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ANALYSIS OF A NEWTON-SABATIER SCHEME FOR INVERTING FIXED-ENERGY PHASE SHIFTS

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Abstract Suppose that the inverse scattering problem is understood as follows: given fixed-energy phase shifts, corresponding to an unknown potential $q = q(r)$ from a certain class, for example, $q \in L_{1,1}$, recover this potential. Then it is proved that the Newton-Sabatier (NS) procedure does not solve the above problem. It is not a valid inversion method, in the following sense: 1) it is not possible to carry this procedure through for the phase shifts corresponding to a generic potential $q \in L_{1,1}$, where $L_{1,1} := \{q : q = \overline{q}, \int_0^\infty r|q(r)|dr < \infty\}$ and recover the original potential: the basic integral equation, introduced by R. Newton without derivation, in general, may be not solvable for some $r > 0$, and if it is solvable for all $r > 0$, then the resulting potential is not equal to the original generic $q \in L_{1,1}$. Here a generic q is any q which is not a restriction to $(0, \infty)$ of an analytic function. 2) the ansatz $(*)$ $K(r, s) = \sum_{l=0}^{\infty} c_l \varphi_l(r) u_l(s)$, used by R. Newton, is incorrect: the transformation operator $I - K$, corresponding to a generic $q \in L_{1,1}$, does not have K of the form $(*)$, and 3) the set of potentials $q \in L_{1,1}$, that can possibly be obtained by NS procedure, is not dense in the set of all $L_{1,1}$ potentials in the norm of $L_{1,1}$. Therefore one cannot justify NS procedure even for approximate solution of the inverse scattering problem with fixed-energy phase shifts as data. Thus, the NS procedure, if considered as a method for solving the above inverse scattering problem, is based on an incorrect ansatz, the basic integral equation of NS procedure is, in general, not solvable for some $r > 0$, and in this case this procedure breaks down, and NS procedure is not an inversion theory: it cannot recover generic potentials $q \in L_{1,1}$ from their fixed-energy phase shifts. Suppose now that one considers another problem: given fixed-energy phase shifts, corresponding to some potential, find a potential which generates the same phase shifts. Then NS procedure does not solve this problem either: the basic integral equation, in general, may be not solvable for some $r > 0$, and then NS procedure breaks down.

KEY WORDS: inverse scattering, fixed-energy phase shifts, integral equations.

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1. Introduction and conclusions

The NS procedure is described in [4] and [2]. In the sixties P. Sabatier published several papers concerning this procedure, and there are quite a few papers of several authors using this procedure and generalizing it. A vast bibliography of this topic is given in [2] and [4], and by this reason we do not include references to many papers treating this topic.

In our arguments below two cases are discussed. The first case deals with the inverse scattering problem with fixed-energy phase shifts as the data. This problem is understood as follows: an unknown spherically symmetric potential q from an a priori fixed class, say $L_{1,1}$, a standard scattering class, generates fixed-energy phase shifts $\delta_l, l = 0, 1, 2, \dots,$. The inverse scattering problem consists of recovery of q from these data.

The second case deals with a different problem: given some numbers $\delta_l, l = 0, 1, 2, \dots,$, which are assumed to be fixed-energy phase shifts of some potential q, from a class not specified, find some potential q_1, which generates fixed-energy phase shifts equal to $\delta_l, l = 0, 1, 2, \dots,$. This potential q_1 may have no physical interest because of its non-physical" behavior at infinity or other undesirable properties.

We first discuss NS procedure assuming that it is intended to solve the inverse scattering problem in case 1. Then we discuss NS procedure assuming that it is intended to solve the problem in case 2.

Discussion of case 1:

In [5] and [4] a procedure was proposed by R. Newton for inverting fixed-energy phase shifts $\delta_l, l = 0, 1, 2, \dots,$ corresponding to an unknown spherically symmetric potential $q(r)$. R. Newton did not specify the class of potentials for which he tried to develop an inversion theory and did not formulate and proved any results which would justify the inversion procedure he proposed (NS procedure). His arguments are based on the following claim N1, which is implicit in his works, but crucial for the validity of NS procedure:

Claim N1: the basic integral equation

$$K(r,s) = f(r,s) - \int_0^r K(r,t) f(t,s) \frac{dt}{t^2}, \quad 0 \le s \le r < \infty, \tag{1.1}$$

is uniquely solvable for all $r > 0$.

