs our gaze to noncommutative geometry.

Let's begin with some simple algebraic arguments and definitions. They will turn out to be sufficient.

Define an algebra $U(\mathfrak{sl}_2)$ by generators and relations,

Definition 1.4.1. $U(\mathfrak{sl}_2) := \mathbb{C}\langle e, f, h\rangle \,/ < he - eh - 2e, hf - fh + 2f, ef - fe - h >$ which we call the *enveloping algebra of* $\mathfrak{sl}_2$.

And let us think of this algebra as its image embedded in $D(\mathbb{C}[x, y])$,

Lemma 1.4.2. $U(\mathfrak{sl}_2) \hookrightarrow D(\mathbb{C}[x, y]^r) \hookrightarrow D(\mathbb{C}[x, y])$.

Proof. Consider $e \mapsto x\partial_y$, $f \mapsto y\partial_x$, $h \mapsto x\partial_x - y\partial_y$. Then we need check that the following relations hold:

$$\begin{aligned}
x\partial_x x\partial_y - y\partial_y x\partial_y - x\partial_y x\partial_x + x\partial_y y\partial_y - 2x\partial_y &= 0,\\
x\partial_x y\partial_x - y\partial_y y\partial_x - y\partial_x x\partial_x + y\partial_x y\partial_y + 2y\partial_x &= 0,\\
x\partial_y y\partial_x - y\partial_x x\partial_y - x\partial_x + y\partial_y &= 0.
\end{aligned}$$

so we are finished. □

Remark 1.4.3. If we assume some ordering with $x < y$, this representation of $U(\mathfrak{sl}_2)$ on the polynomial algebra $\mathbb{C}[x, y]$ recovers the familiar terminology that e is a *raising operator*, f is a *lowering operator*, h preserves homogenous polynomial degree.

Now we make an observation about this representation:

Remark 1.4.4. $e \cdot x^n y^0 = 0$ for all n, $f \cdot x^0 y^m = 0$ for all m, and $h \cdot x^n y^m = (n-m)x^n y^m$ for all n and m. So for homogeneous degree r polynomials, $\mathbb{C}[x,y]^r$ for any r, x^r is the highest-weight vector of weight r, y^r is the lowest-weight vector of weight r, and $x^n y^{r-n}$ is an eigenvector for h for all n.

Moreover, let us recall that there is a geometric interpretation of $\mathbb{C}[x,y]$:

Remark 1.4.5. $\mathbb{C}[x,y] = \bigoplus_n H^0(\mathbb{P}^1, \mathcal{O}(n))$, i.e., the *homogeneous coordinate ring*, i.e., $\mathbb{P}^1 = \mathrm{Proj}(\mathbb{C}[x,y])$. This is true more generally for projective spaces [7].

Lemma 1.4.6. *$\mathbb{P}^1$ has an affine covering by two affine lines.*

Proof. Consider $A = \mathbb{C}[x,y]$. Then $A_x = \mathbb{C}[x,y][x^{-1}]$, the localization by $< x >$, the multiplicative set generated by x is graded. Taking the degree zero component yields $A_{(x)} = \mathbb{C}[y/x] =: \mathbb{C}[t]$. Similarly, $A_y = \mathbb{C}[x,y][y^{-1}]$ is the localization by $< y >$ so $A_{(y)} = \mathbb{C}[x/y] =: \mathbb{C}[t^{-1}]$. Both spaces constructed by localizations are affine and are glued by common localization, with the association $t = y/x$. $\square$

Definition 1.4.7. Define a $\mathbb{C}$-algebra homomorphism$\psi : U(\mathfrak{sl}_2) \to D(\mathbb{C}[t])$ generated by

$$\begin{aligned} e &\mapsto \partial_t, \\ f &\mapsto -t^2\partial_t, \\ h &\mapsto -2t\partial_t, \end{aligned}$$

so $e(t) = 1$, $f(t) = -t^2$, and $h(t) = -2t$.

We check that it is well defined.

Lemma 1.4.8. *$\psi : U(\mathfrak{sl}_2) \hookrightarrow D(\mathbb{C}[t])$ by $e \mapsto \partial_t, f \mapsto -t^2\partial_t, h \mapsto -2t\partial_t$ is an $\mathbb{C}$-algebra morphism.*

Proof. Let us check the relations:

$$\begin{aligned} he - eh - 2e &\mapsto (-2t\partial_t)(\partial_t) - (\partial_t)(-2t\partial_t) - 2(\partial_t) = 0, \\ hf - fh + 2f &\mapsto (-2t\partial_t)(-t^2\partial_t) - (-t^2\partial_t)(-2t\partial_t) + 2(-t^2\partial_t) = 0, \\ ef - fe - h &\mapsto (\partial_t)(-t^2\partial_t) - (-t^2\partial_t)(\partial_t) - (-2t\partial_t) = 0. \end{aligned}$$

□

It is useful to investigate the kernel of ψ:

Lemma 1.4.9. *Recall the Casimir element is $\mathfrak{c} = (h-1)^2 + 4ef$; $\mathfrak{c} - 1$ lies in the kernel of ψ.*

Proof. Simply consider the image:

$$\mathfrak{c} - 1 \mapsto (-2t\partial_t - 1)^2 + 4(\partial_t)(-t^2\partial_t) - 1 = 0.$$

□

We have similar computations for the other affine patch, with parameter t^{-1}. By our earlier computation, we see that these patches glue together and respect our representations. The *global sections of the sheaf of algebras of differential operators, $\mathcal{D}$, on* $\mathbb{P}^1$ comprise a subring of $D(\mathbb{C}[t]) \oplus D(\mathbb{C}[t^{-1}])$ with an identification by gluing. In particular, we write R_t for the ring of differential operators on the affine line associated to localizing by the multiplicative set generated by t (and $R_{t^{-1}}$ vice-versa),

$$\Gamma(\mathcal{D}, \mathbb{P}^1) = R_t \oplus R_{t^{-1}} \hookrightarrow D(\mathbb{C}[t]) \oplus D(\mathbb{C}[t^{-1}]).$$

This is precisely what we need to connect differential operators on open sets of the flag variety with representations of the Lie algebra. Thus, the simplest example of the localization theorem of Beilinson-Bernstein has been illustrated:

Theorem 1.4.10. $U(\mathfrak{sl}_2)/\langle \mathfrak{c}-1\rangle \simeq R_t \oplus R_{t^{-1}}$.

We have seen a low-brow approach to the Beilinson-Bernstein theorem which lifts this isomorphism to the categories of modules. Now we move towards quantum groups.

1.4.11 Introduction to quantum algebras

Let $q \in \mathbb{C} \setminus \{0, \pm 1\}$. For a systematic introduction and proofs, one may consult a book on quantum groups such as [19]. Throughout the thesis we use the notation $[a, b]_c := ab - cba$.

Definition 1.4.12. Consider $\mathbb{C}\langle E, F, K^{\pm 1}\rangle$ with relations

$$\begin{aligned} [K, E]_{q^2} &= 0, \\ [K, F]_{q^{-2}} &= 0, \\ [E, F] &= \frac{K - K^{-1}}{q - q^{-1}}, \\ KK^{-1} = K^{-1}K &= 1, \end{aligned}$$

and call this algebra $U_q(\mathfrak{sl}_2)$.

Proposition 1.4.13. $\left\{E^n K^m F^l \mid m \in \mathbb{Z}, l, n \in \mathbb{Z}_{\geq 0}\right\}$ *is a linear basis for* $U_q(\mathfrak{sl}_2)$.

Definition 1.4.14. The *quantum casimir* is defined by $\mathfrak{c}_q := \frac{Kq^{-1}+K^{-1}q}{(q-q^{-1})^2} + EF$. It lies in the center and for q not a root of unity it generates the center of $U_q(\mathfrak{sl}_2)$.

Now we consider a couple of the automorphisms of $U_q(\mathfrak{sl}_2)$,

Definition 1.4.15. Define a $\mathbb{C}$-algebra automorphism $\Theta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)$ such that

$$E \mapsto F, \qquad F \mapsto E, \qquad K \mapsto K^{-1},$$

so $\Theta^{-1} = \Theta$, and $\vartheta_{\alpha,n,\nu} : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)$ for $\alpha \in \mathbb{C}, n \in \mathbb{N}, \nu \in \mathbb{C}$ such that

$$E \mapsto \alpha K^n E, \qquad F \mapsto \nu\alpha^{-1}q^{-2n}K^{-n}F, \qquad K \mapsto \nu K$$

thus $\vartheta_{\alpha,n,\nu}^{-1} = \vartheta_{\nu^n\alpha^{-1},-n,\nu}$.

Lemma 1.4.16. *If q is not a root of unity,* $\mathrm{Aut}(U_q(\mathfrak{sl}_2)) = \mathbb{C}\langle\Theta,\vartheta\rangle$*, generated by two elements [4].*

Definition 1.4.17. $U_q(\mathfrak{sl}_2)$ is a Hopf algebra with Hopf structure (see [19] for a complete definition),

$$
\begin{aligned}
\Delta(E) &= E\otimes K + 1\otimes E, \\
\Delta(F) &= F\otimes 1 + K^{-1}\otimes F, \\
\Delta(K) &= K\otimes K, \\
S(E) &= -EK^{-1}, \\
S(F) &= -KF, \\
S(K) &= K^{-1}, \\
\varepsilon(E) &= \varepsilon(F) = 0, \\
\varepsilon(K) &= 1.
\end{aligned}
$$

Remark 1.4.18. Recall that we call elements such that $\Delta(g) = g\otimes g$ *grouplike elements*.

Proposition 1.4.19. *The only automorphisms preserving the Hopf structure are $\vartheta_{\alpha,0,1}$. [4]*

Proposition 1.4.20. $U_q(\mathfrak{sl}_2) \simeq U_p(\mathfrak{sl}_2)$ *iff* $q \in \{\pm p^{\pm 1}\}$.

Proof. One can check that the only grouplike elements are K^n for $n \in \mathbb{Z}$. Then any purported automorphism must send K to K^n and it's inverse likewise. By checking the adjoint action by K, and its eigenvalues, the result follows. □

Remark 1.4.21. For $q^4 \neq 1$, consider $\mathbb{C}\langle E, F, K^{\pm 1}\rangle$ with relations

$$
\begin{aligned}
[K,E]_q &= 0, \\
[K,F]_{q^{-1}} &= 0, \\
[E,F] &= \frac{K^2 - K^{-2}}{q - q^{-1}}, \\
KK^{-1} = K^{-1}K &= 1,
\end{aligned}
$$

and call this algebra $\hat{U}_q(\mathfrak{sl}_2)$. Then $U_q(\mathfrak{sl}_2) \hookrightarrow \hat{U}_q(\mathfrak{sl}_2)$.

Remark 1.4.22. Another way to elucidate the relationship is to compare relations, note $U_q(\mathfrak{sl}_2)$ with relations:

$$
\begin{aligned}
EF - FE &= L, \\
LE - EL &= q(EK + K^{-1}E), \\
LF - FL &= -q^{-1}(FK + K^{-1}F), \\
(q - q^{-1})L &= K - K^{-1}, \\
KE &= q^2 EK, \\
KF &= q^{-2} FK,
\end{aligned}
$$

which are the same as our previous presentation. Then, $q = 1$ and $K = 1$ yields

$$
\begin{aligned}
EF - FE &= L, \\
LE - EL &= (2E), \\
LF - FL &= -(2F),
\end{aligned}
$$

which are the relations of $U(\mathfrak{sl}_2)$.

Now we return to seeing $U_q(\mathfrak{sl}_2)$ in terms of a faithful representation of differential operators on some polynomial ring. Since we quantized our algebra, now we quantize our polynomials.

Definition 1.4.23. Call $\mathbb{C}_q[x,y]$ the *quantum plane*, the $\mathbb{C}$-algebra generated by x and y and relation

$$[x,y]_q = 0.$$

Remark 1.4.24. As $U_q(\mathfrak{sl}_2)$ descends to $U(\mathfrak{sl}_2)$ by specializing $q=1$, $\mathbb{C}_q[x,y]$ becomes $\mathbb{C}[x,y]$.

Proposition 1.4.25. *There is an algebra homomorphism $U_q(\mathfrak{sl}_2) \hookrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C}_q[x,y])$ that gives a faithful $U_q(\mathfrak{sl}_2)$-module algebra.*

Proof.

$$\begin{gathered} K(1)=1, \qquad K(x)=qx, \qquad K(y)=q^{-1}y, \\ E(1)=0, \qquad E(x)=0, \qquad E(y)=x, \\ F(1)=0, \qquad F(x)=y, \qquad F(y)=0, \end{gathered}$$

yields the map. The action respects Δ, so for $a \in U_q(\mathfrak{sl}_2)$,

$$\Delta(a) =: \sum a_{(1)} \otimes a_{(2)},$$

the above action on generators is enough to prove our claim. To check faithfulness, we check on a basis element. First, observe that

$$E(x^i y^j) = q^i \{j\}_{-2} x^{i+1} y^{j-1}, \qquad F(x^i y^j) = q^j \{i\}_{-2} x^{i-1} y^{j+1}, \qquad K(x^i y^j) = q^{i-j} x^i y^j,$$

where $\{n\}_a := \frac{q^{na}-1}{q^a-1}$. Then,

$$(E^l K^m F^n)(xy) = q^{-2n+3} q^{-2m} (1+q^{-2}) q^{-2(n+m)-3} x^2,$$

so if $E^l K^m F^n = id$, then $l = m = n = 0$. $\square$

Remark 1.4.26. Lest the reader think this is the only representation, in [4], the authors show that there are several classes of non-isomorphic $U_q(\mathfrak{sl}_2)$-module subalgebras in the quantum plane.

Example 1.4.27. Let us check that this representation recovers our representation of $U(\mathfrak{sl}_2)$ on the affine plane. We see that when $q \to 1$ both E and F act the way we expected before. Further, if we consider $K = e^{\hbar H}$, then apply(as before) $\frac{\partial}{\partial \hbar}$ to the K action we indeed recover our analogous action of H.

We have made the necessary connections between our simple geometric interpretation of $U(\mathfrak{sl}_2)$ as differential operators on the space $\mathbb{P}^1$ and our new algebra $U_q(\mathfrak{sl}_2)$. We could work harder and even recollect our localization picture, but it will appear more naturally later. An important(and later essential) difference between the two pictures, is that the first picture had a real geometry, $\mathbb{P}^1$ and its line bundles, whereas our second only had an implied geometry by an algebra that looked like line bundles on something.

1.4.28 Serre relations for $U(\mathfrak{sl}_n)$

The Serre relations form the relations between generators for $U(\mathfrak{g})$ for higher rank Lie algebras. We will focus on $\mathfrak{sl}_n$ but the definitions presented here, which depend only on the generalized Cartan matrix, are the same for all Lie algebras.

Given $\mathfrak{A} = (a_{ij})_{i,j\in I}$ a symmetrizable generalized Cartan matrix, with entries $a_{ii} = 2$ and $a_{ij} = 0, -1, -2, -3$ when $i \neq j$,

Definition 1.4.29. $\mathfrak{g}_{\mathfrak{A}}$ is the Lie algebra associated to $\mathfrak{A}$ is generated by $\{E_i, F_i, H_i\}_{i\in I}$ and relations

$$[H_i, H_j] = 0, \quad [E_i, F_j] = \delta_{ij} H_i,$$

$$[H_i, E_j] = a_{ij} E_j, \quad [H_i, F_j] = -a_{ij} F_i, \quad i, j \in I,$$

and for $i \neq j$, $\mathrm{ad}(E)(x) := [E, x]$,

$$\mathrm{ad}(E_i)^{1-a_{ij}}(E_j) = 0$$

These iterated commutators form the so-called *Serre relations*.

