ADVANCED STUDY ON THE MOMENT DISTRIBUTION METHOD AND ITS APPLICATION TO SIDESWAY PROBLEMS

by

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SYNOPSIS

The process of moment distribution is based upon the repeated application of the principle of superposition. Artificial restraints are imposed on the structure and subsequently eliminated. The combined effect of these is the same as would exist in the actual structure without any of the assumed restraints.

First, the fixed-end moments are computed for the loaded span (or spans); secondly, the fixed-end moments occurring at joints not permanently fixed are balanced by distributing equal and opposite moments in accordance with the stiffness of component members. These distributing moments are finally carried-over to opposite ends of members, except where these ends are permanently hinged.

Under the application of loads, the joints of the structural members undergo either joint rotation or joint translation. In some cases, both occur simultaneously. In this case,

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it is better to divide the joint movement into two components, i.e., joint translation and joint rotation. By applying the theory of superposition, the effect of the two components are combined together.

The moment distribution method is well applied in the solution of the sidesway problems. In the sidesway problem, first, joint rotation is considered; secondly, the joint translation is considered.

In the process of joint rotation, all joints, except hinged ends are held against rotation and all moments at the fixed ends produced by the actual loads are computed. These moments are balanced and distributed repeatedly until the desired accuracy is obtained. The final effects are combined to get the exact moment due to joint rotation. From these moments, the reactional forces in the structure are obtained which are not true because in the joint rotation the joints have been prevented from translating, and, hence, additional forces are added. In the study of the effects of the additional moments, the forces with some magnitude but opposite in direction are applied. The joint translation problems are involved in calculating the effects of the additional forces. In the joint translation problem, first, the different joints translation should be known. This can be done by constructing a Williot diagram so that the displacement of the joint can be visualized. Moments due to joint translation are computed according to the lateral stiffness and are balanced and distributed until the desired accuracy is obtained.

INTRODUCTION

The structural engineer of today is much concerned with the statically indeterminate structures. There are many methods introduced for the solution of highly complicated structural problems. Since the development of electronic computor machines, which have proved highly efficient in the solution of structural problems with high degree of statical indeterminancy, the techniques of the short-cut and the approximation method seem to have the tendency of losing their merits. Nevertheless, the study of these methods for the solution of structural problems is very important for the students of structural engineering in order that they can have the ability to solve complicated structural problems easily and quickly without the help of computor machines. Among the methods, the moment distribution method is considered to be one of the best. It is a method of solving for the moment in continuous beams and frames by successive approximations. It became generally known to structural engineers through the publication of Hardy Cross' classic paper in 1932. It may be described as a solution by successive approximation of the slope-deflection equations. It may be applied very quickly and simply for an approximate solution or, with a little more labor, extended to any degree of exactness desired.

SIGN CONVENTIONS AND NOTATIONS

The following sign conventions and notations are employed in this report. Positive moments act counterclockwise at one end of a member. If they act clockwise, they are called negative moments.

The sign conventions employed in the moment diagrams are somewhat different. Positive moments are drawn on the compression side of the member and negative moments are on the tension side.

The notations employed are as follows:

E = Modulus of Elasticity

I = Moment of Inertia

M = Moment

M_F = Fixed-end Moment

A, B, C, ... X, Y, Z = Joint

L = Length

D. F. = Distribution Factor

K_{AB} = Rotational Stiffness for the Member A B, etc.

C. O. F. = Carry-over Factor

C. O. M. = Carry-over Moment

Q = Auxiliary Force

P = Concentrated External Force

Σ = Summation

Δ = Displacement

m, n = Correction Factor to the Moments

α, β θ, = Angle of Rotation

BASIC ASSUMPTIONS MADE IN ENGINEERING STRUCTURAL ANALYSIS

The ordinary methods of structural analysis are based upon the following fundamental assumptions:

1. The material is homogeneous and isotropic.

2. The material is stressed within the elastic limit.

3. Displacements are small enough that stresses, deflections, etc., computed from applied loads assumed acting through the undeflected positions will not materially change as the displacements take place.

4. The surface effects of external loads are negligible.

BASIC CONCEPTS: STIFFNESS, DISTRIBUTION FACTOR, AND CARRY-OVER FACTOR

(a) Beam with one end fixed and the other end hinged.

(b) Deflection diagram.

Fig. 1. A beam with a moment at one end.

In Fig. 1 (a) is shown a beam of Modules of Elasticity, E, and Moment of Inertia, I, loaded with a moment (or couple) at the hinged end while the other end is fixed. In Fig. 1 (b) is shown the deflection curve of the beam. Since B end is fixed, there is no rotation of the beam at B. From the slope-deflection equation:

$$M = 2\,\frac{EI}{L}\,(2\theta a) = 4E\,\frac{I}{L}\,\theta a \qquad (1)$$

Apply a moment with a magnitude K such that it produces a unit rotation at the end A. Then:

$$M = K = \frac{4EI}{L} \qquad (2)$$

Define the term K as the stiffness factor of a beam and determine this for each member of the structure. The stiffness factor, as defined, is the moment that must be applied at one end of a constant section member to produce a unit rotation of that end when the other end is fixed.

In Fig. 2. a clockwise moment, M_O, is applied at the rigid joint X producing a clockwise rotation θ_X. The equilibrium equation of joint X is:

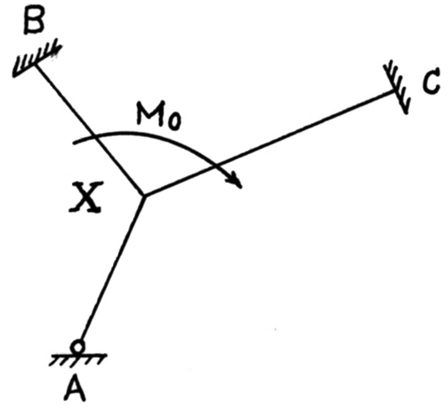


Fig. 2. Frame with a moment at the joint, X.

$$M_{XA} + M_{XB} + M_{XC} + M_O = 0 \qquad (3a)$$

From the slope-deflection equation:

$$M_{XA} = K_{XA} (\theta_X + \tfrac{1}{2}\theta_A) \qquad (3b)$$

$$M_{XB} = K_{XB}\, \theta_X \qquad (3c)$$

$$M_{XC} = K_{XC}\, \theta_X \qquad (3d)$$

But, $M_{AX} = K_{XA} (\tfrac{1}{2}\theta_X + \theta_A) = 0$ (3e)

From which, $\theta_A = -\tfrac{1}{2}\theta_X$ (3f)

Substituting equation 3 (f) in equation 3 (b):

$$M_{XA} = K_{XA}\left(\tfrac{3}{4}\theta_X\right) \qquad (3g)$$

Substituting equation 3 (c), (d), (g) in equation (3a):

$$K_{XA}\left(\tfrac{3}{4}\theta_X\right) + K_{XB}\,\theta_X + K_{XC}\,\theta_{XC} - M_O = 0$$

Or: $$\theta_X = \frac{M_O}{\tfrac{3}{4}K_{XA} + K_{XB} + K_{XC}} \qquad (3h)$$

Substituting equation (3h) back into the equations for the moments:

$$M_{XA} = \frac{-\tfrac{3}{4}K_{XA}}{\tfrac{3}{4}K_{XA} + K_{XB} + K_{XC}}\; M_O \qquad (3i)$$

$$M_{XB} = \frac{-K_{XB}}{\tfrac{3}{4}K_{XA} + K_{XB} + K_{XC}}\; M_O \qquad (3j)$$

$$M_{XC} = \frac{-K_{XC}}{\tfrac{3}{4}K_{XA} + K_{XB} + K_{XC}}\; M_O \qquad (3k)$$

The stiffness factor ratio in equations (3i), (3j), and (3k) is defined as the distribution factor. Thus, for example:

$$D_{XA} = \frac{\tfrac{3}{4}K_{XA}}{\tfrac{3}{4}K_{XA} + K_{XB} + K_{XC}} \qquad (3m)$$

is the distribution factor for end X of beam XA and is a measure of the portion of the applied external moment at X which is taken by beam XA at end X. Then:

$$M_{XA} = -D_{XA}\,M_O \qquad (4a)$$

$$M_{XC} = -D_{XC}\,M_O \qquad (4b)$$

$$M_{XB} = -D_{XB}\,M_O \qquad (4c)$$

From the slope-deflection equations, the moments M_{AX}, M_{BX}, and M_{CX} in Fig. 2 are found as follows:

$$M_{AX} = \tfrac{1}{2}K_{AX}\,(2\theta_A + \theta_X) = 0 \qquad (5a)$$

$$M_{BX} = \tfrac{1}{2}K_{BX}\,(\theta_X) = \tfrac{1}{2}M_{XB} \qquad (5b)$$

$$M_{CX} = \tfrac{1}{2}K_{CX}\,(\theta_X) = \tfrac{1}{2}M_{XC} \qquad (5c)$$

The ratio of the far end moment to the rigid-joint moment is defined as a carry-over factor and is denoted by the letter C. Thus:

$$C_{XA} = \frac{M_{AX}}{M_{XA}} = 0 \qquad (6a)$$

$$C_{XB} = \frac{M_{BX}}{M_{XB}} = \tfrac{1}{2} \qquad (6b)$$

$$C_{XC} = \frac{M_{CX}}{M_{XC}} = \tfrac{1}{2} \qquad (6c)$$

ROTATIONAL STIFFNESS

In Fig. 3 (a) is shown a beam having one end fixed while the other end is allowed to rotate under the influence of a couple, M. In order to find the angle of rotation θ, the effect of the moment M and the reaction R may be considered separately. In Fig. 3 (b) the effect of the moment M and in Fig. 3 (c) the effect of the reaction R are considered separately. The deflections and angles of rotation can be found by the moment-area method. The shaded areas in the Fig. 3 (a), (b), and (c) represent the moment diagrams.

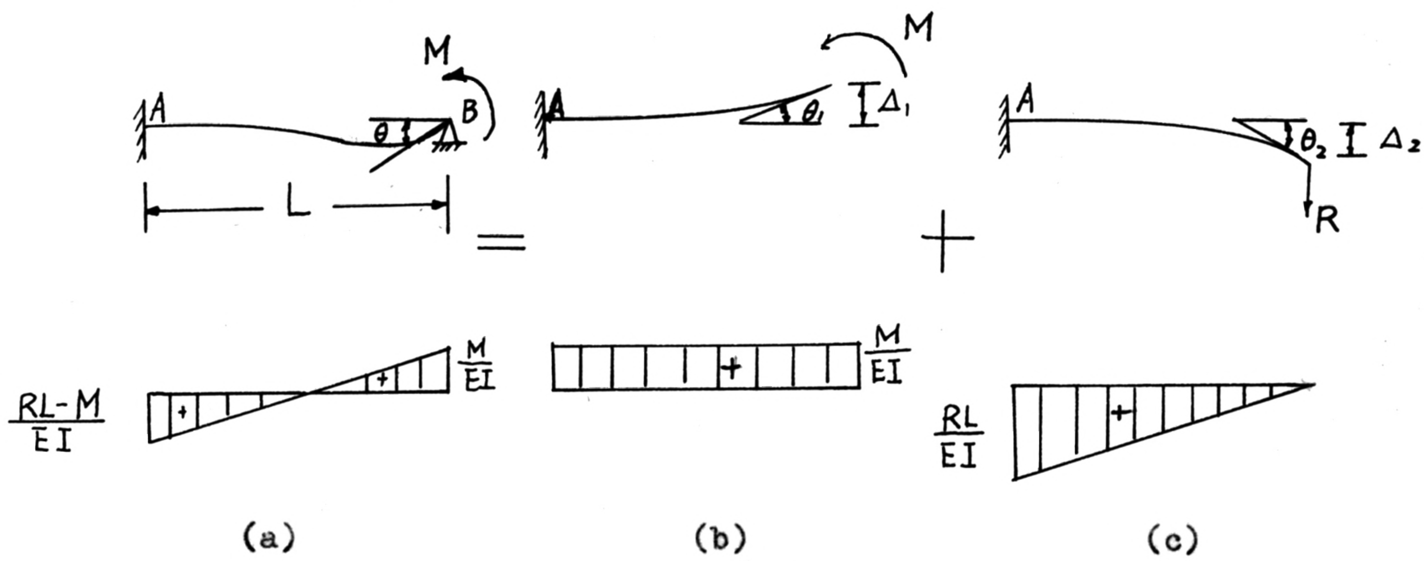


Fig. 3. Rotational stiffness.

The deflection in Fig. 3 (b) is $\Delta_1 = \frac{M}{EI} \times L \times \frac{L}{2} = \frac{ML^2}{2}$

The deflection in Fig. 3 (c) is $\Delta_2 = \frac{-RL}{2EI} \times L \times \frac{2L}{3} = -\frac{RL^3}{3}$

Since $\Delta_1 + \Delta_2 = 0$, therefore, $\frac{ML^2}{2EI} - \frac{RL^3}{3EI} = 0$

$$R = \frac{3M}{2L} \qquad (7a)$$

Similarly, $\theta = \theta_1 + \theta_2$

By moment-area method: $\theta_1 = \frac{ML}{EI} \quad \theta_2 = -\frac{RL^2}{2EI}$

$$\theta = \frac{ML}{EI} - \frac{RL^2}{2EI} \qquad (7b)'$$

Substituting equation (7a) in equation (7b)':

$$\theta = \frac{ML}{4EI} \qquad (7b)$$

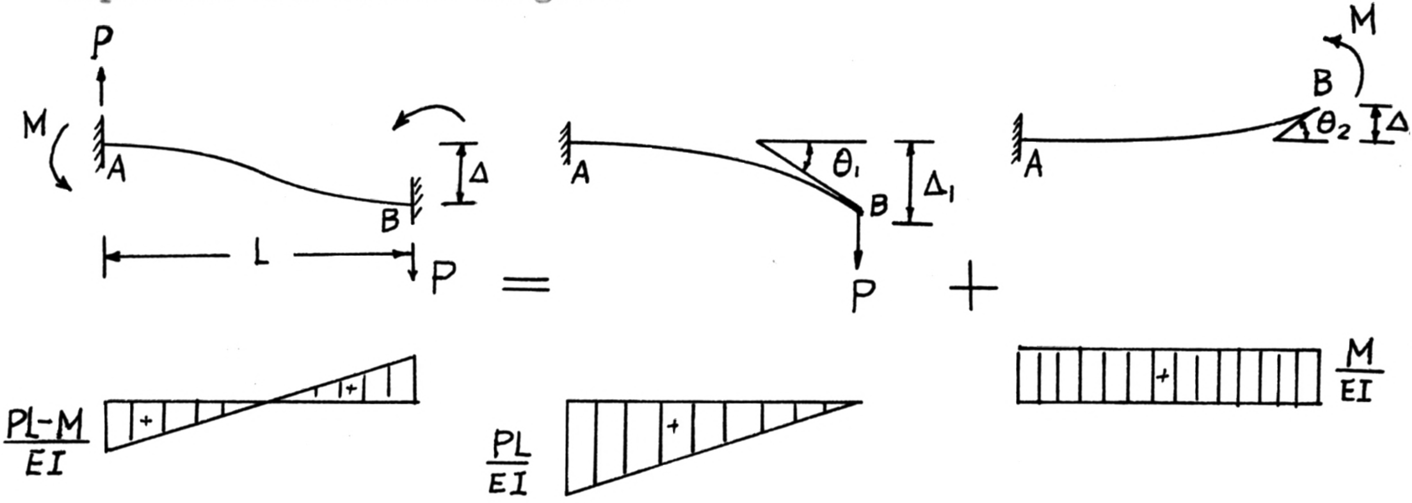
Then: $$M = 4E\frac{I}{L}\theta \qquad (7c)$$

When θ is taken as a unit rotation, then $M = 4E\frac{I}{L}$, let K be equal to $4E\frac{I}{L}$, then $M = K = 4E\frac{I}{L}$ where K is identical with equation (2).