Here

$$f(r,s) := \sum_{l=0}^{\infty} c_l u_l(r) u_l(s), \quad u_l := \sqrt{\frac{\pi r}{2}} J_{l+\frac{1}{2}}(r), \tag{1.2}$$

c_l are real numbers, the energy k^2 is fixed: $k = 1$ is taken without loss of generality, $J_{l+\frac{1}{2}}(r)$ are the Bessel functions. If equation (1.1) is uniquely solvable for all $r > 0$, then the potential q_1, that NS procedure yields, is defined by the formula:

$$q_1(r) = -\frac{2}{r} \frac{d}{dr} \frac{K(r,r)}{r}. \tag{1.3}$$

The R. Newton's ansatz (1.1)-(1.2) for the transformation kernel $K(r,s)$ of the Schroedinger operator, corresponding to some $q(r)$, namely, that $K(r,s)$ is the unique solution to (1.1)-(1.2), is not correct for a generic potential, as follows from our argument below (see the justification of Conclusions).

If for some $r > 0$ equation (1.1) is not uniquely solvable, then NS procedure breaks down: it leads to locally non-integrable potentials for which the scattering theory is, in general, not available (see [9]and [1] for a proof of the above statement) .

In the original paper [5] and in his book [4] R. Newton did not study the question, fundamental for any inversion theory: does the reconstructed potential q_1 generate the data from which it was reconstructed?

In [2], p. 205, there are two claims:

i) that $q_1(r)$ generates the original shifts $\{\delta_l\}$ "provided that $\{\delta_l\}$ are not "exceptional"", and ii) that NS procedure "yields one (only one) potential which decays faster than $r^{-\frac{3}{2}}$" and generates the original phase shifts $\{\delta_l\}$.

If one considers NS procedure as a solution to inverse scattering problem of finding an unknown potential q from a certain class, for example $q(r) \in L_{1,1} := \{q : q = \overline{q}, \int_0^\infty r|q(r)|dr < \infty\}$, from the fixed-energy phase shifts, generated by this q, then the proof, given in [2], of claim i) is not convincing: it is not clear why the potential q_1, obtained by NS procedure, has the transformation operator generated by the potential corresponding to the original data, that is, to the given fixed-energy phase shifts. In fact, as follows from Proposition 1 below, the potential q_1 cannot generate the kernel $K(r,s)$ of the transformation operator corresponding to a generic original potential $q(r) \in L_{1,1} := \{q : q = \overline{q}, \int_0^\infty r|q(r)|dr < \infty\}$.

Claim ii) is incorrect because the original generic potential $q(r) \in L_{1,1}$ generates the phase shifts $\{\delta_l\}$, and if $q_1(r)$, the potential obtained by NS procedure and therefore not equal to $q(r)$ by Proposition 1, generates the same phase shifts $\{\delta_l\}$, then one has two different potentials $q(r)$ and $q_1(r)$, which both decay faster than $r^{-\frac{3}{2}}$ and both generate the original phase shifts $\{\delta_l\}$, contrary to claim ii).

The purpose of this paper is to formulate and justify the following

Conclusions:

Claim N1 and ansatz (1.1)-(1.2) are not proved by R.Newton and, in general, are wrong. Moreover, one cannot approximate with a prescribed accuracy in the norm $||q|| := \int_0^\infty r|q(r)|dr$ *a generic potential* $q(r) \in L_{1,1}$ *by the potentials which might possibly be obtained by the NS procedure. Therefore NS procedure cannot be justified even as an approximate inversion procedure.*

Let us justify these conclusions:

Claim N1, formulated above and basic for NS procedure, is wrong, in general, for the following reason:

Given fixed-energy phase shifts, corresponding to a generic potential $q \in L_{1,1}$, one either cannot carry through NS procedure because:

a) the system (12.2.5a) in [2], which should determine numbers c_l in formula (1.2), given the phase shifts δ_l, may be not solvable, or

b) if the above system is solvable, equation (1.1) may be not (uniquely) solvable for some $r > 0$, and in this case NS procedure breaks down since it yields a potential which is not locally integrable (see [9] for a proof).