Definition 1.4.30. Let $U(\mathfrak{g}_{\mathfrak{A}})$ be the $\mathbb{C}$-algebra defined by generators $\{E_i, F_i, H_i\}_{i\in I}$ and relations

$$[H_i, H_j] = 0, \quad [E_i, F_j] = \delta_{ij} H_i, \quad [H_i, E_j] = a_{ij} E_j, \quad [H_i, F_j] = -a_{ij} F_j,$$

and

$$\begin{aligned} \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} E_i^k E_j E_i^{1-a_{ij}-k} &= 0, \\ \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} F_i^k F_j F_i^{1-a_{ij}-k} &= 0. \end{aligned}$$

This pair of relations are the *Serre relations* as they appear in the universal enveloping algebra. See [11].

Remark 1.4.31. To arrive at the special linear Lie algebras, one should take the $n \times n$ Cartan matrix

$$A_n : \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & \ddots & \dots & 0 & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & -1 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}$$

Example 1.4.32. Let $A_1 = (2)$ so $U(\mathfrak{sl}_2)$ has $\{e, f, h\}$ and relations

$$[e, f] = 2h, \quad [h, e] = 2e, \quad [h, f] = -2f,$$

which was what we saw before. Notice that we have one simple root, and $W = \langle s_1 \rangle = \{e, s_1\} = \mathfrak{S}_2$ the Weyl group. The Serre relations are trivial in this case.

Now we build our second standard example $U(\mathfrak{sl}_3)$.

Example 1.4.33. Let $A_2 := \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, so we have $\{E_1, E_2, F_1, F_2, H_1, H_2\}$ as generators, the relations of before, and we see that the Serre relations are,

$$[E_i, [E_i, E_j]] = 0, \qquad [F_i, [F_i, F_j]] = 0, \qquad i \neq j$$

For the simple roots we write α, β. Let

$$W = \langle s_1, s_2 \rangle = \{id, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\} = \mathfrak{S}_3,$$

be the Weyl group. Further, we have $U(\mathfrak{sl}_3)$, defined by the relations of before where the Serre relations become

$$\sum_{k=0}^{2} (-1)^k \binom{2}{k} E_i^k E_j E_i^{2-k} = 0 \quad \Longrightarrow \quad E_i E_j E_i = \frac{E_j E_i^2 + E_i^2 E_j}{2},$$
$$\sum_{k=0}^{2} (-1)^k \binom{2}{k} F_i^k F_j F_i^{2-k} = 0 \quad \Longrightarrow \quad F_i F_j F_i = \frac{F_j F_i^2 + F_i^2 F_j}{2}.$$

Serre Relations for $U_q(\mathfrak{sl}_n)$

As before, the quantum Serre relations will provide the presentations of our algebras. First we need a few definitions.

Definition 1.4.34. $[m]_q := \frac{q^m - q^{-m}}{q - q^{-1}}$ the *quantum integers*, $[m]_q^! := [m]_q[m-1]_q \dots [2]_q[1]_q$ the *q-factorials.*

Definition 1.4.35. Let $U_q(\mathfrak{g}_{\mathfrak{A}})$ be the $\mathbb{C}$-algebra defined by generators $\left\{E_i, F_i, K_i^{\pm 1}\right\}_{i \in I}$, and let $q_i = q^{d_i}$(from the diagonalization of the Cartan matrix) with relations

$$[K_i^{\pm 1}, K_j^{\pm 1}] = 0, \quad [E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \quad [K_i, E_j]_{q^{a_{ij}}} = 0, \quad [K_i, F_j]_{q^{-a_{ij}}} = 0,$$

and

$$\begin{aligned} \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_i} E_i^k E_j E_i^{1-a_{ij}-k} &= 0, \\ \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_i} F_i^k F_j F_i^{1-a_{ij}-k} &= 0. \end{aligned}$$

This pair of relations are the *quantum Serre relations* for the quantized universal enveloping algebras $U_q(\mathfrak{g}_{\mathfrak{A}})$.

Remark 1.4.36. Since roots, Weyl groups, weights, etc. are all built from generalized Cartan matrix, they are shared by $U(\mathfrak{g})$ and $U_q(\mathfrak{g})$.

Example 1.4.37. We have $U_q(\mathfrak{sl}_3)$, defined by the relations of before where the quantum Serre relations become:

$$\begin{aligned} \sum_{k=0}^{2} (-1)^k \binom{2}{k}_{q_i} E_i^k E_j E_i^{2-k} = 0 \quad &\Longrightarrow \quad E_i E_j E_i = \frac{E_j E_i^2 + E_i^2 E_j}{q + q^{-1}}, \\ \sum_{k=0}^{2} (-1)^k \binom{2}{k}_{q_i} F_i^k F_j F_i^{2-k} = 0 \quad &\Longrightarrow \quad F_i F_j F_i = \frac{F_j F_i^2 + F_i^2 F_j}{q + q^{-1}}, \end{aligned}$$

when $i \neq j$.

Remark 1.4.38. Notice that when $q = 1$ these relations become the Serre relations for $U(\mathfrak{sl}_3)$.

1.4.39 Building the quantum geometries

Recall from the previous section, for a complex reductive algebraic group G the flag variety $\mathrm{Fl} := G/B$ and it is well known from Bruhat decomposition that $\mathrm{Fl} = \coprod_{w \in W} BwB/B$. In particular, the Zariski closure $\overline{BwB/B}$ are called the Schubert varieties. If one considers w_0 the longest element of W, the Weyl group, this the cell Bw_0B/B is called the big cell which is an affine open and dense algebraic variety in G/B.. The elements $w \in W$ of the Weyl group translate this big cell to produce an affine cover of Fl. These translates of the big cell denoted wBw_0B/B.

Each big cell translate is isomorphic to $\mathbb{C}^{|\Lambda^+|}$. Thus, if $\mathcal{D}$ denotes the sheaf of differential operators, then $\mathcal{D}(wBw_0B/B) \simeq A_n$, the n'th Weyl algebra (here $n = |\Lambda^+|$). As mentioned above, A_n are generalized Weyl algebras and have a convenient structure for building representations.

In the $U_q(\mathfrak{g})$ setting, things are more complicated. First, there is not an obvious analog of G. Thus, to obtain Fl_q we take instead the Gel'fand model $\bigoplus_{\lambda \in P^+} R_\lambda$ with multiplication given by projection to highest weight vectors as our ring of functions on Fl_q. This completely algebraic description of the flag variety is due to Joseph [17]. The algebraic analog of our Schubert cells, BwB/B, is a construction of Lusztig/De Cocini-Procesi([23], [3]), and is constructed as an iterated Ore extension [15], denoted $U_q[w]$. Note that an iterated Ore extension is, in some sense, two steps in complexity from a polynomial ring. For translates of the big cell, Joseph constructed the appropriate algebras by localization [17], denoted S_q^w. Further, Joseph and Gorelik showed that for $w_1 \neq w_0 w_2$, that $S_q^{w_1} \not\simeq S_q^{w_2}$ [5].

Analogously to the classical case, [21] and [22] defined D_q as a sheaf of quantum differential operators on Fl_q which satisfies a "quantum Beilinson-Bernstein localization" [26]. Whence we have returned to deformed algebras, from deformed spaces, which are associated to deformed Lie algebras.

Chapter 2

Deformations of differential operators

2.1 Hayashi's quantized Weyl algebra

In [8], Hayashi introduces an algebra which is called the q-analog of the n-th Weyl algebra. In particular, this is a one-parameter family of deformations of the Weyl algebra. We recall the definition here for later use. We assume q is a non-zero complex number, *not* a root of unity.

Definition 2.1.1. The *q-analog of the Weyl algebra*, $\mathcal{A}_q(n)$, is a $\mathbb{C}$-algebra defined by generators $\psi_i, \psi_i^\dagger, \omega_i, \omega_i^{-1}$ for $1 \le i \le n$, with relations: $i \neq j$,

$$\begin{aligned} \omega_i\omega_j &= \omega_j\omega_i, & \omega_i\omega_i^{-1} &= \omega_i^{-1}\omega_i = 1, \\ \omega_i\psi_j\omega_i^{-1} &= q^{-\delta_{ij}}\psi_j, & \omega_i\psi_j^\dagger\omega_i^{-1} &= q^{\delta_{ij}}\psi_j^\dagger, \\ \psi_i\psi_j - \psi_j\psi_i = \psi_i^\dagger\psi_j^\dagger - \psi_j^\dagger\psi_i^\dagger &= 0, & \psi_i\psi_j^\dagger - \psi_j^\dagger\psi_i &= 0, \\ \psi_i\psi_i^\dagger - q\psi_i^\dagger\psi_i &= \omega_i^{-1}, & \psi_i\psi_i^\dagger - q^{-1}\psi_i^\dagger\psi_i &= \omega_i. \end{aligned}$$

Notice that like the Weyl algebra, $A_q(n) = \underbrace{A_q(1) \otimes \ldots \otimes A_q(1)}_{n}$ as algebras.

Remark 2.1.2. We have slightly modified the original notation by eschewing the minus superscript.

As a generalization of the n-th Weyl algebra, $\mathcal{A}_q(n)$ acts on $\mathbb{C}[x_1, \dots, x_n]$ as a vector space, analagously to differential operators.

Definition 2.1.3. For a monomial $x(\mathbf{m}) := x_1^{m_1} \dots x_n^{m_n}$ in $\mathbb{C}[x_1, \dots, x_n]$, with $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}_{\geq 0}^n$, we define the action of $\mathcal{A}_q(n)$ as

$$\begin{aligned} \omega_i(x(\mathbf{m})) &= q^{m_i} x(\mathbf{m}), \\ \psi_i(x(\mathbf{m})) &= [m_i]_{q^2} x(\mathbf{m} - \mathbf{e}_i), \\ \psi_i^\dagger(x(\mathbf{m})) &= x(\mathbf{m} + \mathbf{e}_i), \\ \psi_i(1) &= 0, \end{aligned}$$

where $\mathbf{e}_i := (\delta_{1i}, \dots, \delta_{ni})$. If $m_i < 0$ for any i then we fix $x(\mathbf{m}) = 0$, and $[m_i]_{q^2}$ the quantum number for q^2.

One can check that the relations in Definition 2.1.1 are satisfied so $\mathbb{C}[x_1, \dots, x_n]$ is an $A_q(n)$-module. We note that ω_i acts as a $\mathbb{C}$-algebra homomorphism.

Theorem 2.1.4 ([8], §2.1)**.** *$\mathbb{C}[x_1, \dots, x_n]$ is an irreducible $\mathcal{A}_q(n)$-module.*

Example 2.1.5. When $q \to 1$ and $\omega_i \mapsto id$ we see that $\mathcal{A}_1(n)$ coincides with A_n the classical Weyl algebras.

2.1.6 Action of $U_q(\mathfrak{sl}_n)$

Note that, for each n, homomorphism are also constructed in [8]: $U_q(\mathfrak{sl}_n) \to \mathcal{A}_q(n)$ for the purpose of producing representations of the quantized enveloping algebras (actually, for Lie algebras of types A, B, C, D). These homormophism will appear again in the sequel chapter.

Theorem 2.1.7 ([8], §3.2)**.** *There exists a homomorphism of algebras $\pi_n : U_q(\mathfrak{sl}_n) \to \mathcal{A}_q(n)$*

given by:

$$E_i \mapsto \psi_i \psi_{i+1}^{\dagger}, \quad F_i \mapsto \psi_{i+1} \psi_i^{\dagger}, \quad K_i \mapsto \omega_{i+1} \omega_i^{-1},$$

for all $1 \leq i \leq n$*, thus* $\mathbb{C}[x_1, \ldots, x_n]$ *is a representation of* $U_q(\mathfrak{sl}_2)$*.*

Definition 2.1.8. The representation of $U_q(\mathfrak{sl}_n)$ on $\mathbb{C}[x_1, \ldots, x_n]$ given by the morphism π_n is called the *oscillator representation*, denoted V_n.

This representation is not irreducible.

Theorem 2.1.9 ([8])**.** *Let* V_n *be the* $U_q(\mathfrak{sl}_n)$*-module given by* π_n *then*

$$V_n = \bigoplus_{r=0}^{\infty} V_n^r, \text{ with } V_n^r = \bigoplus_{|\boldsymbol{m}|=r} \mathbb{C}x(\boldsymbol{m})$$

for $\boldsymbol{m} = (m_i) \in \mathbb{Z}_{\geq 0}^n$ *and* $|\boldsymbol{m}| = \sum_i m_i$*.* V_n^r *is an irreducible* $U_q(\mathfrak{sl}_n)$*-module with lowest weight vector* $x(r\mathbf{e}_n)$ *of lowest weight* $(0, \ldots, 0, r)$*.*

2.2 Quantum differential operators

Definition 2.2.1. (cf. 1.3) For a commutative $\mathbb{K}$-algebra R, let $\mathrm{End}_{\mathbb{K}}(R)$ be the set of all $\mathbb{K}$-linear abelian-group homomorphisms from R to itself. The *algebra of differential operators on* R is the set of $D \in \mathrm{End}_{\mathbb{K}}(R)$ such that there exists some $n \in \mathbb{N}$ such that

$$[\ldots[[D, r_0], r_1], \ldots, r_n] = 0$$

for any sequence $\{r_i\}_{0 \leq i \leq n}$ of $r_i \in R$. We write this algebra $D(R)$.

Definition 2.2.2. Let $D^n(R)$ be the subspace consisting of those $D \in D(R)$ with n minimal such that

$$[\ldots[[D, r_0], r_1], \ldots, r_n] = 0$$

for any sequence $\{r_i\}_{1\leq i\leq n}$ of $r_i \in R$. We call $D \in D^n(R)$ a *differential operator of degree* $\leq n$ and the filtration $D^0(R) \subseteq D^1(R) \subseteq \ldots \subseteq D(R)$ is called the *D-filtration* $D(R)$.

The algebra of differential operators can be generalized to a noncommutative $\mathbb{K}$-algebra R:

Definition 2.2.3. For $r \in R$, let $\lambda_r \in \mathrm{End}_{\mathbb{K}}(R)$ and $\rho_r \in \mathrm{End}_{\mathbb{K}}(R)$ be the operators:

$$\lambda_r : s \mapsto rs \qquad \rho_r : s \mapsto sr$$

for all $s \in R$, and define $\mathrm{ad}_r(z) = [\lambda_r, z]$ for all $z \in \mathrm{End}_{\mathbb{K}}(R)$.

Remark 2.2.4. Notice we are using left actions and left adjoints here. This indicates we are constructing left differential operators. We will assume this throughout the thesis. This is especially important later when we have noncommutative rings, bimodules, and deformed multiplications. Many of the definitions and constructions could be extended to right actions, and in the bimodule cases left-right, and right-left actions.

For r_1, $r_2 \in R$ and $\varphi \in \mathrm{End}_{\mathbb{K}}(R)$ the actions by R on $\mathrm{End}_{\mathbb{K}}(R)$, $r_1\varphi r_2 = \lambda_{r_1} \circ \varphi \circ \rho_{r_2}$ make $\mathrm{End}_{\mathbb{K}}(R)$ an R-bimodule.

Definition 2.2.5. Let $\bar{D}^0 = \bar{D}^0(R) := \{\delta \in \mathrm{End}_{\mathbb{K}}(R) \mid \mathrm{ad}_r(\delta) = 0, \forall r \in R\}$. We know that $\bar{D}^0$ is an $\mathbb{K}$-vector subspace of $\mathrm{End}_{\mathbb{K}}(R)$. However, $\mathrm{End}_{\mathbb{K}}(R)$ is also an R-bimodule under the R-action obtained via λ_r and ρ_r, thus we want to consider D^0, the R-subbimodule generated by $\bar{D}^0$.

Definition 2.2.6. $\bar{D}^i(R) := \{\delta \in \mathrm{End}_{\mathbb{K}}(R) \mid \mathrm{ad}_r(\delta) \in D^{i-1}, \forall r \in R\}$ for $i \geq 1$, and $D^i(R) = R\bar{D}^i(R)R$ an R-bimodule. This array of sub-bimodules, $D^i(R) \subset D^{i+1}(R)$, is called a *D-filtration* of $\mathrm{End}_{\mathbb{K}}(R)$, and the R-subbimodule $D(R) = \bigcup D^i(R)$ is the *space of differential operators of* R. The elements are called differential operators.

