APPLICATIONS OF THE DISCRETE MAXIMUM PRINCIPLE TO SOME PROBLEMS IN INDUSTRIAL MANAGEMENT

by

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INTRODUCTION

This report is concerned with demonstrating the wide applicability of the discrete maximum principle for solving different management problems. We have chosen fairly simple problems for this report in order to illustrate the manner of problem formulation and solution. Of importance is the fact that the discrete maximum principle provides another tool to solve management problems and may have some advantages in solving some classes of problems.

The maximum principle is an optimization technique which was originally developed for continuous processes by Pontryaging [11] in 1956. Various discrete versions of the maximum principle were proposed by Rozonoer [13], Chang [5], Katz [6], and by Fan and Wang [4]. The application of the maximum principle in the field of management and operations research is still not extensive. Transportation problems [3,7], a capital investment problem (allocation of a resource)[8], and one-dimensional production planning problem [9] were investigated recently.

To demonstrate the use of the algorithm several well known problems familiar to students in industrial engineering are formulated and solved. The six cases presented in this report are detailed numerical problems worked out to show the standard way of solving problems by the discrete maximum principle. They embrace problems with one, three, five, and twelve stages. The first case is the determination of economical lot sizes for inventory systems where the replenishment of stock is instantaneous and over a finite period of time. This problem is solved first by an empirical and then a graphical approach both of which will assist us in making quite clear the solution by the discrete maximum principle. This case provides an illustration for a single stage problem. The second case is a personnel scheduling problem which serves as a typical illustration for a process with memory in decision. A transportation problem with a non-linear cost function is discussed next, to illustrate a class of problems in which the equivalence of the objective function and the Hamiltonian function exists. A cattle breeding marketing problem is given next; this provides an illustration for a fixed end point process and a general N-stage optimal solution. A personnel and production scheduling problem with a non-linear cost function is set up; the formulation develops the optimal policy structure and serves to illustrate a two decision variable problem. Finally, a solution of a warehousing decision problem is attempted by the discrete maximum principle which involves much more computation than the simple analytical approach. However, this problem may serve to illustrate the importance of choosing state variables and decision variables correctly in a multistage process problem.

Some comparisons with other methods are mentioned; however, the efficiency of the discrete maximum principle in solving these problems is not compared with that of the other methods. Since the application of the maximum principle to this sort of problems is still in a stage of infancy, it is felt that much more experience in applying this principle is necessary before any intelligent judgment or comparison can be made.

THE ALGORITHM OF THE DISCRETE MAXIMUM PRINCIPLE [8]

A schematical representation of a simple multistage process is shown in Fig. 1. The process consists of N stages connected in series. The state of the process stream denoted by an s-dimensional vector, $x = (x_1, x_2, \ldots, x_3)$, is transformed at each stage according to an r-dimensional decision vector, $\theta = (\theta_1, \theta_2, \ldots, \theta_r)$, which represents the decisions made at that stage. The transformation of the process stream at the n-th stage is described by a set of performance equations,

$$x_i^n = T_i^n(x_1^{n-1}, x_2^{n-1}, \ldots, x_s^{n-1}; \theta_1^n, \theta_2^n, \ldots, \theta_r^n) ,$$

$$x^o = \alpha_i$$

$$i = 1, 2, \ldots, s; \qquad n = 1, 2, \ldots, N$$

or in vector form

$$x^n = T^n (x^{n-1}; \theta^n) \qquad n = 1, 2, \ldots, N \qquad (1)*$$

$$x^o = \alpha$$

A typical optimization problem associated with such a process is to find a sequence of θ^n, $n = 1, 2, \ldots, N$, subject to constraints

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*The superscript n indicates the stage number. The exponents are written with parentheses or brackets such as $(x^n)^2$ or $[T^n(x^{n-1}; \theta^n)]^2$.

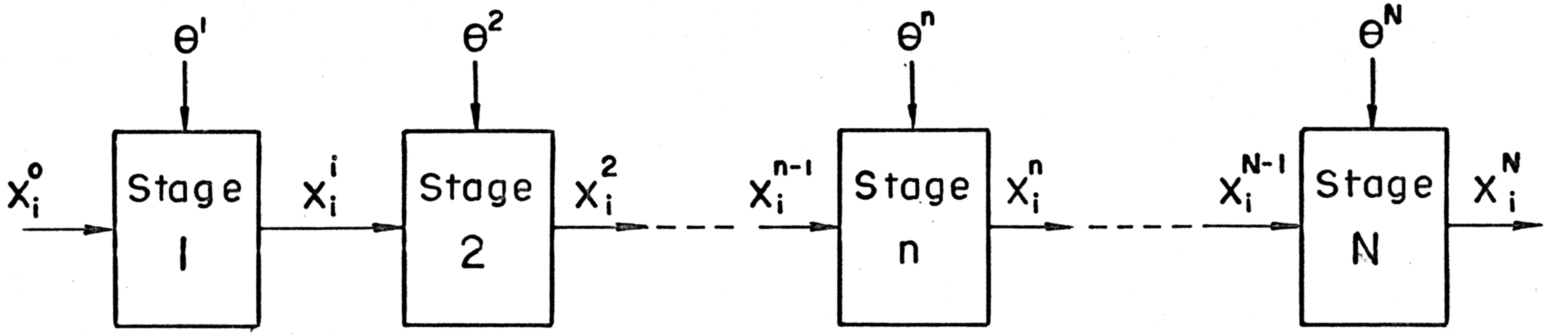

Fig. 1. Multistage decision process

$$\phi_i^n\ (\theta_1^n,\ \theta_2^n,\ldots.,\ \theta_r^n) \leq 0 \quad , \qquad \begin{array}{l} n = 1,\ 2,\ldots.,\ N, \\ i = 1,\ 2,\ldots.,\ r, \end{array} \tag{2}$$