LATERAL STIFFNESS

The lateral stiffness is the measure of resistance to lateral displacement of one end of a member while both ends are fixed against rotation.

In Fig. 4 (a), (b), and (c) are shown a beam of span L and of constant EI. The summation of the effect in Fig. 4 (b) and Fig. 4 (c) is identical with Fig. 4 (a). The shaded areas represent the moment diagram.



(a) (b) (c)

Fig. 4. Lateral stiffness.

By moment-area method:

$$\theta_1 = -PL \times \frac{L}{2\ EI} = \frac{-PL^2}{2EI} , \qquad _1 = \frac{-PL^2}{2EI} \times \frac{2}{3} L = \frac{-PL^3}{3EI}$$

$$\theta_2 = \frac{ML}{EI} , \qquad _2 = \frac{ML}{EI} \times \frac{L}{2} = \frac{ML^2}{2EI}$$

Since, $\theta_1 + \theta_2 = 0$ then, $\frac{ML}{EI} + \frac{-PL^2}{2EI} = 0$

$$M = \frac{P}{2} L \qquad (8a)$$

$$\Delta = \Delta_1 + \Delta_2 = \frac{ML^2}{2EI} + \frac{-PL^3}{3EI} = \frac{PL^3}{4EI} - \frac{PL^3}{3EI}$$

$$= \frac{-PL^3}{12EI} = \frac{-ML^2}{6EI} \qquad (8b)$$

therefore, $$M = -6E\Delta \frac{I}{L^2} \qquad (8c)$$

If the downward displacement is defined as positive, then,

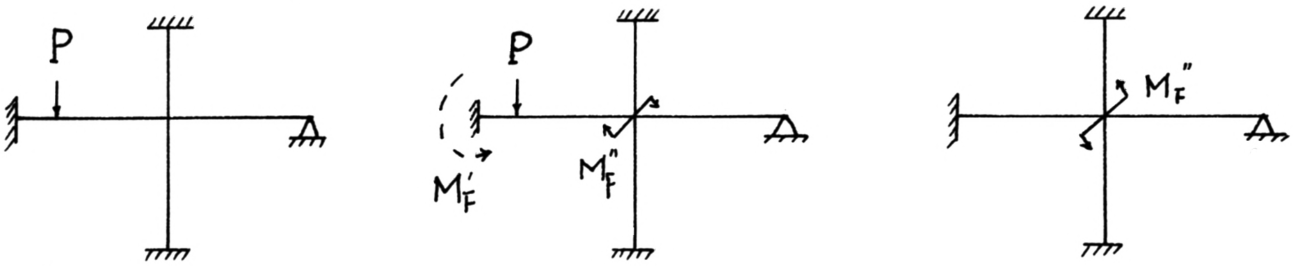
$$M = 6E\Delta \frac{I}{L^2} \qquad (8c)'$$

The lateral stiffness can be defined from equation (8c)', as the moment developed at the ends of a member due to the unit lateral displacement of one end. Thus,

$$M = 6E \frac{I}{L^2} \qquad (8d)$$

SUMMARY

The process of moment distribution is based upon the repeated application of the principle of superposition. Artificial restraints are imposed on the structure and subsequently eliminated. The combined effect of these is the same as would exist in the actual structure without any of the assumed restraints. This is illustrated in Fig. 5, which shows a structure loaded in one of its spans, the left end of which is permanently fixed.



(a) Frame.   (b) Restrain the joint.   (c) Apply a moment at the joint.

Fig. 5. Principle of superposition used in moment distribution.

The ends of the two vertical members are also permanently fixed, while the end of the right horizontal member is hinged.

If to the Fig. 5 (a) is added a couple at the center joint, see Fig. 5 (b), this addition should be compensated for by including the effect of an opposite couple, see Fig. 5 (c). This couple is made equal to the fixed-end moment M_F'' at the right end of the span. The simultaneous moment at the left end, M_F', will be the fixed-end moment there.

The effect of the opposite moment M_F'' will be to set up resisting moments in the four connecting members in accordance with their rotational stiffnesses. The resisting moments will, in turn, cause moments of one-half their magnitude at the fixed end (for the constant EI).

The method of moment distribution as applied to structures in which the joints rotate but do not translate may be summarized as follows:

1. Assume all joints held against rotation and compute all moments at the fixed ends produced by the actual loads.

2. At each joint distribute the unbalanced fixed-end

moments with opposite sign among the connecting members in proportion to their relative stiffness.

3. Multiply the moments distributed to each member at a joint by the carry-over factor and set this product at the other end of that member.

4. Distribute the moments thus carried-over among connecting members.

5. Repeat the process until the moments to be carried over are sufficiently small to be disregarded.

6. Add all moments, with due regard for signs, fixed-end moments, distributed moments, caried-over moments--at each end of each member to obtain the true moments.

JOINT TRANSLATION

In a great number of actual structures, the joints can depart from their positions in the unloaded structure, and the movement will influence the magnitude of the bending moments caused by the load.

Problems involving joint translations usually fall in two categories: those in which the original fixed-end moments are due solely to the joint movement; and those in which there exist, in addition, fixed-end moments due to loaded spans.

An example of the first type is shown in Fig. 7 (a). A constant moment of inertia has been assumed for all members. In Fig. 7 (b) the bent is assumed deflected laterally, but with the top joints fixed against rotation. This movement will cause moment M at the tops and bottoms of the two vertical members, AB and CD.

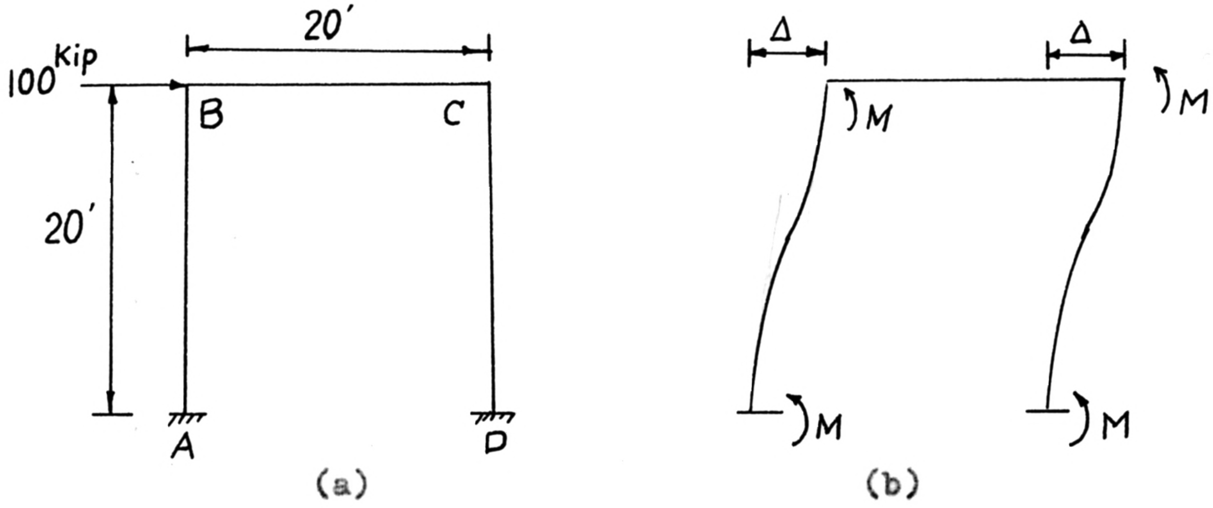


Fig. 6. Moment due to joint translation.

From equation (8c)'
$$M^F_{AB} = 6E\frac{I}{L^2}\Delta = M^F_{BA} = M$$
$$M^F_{DC} = 6E\frac{I}{L^2}\Delta = M^F_{CD} = M$$

Any convenient value can be used to replace M and the lateral force can be found by a shear equation. For example, replace M by 1000 kip-ft:

	Kip-ft	Kip-ft	Kip-ft	Kip-ft
Σ	+601.6	-601.6	-601.6	+601.6
	+7.8	+7.8	+7.8	+7.8
		-15.6	-15.6	
	-31.3	-31.3	-31.3	-31.3
		+62.5	+62.5	
	+125	+125	+125	+125
		-250	-250	
	-500	-500	-500	-500
	+1000			+1000
D.F	0.5	0.5	0.5	0.5

+1000
−½(1000−601.6)

+800.8 Kip-ft

+1000
−½(1000−601.6)

+800.8 Kip-ft

Fig. 6 (c) Moment distribution.

The shear forces corresponding to the moments in Fig. 6 (c) is:

$$2 \times \frac{800.8 + 601.6}{20} = 140.24 \text{ kips}$$

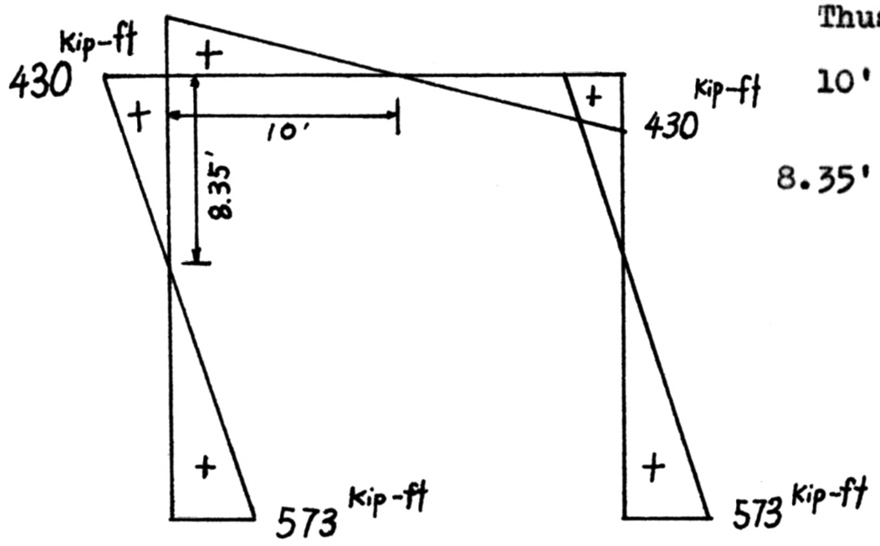
The true moments can be obtained by multiplying the moments in Fig. 6 (c) by the ratio of shear forces, i.e., $\frac{100}{140.24}$

Thus,

$$M_{BA} = 601.6 \times \frac{100}{140.24} = 430 \text{ kip-ft}$$

$$M_{AB} = 800.8 \times \frac{100}{140.24} = 573 \text{ kip-ft}$$

The point of zero moment can be obtained by direct proportion of the moments shown in Fig. 6 (e).



Thus:

$$10' = 20 \times \frac{430}{430 + 430}$$

$$8.35' = 20 \times \frac{430}{430 + 573}$$

Fig. 6 (e). Moment diagram. Positive moments on the compression side.

Another example of an unsymmetrical bent is shown in Fig. 7 (a). The rotational stiffnesses at the two joints are proportional to $\frac{I}{L}$, and the lateral stiffnesses are proportional to $\frac{I}{L^2}$. The dimension of the cross-section of each member is noted in Fig. 7 (a).

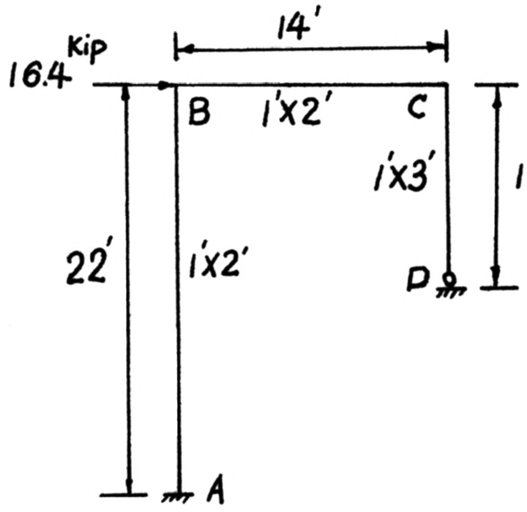


Fig. 7 (a). Unsymmetrical bent.

$$K_{BA} = \frac{4 \times (2)^3}{12} E \times \frac{1}{22} = \frac{4}{12} E \frac{(2)^3}{22} = \frac{4}{12} E \times 0.364$$

let $\frac{4E}{12}$ be denoted as C.

then $K_{BA} = C \times 0.364$

$$K_{BC} = K_{CB} = C \times \frac{(2)^3}{14} = 0.571\ C$$

$$K_{CD} = \frac{3}{4} C \times \frac{3^3}{12} = 1.688\ C$$

$$D_{BA} = \frac{K_{BA}}{K_{BA} + K_{BC}} = \frac{0.364C}{0.364C + 0.571C} = 0.39$$

$$D_{BC} = 1 - 0.39 = 0.61$$

$$D_{CD} = \frac{1.688C}{0.571C + 1.688C} = 0.75$$

$$D_{CB} = 1 - 0.75 = 0.25$$

Lateral stiffness for the member AB: (from equation 8 (d))

$$6E \frac{I}{L^2} = 6E \frac{(2)^3}{12} \frac{1}{(22)^2}$$

For the hinged member CD, the lateral stiffness is:

$$3E \frac{I}{L^2} = 3E \frac{(3)^3}{12} \frac{1}{12^2}$$

The corresponding lateral moments at the joints A, B, and C due to Δ lateral displacement at joints B and C are:

$$M_{BA} = M_{AB} = 6E \frac{(2)^3}{12} \frac{1}{22^2} \Delta$$

$$M_{CD} = 3E \frac{(3)^3}{12} \frac{1}{12^2} \Delta$$

$$M_{BA} : M_{CD} = 6E \frac{(2)^3}{12} \frac{1}{22^2} \Delta : 3E \frac{(3)^3}{12} \frac{1}{12^2} \Delta$$

$$M_{BA} : M_{CD} = 100 : 563$$

Since Δ is an assumed magnitude of the displacement, make it equal to the value which makes M_{BA} equal to 100 kip-ft. Then, M_{CD} = 563 kip-ft. Next, balance the moments due to the lateral displacement by distributing and carrying-over. This is illustrated in Fig. 7 (b).

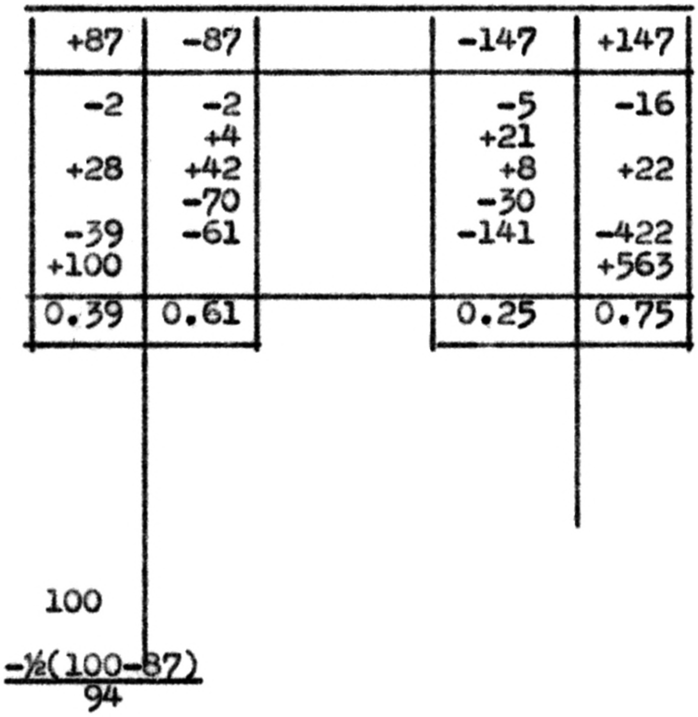


Fig. 7 (b). Moment distribution.