Remark 2.2.7. It is routine to verify that $D^i(R)\, D^j(R) \subseteq D^{i+j}(R)$, thus $D(R)$ is a filtered algebra.

Recall that the set of *derivations*, $\mathrm{Der}(R)$, consists of those elements of $D(R)$ satisfying the Leibniz rule,

$$\partial(r_1 r_2) = \partial(r_1) r_2 + r_1 \partial(r_2),$$

and so $\mathrm{Der}(R) \subset D^1(R)$. Finally, observe that λ_r and ρ_r generate D^0, which is analogous to the case of commutative rings. These facts are shown in [21].

We see that one can extend these notions to general R-bimodules M. $M = \mathrm{End}_{\mathbb{K}}(R)$ will be important for defining the quantum version.

Let R be a not necessarily commutative, associative algebra over a field $\mathbb{K}$, and M be an R-bimodule.

Definition 2.2.8. The *center* of M, $\mathcal{Z}(M)$, is defined as the R-bimodule generated by the set $\{m \in M \mid mr = rm\ \forall r \in R\}$.

In the commutative case, this set is automatically an R-bimodule.

Definition 2.2.9. Let $M_0 := \mathcal{Z}(M)$. The *D-filtration* of M is defined as the filtration of R-bimodules $M_0 \subseteq M_1 \subseteq ...$, with

$$M_i = R\,\{m \in M \mid mr - rm \in M_{i-1} \forall r \in R\}\, R.$$

we call $\bigcup M_i$ the *differential bimodule of* M, written M_{diff}.

Remark 2.2.10. This can be seen as a more familiar construction; consider the iterated pullbacks

$$\begin{tikzcd}
\mathcal{Z}(M/M_{i-1}) \arrow[r, hook] & M/M_{i-1} \\
M_i \arrow[u, "\pi"] \arrow[r, hook] \arrow[ur, phantom, "\text{cart}"] & M \arrow[u, "\pi"']
\end{tikzcd}$$

Lemma 2.2.11. *If R is commutative,* $\mathrm{Hom}_{\mathbb{K}}(R,R)_{\mathit{diff}} = D(R)$.

2.2.12 q-analog of differential operators

Definition 2.2.13. ([21]) Let Γ be an abelian group. For a Γ-graded $\mathbb{K}$-algebra $R = \bigoplus_{\gamma\in\Gamma} R_\gamma$, $M = \bigoplus_{\gamma\in\Gamma} M_\gamma$ a Γ-graded R-bimodule, and a bicharacter(see 3.1) $\beta : \Gamma\times\Gamma \to \mathbb{K}^*$, define the *$\beta$-center of M* as:

$$\mathcal{Z}_\beta(M) := R\left\{m \in M_a, a \in \Gamma \mid a \in \Gamma, mr = \beta(a,b)rm \text{ for } r \in R_b, b \in \Gamma\right\} R.$$

Definition 2.2.14. We define $M_{\beta,0} := \mathcal{Z}_\beta(M)$ and for $i \geq 1$ the R-bimodule, $M_{\beta,i}$, is the R-bimodule

$$R\left\{m \in M \mid \exists a \in \Gamma, mr - \beta(|a|, |r|)rm \in M_{\beta,i-1} \text{ for } r \in R\right\} R,$$

Definition 2.2.15. If $[m,r]_\beta$ is written, it means that m and r are homogeneous and $[m,r]_\beta := mr - \beta(|m|, |r|)rm$.

These $M_{\beta,i}$'s give a filtration. We call it the *β D-filtration*.

Definition 2.2.16. Since $M_{\beta,0} \subset M_{\beta,1} \subset \ldots$, we call $D^i_\beta(M) := M_{\beta,i}$ and $D_\beta(M) := \varinjlim M_{\beta,i}$ and call these the *β-differential bimodule of M*, $M_{\beta-\text{diff}}$. We define *β-differential operators* as $\mathrm{grHom}_{\mathbb{K}}(R,R)_{\beta-\text{diff}}$(see 3.1) written $D_\beta(R)$, where $\mathrm{grHom}_{\mathbb{K}}(R,R)$ denotes the $\mathbb{K}$-submodule of $\mathrm{Hom}_{\mathbb{K}}(R,R)$ spanned by homogeneous elements.

Remark 2.2.17. As before, we can use iterated pullbacks to define these $M_{\beta,i}$ in a more systematic way:

$$\begin{tikzcd}
\mathcal{Z}_\beta(M/M_{i-1}) \arrow[r, hook] & M/M_{\beta,i-1} \\
\tilde{M}_{\beta,i} \arrow[u, "\pi"] \arrow[r, hook] \arrow[ur, phantom, "\text{cart}"] & M \arrow[u, "\pi"']
\end{tikzcd}$$

with $M_{\beta,i} := R\tilde{M}_{\beta,i}R$ and $\varinjlim_i M_{\beta,i} =: M_{\beta-\text{diff}} \hookrightarrow M$.

Example 2.2.18. We expect that, as in the standard case, "derivations" are elements of

D^0. This carries over to the quantum case as follows: for $a \in \Gamma$ and $\varphi \in \operatorname{gr}\operatorname{Hom}_{\mathbb{K}}(R,R)$ such that

$$\varphi(rs) = \varphi(r)s + \beta(a, d_r)r\varphi(s),$$

then $\varphi \in D^1_\beta(R)$.

Note that these definitions are for a general bicharacter β; we specify this next, but will return to the general case in Chapter 3.

2.2.19 Quantum differential operators on $\mathbb{C}[x_1, \ldots, x_n]$

Remark 2.2.20. In [13] it is claimed that the algebra of differential operators on $\mathbb{K}[x]$ is defined by the generators and relations given below. However, the result there is not true as claimed. The relations claimed are contained in the ideal, but do not form a complete set of relations for the algebra; the result is true for the algebra $\mathbb{K}[x, x^{-1}]$. The authors of [13] have notified me of progress towards repairing this result. We state what is known.

Equip $\mathbb{K}[x]$ with the grading $\Gamma = \mathbb{Z}$ such that $|x^a| = a$ and bicharacter $\beta(n,m) = q^{nm}$. Let $\{n\}_q := \dfrac{q^n - 1}{q - 1}$, here $D_\beta(R)$ will be written $D_q(R)$ based on this bicharacter.

Definition 2.2.21. Consider the operators $\left\{x, \partial, \mathfrak{d}^\beta, \mathfrak{d}^{\beta^{-1}}\right\}$ on $\mathbb{C}[x]$ such that

$$\begin{aligned}
x(x^n) &:= x^{n+1}, \\
\partial(x^n) &:= n x^{n-1}, \\
\mathfrak{d}^\beta(x^n) &:= (1 + q + \ldots + q^{n-1})x^{n-1} = \{n\}_q x^{n-1}, \\
\mathfrak{d}^{\beta^{-1}}(x^n) &:= (1 + q^{-1} + \ldots + q^{-(n-1)})x^{n-1} = \{n\}_{q^{-1}} x^{n-1}.
\end{aligned}$$

Remark 2.2.22. We modify the notation δ^{β^a} to $\mathfrak{d}^a$ to avoid confusion.

The main result of [13], after the correction, is that:

Theorem 2.2.23. *([13]) There is a morphism of algebras $D_q(\mathbb{K}[x]) \to \mathbb{K}[x, \partial, \mathfrak{d}, \mathfrak{d}^{-1}]$.*

The erroneous claim in [13] deals with the relations that hold in this latter algebra. For our purposes, we need only the morphisms existence, not a presentation of the algebra.

Definition 2.2.24. We need the additional operators $\sigma_a, \mathfrak{d}^a, \mathfrak{d}_a$ for $a \in \mathbb{Z}$ such that

$$\mathfrak{d}^a \ := \ \left(\frac{1-q}{1-q^a}\right)\mathfrak{d}^1(1+\sigma_1+\ldots+\sigma_{a-1}),$$

which have actions

$$\begin{aligned} \sigma_a(x^b) &:= \beta(a,b)x^b, \\ \mathfrak{d}^a(x^b) &:= (1+q^a+\ldots+q^{a(b-1)})x^{b-1}, \\ \mathfrak{d}_a(x^b) &:= \beta(a,b)x^{b-1}. \end{aligned}$$

Theorem 2.2.25 ([13], §3.1)**.** *The following relations hold in $D_\beta(\mathbb{K}[x])$ for $\beta(a,b) = q^{ab}$:*

$$\begin{aligned} \mathfrak{d}^a x - q^a x \mathfrak{d}^a &= 1, \\ \mathfrak{d}^a x \mathfrak{d}^b &= \mathfrak{d}^b x \mathfrak{d}^a, \\ \mathfrak{d}^{-1} - q\mathfrak{d} &= (1-q)\mathfrak{d}^{-1}x\mathfrak{d}. \end{aligned}$$

Remark 2.2.26. As stated before, other relations may exist. We will show later (2.4.21) that $\mathfrak{d}_a$ is indeed a β-differential operator in $D_\beta(\mathbb{K}[x_1,\ldots,x_n])$ with $\mathbb{Z}^n$ gradation, and will write it in the aforementioned basis.

Theorem 2.2.27 ([13], §3.1)**.** *$D_q(\mathbb{K}[x,x^{-1}])$ is the algebra with generating set given by $\{x, \partial, \mathfrak{d}, \mathfrak{d}^{-1}\}$ and relations $\mathcal{R}$.*

These results are extended to $\mathbb{K}[x_1,\ldots,x_n]$ in that same paper. For $\{q_i\}_{i\leq n}$ trancendental over $\mathbb{Q}$ and $\mathbb{K}$ containing $\mathbb{Q}(q_1,q_2,\ldots,q_n)$, consider $\mathbb{K}[x_1,\ldots,x_n]$ with $\mathbb{Z}^n$ grading, $|x_1^{a_1}x_2^{a_2}\ldots x_n^{a_n}| = (a_1,a_2,\ldots,a_n)$ and bicharacter $\beta((a_1,a_2,\ldots,a_n),(b_1,b_2,\ldots,b_n)) = q_1^{a_1b_1}q_2^{a_2b_2}\ldots q_n^{a_nb_n}$. Define similarly differential operators as described in [13] where index on a differential corresponds to an element of the ordered basis.

Corollary 2.2.28. *$D_q(\mathbb{K}[x_1, \ldots, x_n]) \twoheadleftarrow \mathbb{K}[x_i, \partial_i, \mathfrak{d}_i, \mathfrak{d}_i^{-1}]$ for $i \leq n$. The kernel is similar to $\mathcal{R}$ for indices up to n.*

2.2.29 Quantum differential operators on the quantum plane

Remark 2.2.30. In the classical setting, a Lie algebra acts on the coordinate ring of the flag variety of it's Lie group by differential operators, and in the quantum case, by quantum differential operators [22]. For the case of $U_q(\mathfrak{sl}_2)$ that analogously defined coordinate ring is the quantum plane [17]. Hence, we get a representation of $U_q(\mathfrak{sl}_2)$, by differential operators on the quantum plane. The $U_q(\mathfrak{sl}_2)$ example is worked out in [14], but we reproduce their main results here to fix notation for our later comparison of this algebra to Hayashi's 2-dimensional quantum Weyl algebra. Compare to 1 where we described Joseph's construction of these spaces.

Definition 2.2.31. The algebra $\mathcal{Q} := \mathbb{K}\langle x, y\rangle / (xy - qyx)$ is the so-called quantum plane, making $D_q(\mathcal{Q})$ the algebra of quantum differential operators on the quantum plane. The action of $D_q(\mathcal{Q})$ on $\mathcal{Q}$ is defined by

$$\mathfrak{d}_x^a(x^i y^j) = \{i\}_{q^a} x^{i-1} y^j, \qquad \mathfrak{d}_y^b(x^i y^j) = \{j\}_{q^b} x^i y^{j-1}.$$

Remark 2.2.32. We diverge from the notation in [14] to be consistent with those in this thesis.

Remark 2.2.33. Since $\mathcal{Q}$ is noncommutative, recall that $\lambda_x(p) = x \cdot p$ and $\rho_y(p) = p \cdot y$, and ∂_x, ∂_y are the partial derivatives with left action.

Lemma 2.2.34. *$[n]_q q^{n-1} = \{n\}_{q^2}$ and $\{m\}_{q^a} + q^{ma} \{n\}_{q^a} = \{n+m\}_{q^a}$.*

Proof. A simple computation. □

Theorem 2.2.35 ([14], §4.0.3). *As graded $\mathbb{K}$-algebras, $D_q(\mathcal{Q}) \simeq D_x \otimes_{\mathbb{K}} D_y$ for*

$$\begin{aligned} D_x &= \mathbb{K}[\lambda_x, \mathfrak{d}_x^a \mid a = -1, 0, 1] \simeq D_q(\mathbb{K}[x]), \\ D_y &= \mathbb{K}[\rho_y, \mathfrak{d}_y^a \mid a = -1, 0, 1] \simeq D_q(\mathbb{K}[y]). \end{aligned}$$

Remark 2.2.36. Observe that this is a result analagous to the classical case of Weyl algebras where $A_2 \simeq A_1 \otimes_{\mathbb{K}} A_1$.

2.2.37 Action of $U_q(\mathfrak{sl}_2)$ on $\mathcal{Q}$

In [14] the authors construct a homomorphism $U_q(\mathfrak{sl}_2) \to D_q(\mathcal{Q})$ for the purpose of producing representations of the quantized enveloping algebras.

Theorem 2.2.38 ([14], §5.5). *There exists a homomorphism of algebras $\tilde{\pi}_2 : U_q(\mathfrak{sl}_2) \to D_q(\mathcal{Q})$ given by:*

$$E \mapsto \sigma_x \sigma_y^{-1} \lambda_x \mathfrak{d}_y (1 + \sigma_y^{-1}), \quad F \mapsto \sigma_y \sigma_x^{-1} \rho_y \mathfrak{d}_x (1 + \sigma_x^{-1}), \quad K \mapsto \sigma_x \sigma_y^{-1},$$

for all $1 \leq i \leq n$, where $\sigma_x(x^n y^m) = q^n x^n y^m$ and $\sigma_y(x^n y^m) = q^m x^n y^m$.

Definition 2.2.39. The representation of $U_q(\mathfrak{sl}_2)$ on $\mathcal{Q}$ given by $\tilde{\pi}_2$ is called the *quantized standard representation*, which we denote by $\tilde{V}_2$.

Corollary 2.2.40.

$$\tilde{V}_2^r \hookrightarrow \mathbb{K}[x, y]$$

as homogeneous polynomials of homogeneous degree r. $\tilde{V}_2^r$ is an irreducible highest-weight representation of $U_q(\mathfrak{sl}_2)$ with highes-weight vector x^r and highest weight $(r, 1)$.

Proof. Compute the action of E and K on this vector. See [16] for information on the representations of $U_q(\mathfrak{sl}_n)$ and their weight theory. □

2.3 Comparing the algebras of Lunts-Rosenberg and Hayashi

2.3.1 Comparing the algebras

When comparing the algebras $\mathcal{A}_q(n)$ and $D_q(\mathbb{K}[x_1,\ldots,x_n])$ we wish for the map to be compatisble with the respective representations. We construct a map from Hayashi's algebra to quantum differential operators on polynomials.

