GRID STABILIZATION FOR THE ONE-DIMENSIONAL ADVECTION EQUATION USING BIASED FINITE DIFFERNCES OF ODD ORDERS AND ORDERS HIGHER THAN TWENTY-TWO

by

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Abstract

This work utilizes finite differences to approximate the first derivative of non-periodic smooth functions. Math literature indicates that stabilizing Partial Differential Equation solvers based on high order finite difference approximations of spatial derivatives of a non-periodic function becomes problematic near a boundary. Hagstrom and Hagstrom have discovered a method of introducing additional grid points near a boundary, which has proven to be effective in stabilizing Partial Differential Equation solvers. Hagstrom and Hagstrom demonstrated their method for the case of the one-dimensional advection equation using spatial derivative approximations of even orders up to twenty-second order. In this dissertation, we explore the efficacy of the Hagstrom and Hagstrom method for the same Partial Differential Equation with spatial derivative approximations of odd orders and orders higher than twenty-two and report the number and locations of additional grid points required for stability in each case.

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Dedication

I dedicate this work to my family, friends and anyone who has supported me throughout the years.

Chapter 1

Introduction

Definition 1. A *partial differential equation* (PDE) is an equation involving a function u of several variables and its partial derivatives [11].

Definition 2. A PDE is *linear* if it depends linearly on u and its derivatives [11].

To many, the advection equation is the simplest first-order PDE. The advection equation acts as a transport instrument that models a substance being carried along by a flow of liquid [9, 11]. Consider the following PDE defined as

$$u_t + a(u)u_x = 0 \quad t > 0 \quad and \quad x \in (x_l, x_r) \tag{1.1}$$

where (x_l, x_r) represents the left and right endpoints for the interval on which x is defined on. When $a(u) \equiv a$, where a is a constant, (1.1) is called the advection equation (aka the transport equation). In general, when $a(u) \equiv u$, (1.1) is categorized as *quasilinear* equation and is called the inviscid Burgers equation. Here quasilinear means that all the highest-order derivatives of u occur linearly with coefficients depending only on x, u, and the lower order derivatives of u [11].

In our research we will be solving the one dimensional advection equation with $a(u) = 1$, which is

$$u_t + u_x = 0. \tag{1.2}$$

In general, with a bounded domain, the advection equation can have only one boundary condition. Since $a > 0$, a boundary condition is required at $x = 0$. The boundary condition

as well as the initial condition for (1.2) are displayed below:

$$u(x,0) = u_0(x) \quad \forall x \in (0,1) \tag{1.3}$$
$$u(0,t) = f(t) \quad \forall t > 0$$

for some arbitrary function f. The conditions provided above indicate that $x = 0$ represents an *inflow* boundary. Thus, this makes the boundary at $x = 1$ the *outflow* boundary, meaning that the solution at the boundary $x = 1$ is determined solely from what is flowing to the right from the interior. If it were the case where $a < 0$, then a boundary condition would be required at $x = 1$ pertaining to the inflow boundary. Using the *Method of Characteristics*, the exact solution of (1.2) and (1.3) is found to be

$$u(x,t) = \begin{cases} u_0(x-t), & \text{if } x > t; \\ f(t-x), & \text{if } x < t \end{cases} \quad [9].$$

Recall what was previously mentioned about the one dimensional advection equation being the most simple first-order PDE. Part of this simplicity is due to the fact that the solution to this problem is already known. In general, PDEs are not so simple and the exact solution may not be known. These types of PDEs require some kind of numerical solver.

The method of lines (MOL) provides a general framework for constructing numerical solvers for time dependent PDEs. The MOL process will discritize a PDE in both space and time. To keep the analysis simple, we will consider a MOL discretization for the one dimensional advection equation initial boundary value problem with periodic boundary conditions:

$$u_t + u_x = 0 \tag{1.4}$$
$$u(x,0) = u_0(x) \quad \forall x \in (0,1)$$
$$u(0,t) = u(1,t) \quad \text{for} \quad t \geq 0.$$

The physical description of the system in the case of a periodic boundary would be that whatever leaves the interior through the outflow boundary reenters the system through the inflow boundary [9].

We seek an approximate solution vector, v,

$$v_i(t) \approx u(x_i, t), \quad x_i = ih \quad \text{for } i = 0, 1, 2, ..., N+1,$$

where h is the distance between points on the uniform grid, also known as, the grid mesh. That is, v is a finite vector approximating an infinite collection of function values of u. The MOL consists of first discretizing in space. The vector v will be represented in the spatial discretization. The spatial discretization of problem (1.4) will form an approximation of the spatial derivative, u_x. More specifically, the process of transforming $\frac{\partial}{\partial x}$ into a discrete approximation of a finite difference operator D. Here finite difference methods will be used to discretize (1.4) in space and produce a system of ODEs. Finite differences will be discussed in greater detail later.

In this problem, a uniform grid with $N+2$ grid points including the boundary points is being used. Notice that in the periodic boundary case $v_0(t) = v_{N+1}(t)$, so we will not have to find v_{N+1} because we already know v_0. Either v_0 or v_{N+1} needs to be represented in the vector of grid values $v(t)$. By choosing to represent v_{N+1}, the vector of grid values is

$$v(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_{N+1}(t) \end{bmatrix}.$$

For the interior of the system, that is, for $i = 2, 3, ..., N$, the following finite difference approximation is used:

$$v_i'(t) = -\frac{1}{2h}(v_{i+1}(t) - v_{i-1}(t)).$$

Then considering the periodicity of the system, the first and last finite difference approximations must be modified. For the first finite difference approximation in the system, we get

$$\begin{aligned} v_1'(t) &= -\frac{1}{2h}(v_2(t) - v_0(t)) \\ &= -\frac{1}{2h}(v_2(t) - v_{N+1}(t)). \end{aligned}$$

Moreover the last finite difference approximation in the system is

$$\begin{aligned} v'_{N+1}(t) &= -\frac{1}{2h}(v_{N+2}(t) - v_N(t)) \\ &= -\frac{1}{2h}(v_1(t) - v_N(t)). \end{aligned}$$

Having all the ODEs in the system allows one to form the $(N+1)\times(N+1)$ finite difference operator matrix, D, which in this case looks like

$$D = \frac{1}{h}\begin{bmatrix} 0 & \frac{1}{2} & 0 & \cdots & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} & \ddots & & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \cdots & 0 & -\frac{1}{2} & 0 \end{bmatrix} \text{[9]}.$$

Having discretized in space and formed the matrix D, the one dimensional advection equation (1.4) now takes the form

$$\begin{aligned} &v_t + Dv = 0 \qquad (1.5) \\ &v_i(0) = u_0(x_i) \quad \text{for } i = 0, 1, 2, ..., N+1 \end{aligned}$$

Now we seek

$$U_i^n \approx v_i(t_n) \approx u(x_i, t_n),$$

a numerical approximation of u at the grid point x_i at the n^{th} time for $i = 0, 1, ..., N+1$ and $n \geq 0$ [9]. After discretizing in space and obtaining the system of ODEs (1.5), the next step in the MOL process is to discretize in time. As indicated above, the vector U is also a finite collection of approximations of the function u and is part of the time discretization. To discretize in time means that we want to solve the system of ODEs (1.5) using an ODE solver. Keep in mind that in general, we have a number of options for solving this system of ODEs. A reasonable choice however, is to use the midpoint point to solve this system of ODEs. Consider the generic ODE, $\frac{dv}{dt} = g(v(t), t)$ for the purpose of defining the midpoint method. The midpoint method is

$$\frac{U_i^{n+1} - U_i^n}{2k} = g(U_i^n) \quad \text{for } i = 0, 1, 2, ..., N+1 \text{[9]}.$$

If one were to use the midpoint method to solve the system of ODEs in (1.5), a recurrence relation could be obtained. This recurrence relation could then be solved to provide the solution to the initial boundary value problem in (1.4).

For linear PDEs, the Lax equivalence theorem tells us that consistency and stability of a method is equivalent to convergence of the numerical solution to the true solution. When solving a PDE initial boundary value problem, such as the one in (1.4), one hopes that the solution obtained through using MOL is a good approximation of the real solution. Roughly speaking, a numerical method is consistent if the truncation error of the approximation goes to zero as grid mesh, h, and time step size, k, go to zero. Absolute stability requires that $k \cdot \text{spec}(-D) \subseteq S$, where spec is the spectrum of the finite difference operator D, which is the set containing the eigenvalues of D and S denotes the stability region of the time integration method.

Some additional analysis on matrix D will show that D is a *skew-symmetric* matrix meaning that $D^T = -D$ and the eigenvalues of D are purely imaginary. The eigenvalues are contained in the interval $(-\frac{i}{h}, \frac{i}{h})$. In order to obtain stability after solving the time dependent ODE system (1.5), the stability region must include the interval $k(-\frac{i}{h}, \frac{i}{h})$. The midpoint method was an appropriate choice for the time integration method because it yielded a stability region containing the interval iy of the imaginary axis with $|y| < b$. Thus, the method will be stable provided that $|k/h| \leq b$ [9]. The MOL and stability analysis of the spec(D) are both important factors that tie into our research.