The horizontal components of the reaction at the base are:

$$\frac{M_{BA} + M_{AB}}{L} \text{ and } \frac{M_{CD}}{L} \text{ or } \frac{94 + 87}{22} \text{ and } \frac{147}{12} .$$

The sum of the horizontal components is $\frac{94 + 87}{22} + \frac{147}{12} = 20.6$ kips and is much greater than the actual external force, 16.4 kips. Hence, the multiplier $\frac{16.4}{20.6}$ was used and the moments in Fig. 7 (c) multiplied by $\frac{16.4}{20.6}$ to get the exact moments.

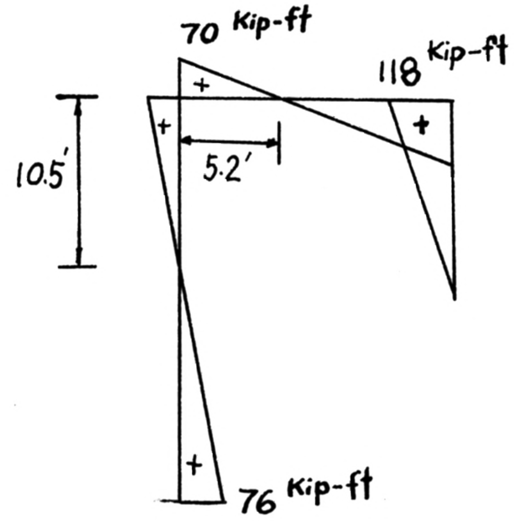
Thus:

$$M_{AB} = 94 \times \frac{16.4}{20.6} = 76 \text{ kip-ft}$$

$$M_{BA} = 87 \times \frac{16.4}{20.6} = 70 \text{ kip-ft}$$

$$M_{CD} = 147 \times \frac{16.4}{20.6} = 118 \text{ kip-ft}$$

The moment diagram due to the loading in Fig. 7 (a) is shown in Fig. 7 (c).



The points of zero moment are:

$$22 \times \frac{70}{70 + 76} = 10.5'$$

$$14 \times \frac{70}{118 + 70} = 5.2'$$

Fig. 7 (c). Moment diagram.

THE MOMENT DISTRIBUTION METHOD APPLIED TO ONE-STORY BENT

The general case of a one-story bent is shown in Fig. 9 (a).

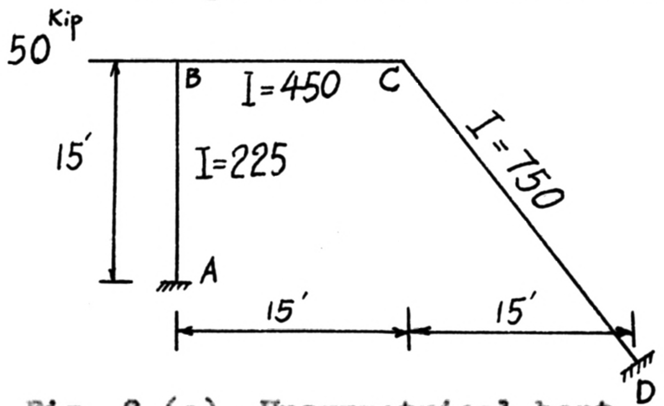


Fig. 8 (a). Unsymmetrical bent with sloping member.

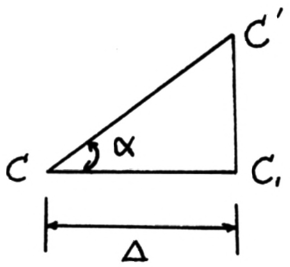


Fig. 8 (b). A Williot diagram for displacement of joint C.

Due to a 50 $^{\text{kips}}$ lateral force, joint B is assumed to have a lateral displacement . The displacement of the joint C is obtained by constructing a Williot diagram in Fig. 8 (b). From Fig. 8 (b):

$$\tan\alpha = \frac{C'C_1}{CC_1} = \frac{C'C_1}{\Delta} \qquad C'C_1 = \Delta \tan\alpha$$

$$\tan\alpha = \frac{15}{20} \quad \text{therefore} \quad C'C = \Delta \times \frac{15}{20} = 0.75\,\Delta$$

$$\text{similarly} \quad CC' = 1.25\,\Delta$$

From equation 8 (d), the moments due to the lateral displacements are:

$$M^F_{BA} = M^F_{AB} = 6E\,\frac{225}{(15)^2}\,\Delta$$

$$M^F_{BC} = M^F_{CB} = 6E\,\frac{450}{(15)^2}\,0.75\,\Delta$$

$$M^F_{CD} = M^F_{DC} = 6E\,\frac{750}{(25)^2}\,1.25\,\Delta$$

Then: $M^F_{BA} : M^F_{BC} : M^F_{CD} = 100 : -150 : 150$

Since is an assumed magnitude of displacement, can be adjusted so that the moment, M^F_{BA}, is equal to 100 kip-ft. The moments due to Δ displacement at B are distributed and carried-over as shown in Fig. 8 (c).

Approximately	+118	-118	-142	+142
Σ	+117.7	-117.7	-141.6	+141.6
	+1.2	+2.9		
		-4.1		
			-8.3	-8.4
			+16.7	
	+16.5	+33.5		
	+100	-150	-150	+150
D. F.	0.33	0.07	0.5	0.5

100
+½(118-100)
+109 kip-ft

$$15.1 = \frac{109 + 118}{15}$$

$$11.5 = \frac{142+146}{25}$$

+150
-½(150-142)
+146 kip-ft

Fig. 8 (c). Moment distribution.

The distribution factors shown in Fig. 8 (c) are obtained as follows:

$$K_{BA} = \frac{225}{15}, \quad K_{BC} = \frac{450}{15}, \quad \text{therefore,} \quad D_{BA} = \frac{\frac{225}{15}}{\frac{225}{15} + \frac{450}{15}} = 0.33$$

$$D_{BC} = 1 - 0.33 = 0.67$$

$$K_{CB} = \frac{450}{15}, \quad K_{CD} = \frac{750}{25}, \quad D_{CB} = \frac{\frac{450}{15}}{\frac{450}{15} + \frac{750}{25}} = 0.5$$

$$D_{CD} = 1 - 0.5 = 0.5$$

Assume a horizontal force, P, applied at joint B which cause counterclockwise moments 109 kip-ft at A and 146 kip-ft at D as shown in Fig. 8 (d).

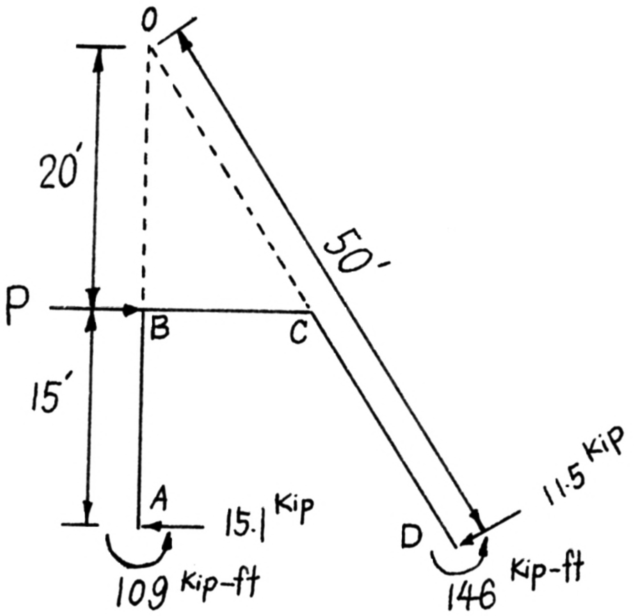


Fig. 8 (d). A free body diagram for the computation of P.

From equilibrium equation ΣM at O = 0

then $-P \times 20 + 15.1 \times 35 + 11.5 \times 50 - 109 - 146 = 0$

$P = 42.5$ kips

Since the actual force P is 50 kips so that the moments in Fig. 8 (c) should be multiplied by the factor $\frac{50}{42.5}$ or 1.17.

Thus:

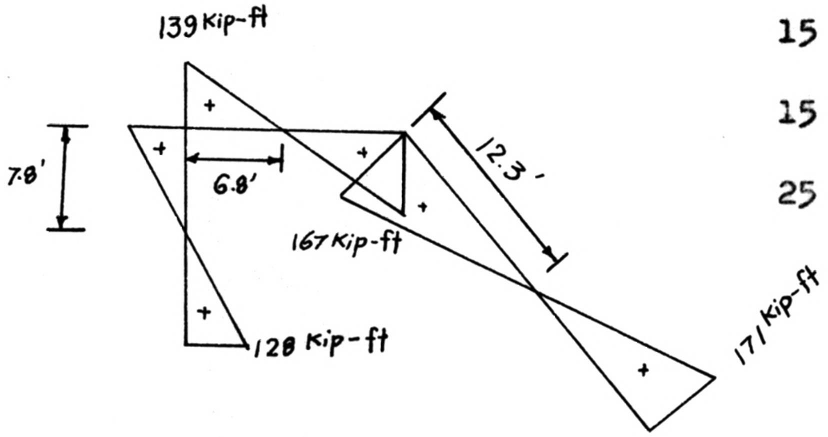
$M_{AB} = 109 \times 1.17 = 128$ kip-ft

$M_{BA} = 118 \times 1.17 = 139$ kip-ft $= -M_{BC}$

$M_{CD} = 142 \times 1.17 = 167$ kip-ft $= -M_{CB}$

$M_{DC} = 146 \times 1.17 = 171$ kip-ft

The moment diagram is shown in Fig. 8 (e). The points of zero moments are obtained by direct proportion.



$$15 \times \frac{139}{139 + 128} = 7.8'$$

$$15 \times \frac{139}{139 + 167} = 6.8'$$

$$25 \times \frac{167}{167 + 171} = 12.3'$$

Fig. 8 (e). Moment diagram.

APPLICATION OF THE MOMENT DISTRIBUTION METHOD TO THE GABLE BENT

The simplest possible example of such a structure is shown in Fig. 9 (a). If subjected to the action of a concentrated load acting at the top joint, this joint will, due to symmetry of shape and loading, move down vertically while the two column tops will move horizontally as shown in Fig. 9 (a). All the members are of same EI.

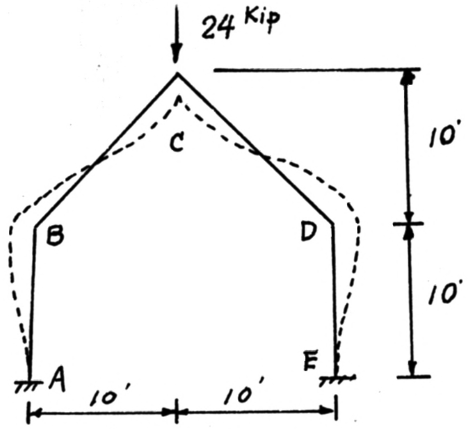


Fig. 9 (a) Gable bent.

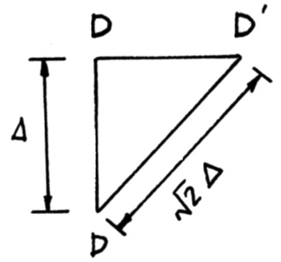


Fig. 9 (b). The displacement of joint, D, relative to joint, C.

In Fig. 9 (b) the relative displacement of D with respect to C is equal to $\sqrt{2}\Delta$.

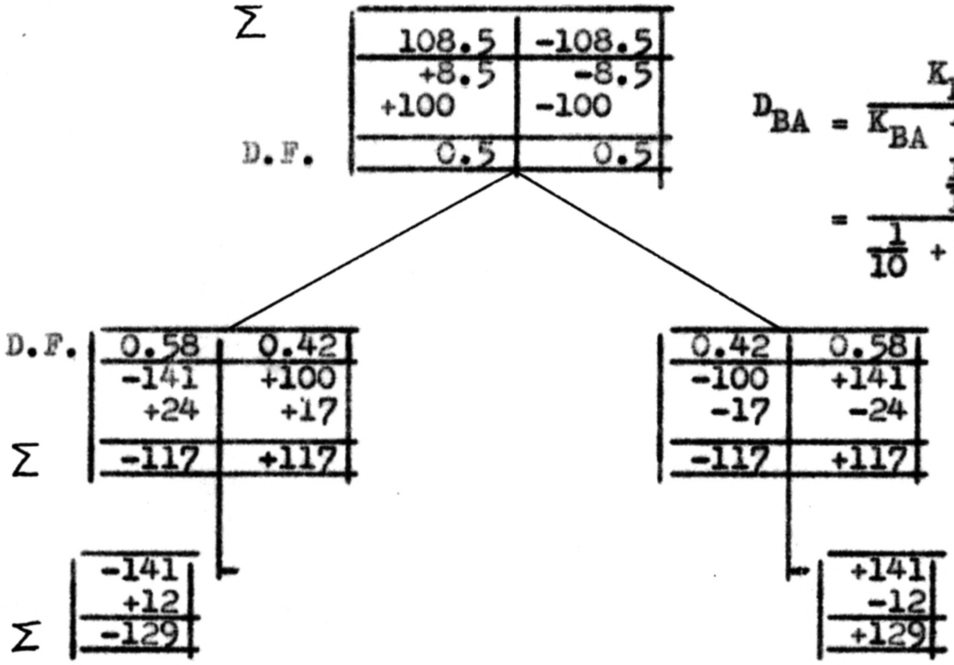
From equation (8c)' the moments due to the lateral displacement in Fig. 9 (a) and Fig. 9 (b) are obtained as follows:

$$M^F_{BA} = M^F_{AB} = 6E\frac{I}{L^2}\Delta = -6EI\frac{1}{10^2}$$

$$M^F_{BC} = M^F_{CB} = 6E\frac{I}{L^2}\sqrt{2}\,\Delta = 6EI\frac{I}{(10\sqrt{2}\,)^2}\sqrt{2}\,\Delta$$

Therefore, $M^F_{BA} : M_{BC}{}^F = -141 : 100$

As before, adjust the value of Δ so that M^F_{BA} and M^F_{AB} are equal to 141 kip-ft and 100 kip-ft respectively. Due to symmetry, the moments at joints D and E are the same as in B and A. Next, distribute and carry-over the moments as shown in Fig. 9 (c).

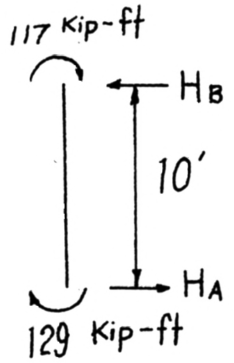


$$D_{BA} = \frac{K_{BA}}{K_{BA} + K_{BC}} = \frac{\frac{1}{10}}{\frac{1}{10} + \frac{1}{10\ 2}} = 0.58$$

$$D_{BC} = 1-0.58=0.42$$

Fig. 9 (c). Moment distribution.

The horizontal component of reaction at A and E is:



$$H_A = \frac{(129 + 117)}{10} = 24.6^{\text{ kips}}$$

Fig. 9 (d). Free body diagram of member BA.

The vertical component of reaction, R, is found in Fig. 9 (e).

$$\Sigma M \text{ at }_C = 0$$

$$R \times 10 - 24.6 \times 20 + 129 - 108.5 = 0$$

$$R = 47^{\text{ kips}}$$

Since symmetry, the vertical component of reaction at E is 47 $^{\text{kips}}$.

