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COLLOCATION METHOD FOR SOLVING SOME INTEGRAL EQUATIONS OF ESTIMATION THEORY

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Abstract: A class of integral equations $Rh = f$ basic in estimation theory is introduced. The description of the range of the operator R is given. The operator R is a positive rational function of a selfadjoint elliptic operator $\mathcal{L}$. This operator is defined in the whole space $\mathbb{R}^r$, it has a kernel $R(x, y)$, and $Rh := \int_D R(x, y)h(y)dy$, where $D \subset \mathbb{R}^r$ is a bounded domain with a sufficiently smooth boundary S. Example of the equation of this type is $\int_{-1}^{1} e^{-|x-y|}h(y)dy = f(x), \quad -1 \le x \le 1$. This equation has, in general, only distributional solutions. In estimation theory one is interested in the MOS (minimal order of singularity) solution to equation $Rh = f$. It is proved that such solution does exist and is unique for the class of equations defined by the author. A collocation method for numerical solution of equation $Rh = f$ in distributions is formulated and its convergence is proved.

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1. Introduction

Optimal statistical estimation of a random process or field $s(x)$ from the observation of noisy field $u(x) = s(x) + n(x)$, where $n(x)$ is noise, x is in a given

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domain $D \subset \mathbb{R}^r$, and the optimality criterion is minimum of the variance of the error of the estimate, leads to the basic integral equation of estimation theory (see [4] for details):

$$\int_D R(x,y)h(y,z)dy = f(x,z), \quad x,z \in \overline{D} = D \cup S, \tag{1}$$

where $\overline{D}$ is the closure of D,

$$R(x,y) = \overline{u^*(x)u(y)}, \ f(x,y) = \overline{u^*(x)s(y)}, \ \overline{s(x)} = \overline{n(x)} = 0, \tag{2}$$

the bar stands for mean value, the star stands for complex conjugate, S is the boundary of D which is assumed sufficiently smooth.

We make no assumptions about the distribution law of $s(x)$ and $n(x)$: neither Gaussian nor Markovian properties are assumed. Our estimation theory is based entirely on the knowledge of two first moments: the mean values of $n(x)$ and $s(x)$ and the covariance functions $R(x,y)$ and $f(x,y)$. The optimal estimate of $s(x)$ is

$$Lu := \int_D h(x,y)u(y)dy, \tag{3}$$

where $h(x,y)$ may be a distribution. The optimality of estimate (3) means

$$\epsilon(x) := \overline{|(Lu)(x) - s(x)|^2} = \min, \tag{4}$$

where the overline denotes mean value (mathematical expectation), and the minimization is taken over all linear estimates (3), i.e., over all filters (kernels) $h(x,y)$, including distributional kernels. Equation (1) is a necessary condition for h to minimize the error functional (4). Since z enters in (1) as a parameter, it is sufficient to study the equation

$$Rh := \int_D R(x,y)h(y)dy = f(x), \quad x \in \overline{D}. \tag{5}$$

Equation (5) may have many solutions. However, if $R(x,y)$ belongs to the class $\mathcal{R}$ of random fields (and processes) introduced and studied in [4], then equation (5) has only one solution of *minimal order of singularity (MOS solution).* The unique MOS solution to equation (1) is the unique solution to the estimation problem (4): any other solution to equation (1) gives infinite value to the error of the estimate (see [4]). The class $\mathcal{R}$ of the covariance functions (or random fields or processes) is defined in [4] as the class of kernels of positive rational functions of elliptic differential operators $\mathcal{L}$ in $\mathbb{R}^r$, or a system of commuting operators $\mathcal{L}_j$ of such type. Such kernels $R(x,y)$ can be written as (see [4], p. 10)

$$R(x,y) = \int_\Lambda \frac{P(\lambda)}{Q(\lambda)} \Phi(x,y,\lambda) d\rho(\lambda), \tag{6}$$

where $P(\lambda)$ and $Q(\lambda)$ are positive polynomials, $\Phi(x,y,\lambda)$ is the spectral kernel and $d\rho(\lambda)$ is the spectral measure corresponding to the selfadjoint elliptic operator $\mathcal{L}$. For example, if $\mathcal{L}_j = -i\frac{\partial}{\partial x_j}$, $1 \le j \le r$, then formula (6) yields (see [4])

$$R(x,y) = R(x-y) = \frac{1}{(2\pi)^r}\int_{\mathbb{R}^r} e^{i\lambda\cdot(x-y)}\frac{P(\lambda)}{Q(\lambda)}d\lambda, \quad \lambda = (\lambda_1,\dots,\lambda_r). \tag{7}$$