which makes a function of the state variable of the final stage

$$S = \sum_{i=1}^{s} c_i x_i^N \qquad\qquad c_i = \text{constant} \tag{3}$$

an extremum when the initial condition $x^o = \alpha$ is given. The function

$$S = \sum_{i=1}^{s} c_i x_i^N$$

which is to be maximized (or minimized), is the objective function of the process.

The procedure for solving such an optimization problem by the discrete maximum principle is to introduce an s-dimensional adjoint vector z^n and a Hamiltonian function H^n which satisfy the following relations:

$$H^n = \sum_{i=1}^{s} z_i^n T_i^n(x^{n-1};\ \theta^n) \quad , \quad n = 1,\ 2,\ldots,\ N, \tag{4}$$

$$z_i^{n-1} = \frac{\partial H^n}{\partial x_i^{n-1}} \quad , \quad i = 1,\ 2,\ldots,\ s; \quad n = 1,\ 2,\ldots,\ N, \tag{5}$$

and

$$z_i^N = c_i \quad , \qquad i = 1,\ 2,\ldots,\ s. \tag{6}$$

If the optimal decision vector function $\bar{\theta}^n$, which makes the objective function S an extremum (maximum or minimum), is interior to the set of admissible decisions θ^n, the set given by equation (2), a necessary condition for S to be a (local) extremum with respect to θ^n, is

$$\frac{\partial H^n}{\partial \theta^n} = 0 \quad , \quad n = 1, 2, \ldots, N \tag{7}$$

If $\bar{\theta}^n$ is at a boundary of the set, it can be determined from the condition that H^n is (locally) extremum. The following special cases can be considered:

(i) A necessary condition for S to be a (local) extremum with respect to θ^n , is

$$\frac{\partial H^n}{\partial \theta^n} = 0 \quad , \quad n = 1, 2, \ldots, N$$

(ii) When the performance equation is linear in state variables x_i^{n-1} , namely

$$T_i^n(x^{n-1}; \theta^n) = \sum_{j=1}^{s} A_{ji}^n(\theta^n) x_j^{n-1} + f_i^n(\theta^n) \tag{8}$$

A local maximum (or minimum) of the objective function corresponds to a local maximum (or minimum) of the Hamiltonian function. In other words,

$$H^n = \text{maximum (or minimum)}$$

is the necessary condition for the objective function to be locally maximum (or minimum).

(iii) When A_{ji}^n in equation (8) is constant, or when the optimal decision is always known to be on the boundary of its admissible decision, the objective function is absolutely maximum (or minimum) if and only if H^n is absolutely maximum (or minimum).

(iv) When the performance equation is linear in its arguments, that is,

$$T_i^n(x^{n-1};\theta^n) = \sum_{j=1}^{s} A_{ji}^n x_j^{n-1} + \sum_{j=1}^{r} B_{ji}^n \theta_j^n \quad , \tag{8a}$$

then,

$$H^n = \text{maximum (or minimum)},$$

is necessary as well as sufficient for the objective function, S, to be absolutely (or globally) maximum (or minimum).

The algorithm of the discrete maximum principle can be extended to handle a variety of problems usually encountered in practice, such as processes with fixed end points, processes with choice of extra parameters, processes with memory in decisions, processes with arbitrary final measures as the objective function, and processes with cumulated measures as objective functions. The details of these extensions can be seen in references [4,8].

CASE 1. ECONOMICAL LOT SIZE MODEL (DETERMINATIONS WHEN REPLENISHMENT IS INSTANTANEOUS)

The general solution of the economical lot-size depends on the method of replenishing stock. Replenishment can be instantaneous, or it can occur over a finite period of time. The economical lot-size determination involves the consideration of two major factors, cost and quantity. Three kinds of costs are significant and are subject to control:

1. The cost of replenishing inventory (order cost),
2. The cost of carrying inventory,
3. The cost of incurring shortage.

In this model, we shall consider only the first two costs. These costs are a function of lot size, and yearly requirements. The setup costs arise from the internal and external costs associated with a purchase order. The expense of carrying inventory includes such costs as property taxes, insurance, storage, handling, and interest. The total variable cost per year is the objective function to be minimized.

The inventory fluctuations are illustrated in Fig. 1-1. In this situation we have the ideal conditions, of constant rate of usage, zero inventory at each replenishment point and instantaneous replenishment. A specific problem will be used as an illustration.