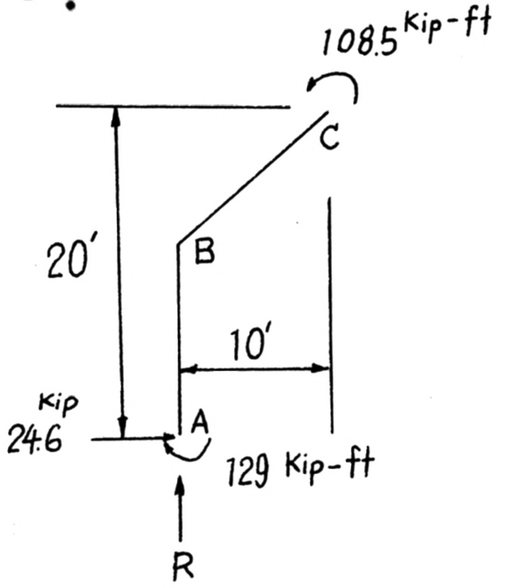


Fig. 9 (e). Free body diagram from joint, C, to joint, A.

The actual external force is 24 kips downward, hence, the obtained moments in Fig. 9 (c) should be multiplied by a factor of $\frac{24}{2x(47)}$ in order that the sum of the vertical reaction at the base is equal to 24 kips.

Thus:

$$M_{BA} = -117 \times \frac{24}{2 \times 47} = -30 \text{ kip-ft}$$

$$M_{AB} = -129 \times \frac{24}{2 \times 47} = -33 \text{ kip-ft}$$

$$M_{CB} = +108.5 \times \frac{24}{2 \times 47} = 28 \text{ kip-ft}$$

The moment diagram is shown in Fig. 9 (f).

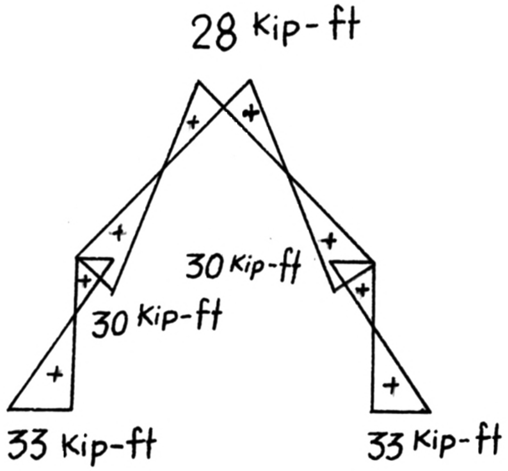


Fig. 9 (f). Moment diagram.

In the case of an unsymmetrical gable bent, as shown in Fib. 10 (a), the displacements are functions of two unknowns. If X and Y denote the relative displacements of the top joint C with respect to the ends of AC and BC, and if it is assumed that the displacements are so small that A'C'=AC and B'C'=BC, then it can be shown that:

$$X = \frac{\Delta_2 \cos\alpha_2 + \Delta_1 \cos\alpha_1 \cos(\alpha_1 + \alpha_2)}{\sin(\ _1 + \ _2)} + \Delta_1 \sin\alpha_1$$

$$Y = \frac{\Delta_1 \cos\alpha_1 + \Delta_2 \cos\alpha_2 \cos(\alpha_1 + \alpha_2)}{\sin(\alpha_1 + \alpha_2)} + \Delta_2 \sin\alpha_2$$

The above equations as shown in Fig. 11 (b) by constructing a Williot diagram.

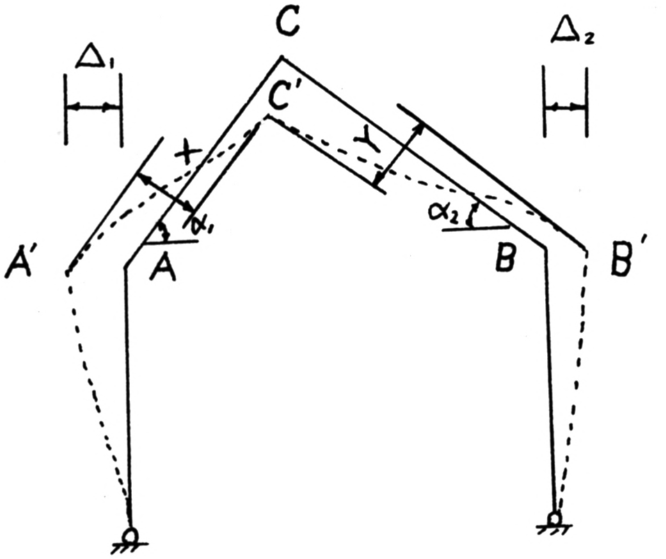


Fig. 10 (a). Consistent displacement of gable bent.

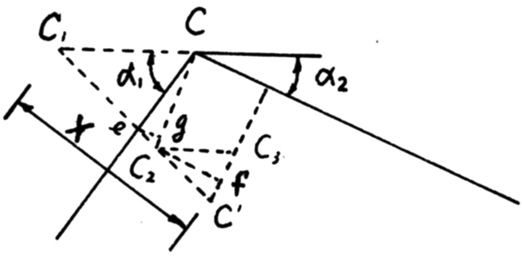


Fig. 10 (b). A Williot diagram showing the displacement of joint C. C' is the final position of C after the deflection of the structure.

From Fig. 10 (b):

$$X = \overline{C_1e} + \overline{eC_2} + \overline{C_2C'}$$

$$\angle C_1\ Ce = \alpha_1$$

$$\angle e\ Cg = 90 - (\alpha_1 + \alpha_2)$$

$$\angle C_3\ C_2f = \alpha_2$$

$$\overline{C_1e} = \Delta_1 \sin \alpha_1$$

$$\overline{eC_2} = \overline{eg} \div \sin g\ (\alpha_1 + \alpha_2) \quad , \quad \overline{eg} = \Delta_1 \cos \alpha_1 \sin\ 90-(\alpha_1+\alpha_2)$$

$$= \Delta_1 \cos \alpha_1 \cos\ (\alpha_1 + \alpha_2)$$

Therefore: $$\overline{eC_2} = \frac{\Delta_1 \cos \alpha_1 \cos (\alpha_1 + \alpha_2)}{\sin (\alpha_1 + \alpha_2)}$$

$$\overline{C_2C'} = \overline{C_2f} \div \sin (\alpha_1 + \alpha_2)$$

$$\overline{C_2f} = \Delta_2 \cos \alpha_2$$

Therefore: $$\overline{C_2C'} = \frac{\Delta_2 \cos \alpha_2}{\sin (\alpha_1 + \alpha_2)}$$

$$X = \overline{C_1e} + \overline{eC_2} + \overline{C_2C'}$$

By substituting:

$$X = \Delta_1 \sin \alpha_1 + \frac{\Delta_1 \cos \alpha_1 \cos (\alpha_1 + \alpha_2)}{\sin (\alpha_1 + \alpha_2)} + \frac{\Delta_2 \cos \alpha_2}{\sin (\alpha_1 + \alpha_2)}$$

$$\text{or } X = \frac{\Delta_2 \cos \alpha_2 + \Delta_1 \cos \alpha_1 \cos (\alpha_1 + \alpha_2)}{\sin (\ _1 +\ _2)} + \Delta_1 \sin \alpha_1$$

Similarly, by replacing Δ_1 and α_1 by Δ_2 and α_2

$$Y = \frac{\Delta_1 \cos \alpha_1 + \Delta_2 \cos \alpha_2 (\cos \alpha_2 + \alpha_1)}{\sin (\ _2 +\ _1)} + \Delta_2 \sin \alpha_2$$

$$\text{or } Y = \frac{\Delta_1 \cos \alpha_1 + \Delta_2 \cos \alpha_2 \cos (\alpha_1 + \alpha_2)}{\sin (\alpha_1 + \alpha_2)} + \Delta_2 \sin \alpha_2$$

A numerical example for the case of an unsymmetrical gable bent is given in Fig. 11 (a).

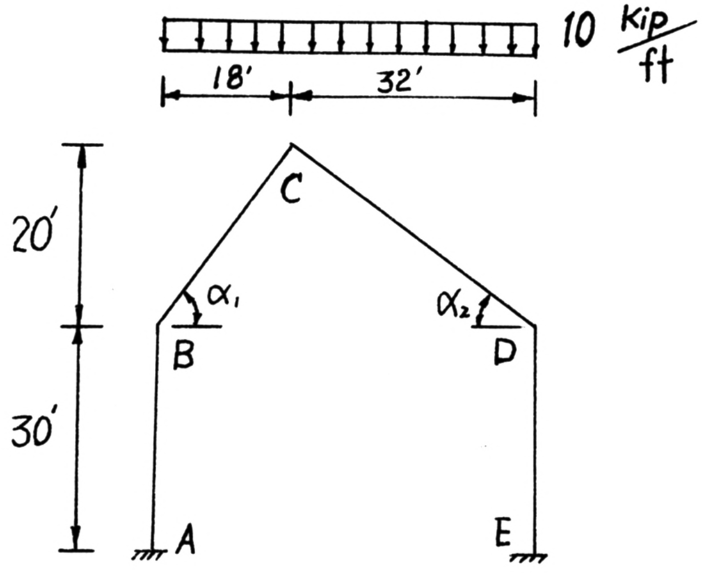


Fig. 11 (a). Unsymmetrical gable bent.

The structure shown in Fig. 11 (a) has a constant EI with α_1 = 53°08' α_2 = 36°52'.

Substituting α_1 and α_2 values in the equations for X and Y:

$$X = \Delta_{BC} = 0.8\ (\Delta_1 + \Delta_2).$$

$$Y = \Delta_{CD} = 0.6\ (\Delta_1 + \Delta_2).$$

From equation (8c)' the lateral moments are:

$$M_{AB} = M_{BA} = 6EI \frac{\Delta 1}{L^2} = 6EI \frac{\Delta 1}{(30)^2}$$

$$M_{BC} = M_{CB} = 6EI \frac{0.8\ (\Delta 1 + \Delta 2)}{(30)^2}$$

$$M_{CD} = M_{DC} = 6EI \frac{0.6\ (\Delta_1 + \Delta_2)}{(40)^2}$$

$$M_{DE} = M_{ED} = 6EI \frac{\Delta 2}{(30)^2}$$

Therefore: $M_{AB} : M_{CD} : M_{DC} : M_{DE} = M_{AB} : 0.8\ (M_{AB} + M_{DE}) :$

$0.34\ (M_{AB} + M_{DE}) : M_{ED}$.

let M_1 denotes M_{AB} and M_2 denotes M_{DE}

Then:

$$M_{AB} : M_{CD} : M_{DC} : M_{DE} = M_1 : 0.8\ (M_1 + M_2) : 0.34\ (M_1 + M_2) : M_2$$

The distribution factors are:

$$D_{CB} = \frac{\frac{1}{30}}{\frac{1}{30} + \frac{1}{40}} = 0.57 \qquad D_{CD} = 0.43$$

$$D_{BC} = \frac{\frac{1}{30}}{\frac{1}{30} + \frac{1}{30}} = 0.5 \qquad D_{BA} = 0.5$$

$$D_{DC} = \frac{\frac{1}{40}}{\frac{1}{30} + \frac{1}{40}} = 0.43 \qquad D_{DE} = 0.57$$

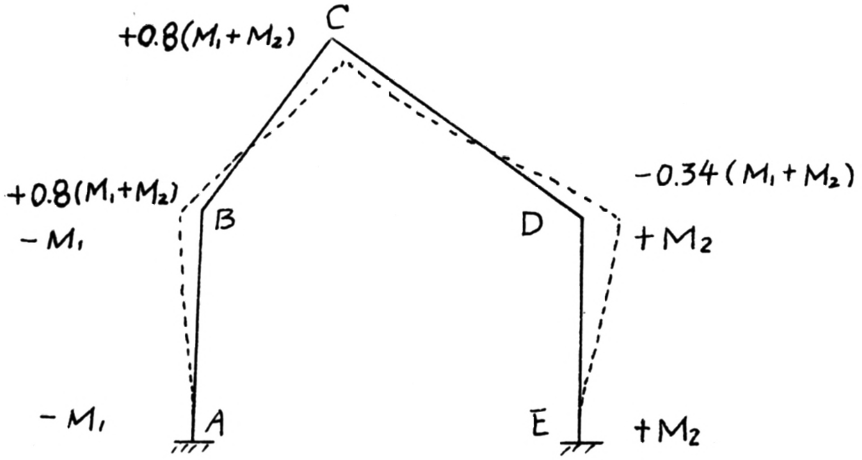


Fig. 11 (b). Deflected structure and fixed-end moments due to joint translation.

Assume the deflected shape of the structure due to lateral moments as in Fig. 11 (b) with the lateral moments as shown.

The moment distribution corresponding to Fig. 11 (b) is shown in Table 2.

Table 1. Moment Distribution

Joint	A	B		C		D		E
D. F.	B 0	A 0.5	C 0.5	B 0.57	D 0.43	C 0.43	E 0.57	D 0
	$-M_1$	$-M_1$	$0.8(M_1+M_2)$	$0.8(M_1+M_2)$	$-0.34(M_1+M_2)$	$-0.34(M_1+M_2)$	M_2	M_2
		$0.1M_1-0.4M_2$	$0.1M_1-0.4M_2$	$-0.26(M_1+M_2)$	$-0.20(M_1+M_2)$	$-0.284M_2$ $+0.143M_1$	$-0.376M_2$ $+0.197M_1$	
			$-0.13(M_1+M_2)$	$+0.05M_1$ $-0.2M_2$	$-0.142M_2$ $+0.071M_1$	$-0.1(M_1+M_2)$		
		$+0.065(M_1+M_2)$	$+0.065(M_1+M_2)$	$+0.195M_2$ $-0.07M_1$	$+0.147M_2$ $-0.051M_1$	$+0.043(M_1+M_2)$	$0.057(M_1+M_2)$	
			$+0.098M_2$ $-0.035M_1$	$+0.0325(M_1+M_2)$	$+0.02(M_1+M_2)$	$+0.073M_2$ $-0.026M_1$		
	$\frac{1}{2}[(1-0.818)M_1$ $-384M_2]$	$-0.049M_2$ $+0.017M_1$	$-0.049M_2$ $+0.018M_1$	$-0.03(M_1+M_2)$	$.0235(M_1+M_2)$	$-0.033M_2$ $+0.011M_1$	$-0.040M_2$ $+0.015M_1$	$-\frac{1}{2}(1.0-0.64)M_2$ $+\frac{1}{2}\,0.269M_1$
Σ	$-0.909M_1$ $-0.192M_2$	$-0.818M_1$ $-0.384M_2$	$+0.818M_1$ $+0.384M_2$	$0.522M_1$ $+0.537M_2$	$-0.522M_1$ $-0.537M_2$	$-0.269M_1$ $-0.64M_2$	$+0.269M_1$ $+0.64M_2$	$+0.134M_1$ $+0.82M_2$

Next, put two fictious forces P_1 and P_2 at B and D to prevent the joints from translating laterally. In this case, the moments due to the uniformly distributed, vertical load are:

$$M^F_{BC} = - M^F_{CB} = \frac{WL^2}{12} = \frac{10 \times (18)^2}{12} = 270$$

$$M^F_{CD} = - M^F_{DC} = WL^2 = \frac{10(32)^2}{12} = 855$$

The balancing of the above moments is shown in Fig. 11 (c).