Chapter 2

Accuracy and Stability

2.1 Finite Difference Methods

For the purpose of our research, we aim to use finite difference formulas to approximate the first derivative of a specified function given a finite number of discrete function values. Suppose that $f(x)$ is a univariate function assumed to be smooth, unless otherwise stated. Finite difference methods require the smoothness characteristic of functions so that the required function derivatives of particular orders exist and are bounded on an interval about a distinct point of interest x_0.

Definition 3. The right one-sided first order accurate finite difference approximation to $f'(x_0)$ is

$$D_{+,1}f(x_0) = \frac{f(x_0+h) - f(x_0)}{h}, \quad \text{for } h > 0, \tag{2.1}$$

where the subscript + denotes a right sided approximation and the subscript 1 represents first order accuracy.

Notice that as h $\to$ 0, $D_{+,1}f(x_0)$ is the definition of the derivative of the function $f(x)$. $D_{+,1}f(x_0)$ is the slope of the interpolating line to the graph of $f(x)$ at the points x_0 and x_0+h. The formula given in (2.1) is considered first order accurate because $D_{+,1}f(x_0) - f'(x_0) = O(h)$.

Definition 4. The left one-sided first order accurate finite difference approximation to

$f'(x_0)$ is

$$D_{-,1}f(x_0) = \frac{f(x_0) - f(x_0 - h)}{h}, \quad \text{for } h > 0, \tag{2.2}$$

where the subscript - denotes a left sided approximation and the subscript 1 represents first order accuracy.

The left one-sided finite difference approximation (2.2) is also first order accurate because $D_{-,1}f(x_0) - f'(x_0) = O(h)$. It turns out that taking an average of the two one-sided approximation formulas will yield a more desirable approximation of $f'(x_0)$. The resulting approximation is the centered formula, which represents the slope of the line interpolating f at the points $x_0 - h$ and $x_0 + h$:

$$D_{0,2}f(x_0) = \frac{f(x_0 + h) - f(x_0 - h)}{2h} \quad \text{for } h > 0. \tag{2.3}$$

Furthermore, the centered finite difference approximation of $f'(x_0)$ is second order accurate, meaning $D_{0,2}f(x_0) - f'(x_0) = O(h^2)$. The notation $D_{0,2}$ with subscript 0 denotes a centered approximation and the subscript 2 represents second order accuracy. So for small h, the error for the centered approximation will be much less than the error made with the one-sided approximations [9].

$D_{+,1}$, $D_{-,1}$ and $D_{0,2}$ are the canonical examples of the three basic types of finite difference approximations to the first derivative, namely *right biased approximations*, *left biased approximations* and the *centered approximations*. The term *biased approximations* indicates that there are more sample points lying to the left or right of the point of interest x_0.

We will be using finite difference methods to replace the spatial derivative in the one-dimensional advection equation $u_t + u_x = 0$. Considering that we started working with a second order accurate finite difference method, we have chosen to only display the derivation of such methods below. A derivation of the centered finite difference formula (2.3) along with the second order right and left biased finite difference formulas will be provided.

2.1.1 Derivation of second order finite difference formulas

Taylor series expansion and the *method of undetermined coefficients* are used to derive each approximation formula.

Example 1. Derive a second order left one-sided approximation to $f'(x_0)$ utilizing the points $f(x_0)$, $f(x_0-h)$ and $f(x_0-2h)$. First, form the Taylor series expansion for $f(x_0-h)$ and $f(x_0-2h)$, both of which are centered at x_0:

$$f(x_0-h) = f(x_0) - hf'(x_0) + \frac{h^2}{2}f''(x_0) - \frac{h^3}{6}f'''(x_0) + O(h^4) \tag{2.4}$$

$$f(x_0-2h) = f(x_0) - 2hf'(x_0) + \frac{(2h)^2}{2}f''(x_0) - \frac{(2h)^3}{6}h^3f'''(x_0) + O(h^4). \tag{2.5}$$

Then substitute the Taylor series expansions of $f(x_0-h)$ (2.4) and $f(x_0-2h)$ (2.5) into the following equation

$$D_{-,2}f(x_0) = af(x_0) + bf(x_0-h) + cf(x_0-2h) \tag{2.6}$$

where a,b and c are coefficients, often referred to as weights. A simple rearrangement of terms results in obtaining

$$D_{-,2}f(x_0) = (a+b+c)f(x_0) - (b+2c)hf'(x_0) + (b+4c)\frac{h^2}{2}f''(x_0) - (b+8c)\frac{h^3}{6}f'''(x_0) + (b+c)O(h^4)$$

Since we hope to have $D_{-,2}f(x_0)$ agree with $f'(x_0)$, we need to require

$$\begin{aligned} a+b+c &= 0 \\ b+2c &= -\frac{1}{h} \\ b+4c &= 0. \end{aligned}$$

In attempting to make a more accurate approximation, we would want the higher order coefficients to be zero as well. However, working in the context of the given formula for $D_{-,2}f(x_0)$, we are restricted to the system of three equations with the three unknowns, a,b

and c. Next, use back-substitution to get

$$\begin{aligned} a &= \frac{3}{2h} \\ b &= -\frac{2}{h} \\ c &= \frac{1}{2h}. \end{aligned}$$

Now that a,b and c have have been identified, return to the arbitrary formula for $D_{-,2}f(x_0)$ (2.6) and make the necessary substitutions, that is,

$$D_{-,2}f(x_0) = \frac{1}{h}[\frac{3}{2}f(x_0) + -2f(x_0 - h) + \frac{1}{2}f(x_0 - 2h)].$$

Then the error in the approximation is found by

$$\begin{aligned} D_{-,2}f(x_0) - f'(x_0) &= -\frac{h^2}{3}f'''(x_0) - \frac{3}{2h}O(h^4) \\ &= O(h^2). \end{aligned}$$

A simple algebraic rearrangement of the latter, will result in the equality

$$f'(x_0) = D_{-,2}f(x_0) + O(h^2).$$

If one were to use the method of undetermined coefficients to find a second order right one-sided finite difference approximation to $f'(x_0)$ using the grid points $f(x_0)$, $f(x_0 + h)$ and $f(x_0 + 2h)$, they would discover a connection with the weights found in Example 1. Roughly speaking, the weights between a right and left biased finite difference approximation are negative of each other. To establish a second order right biased finite difference approximation to $f'(x_0)$ we will need the Taylor series expansion of $f(x_0+h)$ and $f(x_0+2h)$ centered at x_0:

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{(h)^2}{2}f''(x_0) + \frac{(h)^3}{6}h^3f'''(x_0) + O(h^4) \tag{2.7}$$

$$f(x_0 + 2h) = f(x_0) + 2hf'(x_0) + \frac{(2h)^2}{2}f''(x_0) + \frac{(2h)^3}{6}h^3f'''(x_0) + O(h^4). \tag{2.8}$$

Consider the formula for a second order right biased finite difference approximation to $f'(x_0)$ denoted $D_{+,2}f(x_0)$:

$$D_{+,2}f(x_0) = af(x_0) + bf(x_0 + h) + cf(x_0 + 2h).$$

Substituting the Taylor series expansions of $f(x_0 + h)$ (2.7) and $f(x_0 + 2h)$ (2.8) into the latter formula and rearranging terms will result in obtaining

$$D_{+,2}f(x_0) = (a+b+c)f(x_0)+(b+2c)hf'(x_0)+(b+4c)\frac{h^2}{2}f''(x_0)+(b+8c)\frac{h^3}{6}f'''(x_0)+(b+c)O(h^4).$$

At this point, one could set-up and solve a similar system of three equations (as in Example 1) of the three unknowns, a, b and c, but that is not necessary considering the relationship mentioned above. Thus, the weights for this approximation are negative the weights that are depicted in Example 1. So

$$\begin{aligned} a &= \frac{-3}{2h} \\ b &= \frac{2}{h} \\ c &= \frac{1}{2h} \end{aligned}$$

and the formula for $D_{+,2}f(x_0)$ is

$$D_{+,2}f(x_0) = \frac{1}{h}[-\frac{3}{2}f(x_0 + 2f(x_0 + h) - \frac{1}{2}f(x_0 + 2h)].$$

Moreover, the error in the approximation is found by

$$\begin{aligned} D_{+,}f(x_0) - f'(x_0) &= -\frac{h^2}{3}f'''(x_0) + \frac{3}{2h}O(h^4) \\ &= O(h^2). \end{aligned}$$

Again, a simple rearrangement of the latter, will result in the equality

$$f'(x_0) = D_+f(x_0) + O(h^2).$$

Example 2. Now let's find a second order centered approximation to $f'(x_0)$ utilizing the points $f(x_0)$, $f(x_0 - h)$ and $f(x_0 + h)$. This requires one to substitute the Taylor series expansions of $f(x_0 - h)$ (2.4) and $f(x_0)$ (2.7) into the following equation:

$$D_{0,2}f(x_0) = af(x_0) + bf(x_0 - h) + cf(x_0 + h).$$

A simple rearrangement of terms results in obtaining

$$D_{0,2}f(x_0) = (a+b+c)f(x_0) + (-b+c)hf'(x_{0)} + (b+c)\frac{h^2}{2}f''(x_0) + (-b+c)\frac{h^3}{6}f'''(x_0) + \\ +(b+c)\frac{h^4}{24}f^4(x_0) + (-b+c)O(h^5).$$