Proposition 2.3.2. *There is an algebra homomorphism $\Omega : \mathcal{A}_q(1) \to D_q(\mathbb{K}[x])$.*

Proof. Consider the extension by linearity of the map of the generators of $\mathcal{A}_q(1)$:

$$\begin{aligned} \omega^{\pm 1} &\mapsto \sigma_{\pm 1} \\ \psi^\dagger &\mapsto x \\ \psi &\mapsto \frac{\mathfrak{d}_1 - \mathfrak{d}_{-1}}{q - q^{-1}}. \end{aligned}$$

One needs to show that $\mathfrak{d}_a$ is indeed a q-differential operator, for $\mathfrak{d}_a$ the operator acting as

$$\mathfrak{d}_a(x^m) = \beta(a,m)x^{m-1},$$

with $\beta(a,b) = q^{ab}$, but we see that since $|\mathfrak{d}_a| = -1$ then

$$\left[\mathfrak{d}_a, x\right]_\beta (x^m) = \mathfrak{d}_a(x^{m+1}) - x\mathfrak{d}_a(x^m) = (\beta(a+1,1) - 1)\sigma_a(x^m).$$

Thus $[\mathfrak{d}_a, x]_\beta \in D^0_\beta(\mathbb{K}[n])$. When $\beta(a,b) = q^{ab}$ we have our result. □

We wish to describe a similar result for differential operators on the quantum plane:

Proposition 2.3.3. *There exists a $\mathbb{K}$-algebra homomorphism $\Omega_{xy} : \mathcal{A}_q(2) \to D_q(\mathcal{Q})$.*

Proof. We construct Ω_{xy} by restricting to the two sets which together form a generating set for $\mathcal{A}_q(2)$. Then $\omega_1^{\pm 1}, \psi_1^\dagger, \psi_1$ map to D_x and $\omega_2^{\pm 1}, \psi_2^\dagger, \psi_2$ to D_y by Ω projecting onto the respective sets. $\square$

We anticipate that these should be related since $\mathcal{A}_q(2)$ can be thought of as the quantization of the differential operators on the commutative affine plane, and $D_q(\mathcal{Q})$ can be thought of as quantum differential operators on the noncommutative affine plane. However, this result is an indication that deforming the geometry and deforming the differential geometry are independent for schemes.

2.3.4 Comparing the representations

A natural question to ask is "How does our algebra morphism, Ω, behave with respect to the natural representations of these algebras?". An immediate observation is that since both $A_q(n)$ and D_q are built with representations in mind (their natural representations on the polynomial algebras on which they act), the morphism between these algebras should at least respect these representations.

Lemma 2.3.5. $\Omega : \mathcal{A}_q(1) \to D_q(\mathbb{K}[x])$ *defines a* $U_q(\mathfrak{sl}_2)$*-module structure on* $\mathbb{K}[x]$*.*

And similarly,

Lemma 2.3.6. $\Omega_{xy} : \mathcal{A}_q(2) \to D_q(\mathcal{Q})$ *intertwines the standard representations by differential action on* $\mathbb{K}[x, y]$*.*

Lemma 2.3.7. *The representations* (π_2, V_2^r) *and* $(\tilde{\pi}_2, \tilde{V}_2^r)$ *of* $U_q(\mathfrak{sl}_2)$ *are ismorphic* $U_q(\mathfrak{sl}_2)$*-modules.*

This has the following consequence:

Proposition 2.3.8. Ω_{xy} *commutes with an iso-intertwiner of* $U_q(\mathfrak{sl}_2)$ *representations.*

Proof. One simply checks that the map Ω_{xy} respects the action of $U_q(sl_2)$. We see that since the quantum group acts through $A_q(2)$ and $D_q(\mathcal{Q})$, respectively. Recall (cf. 2.2.38) that

$$\begin{aligned} E &\mapsto \sigma_x\sigma_y^{-1}\lambda_x\mathfrak{d}_y(1+\sigma_y^{-1}), \\ F &\mapsto \sigma_y\sigma_x^{-1}\rho_y\mathfrak{d}_x(1+\sigma_x^{-1}), \\ K^{\pm 1} &\mapsto (\sigma_x\sigma_y^{-1})^{\pm 1}, \end{aligned}$$

so if we evaluate, recalling $\{a\}_b = \frac{q^{ab}-1}{q^b-1}$,

$$\begin{aligned} \tilde{\pi}_2(E)(x^n y^m) &= \{m\}_{-2}\, q^n x^{n+1} y^{m-1}, \\ \tilde{\pi}_2(F)(x^n y^m) &= \{n\}_{-2}\, q^m x^{n-1} y^{m+1}, \\ \tilde{\pi}_2(K)(x^n y^m) &= q^{n-m} x^n y^m. \end{aligned}$$

We should recall that

$$\begin{aligned} \pi_2(E)(x^n y^m) &= [n]_q x^{n-1} y^{m+1}, \\ \pi_2(F)(x^n y^m) &= [m]_q x^{n+1} y^{m-1}, \\ \pi_2(K)(x^n y^m) &= q^{n-m} x^n y^m. \end{aligned}$$

Remember that $[n]_q := (\frac{q^n-q^{-n}}{q-q^{-1}})$. We want to construct an endomorphism, a, of $U_q(\mathfrak{sl}_2)$ making the following diagram commute:

$$\begin{tikzcd} U_q(\mathfrak{sl}_2) \arrow[r, "\pi_2"] \arrow[d, "a"] & \mathcal{A}_q(2) \arrow[d, "\Omega_{xy}"] \\ U_q(\mathfrak{sl}_2) \arrow[r, "\tilde{\pi}_2"] & D_q(\mathcal{Q}) \end{tikzcd}$$

and produce the same action on $\mathbb{C}[x,y]^r$ the r-homogeneous components. This map a is easily described in terms of generators of $U_q(\mathfrak{sl}_2)$:

$$\begin{aligned} a(E) &\mapsto qK^{-1}F, \\ a(F) &\mapsto qKE, \\ a(K^{\pm 1}) &\mapsto K^{\pm 1}. \end{aligned}$$

$\square$

Remark 2.3.9. This recovers the ring-theoretic result obtained in [9] and [17] in the framework of [21] and [26].

Remark 2.3.10. We notice that when $q \to 1$, both the oscillator representation and the quantum standard representation coincide with the standard representation of $\mathfrak{sl}_2$. This is essentially an artifact of the above and the fact that D_q and $\mathcal{A}_q$ coincide in the classical limit.

We recapitulate this section with the following commutative diagram:

$$\begin{tikzcd} U_q(\mathfrak{sl}_2) \arrow[rrr, "a"] \arrow[dd, "O.R."'] \arrow[dr, "\pi_2"] & & & U_q(\mathfrak{sl}_2) \arrow[dd, "Q.S.R."] \arrow[dl, "\tilde{\pi}_2"'] \\ & \mathcal{A}_q(2) \arrow[r, "\Omega_{xy}"] \arrow[dl] & D_q(\mathcal{Q}) \arrow[dr] & \\ \mathrm{End}_{\mathbb{C}}(\mathbb{C}[x,y]^r) \arrow[rrr, "(-)^*"'] & & & \mathrm{End}_{\mathbb{C}}(\mathbb{C}[x,y]^r) \end{tikzcd} .$$

2.4 Generalizing to $U_q(\mathfrak{sl}_n)$

2.4.1 Noncommutative projective varieties associated to quantum groups

In [22] the authors construct a localization theory for $U_q(\mathfrak{g})$ analogous to the celebrated Beilinson-Bernstein localization. In that work they construct the noncommutative flag variety for $U_q(\mathfrak{g})$ using noncommutative **Proj** [20]. They also show that there is a canonical covering of the noncommutative flag variety by big cells, i.e., one that is *quasi-affine*. They construct these cells using localization by Joseph sets. For this reason, one can think of the quantum standard representation as the representation given by the quantized enveloping algebra, acting on the big cells of its flag varieties through quantum differential operators. Thus, the coincidence of the quantum standard representation, and the standard representation when $q \to 1$, is even more natural considering that the standard representation emerges from this process on the classical flag variety.

There are different notions of what noncommutative projective space should be ([1], [25]), but it is satisfactory to work with the underlying graded noncommutative algebras that replace rings of polynomials in commuting variables.

Remark 2.4.2. In the classical case, the covering by big cells (Weyl shifts of the big cell) gives a nice affine cover of G/B by spaces isomorphic to $\mathbb{A}^N$ for N the number of positive roots of $\mathfrak{g}$. In [17], the author constructs $\mathcal{S}^{w_0}$, the ring of functions of a Schubert cell, by localization of $R_q[G]$, the q-coordinate ring, followed by taking the degree zero component. It is shown that $U_q(\mathfrak{n}^-)$ is isomorphic to $\mathcal{S}^{w_0}$ as $U_q(\mathfrak{g})$-Hopf module algebras. However, it was also shown by Joseph that the other cells, $\mathcal{S}^w$ for $w \in W$, are localizations of R_q and

$$\mathcal{S}^{w_1} \not\cong \mathcal{S}^{w_2} \tag{2.1}$$

as algebras for $w_1 \neq w_2$ and $w_1 \neq w_0 w_2$. For the puposes of this paper, we restrict ourselves to $\mathcal{S}^{w_0}$, and will discuss the more general case (and gluing) in future work ([2]). Relations and ring properties for $\mathcal{S}^{w_0}$ are discussed in [18] and [5].

Definition 2.4.3. Call $\mathbb{C}_q[n] := \mathbb{C}\langle x_1, \dots, x_n\rangle / (x_i x_j = q_{ij} x_j x_i)$ *quantum affine n-space* where $q_{ij} = q_{ji}^{-1}$. We will assume in this thesis that $q_i j = q$ for $i < j$.

Remark 2.4.4. Observe $C_q[2] = \mathcal{Q}$.

We wish to embed $U_q(\mathfrak{sl}_n) \hookrightarrow D_q(\mathbb{C}_q[n])$, but we recall that there are $n-1$ embeddings $U_q(\mathfrak{sl}_2) \hookrightarrow U_q(\mathfrak{sl}_n)$:

Definition 2.4.5. For all $1 \leq m \leq n-1$ we define $i_m : U_q(\mathfrak{sl}_2) \hookrightarrow U_q(\mathfrak{sl}_n)$ by

$$\begin{aligned} E &\mapsto E_m, \\ F &\mapsto F_m, \\ K^{\pm 1} &\mapsto K_m^{\pm 1}. \end{aligned}$$

Definition 2.4.6. For all $1 \leq m \leq n-1$ define $\mathbb{K}$-algebra morphisms $e_m : \mathcal{A}_q(2) \hookrightarrow \mathcal{A}_q(n)$ by

$$\begin{aligned} \psi_1 &\mapsto \psi_m, \\ \psi_2 &\mapsto \psi_{m+1}, \\ \psi_1^\dagger &\mapsto \psi_m^\dagger, \\ \psi_2^\dagger &\mapsto \psi_{m+1}^\dagger, \\ \omega_1^{\pm 1} &\mapsto \omega_m^{\pm 1}, \\ \omega_2^{\pm 1} &\mapsto \omega_{m+1}^{\pm 1}. \end{aligned}$$

Then by 2.1.9 we have the commuting diagrams for all $1 \leq m \leq n-1$

$$\begin{tikzcd} U_q(\mathfrak{sl}_2) \arrow[r, "i_m"] \arrow[d, "\pi_2"] & U_q(\mathfrak{sl}_n) \arrow[d, "\pi_n"] \\ \mathcal{A}_q(2) \arrow[r, "e_m"] & \mathcal{A}_q(n) \end{tikzcd} .$$

2.4.7 Quantum differential operators on the quantum affine space

Similar to how we built the higher oscillator representations by completing our commutative squares above, we will do the same for our quantum standard representations.

Definition 2.4.8. We define $n-1$ $\mathbb{K}-algebra$ embeddings,

$$\{s_m : D_q(\mathbb{C}_q[2]) \hookrightarrow D_q(\mathbb{C}_q[n]), (1 \leq m \leq n-1)\},$$

by

$$\begin{aligned}
\partial_1 &\mapsto \partial_m, \\
\partial_2 &\mapsto \partial_{m+1}, \\
\mathfrak{d}_1^{\pm} &\mapsto \mathfrak{d}_m^{\pm}, \\
\mathfrak{d}_2^{\pm} &\mapsto \mathfrak{d}_{m+1}^{\pm}, \\
\lambda_1 &\mapsto \lambda_m \prod_{j=1}^{m} \sigma_j, \\
\lambda_2 &\mapsto \lambda_{m+1} \prod_{j=1}^{m} \sigma_j, \\
\rho_1 &\mapsto \rho_m \prod_{j=m+2}^{n} \sigma_j, \\
\rho_2 &\mapsto \rho_{m+1} \prod_{j=m+2}^{n} \sigma_j, \\
\sigma_1^{\pm 1} &\mapsto \sigma_m^{\pm 1}, \\
\sigma_2^{\pm 1} &\mapsto \sigma_{m+1}^{\pm 1}.
\end{aligned}$$

Remark 2.4.9. Recall from 2.2.3, $\lambda_i := \lambda_{x_i}$, and $\rho_i := \rho_{x_i}$: left and right multiplication respectively by x_i.

Lemma 2.4.10. *We have commutative diagrams for all* $1 \leq m \leq n-1$*:*

$$\begin{tikzcd}
U_q(\mathfrak{sl}_2) \arrow[r, "i_m"] \arrow[d, "\tilde{\pi}_2"] & U_q(\mathfrak{sl}_n) \arrow[d, "\tilde{\pi}_n"] \\
D_q(\mathbb{C}_q[2]) \arrow[r, "s_m"] & D_q(\mathbb{C}_q[n])
\end{tikzcd}$$

where $\tilde{\pi}_n$ is defined on generators by the images from these commutative diagrams. In particular,

$$\begin{aligned} E_m &\mapsto \sigma_m \sigma_{m+1}^{-1} \lambda_m \prod_{j=1}^{m} \sigma_j \mathfrak{d}_{m+1}(1+\sigma_{m+1}^{-1}), \\ F_m &\mapsto \sigma_{m+1}\sigma_m^{-1}\rho_{m+1} \prod_{j=m+2}^{n} \sigma_j \mathfrak{d}_m (1+\sigma_m^{-1}), \\ K_m^{\pm 1} &\mapsto (\sigma_m \sigma_{m+1}^{-1})^{\pm 1}. \end{aligned}$$

The extra σ's are simply there to make this morphism compatable with our embeddings into quantum polynomials.