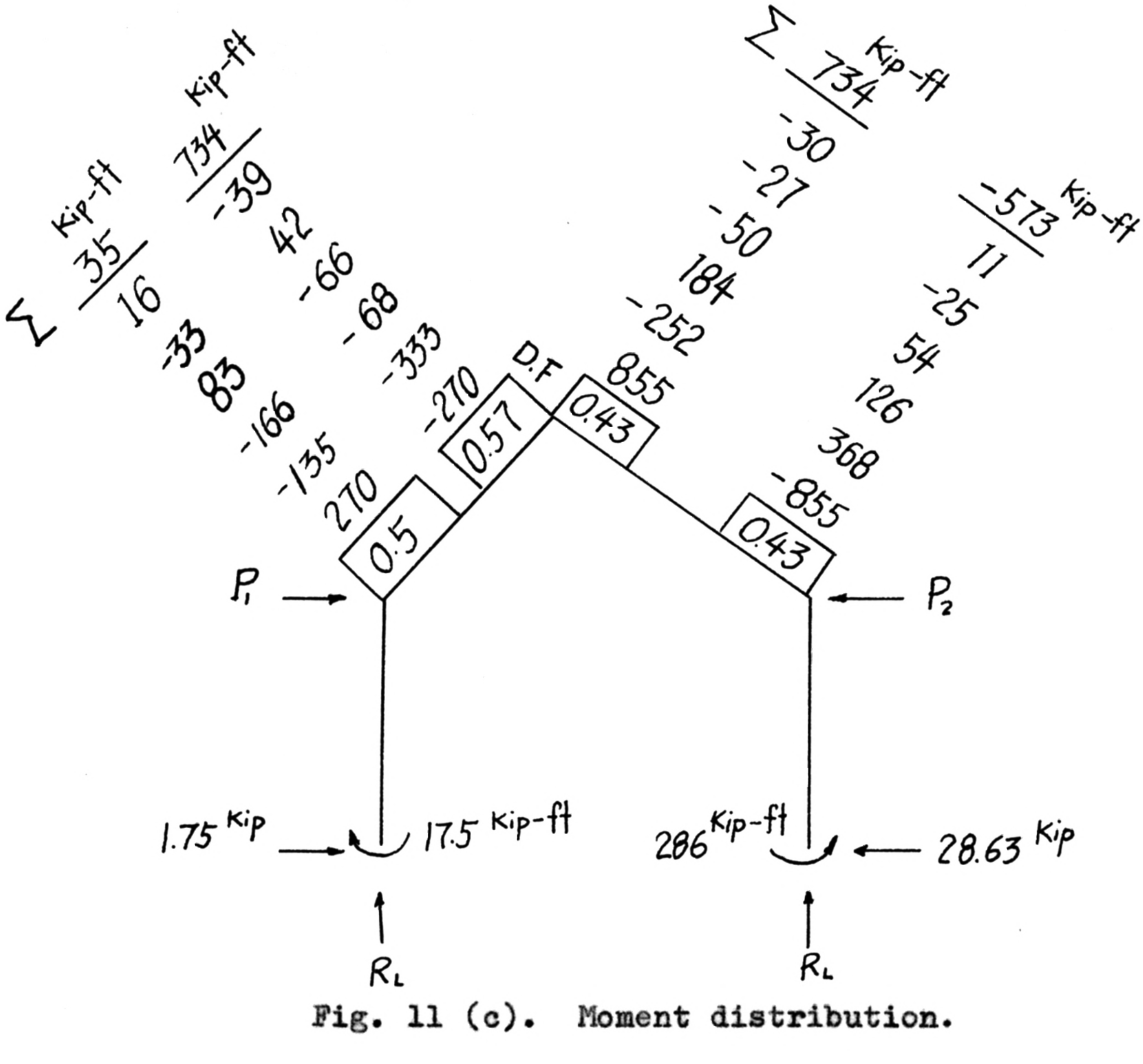


Fig. 11 (c). Moment distribution.

The horizontal reaction at A = $\frac{M_{AB} + M_{BA}}{L} = \frac{17.5 + 35}{30} = 1.75^{\text{kips}}$

The horizontal reaction at E = $\frac{M_{DE} + M_{ED}}{L} = -(\frac{573 + 286}{30}) = -28.63^{\text{kips}}$

Taking D as a center of moment (See Fig. 11 (c).) and from

$\Sigma \underset{at}{M_D} = 0$

$$R_L = \frac{10 \times 50 \times 25 - (28.63 - 1.75) \times 30 + 286 - 17.5}{50} = 239 \text{ kips}$$

$R_R = 500 - 239 = 261$ kips

P_1 is found by taking moment at C (in Fig. 11 (d).)

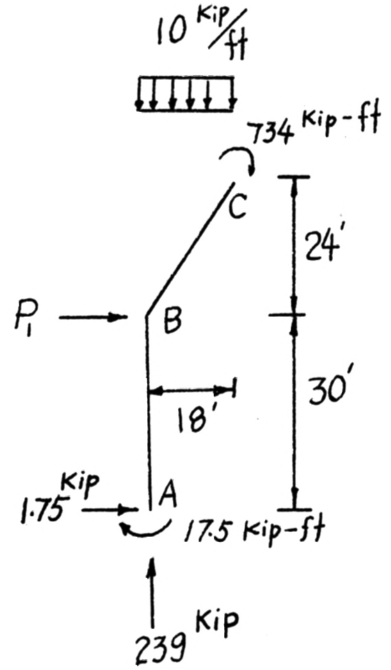


Fig. 11 (d). Free body diagram from joint, C, to joint, A.

$$P_1 = 139.5^{\text{kips}} = \frac{734 - 17.5 \times 54 + 1.75 + 239 \times 18 - 180 \times 9}{24}$$

$P_2 = 139.5 + 1.75 - 28.63 = 112.5$ kips

There remains now the correction for the lateral restraint of points B and D. The consistent fixed-end moments indicated in Fig. 11 (b) are balanced by a series of distributions and carry-overs as shown in Table 1. The restraints at B and D and the vertical reactions at A and E, corresponding to the moments in the last row of Table 2, can be found as shown in Fig. 11 (e) and Fig. 11 (f).

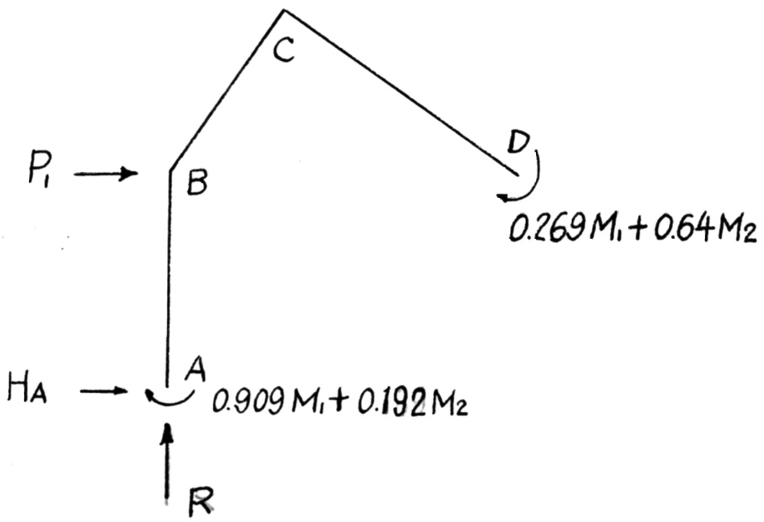


Fig. 11 (e). Free body diagram from joint, D to joint, A.

$$H_A = \frac{M_{BA} + M_{AB}}{L} = \frac{0.909M_1 + 0.192M_2 + 0.818M_1 + 0.384M_2}{30}$$

$$= 0.0576M_1 + 0.0192M_2$$

$$R = \frac{(0.0576M_1 + 0.0192M_2) \times 30 - (0.269M_1 + 0.64M_2) - (0.909M_1 + 0.189M_2)}{50}$$

$$= 0.011M_1 - 0.0051M_2$$

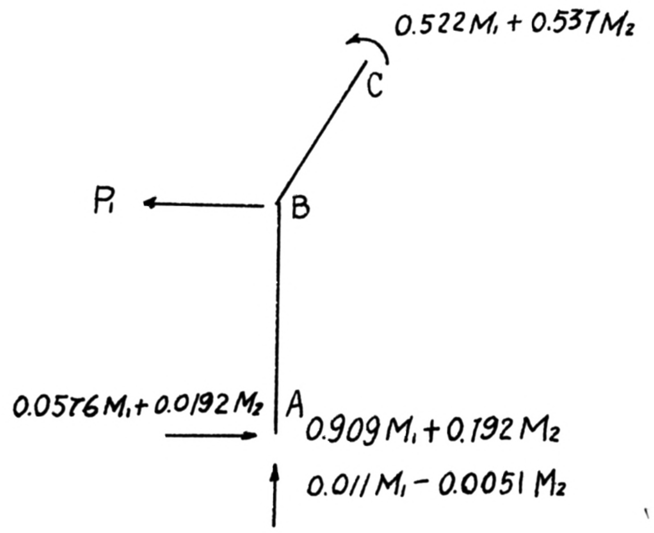


Fig. 11 (f). Free body diagram from joint, C to joint, A.

$$\sum_{at} M_C = 0$$

$$P_1 = \frac{(0.909M_1 + 0.192M_2)-(0.522M_1 + 0.537M_2)-(0.0576M_1+0.0192M_2)x54}{24}$$

$$+ \frac{(0.011M_1 - 0.0051M_2) \times 18}{24}$$

$$P_1 = 0.1052M_1 + 0.0607M_2$$

Similarly, $P_2 = 0.0608M_1 + 0.0897M_2$

Since $P_1 = 139.5$ kips $P_2 = 112.5$ kips, the following two equations are obtained:

$$0.1052M_1 + 0.0607M_2 = 139.5$$

$$0.0608M_2 + 0.0897M_2 = 112.5$$

Then: $M_1 \cong 986$ kip-ft $\quad M_2 = 586$ kip-ft

The final moments are obtained as the sum of the last row of Table 1, substituting M_1 and M_2 the numerical values, and the final moments in Fig. 11 (c).

Thus:

$M_{AB} = -0.909M_1 - 0.192M_2 - 17.5 = -(0.909x986 + 0.192x586) - 17.5$

$= -1026.5$ kip-ft

$M_{BA} = -0.818M_1 - 0.384M_2 - 35 = -818x986 - o.384x586 - 35 =$

$= -1066$ kip-ft $= -M_{BC}$

$M_{CB} = 0.522M_1 + 0.537M_2 - 734 = 96$ kips $= -M_{CD}$

$M_{DE} = 0.269M_1 + 0.64M_2 + 573 = 1213$ kip-ft

$M_{ED} = 0.134M_1 + 0.82M_2 + 286 = 898$ kip-ft

The points of zero moment in the sloping members are found as follows. (See Fig. 11 (g)).

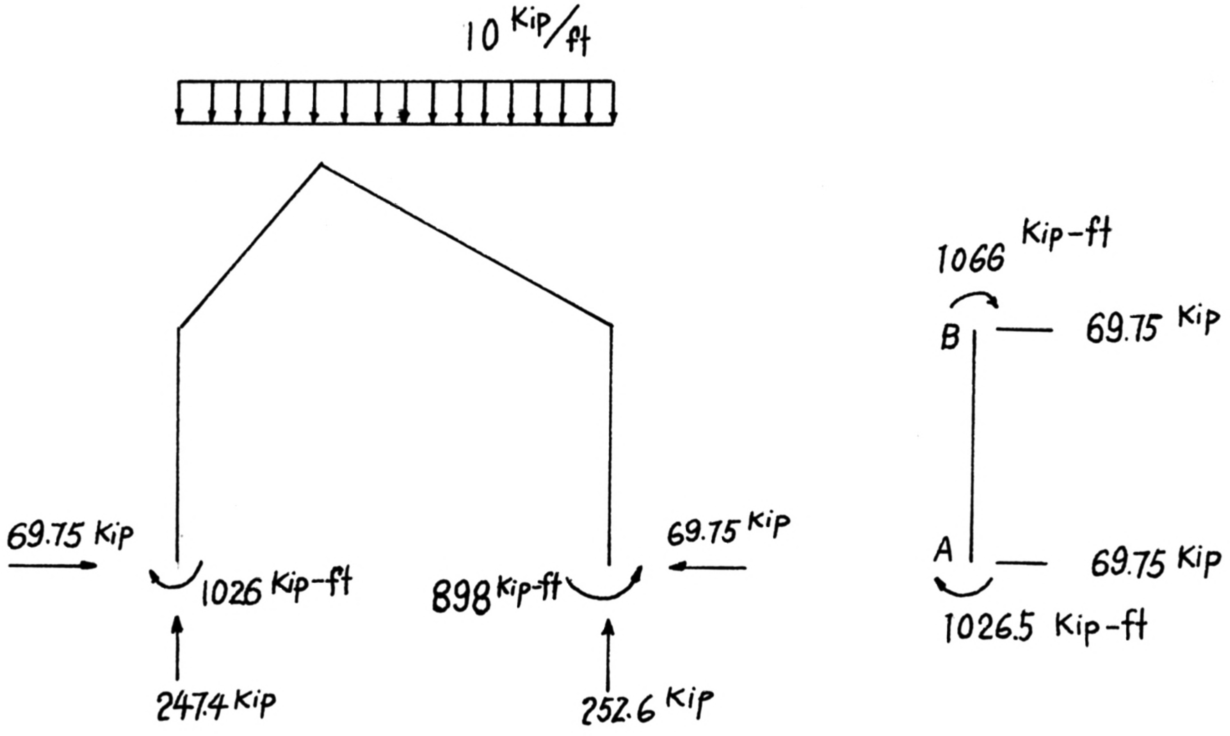


Fig. 11 (g). Free body diagram and the reactions at the base.

$\Sigma_{at} M_E = 0 \quad V_L = \dfrac{50 \times 10 \times 25 + 898 - 1026.5}{50} = 247.4$ kips

let $M_x = 0$ for the member BC

$$M_x = 1026.5 + 247.4x - 69.75\left(30 + x \times \frac{24}{18}\right) - 10 \times \frac{x^2}{2} = 0$$

$$x = \frac{77.2 \pm \sqrt{(77.2)^2 - 5330}}{5} \qquad \text{then} \quad x = 10.4' \text{ from pt. B}$$

let $M = 0$ for the member DE.

$$M = 898 + 252.6x - 69.75\left(30 + x\,\frac{24}{32}\right) - 5x^2 = 0$$

$$M = 5x^2 - 200x + 1194.5 = 0 \qquad x \doteq 7.5' \text{ from D.}$$

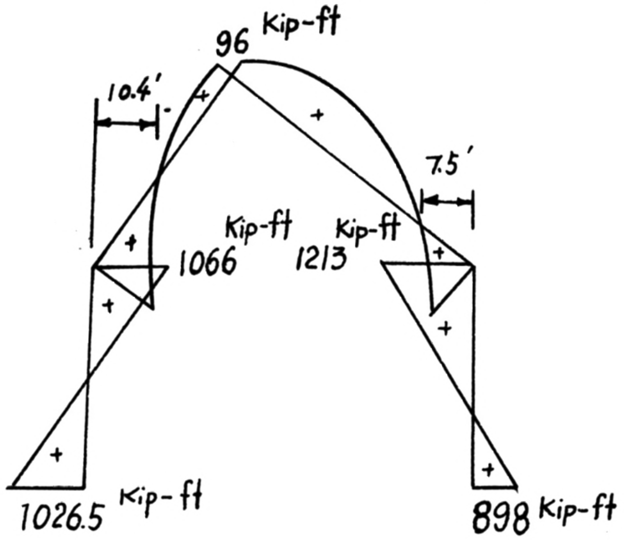
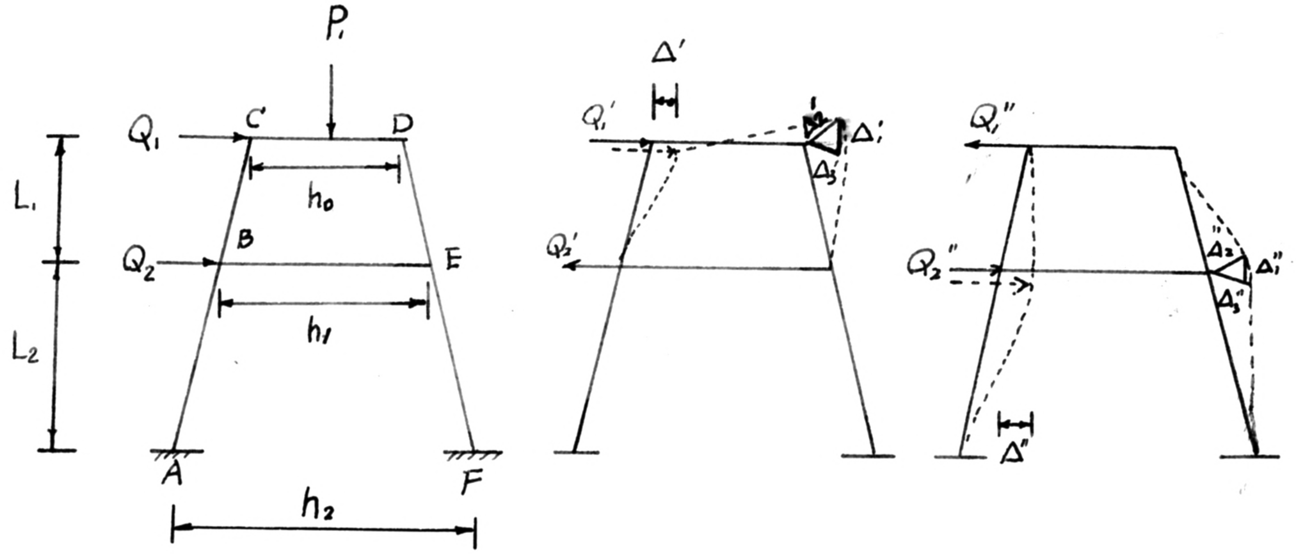


Fig. 11 (h). Moment diagram. Positive moment on the compression side.