This is the class of homogeneous random fields with rational spectral density $\frac{P(\lambda)}{Q(\lambda)} \ge 0$.

The theory developed in [4] can be considered as a far-reaching generalization of the Wiener filtering theory [9], [8], for stationary random processes. The generalization consists of the following items:

a) The stationarity assumption is dropped.

b) Random fields estimation theory is developed. This means that $r > 1$. There was no results of this type in the published literature, to the author's knowledge.

c) The singular support and order of singularity of the optimal h are calculated.

d) Analytical formulas for the solution to equation (1) are obtained.

e) Numerical methods for solving equation (1) in distributions are developed.

The goal of this paper is to formulate a collocation method for solving equation (1) in distributions and to prove the convergence of this method. In [6], [7], and in [4] a projection method for solving equation (1) in distributions was developed. The collocation method was earlier developed in the literature for solving differential and integral equations for continuous functions (see e.g., [1], [3], [5]). The essentially new feature of this paper is the development of a new version of the collocation method for finding distributional solutions of integral equations. This work apparently is the first one in this direction.

In Section 2, the collocation method is formulated and its convergence is proved. These results depend heavily on the theory of the basic equation (5) of linear estimation and filtering, developed in [4]. From the statistical point of view our justification of the collocation method for solving equation (5) is significant because it allows one to use the values of the covariance functions $R(x,y)$ and $f(x,y)$ at a discrete set of points.

We illustrate the necessity to deal with the distributional solutions to the equation (5) by an example, which also clarifies our methodology.

Consider a particular example of equation (5) in the one-dimensional case

$$\int_{-1}^{1} e^{-|x-y|} h(y)dy = f(x), \quad x \in [-1,1]. \tag{8}$$

Note that $e^{-a|x-y|}$, where a =const>0, is a covariance function used in many examples in various texts on estimation theory. Without loss of generality one may assume $a=1$, and this is done below. The unique solution of minimal order of singularity to equation (8) is

$$h(x) = \frac{-f''+f}{2} + \delta(x-1)\frac{f'(1)+f(1)}{2} + \delta(x+1)\frac{-f'(-1)+f(-1)}{2}. \tag{9}$$

This h is a distribution. Its order of singularity equals to one, and h solves uniquely estimation problem (4). Here $r=1$, $D=[-1,1]$, $R(x,y)=e^{-|x-y|} \in \mathcal{R}$,

$$e^{-|x|} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{(\lambda^2+1)/2} e^{i\lambda x} d\lambda. \tag{10}$$

Thus, in formula one has (7)

$$P(\lambda)=1, \quad Q(\lambda)=\frac{\lambda^2+1}{2}, \quad \mathcal{L}=-i\frac{d}{dx}.$$

2. Formulation of the Collocation Method and its Convergence

2.1. Auxiliary Results

Let us formulate some theoretical results from [4], which are necessary for the construction and justification of the collocation method for solving equation (5). Throughout this paper we assume that $R(x,y) \in \mathcal{R}$,

$$\deg P(\lambda)=p, \quad \deg Q(\lambda)=q \geq p, \quad \operatorname{ord}\mathcal{L}=s, \quad \alpha := \frac{(q-p)s}{2}. \tag{11}$$