Proposition 2.4.11. *$\tilde{\pi}_n$ is a homomorphism of algebras.*

Proof. It suffices to check the quantum Serre relation in 1.4.37

$$E_i E_{i+1} E_i = \frac{E_{i+1}E_i^2 + E_i^2 E_{i+1}}{q+q^{-1}}$$

as operators on $\mathbb{C}_q[n]$. We recall the action

$$E_i(x(\mathbf{m})) := \{m_{i+1}\}_{q^{-2}}\, q^{m_i}(x(\mathbf{m}+\mathbf{e}_i-\mathbf{e}_{i+1}))$$

under the map $\tilde{\pi}_n$. For the duration of this proof, we omit the subscript of q^{-2} from the $\{\ \ \}_{q^{-2}}$. Now

$$\begin{aligned} \tilde{\pi}_n(E_iE_{i+1}E_i(x(\mathbf{m})) &= q^{2m_i+m_{i+1}}\{m_{i+1}\}^2\{m_{i+2}\}(x(\mathbf{m}+2\mathbf{e}_i-\mathbf{e}_{i+1}-\mathbf{e}_{i+2})), \\ \tilde{\pi}_n(E_{i+1}E_i^2(x(\mathbf{m})) &= q^{2m_i+m_{i+1}-1}\{m_{i+1}\}\{m_{i+2}\}\{m_{i+1}-1\}(x(\mathbf{m}+2\mathbf{e}_i-\mathbf{e}_{i+1}-\mathbf{e}_{i+2})), \\ \tilde{\pi}_n(E_i^2E_{i+1}(x(\mathbf{m})) &= q^{2m_i+m_{i+1}+1}\{m_{i+1}\}\{m_{i+2}\}\{m_{i+1}+1\}(x(\mathbf{m}+2\mathbf{e}_i-\mathbf{e}_{i+1}-\mathbf{e}_{i+2})). \end{aligned}$$

We see that

$$\{m_{i+1}\}^2\{m_{i+2}\} = \frac{q^{-1}\{m_{i+1}\}\{m_{i+2}\}\{m_{i+1}-1\} + q\{m_{i+1}\}\{m_{i+2}\}\{m_{i+1}-1\}}{q+q^{-1}}$$

using

$$\begin{aligned}\{m_{i+1}-1\} &= \{m_{i+1}\} - q^{-2(m_{i+1}-1)},\\ \{m_{i+1}+1\} &= \{m_{i+1}\} + q^{-2(m_{i+1})}.\end{aligned}$$

The version for F_i's follows similarly. □

2.4.12 Comparing the algebras for $U_q(sl_n)$

For $\Gamma = \mathbb{Z}^n$, with standard basis $\mathbf{e}_i$, as an additive group and $|x_i| = \mathbf{e}_i$, with $\beta(\mathbf{e}_i, \mathbf{e}_j) = q_{ij}$, we have

Proposition 2.4.13. *The embedding $\Omega_n : \mathcal{A}_q(n) \hookrightarrow D_q(\mathbb{C}[n])$ is an algebra morphism sending*

$$\begin{aligned}\omega_i^{\pm 1} &\mapsto \sigma_{\pm e_i},\\ \psi_i^\dagger &\mapsto \lambda_{x_i},\\ \psi_i &\mapsto \frac{\mathfrak{d}_{e_i} - \mathfrak{d}_{-e_i}}{q-q^{-1}}.\end{aligned}$$

Proof. This follows as before: the operators $\mathfrak{d}_{\mathbf{e}_i}$ send $x(\mathbf{m})$ to $\beta(\mathbf{e}_i, m_i)x(\mathbf{m}-\mathbf{e}_i)$ and we see that they are elements of our ring. □

Corollary 2.4.14. *Our representations commute with the morphism Ω between algebras.*

Proof. Follows from the definitions of our maps. □

We see there should also be a "twisted Ω" for $\mathcal{A}_q(n) \hookrightarrow D_q(\mathbb{C}_q[n])$ such that $\Omega_{xy} = \Omega_2^q$,

Theorem 2.4.15. *There is a homomorphism $\Omega_n^q : \mathcal{A}_q(n) \hookrightarrow D_q(\mathbb{C}_q[n])$ realizing the diagram below for all $1 \leq m \leq n-1$:*

$$\begin{tikzcd}
 & U_q(\mathfrak{sl}_2) \arrow[rrr, "a"] \arrow[dl, "i_m"'] \arrow[dr, "\pi_2"] & & & U_q(\mathfrak{sl}_2) \arrow[dl, "\tilde{\pi}_2"'] \arrow[dr, "i_m"] & \\
U_q(\mathfrak{sl}_n) \arrow[dr, "\pi_n"'] & & \mathcal{A}_q(2) \arrow[r, "\Omega_{xy}"] \arrow[dl, "e_m"'] & D_q(\mathcal{Q}) \arrow[dr, "s_m"] & & U_q(\mathfrak{sl}_n) \arrow[dl, "\tilde{\pi}_n"] \\
 & \mathcal{A}_q(n) \arrow[rrr, hook, dashed, "\Omega_n^q"'] & & & D_q(\mathbb{C}_q[n]) &
\end{tikzcd}$$

Proof. We simply consider the images of our previously defined maps. We see, by considering the diagram for all m, that this map will be well defined. $\square$

2.4.16 $\tilde{D}_q(\mathcal{B}_q(\mathfrak{g}))$ and [14]'s generators and relations

Recalling the subalgebra $\tilde{D}_q(\mathcal{B}_q(\mathfrak{g}))$ defined in [26], we compare this with our aforementioned observations.

Using the definition in [26] we can write $\tilde{D}_q(\mathcal{B}_q(\mathfrak{sl}_2))$, thought of as the global quantum differential operators on the quantum flag variety of $\mathfrak{sl}_2$, in the generators and relations of $D_q(\mathcal{Q})$ computed in [14], the ring of quantum differential operators on the quantum plane.

Remark 2.4.17. Tanisaki uses $\mathbb{C}_q[G/N^-]$ for the Gel'fand model $\bigoplus_{\lambda \in P^+} R_\lambda$, and l_x for left multiplication. For $\lambda, \mu \in P^+$, denote (λ, μ) their inner product (see [16].)

Definition 2.4.18. We define $\tilde{D}_q(\mathcal{B}_q(\mathfrak{g}))$ as the algebra

$$\mathbb{C}\left\langle l_\varphi, \partial_u, \sigma_\lambda \mid \varphi \in \mathbb{C}_q[G/N^-], u \in U_q(\mathfrak{g}), \lambda \in P^+ \right\rangle \subset \mathrm{End}_{\mathbb{C}}(CC_q[G/N^-])$$

where

$$l_\varphi(\psi) = \varphi\psi, \qquad \partial_u(\psi) = u \cdot \psi, \qquad \sigma_\lambda(\psi) = q^{(\lambda,\mu)}\psi$$

for $\psi \in \mathbb{C}_q[G/N^-]_\mu$.

Now, recall that $\mathbb{C}_q[G/N^-] = R_q^-[G]$ of [18] in (9.5.5). We take now the special case of $\mathfrak{g} = \mathfrak{sl}_2$.

Now, from [14] we have that for $\mathcal{Q}$ with $|x| = (1,1)$, $|y| = (-1,1)$,

$$\begin{aligned} E(x^i y^j) &= q^i \{j\}_{-2} x^{i+1} y^{j-1}, \\ F(x^i y^j) &= q^j \{i\}_{-2} x^{i-1} y^{j+1}, \\ K(x^i y^j) &= q^{(i-j)} x^i y^j \end{aligned}$$

and written in the generators of $D_q(\mathcal{Q})$, these are (cf. 2.2.38)

$$K \mapsto \sigma_x \sigma_y^{-1}, \quad E \mapsto \sigma_x \sigma_y^{-1} \lambda_x \mathfrak{d}_y (1 + \sigma_y^{-1}), \quad F \mapsto \sigma_y \sigma_x^{-1} \rho_x \mathfrak{d}_x (1 + \sigma_x^{-1})$$

Whence, we see that

Corollary 2.4.19. *$\tilde{D}_q(\mathcal{B}_q(\mathfrak{sl}_2)) \hookrightarrow D_q(\mathcal{Q})$ is an algebra morphism.*

Remark 2.4.20. This shows that the algebra that Tanisaki used to confirm a quantum version of (II) sits inside the larger algebra of quantum differential operators on the ring of functions: it picks up the operators corresponding to multiplication by functions, by torus action, and those differentials coming from quantum group action. In this case, its image *does not* generate this larger ring.

2.4.21 $\mathcal{A}_q(2)$, $\tilde{D}_q(\mathcal{B}_q(\mathfrak{sl}_2))$, and the localization of the theorem of Beilinson-Bernstein

We recall that Hayashi's quantum Weyl algebra embeds into quantum differential operators. In particular,

$$\begin{aligned} \omega_i^{\pm 1} &\mapsto \sigma_{\pm|i|} \\ \psi_i^\dagger &\mapsto x_i \\ \psi_i &\mapsto \frac{\mathfrak{d}_{e_i} - \mathfrak{d}_{-e_i}}{q - q^{-1}}. \end{aligned}$$

for $i \in \{1, 2\}$ and $|1| = (1, 1), |2| = (-1, 1)$.

Proposition 2.4.22. $\mathcal{A}_q(2) \hookrightarrow \tilde{D}_q(\mathcal{B}_q(\mathfrak{sl}_2))$ *compatable with* Ω_2.

Proof. We have shown that σ's and ring elements are in the subalgebra, so we are left to show that $\mathfrak{d}_{2e_i}$ is. Observe that

$$\begin{aligned} \mathfrak{d}_{e_1} &= (1 - q^{-2})\rho_y E \sigma_x - \lambda_x \\ \mathfrak{d}_{e_2} &= (1 - q^{-2})\lambda_y E \sigma_y - \rho_y \end{aligned}$$

as operators. By Lemma 4.1 of [26],

$$\rho_\psi = \sum_p \lambda_{a_p \psi} \delta_{b_p \kappa_\nu} \sigma_{-\mu}$$

for all $\psi \in \mathcal{B}_q(\mathfrak{sl}_2)(\mu)_\nu$, so $\mathfrak{d}_{e_i} \in \tilde{D}_q(\mathcal{B}_q(\mathfrak{sl}_2))$. □

Remark 2.4.23. This gives an alternative proof that $\mathfrak{d}_{e_i} \in D_q(\mathcal{B}_q(\mathfrak{sl}_2))$.

In light of this fact, one may wonder if the previous proposition holds for the higher-rank quantum Weyl algebras, and these subalgebras of the quantum differential operators on quantum flag varieties. The main difficulty in checking this fact, is that there is not a nice description of the ring of functions for quantum flag varieties of higher rank. In the case of $\mathfrak{sl}_2$, we can easily compute the ring of functions to be the quantum plane. These algebras are noncommutative and are described with multiplication by the projection to highest-weight component of weight representations for the corresponding quantum groups. One can get an idea for the ring structure of these algebras in ([[18], §9.1], [[17]]). Without a description of the rings of functions, we are unable to compute the images of the generators for $U_q(\mathfrak{g})$ in the algebras of quantum differential operators, and thus unable to check if the higher-rank quantum Weyl algebras stay inside these rings.

Despite these difficulties,

Conjecture 2.4.24. $\mathcal{A}_q(n) \hookrightarrow \tilde{D}_q(\mathcal{B}_q(\mathfrak{sl}_n))$ *commutes with* Ω_n.

Theorem 2.4.25. *([Conjecture 5.3, [22]], [Theorem 0.6, [26]]) The quantum analog of the localization theorem of Beilinson-Bernstein is true for* $\tilde{D}_q(\mathcal{B}_q(\mathfrak{g}))$.

Remark 2.4.26. This theorem combined with the previous proposition is suggestive about the geometric nature of $\mathcal{A}_q(2)$.

Chapter 3

Twisting and untwisting for differential operators

In the last chapter we have seen that $D_q(\mathbb{K}[x_1, \ldots, x_n])$ and $D_q(\mathbb{K}_q[x_1, \ldots, x_n])$ are different, but we wish to see how they can be related.

Example 3.0.27. As a simple example, if we recall that $\mathbb{C}_q(n)$ is a twist of $\mathbb{C}[x_1, \ldots, x_n]$ then we wish to give some idea of the question

$$D_\beta(\mathbb{C}[x_1, \ldots, x_n]) \simeq^? D_{\beta'}(\mathbb{C}_q(n))$$

which is obviously very related to our considerations for quantum differential operators. Notice here that the twist that defines these "quantum polynomial algebras" corresponds to the 2-cocycle given by $\gamma(a, b) = q^{ab}$. We should be careful with our assumptions on q, or at the least, tell where it lives, but we will shirk that discussion as of now.

3.1 Cocycle deformations of rings and modules

In 2.2.12 we introduced those differential operators defined by a β-twist. We will utilize these differential operators in a more general context now.

Let Γ be an abelian group with operation written as addition and $R := \bigoplus_{\lambda\in\Gamma} R_\lambda$ a graded $\mathbb{K}$-algebra, and write $|x| = \lambda$ for the degree of x for $x \in R_\lambda$. We will consider a bicharacter

$$\beta : \Gamma \times \Gamma \to \mathbb{K}^*$$

and we will consider a 2-cocycle

$$\gamma : \Gamma \times \Gamma \to \mathbb{K}^*$$

in group cohomology.

The main idea is to define R^γ as a deformed $\mathbb{K}$-algebra of R. This means another $\mathbb{K}$-algebra with a finitely deformed multiplication, but the same $\mathbb{K}$-linear structure. In particular,

$$x \star y := \gamma(|x|, |y|)xy$$

where we assume (and for the rest of this discussion) that concatenation of ring elements is multiplication in the original ring, $x, y \in R$ homogeneous elements. The standard exercise is to write down the associativity axiom for $- \star -$ and arrive at the relation:

$$\gamma(|x|, |y|)\gamma(|xy|, |z|) = \gamma(|y|, |z|)\gamma(|x|, |yz|)$$

which is exactly the 2-cocycle condition. Note that we also can write this relation additively:

$$\gamma(a, b)\gamma(a + b, c) = \gamma(b, c)\gamma(a, b + c)$$

for $a, b, c \in \Gamma$.

Problem 3.1.1. (cf. 3.2.15) For β and β' two bicharacters, is

$$D_\beta(R) \simeq^? D_{\beta'}(R^\gamma)$$

for some choices of β, β', γ? Here $D_\beta(R)$ are the β-differential operators as defined by Lunts and Rosenberg ([21]).

3.1.2 Twisting modules

Let $M \in R - \mathbf{GrMod}$, with grading $\bigoplus_{a\in\Gamma} M_a$.

Definition 3.1.3. M^γ is defined as a *twisted R^γ-module* for $M \in R - mod$ with the same group structure, but twisted action

$$r \star x = \gamma(|r|, |x|)rx$$

for both $r \in R$ and $x \in M$ homogeneous elements.

Definition 3.1.4. Let M be an R-bimodule, then we can define $M^{\gamma,\gamma}$ for the bi-twist on both sides by γ as

$$(r \star x) \star s = \gamma(|r| + |x|, |s|)\gamma(|r|, |x|)rxs = \gamma(|r|, |s| + |x|)\gamma(|s|, |x|)rxs = r \star (x \star s)$$

Thus $M^{\gamma,\gamma}$ is an R^γ-bimodule. We sometimes will only write M^γ for $M^{\gamma,\gamma}$. We will always assume both twists are the same.

Remark 3.1.5. Let $R - \mathbf{GrBiMod}$ be the category of graded R-bimodules which are graded on both left and right by the same grading group.

Definition 3.1.6. We can think of this twisting as a functor. Let $R - \mathbf{GrBiMod}$ be those graded bimodules over R and the morphisms of degree 0. Then define

$$(\quad)^{\gamma,\gamma} : R-\textbf{GrBiMod} \to R^{\gamma}-\textbf{GrBiMod}.$$

Now we move to the level of Hom:

Definition 3.1.7. For $M, N \in R-\textbf{GrMod}$, we define

$$\mathrm{Hom}^{\mathrm{res}}_{\mathbb{K}}(M,N) := \bigoplus_{a\in\Gamma} \mathrm{Hom}^{a}_{\mathbb{K}}(M,N)$$

for $\mathrm{Hom}^{a}_{\mathbb{K}}(M,N)$ those maps of degree a, i.e., $f(M_b) = N_{a+b}$ for all $b \in \Gamma$.

Remark 3.1.8. $\mathrm{Hom}^{\mathrm{res}}_{\mathbb{K}}(M,N)$ is a graded R-bimodule. Then $(\mathrm{Hom}_{\mathbb{K}}(M,N))^{\gamma,\gamma}$ is a R^{γ}-bimodule.

Similarly we also define another relevant R^{γ}-bimodule, $\mathrm{Hom}^{\mathrm{res}}_{\mathbb{K}}(M^{\gamma},N^{\gamma})$. We want to know how $\mathrm{Hom}^{\mathrm{res}}_{\mathbb{K}}(M^{\gamma},N^{\gamma})$ and $\mathrm{Hom}^{\mathrm{res}}(M,N)^{\gamma,\gamma}$ are related.

In summary, we have three bimodules that we are considering

$$\begin{array}{ccc} \mathrm{Hom}^{\mathrm{res}}_{\mathbb{K}}(M,N) & \in & R-\textbf{GrBiMod}, \\ \mathrm{Hom}^{\mathrm{res}}_{\mathbb{K}}(M^{\gamma},N^{\gamma}) & \in & R^{\gamma}-\textbf{GrBiMod}, \\ (\mathrm{Hom}^{\mathrm{res}}_{\mathbb{K}}(M,N))^{\gamma,\gamma} & \in & R^{\gamma}-\textbf{GrBiMod} \end{array}$$

and we already have a relation from the first to the last:

$$(\quad)^{\gamma,\gamma} : \mathrm{Hom}^{\mathrm{res}}_{\mathbb{K}}(M,N) \mapsto (\mathrm{Hom}^{\mathrm{res}}_{\mathbb{K}}(M,N))^{\gamma,\gamma}.$$

We also have a relation from the first to the second:

$$(\quad^{\gamma}, \quad^{\gamma}) : \mathrm{Hom}^{\mathrm{res}}_{\mathbb{K}}(M, N) \mapsto \mathrm{Hom}^{\mathrm{res}}_{\mathbb{K}}(M^{\gamma}, N^{\gamma}).$$

3.1.9 Returning to $\mathrm{Hom}^{\mathbf{res}}_{\mathbb{K}}(M, N)$

For $f \in \mathrm{Hom}^{\mathrm{res}}_{\mathbb{K}}(M, N)$, we have bimodule actions:

$$(rfs)(x) := rf(sx)$$

for $x \in M$, $r, s \in R$, which gives an $R-$**GrBiMod** structure. We will deform this structure:

Definition 3.1.10. On $(\mathrm{Hom}^{\mathrm{res}}_{\mathbb{K}}(M, N))^{\gamma,\gamma}$, the R^{γ}-bimodule structure is given:

$$(r \star f \star s)(x) := \gamma(|r|, |f|)\gamma(|r| + |f|, |s|)(rfs)(x)$$

for homogeneous elements.