If equation (1.1) is solvable for all $r > 0$ and yields a potential q_1 by formula (1.3), then this potential is not equal to the original generic potential $q \in L_{1,1}$, as follows from Proposition 1, which is proved in [9] (see also [1]):

Proposition 1. *If equation (1.1) is solvable for all* $r > 0$ *and yields a potential* q_1 *by formula (1.3), then this* q_1 *is a restriction to* $(0, \infty)$ *of a function analytic in a neighborhood of* $(0, \infty)$.

Since a generic potential $q \in L_{1,1}$ is not a restriction to $(0, \infty)$ of an analytic function, one concludes that even if equation (1.1) is solvable for all $r > 0$, the potential q_1, defined by formula (1.3), is not equal to the original generic potential $q \in L_{1,1}$ and therefore the inverse scattering problem of finding an unknown $q \in L_{1,1}$ from its fixed-energy phase shifts is not solved by NS procedure.

The ansatz (1.1)-(1.2) for the transformation kernel is, in general, incorrect, as follows also from Proposition 1.

Indeed, if the ansatz (1.1)-(1.2) would be true and formula (1.3) would yield the original generic q, that is $q_1 = q$, this would contradict Proposition 1. If formula (1.3) would yield a q_1 which is different from the original generic q, then NS procedure does not solve the inverse scattering problem formulated above. Note also that it is proved in [10] that independent of the angular momenta l transformation operator, corresponding to a generic $q \in L_{1,1}$ does exist, is unique, and is defined by a kernel $K(r, s)$ which cannot have representation (1.2), since it yields by the formula similar to (1.3) the original generic potential q, which is not a restriction of an analytic in a neighborhood of $(0, \infty)$ function to $(0, \infty)$.

The conclusion, concerning impossibility of approximation of a generic $q \in L_{1,1}$ by potentials q_1, which can possibly be obtained by NS procedure, is proved in section 2, see proof of Claim 1 there.

Thus, our conclusions are justified. $\Box$

Let us give some additional comments concerning NS procedure.

Uniqueness of the solution to the inverse problem in case 1 was first proved by A.G.Ramm in 1987 (see [7] and references therein) for a class of compactly supported potentials, while R. Newton's procedure was published in [5], when no uniqueness results for this inverse problem were known. It is still an open problem if for the standard in scattering theory class of $L_{1,1}$ potentials the uniqueness theorem for the solution of the above inverse scattering problem holds.

We discuss the inverse scattering problem with fixed-energy phase shifts (as the data) for potentials $q \in L_{1,1}$, because only for this class of potentials a general theorem of existence and uniqueness of the transformation operators, independent of the angular momenta l, has been proved, see [10]. In [5], [4], and in [2] this result was not formulated and proved, and it was not clear for what class of potentials the transformation operators, independent of l, do exist. For slowly decaying potentials the existence of the transformation operators, independent of l, is not established, in general, and the potentials, discussed in [2] and [4] in connection with NS procedure, are slowly decaying.

Starting with [5], [4], and [2] claim N1 was not proved or the proofs given (see [3] were incorrect (see [11]). This equation is uniquely solvable for sufficiently small $r > 0$, but, in general, *it may be not solvable for some $r > 0$, and if it is solvable for all $r > 0$, then it yields by formula (1.3) a potential q_1, which is not equal to the original generic potential $q \in L_{1,1}$, as follows from Proposition 1.*

Existence of "transparent" potentials is often cited in the literature. A "transparent" potential is a potential which is not equal to zero identically, but generates the fixed-energy shifts which are all equal to zero.

In [2], p.207, there is a remark concerning the existence of "transparent" potentials. This remark is not justified because it is not proved that for the values c_l, used in [2], p.207, equation (1.1) is solvable for all $r > 0$. If it is not solvable even for one $r > 0$, then NS procedure breaks down and the existence of transparent potentials is not established.

