Spectral Properties of Schrödinger-type Operators and Large-time Behavior of the Solutions to the Corresponding Wave Equation

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Abstract. Let $L$ be a linear, closed, densely defined in a Hilbert space operator, not necessarily selfadjoint. Consider the corresponding wave equations

\begin{align}
(1) \quad \ddot{w} + Lw &= 0, \quad w(0) = 0, \quad \dot{w}(0) = f, \quad \dot{w} = \frac{dw}{dt}, \quad f \in H.
\end{align}

\begin{align}
(2) \quad \ddot{u} + Lu &= fe^{-ikt}, \quad u(0) = 0, \quad \dot{u}(0) = 0,
\end{align}

where $k > 0$ is a constant. Necessary and sufficient conditions are given for the operator $L$ not to have eigenvalues in the half-plane $\text{Re}z < 0$ and not to have a positive eigenvalue at a given point $k^2 > 0$. These conditions are given in terms of the large-time behavior of the solutions to problem (1) for generic $f$.

Sufficient conditions are given for the validity of a version of the limiting amplitude principle for the operator $L$.

A relation between the limiting amplitude principle and the limiting absorption principle is established.

Keywords and phrases: elliptic operators, wave equation, limiting amplitude principle, limiting absorption principle

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1. Introduction

Let $L$ be a linear, densely defined, closed operator in a Hilbert space $H$. Our results and techniques are valid in a Banach space also, but we wish to think about $L$ as of a Schrödinger-type operator in a Hilbert space and, at times, think that $L$ is selfadjoint. For a Schrödinger operator $L = -\nabla^2 + q(x)$ the resolvent $(L - k^2)^{-1}, \text{Im}k > 0$, is an integral operator with a kernel $G(x,y,k)$, its resolvent kernel. If $q$ is a real-valued function, sufficiently rapidly decaying then $L$ is selfadjoint, $G(x,y,k)$ is analytic with respect to $k$ in the half-plane $\text{Im}k > 0$, except, possibly, for a finitely many simple poles $ik_j, k_j > 0$, the

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semiaxis \( k \geq 0 \) is filled with the points of absolutely continuous spectrum of \( L \), and there exists a limit
\[
\lim_{\epsilon \to 0} G(x, y, k + i\epsilon) = G(x, y, k)
\]
for all \( k > 0 \).

Sufficient conditions for \( k^2 = 0 \) not to be an eigenvalue of \( L \) are found in papers [5], [6]. Spectral analysis of the Schrödinger operators is presented in many books (see, for example, [2] and [11]). In papers [3], [4], such an analysis was given in a class of domains with infinite boundaries apparently for the first time, see also [8]. In [7] an eigenfunctions expansion theorem was proved for non-selfadjoint Schrödinger operators with exponentially decaying complex-valued potential \( q \). The operator \( L \) in this paper is not necessarily assumed to be selfadjoint.

In [1] the validity of the limiting amplitude principle for some class of selfadjoint operators \( L \) has been established.

This principle says that, as \( t \to \infty \), the solution to problem
\[
\ddot{u} + Lu = fe^{-ikt}, \quad u(0) = 0, \quad \dot{u}(0) = 0, \quad \dot{u} = \frac{du}{dt}, \quad (1.1)
\]
has the following asymptotics
\[
u = e^{-ikt}v + o(1), \quad t \to \infty,
\]
where \( k \) is a real number and \( v \in H \) solves the equation
\[
Lv - k^2 v = f. \quad (1.3)
\]
The \( v \) is called the limiting amplitude. It turns out that a more natural definition of the limiting amplitude is:
\[
v = \lim_{t \to \infty} \frac{1}{t} \int_0^t u(s) e^{iks} ds, \quad (1.4)
\]
if this limit exists and solves equation (1.3).

Why is this definition more natural than (1.2)? There are good reasons for this. One of the reasons is: if (1.2) and (1.3) hold, then the limit (1.4) exists and solves equation (1.3). The other reason is: the limit (1.4) may exist and solve equation (1.3) although the limit (1.2) does not exist.