In order to have $D_{0,2}f(x_0)$ agree with $f'(x_0)$, this time we need to require

$$\begin{aligned} a+b+c &= 0 \\ -b+c &= \frac{1}{h} \\ b+c &= 0. \end{aligned}$$

To solve this system of three equations use back-substitution and get

$$\begin{aligned} a &= 0 \\ b &= -\frac{1}{2h} \\ c &= \frac{1}{2h}. \end{aligned}$$

Having found a,b, and c, the formula for $D_{0,2}f(x_0)$ is

$$D_{0,2}f(x_0) = \frac{1}{2h}[-f(x_0 - h) + f(x_0 + h)].$$

Notice this was the derivation for the centered finite difference approximation defined in (2.3). The error in the approximation is found by

$$\begin{aligned} D_{0,2}f(x_0) - f'(x_0) &= \frac{h^2}{6}f'''(x_0) + \frac{1}{h}O(h^5) \\ &= O(h^2). \end{aligned}$$

Notice the error term for this centered finite difference approximation of $f'(x_0)$. i.e. $\frac{h^2}{6}f'''(x_0)+\frac{1}{h}O(h^5)$. The error for this centered finite difference approximation is still $O(h^2)$, consequently making the approximation second order accurate. In fact, here the error acts like $\frac{h^2}{6}$ as compared to $-\frac{h^2}{3}$ as seen with the left biased approximation from Example 1. This observation suggests the error for the centered approximation is approximately two times as small as compared to the error seen in the left biased approximation. Considering the derivation given above, we obtain the equality

$$f'(x_0) = D_{0,2}f(x_0) + O(h^2).$$

2.2 ODE Solvers

Working with an initial boundary value problem means that one is given data at the initial time, $t_0 = 0$. For example let us consider the following initial boundary value problem with the PDE given by (1.2) along with the boundary condition and initial condition in (1.3). After discretizing in space, this initial boundary value problem takes the form

$$\begin{aligned} &v_t + \tilde{D}v = 0 \\ &v_i(0) = u_0(x_i) \quad \forall x \in (0,1) \\ &v_0(t) = f(t) \quad \forall t > 0, \end{aligned} \tag{2.9}$$

again where $v_i(t) \approx u(x_i, t), x_i = ih$ for $i = 0, 1, 2, ..., N$. Now, we wish to solve the ODE system in (2.9). The given information allows one to march forward in time using a particular time-stepping method to compute approximations to the solution at subsequent times. These time-stepping methods are also referred to as ODE solvers. Let k represent the time step size, so that the n^{th} time step is given by $t_n = nk$ for $n \geq 0$ [9].

Notice that the system of ODEs in (2.9) differs from the system of ODEs presented in (1.4) in the sense that (2.9) does not have a periodic boundary. So the spectrum of the differential operator $\tilde{D}$ will have a different pattern as compared to the differential operator

D in the introduction. Applicable time step methods such as Runge-Kutta 4 and Adams-Bashforth 4 that accommodate the spectrum of $\tilde{D}$ are presented.

First, we will discuss Runge-Kutta 4 (RK4). For purposes of defining RK4, let $\frac{dv}{dt} = f(v(t), t)$ be a generic ODE. RK4 is a four-stage iterative method that approximates the solutions of ODEs and is given by

$$\begin{aligned}
Y_1 &= U^n \\
Y_2 &= U^n + \frac{1}{2}kf(Y_1, t_n) \\
Y_3 &= U^n + \frac{1}{2}kf(Y_2, t_n + \frac{k}{2}) \\
Y_4 &= U^n + kf(Y_3, t_n + \frac{k}{2}) \\
U^{n+1} &= U^n + \frac{k}{6}[f(Y_1, t_n) + 2f(Y_2, t_n + \frac{k}{2}) + 2f(Y_3, t_n + \frac{k}{2}) + f(Y_4, t_n + k)].
\end{aligned}$$

A key feature of the Runge-Kutta methods is that each successive Y_m value is computed using only the previous Y_l value, where in the case of RK4, $m = 1, 2, 3, 4$ and $l = 1, 2, 3$. Notice that in the first step, when computing Y_1, this is the first Y_m generated, which does not have a previous Y_l value to use.

Now we will move on to discuss the Adams-Bashforth 4 (AB4) method. A generic r-step linear multistep method (LMM) is of the form

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k \sum_{j=0}^{r} \beta_j f(U^{n+j}, t_{n+j}). \tag{2.10}$$

The value U^{n+r} is computed using the formula 2.10 and is based on values $U^{n+r-1}, U^{n+r-2}, \cdots, U^n$ as well as the function values at these points. AB4 is a four step LMM that also approximates the solution of ODEs. Let $\frac{dv}{dt} = f(v(t), t)$ be a generic ODE, such that, AB4 is given by

$$U^{n+4} = U^{n+3} + \frac{k}{24}(-9f(U^n) + 37f(U^{n+1}) - 59f(U^{n+2}) + 55f(U^{n+3})).$$

Considering that AB4 is a LMM, U^{n+4} is computed in terms of the four preceding values: $U^{n+3}, U^{n+2}, U^{n+1}, U^n$ [9].

In the results section we will see that the eigenvalues of the differential operator, $\tilde{D}$, pertaining to the system of ODEs for the one dimensional advection equation are in general complex valued. Furthermore, these eigenvalues have negative real part and are often times close to the imaginary axis. In the next section we will see that RK4 and AB4 yield desirable stability regions for the initial boundary value problem seen in (2.9).

2.3 Stability and Convergence

The convergence of a numerical method for an initial value problem is determined through analyzing the error in the approximation of $u(T)$, for some arbitrarily fixed time $T > 0$, with time step k. Intuitively, the method will converge if the error in the approximation goes to zero as $k \to 0$. Thus, convergence is defined in the sense of the following limit,

$$\lim_{k \to 0} U^N = u(T),$$

where the number of time steps needed to reach time T is given by $N = T/k$.

In general, referring to a convergent r-step method means that it converges on all problems (of a relatively large collection) and with all reasonable starting values. The given notion of convergence typically assumes that the starting values converge to the correct initial value η as $k \to 0$, that is,

$$\lim_{k \to 0} U^{\nu}(k) = \eta \quad for \quad \nu = 0, 1, \cdots, r-1 \quad [9].$$

Proving convergence of a method can be difficult. In [9], there are some examples of convergence proofs. In one of these examples, LeVeque has proved the convergence of general one-step methods. This proof is the first instance when we see that when dealing with one-step methods, the error formed for each single step ties into the global error of the method. In general, the global error can be bounded by the sum of all the one-step errors. This implies that the global error will be of the same order as the local truncation error as $k \to 0$. These properties make-up our first concept of stability, more specifically, called zero-stability.

Definition 5. An r-step LMM (2.10) is said to be zero-stable if the roots of the characteristic polynomial $\rho(\xi)$ defined by $\sum_{j=0}^{r} \alpha_j \xi^j$ satisfy the following conditions:

$$|\xi_j| \leq 1 \quad \text{for} \quad j = 1, 2, \cdots, r.$$

$$\text{If} \quad \xi_j \quad \text{is a repeated root, then} \quad \xi_j < 1 \quad [9].$$

If all the roots of the polynomial ρ satisfy the conditions given in definition 5, then the polynomial is said to satisfy the root condition. Considering that proving convergence of a method can be a difficult task, if one can instead show that the method is consistent and zero-stable, then the method is convergent. A method is consistent if the local truncation error is $o(1)$ as $k \to 0$.

In comparison to zero stability, absolute stability is way of analyzing how the error in an approximation behaves for a chosen time step $k > 0$. We will obtain stability of our semidiscrete finite difference approximations to the one dimensional advection equation by utilizing absolute stability regions. A region of absolute stability is found by applying a generic LMM 2.10 to $u' = \lambda u$, resulting in the formula

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = k \sum_{j=0}^{r} \beta_j \lambda U^{n+j}.$$

By rearranging the line above, we arrive at the recurrence relation

$$\sum_{j=0}^{r} (\alpha_j - z\beta j) U^{n+j} = 0.$$

In particular, this is a homogeneous linear difference equation and has a general solution of the form

$$U^n = c_1 \xi_1^n + c_2 \xi_2^n + \cdots + c_r \xi_r^n,$$

where c_j are constant coefficients and ξ_j are roots of the characteristic polynomial $\sum_{j=0}^{r} (\alpha j - k\lambda\beta j)\xi^j$. Let $z = k\lambda$. The previous polynomial is referred to as the stability polynomial and denoted $\pi(\xi; z)$. π is a polynomial in terms of ξ with coefficients dependent on the value of z [9].

Definition 6. The region of absolute stability for the LMM (2.10) is the set of points z in the complex plane for which the polynomial $\pi(\xi; z)$ satisfies the root condition in the definition of zero stability 5 [9].

RK4 is an appropriate ODE solver for the problem in (2.9) because it has a suitable stability region, S, such that $k \cdot \text{spec}(\tilde{D}) \subseteq S$. That is, there exists a time step size k that can scale each eigenvalue of $\tilde{D}$ to be included inside of S. The stability region for RK4 is depicted in Figure 2.1.