PROBLEM [10]

A customer orders parts from a manufacturing company at a uniform rate of 3650 parts per year. The company makes the parts on a machine that can produce any number of parts at a time. The cost of setting up the machine for a production run is $10. The

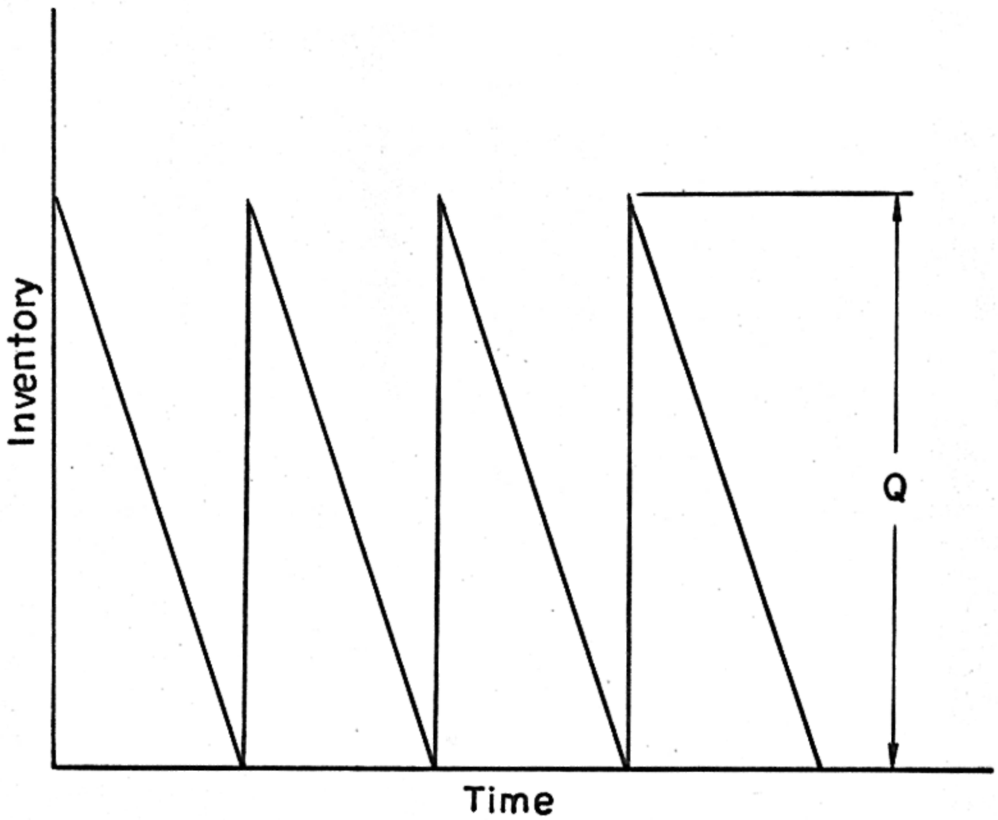

Fig. I-I. Inventory under constant usage and instantaneous replenishment cost of carrying inventory is \$ 0.10 per part per year. No shortages are allowed to occur. The manufacturing company's inventory problem is to find how many parts should be produced for each production run.

Let:

R = yearly requirements = 3650

U = setup cost = \$10.

I = carrying cost per piece per year = \$ 0.10

(i) Solution by a Tabular Analysis.

A tabular analysis will first be used. Different lot size decisions will be evaluated and compared as shown in table 1. The table indicates that a lot size of 800 parts will minimize the total cost of the system and this minimum is \$90. This method of approach, though suitable in some instances, is generally not the best method for finding the optimal lot size (a) becuase it does not insure that the best alternative is actually included in the list of alternatives and (b) because it is lengthy and time consuming.

(ii) A Graphical Solution.

The replenishing cost and the inventory carrying cost are plotted against order quantity. The sum of these two costs constitute the total cost. These curves are shown in Fig. 1-2. This graph shows that carrying costs increase linearly with an increase in the lot size. The ordering cost decreases at a decreasing rate with an increase in the lot size. The total cost decreases at first with an increase in the lot size and then increases. The

TABLE 1-1. TOTAL COST PER YEAR OF AN INVENTORY SYSTEM FOR VARIOUS LOT SIZES

LOT SIZE Q	AVERAGE INVENTORY	REPLENISHING COST PER YEAR	CARRYING COST PER YEAR	TOTAL COST PER YEAR
200	100	190	10	200
500	250	80	25	105
700	350	60	35	95
800	400	50	40	90
900	450	50	45	95
1200	600	40	60	100

problem is to find the lowest point on the total cost line. This would indicate that the lot size is Q and the total cost is P.

(iii) Solution by the Discrete Maximum Principle.

Let us consider the problem as a one stage problem, and let

θ^1 = lot size, the decision variable,

x_1^1 = carrying cost (average inventory x carrying cost per piece per year),

x_2^1 = ordering cost (number of orders x cost per order),

R = annual requirements,

I = carrying cost per piece per year,

U = setup cost per order,

The performance equations for x_1^1 and x_2^1 are

$$x_1^1 = T_1(x^o \; ; \; \theta^1) = x_1^o + \frac{\theta^1 I}{2} \tag{1-1}$$

$$x_1^o = 0 \quad , \tag{1-1a}$$

$$x_2^1 = T_2(x^o \; ; \; \theta^1) = x_2^o + \frac{RU}{\theta^1} \quad , \tag{1-2}$$

$$x_2^o = 0 \quad , \tag{1-2a}$$

The objective function is to minimize:

$$S = \sum_{i=1}^{2} c_i x_i^1 = c_1 x_1^1 + c_2 x_2^1 = x_1^1 + x_2^1 \quad . \tag{1-3}$$