**Example.** If \( u = e^{ikt}v + e^{ik_1t}v_1 \), then the limit (1.2) does not exist, while the limit (1.4) does exist and is equal to \( v \).

To describe our assumptions and results, some preparation is needed.

Consider the problem
\[
\ddot{w} + Lw = 0, \quad w(0) = 0, \quad \dot{w}(0) = f. \quad (1.5)
\]
Assuming that \( ||u(t)|| \leq ce^{at} \), where \( c > 0 \) stands throughout the paper for various generic constants, and \( a \geq 0 \) is a constant, one can define the Laplace transform of \( u(t) \),
\[
\mathcal{U} := \mathcal{U}(p) := \int_0^\infty e^{-pt} u(t) dt, \quad \sigma > a,
\]
where \( p = \sigma + i\tau \), \( \text{Re} \ p = \sigma \).

Let us take the Laplace transform of (1.1) and of (1.5) to get
\[
L\mathcal{U} + p^2 \mathcal{U} = \frac{f}{p + ik}, \quad (1.6)
\]
and
\[
L\mathcal{W} + p^2 \mathcal{W} = f, \quad (1.7)
\]
where
\[ W = W(p) = \int_0^\infty w(t)e^{-pt}dt. \]

We also denote \( W(p) := \bar{w}(t) \).

The complex plane \( p \) is related to the complex plane \( k \) by the formula
\[ p = -ik, \quad k = k_1 + ik_2, \quad k_2 \geq 0, \quad \sigma = k_2 \geq 0. \tag{1.8} \]

We assume throughout that \( f \) is generic in the following sense:

If \( I \) is the identity operator and a point \( p \) is a pole of the kernel of the operator \((L + p^2 I)^{-1}\), then it is a pole of the same order of the element \((L + p^2 I)^{-1} f = W \).

If \( k^2 \) is an eigenvalue of \( L \) and \( \text{Re} k^2 < 0 \), then \( \text{Im} k > 0 \), where \( k = |k|e^{i\arg k^2} \), \( p = -ik \), so \( \text{Re} p > 0 \).

Let us now formulate the main Assumptions A and B standing throughout this paper.

**Assumption A.** For a generic \( f \) the \( W(p) = (L + p^2 I)^{-1} f \) is analytic in the half-plane \( \sigma > \sigma_0 \geq 0 \), except, possibly, at a finitely many simple poles at the points \(-ik_j, 1 \leq j \leq J, \) \( k_j \) are real numbers, and at the points \( \kappa_m, \text{Re} \kappa_m > 0, \)
\[ W(p) = \sum_{j=1}^J \frac{v_j}{p + ik_j} + W_1(p) + \sum_{m=1}^M \frac{b_m}{p - \kappa_m}, \tag{1.12} \]

where \( v_j \) and \( b_m \) are some elements of \( H, \) \( W_1(p) \) is an analytic function in the half-plane \( \text{Re} p = \sigma > 0, \) continuous up to the imaginary axis \( \sigma = 0, \) and satisfying the following estimate
\[ ||W_1(p)|| \leq \frac{c}{1 + |p|^\gamma}, \quad \gamma > \frac{1}{2}. \tag{1.13} \]

**Assumption B.** There exists the limit
\[ \lim_{\sigma \to 0} ||W_1(\sigma - ik) - W_1(-ik)|| = 0 \tag{1.14} \]
for all real numbers \( k. \)
Theorem 1.3. Let the Assumption A hold. Then a necessary and sufficient condition for the operator $L$ to have no eigenvalues in the half-plane $\Re k^2 < 0$ is the validity of the estimate

$$\left\| \int_0^t w(s)ds \right\| = O(e^{\epsilon t}), \quad t \to \infty,$$

(1.15)

for an arbitrary small $\epsilon > 0$.

A necessary and sufficient condition for the operator $L$ not to have any positive eigenvalues $k^2 > 0$ is the validity of the estimate

$$\left\| \frac{1}{t} \int_0^t e^{iks}w(s)ds \right\| = o(1), \quad t \to \infty, \quad \forall k \in \mathbb{R}.$$  

(1.16)

A point $ik_0 > 0, k_0 > 0$, is not a pole of the resolvent kernel of the operator $(L - k^2 - i0)^{-1}$ if and only if estimate (1.16) holds with $k = k_0 > 0$.