MULTIPLE STORY BENT



(a) (b) (c)

Fig. 12. Two-story bent and the auxiliary force systems, Q_1', Q_2', Q_1'', Q_2''.

The bent of Fig. 12 will be analyzed by using auxiliary force systems Q_1', Q_2', Q_1'', Q_2''. The procedures to solve this kind of bent are:

1. Fix all the joints against translation.
2. Compute the required forces for (1).
3. Analyze the counter-forces of (2) and compute their moments.
4. Add all the moments due to (1) and (2).

As in the usual procedure, find Δ_1', Δ_2', and Δ_3' in terms of Δ' in order to solve for the fixed-end moments due to the translation without rotation shown in Fig. 12 (b). Next, express Δ_1'', Δ_2'', and Δ_3'' in terms of Δ'' (Fig. 12 (c)).

From the Williot type construction, Δ_1' and Δ_2' are obtained in terms of Δ'.

$$\Delta 1' = \Delta' (\tan \alpha + \tan \beta) = \Delta' \frac{(h_1 - h_0)}{L_1}$$

$$\Delta 2' = \frac{\Delta'}{\cos B} = \Delta' \frac{L_{DE}}{L_1}$$

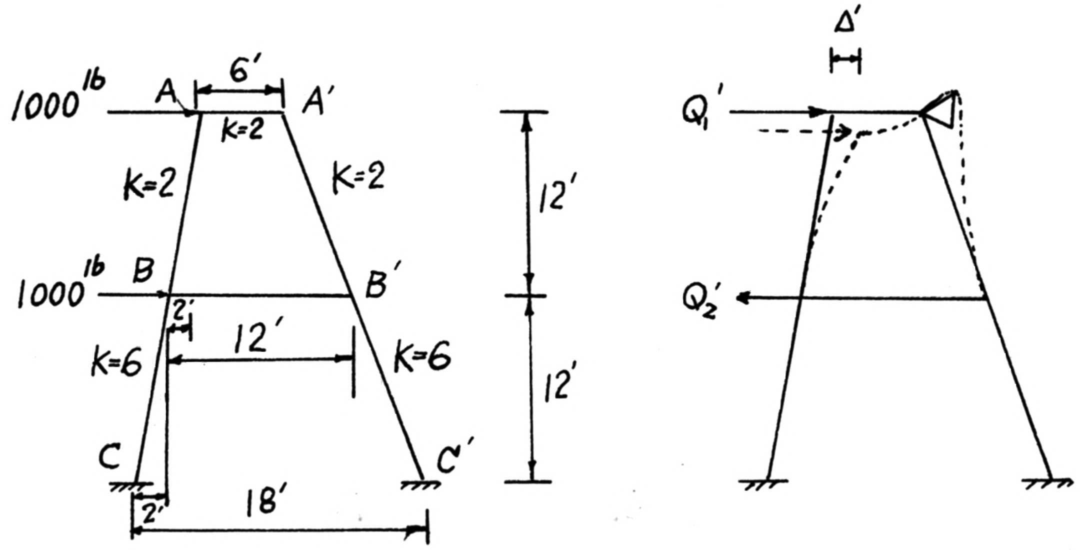
$$\Delta 3' = \frac{\Delta'}{\cos} = \Delta' \frac{L_{BC}}{L_1}$$

$$\Delta 1'' = \Delta'' \frac{(h_2 - h_1)}{L_2}$$

$$\Delta 2'' = \Delta'' \frac{L_{EF}}{L_2}$$

$$\Delta 3'' = \Delta'' \frac{L_{AB}}{L_2}$$

Numerical examples are given as in Fig. 13 (a).



(a) (b)

Fig. 13. Two-story bent with dimentions and auxiliary force system.

Assume $\Delta' = 10$,

$$M_{BA} = M_{AB} = -6E\Delta \frac{I}{L^2} = -6E \frac{\Delta' K}{L} = -(6)(2)\,\frac{10}{12} = -10.0$$

$$M_{B'A'} = M_{A'B'} = -(6)(2)\,(\frac{10}{12}) = -10.0$$

$$M_{AA'} = M_{A'A} = +(6)(2)\,(\frac{10}{12}) = +10.0$$

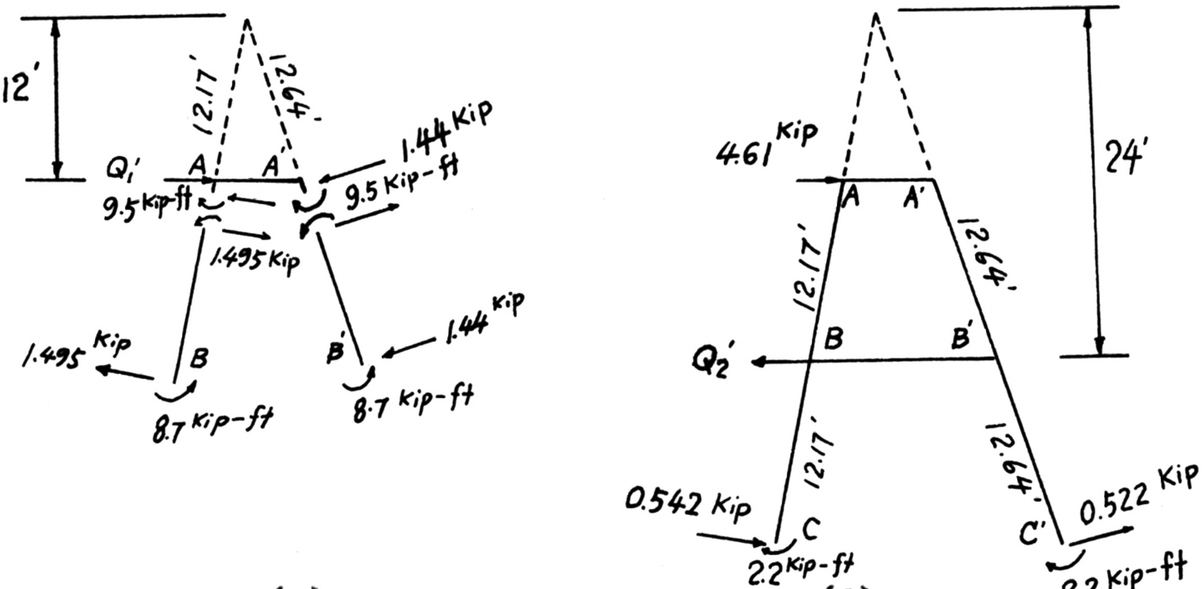
Solving the structure for these initial fixed-end moments, obtain Table 2. The last line in this Table 2 was filled after m had been obtained at the end of the solution.

Now solving Q_1' and Q_2' (See Fig. 13 (c) and (d).

$$12\,Q_1' = 2(9.5) + 1.495\,(12.17) + 1.44\,(12.64)$$

$$Q_1' = 4.61 \text{ kip}$$

Similarly, from Fig. 13 (d), $Q_2' = 3.78$



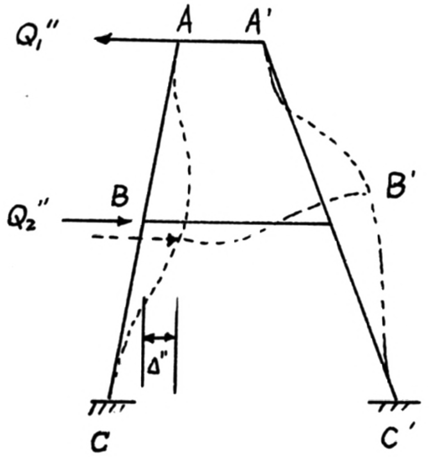
(c) (d)

Fig. 13. Free body diagram.

Table 2.

	C	B			A		A'		B			C'
	B	C	B'	A	B	A'	A	B'	A'	B	C'	B'
D. F.	0	0.5	0.333	0.167	0.50	0.50	0.50	0.50	0.167	0.333	0.50	0
F.E.M.				-10.0	-10.0	+10.0	+10.0	-10.0	-10.0			
	+2.5	+5.0	+3.3	+1.7	+0.8			+0.7		+1.7		+2.1
			+1.4		-0.1	-0.2	-0.4	-0.3	+1.4	+2.8	+4.1	+0.1
	-0.3	-0.7	-0.5	-0.2	-0.2	-0.3	-0.1		-0.1	-0.2		
				-0.1				+0.1		+0.1	+0.2	
		+0.1										
Σ	+2.2	+4.4	+4.3	+8.7	-9.5	+9.5	+9.5	-9.5	-8.7	+4.3	+4.4	+2.2
$\frac{m \times \Sigma}{1000}$	0.75	+1.49	+1.46	-2.95	-3.22	+3.22	+3.22	-3.22	-2.95	+1.46	+1.49	+0.75

Now consider the deflection Δ", see Fig. 13 (e).



Assume $\Delta'' = 10'$

Fig. 13 (e). Deflections of joint B and B' while the other joints are fixed.

$$M_{AB} = M_{BA} = \frac{6\ K\Delta''}{12} = \frac{6\ (2)(10)}{12} = +10$$

$$M_{BC} = M_{CB} = -\frac{6\ (6)(10)}{12} = -30$$

$$M_{A'B'} = M_{B'A'} = \frac{6\ (2)(10)}{12} = +10$$

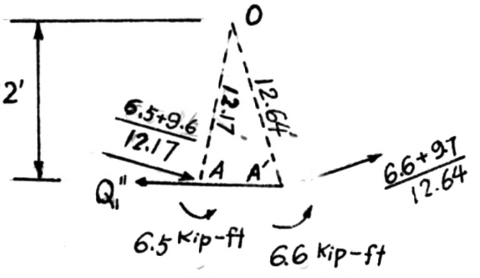
$$M_{B'C'} = M_{C'B'} = -\frac{6\ (6)(10)}{12} = -30$$

$$M_{B'B} = M_{BB'} = \frac{6\ (4)(10)}{24} = +10$$

The list of balancing moments is as follows:

Table 3.

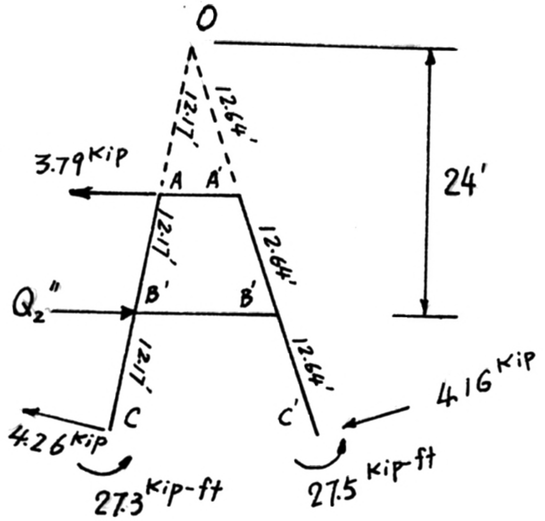
Joints	C	B			A		A'		B'			C'
D. F.	B 0	C 0.50	B' 0.333	A 0.167	B 0.5	A' 0.5	A 0.5	B' 0.5	A' 0.167	B 0.333	C' 0.5	B' 0
	-30	-30	+10	+10	+10	0	0	+10	+10	+10	-30	-30
	+2.5	+5	+3.3	+1.7	+0.8		-2.7		-1.8	+1.7		+2.5
	+0.2		+1.7	-2.7	-5.4	-5.4	-3.6	-3.7	+1.7	+3.4	+5.0	
		+0.5	+0.3	+0.2	+0.1	-1.8	+0.2	+0.8	-0.2	+0.1	+0.1	
				+0.4	+0.9	+0.8	-0.5	-0.5				
						-0.2						
					+0.1	+0.1						
Σ	-27.3	-24.5	+15.3	+9.6	+6.5	-6.5	-6.6	+6.6	+9.7	+15.2	-24.9	-27.5
$\frac{n \times \Sigma}{1000}$	-4.06	-3.65	+2.28	+1.43	+0.97	-0.97	-0.98	+0.98	+1.45	+2.26	-3.70	-4.10



Now, solving for Q_1'' and Q_2''

$$Q_1'' = \frac{6.5+6.6+16.1+16.3}{12} = 3.79^{\text{kip}}$$

Fig. 13 (f). Free body diagram for member AA'.



Take moment at O.

$\Sigma M = 0$

Then, $Q_2'' = 15.34$ kip

Fig. 13 (g). Free body diagram above the member, BB'.

To determine n and m:

$$\begin{aligned} mQ_1' + nQ_1'' &= 1000 \\ mQ_2' + nQ_2'' &= 1000 \end{aligned} \quad \text{or} \quad \begin{aligned} 4.61m + 3.79n &= 1000 \\ 3.78m + 15.34n &= 1000 \end{aligned}$$

which give

$$m = +339$$

$$n = +148.8$$

The final joint moments are obtained by using the values in the last rows of Tables 2 and 3. (Note: All values are in kip-ft.)

M_{CB} = +0.75 - 4.06 = -3.31

M_{BC} = +1.49 - 3.65 = -2.16

$M_{BB'}$ = +1.46 + 2.28 = +3.74

M_{BA} = -2.95 + 1.43 = -1.52

M_{AB} = -3.22 + 0.97 = -2.25

$M_{AA'}$ = +3.22 - 0.97 = +2.25

$M_{A'A}$ = +3.22 - 0.98 = +2.24

$M_{A'B'}$ = -3.22 + 0.98 = -2.24

$M_{B'A'}$ = -2.94 + 1.45 = -1.50

$M_{B'B}$ = +2.26 + 1.46 = +3.72

$M_{B'C'}$ = +1.49 - 3.70 = -2.21

$M_{C'B'}$ = +0.75 -4.10 = -3.35

TWO-STAGE FRAMES

To determine the moments for the frame of Fig. 14 (a).

1. Prevent translation of points C and F by applying holding forces Q_1 and Q_2; find fixed-end moments. Solve by using moment distribution. Check the shear equation to find the holding forces Q_1 and Q_2. (See Fig. 14 (b).)