Let $H^\alpha = H^\alpha(D)$ be the Sobolev space, $\alpha \geq 0$, and $\dot{H}^{-\alpha}$ be its dual with respect to $H^0=L^2(D)$ pairing. The space $\dot{H}^{-\alpha}$ is the closure of $C_0^\infty(D)$ in the norm of the space $H^{-\alpha}(\mathbb{R}^r)$,

$$\|h\|_{H^{-\alpha}(\mathbb{R}^r)} := \int_{\mathbb{R}^r} (1+|\lambda|^2)^{-\alpha} |\tilde{h}(\lambda)|^2 d\lambda. \tag{12}$$

Theorem 2.1. (see [4], p. 15) *If $R(x,y) \in \mathcal{R}$, then $R: \dot{H}^{-\alpha} \to H^\alpha$ is an isomorphism and* singsupp $h = \partial D := S$. *The order of singularity of h is equal* *to α. Equation (5) with $f \in H^\alpha$ has a unique solution h in $\dot{H}^{-\alpha}$. This solution is the unique solution of minimal order of singularity to equation (5). It can be calculated by the formula*

$$h = Q(\mathcal{L})G, \quad G := \begin{cases} g + v, & \text{in } D, \\ u, & \text{in } D' := \mathbb{R}^r \setminus D, \end{cases} \tag{13}$$

where g is an arbitrary fixed solution to the equation

$$P(\mathcal{L})g = f \quad \text{in } D, \quad g \in H^{(p+q)s/2}, \tag{14}$$

u and v solve uniquely the following problem

$$Q(\mathcal{L})u = 0 \text{ in } D', \quad u(\infty) = 0, \quad P(\mathcal{L})v = 0 \text{ in } D, \tag{15}$$

$$\partial_N^j u|_S = \partial_N^j (g + v)|_S, \quad 0 \le j \le \frac{(p+q)s}{2} - 1, \tag{16}$$

and N is the normal to S pointing into D'.

Problem (15)-(16) is a non-classical boundary-value problem. It is an interface problem for elliptic operators the orders of which are different in D and in D', namely, this order equals to qs in D and to ps in D'. Uniqueness and existence of the solution to problem (15)-(16) are established in [4].

The following two estimates, proved in [4], p. 47, are useful:

$$\|R\|_{\dot{H}^{-\alpha} \to H^\alpha} \le \sup_{\lambda \in \Lambda} \left\{ \frac{P(\lambda)}{Q(\lambda)} (1 + \lambda^2)^{\frac{q-p}{2}} \right\} := \gamma_2, \tag{17}$$

$$\|R^{-1}\|_{H^\alpha \to \dot{H}^{-\alpha}} \le \gamma_1^{-1}, \quad \gamma_1 := \inf_{\lambda \in \Lambda} \left\{ \frac{P(\lambda)}{Q(\lambda)} (1 + \lambda^2)^{\frac{q-p}{2}} \right\}. \tag{18}$$

2.2. Description of the Collocation Method for Solving Equation (5)

The standard collocation method for solving operator equation $Rh = f$ can be described as follows. If $R : X \to Y$ is a linear isomorphism between Banach spaces X and Y, $\{w_j\}$ is a complete in X linearly independent system, and $Rw_j = \psi_j$, then one solves a linear algebraic system

$$\sum_{j=1}^{n} c_j^{(n)} \psi_j(x_i) = f(x_i), \quad 1 \le i \le n; \quad \psi_j = Rw_j, \tag{19}$$

where x_i are some points in D and finds an approximate solution to the equation $Rh = f$ by the formula:

$$h^{(n)} := \sum_{j=1}^{n} c_j^{(n)} w_j(x). \tag{20}$$

Since $f \in H^\alpha$, we assume that $\alpha > \frac{r}{2}$, because in this case the embedding $H^\alpha \to C(D)$ is compact. The constants $c_j^{(n)}$ are uniquely determined from the linear algebraic system (19) provided that

$$\det\psi_j(x_i) \neq 0, \quad 1 \leq i, j \leq n. \tag{21}$$

This condition can be satisfied by a suitable choice of the points x_i and the functions $w_j(x)$. The system $w_j(x)$ such that condition (21) holds for any choice of x_i such that $x_i \neq x_j$ for $i \neq j$ is called Chebyshev system (see [2], where examples of such systems are given).