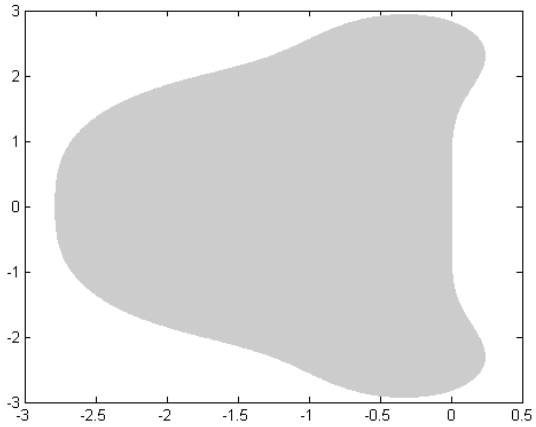

Figure 2.1: *Stability region for RK4.*

AB4 is also a good method to solve the system of ODEs in (2.9). Just like with RK4, there exists a time step size k when using AB4 such that $k \cdot \text{spec}(\tilde{D}) \subseteq S$, where this time S represents the stability region of AB4. The stability region for AB4 is shown in Figure 2.2.

2.3.1 The Courant-Friedrichs-Lewy (CFL) condition

In [3], the English translation of Courant, Friedrichs and Lewy's 1928 paper on PDEs, the authors investigate how the approximation to the solution of a linear PDE reacts as the mesh width, h, goes to zero. This study encompasses working with a system of linear difference equations constructed from rectangular grids that approximate a particular hyperbolic PDE

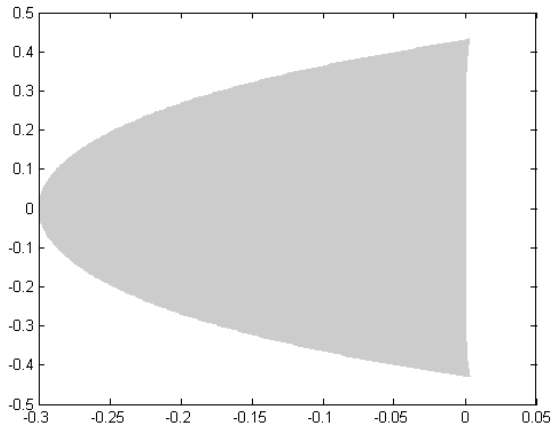

Figure 2.2: *Stability region for AB4.*

initial value problem. Courant, Friedrichs and Lewy focus on solving the wave equation where the solution u and the derivatives of u are given on the line $t = 0$. Each difference equation corresponding to a particular grid point, relates the value of the function at the grid point of interest and the function values of surrounding grid points.

The explicit solution of the specified initial value problem can be determined at any point in terms of the initial values lying in the first two rows of the grid. That is, each point on the grid has a certain *domain of dependence* in the first two rows of the grid. In addition to working with a rectangular grid, the triangle of determination is introduced. The triangle of determination contains values of u that remain unchanged under the assumption that the first two rows of the triangle are fixed. Courant, Friedrichs and Lewy go further to say that the solution of a linear hyperbolic PDE at a point is dependent on a portion of the line containing initial values. This initial value line lies between the sides of the triangle, which are referred to as the lines of determination. It is also true that the solution of the difference equation at the specified point has its own domain of dependence indicated by the lines of determination [3].

In order to illustrate the definition of a domain of dependence, we will consider the domain of dependence of a finite difference method. For a fixed grid, the domain of dependence

at a point $(x_i, t_n$ is defined to be the collection of grid points x_r at the initial time, t_0, such that the data U_r^0 at x_r has an effect on the solution U_i^n [9].

Definition 7. The CFL condition says that a numerical method can be convergent only if its numerical domain of dependence contains the true domain of dependence of the PDE, at least in the limit as k and h go to zero [9].

The CFL condition acts as a necessary condition for convergence. By necessary condition, it is meant that if a method disregards the CFL condition then the method will not converge. However, if a method fulfills the CFL condition then the method may or may be convergent [9].

2.3.2 Upwind methods

Considering that the advection equation models a substance being carried along by a flow of liquid, naturally the equation is not symmetric. Recall if $a > 0$, the fluid mixture is moving to the right and moving to the left respectively if $a < 0$. Consequently, this situation gives reason to use biased approximations to the function's derivative(s). For simplicity, we will present both one-sided approximations of the spatial derivative of u (in the advection equation below):

$$u_x(x_i, t) = \frac{1}{h}(U_i - U_{i-1})$$
$$u_x(x_i, t) = \frac{1}{h}(U_{i+1} - U_i).$$

Applying forward Euler to either method from above, results in the following two schemes for the advection equation:

$$U_i^{n+1} = U_i^n - \frac{ak}{h}(U_i^n - U_{i-1}^n) \tag{2.11}$$

or

$$U_i^{n+1} = U_i^n - \frac{ak}{h}(U_{i+1}^n - U_i^n). \tag{2.12}$$

Note that both (2.11) and (2.12) are first order accurate in both space and time.

The sign of a will determine which of the two schemes,(2.11) or (2.12), are to be used. The true solution of the advection equation after one time step is

$$u(x_i, t+k) = u(x_i - ak, t).$$

Thus, for $a > 0$, the solution at the point x_i for the succeeding time step is provided using data to the left of x_i and then to the right of x_i provided $a < 0$. Therefore, (2.11) should be used when $a > 0$ and (2.12) when $a < 0$ [9]. According to [9], (2.11) is stable under the condition that

$$0 \leq \frac{ak}{h} \leq 1$$

and (2.12) is stable if

$$-1 \leq \frac{ak}{h} \leq 0.$$

2.4 The Importance of High-order Methods

2.4.1 Dispersion and dissipation errors

When a discrete model (e.g. a finite difference method) is used to approximate the solution of a PDE, almost always there are errors associated with this approximation. One can consider truncation analysis to find bounds on discretization errors, which will be dependent on the size of the grid mesh, h, and the size of the time step, k. The study of the behavior of discretization errors is done through numerical dispersion and dissipation [13].

We see in [13], that any time-dependent scalar, linear PDE with constant coefficients on an unbounded space domain admits plane wave solutions

$$u(x,t) = e^{(i(\xi x + \omega t))}, \xi \in \Re, \tag{2.13}$$

where ξ is the wave number and ω is the frequency. There is a relationship between ξ and ω respective to each PDE equation,

$$\omega = \omega(\xi), \tag{2.14}$$

which is referred to as the dispersion relation.

Taking one time and one spatial derivative of (2.13) gives

$$u_t = -i\omega u$$

$$u_x = i\xi u$$

Now substitute u_t and u_x into the one dimensional advection equation (1.2) which becomes

$$\begin{aligned} -i\omega u + i\xi u &= 0 \\ -iu(\omega - \xi) &= 0 \quad \text{such that} \\ \Rightarrow \text{either} \quad -iu &= 0 \quad \text{or} \quad \omega - \xi = 0 \\ \Rightarrow u &= 0 \quad \text{or} \quad \omega = \xi. \end{aligned}$$

This computation provides us with the dispersion relation, $\omega = \xi$ for the specified one dimensional advection equation. This dispersion relation is plotted in Figure 2.3.

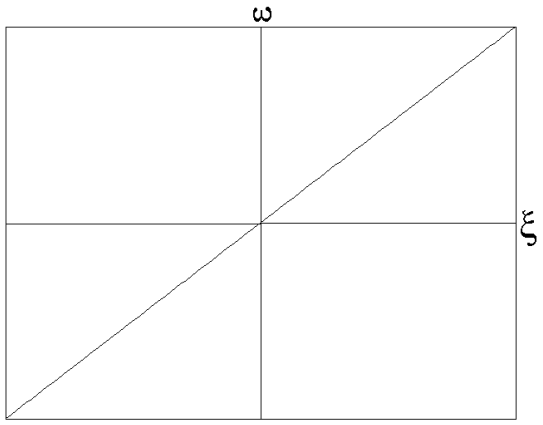

Figure 2.3: *The dispersion relation for* $u_t + u_x = 0$.

In his book, Trefethen states that discrete approximations to differential equations also experience dispersion relations. Trefethen reports that the dispersion relation for the centered finite difference approximation of the specified one dimensional advection equation is

$$\omega = \frac{1}{h}\sin(\xi h).$$

This dispersion relation is $2\pi/h$-periodic in ξ and shown in Figure 2.4 for $\xi \in [-\pi/h, \pi/h]$. After a comparison of the curves presented in Figure 2.4, notice that the approximation dispersion relation is accurate for a small values of ξ. Trefethen identifies the number of points per wavelength for the wave (2.13), to be given by $2\pi/\xi h$. The small values of ξ that yield an approximate dispersion relation depicted in Figure 2.4 correspond to many grid points per wavelength.

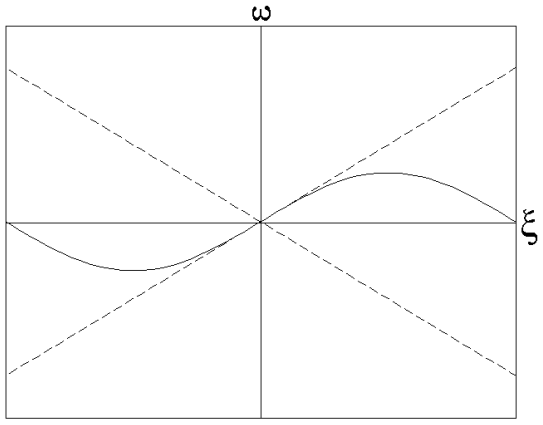

Figure 2.4: *The dispersion relation for* $u_t + u_x = 0$.