In the proof, given for the existence of the "transparent" potentials in [2], p.197, formula (12.3.5), is used. This formula involves a certain infinite matrix M. It is claimed in [2], p.197, that this matrix M has the property $MM = I$, where I is the unit matrix, and on p.198, formula (12.3.10), it is claimed that a vector $v \neq 0$ exists such that $Mv = 0$. However, then $MMv = 0$ and at the same time $MMv = v \neq 0$, which is a contradiction. The difficulties come from the claims about infinite matrices, which are not formulated clearly: it is not clear in what space M, as an operator, acts, what is the domain of definition of M, and on what set of vectors formula (12.3.5) holds.

The construction of the "transparent" potential in [2] is based on the following logic: take all the fixed-energy shifts equal to zero and find the corresponding c_l from the infinite linear algebraic system (12.2.7) in [2]; then construct the kernel $f(r, s)$ by formula (1.2) and solve equation (1.1) for all $r > 0$; finally construct the "transparent" potential by formula (1.3). As was noted above, it is not proved that equation (1.1) with the constructed above kernel $f(r, s)$ is solvable for all $r > 0$. Thefore the existence of the "transparent" potentials is not established.

The physicists have been using NS procedure without questioning its validity for several decades. Apparently the physicists still believe that NS procedure is "an analog of the Gel'fand-Levitan method" for inverse scattering problem with fixed-energy phase shifts as the data. In this paper the author explains why such a belief is not justified and why NS procedure is not a valid inversion method. Since modifications of NS procedure are still used by some physicists, who believe that this procedure is an inversion theory, the author pointed out some questions concerning this procedure in [1] and [9] and wrote this paper.

This concludes the discussion of case 1. □

Discussion of case 2:

Suppose now that one wants just to construct a potential q_1, which generates the phase shifts corresponding to some q.

This problem is actually *not an inverse scattering problem* because one does not recover an original potential from the scattering data, but rather wants to construct some potential which generates these data and may have no physical meaning. Therefore this problem is much less interesting practically than the inverse scattering problem.

However, NS procedure does not solve this problem either: there is no guarantee that this procedure is applicable, that is, that the steps a) and b), described in the justification of the conclusions, can be done, in particular, that equation (1.1) is uniquely solvable for all $r > 0$.

If these steps can be done, then one needs to check that the potential q_1, obtained by formula (1.3), generates the original phase shifts. This was not done in [5] and [4].

This concludes the discussion of case 2. □

The rest of the paper contains formulation and proof of Remark 1 and Claim 1.

It was mentioned in [6] that if $Q := \int_0^\infty rq(r)dr \neq 0$, then the numbers c_l in formula (1.2) cannot satisfy the condition $\sum_0^\infty |c_l| < \infty$. This observation can be obtained also from the following

Remark 1. *For any potential $q(r) \in L_{1,1}$ such that $Q := \int_0^\infty rq(r)dr \neq 0$ the basic equation (1.1) is not solvable for some $r > 0$ and any choice of c_l such that $\sum_{l=0}^\infty |c_l| < \infty$.*

Since generically, for $q \in L_{1,1}$, one has $Q \neq 0$, this gives an additional illustration to the conclusion that equation (1.1), in general, is not solvable for some $r > 0$. Conditions $\sum_{l=0}^\infty |c_l| < \infty$ and $Q \neq 0$ are incompatible.

In [2], p. 196, a weaker condition $\sum_{l=0}^\infty l^{-2}|c_l| < \infty$ is used, but in the examples ([2] pp. 189-191), $c_l = 0$ for all $l \geq l_0 > 0$, so that $\sum_{l=0}^\infty |c_l| < \infty$ in all of these examples.

Claim 1. *The set of the potentials $v(r) \in L_{1,1}$, which can possibly be obtained by the NS procedure, is not dense (in the norm $\|q\| := \int_0^\infty r|q(r)|dr$) in the set $L_{1,1}$.*

In section 2 proofs are given.

2. Proofs

Proof of Remark 1.