Remark. If condition (1.16) holds for $k = 0$, then $\| \int_0^t w(s)ds \| = o(t), so condition (1.15) holds, and the operator $L$ has no eigenvalues in the half-plane $\Re k^2 < 0$.

Theorem 1.4. Let the Assumptions A and B hold. Suppose that estimates (1.14) and (1.15) hold. Then the limiting amplitude principle (1.4) holds for every $k \in \mathbb{R}, k \neq k_j, 1 \leq j \leq J$.

In section 2, proofs are given.

2. Proofs

2.1. Proof of Theorem 1.3

From the Assumption A and Proposition 1.1, it follows that $W(p)$ is a Laplace transform of a function $w(t)$ such that

$$w(t) = \sum_{j=1}^J v_j e^{-ik_j t} + \sum_{m=1}^M b_m e^{\kappa_m t} + w_1(t),$$

(2.1)

where

$$w_1(t) = \frac{1}{2\pi i} \int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} e^{pt}W_1(p)dp,$$

(2.2)

and the integral in (2.2) converges in $L^2$-sense due to the assumption (1.13). It is clear from formula (2.1) that all $b_m = 0$ if and only if estimate (1.15) holds with $0 < \epsilon < \min_{1 \leq m \leq M} \Re \kappa_m$. This proves the first conclusion of Theorem 1.3.

Let us calculate the expression on the left side of formula (1.16) and show that this expression is $o(1)$ unless $k = k_j$ for some $1 \leq j \leq J$. In this calculation it is assumed that $L$ does not have any eigenvalues in the half-plane $\Re k^2 < 0$, in other words, that all $b_m = 0$. Otherwise the expression on the left of formula (1.16) tends to infinity as $t \to \infty$ at an exponential rate.

If all $b_m = 0$ in (2.1), then

$$\sum_{j=1}^J v_j \frac{1}{t} \int_0^t e^{i(k-k_j)t}dt + \frac{1}{t} \int_0^t w_1(t)e^{ikt}dt := I_1 + I_2.$$  

(2.3)

If $k$ and $k_j$ are real numbers, then

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t e^{i(k-k_j)t}dt = \begin{cases} 1, & k = k_j, \\ 0, & k \neq k_j. \end{cases}$$

(2.4)
Thus, \( I_1 = 0 \) if and only if \( k \) does not coincide with any of \( k_j, 1 \leq j \leq J \).

Let us prove that
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t w_1(t) e^{ikt} dt = 0.
\] (2.5)

By proposition (1.2) and the Mellin inversion formula, one has
\[
I := \frac{1}{t} \int_0^t w_1(t) e^{ikt} dt = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{W}_1(p-ik) \frac{e^{pt}}{pt} dp,
\] (2.6)

where \( \text{Re} \sigma > 0 \) can be chosen arbitrarily small.

Let \( pt = q \), take \( \sigma = \frac{1}{t} \), write \( q = 1 + is \), and write the integral on the right side of (2.6) as:
\[
I = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \mathcal{W}_1(q/t - ik) \frac{q^s}{t q^s} dq.
\] (2.7)

If one uses estimate (1.13) and formula \( |q| = (1 + s^2)^{1/2} \), then one obtains the following inequality
\[
||I|| \leq \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{(1 + s^2)^{1/2} [1 + |1/t^2 - ik|]} ds = \frac{c}{2\pi t^{1-\gamma}} \int_{-\infty}^{\infty} \frac{1}{(1 + s^2)^{1/2} (t^n + |1 + (s - kt)^2|^{1/2})} ds.
\] (2.8)

Let \( s = ty \). Then the integral on the right side of (2.8) can be written as
\[
\frac{ct}{2\pi t^{1-\gamma}} \int_{-\infty}^{\infty} \frac{dy}{(1 + t^2 y^2)^{1/2} (t^n + |1 + t^2(y - k)^2|^{1/2})} \leq \frac{c}{2\pi} \int_{-\infty}^{\infty} \frac{dy}{(1 + t^2 y^2)^{1/2} [1 + (y - k)^\gamma]} \to 0, \text{ as } t \to \infty,
\] (2.9)

and the convergence of the last integral to zero is uniform with respect to \( k \in \mathbb{R} \).