Dissipation error is responsible for the decay of a wave form. To examine this kind of error, suppose we have a linear PDE or finite difference approximation that introduces waves (2.13) with ω given by the dispersion realtion in (2.14). Considering that $\xi \in \Re$, the absolute value of the wave is given by

$$|e^{(i(\xi x+\omega t))}| = e^{-tIm(\omega)},$$

and is a function of t. The following definition identifies when a finite difference approximation will be nondissipative and dissipative.

Definition 8. A finite difference formula is nondissipative if Im(ω)=0 for all ξ. It is dissipative if Im(ω)$>$ 0 for all $\xi \neq 0$. It is dissipative of order $2r$ if ω satisfies

$$Im(\omega k) \geq \gamma_1(\xi h)^{2r}, \quad \text{i.e.,} \quad |e^{i\omega k}| \leq 1 - \gamma_2(\xi h)^{2r}$$

for some constants $\gamma_j > 0$. In each of these statements, ξ varies over the interval $[-\pi/h, \pi/h]$, and ω represents all possible values ω corresponding to a given ξ. For problems in mutiple space dimensions, (ξh) is replaced by $\|\xi h\|$ in any norm [13].

In the case where a multistep method is used, one should interpret the defintion of dissipative as requiring $\mathrm{Im}(\omega) > 0$ for every value of ω corresponding to some $\xi \neq 0$. Note that dissipative and nondissipative are mutually exclusive entities. For instance, a finite difference approximation can be neither dissipative nor nondissipative [13].

Kreiss and Oliger investigated the consequences of using high order methods in their 1972 paper [8]. Kreiss and Oliger's primary study was on dispersion error, or what they called phase error. Kreiss and Oliger managed to place bounds on the maximum dispersion error allowed in the computation of a wave for a specified amount of time. Kreiss and Oliger then observed the advantage of using a fourth and sixth order finite difference method over the second order finite difference method. Moreover, when considering a maximum error of 0.1, the fourth order method was approximately two orders of magnitude smaller than the error found with the second order method. Then the sixth order method was approximately 3/2 times the order of magnitude seen in the fourth order method. This means that the error found using the fourth and sixth order methods was significantly smaller than the error found in the second order method. As the bound on the allowed error is decreased and the computation of the wave is carried for a longer amount of time, the superiority of the fourth and sixth order methods is clear [8].

In [7], Jameson explains that the error in an approximation is not constant and depends on the order of the method as well as the wavenumber. This paper indicates that it is advantageous to use high order methods in comparison to low order methods because with the use of a high order method the approximation error will be orders of magnitudes smaller. Jameson proves this notion through the use of truncation errors. The algebraic manipulation and computations in this paper are done through using Fourier series. Jameson works with

different computational fields defined as

$$f_m(x,t) = \sum_k c_k^m(t) e^{ikx}, \tag{2.15}$$

where m denotes the field and c_k^m is a set of coefficients with respect to each field. Furthermore, the time element is dependent on the PDE in a given problem.

In general, we find that errors made in a spatial discretization typically dominate the errors that occur in a temporal discretization. With that said, the spatial derivative of (2.15) involves taking the spatial derivative of a complex exponential, that is,

$$\begin{aligned}\frac{d}{dx} f_m(x,t) &= \sum_k c_k^m \frac{d}{dx}(e^{ikx}) \\ &= \sum_k c_k^m \frac{d}{dx}(\cos(kx) + i\sin(kx)).\end{aligned}$$

Consequently, errors arise from differentiating the sine waves. Jameson provides the following truncation error formula,

$$TE = \text{const} \cdot (\triangle x)^{p-1} k^p, \tag{2.16}$$

where k is the kth Fourier mode, p represents the method order, and $\triangle x$ is the distance between adjacent grid points (equivalent to the mesh width h). Jameson says that *const* shown in the above equation will be dependent upon the method used, but is not important for the point being made. Next, a relationship between $\triangle x$ and k is presented with $\triangle x$ being small as compared to $1/k$, such that,

$$\triangle x < \frac{1}{2k_{\max}}.$$

We see that $\triangle x < 1$. It is clear that for a fixed k, increasing the method order, p, will result in a dramatic decrease of the truncation error (2.16).

The error at the final time should be the one that takes precedence over all others. When using a time-domain PDE solver, errors will occur in each step of the method. Furthermore, these numerical errors can sometimes accumulate linearly with time. Naturally, the more

steps taken in an approximation means more errors are being introduced. Jameson provides a rough formula for the final time error at the kth sine mode,

$$E^k_{\text{final}} = T_{\text{final}} \cdot TE_k \quad [7].$$

2.5 Stable High-order Methods

One need be aware of the difficulty that arises in stabilizing finite difference based PDE solvers. Note that the absence of data outside the computational domain forces finite difference stencils to become increasingly biased as the domain boundaries are approached. The simple procedure of using centered finite difference methods in the interior of the domain and biased stencils near the boundaries typically does not result in a stable method. This issue has been investigated and resulted in a number of stable high-order PDE solvers. Padé methods involve inverting banded matrices and reducing the stencil size as the domain boundary is approached. There are also the summation-by-parts (SBP) methods which incorporate energy estimates for semidiscrete problems to obtain derivative operators near the domain boundary. A shortcoming of both the Padé and SBP methods is the restriction placed on the method order near a boundary. By decreasing the method order near a boundary, this introduces a tradeoff between accuracy and stability. Essentially, both the Padé and SBP methods reduce accuracy near a boundary to in turn obtain stability. The Fourier Continuation (FC) method also results in a stable approximation[1]. The FC method consists of taking a non-periodic function and extending the function on a larger domain, such that, a periodic extension is obtained. The FC method has spectral accuracy, but restricted method order near the boundary [1, 2, 10]. An additional fact about the Padé, SBP and FC methods are that the time step size, k, is scaled like the mesh size, h.

There is another high-order method called the Chebyshev polynomial method, which can achieve good accuracy using a moderate number of grid points as compared to the substantial number of points required for a high-order finite difference method. The Chebyshev polynomial methods have non-uniform grids that use a clustering of grid points to avoid

using function values near the domain boundary that have singularities at these points. These methods consist of representing the unknown solution of a system of ODEs as a large series of Chebyshev polynomials. This series is then truncated and the approximation is substituted back into the ODE equation so that an equation that describes the coefficients of the solution can be obtained [12]. The Chebyshev polynomial methods maintain high order near the domain boundary and k is scaled like h^2.

In the next chapter we will discuss the Hagstrom and Hagstrom method. As an introduction to the Hagstrom and Hagstrom method, it is a stable method that incorporates finite differences and does not require reduction of the method order near a boundary. The Hagstrom and Hagstrom method also has k scaled like h.

Chapter 3

Understanding *Grid Stabilization of High-order One-sided Differencing I: First-order Hyperbolic Systems* By T. Hagstrom and G. Hagstrom

In [6], a technique is presented that is utilized to avoid order restrictions of biased finite difference approximations near a boundary. In effort to counteract the instability problem near a boundary, a number of additional grid points are introduced, close to each boundary. This idea of introducing additional gird points is somewhat similar to the clustering of points as seen in the Chebyshev polynomial methods. This technique finds the smallest number of additional points required to yield stability near each boundary.

3.0.1 Formulation and usage of the grid

We require approximations of a functions first spatial derivative. Suppose the function f is defined for x in the interval $(0, 1)$ with N linearly spaced points between 0 and 1. The grid, G_h is defined to be the union of the uniform grid with grid width $h = 1/N$ and the finite set of additional grid points. That is,

$$G_h = \{jh, 0 \leq j \leq N\} \bigcup \{z_k h, 1 - z_k h, k = 1, ..., p\},$$

where z_k is not an integer and p represents half the number of additional grid points.

The finite difference operator, D_{ord}, of order ord, is a matrix of size $N + 2p \times N + 2p$, containing the coefficients or weights of the finite difference approximation at each $x \in G_h$. That is, the i^{th} row in D_{ord} contains the finite difference weights for the first derivative approximation of f at the i^{th} grid point in G_h for $i = 1, 2, ..., N + 2p$. Considering the additional grid points and accommodating the boundary, the points used in a stencil that are somewhere in the middle of the grid will differ from the points in a stencil near the boundary. As a result of the differing stencil points, the finite difference weights will be different in these locations. When computing the finite difference weights of a particular order at a point x_0 on the uniform grid, with ord/2 points on either side of x_0, all of which still lie on the uniform grid, the weights of the centered finite difference formula is used. Hagstrom and Hagstrom indicate that for the other grid points in G_h, they used the the derivative of the degree ord interpolating polynomial (e.g. Lagrange polynomial) formed from the point of interest x_0 and ord/2 points to the left and right of x_0. Notice sometimes x_0 is a grid point close enough to either boundary and either ord/2 points to the left or right of x_0 are not attainable. That is, one of the boundary points has been reached and is one of the grid points included in the stencil [6]. When ord/2 grid points are not available to the left and right of x_0, the only choice is to use a biased stencil. Keep in mind that, in general, we want the finite difference stencils to be as centered as possible.

