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A counterexample to a uniqueness result

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Abstract

A counterexample is given to the uniqueness result given in the paper by J.Cox and K.Thompson, "Note on the uniqueness of the solution of an equation of interest in inverse scattering problem", J. Math.Phys., 11, N3, (1970), 815-817.

1 Introduction

In [3] the authors claimed that the integral equation

$$h(s) = -\int_0^r g(s,t)h(t)t^{-2}dt, \tag{1.1}$$

where $h(0) = 0$, has only the trivial solution for all $r > 0$. Here

$$g(s,t) = \sum_{m\in S}\gamma_m u_m(t)v_m(s), \quad s > t,$$

the set S is a finite set of distinct real numbers from the interval $(-0.5, \infty)$, u_m and v_m are the regular and irregular Bessel-Riccati functions, defined e.g. in [2], $g(s,t)$ is symmetric, and γ_m satisfy the following equation:

$$\sum_{m\in S}\gamma_m[m(m+1) - l(l+1)]^{-1} = 1, \quad l \in T, \tag{1.2}$$

where T is a finite set of distinct real numbers from the interval $(-0.5, \infty)$, and the sets T and S are disjoint. This uniqueness result is crucial for the arguments in [4]. For references on the inverse scattering with fixed-energy data see [1]-[9]. In this note a counterexample to the uniqueness claim from [3] is constructed. This counterexample invalidates the arguments in [3]. The uniqueness of the solution of similar equations in [2] (see equations (12.1.2) and (12.2.1) on pp.195-196 in [2]) and [5] does not hold for some $r > 0$, in general, also, as follows from the results in [10] and [8].

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2 A counterexample

Take the single-element sets $S = \{0\}$ and $T = \{2\}$. In this case equation (1.2) yields $\gamma_0 = -6$, $g(s,t) = -6u_0(t)v_0(s) := -6g_0(s,t)$ for $t < s$ and $g_0 = u_0(s)v_0(t)$ for $t > s$. Note that $(d^2 + 1)g_0 = \delta(s - t)$, where $\delta(s - t)$ is the delta-function, and d^2 stands for the second derivative. Therefore, applying the operator $d^2 + 1$ to equation (1.1), one gets $(d^2 + 1)h = 6s^{-2}h$. This equation has a nontrivial, regular at zero, solution $cu_2(s)$, where $c \neq 0$ is a constant. Without loss of generality we take $c = 1$ in what follows.

Note that $u_0(r) = \sin r$, $v_0(r) = -\cos r$, $u_2(r) = (3r^{-2} - 1)\sin r - 3r^{-1}\cos r$, and the Wronskian $u_0 v_0' - u_0' v_0 = 1$.

Define the function

$$p(r) := v_0(r)u_2'(r) - v_0'(r)u_2(r).$$

One may check, using the explicit formulas for v_0 and u_2, that

$$p(r) = 1 - \frac{3 + 3\cos^2(r)}{r^2} + \frac{3\sin(2r)}{r^3}.$$

Using this explicit formula, one checks that $p(r) = -\frac{r^2}{5} + o(r^2)$ as $r \to 0$, and that $p(r) > 0$ as $r \to \infty$. In fact, $p(r) \to 1$ as $r \to \infty$. Since $p(r)$ is continuous on $(0, \infty)$, one concludes that:

There is a number $R > 0$ such that $p(R) = 0$.

Let us now prove the following:

Claim: Equation (1.1) has a nontrivial solution $u_2(s)$ if $r = R$.

Proof of the claim:

One has:

$$-\int_0^r g(s,t)u_2(t)t^{-2}dt = \int_0^r g_0(s,t)6u_2(t)t^{-2}dt = \int_0^r g_0(s,t)(d^2 + 1)u_2(t)dt := J.$$

Integrating by parts twice and taking into account that u_0 and u_2 vanish at the origin, one gets:

$$J = u_2(s)[u_0(s)v_0'(s) - u_0'(s)v_0(s)] + u_0(s)[v_0(r)u_2'(r) - v_0'(r)u_2(r)] = u_2(s) + p(r)u_0(s).$$

We have used above the formula for the Wronskian: $u_0 v_0' - u_0' v_0 = 1$.

It follows from the foregoing equation for J that if, for some $r > 0$, one has $p(r) = 0$, then $u_2(s)$ solves equation (1.1) with this $r > 0$.

Since we have already proved that $p(R) = 0$, the claim follows. $\square$

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