Therefore,

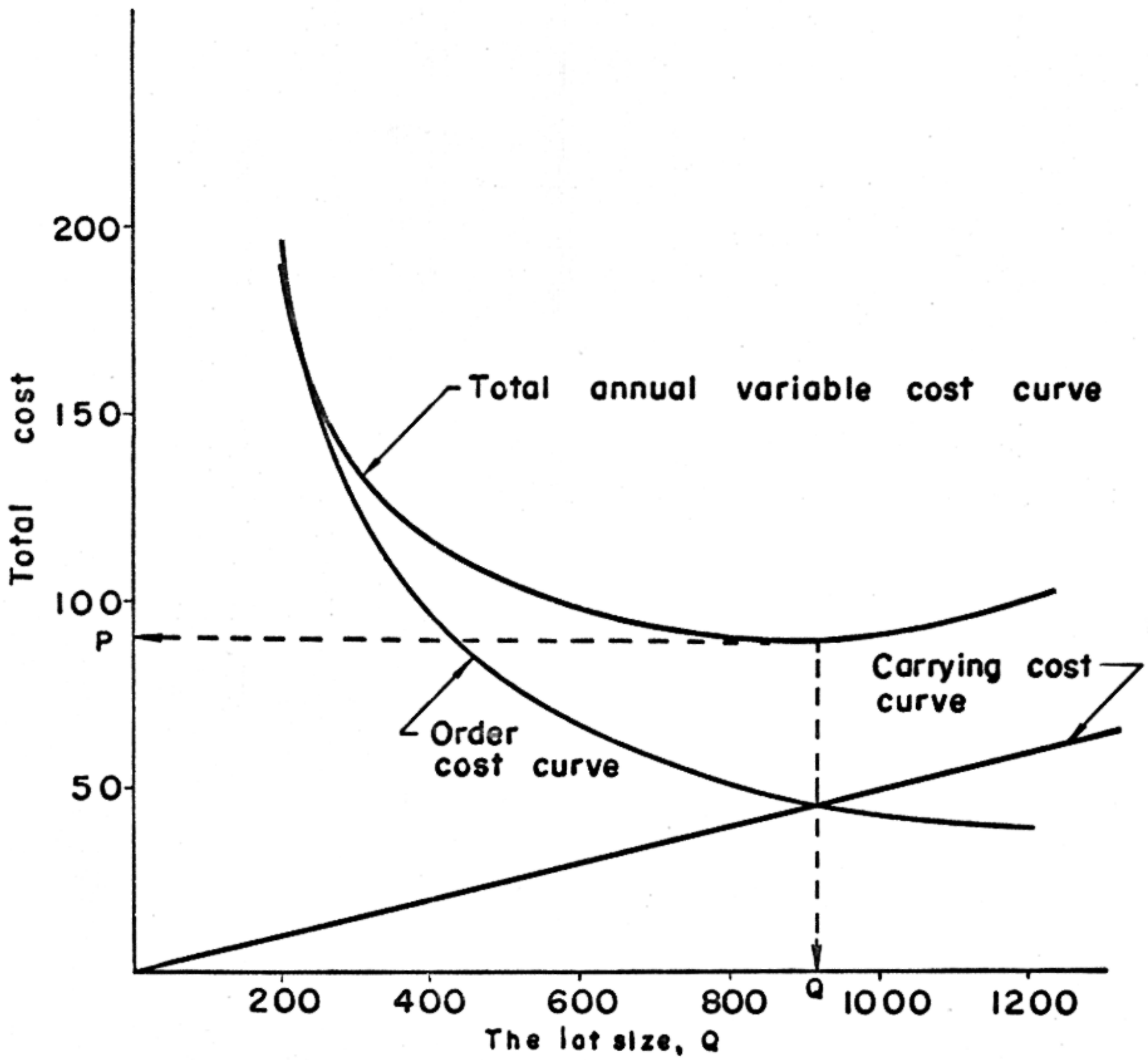

Fig. 1-2. INVENTORY AND COST BEHAVIOR

$$c_1 = 1 \quad , \tag{1-3a}$$

$$c_2 = 1 \quad . \tag{1-3b}$$

Introducing a 2-dimensional adjoint vector z and a Hamiltonian function H^1 which satisfy the following relations (see equations [4], [5], and [6]):

$$H^1 = z_1^1(x_1^o + \frac{\theta^1 I}{2}) + z_2^1(x_2^o + \frac{RU}{\theta^1}) \quad , \tag{1-4}$$

$$z_1^o = \frac{\partial H^1}{\partial x_1^o} = z_1^1 \quad , \tag{1-5}$$

$$z_1^1 = c_1 = 1 \quad , \tag{1-5a}$$

$$z_2^o = \frac{\partial H^1}{\partial x_2^o} = z_2^1 \quad , \tag{1-6}$$

$$z_2^1 = c_2 = 1 \quad . \tag{1-6a}$$

Hence the Hamiltonian function can be rewritten as

$$H^1 = (x_1^o + \frac{\theta^1 I}{2}) + (x_2^o + \frac{RU}{\theta^1}) \quad . \tag{1-7}$$

A necessary condition for S to be a (local) extremum with respect to θ^1 is (see equation [7])

$$\frac{\partial H^1}{\partial \theta^1} = \frac{I}{2} - \frac{RU}{(\theta^1)^2} = 0 \quad . \tag{1-8}$$

Therefore,

$$\bar{\theta}^1 = \sqrt{\frac{2RU}{I}} \tag{1-9}$$

The value of θ^1 is the answer to the problem posed. It gives the economical lot size with the advantage of a general solution. Equation (1-9) turns out to be exactly the same expression for the economical lot size obtained by the differential calculus [10].