Hagstrom and Hagstrom considered the spectra of matrices D_{ord} to determine that the construction of the grid G_h is in fact a solution to stabilizing boundary closures. In particular, Hagstrom and Hagstrom focused on stabilizing finite difference approximations to the following hyperbolic partial differential equation initial value problem:

$$u_t = u_x + F, u(x, 0) = g(x), u(1, t) = 0.$$

The method used in solving this partial differential equation required a spatial discretization and yielded a stability region in which the eigenvalues of D_{ord} need to have negative real part. In an effort to minimize the time step stability constraints without being restrained to a particular solver, Hagstrom and Hagstrom alternatively minimize the spectral radius

of D_{ord} [6].

Consider the following problem proposed by Hagstrom and Hagstrom:

Problem 1. For fixed $N, p, \text{ord}/2$ choose z_k, for $k = 1, \cdots, p$ to minimize $\rho(D_{\frac{\text{ord}}{2},h})$ subject to $\sigma(D_{\frac{\text{ord}}{2},h}) \subset \mathbb{C}^-$.

Prior to obtaining results, it was important to know how to find the points z_k for any value of N. Thus, Hagstrom and Hagstrom had to check how the spectrum of D_{ord} was effected by the value of N. It was found that stability could still be achieved for large values of N, such as, $N = 100$ and $N = 200$. Furthermore, it was noted that the spectral radius of D_{ord} for the most part appeared unaffected by N, except that as N increased, the distance between the spectrum and imaginary axis decreased. A few cases where performed to check how changing the number and location of additional grid points effected the spectrum of D_{ord}. In each case, the discretization is stabilized for a range of z_k values and the spectral radius of D_{ord} remains small if the first additional grid point, z_1, is not to small [6]. In particular, Hagstrom and Hagstrom investigated the stability of finite difference methods of even orders ranging from four to twenty-two. Hagstrom and Hagstrom went up to incorporating three additional grid points near the boundary in their search for stability. This left us with the possibility to analyze stability in finite difference methods of even order greater that twenty-two as well as methods of odd order. When it came to stabilizing odd order methods, we needed to incorporate upwinding. Hence, the biased stencils being used are biasing to the left.

Chapter 4

Results

Using the technique presented in the Hagstrom and Hagstrom paper, we too wanted to stabilize boundary closures for the one dimensional advection equation. That is, we wanted to choose additional grid points z_k, for $k = 1, \cdots, p$ that would minimize the spectral radius of D as specified in problem 1. Our primary objective was to identify the minimum number of additional grid points required to stabilize biased finite difference approximations of the spatial derivative of a function near each boundary. With that said we aimed to achieve this goal using finite difference methods of odd order and orders higher than twenty-two. The finite difference weights are calculated using R. LeVeque's MATLAB script *fdcoeffF.m*, which is based on Fornberg's code titled *weights*. Fornberg's code generates the finite difference weights by taking derivatives of Lagrangian polynomials and can be found in [4] and [5].

We found that adding one additional grid point to the first cell results in a stable biased finite difference approximation near the boundary for orders up to ninth order. Then using two additional points (in the first cell), stable approximations can be obtained for orders up through seventeenth order. Moreover, three additional grid points (two in the first cell and one in the second cell) will allow for stable approximations up through twenty-third order. We went as far as finding that these biased finite difference approximations of orders up to the twenty-ninth order can be stabilized using four additional grid points (three of which are in the first cell and one in the second cell). See Table 4.1 for the list of the nodes z_k, for $k = 1, \cdots, p$ corresponding to a particular order finite difference method, spectral radius,

and the maximum negative real eigenvalue. Notice that just as before in the Hagstrom and Hagstrom paper, for methods that can be stabilized for a certain value of p, means that all methods of lower order can also be stabilized for the same value of p. Another observation to be made is that the maximum real eigenvalues for odd order methods presented in 4.1 differ by certain orders of magnitude in comparison to the even order maximum real eigenvalues presented in [6].

We have selected methods of orders eleven, seventeen, twenty-three, and twenty-nine and displayed their spectra in Figures (4.1,4.2,4.3, and 4.4). Again, we see that each spectrum displayed is formed from mostly eigenvalues lying on what looks like a smooth curve and a number of outlying eigenvalues. The outlying values determine the spectral radius, where as, the rightmost part of the curve should give rise to the maximum real part of the eigenvalue [6].

Note that the additional grid points z_k can be chosen independently from the sample size, N. In determining the placement of the additional grid points, we developed a multistepper MATLAB script that looped over the points z_k, for $k = 1, \cdots, p$. Then we identified the minimum number of additional grid points necessary to find stability in each method by selecting the outputted z_k values which yielded the minimum spectral radius. Next, we will discuss how the spectra of D are effected by varying the value of N. Again we will consider the methods of orders 11,17,23, and 29. Figures (4.5,4.6,4.7 and 4.8) show the sensitivity of spectral quantities for $hD_{11,h}, hD_{17,h}, hD_{23,h}$ and $hD_{29,h}$ with the respective number p and varying N. In all of these figures, the subfigure to the left, shows how close the spectrum is to the imaginary axis. We see that each spectrum reaches a certain distance from the imaginary axis for many values of N. The subfigures to the right, show the spectral radius times h converging to some value for many values of N. In each case the spectral quantities experience asymptotic behavior for increasing values of N. Thus, the spectral quantities are independent of N.

z_1	z_2	z_3	z_4	ord	N	$N^{-1}\rho(D)$	$N^{-1}max_{\lambda\in\sigma(D)}\Re\lambda$
0.206800	-	-	-	5	100	1.681970	-3.004915(-1)
0.206800	-	-	-	5	200	1.690465	-1.742913(-1)
0.189000	-	-	-	7	100	1.810830	-2.286831(-1)
0.189000	-	-	-	7	200	1.819976	-1.733495(-1)
0.189000	-	-	-	9	100	1.957880	-7.585413(-2)
0.189000	-	-	-	9	200	1.967769	-7.623723(-2)
0.135600	0.972200	-	-	11	100	2.862939	-3.666573(-2)
0.135600	0.972200	-	-	11	200	2.877399	-3.685091(-2)
0.117800	0.740800	-	-	13	100	2.993467	-1.341673(-1)
0.117800	0.740800	-	-	13	200	3.008586	-1.344209(-1)
0.100000	0.572200	-	-	15	100	3.013943	-1.141864(-1)
0.100000	0.572200	-	-	15	200	3.029165	-1.147632(-1)
0.100000	0.545000	-	-	17	100	2.956401	-2.979540(-3)
0.100000	0.545000	-	-	17	200	2.971332	-2.994588(-3)
0.100000	0.440000	1.952200	-	19	100	3.167522	-9.527889(-2)
0.100000	0.440000	1.952200	-	19	200	3.183520	-9.524120(-2)
0.100000	0.440000	1.950600	-	21	100	3.172817	-8.674499(-2)
0.100000	0.440000	1.950600	-	21	200	3.188841	-8.680982(-2)
0.100000	0.440000	1.911200	-	23	100	3.282088	-2.361086(-2)
0.100000	0.440000	1.911200	-	23	200	3.298665	-2.373010(-2)
0.079400	0.317000	0.965600	1.871800	24	100	4.389232	-5.679191(-4)
0.079400	0.317000	0.965600	1.871800	24	200	4.411400	-5.662071(-5)
0.079400	0.317000	0.965600	1.871800	25	100	4.381588	-7.402243(-2)
0.079400	0.317000	0.965600	1.871800	25	200	4.403718	-5.381928(-2)
0.079400	0.317000	0.965600	1.871800	26	100	4.372633	-6.204306(-4)
0.079400	0.317000	0.965600	1.871800	26	200	4.394717	-5.954113(-5)
0.079400	0.317000	0.965600	1.871800	27	100	4.366301	-3.224161(-2)
0.079400	0.317000	0.965600	1.871800	27	200	4.388353	-3.240444(-2)
0.079400	0.317000	0.886800	1.950600	28	100	4.888664	-7.111271(-4)
0.079400	0.317000	0.886800	1.950600	28	200	4.913354	-6.567527(-5)
0.079400	0.317000	0.768600	1.950600	29	100	5.491674	-6.166857(-3)
0.079400	0.317000	0.768600	1.950600	29	200	5.519410	-6.198002(-3)