The crucial point in a construction of a collocation method for solving equation (5) is to include the distributional parts of h in the set $\{w_j(x)\}$.

Let us first illustrate the idea using example (8)-(9). It is seen in this example that

$$h = h_{sing} + h_{sm}, \tag{22}$$

where h_{sing} is the singular part of h, a distribution,

$$\text{singsupp} h_{sing} = \partial D, \tag{23}$$

and h_{sm} is the smooth part of h, assuming that $f(x)$ is a smooth function. In example (8) ∂D consists of two points -1 and 1, $\alpha = 1$.

2.3. Discussion of the Collocation Method for Solving Equation (8)

Let us formulate a *new version of the collocation method* which allows one not to replace condition (21) by another condition.

This version consists of the minimization of the following functional:

$$E_n := \min_{c_j^{(n)}} \sum_{k=1}^{n} (x_{k+1}^{(n)} - x_k^{(n)}) \times \left[|f(x_k) - \sum_{j=1}^{n} c_j^{(n)} \psi_j(x_k)|^2 + |f'(x_k) - \sum_{j=1}^{n} c_j^{(n)} \psi_j'(x_k)|^2 \right], \tag{24}$$

where $x_k := x_k^{(n)}$, $x_k \in [-1, 1]$, $1 \leq k \leq n$, are some points,

$$\lim_{n\to\infty} \max_k |x_{k+1}^{(n)} - x_k^{(n)}| = 0. \tag{25}$$

Let us assume that $\psi_j(x_k) \neq 0$ for $1 \leq k \leq n$ and $1 \leq j \leq n$. Then E_n tends to infinity as $\sum_{j=1}^{n} |c_j^{(n)}| \to \infty$. Since $E_n \geq 0$ is a continuous function of finitely many variables $c_j^{(n)}$, it has to attain its unique global minimum at some values of $c_j^{(n)}$. Without an additional assumption, for example, assumption (21), the global minimizer may be non-unique. However, any global minimizer of E_n suffices for our argument. The functional E_n is a nonnegative quadratic function of the variables $c_j^{(n)}$. A necessary condition for the minimizer of such function is a linear algebraic system for $c_j^{(n)}$. If the determinant of this system is not equal to zero, then the necessary condition for the minimizer is also sufficient, and the global minimizer is unique.

Let us assume that f'^2 is Riemann-integrable and define

$$f_n(x) := \sum_{j=1}^{n} c_j^{(n)} \psi_j(x),$$

where $\{c_j^{(n)}\}$ is a minimizer of (24). This assumption allows one to consider the sum (24) as the Riemann sum for the integral

$$\int_0^1 dx[|f(x) - f_n(x)|^2 + |f'(x) - f_n'(x)|^2] = \|f - f_n\|_1^2, \tag{26}$$

and to claim that

$$E_n - d_n \leq \|f - f_n\|_1^2 \leq E_n + d_n, \quad \lim_{n\to\infty} d_n = 0, \tag{27}$$

because the Riemann sum E_n converges to the integral (26) if condition (25) holds, and we assume that condition (25) holds.

Assume now that the set $\{\psi_j(x)\}$ forms an orthonormal basis of $H^1 := H^1([-1, 1])$, denote by a_j the Fourier coefficients of f in this basis and by g_n the n-th Fourier sum of f in this basis:

$$g_n := \sum_{j=1}^{n} a_j \psi_j(x), \quad \lim_{n\to\infty} \|f - g\|_1 = 0. \tag{28}$$

Then one has

$$\|f - g_n\|_1^2 \leq \|f - f_n\|_1^2, \quad \lim_{n\to\infty} \|f - g_n\|_1 = 0, \tag{29}$$

because of the known extremal property of the Fourier sum.