Since we have already solved for θ^1 we can determine the equation for the number of orders directly:

$$\text{Number of orders} = \frac{R}{\theta^1} = \sqrt{\frac{RI}{2U}} \tag{1-10}$$

Returning to equation (1-9), let us illustrate its use on problem 1:

$$\theta^1 = \sqrt{\frac{2(3650)(10)}{0.10}} = 854 \text{ units.}$$

Substituting the optimal decision value into equations (1-1) and (1-2) yields

$$x_1^1 = 42.7 \quad ,$$

$$x_2^1 = 42.8 \quad .$$

Substituting the values of x_1^1 and x_2^1 into equation (1-3) yields

$$S = \$\ 85.50 \quad .$$

Therefore the total variable cost per year will be $85.50.

CASE 1a. ECONOMICAL LOT SIZE MODEL (DETERMINATIONS WHEN REPLENISHMENT OCCURS OVER A FINITE PERIOD OF TIME)

In the previous case we have assumed that stock is replenished immediately. In some cases, particularly in manufacturing, production batches take a considerable time to complete, and inventory replenishment is being manufactured while demands are being met. Fig. 1-3 shows this situation. In this case, replenishment of the inventory occurs over the time period t_1 of each cycle, and usage occurs during $t_1 + t_2$ or the entire cycle.

The required definitions are:

R = yearly sales requirements,

U = setup cost,

p = production rate per year,

θ^1= lot size manufactured.

With reference to Fig. 1-3, the inventory is increased during the production period at a rate of $(p-R)$ units per year. After t_1 years, production ceases, and inventory is depleted at a rate of s units per year. The maximum inventory will be $(p-R)t_1$, where t_1 is the production period, with an average inventory per cycle of $(p-R)t_1/2$. Now, $t_1 = \frac{\theta^1}{p}$ since, during the replenishment period, a lot size is produced. Substituting t_1 in the above, and multiplying by the number of cycles per year, we obtain:

Average inventory per year = (avg. inventory per cycle)(# of runs)

$$= ((p-R)t_1/2)(\tfrac{1}{t})t$$

$$= (p-R)\theta^1/2p = \frac{\theta^1}{2}(1-R/p) \quad .$$

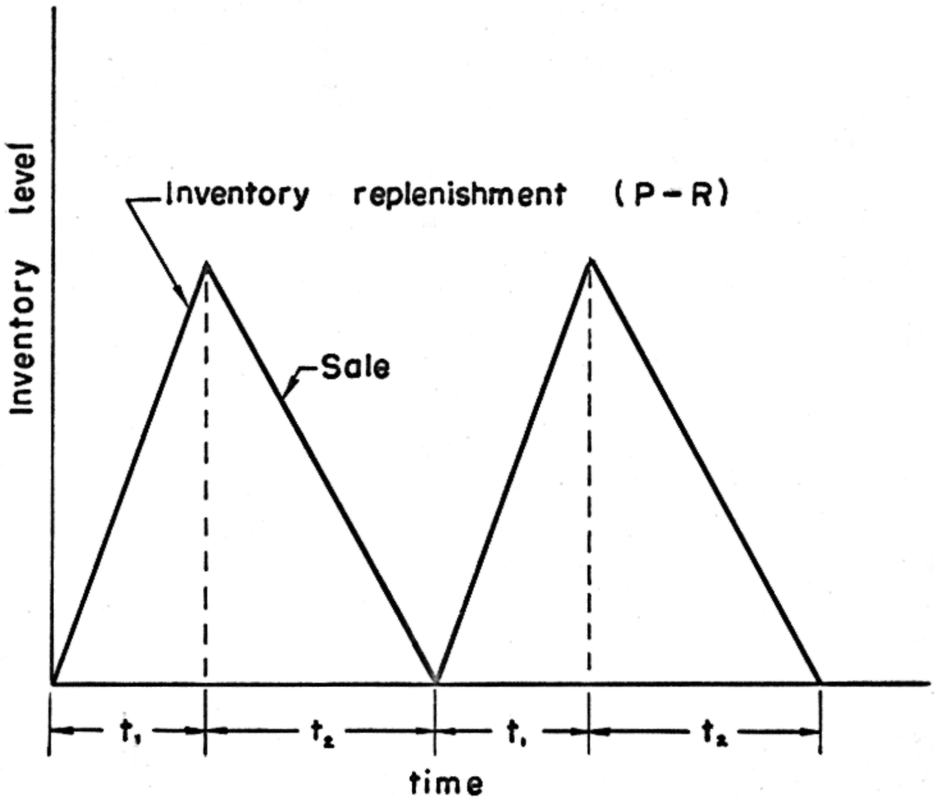

Fig 1-3. Inventory under constant usage and replenishment over a finite period of time.