Thus
\[
\lim_{t \to \infty} ||I|| = 0.
\] (2.10)

From (2.3)-(2.5) the last two conclusions of Theorem 1.3 follow. Theorem 1.3 is proved. \( \square \)

### 2.2. Proof of Theorem 1.4

Using Proposition 1.2, equation (1.6), and the Mellin formula, one gets
\[
\frac{1}{t} \int_0^t u(t) e^{ikt} dt = \frac{1}{t} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{U}(p-ik) \frac{e^{pt}}{p} dp,
\] (2.11)

where, according to (1.6),
\[
\mathcal{U}(p-ik) = \frac{W(p - ik)}{p}.
\] (2.12)

Let \( \sigma = \frac{1}{t} \) and \( pt = q \). Then
\[
\frac{1}{t} \int_0^t u(t) e^{ikt} dt = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \mathcal{W} \left( \frac{q}{t} - ik \right) \frac{e^q}{q^s} dq.
\] (2.13)

Estimate (1.15) and Theorem 1.3 imply that all \( b_m = 0 \) in formula (2.1). Therefore, using formula (2.1) with \( b_m = 0 \), one gets
\[
\mathcal{W} = \sum_{j=1}^J \frac{v_j}{p + ik_j} + \mathcal{W}_1,
\]
and
\[ W \left( \frac{q}{t} - ik \right) = W_1 \left( \frac{q}{t} - ik \right) + \sum_{j=1}^{J} v_j \frac{1}{t - i(k - k_j)}, \]  
(2.14)

One has \( \frac{n!}{p^p} \). Therefore
\[ \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} e^q q^2 dq = 1, \]
and
\[ \lim_{t \to \infty} v_j \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} e^q q^2 dq = \left\{ \begin{array}{ll}
\frac{iv_j}{k - k_j}, & k \neq k_j, \\
\infty, & k = k_j.
\end{array} \right. \]  
(2.15)

Furthermore,
\[ \lim_{t \to \infty} \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} W_1 \left( \frac{q}{t} - ik \right) e^q q^2 dq = W_1(-ik), \]  
(2.16)
as follows from assumption (1.14) and the Lebesgue’s dominated convergence theorem if one passes to the limit \( t \to \infty \) under the sign of the integral (2.16). Let us check that this \( v \) solves equation (1.3). This would conclude the proof of Theorem 1.4. We need a lemma.

**Lemma 2.1.** If \( h \in L_{\text{loc}}^{1}(0, \infty) \) and the limit \( \lim_{t \to \infty} t^{-1} \int_{0}^{t} h(s) ds \) exists, then the limit \( \lim_{p \to 0} p \int_{0}^{\infty} e^{-pt} h(t) dt \) exists, and
\[ \lim_{t \to \infty} t^{-1} \int_{0}^{t} h(s) ds = \lim_{p \to 0} p \int_{0}^{\infty} e^{-pt} h(t) dt. \]  
(2.17)

**Proof of Lemma 1.** One has
\[ p \int_{0}^{\infty} e^{-pt} h(t) dt = pe^{-pt} \int_{0}^{t} h(s) ds|_{0}^{\infty} + \frac{p^2}{2} \int_{0}^{\infty} te^{-pt} t^{-1} \int_{0}^{t} h(s) ds dt. \]

For any \( p > 0 \) one has
\[ pe^{-pt} \int_{0}^{t} h(s) ds|_{0}^{\infty} = 0. \]

Let \( q = pt \) and denote \( H(t) := t^{-1} \int_{0}^{t} h(s) ds, J := \lim_{t \to \infty} H(t) \). Then
\[ \lim_{p \to 0} p^2 \int_{0}^{\infty} te^{-pt} t^{-1} \int_{0}^{t} h(s) ds dt = \lim_{p \to 0} \int_{0}^{\infty} qe^{-q} H(q(p^{-1})) dq. \]

Passing in the last integral to the limit \( p \to 0 \) one obtains (2.17). Lemma 1 is proved. \( \square \)

Using equation (2.17), one writes \( v = \lim_{p \to 0} p \mathcal{U}((p - ik)) \), where \( \mathcal{U} \) solves equation (1.6). Thus,
\[ L \mathcal{U}(p - ik) + (p - ik)^2 \mathcal{U}(p - ik) = p^{-1} f. \]

Multiplying both sides of this equation by \( p \) and passing to the limit \( p \to 0 \), one obtains equation (1.3). In the passage to the limit under the sign of the unbounded operator \( L \) the assumption that \( L \) is closed was used.