Table 4.1: *Optimized grid locations and spectra of* D_{ord}

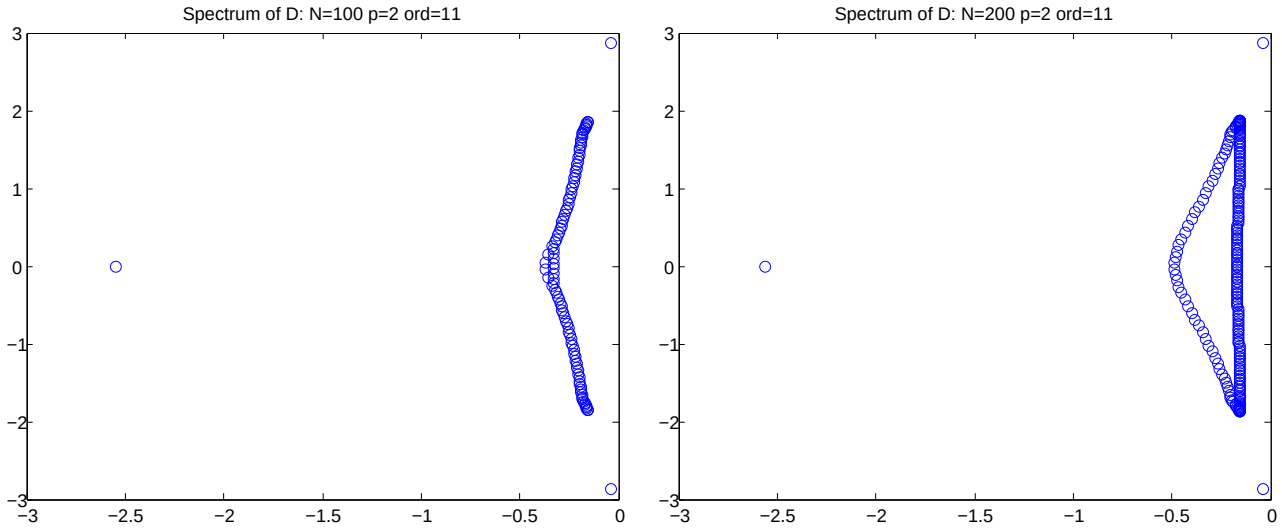

Figure 4.1: *Spectra of* $hD_{11,h}$ *for* $p = 2$.

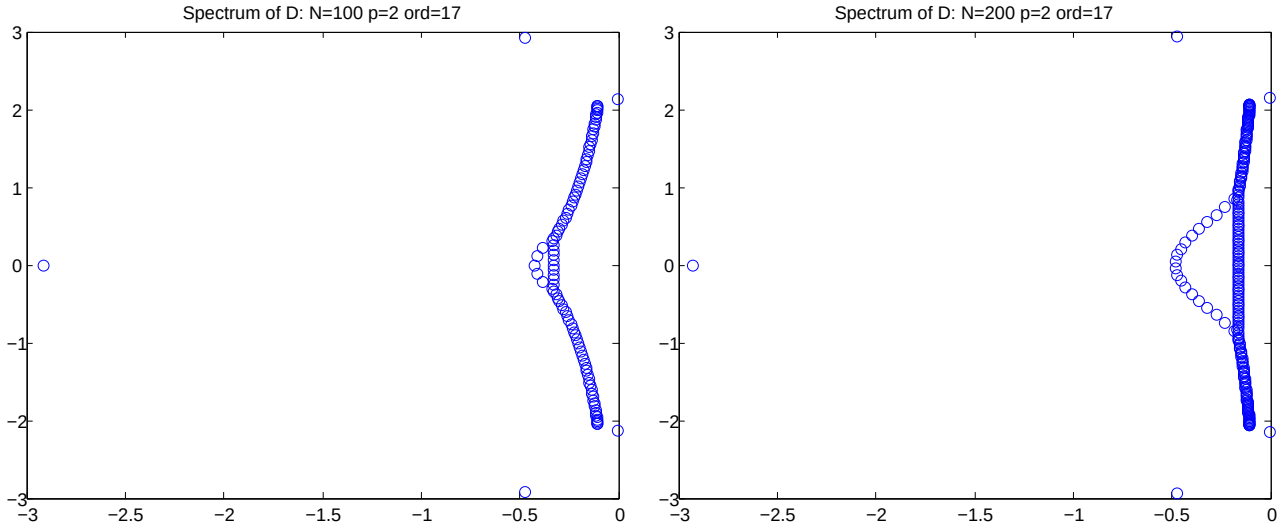

Figure 4.2: *Spectra of* $hD_{17,h}$ *for* $p = 2$.

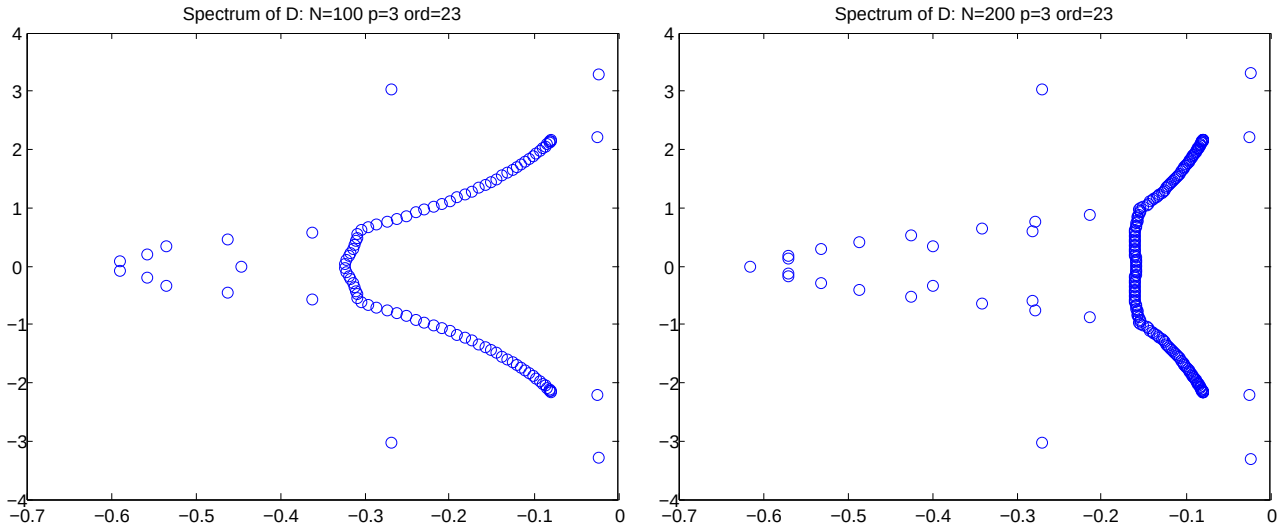

Figure 4.3: *Spectra of* $hD_{23,h}$ *for* $p = 3$.

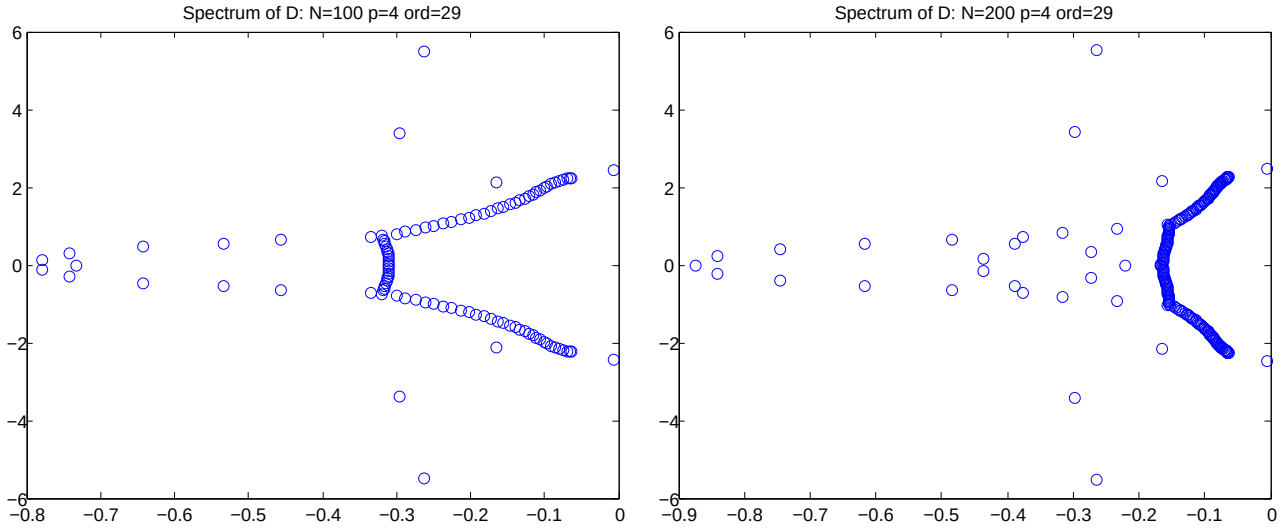

Figure 4.4: *Spectra of* $hD_{29,h}$ *for* $p = 4$.