Then, by following the procedure used in the previous section, we obtain (see equation [1]):

$$x_1^1 = x_1^o + \frac{\theta^1}{2}(1 - R/p)I = T_1(x^o \; ; \; \theta^1) \quad , \qquad (1a\text{-}1)$$

$$x_1^o = 0 \quad , \qquad (1a\text{-}1a)$$

$$x_2^1 = x_2^o + \frac{RU}{\theta^1} = T_2(x^o \; ; \; \theta^1) \quad , \qquad (1a\text{-}2)$$

$$x_2^o = 0 \quad . \qquad (1a\text{-}2a)$$

Following the same steps used previously in the determination of the economical lot size when the replenishment of stock is instantaneous (see equations [1-3] to [1-7]), we follow the steps described in the algorithm (equations [3] to [7]):

The objective function is to minimize

$$S = \sum_{i=1}^{2} c_i x_i^1 = c_1 x_1^1 + c_2 x_2^1 = x_1^1 + x_2^1 \quad . \qquad (1a\text{-}3)$$

Therefore,

$$c_1 = 1 \quad , \qquad (1a\text{-}4a)$$

$$c_2 = 1 \quad . \qquad (1a\text{-}4b)$$

The Hamiltonian is

$$H^1 = z_1^1 x_1^1 + z_2^1 x_2^1$$

$$= z_1^1 \left[\frac{\theta^1}{2}(1-\frac{R}{p})I\right] + z_2^1 \left[\frac{RU}{\theta^1}\right] \quad . \qquad (1a\text{-}5)$$

The adjoint vector z^1 is

$$z_1^1 = c_1 = 1 \quad , \tag{1a-6a}$$

$$z_2^1 = c_2 = 1 \quad . \tag{1a-6b}$$

Hence the Hamiltonian becomes

$$H^1 = \frac{\theta^1}{2}(1-\frac{R}{p})I + \frac{RU}{\theta^1} \quad . \tag{1a-7}$$

The necessary condition for S to be an extremum with respect to θ^1 is

$$\frac{\partial H^1}{\partial \theta^1} = 0 = \frac{1}{2}(1-\frac{R}{p})I - \frac{RU}{(\theta^1)^2} \tag{1a-8}$$

or

$$\theta^1 = \sqrt{\frac{2RU}{I(1-R/p)}} \tag{1a-9}$$

which represents the economical manufactured lot size when the replenishment of stock is over a finite period of time.

NUMERICAL EXAMPLE [12].

A contractor has to supply 10,000 bearings per day to an automobile manufacturer. He finds that, when he starts a production run, he can produce 25,000 bearings per day. The cost of holding a bearing in stock for one year is $0.02, and the setup cost of a production run is $18.00. How frequently should production runs be made?

Here

$R = 10{,}000x(360)$ bearings/year,

$U = \$\ 18.00$ /setup,

$I = \$\ 0.02$ per year,

$p = 25{,}000\ (360)$ per year.

To obtain the optimum run size we can use equation (1a-9). The result is

$$\theta^1 = \sqrt{\frac{2RU}{(1-R/p)I}} = \sqrt{\frac{2(18)10^4(360)}{0.02(1-10/25)}} = 104{,}000 \text{ bearings,}$$

and

$$T = \frac{\theta^1}{R} = \frac{104{,}000}{10{,}000} = 10.4 \text{ days.}$$

CASE 2. A PRODUCTION SCHEDULING PROBLEM-ILLUSTRATION OF A PROCESS WITH MEMORY IN DECISION

PROBLEM [1]

The following data are a sales forecast, initial inventory and production, and cost functions in a situation in which it is desired to obtain the lowest cost production rates.

Sales by period (must be satisfied): $S_1 = 30$,

$S_2 = 10$,

$S_3 = 40$.

Initial inventory, $I_o = 12$.

Production in period 0, $P_o = 15$.

Inventory required at end of period 3, $I_3 = 10$.

Costs: $\$100 \ (P_t - P_{t-1})^2$,

$\$\ 20 \ (10 - I_t)^2$ per period.

Required are P_1, P_2, and P_3 which minimize cost.

FORMULATION AND SOLUTION BY THE DISCRETE MAXIMUM PRINCIPLE

This is a three stage problem. The state variables and the decision variable are defined as follows:

θ^n = production at the n^{th} period,

x_1^n = inventory at the end of the n^{th} period,

x_2^n = cost up to and including the n^{th} period.

A material balance at each stage gives:

$$I^{n-1} + p^n = S^n + I^n \quad .$$

Therefore,

$$x_1^n = x_1^{n-1} + \theta^n - S^n \quad , \qquad n = 1, 2, 3 \quad . \tag{2-1}$$

$$x_1^o = I_o = 12 \quad , \tag{2-1a}$$

$$x_1^3 = I_3 = 10 \quad , \tag{2-1b}$$

$$x_2^n = x_2^{n-1} + 100(\theta^n-\theta^{n-1})^2 + 20(10-x_1^n)^2 \quad , \tag{2-2}$$

$$x_2^o = 0 \quad . \tag{2-2a}$$

Substituting the equation (2-1) into equation (2-2) yields

$$x_2^n = x_2^{n-1} + 100(\theta^n-\theta^{n-1})^2 + 20(10-x_1^{n-1}-\theta^n+S^n) \quad , \tag{2-3}$$

$$= T_2(x^{n-1} \; ; \; \theta^n \; ; \; \theta^{n-1}) \quad , \qquad n = 1, 2, 3,$$

$$x_2^o = 0 \quad . \tag{2-3a}$$

Equation (2-3) shows that the transformation at each stage is not only a function of the decision variable θ^{n-1}, that is, the previous decision has an effect on the subsequent stage. This process is defined as a process with memory in decisions. To solve this problem by the discrete maximum principle the following transformation must be done.