Definition 3.1.11. So we wish to relate this structure to the structure on $\mathrm{Hom}^{\mathrm{res}}_{\mathbb{K}}(M^{\gamma}, N^{\gamma})$:

$$\begin{aligned}
(r \cdot f \cdot s)(x) &:= r \star (f(s \star x)) = r \star f(\gamma(|s|, |x|)sx) \\
&= \gamma(|s|, |x|)\gamma(|r|, |f(sx)|)(rf(sx)) \\
&= \gamma(|s|, |x|)\gamma(|r|, |f| + |s| + |x|)rf(sx) \\
&= \gamma(|s|, |x|)\gamma(|r|, |f| + |s| + |x|)(rfs)(x).
\end{aligned}$$

To relate these structures, we use an automorphism on $\mathrm{Hom}^{\mathrm{res}}_{\mathbb{K}}(M, N)$:

Lemma 3.1.12. *For all $f \in \mathrm{Hom}^{res}_{\mathbb{K}}(M, N)$ homogenous, define*

$$f^{\gamma}(x) := \gamma(|f|, |x|)f(x).$$

for all $x \in M$ homogenous, then

$$(\ ^{\gamma}) : f \mapsto f^{\gamma}$$

is an $\mathbb{K}$-linear automorphism of $\mathrm{Hom}^{res}_{\mathbb{K}}(M, N)$.

Proof. This morphism has an inverse given by $(\ ^{\gamma^{-1}})$. □

Now, if we incorporate this morphism into our previous computation,

Lemma 3.1.13. $(f \star s)^{\gamma}(x) = (f^{\gamma} \cdot s)(x)$.

Lemma 3.1.14. $(r \star f)^{\gamma}(x) = (r \cdot f^{\gamma})(x)$.

Theorem 3.1.15. *There exists a $\mathbb{K}$-linear homomorphism of R^{γ}-bimodules*

$$\tau^{\gamma} : (\mathrm{Hom}^{res}_{\mathbb{K}}(M, N))^{\gamma,\gamma} \to \mathrm{Hom}^{res}_{\mathbb{K}}(M^{\gamma}, N^{\gamma})$$

given by 3.1.12 and it is a natural isomorphism.

Proof. This is precisely the implication of the last two lemmas. Our inverse gives the isomorphism. □

3.2 Twisting and untwisting

We will define a set similar to the β-differential operators of [21]. The goal is to use these as a prototype for the aforementioned comparison of twistings. Define β as a bicharacter for Γ over $\mathbb{K}$ throughout. R is a $\mathbb{K}$-algebra, M and N are $R-$**GrMod**s.

Definition 3.2.1. We consider the *set of β-differential morphisms from M to N* as defined inductively for $n \geq 0$,

$$D^{'}_{\beta}(M, N)^{i} := \left\{ f \in \mathrm{Hom}^{\mathrm{res}}_{\mathbb{K}}(M, N) \mid \lambda^{\beta}_{r}(f) \in D^{'}_{\beta}(M, N)^{i-1}, \forall r \in R \right\}$$

$D'_\beta(M,N)^{-1} := 0$ where λ_r^β we call the β-*adjoint action by* r:

$$\lambda_r^\beta(f) := rf - \beta(|r|, |f|)fr$$

for all homogenous $r \in R$.

Definition 3.2.2. We could actually define a slightly more general version of the above for $M \in R - \mathbf{GrBiMod}$:

$$D'_\beta(M)^i := \left\{ x \in M \mid \lambda_r^\beta(x) \in D'_\beta(M)^{i-1} \right\}$$

for all homogenous $r \in R$, this set we refer to as β-*differential elements of* M.

Remark 3.2.3. The definition of Lunts and Rosenberg in [21] coincides with

$$D_\beta(M,N)^i := RD'_\beta(M,N)^i R \subset \mathrm{Hom}^{\mathrm{res}}(M,N).$$

the main difference is that they generate an R-bimodule at every step, whereas D'_β will be considered set-theoretically.

Example 3.2.4. For $M = R$ we can define $D'_\beta(R)$:

$$D'_\beta(R)^i \subset D'_\beta(R)^{i+1}$$

and

$$D'_\beta(R) := \bigcup_i D'_\beta(R)^i$$

with

$$D'_\beta(R)^i D'_\beta(R)^j \subset D'_\beta(R)^{i+j}$$

where $D'_\beta(R)^0$ is a $\mathbb{K}$-algebra of R called the β-*center of* R. We observe that this $D'_\beta(R)$ is a filtered subring of R by this construction.

Remark 3.2.5. We notice that $D'_\beta(M)$ is a filtered $D'_\beta(R)$-bimodule for M a $R-$**GrBiMod**. Additionally, there exists the associated graded $\mathbb{K}$-algebra:

$$gr_\beta(R) := \bigoplus_{i\geq 1} D'_\beta(R)^i / D'_\beta(R)^{i-1}.$$

Remark 3.2.6. We will use the notation to write $M^i_\beta := D'_\beta(M)^i$.

We proceed using both bicharacter and cocycle twists and approach this question:

Lemma 3.2.7. *If γ is a 2-cocycle, and Γ is an abelian group, then*

$$(a,b) \mapsto \gamma^{-1}(a,b)\gamma(b,a)$$

is a bicharacter. For β a bicharacter,

$$\beta^\gamma := \beta(a,b)\gamma^{-1}(a,b)\gamma(b,a)$$

is also a bicharacter.

Proof. This follows easily. □

Definition 3.2.8. $\lambda_r^{(\gamma,\beta)}(x) := r\star x - \beta(|r|,|x|)x\star r$ for $M^{\gamma,\gamma}$ if $M \in R-$ **GrBiMod** and R^γ. We call this the *γ-twisted β-adjoint action by homogenous r in R.*

Lemma 3.2.9. *For $r\in R$, $x \in M \in R-$ **BiMod**, and $\lambda_r^{(1,\beta)} := \lambda_r^\beta$,*

$$\lambda_r^{(\gamma,\beta)}(x) = \gamma(|r|,|x|)\lambda_r^{(1,\beta^\gamma)}(x).$$

Proof. Note only that for a Γ-graded R-bimodule, M with γ-twist,$M^{\gamma,\gamma}$, has R^γ:

$$x\star r = \gamma(|r|,|x|)xr.$$

So now we can ask the question: as $R^\gamma - \mathbf{BiMod}$, do we have

$$D^{'}_{\beta}((\mathrm{Hom}^{\mathrm{res}}_{\mathbb{K}}(M,N))^{\gamma,\gamma}) \simeq D^{'}_{\beta}(\mathrm{Hom}^{\mathrm{res}}_{\mathbb{K}}(M^\gamma,N^\gamma))?$$

Example 3.2.10. We have

$$D^{'}_{\beta\gamma}(\mathrm{Hom}^{\mathrm{res}}_{\mathbb{K}}(M,N)) \simeq D^{'}_{\beta}((\mathrm{Hom}^{\mathrm{res}}_{\mathbb{K}}(M,N))^{\gamma,\gamma})$$

and now

$$D^{'}_{\beta\gamma}(\mathrm{Hom}^{\mathrm{res}}_{\mathbb{K}}(M,N)) \simeq D^{'}_{\beta}(\mathrm{Hom}^{\mathrm{res}}_{\mathbb{K}}(M^\gamma,N^\gamma)).$$

Theorem 3.2.11.

$$D^{'}_{\beta}(M^\gamma)^i := ((M^\gamma)^{'}_{\beta})^i = (M^{'}_{\beta\gamma})^i =: D^{'}_{\beta\gamma}(M)^i$$

For $M^{'}$ defined similar to $D^{'}$, i.e. not generating a module at each step.

Proof. Follows directly from the lemma. □

We have now shown how twisting interacts with the differential twisting for $D^{'}$. We need to put this in the context of Lunts and Rosenberg's D.

3.2.12 The functor $D^{'}_{\beta}$

We can think of the construction above in terms of a functor from $R - \mathbf{GrBiMod}$ to $R^\gamma_\beta - \mathbf{GrBiMod}$. Note that here we mean $R^\gamma_\beta := D^{'}_{\beta}(R^\gamma)$.

In particular, this functor should send $R \mapsto R^\gamma_\beta = D^{'}_{\beta}(R^\gamma)$. Keeping in mind that we have defined the functor $(\quad)^{\gamma,\gamma}$ from $R - \mathbf{GrBiMod}$ to $R^\gamma - \mathbf{GrBiMod}$, we can see how to define this functor:

Definition 3.2.13. $D^{'}_{\beta}(\quad^{\gamma,\gamma}) : R - \mathbf{BiMod} \to \mathbb{K} - \mathbf{Vect}$ by

$$M \mapsto D^{'}_{\beta}(M^{\gamma,\gamma}).$$

which is $\mathbb{K}$-linear, with morphisms being sent to composition with operators. Similar to the classical functor of differential operators on an algebra.

3.2.14 Twisting LR D_q

We have set our two objects up very similarly, $D'_\beta(M)$ and $D_\beta(M)$. For $D'_\beta(M)$ we were able to show 3.2.11 which essentially says that we can twist differential operators, or the module. This however, is at the level of $\mathbb{K}$-vector spaces. Recall that our functor D'_β was only a functor into $\mathbb{K}$-vector spaces. By considering instead the modules as described by Lunts and Rosenberg, we hope to get an analogue of our twisting and untwisting theorem.

The main difficulty appears in that when considering the bimodule generated by these sets, we do not expect to maintain functoriality. Moreover, when we write down these modules, we quickly see that they live in two different categories, $R-\mathbf{GrBiMod}$ and $R^\gamma-\mathbf{GrBiMod}$. However, we have a tool already for passing from one to the other. We see that this tool will yield what we wish. We prove

Proposition 3.2.15. *For any $M, N \in R-\mathbf{GrMod}$*

$$(D_{\beta^\gamma}(\mathrm{Hom}^{res}_{\mathbb{K}}(M,N)))^\gamma \simeq D_\beta(\mathrm{Hom}^{res}_{\mathbb{K}}(M^\gamma, N^\gamma))$$

where we are thinking of the left side as the image under the functor $(\quad)^\gamma : R-\mathbf{BiMod} \to R^\gamma-\mathbf{BiMod}$.

Proof. This follows from our previous work. $\square$

Example 3.2.16. Consider the 0'th β^γ-differential part of a module M, for M an $R-\mathbf{GrBiMod}$:

$$D_{\beta^\gamma}(M)^0 = R\left\{m \in M \mid rm - \beta(|r|,|m|)\gamma^{-1}(|r|,|m|)\gamma(|m|,|r|)mr = 0, \forall r \in R\right\}R.$$

Similarly

$$\begin{aligned}
D_\beta(M^\gamma)^0 &= R^\gamma \{m \in M^\gamma \mid r \star m - \beta(|r|,|m|)m \star r = 0, \forall r \in R^\gamma\} R^\gamma \\
&= R^\gamma \{m \in M^\gamma \mid \gamma(|r|,|m|)rm - \beta(|r|,|m|)\gamma(|m|,|r|)mr = 0, \forall r \in R^\gamma\} R^\gamma \\
&= R^\gamma \left\{m \in M^\gamma \mid rm - \beta(|r|,|m|)\gamma^{-1}(|r|,|m|)\gamma(|m|,|r|)mr = 0, \forall r \in R^\gamma\right\} R^\gamma.
\end{aligned}$$

Recall that these are the smallest R and R^γ bimodules generated by $D'_\beta(M)^0$ respectively. Further, $(\quad)^\gamma : D_{\beta^\gamma}(M)^0 \mapsto D_\beta(M^\gamma)^0$.

Lemma 3.2.17. $(\quad)^\gamma : D_{\beta^\gamma}(M)^1 \mapsto D_\beta(M^\gamma)^1$.

Proof. We need only consider that $D_{\beta^\gamma}(M)^1$ is

$$R\left\{m \in M \mid rm - \beta(|r|,|m|)\gamma^{-1}(|r|,|m|)\gamma(|m|,|r|)mr \in D_{\beta^\gamma}(M)^0, \forall r \in R\right\} R$$

so $(D_{\beta^\gamma}(M)^1)^\gamma$ is

$$R^\gamma\left\{m \in M^\gamma \mid rm - \beta(|r|,|m|)\gamma^{-1}(|r|,|m|)\gamma(|m|,|r|)mr \in (D_{\beta^\gamma}(M)^0)^\gamma, \forall r \in R^\gamma\right\} R^\gamma$$

which we saw in the example to be

$$R^\gamma\left\{m \in M^\gamma \mid rm - \beta(|r|,|m|)\gamma^{-1}(|r|,|m|)\gamma(|m|,|r|)mr \in D_\beta(M^\gamma)^0, \forall r \in R^\gamma\right\} R^\gamma.$$

$\square$

Theorem 3.2.18. *$(D_{\beta^\gamma}(M))^\gamma \simeq D_\beta(M^\gamma)$ for any M a $R-$* **GrBiMod**. *In particular,*

$$\begin{tikzcd}
R-\mathbf{GrBiMod} \arrow[r, "D_{\beta^\gamma}"] \arrow[d, "(\quad)^{\gamma,\gamma}"] & R-\mathbf{GrBiMod} \arrow[d, "(\quad)^{\gamma,\gamma}"] \\
R^\gamma-\mathbf{GrBiMod} \arrow[r, "D_\beta"] & R^\gamma-\mathbf{GrBiMod}
\end{tikzcd},$$

as a diagram of functors, commutes up to natural isomorphism.

Proof. Induction on $(D_{\beta^\gamma}(M)^i)^\gamma$ using the last lemma. $\square$

3.3 A twisting functor

3.3.1 Functoriality of $D'_\beta(-^\gamma)$

Lemma 3.3.2. $D'_\beta(\quad^\gamma)$ *is a functor from* $R-$ **GrBiMod** $\to$ **Vect**.

Proof. First observe that $(\quad)^\gamma : \phi \to \phi^\gamma$ for $\phi : R \to R'$ and $\phi^\gamma : R^\gamma \to R'^\gamma$, as a Γ-graded $\mathbb{K}$-algebra homomorphism, behave the same on underlying vector spaces. Now observe that

$$D'_\beta(R)^0 := \{s \in R \mid r \star s - \beta(|r|\,, |s|)s \star r = 0, \forall r \in R\}$$

and notice that $\phi(D'_\beta(R)^0)$ is the set where

$$\phi(r) \star' \phi(s) - \beta(|r|\,, |s|)\phi(s) \star' \phi(r) = 0$$

which is $D'_\beta(\phi^\gamma(R^\gamma))^0$. So we see that

$$\phi(D'_\beta(R)^0) = D'_\beta(\phi^\gamma(R^\gamma))^0 = D'_\beta(\phi(R)^\gamma)^0.$$

By induction we see this holds for all i and thus for $D'_\beta(R)$. A similar proof confirms it for any R-bimodule M. $\square$

So the answer to the question of "how does it behave on morphisms", is that the functor ignores them. Since the morphisms are applied on the underlying set, and the functor forgets the structure down to a subset, functoriality requires that it acts trivially on the morphisms.