Writing (1.3) as $K(r,r) = -\frac{r}{2}\int_0^r sq_1(s)ds$ and assuming $Q \neq 0$, one gets the following relation:

$$K(r,r) = -\frac{Qr}{2}\,[1 + o(1)] \to \infty \text{ as } r \to \infty. \tag{2.1}$$

If (1.1) is solvable for all $r > 0$, then from (1.2) and (1.1) it follows that $K(r,s) = \sum_{l=0}^\infty c_l \varphi_l(r) u_l(s)$, where $\varphi_l(r) := u_l(r) - \int_0^r K(r,t)u_l(t)\frac{dt}{t^2}$, so that $I - K$ is a transformation operator, where K is the operator with kernel $K(r,s)$, $\varphi_l''+\varphi_l-\frac{l(l+1)}{r^2}\varphi_l-q(r)\varphi_l=0$, $q(r)$ is given by (1.4), $\varphi_l=O(r^{l+1})$, as $r\to 0$,

$$u_l(r)\sim \sin\left(r-\frac{l\pi}{2}\right),\quad \varphi_l(r)\sim |F_l|\sin\left(r-\frac{l\pi}{2}+\delta_l\right)\text{ as } r\to\infty,$$

where δ_l are the phase shifts at $k=1$ and F_l is the Jost function at $k=1$. It can be proved that $\sup_l|F_l|<\infty$. Thus, if $\sum_{l=0}^{\infty}|c_l|<\infty$, then

$$K(r,r)=O(1)\text{ as } r\to\infty. \tag{2.2}$$

If $Q\neq 0$ then (2.2) contradicts (2.1). It follows that if $Q\neq 0$ then equation (1.1) cannot be uniquely solvable for all $r>0$, so that NS procedure cannot be carried through if $Q\neq 0$ and $\sum_{l=0}^{\infty}|c_l|<\infty$. This proves Remark 1. □

Proof of Claim 1.

Suppose that $v(r)\in L_{1,1}$ and $Q_v:=\int_0^\infty rv(r)dr=0$, because otherwise NS procedure cannot be carried through as was proved in Remark 1.

If $Q_v=0$, then there is also no guarantee that NS procedure can be carried through. However, we claim that if one assumes that it can be carried through, then the set of potentials, which can possibly be obtained by NS procedure, is not dense in $L_{1,1}$ in the norm $\|q\|:=\int_0^\infty r|q(r)|dr$. In fact, any potential q such that $Q:=\int_0^\infty rq(r)dr\neq 0$, and the set of such potentials is dense in $L_{1,1}$, cannot be approximated with a prescribed accuracy by the potentials which can be possibly obtained by the NS procedure.

Let us prove this. Suppose that $q\in L_{1,1}$,

$$Q_q:=\int_0^\infty rq(r)dr\neq 0,\text{ and } \|v_n-q\|\to 0\text{ as } n\to\infty,$$

where the potentials $v_n\in L_{1,1}$ are obtained by the NS procedure, so that

$$Q_n:=\int_0^\infty rv_n(r)dr=0.$$

We assume $v_n\in L_{1,1}$ because otherwise v_n obviously cannot converge in the norm $||\cdot||$ to $q\in L_{1,1}$. Define a linear bounded on $L_{1,1}$ functional

$$f(q):=\int_0^\infty rq(r)dr,\quad |f(q)|\leq\|q\|,$$

where $\|q\|:=\int_0^\infty r|q(r)|dr$. The potentials $v\in L_{1,1}$, which can possibly be obtained by the NS procedure, belong to the null-space of f, that is $f(v)=0$.

If $\lim_{n\to\infty}\|v_n-q\|=0$, then $\lim_{n\to\infty}|f(q-v_n)|\leq\lim_{n\to\infty}\|q-v_n\|=0$. Since f is a linear bounded functional and $f(v_n)=0$, one gets: $f(q-v_n)=f(q)-f(v_n)=f(q)$. So if $f(q)\neq 0$ then

$$\lim_{n\to\infty}|f(q-v_n)|=|f(q)|\neq 0.$$

Therefore, no potential $q\in L_{1,1}$ with $Q_q\neq 0$ can be approximated arbitrarily accurately by a potential $v(r)\in L_{1,1}$ which can possibly be obtained by the NS procedure. Claim 1 is proved. □

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