Let

$$x_3^n = \theta^n \quad , \qquad n = 0, 1, 2, 3 \tag{2-4}$$

$$w^n = \theta^n - \theta^{n-1} \quad , \qquad n = 1, 2, 3 \tag{2-5}$$

$$x_3^n = x_3^{n-1} + w^n \quad , \qquad n = 1, 2, 3 \tag{2-6}$$

$$x_3^o = 15 \quad . \tag{2-6a}$$

The new decision variable is w^n. Substituting equations (2-4), (2-5), and (2-6) into equations (2-1) and (2-3) gives

$$x_1^n = x_1^{n-1} + x_3^{n-1} + w^n - S^n \quad , \quad n = 1, 2, 3 \tag{2-7}$$

$$x_2^n = x_2^{n-1} + 100(w^n)^2 + 20(10-x_1^{n-1}-x_3^{n-1}-w^n+S^n)^2 \quad , \qquad n = 1, 2, 3 \quad . \tag{2-8}$$

Then the Hamiltonian and the adjoint vector are

$$H^n = z_1^n(x_1^{n-1}+x_3^{n-1}+w^n-S^n) + z_2^n \{x_2^{n-1}+100(w^n)^2+ 20(10-x_1^{n-1}-x_3^{n-1}-w^n+S^n)^2\} + z_3^n(x_3^{n-1}+w^n) \quad , \qquad n = 1, 2, 3 \tag{2-9}$$

$$z_1^{n-1} = \frac{\partial H^n}{\partial x_1^{n-1}} = z_1^n - 40z_2^n(10-x_1^{n-1}-x_3^{n-1}-w^n+S^n) \quad , \tag{2-10a}$$

$$z_2^{n-1} = \frac{\partial H^n}{\partial x_2^{n-1}} = z_2^n \quad , \tag{2-10b}$$

$$z_3^{n-1} = \frac{\partial H^n}{\partial x_3^{n-1}} = z_1^n - 40z_2^n(10-x_1^{n-1}-x_3^{n-1}-w^n+S^n) + z_3^n \tag{2-10c}$$

The objective function is to minimize

$$S = \sum_{i=1}^{3} c_i x_i^N = c_1 x_1^N + c_2 x_2^N + c_3 x_3^N = x_2^N \tag{2-11}$$

Therefore,

$$c_1 = 0,$$

$$c_2 = 1,$$

$$c_3 = 0,$$

and

$$z_1^N \neq c_1 \text{ because } x_1^N \text{ is fixed,} \tag{2-12a}$$

$$z_2^N = c_2 = 1, \tag{2-12b}$$

$$z_3^N = c_3 = 0. \tag{2-12c}$$

Combination of equations (2-10b) and (2-12b) yields

$$z_2^n = 1 \quad , \qquad n = 1, 2, 3 \quad . \tag{2-13}$$

According to the maximum principle, the optimal choice of the decision variable will be found where

$$\frac{\partial H^n}{\partial w^n} = 0 \quad . \tag{2-14}$$

Equation (2-14) gives

$$z_1^n \frac{\partial x_1^n}{\partial w^n} + z_2^n \frac{\partial x_2^n}{\partial w^n} + z_3^n \frac{\partial x_3^n}{\partial w^n} = 0 \quad , \tag{2-15}$$

Taking partial derivatives of equations (2-6), (2-7), and (2-8) with respect to w^n, and inserting the respective values into equation (2-15) yields

$$z_1^n = 40(10-x_1^{n-1}-x_3^{n-1}-w^n+S^n) - 200(w^n) - z_3^n \quad . \quad (2\text{-}16)$$

Combining equations (2-10a), (2-13), and (2-16) gives

$$z_3^n = 40(10-x_1^{n-1}-x_3^{n-1}-w^n+S^n) - 200w^n + 200w^{n+1} + z_3^{n+1} \quad . \quad (2\text{-}17)$$

Combining equations (2-10c), (2-16), and (2-17) gives

$$z_3^{n+1} = 200w^n - 400w^{n+1} - 40(10-x_1^{n-1}-x_3^{n-1}-w^n+S^n) \quad . \quad (2\text{-}18)$$

Inserting equations (2-18) into (2-17) gives the following recurrence relation for optimal conditions:

$$0 = 240w^{n-1} - 400w^n - 40(10-x_1^{n-2}-x_3^{n-2}+S^{n-1}) + 200w^{n+1} \quad (2\text{-}19)$$

The optimal sequences of w^n can now be obtained by the following procedure, utilizing the recurrence equation (2-19) and the performance equations (2-6) and (2-7).

CALCULATION PROCEDURE

STEP 1. Choose $n = 2$ and substitute the known values into (2-19) to obtain

$$240\ w^1 - 400\ w^2 + 200\ w^3 = 520 \quad . \quad (\ A\ )$$

STEP 2. Assume a value for w^1.

STEP 3. Calculate x_1^1 and x_3^1 from equations (2-7) and (2-6) respectively.

STEP 4. Assume a value for w^2.

STEP 5. Calculate x_1^2 from equation (2-7) and x_3^2 from equation (2-6).

STEP 6. With the values of w^1 and w^2, calculate w^3 from equation (A).

STEP 7. Calculate x_1^3 from equation (2-7).

STEP 8. If x_1^3 so obtained is equal to x_1^3 given, the problem is solved. If the calculated value is negative or less than the given value, assume another value w^2 and repeat from step 5. If the calculated value is greater than the given value, assume another value for w^1 and repeat from step 3.