2. Introduce sidesway Δ_1 (Fig. 14 (c)), determine the corresponding fixed-end moment and distribute moments. Find Q_1' and Q_2'' from the equilibrium equations.

3. Introduce sidesway Δ_2. Find the corresponding fixed-end moments, distribute the moments, and determine Q_1'' and Q_2''.

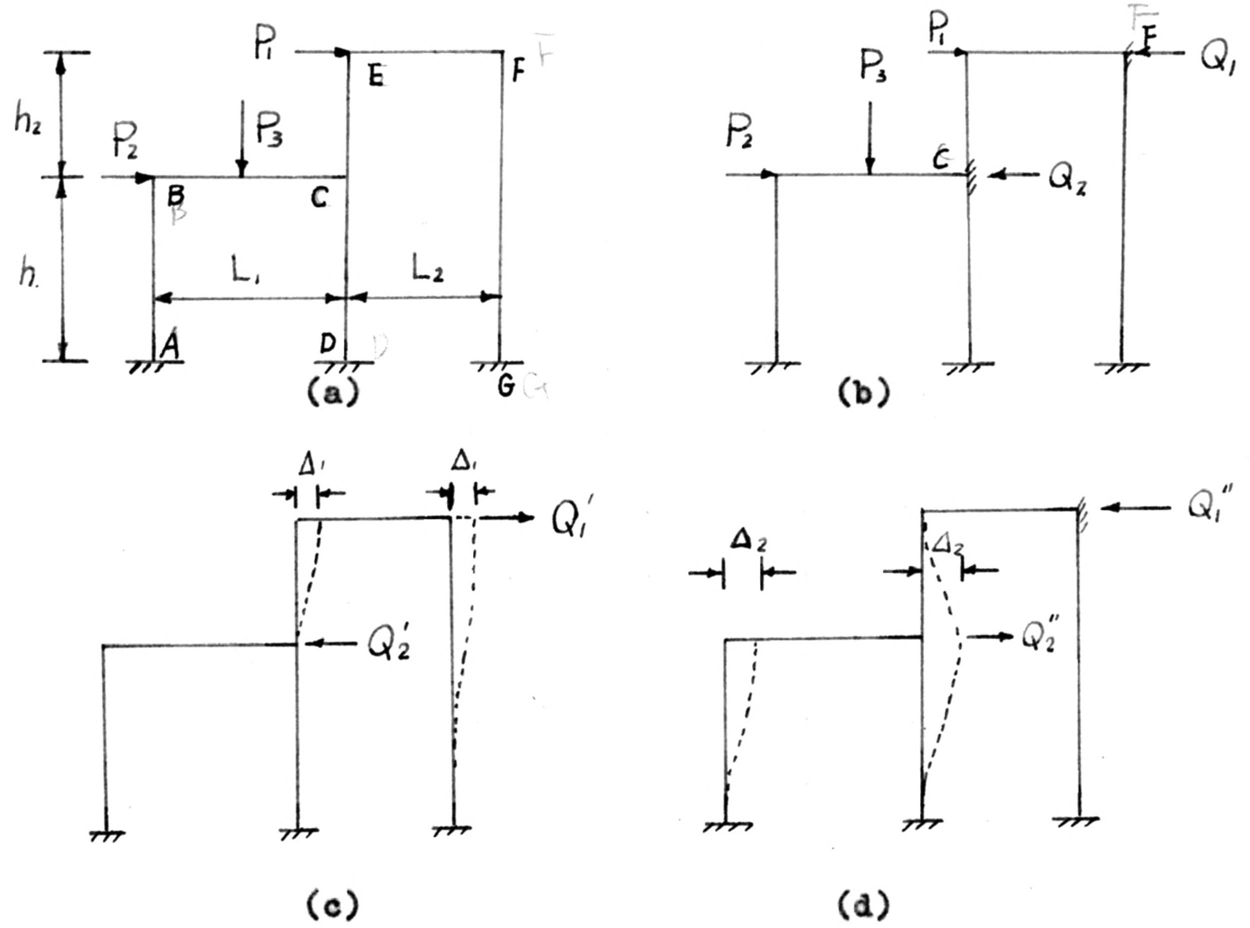


Fig. 14. Two-stage frame with the auxiliary force systems.

4. Introduce factors m and n such that

$$mQ_1' + nQ_1'' + Q_1 = 0$$
$$mQ_2' + nQ_2'' + Q_2 = 0$$

Knowing m and n, obtain the final moments by adding to the moments due to step 1, m times the mmments due to step 2, and n times the moments due to step 3.

Numerical example: Find the joint moments for the two-stage frame shown in Fig. 15 (a).

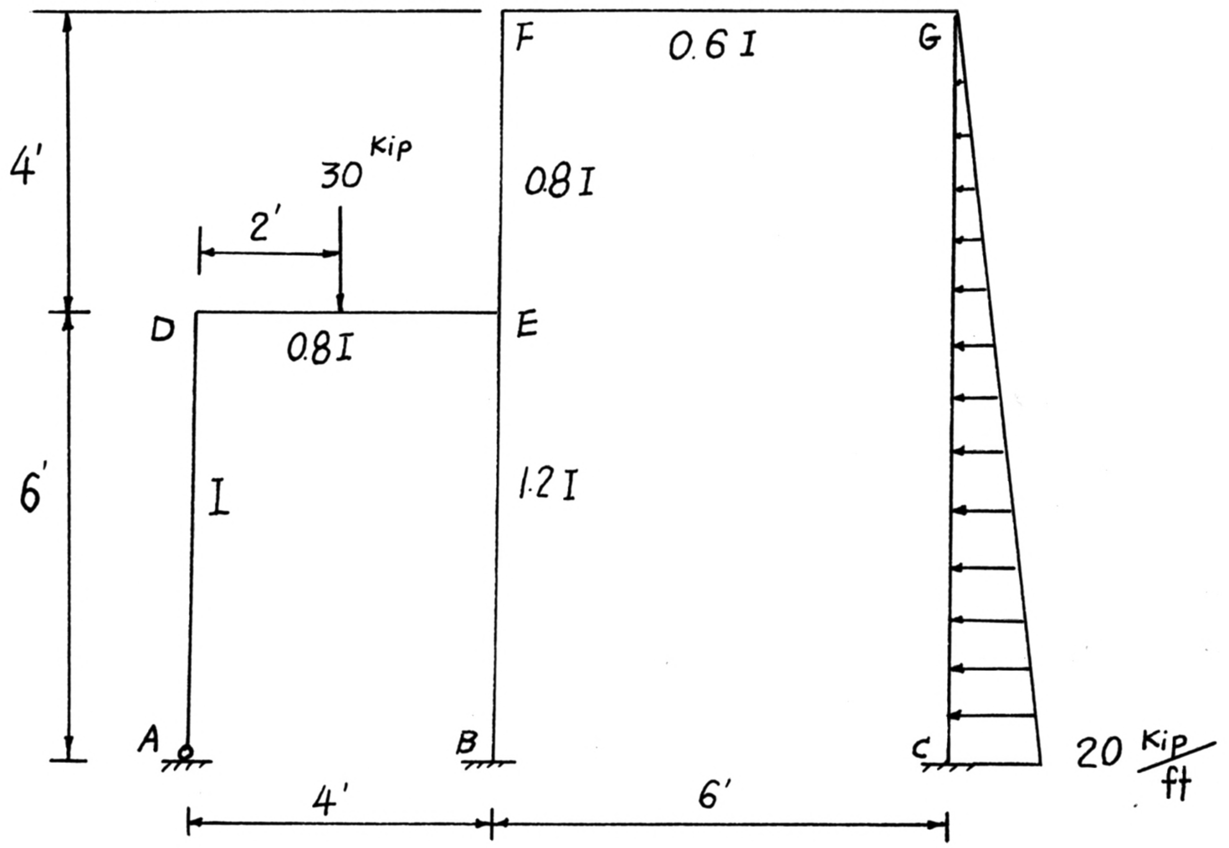


Fig. 15 (a). Two-stage frame with dimensions.

Step 1. Compute the fixed-end moments and distribute them as in Table 4.

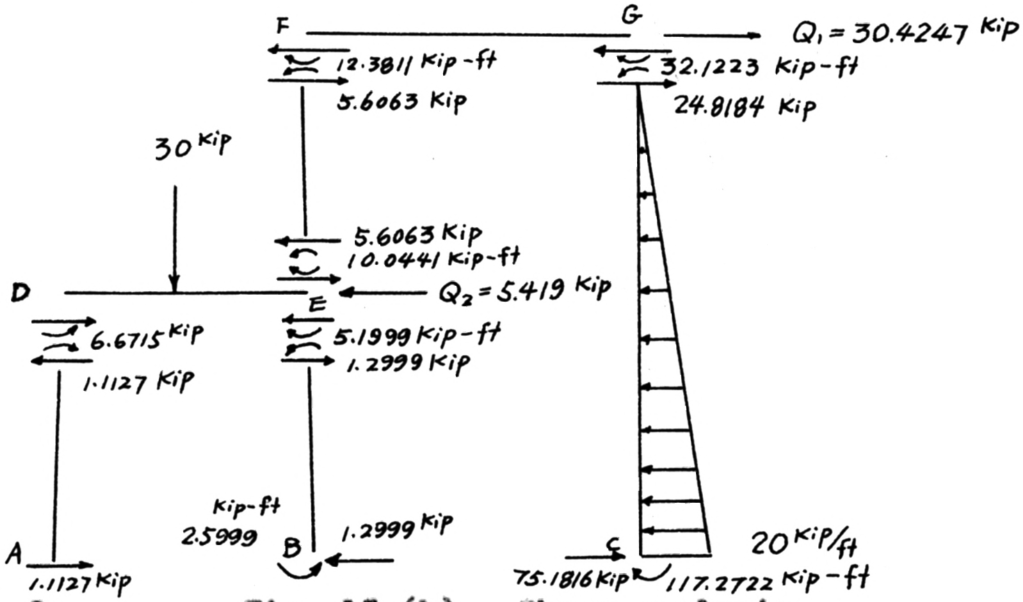
Table 4. Moment distribution.

	D		E			F		G	
	A	E	D	B	F	E	G	F	C
D.F.	0.384615	0.615385	1/3	1/3	1/3	2/3	1/3	1/2	1/2
		+15	-15						+66.66667
	-5.769225	-9.230775	+5	+5	+5			-33.33333	-33.33333
		+2.5	-4.61538			+2.5	-16.66667		
	-0.961537	-1.538463	+1.53846	+1.53846	+1.53846	+9.44444	+4.72222		
		+0.76923	-0.76923		+4.72222	+0.76923		+2.36111	
	-0.29586	-0.473373	-1.31766	-1.31766	-1.31766	-0.51282	-0.25641	-1.18055	-1.18055
		-0.65883	-0.23669		-0.25641	-0.65883	-0.59028	-0.12821	
	+0.25340	+0.40543	+0.16437	+0.16437	+0.16437	+0.83274	+0.41637	+0.06411	+0.06411
		+0.08219	+0.20272		+0.41637	+0.08219	+0.032055	+0.20819	
	-0.031610	-0.05058	-0.20636	-0.20636	-0.20636	-0.07616	-0.03808	-0.10409	-0.10409
		-0.10318	-0.02529		-0.03808	-0.10318	-0.05205	-0.01904	
	+0.03968	+0.0635	+0.02112	+0.02112	+0.02112	+0.10349	+0.05174	+0.00952	+0.00952
Σ	-6.67652	+6.67652	+15.24349	+5.19993	+10.0441	+12.3811	-12.3811	-32.12232	+32.12232

$M_{CG} = -100 - \frac{1}{2}(66.66667 - 32.12232) = -117.2722$

$M_{BE} = \frac{1}{2} M_{EB} = + 5.19993 \times \frac{1}{2} = +2.59996$

Q_1 and Q_2 are found as shown in Fig. 15 (b).



Step 2. Fig. 15 (b). Shear analysis.

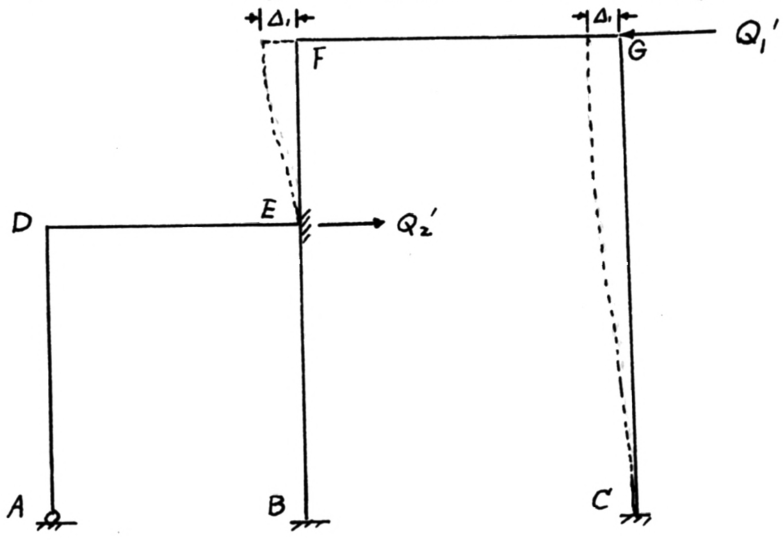


Fig. 15 (c). Assume joints F and G deflect.

Assume sidesway of F and G to be Δ_1 as shown in Fig. 15 (c). The fixed-end moments and moments distribution are listed in Table 5.

Table 5. Moment distribution.

	D		E			F		G	
	A	E	D	B	F	E	G	F	C
D.F.	0.384615	0.615385	1/3	1/3	1/3	2/3	1/3	1/2	1/2
					-1000	-1000			-200
			+333.33333	+333.33333	+333.33333	+666.66667	+333.33333	+100	+100
		+166.66667			+333.33333	+166.66667	+50.0000	+166.66667	
	-64.1025	-102.56417	-111.11111	-111.11111	-111.11111	-144.44444	-72.22222	-83.33333	-83.33333
		-55.55555	-51.28029		-72.22222	-55.55555	-46.16667	-36.11111	
	+21.3675	+34.18805	+41.1681	+41.1681	+41.1681	+68.14814	+34.07407	+18.05555	+18.05555
		+20.58405	+17.09403		+34.07407	+20.58405	+9.027778	+17.03704	
	-7.91693	-12.66712	-17.05603	-17.05603	-17.05603	-19.74122	-9.87061	-8.51852	-8.51852
		-8.52802	-6.33356		-9.87061	-8.52802	-4.25926	-4.93531	
	+3.28000	+5.24802	+5.40139	+5.40139	+5.40139	+8.52486	+4.26243	+2.46766	+2.46766
		+2.70069	+2.62401		+4.26243	+2.70069	+1.23383	+2.13121	
	-1.03873	-1.66196	-2.29548	-2.29548	-2.29548	-2.62301	-1.31151	-1.06561	-1.06561
		-1.14774	-0.83098		-1.31151	-1.14774	-0.53281	-0.65576	
	+0.44144	+0.70630	+0.71416	+0.71416	+0.71416	+1.12036	+0.56018	+0.32788	+0.32788
		+0.35708	+0.35315		+0.56018	+0.35708	+0.16394	+0.28009	
	-0.13734	-0.21974	-0.30443	-0.30443	-0.30443	-0.34735	-0.17367	-0.14004	-0.14004
Σ	-48.10656	+48.10656	+211.47449	+249.84993	-461.3244	-297.61882	+297.61882	+172.20642	-172.20642
m×Σ	-12.93729	+12.93729	+56.87179	+67.19208	-124.06386	-80.03856	+80.03856	+46.31143	-46.31143

$M_{CG} = -200 + \frac{1}{2}(200-172.20642) \times 0.26892977 = -50.04869$

Q_1' and Q_2' are obtained as shown in Fig. 15 (d).