Thus, the conclusion of Theorem 1.4 follows. \( \square \)

If the limit (1.14) exists at a point \( p = iq \) then one says that the limiting absorption principle holds for the operator \( L \) at the point \( k = iq = i(-ik) = k, k > 0. \)

Thus, Assumption B means that the limiting absorption principle holds for \( L \) at the point \( k > 0. \) that is, \( \lim_{\epsilon \to 0} (L - k^2 - i\epsilon)^{-1} f \) exists.
3. Applications

Let $L = -\nabla^2 + q(x)$, where $q(x)$ is a real-valued function, $|q(x)| \leq c(1 + |x|)^{-2-\epsilon}$, $\epsilon > 0$, $x \in \mathbb{R}^3$. Then $L$ is selfadjoint on the domain $H^2(\mathbb{R}^3)$. Its resolvent $(L - k^2 - i0)^{-1}$ satisfies Assumptions A and B if one keeps in mind the following.

Let $G(x, y, k)$ be the resolvent kernel of $L$, that is, the kernel of the operator $(L - k^2 - i0)^{-1},$

$$LG(x, y, k) = -\delta(x - y) \quad \text{in} \quad \mathbb{R}^3,$$

$G \in L^2(\mathbb{R}^3)$ for $\text{Im} \ k > 0$. If $f \in L^2(\mathbb{R}^3)$ is compactly supported, then for $k > 0$ the function

$$v(x) := (L - k^2 - i0)^{-1}f = \int_{\mathbb{R}^3} G(x, y, k)f(y)dy$$

does not necessarily belong to $L^2(\mathbb{R}^3)$.

For example, if $q(x) = 0$, then $G(x, y, k) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$, and the function

$$v(x, k) = \int_{|y| \leq 1} g(x, y, k)dy = O\left(\frac{1}{|x|}\right) \quad (3.1)$$

does not belong to $L^2(\mathbb{R}^3)$ (except for those $k > 0$ for which $x(x, k) = 0$ in the region $|y| \geq 1$. These numbers $k > 0$ are the zeros of the Fourier transform of the characteristic function of the ball $|y| \leq 1$, see [10], Chapter 11.

By this reason the abstract results of theorem (1.3) and (1.4) can be used in applications if one defines some subspace of $H$, for example, a subspace of functions with compact support, denote by $\mathcal{P}$, a projection operator on this subspace, and replaces $W$ and $W_1$ by $\mathcal{P}W$ and $\mathcal{P}W_1$ in equations (1.12) and (1.14). For example, the function (3.1) one replaces by $\eta(x)v(x, k)$, where $\eta(x)$ is a characteristic function of a compact subset of $\mathbb{R}^3$.

The analytic properties of $\eta(x)v(x, k)$ and of $v(x, k)$ as functions of $k$ are the same. A similar suggestion is used in [1].

With the above in mind, one knows (for example, from [2] or [11]) that Assumptions A and B hold for $L = -\nabla^2 + q(x)$.

Consequently, the conclusions of Theorems 1.3 and 1.4 hold.

In addition, the assumptions

$$|q(x)| \leq c(1 + |x|)^{-2-\epsilon}, \quad \epsilon > 0, \quad \text{Im} \ q = 0,$$

imply that $L$ does not have positive eigenvalues, so all $v_j = 0$, and zero is not an eigenvalue of $L \geq 0$ if $\epsilon > 0$ (see [5], [6]).

A new method for estimating of large time behavior of solutions to abstract evolution problems is developed in [9], where some applications of this method are given.

References