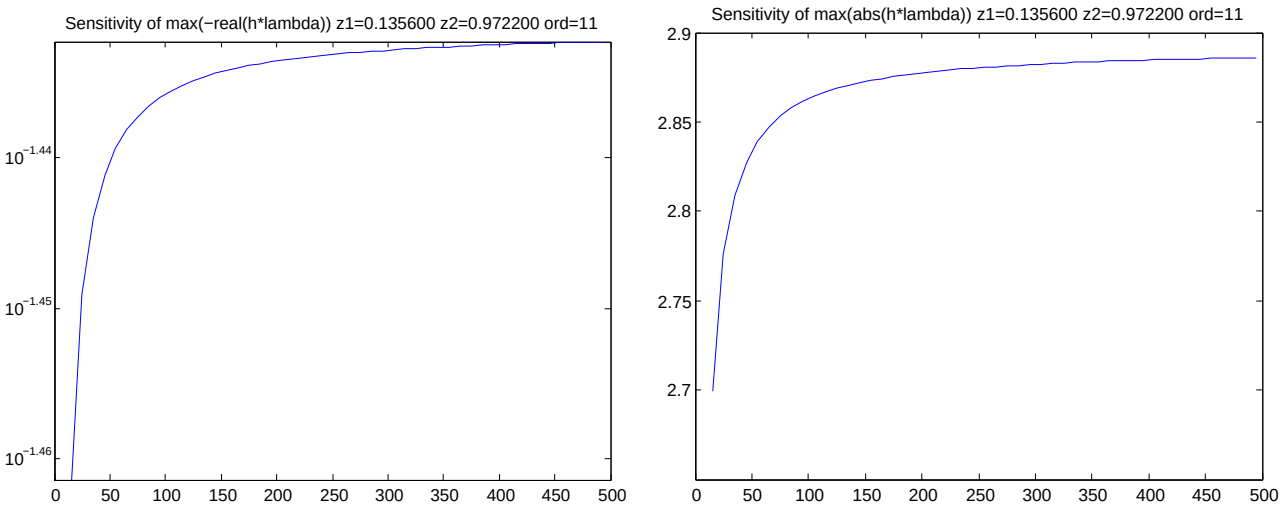

Figure 4.5: *Sensitivity of spectral quantities for $hD_{11,h}$ with $p = 2$ and varying N.*

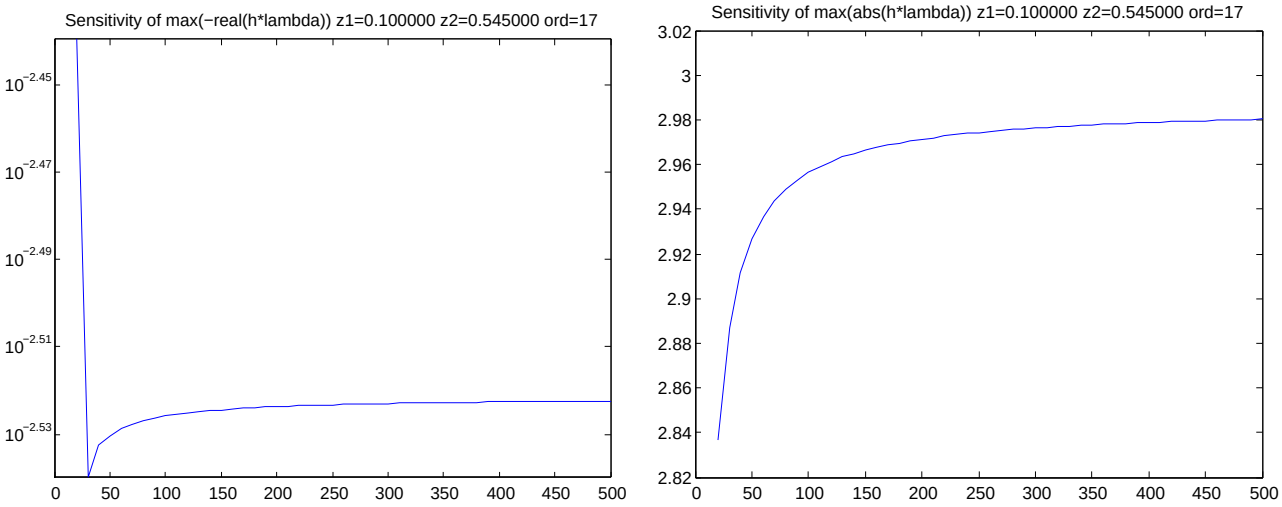

Figure 4.6: *Sensitivity of spectral quantities for $hD_{17,h}$ with $p = 2$ and varying N.*

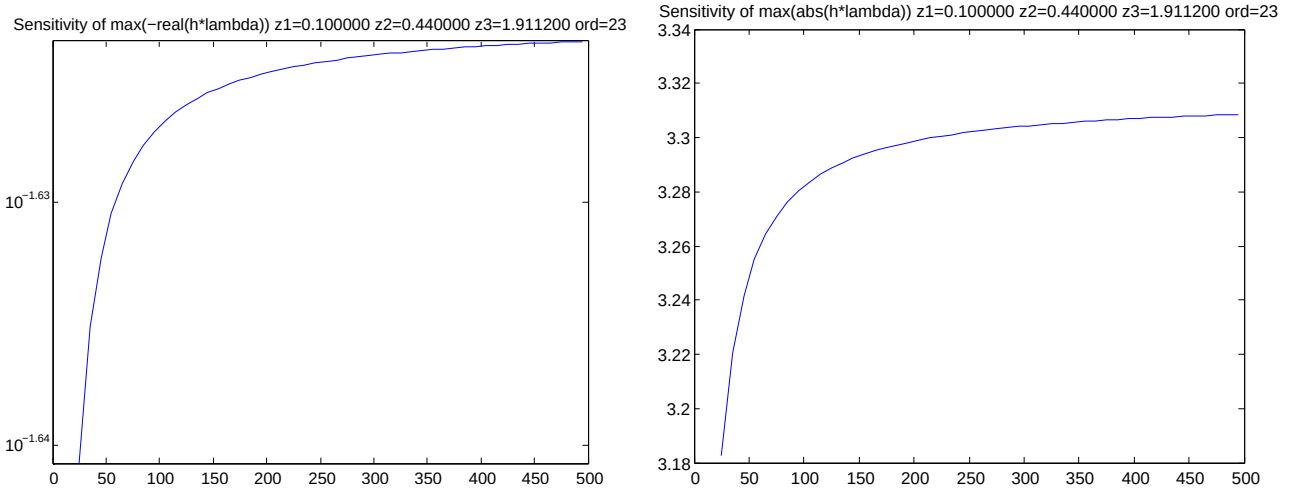

Figure 4.7: *Sensitivity of spectral quantities for $hD_{23,h}$ with $p = 3$ and varying N.*

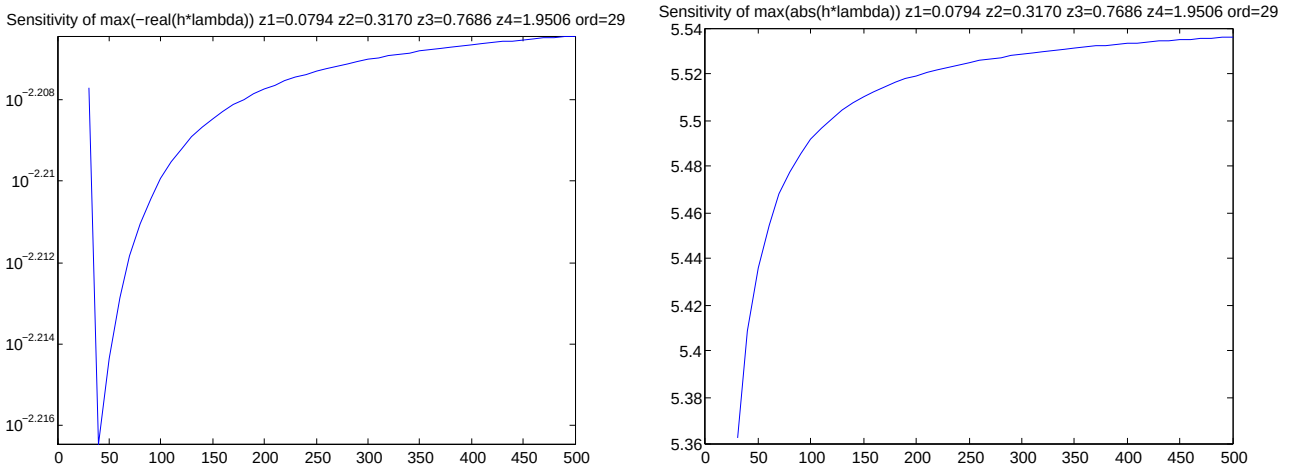

Figure 4.8: *Sensitivity of spectral quantities for $hD_{29,h}$ with $p = 4$ and varying N.*

Chapter 5

Conclusion

In this study, we used the grid formulation and analysis techniques presented in [6] and hoped to answer two main questions. First being, could finite difference approximations of odd order be stabilized via the Hagstrom and Hagstrom method for the one dimensional advection equation? We had also posed the question of whether or not it is possible to stabilize these methods for orders greater than twenty-two. We were able to answer both of these questions positively.

The results in the Hagstrom and Hagstrom paper indicate that one additional grid point is needed to stabilize finite difference methods up to eighth order, two additional points to stabilize methods up to sixteenth order, and three additional points to stabilize methods up to twenty-second order. There are a few subtleties in regards to these additional grid points. The first being that the two additional grid points required to stabilize methods up to sixteenth order are placed in the first cell of the grid. For the three additional points used to stabilize methods of orders seventeen to twenty-two, two points are placed in the first cell and one in the second cell of the grid [6].

By combining our findings with the results provided by Hagstrom and Hagstrom, we can now say the following. With the use of upwinding in the case of odd orders and biasing to the left, orders up through ninth order can be stabilized with the addition of one grid point in the first grid cell. Two additional grid points (both points in the first grid cell) are required to stabilize method orders ten through seventeen. Three additional grid points

(two points in the first cell and one point in the second cell) to stabilize method orders eighteen through twenty-three. We were also able to find that by including four additional grid points (three in the first cell and one in the second cell), finite difference methods for orders between twenty-four and twenty-nine can be stabilized.

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