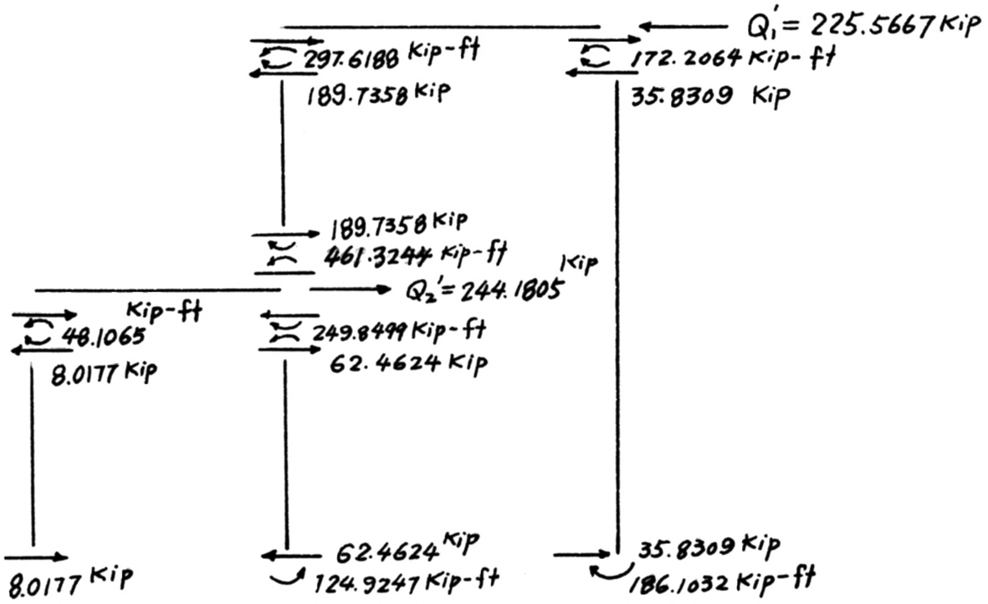


Fig. 16 (d) Shear analysis

Step 3. Assume sidesway of the joints D and E to be Δ_2 as shown in the Fig. 15 (e). The fixed-end moments and the moments distributed are listed in the Table 7.

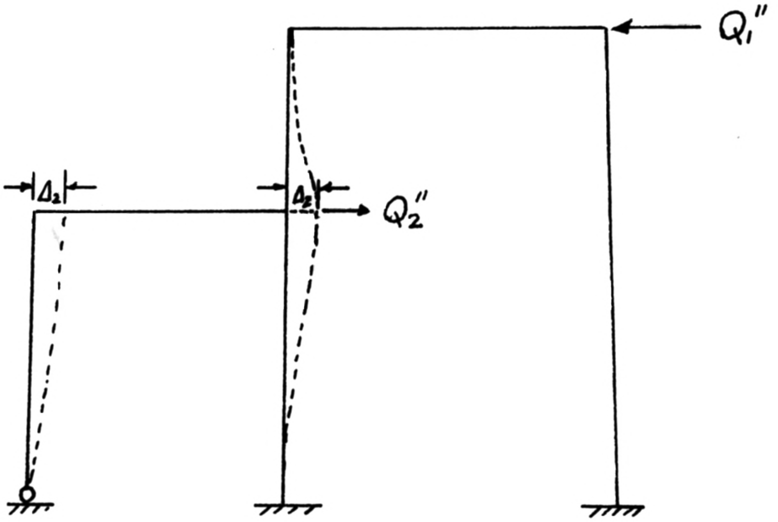


Fig. 15 (e). Assume joints D and E deflect laterally.

Table 6. Moment distribution.

	D		E			F		G	
	A	E	D	B	F	E	G	F	C
D. F.	0.384615	0.615385	1/3	1/3	1/3	2/3	1/3	1/2	1/2
	+1.000			+2.400	-3600	-3.600			
	-384.165	-615.385	+400	+400	+400	+2400	+1200		
		+200.00	-307.6925		+1200	+200		+600	
	-76.923	-123.077	-297.4358	-297.4358	-297.4358	-133.3333	-66.66667	-300	-300
		-148.7179	-61.5385		-66.66667	-148.7178	-150	-33.33333	
	+57.19914	+91.51876	+42.73506	+42.73506	+42.73506	+199.1452	+99.5726	+16.66667	+16.66667
		+21.36753	+45.75938		+99.5726	+21.36753	+8.33333	+49.7863	
	-8.21827	-13.14926	-48.44399	-48.44399	-48.44399	-19.80058	-9.90029	-24.8932	-24.8932
		-24.22199	-6.57463		-9.90029	-24.22199	-12.4466	-4.95014	
	+9.31614	+14.90585	+5.49164	+5.49164	+5.49164	+24.44572	+12.22286	+2.47507	+2.47507
		+2.74582	+7.45292		+12.22286	+2.74582	+1.23753	+6.11143	
	-1.05608	-1.68974	-6.55859	-6.55859	-6.55859	-2.65556	-1.32778	-3.05571	-3.05571
		-3.27929	-0.84487		-1.32778	-3.27929	-1.52785	-0.66389	
	+1.26126	+2.01803	+0.72422	+0.72422	+0.72422	+3.20476	+1.60238	+0.33194	+0.33194
Σ	+596.96419	-596.96419	-226.92566	+2496.51254	-2269.58674	-1081.0995	+1081.0995	+313.42537	-313.42537
$n \times \Sigma$	-20.42028	+20.42028	+7.76242	-85.39788	+77.63546	+36.98103	-36.98103	-10.7213	+10.7213

Q_1'' and Q_2'' are found as shown in Fig. 15 (f).

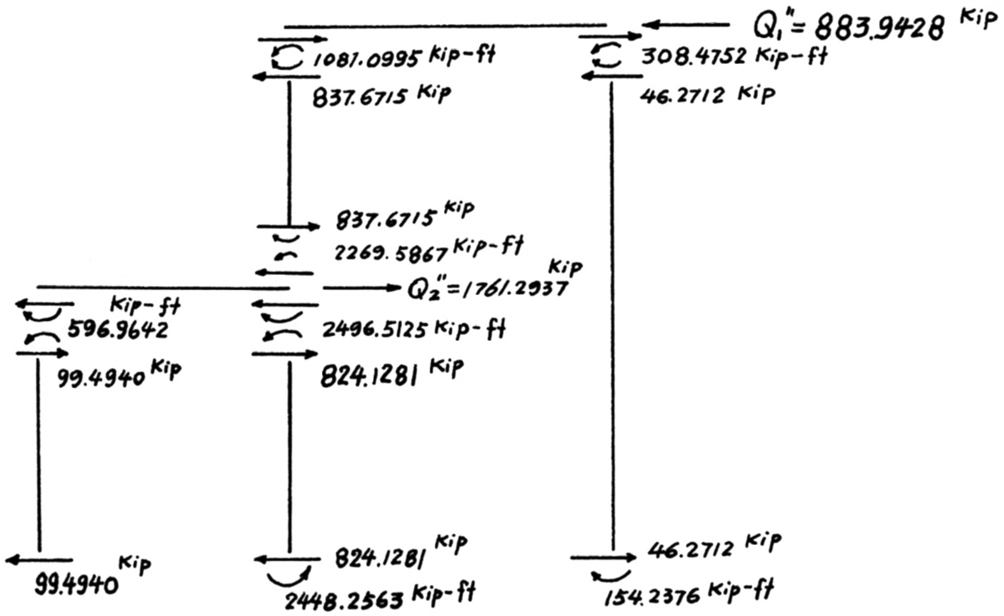


Fig. 15 (f). Shear analysis.

$M_{CG} = \frac{1}{2} M_{GC} \qquad M_{BE} = 2400 + \frac{1}{2}(2496.51254-2400) = 2448.25627$

$n\, M_{BE} = 2448.25627 \times (-0.03420687) = -83.747184$

The following equations are obtained according to the figures from 15 (b) to 15 (f).

$$225.56676m + 883.94284n = 30.4247$$

$$244.18053m + 1761.2937n = 5.41907$$

$$m = 0.268929$$

$$n = -0.03420687$$

The correction moments are listed in the last row of table 5 and 6.

The final joints moments are now given by:

$M_{AD} = 0$

$M_{DA} = -6.67652 - 12.93729 - 20.42028 = -40.0341$

$M_{DE} = +6.67652 + 12.93729 + 20.42028 = +40.0341$

$M_{FD} = -15.24349 + 56.87179 + 7.76242 = +49.39072$

$M_{EB} = +5.19993 + 67.19208 - 85.39788 = -13.00587$

$M_{EF} = +10.0441 - 124.06386 + 77.63546 = -36.38$

$M_{FE} = +12.3811 - 80.03856 + 36.98103 = 30.6764$

$M_{EF} = +30.0764$

$M_{GF} = -32.12232 + 46.31143 - 10.7213 = +3.4677$

$M_{GC} = -3.6371$

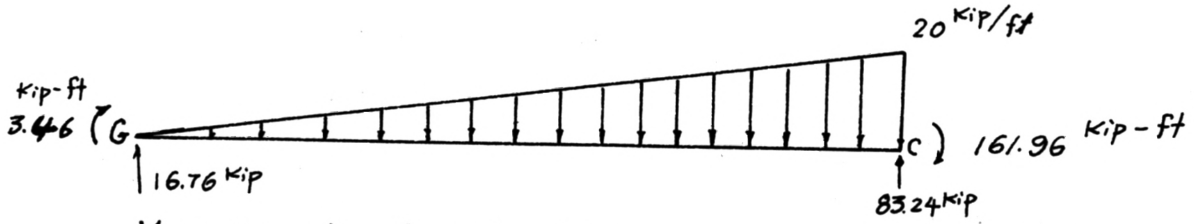
$M_{CG} = -117.2722 - 50.04869 + \frac{1}{2} x 10.7213 = -161.960$

$M_{BE} = +2.59996 + \frac{67.19208}{2} - 83.74718 = -47.55118$

Moment at the point of 30^K load

$= + \frac{30}{2} + \frac{(49.39072 + 40.0341)}{4} \times 2 - 40.0341$

$= +38.6783^K$



M in any section of Member GC is

$M = 16.76\,x + 3.46 - \frac{x^3}{3}$. Let $\frac{dM}{dx} = 0$. then at $x = 4.1'$ $M_{max} = 52.76$ Kip-ft and at $x \doteqdot 7.2'$ $M = 0$

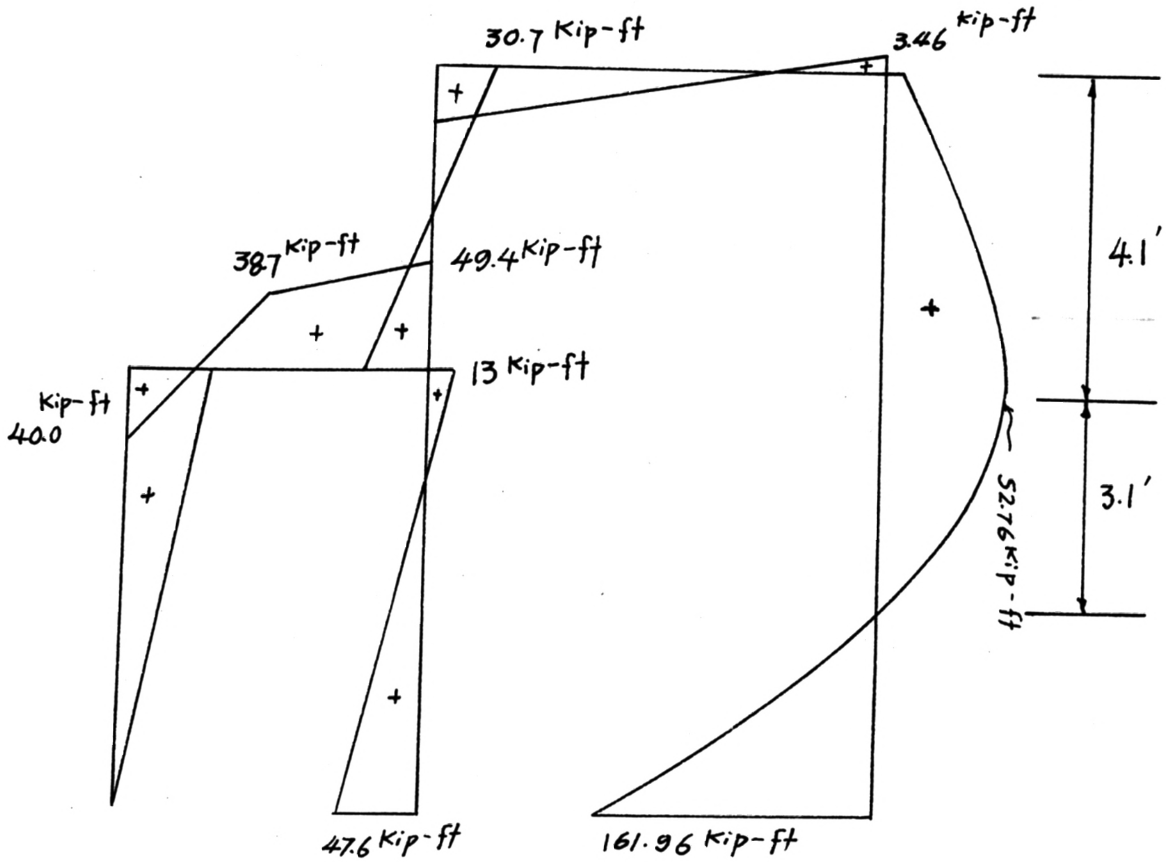


Fig. 15 (g). Moment diagram.

CONCLUSION

The moment distribution method is a method of successive corrections and is either approximate or exact to any desired degree depending on the number of corrections made in the computations.

Problems involving joint translations are usually solved most readily by the indirect method, by which moments throughout the frame, consistent with its dimensions and the make-up of its members, are first determined. In this, the relative value of the sidesway for each member is found by the construction of a Williot type diagram.

In addition to the moment distribution method, there are many other methods, such as the slope-deflection method, energy method, and virtual work method, but the moment distribution method is one of the most convenient toools to solve the statically indeterminate structural problems.

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ADVANCED STUDY ON THE MOMENT DISTRIBUTION METHOD
AND ITS
APPLICATION TO SIDESWAY PROBLEMS

by

CHENG-LIANG WANG

B. S., National Taiwan University, 1954

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The report is mainly concerned with the study of moments in the structural members which compose structural buildings, i.e., bridges and buildings. The structural members carry loads which may be moving live loads or dead loads, or may be both of them. But from the practical point of view, most live loads in civil engineering are converted into the equivalent dead loads so that the analysis of stresses and strain is much simplified.

The structures treated in this report are mostly statically indeterminate, i.e., their change of strain and stress cannot be found by the usual static method. In other words, there are more than three unknowns in the problem where there are only three static equations, i.e., the equilibrium equations of summation of moments, of horizontal components of forces and vertical components of forces, whereas, the three equations are not sufficient for the unknowns which are more than three. Hence, the deflection of the structure comes into consideration. By constructing a Williot diagram, the approximate deflections of the structural members are found. Then the fixed-end moments, corresponding to the deflections obtained from the Williot diagram, are established.

To the study of the structural analysis, the establishment of the moment diagram is pre-requisite. There are many methods for the solution of the moment diagram. Among them, the moment distribution method is one of the most convenient, especially in the region where no computor is available.

After the general procedure of moment distribution method was made clear, the writer illustrates a few problems of typical frames with sidesway. They are simple bent, gable frame, two-story and two-stage frames. With the help of setting auxiliary force systems, the moment distribution method shows its easiness and convenience in solving two-story and two-stage frames.