3.4 Generalizing the algebras of differential operators

3.4.1 β-Hayashi algebra

We wish to write down an extension of Hayashi's algebra for a general bicharacter β. There are two considerations to take into account:

1. The morphism Ω from $\mathcal{A}_q(1)$ to $D_q(\mathbb{K}[x])$.

2. The action of $\mathcal{A}_q(n)$ on $\mathbb{K}[x_1, \ldots, x_n]$.

The first suggests that our generalization to β should be thought of as follows, when $\mathcal{A}_\beta(1)$ coincides with $\mathcal{A}_q(1)$ it should be with $\beta(a,b) = q^{ab}$ with $\Gamma = \mathbb{Z}$, and we can use this map to think of the degree of some elements. The second will give us an immediate representation of $\mathcal{A}_\beta(n)$ analogous to the oscillator representation. This will also be useful in giving us reality conditions.

Example 3.4.2. We define the β-*Weyl Algebra* for a fixed bicharacter β of Γ, $\mathcal{A}_\beta(1)$ as the algebra generated by $\left\langle \psi, \psi^\dagger, \omega^{\pm 1} \right\rangle$ and relations:

$$\begin{aligned} \omega\psi &= \beta(1,-1)\psi\omega, & \omega\psi^\dagger &= \beta(1,1)\psi^\dagger\omega, \\ \psi\psi^\dagger - \beta(2,1)\psi^\dagger\psi &= \omega^{-2}, & \psi\psi^\dagger - \beta(-2,1)\psi^\dagger\psi &= \omega^2. \end{aligned}$$

The action on $\mathbb{C}[x]$ is given by:

$$\begin{aligned} \omega(x^n) &= \beta(1,n)x^n, & \omega^{-1}(x^n) &= \beta(-1,n)x^n, \\ \psi^\dagger(x^n) &= x^{n+1}, & \psi(x^n) &= \left(\frac{\beta(2,n) - \beta(-2,n)}{\beta(2,1) - \beta(-2,1)}\right) x^{n-1}. \end{aligned}$$

We can check that this is a representation of $\mathcal{A}_\beta(1)$ and is compatible with the definition of $A_q(1)$ when $\beta(a,b) = q^{ab}$.

In the original definition of the oscillator representation, quantum integers were used. We first will define a β-integer in the same way:

Definition 3.4.3. For any abelian group Γ and any bicharacter $\beta : \Gamma \times \Gamma \to \mathbb{Z}$ define the *β-integer* by

$$[n]_{\beta(m,-)} := \left(\frac{\beta(m,n) - \beta(-m,n)}{\beta(m,1) - \beta(-m,1)} \right),$$

for all $n, m \in \Gamma$, provided $\beta(m,1) = \beta(m,-1)$ for all $m \neq 0 \in \Gamma$.

So in our last example, we can rewrite, for $\Gamma = \mathbb{Z}$

$$\psi(x^n) = \left(\frac{\beta(1,n) - \beta(-1,n)}{\beta(1,1) - \beta(-1,1)} \right) x^{n-1} = [n]_{\beta(1,-)} x^{n-1}.$$

This leads to our general definition:

Definition 3.4.4. The *β-Weyl algebra* with grading $\Gamma = \mathbb{Z}^n$, $\mathcal{A}_\beta(n)$, is defined by generators $\psi_i, \psi_i^\dagger, \omega_i, \omega_i^{-1}$ for $1 \leq i \leq n$, $i \neq j$, and relations:

$$\begin{aligned} \omega_i \omega_j = \omega_j \omega_i, &\qquad \omega_i \omega_i^{-1} = \omega_i^{-1} \omega_i = 1 \\ \omega_i \psi_j \omega_i^{-1} = \beta(e_i, -e_j) \psi_j, &\qquad \omega_i \psi_j^\dagger \omega_i^{-1} = \beta(e_i, e_j) \psi_j^\dagger \\ \psi_i \psi_j - \psi_j \psi_i = \psi_i^\dagger \psi_j^\dagger - \psi_j^\dagger \psi_i^\dagger = 0, &\qquad \psi_i \psi_j^\dagger - \psi_j^\dagger \psi_i = 0 \\ \psi_i \psi_i^\dagger - \beta(2e_i, e_i) \psi_i^\dagger \psi_i = \omega_i^{-2}, &\qquad \psi_i \psi_i^\dagger - \beta(-2e_i, e_i) \psi_i^\dagger \psi_i = \omega_i^2. \end{aligned}$$

Now we define the *β-Oscillator Representation of $\mathcal{A}_\beta(n)$ on* $\mathbb{C}[x_1, \ldots, x_n]$:

Definition 3.4.5. For $x(\mathbf{m}) := x_1^{m_1} \ldots x_n^{m_n}$ a monomial in $\mathbb{C}[x_1, \ldots, x_n]$, and $\mathbf{m} \in \mathbb{Z}^n_{\geq 0}$, we define the action of $\mathcal{A}_\beta(n)$ as

$$\begin{aligned} \omega_i(x(\mathbf{m})) &= \beta(e_i, \mathbf{m}) x(\mathbf{m}) \\ \psi_i(x(\mathbf{m})) &= [\mathbf{m}]_{\beta(e_i,-)} x(\mathbf{m} - \mathbf{e}_i) \\ \psi_i^\dagger(x(\mathbf{m})) &= x(\mathbf{m} + \mathbf{e}_i) \end{aligned}$$

where $\mathbf{e}_i := (\delta_{1i}, \ldots, \delta_{ni})$ and $m = \sum m_i e_i \in \Gamma$.

Then it is straightforward to check that $\mathcal{A}_\beta(n) \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}[x_1, \ldots, x_n])$ is a $\mathbb{C}$-algebra homomorphism.

3.4.6 Comparing Weyl algebras

We have some different notions of Weyl algebras that still need comparing. Here we discuss the current understanding.

We begin by assuming as in [21], that $q := [q_{ij}]_{n\times n}$, $q_{ij} = q_{ji}^{-1}$, $\Gamma = \mathbb{Z}^n$ with standard basis $\mathbf{e}_i$. Define the bi-character β by

$$\beta(e_i, e_j) = q_{ij}.$$

Further, consider $\mathbb{K}_q[n] := \mathbb{K}\left\langle x_1, \ldots, x_n \right\rangle / (x_i x_j = q_{ij} x_j x_i)$ which is $\mathbb{Z}^n$-graded with $|x_i| = \mathbf{e}_i \in \Gamma$.

Definition 3.4.7. From [21], we have the algebra $W := \mathbb{K}\left\langle x_i, \partial_i, \sigma_i, \sigma_i^{-1} \right\rangle / \mathcal{R}$ where $\mathcal{R}$ is the relations

$$\begin{aligned}
(x_i x_j = q_{ij} x_j x_i), \qquad & (\partial_i \partial_j = q_{ij} \partial_j \partial_i), \\
(\sigma_i \sigma_j = \sigma_j \sigma_i), \qquad & (\partial_i x_j = q_{ij} x_j \partial_i + \delta_{ij}), \\
(\sigma_j x_i = q_{ij} x_i \sigma_j), \qquad & (\partial_i \sigma_j = q_{ji} \sigma_j \partial_i).
\end{aligned}$$

Definition 3.4.8. Let γ be a 2-cocycle of Γ, define the twisted multiplication of $\mathbb{K}[x_1, \ldots, x_n]$ by $x^a \star x^b = \gamma(a, b) x^{a+b}$.

Definition 3.4.9. Let $\alpha : \Gamma \times \Gamma \to \mathbb{K}$, a map, and $\sigma_i(x^a) =: \alpha(e_i, a) x^a$, $a \in \Gamma$ in accordance with our above expectation.

Then we compute:

$$\begin{aligned}[\sigma_i, x_j]_q(x^a) &= \sigma_i(x_j \cdot x^a) - q_{ij}x_j\sigma_i(x^a) \\ &= (\alpha(e_i, a+e_j) - q_{ij}\alpha(e_i, a))\gamma(e_j, a)x^{a+e_j}.\end{aligned}$$

So, for the relation above to hold

$$[\sigma_i, x_j]_q = 0 \Leftrightarrow \alpha(e_i, a+e_j) = q_{ij}\alpha(e_i, a).$$

If we think of $\partial_i(x^a) := a_i x^{\mathbf{a}-\mathbf{e}_i}$ for $\mathbf{a} = (a_1, \ldots, a_n)$ a multi-index, then we can similarly compute:

$$\begin{aligned}[\sigma_i, \partial_j]_q(x^a) &= \sigma_i(\partial_j x^a) - q_{ji}\partial_j\sigma_i(x^a) \\ &= (\alpha(e_i, a-e_j) - q_{ji}\alpha(e_i, a))a_j x^{a-e_j}.\end{aligned}$$

So for the relations, above to hold,

$$[\sigma_i, \partial_j]_q = 0 \Leftrightarrow \alpha(e_i, a-e_j) = q_{ji}\alpha(e_i, a) \quad \forall a \in \Gamma, \quad i, j \in \{1, \ldots, n\}.$$

These conditions give that

$$\alpha(e_i, e_j) = q_{ij},$$

and that $\alpha = \beta$. By the definition of $\sigma_i(x^a)$ we see that σ_i and σ_j commute. However, to get our last relation,

$$(\partial_i x_j = q_{ij}x_j\partial_i + \delta_{ij}),$$

we compute a bit more.

$$\begin{aligned}[\partial_i, x_j]_q(x^a) &= \partial_i x_j(x^a) - q_{ij}x_j\partial(x^a)\\ &= (\gamma(e_j,a)(a_i) - q_{ij}\gamma(e_j,a-e_j)(a_i))x^{a+e_j-e_i}.\end{aligned}$$

Now when $i = j$,

$$\partial_i x_j = q_{ij}x_j\partial_i + 1 \Leftrightarrow 1 = \gamma(e_i,a)a_i - \gamma(e_i,a-e_i)a_i,$$

and for $i \neq j$,

$$\partial_i x_j = q_{ij}x_j\partial_i \Leftrightarrow \gamma(e_j,a) = q_{ij}\gamma(e_j,a-e_i).$$

This last relation shows that

$$\gamma(e_j,e_i) = q_{ji}.$$

So $\beta = \gamma = \alpha$.

We have simply shown what conditions are necessary for W to embed into $D_q(\mathbb{K}_q[n])$, we call this map Ξ. It is left to consider a similar embedding $A_\rho(n)$ the "β"-Hayashi Weyl algebra corresponding to ρ into $D_\rho(\mathbb{K}[n])$, for an appropriate ρ. We recall that 3.2.15 suggests these algebras should be related by the isomorphism of twisted differential operator algebras.

Since we have twisting-untwisting, we have a relation $D_\beta(\mathbb{K}[n]) \simeq D_q(\mathbb{K}_q[n])$ for appropriate β. We have $\Omega_n^\beta : \mathcal{A}_\beta(n) \to D_\beta(\mathbb{K}[n])$ for that same β. We also have $\Omega_n^q : \mathcal{A}_q(n) \to D_q(\mathbb{K}_q[n])$. All together:

$$\begin{tikzcd}
\mathcal{A}_\beta(n) \arrow[d, "\Omega_n^\beta"] & \mathcal{A}_q(n) \arrow[d, "\Omega_n^q"] \\
D_\beta(\mathbb{K}[n]) \arrow[r, "\simeq"] & D_q(\mathbb{K}_q[n])
\end{tikzcd}\quad .$$

Chapter 4

Gluing for twisted coordinate rings

4.1 Localizing rings of differential operators

4.1.1 Localization of differential operators

We still assume that R is a Γ-graded, $\mathbb{K}$-algebra and $\beta : \Gamma \times \Gamma \to \mathbb{K}^*$ a bicharacter. The first question about our modules $D_\beta(R)$, from the context of geometry, is if the respect localizations by homogenous multiplicative (Ore resp.) sets, S_w. We recall from the classical definitions of differential operators that localization preserves them. But we need to check three new cases: β-differential operators for commutative rings, differential operators for noncommutative rings, and β-differential operators for noncommutative rings. In particular, if R is commutative, S_w a multiplicative subset, then for what pairs of bicharacters (β, β') is

$$S_w^{-1} D_\beta(R) \subset D_{\beta'}(S_w^{-1} R)? \tag{4.1}$$

And similarly for R noncommutative with S_w being an Ore subset: multiplicative, left cancellable, and for all $r \in R$, $s \in S_w$, there are r' and s' such that $s'r = r's$. Throughout this section, left multiplication by a ring element will eschew the λ notation, and will instead just write the email concatenated on the left.

4.1.2 Untwisted localization

First we recall a simple computation from the untwisted case.

Lemma 4.1.3. *For R commutative, $S_w \subset R$ a multiplicative set, $\partial \in D^0(R)$ there is a homomorphism $S_w^{-1} : D^0(R) \to D^0(S_w^{-1}R)$, i.e., ∂ extends to the localized ring.*

Proof. We consider $g' = \frac{g}{f^n}$ for $g \in R$, $f \in S_w$, then let $S_w^{-1}(\partial)(g) = \partial(g)$ for all $g \in R$, and

$$\begin{aligned} S_w^{-1}(\partial)(g') &= f^{-n} f^n S_w^{-1}(\partial)(g') \\ &= f^{-n}(S_w^{-1}(\partial)(f^n \cdot g') - S_w^{-1}([\partial, f^n])(g')) \\ &= f^{-n}(\partial(g) - 0) \\ &= f^{-n}\partial(g). \end{aligned}$$

We check for $h' = \frac{h}{k^m} \in S_w^{-1}(R)$, $k \in S_w$, $h \in R$, homogeneous elements, $[S_w^{-1}(\partial), h'](g') =$

$$S_w^{-1}(\partial)(h'g') - h'S_w^{-1}(\partial)(g') = k^{-m}f^{-n}\partial(hg) - h'S_w^{-1}(\partial)(g') = k^{-m}f^{-n}[\partial, h](g) = 0.$$

□

Theorem 4.1.4. *For $\partial \in D^k(R)$, ∂ extends to $D^k(S_w^{-1}(R))$.*

Proof. Exactly the same, but by induction, since $S_w^{-1}([\partial, f^n]) \in D^k(S_w^{-1}R)$. Thus,

$$S_w^{-1}(\partial)(g') := f^{-n}(\partial(g) - S_w^{-1}([\partial, f^n])(g')).$$

Following the computation in the lemma, $S_w^{-1}([\partial, f^n])$ is in $D^{k-1}(S_w^{-1}R)$, so we are finished.

□

We notice that our main technique is to write the image $S_w^{-1}(\partial)$ as an element of $S_w^{-1}R \otimes_R D^k(R)$. In fact, it can be shown that $S_w^{-1}R \otimes_R D^k(R) \simeq D^k(S_w^{-1}R)$ but for our applications, only the inclusion is necessary.

4.1.5 β-Twisted version

Again, we make the assumptions that R is commutative, S_w is a multiplicative set of R, $g \in R$, $f \in S_w$, $g' = \frac{g}{f}$, and now we will assume that $\beta : \Gamma \times \Gamma \to \mathbb{K}^*$ is a bicharacter, consists of S_w homogeneous elements of R. Since our localization is by homogenous elements, we assume that $|S_w^{-1}(\partial)| = |\partial|$, and that $|g'| = |g| - |f^n|$.