Let us write the Riemann sum

$$G_n := \sum_{k=1}^{n} (x_{k+1} - x_k) \times \left[|f(x_k) - \sum_{j=1}^{n} a_j^{(n)} \psi_j(x_k)|^2 + |f'(x_k) - \sum_{j=1}^{n} a_j^{(n)} \psi_j'(x_k)|^2 \right], \tag{30}$$

where $x_k = x_k^{(n)}$. One has, similarly to equation (27), the following estimates

$$G_n - d'_n \le \|f - g_n\|_1 \le G_n + d'_n, \quad \lim_{n\to\infty} d'_n = 0. \tag{31}$$

Furthermore, by the definition of E_n, one has

$$E_n \le G_n. \tag{32}$$

From equations (29), (31), (32) and (27) one gets:

$$\|f - f_n\|_1 \le G_n + d_n \le \|f - g_n\|_1^2 + d'_n + d_n \to 0, \ n \to \infty. \tag{33}$$

We have proved the following theorem.

Theorem 2.2. *Assume that $\{\psi_j\}$ is an orthonormal basis of H^1. Then problem* (24) *has a global minimizer $\{c_j^{(n)}\}$ for any $n > 0$. If condition* (25) *holds then $\lim_{n\to\infty} \|f(x) - f_n(x)\|_1 = 0$. Consequently, the distribution*

$$h_n = \sum_{j=1}^{n} c_j^{(n)} R^{-1}\psi_j(x) \tag{34}$$

is an approximate solution to equation (8),

$$\lim_{n\to\infty} \|h - h_n\|_{-1} = 0. \tag{35}$$

2.4. A Choice of the Basis

Practically it is convenient to choose a basis such that $R^{-1}\psi_j$ is easy to calculate. By formula (9) one has

$$R^{-1}\psi_j = \frac{-\psi''_j + \psi_j}{2} + \delta(x-1)\frac{\psi'_j(1) + \psi_j(1)}{2} + \delta(x+1)\frac{-\psi'_j(-1) + \psi_j(-1)}{2}. \tag{36}$$

Let ψ_j be eigenfunctions of the problem:

$$-\psi''_j = \lambda_j \psi_j, \quad -1 \le x \le 1, \quad \psi_j(1) = \psi_j(-1) = 0. \tag{37}$$

Problem (37) has a discrete spectrum and the set $\{\psi_j\}$ forms an orthogonal basis of H^1:

$$\begin{aligned}(\psi_j, \psi_m)_1 := \int_{-1}^{1} (\psi_j \overline{\psi_m} + \psi'_j \overline{\psi'_m}) dx = \int_{-1}^{1} (\psi_j \overline{\psi_m} - \psi''_j \overline{\psi_m}) dx + \psi'_j \overline{\psi_m}|_{-1}^{1} \\ = (1 + \lambda_j)\delta_{jm},\end{aligned} \tag{38}$$

where $(\psi_j, \psi_m)_0 := \int_{-1}^{1} \psi_j \overline{\psi_m} dx = \delta_{jm}$, and δ_{jm} is the Kronecker symbol. Thus,

$$\left(\frac{\psi_j}{\sqrt{1+\lambda_j}}, \frac{\psi_m}{\sqrt{1+\lambda_m}} \right)_1 = \delta_{jm}. \tag{39}$$

Therefore, an orthonormal basis of H^1 can be written explicitly:

$$\psi_j(x) = \frac{\sin[j\pi(x+1)]}{\sqrt{1+(j\pi)^2}}. \tag{40}$$

For this basis by formula (36) one has:

$$\begin{aligned} h_n = & \sum_{j=1}^{n} c_j^{(n)} \frac{\sqrt{1+(j\pi)^2}}{2} \sin[j\pi(x+1)] + \frac{\delta(x-1)}{2} \sum_{j=1}^{n} c_j^{(n)} \frac{j\pi}{\sqrt{1+(j\pi)^2}} \\ & - \frac{\delta(x+1)}{2} \sum_{j=1}^{n} c_j^{(n)} \frac{j\pi}{\sqrt{1+(j\pi)^2}}. \end{aligned} \tag{41}$$

Formula (41) is convenient for numerical calculations if the coefficients $c_j^{(n)}$ decay fast as j grows and n large.

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