Lemma 4.1.6. *There exists a bicharacter, β', for S_w such that for all $\partial \in D^0_\beta(R)$ then $S_w^{-1}(\partial) \in D^0_{\beta'}(S_w^{-1}R)$, i.e., ∂ extends to the localized ring as a β'-differential operator.*

Proof. Again, we write the image of ∂ under S_w^{-1} as an element of $S_w^{-1}R \otimes_R D^0_\beta(R)$:

$$\begin{aligned} S_w^{-1}(\partial)(g') &= \beta^{-1}(|\partial|\,, |f^n|)f^{-n}(\partial(g) - S_w^{-1}([\partial, f^n]_\beta)(g')) \\ &= \beta(|f^n|\,, |\partial|)f^{-n}\partial(g). \end{aligned}$$

We now want $[S_w^{-1}(\partial), r']_{\beta'} = 0$ for all $r' \in S_w^{-1}R$ and solve for β'. Consider $r' = \frac{r}{h^m}$ for $r \in R$ and $h \in S_w$.

$$\begin{aligned} 0 &= [S_w^{-1}(\partial)(g'), r']_{\beta'}(g') \\ &:= S_w^{-1}(\partial)(r'g') - \beta'(\left|S_w^{-1}(\partial)\right|, |r'|)r'S_w^{-1}(\partial)(g') \\ &= \beta(|h^m f^n|\,, |\partial|)h^{-m}f^{-n}\partial(rg) - \beta'(\left|S_w^{-1}(\partial)\right|, |r'|)r'\beta(|f^n|\,, |\partial|)f^{-m}\partial(g) \\ &= h^{-m}f^{-n}\left(\beta(|h^m f^n|\,, |\partial|)\partial(rg) - \beta'(\left|S_w^{-1}(\partial)\right|, |r'|)\beta(|f^n|\,, |\partial|)r\partial(g)\right) \end{aligned}$$

Recalling $[\partial, r]_\beta(g) = 0$, we get

$$0 = h^{-m}f^{-n}\beta(|h^m f^n|\,, |\partial|)\left(\partial(rg) - \beta'(|r'|\,, \left|S_w^{-1}(\partial)\right|)\beta(|f^n|\,, |\partial|)\beta^{-1}(|h^m f^n|\,, |\partial|)r\partial(g)\right),$$

so

$$\begin{aligned}\beta(|\partial|,|r|) &= \beta'(\left|S_w^{-1}(\partial)\right|,|r'|)\beta(|f^n|,|\partial|)\beta^{-1}(|h^m f^n|,|\partial|)\\ &= \beta'(\left|S_w^{-1}(\partial)\right|,|r'|)\beta^{-1}(|h^m|,|\partial|).\end{aligned}$$

Thus we see that our inclusion holds when

$$\beta(|r'|,|\partial|) = \beta'(|r'|,\left|S_w^{-1}(\partial)\right|).$$

which is satisfied by $\beta = \beta'$. □

Theorem 4.1.7. *For $\partial \in D^k_\beta(R)$ then $S_w^{-1}(\partial) \in D^k_\beta(S_w^{-1}R)$.*

Proof. As before in our lemma

$$S_w^{-1}(\partial)(g') = \beta(|f^n|,|\partial|)f^{-n}(\partial(g) - S_w^{-1}([\partial, f^n]_\beta)(g')),$$

so again we compute

$$\begin{aligned}[S_w^{-1}(\partial), r']_{\beta'}(g') &:= S_w^{-1}(\partial)(r'g') - \beta(\left|S_w^{-1}(\partial)\right|,|r'|)r'S_w^{-1}(\partial)(g')\\ &= \beta(|h^m f^n|,|\partial|)h^{-m}f^{-n}(\partial(rg) - S_w^{-1}([\partial, h^m f^n]_\beta)(r'g'))\\ &- \beta(\left|S_w^{-1}(\partial),|r'|\right|)r'\beta(|f^n|,|\partial|)f^{-n}(\partial(g) - S_w^{-1}([\partial, f^n]_\beta)(g')).\end{aligned}$$

We proceed by induction, then

$$\beta(|r'|,|\partial|) = \beta(|r'|,\left|S_w^{-1}(\partial)\right|).$$

So we have our base case from our lemma, and thus

$$h^{-m}f^{-n}\left(\beta(|h^m f^n|, |\partial|)\partial(rg) - \beta(|\partial|, |r'|)\beta(|f^n|, |\partial|)r\partial(g)\right) \in S_w^{-1}(R)$$

is equal to

$$h^{-m}f^{-n}\beta(|h^m f^n|, |\partial|)([\partial, r]_\beta(g'))$$

of which $[\partial, r]_\beta$ is in $D_\beta^{k-1}(R)$. Hence, $S_w^{-1}(\partial) \in D_\beta^k(S_w^{-1}R)$. □

4.2 Deformed and twisted differentiation rules

With all this twisting, and all this differentiation, we need for a summary of "quantum quotient rules". Here we collect formulas for localizations of differential operators. Note a deviation from our previous notation, where γ is now the bicharacter we twist differential operators by, and β is the bicharacter we twist our ring multiplication by. R is a commutative ring, g, f homogeneous elements of R, $f \in S_w$.

This table summarizes our findings above:

$S_w^{-1}(\partial)(g/f^n)$	R	R^β
$D(R)$	$f^{-n}\partial(g)$	$\beta(\|f^n\|, \|g\|)f^{-n}\partial(g)$
$D_\gamma(R)$	$\gamma(\|f^n\|, \|\partial\|)f^{-n}\partial(g)$	$\gamma(\|f^n\|, \|\partial\|)\beta(\|f^n\|, \|g\|)f^{-n}\partial(g)$

with the assumption that $|S_w^{-1}(\partial)| = |\partial|$.

4.3 Co-localization with respect to deformation

Here we discuss conditions for gluing β-affine spaces as defined above. We compute necessary relations for localizations to agree on overlaps.

4.3.1 Localization of β-affine patches

Consider the noncommutative ring $\mathbb{K}_\beta[x_1, \ldots, x_n]$ as the noncommutative ring generated by variables $x_1, \ldots, x_n$ and relation

$$x_i x_j = \beta(i,j) x_j x_i$$

where $\mathbb{K}_\beta[x_1, \ldots, x_n]$ is graded naturally by $\mathbb{Z}^n$ and $\beta : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{K}^*$. Notice that this indicates $\beta(i,j) = \beta^{-1}(j,i)$. We consider the localizations by $\langle x_i \rangle =: S_i$.

Remark 4.3.2. As before, in $S_i^{-1}\mathbb{K}_\beta[x_1, \ldots, x_n]$ we have

$$x_i^{-1} x_j = \beta(j,i) x_j x_i^{-1}.$$

Now we wish to take the zero-degree component, denoted $(\quad)_0$, of these localized rings.

Lemma 4.3.3. *Consider the morphism which sends $x_j/x_i \mapsto \tilde{x_j}$. There exists a bicharacter $\beta_i : \mathbb{Z} \times \mathbb{Z} \to \mathbb{K}^*$ such that $S_i^{-1}(\mathbb{K}_\beta[x_1, \ldots, x_n])_0 = \mathbb{K}_{\beta_i}[\tilde{x_i}, \ldots, \tilde{x_n}]$ for $\tilde{x_i} = 1$.*

Proof. Observe

$$\begin{aligned}
\tilde{x_j}\tilde{x_k} &= (x_j/x_i)(x_k/x_i) = x_j x_i^{-1} x_k x_i^{-1} \\
&= \beta(i,j) x_i^{-1} x_j x_k x_i^{-1} \\
&= \beta(i,j)\beta(j,k) x_i^{-1} x_k x_j x_i^{-1} \\
&= \beta(i,j)\beta(j,k)\beta(k,i) x_k x_i^{-1} x_j x_i^{-1} \\
&= \beta(i,j)\beta(j,k)\beta(k,i) \tilde{x_k}\tilde{x_j}.
\end{aligned}$$

Define $\beta_i(a,b) := \beta(i,a)\beta(a,b)\beta(b,i)$ for all $a, b \in \mathbb{Z}^n$. We see that β_i is also a skew-symmetric bicharacter. $\square$

Now we need wish to consider conditions on β such that the order of iterated localizations does not matter. This is a property that we expect of localization theories.

Theorem 4.3.4. *$\tilde{S}_j^{-1}(S_i^{-1}(\mathbb{K}_\beta[x_1,\ldots,x_n])_0)_0 \simeq \tilde{S}_i^{-1}(S_j^{-1}(\mathbb{K}_\beta[x_1,\ldots,x_n])_0)_0$ for $\tilde{S}_i := \langle \tilde{x_i} \rangle$ an Ore set.*

Proof. Consider now two sets of generators $\tilde{x_k} := x_k x_i^{-1}$, the local coordinates on $S_i^{-1}(\mathbb{K}_\beta[x_1,\ldots,x_n])_0$ and $\tilde{y_k} := x_k x_j^{-1}$ the local coordinates on $S_j^{-1}(\mathbb{K}_\beta[x_1,\ldots,x_n])_0$. Notice that we localize these rings by Ore sets generated by our local coordinates.

We consider the morphism

$$\psi : \tilde{S}_j^{-1}(S_i^{-1}(\mathbb{K}_\beta[x_1,\ldots,x_n])_0)_0 \to \tilde{S}_i^{-1}(S_j^{-1}(\mathbb{K}_\beta[x_1,\ldots,x_n])_0)_0$$

such that $\psi(\tilde{x_k}) = \tilde{y_k}\tilde{y_i}^{-1}$ and the map

$$\phi : \tilde{S}_i^{-1}(S_j^{-1}(\mathbb{K}_\beta[x_1,\ldots,x_n])_0)_0 \to \tilde{S}_j^{-1}(S_i^{-1}(\mathbb{K}_\beta[x_1,\ldots,x_n])_0)_0$$

such that $\phi(\tilde{y_k}) = \tilde{x_k}\tilde{x_j}^{-1}$ is its inverse.

We need to check the commutativity relations to confirm isomorphism:

$$\begin{aligned}
\beta_j(l,k)\tilde{y_k}\tilde{y_l} &=: \tilde{y_l}\tilde{y_k} = \tilde{x_l}\tilde{x_j}^{-1}\tilde{x_k}\tilde{x_j}^{-1} = \beta_i(j,l)\tilde{x_j}^{-1}\tilde{x_l}\tilde{x_k}\tilde{x_j}^{-1} \\
&= \beta_i(j,l)\beta_i(l,k)\tilde{x_j}^{-1}\tilde{x_k}\tilde{x_l}\tilde{x_j}^{-1} \\
&= \beta_i(j,l)\beta_i(l,k)\beta_i(k,j)\tilde{x_k}\tilde{x_j}^{-1}\tilde{x_l}\tilde{x_j}^{-1} \\
&= \beta_i(j,l)\beta_i(l,k)\beta_i(k,j)\tilde{y_k}\tilde{y_l} \\
&= \beta(i,j)\beta(j,l)\beta(l,k)\beta(k,j)\beta(j,i)\tilde{y_k}\tilde{y_l} \\
&= \beta_j(l,k)\tilde{y_k}\tilde{y_l}.
\end{aligned}$$

□

4.3.5 Localization of γ-Differential operators on β-affine charts

Our goal was to glue differential operators on these spaces. We now check that the relations above provide gluing conditions for a theoretical sheaf of γ-differential operators. In particular, we show that evaluating extended differential operators on co-localized rings produces the same result, i.e., the functor computing twisted differential operators satisfy the gluing condition for the n open charts of $\mathbb{P}^q_n$.

Theorem 4.3.6. *Iterated localization of γ-differential operators on two β-affine charts in alternating order are isomorphic induced by the isomorphism of the charts overlap:*

$$\tilde{S}_j^{-1}(S_i^{-1}(\mathbb{K}_\beta[x_1,\ldots,x_n])_0)_0 \simeq \tilde{S}_i^{-1}(S_j^{-1}(\mathbb{K}_\beta[x_1,\ldots,x_n])_0)_0.$$

In other words,

$$\tilde{S}_j^{-1}(S_i^{-1}(D_\gamma(\mathbb{K}_\beta[x_1,\ldots,x_n])_0)_0) \subseteq D_\gamma(\tilde{S}_i^{-1}(S_j^{-1}(\mathbb{K}_\beta[x_1,\ldots,x_n])_0)_0).$$

Proof. We need to check that $\tilde{S}_j^{-1}(S_i^{-1}(\partial))(\tilde{x}_l) = \tilde{S}_i^{-1}(S_j^{-1}(\tilde{y}_l\tilde{y}_i^{-1}))$ and $\tilde{S}_j^{-1}(S_i^{-1}(\partial))(\tilde{x}_l\tilde{x}_i^{-1}) = \tilde{S}_i^{-1}(S_j^{-1}(\tilde{y}_l))$. We provide only the first, as the second follows similarly.

$$\tilde{S}_j^{-1}(S_i^{-1}(\partial)(\tilde{x}_l) = \gamma(|x_i|,|\partial|)\beta(|x_i|,|x_l|)x_i^{-1}\partial(x_l)$$

and $\tilde{S}_i^{-1}(S_j^{-1}(\tilde{y}_l\tilde{y}_i^{-1})) =$

$$\begin{aligned}
&= \gamma(|\tilde{y}_i|,\left|S_j^{-1}(\partial)\right|)\beta_j(|\tilde{y}_i|,|\tilde{y}_l|)\tilde{y}_i^{-1}S_j^{-1}(\partial(\tilde{y}_l))\\
&= \gamma(|\tilde{y}_i|,\left|S_j^{-1}(\partial)\right|)\beta_j(|\tilde{y}_i|,|\tilde{y}_l|)\gamma(|x_j|,|\partial|)\beta(|x_j|,|x_l|)\tilde{y}_i^{-1}x_j^{-1}\partial(x_l)\\
&= \gamma(|\tilde{y}_i|,\left|S_j^{-1}(\partial)\right|)\beta_j(|\tilde{y}_i|,|\tilde{y}_l|)\gamma(|x_j|,|\partial|)\beta(|x_j|,|x_l|)\beta(|x_j|,\left|x_i^{-1}\right|)x_i^{-1}\partial(x_l)\\
&= \gamma(\left|x_ix_j^{-1}\right|,\left|S_j^{-1}(\partial)\right|)\beta_j(\left|x_ix_j^{-1}\right|,\left|x_lx_j^{-1}\right|)\gamma(|x_j|,|\partial|)\beta(|x_j|,|x_l|)\beta(|x_j|,\left|x_i^{-1}\right|)x_i^{-1}\partial(x_l)\\
&= \gamma(\left|x_j^{-1}\right|,\left|S_j^{-1}(\partial)\right|)\gamma(|x_i|,\left|S_j^{-1}(\partial)\right|)\gamma(|x_j|,|\partial|)\beta(|x_l|,\left|x_i^{-1}\right|)x_i^{-1}\partial(x_l).
\end{aligned}$$

Recall that $\left|\tilde{\partial}\right| = |\partial|$, then

$$\begin{aligned}
\tilde{S}_i^{-1}(S_j^{-1}(\tilde{y}_l\tilde{y}_i{}^{-1})) &= \gamma(|x_i|, \left|S_j^{-1}(\partial)\right|)\beta(|x_l|, \left|x_i^{-1}\right|)x_i^{-1}\partial(x_l) \\
&= \gamma(|x_i|, |\partial|)\beta(|x_i|, |x_l|)x_i^{-1}\partial(x_l).
\end{aligned}$$

□

Since this localization condition holds, we now have the freedom to glue noncommutative charts together and allow the sheaf of γ-differential operators respect this gluing.

Conjecture 4.3.7. *The functor D_γ satisfies the sheaf condition on the open sets of $\mathbb{P}^n_\beta$, where $\mathbb{P}^n_\beta$ has the noncommutative coordinate ring $\mathbb{K}_\beta[x_1, \ldots, x_n]$.*

The point of these gluing theorems, and this conjecture, is to realize the potential of noncommutative projective spaces mimicking $\mathbb{P}$, Gr, G/B, and G/P. The work of Joseph showed us that Fl has a noncommutative version Fl_q with a similarly defined coordinate ring. Here, we thought about $\mathbb{P}^n_\beta$ by deforming the affine charts that build $\mathbb{P}^n$. We then worked on the sheaf of differential operators for the purpose of using twisting-untwisting to relate them. Remember, we can get isomorphisms between algebras of deformed differential operators on coordinate rings, and their deformations. This suggests a result like:

Conjecture 4.3.8. $\mathcal{D}_{\beta'}(\mathbb{P}^n) \simeq \mathcal{D}(\mathbb{P}^n_\beta)$ *as algebras of global differential operators.*

But don't stop there, dear reader; Gr, G/B, and G/P have affine charts too.

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