STEP 9. Substitute the values of w^n, x_1^n and x_3^n into equations (2-8) and obtain the total minimum cost.

To illustrate the computational procedure we shall present only three trials.

	TRIAL I	TRIAL II	TRIAL III
Step 2	$w^1 = 4$		$w^1 = 8$
Step 3	$x_1^1 = 1$		$x_1^1 = 5$
	$x_3^1 = 19$		$x_3^1 = 23$
Step 4	$w^2 = 1$	$w^2 = 6$	$w^2 = 4$

Step 5	$x_1^2 = 11$	$x_1^2 = 16$	$x_1^2 = 22$
	$x_3^2 = 20$	$x_3^2 = 25$	$x_3^2 = 27$
Step 6	$w^3 = -0.2$	$w^3 = 9.8$	$w^3 = 1$
Step 7	$x_1^3 = -9.2$	$x_1^3 = 10.9$	$x_1^3 = 10$
	Repeat from step 4 (next trial).	Repeat from step 1.	x_1^3 (cal.) = x_1^3 (given)

Having found the optimum sequence of w^n values, the total minimum cost can be determined according to step 9. Thus,

$$x_2^3 = \$\ 11{,}480$$

If we examine the recurrence equation (2-19) closely it can be observed that this equation serves for problems with more than three stages. The calculation steps are valid with some modification and still only two values w^1 and w^2 would have to be assumed in the trial and error procedure.

CASE 3. A TRANSPORTATION PROBLEM WITH NON-LINEAR COST FUNCTION

The transportation problem involves a number of shipping sources and a number of destinations. Each source has a certain maximum capacity and each destination has a certain requirement. There is a cost per unit for shipment from each depot to each destination. The objective is to satisfy the destination requirements within the capacity restrictions of the depots with the minimum total cost.

The problem is presented by table 3-1.

FORMULATION OF THE PROBLEM BY THE DISCRETE MAXIMUM PRINCIPLE [4,7]

Let

θ_i^n = the quantity of the resource sent from the i^{th} depot (source) to the n^{th} demand point (destination), and

$F_i^n(\theta_i^n)$ = the cost incurred by this operation.

The problem is to determine the values of θ_i^n, i = 1, 2; n = 1, 2, 3, 4, so as to minimize the total cost of transporting the resources

$$S = \sum_{n=1}^{4} \sum_{i=1}^{2} F_i^n(\theta_i^n)$$

Where the non-linear cost function has the form

$$F_i^n(\theta_i^n) = a_i^n \theta_i^n + b_i^n (\theta_i^n)^2$$

where a_i and b_i are constants.

Table 3-1. Transportation cost and requirements for a two origin and four demand point problem

	depots				
i	1		2		
n	a_1^n	b_1^n	a_2^n	b_2^n	D^n
demand points 1	2.0		4.1		25
2	3.0	0.01	2.1		45
3	1.5		1.1	0.1	15
4	6.0	-0.05	2.0		10
W_i	55		40		95

Constraints to be satisfied are:

(i) $\theta_i^n \geq 0$,

(ii) $\sum_{n=1}^{4} \theta_i^n = W_i$, number of units of the resource available at the i^{th} depot, i = 1, 2,

(iii) $\sum_{i=1}^{2} \theta_i^n = D^n$, number of units of the resource required by the i^{th} demand point, n = 1, 2, 3, 4.

Defining the demand points as stages and the total amount of resource which has been transported from the first depot to the first n stages as a state variable, then

$$x_1^n = x_1^{n-1} + \theta_1^n \quad , \qquad n = 1, 2, 3, 4, \tag{3-2}$$

$$x_1^o = 0 \quad , \tag{3-2a}$$

$$x_1^N = W_1 \quad , \qquad N = 4 \quad .$$

It must be noted that although there are 2 depots in the problem, there is only one state variable. This is because the demand by each stage is preassigned; hence, the number of units supplied from the second depot to the n^{th} stage can be obtained by subtracting the units supplied to the n^{th} stage from the first depot from the total number of units required by the n^{th} stage. That is,

$$\theta_2^n = D^n - \theta_1^n \quad .$$

Since the objective of the problem is to minimize the total cost of transportation, we define this objective as the second state variable which satisfies the following performance equation:

$$x_2^n = x_2^{n-1} + \sum_{i=1}^{2} F_i^n(\theta_i^n) \quad , \quad n = 1, 2, 3, 4 \tag{3-3}$$

$$x_2^o = 0 \quad . \tag{3-3a}$$

It can be shown that x_2^4 is equal to the total cost of transportation. Hence, the problem of minimizing the total cost of transportation becomes that of minimizing the final value of the second state variable, x_2^4, by the proper choice of the sequence of θ_1^n, $n = 1, 2, 3, 4$, for the process described by equations (3-2) and (3-3).

Equations (3-2) and (3-3) with the cost function presented in equation (3-1) are in the form of equation (8) with A_{ij} = constant described in the algorithm of the discrete maximum principle. Therefore, as specified the objective function is an absolute maximum (or minimum) if and only if H^n is absolute maximum (or minimum). In other words, the absolute maximum (or minimum) of H^n implies the absolute maximum (or minimum) of S, the objective function.

The Hamiltonian function and the adjoint vector are

$$H^n = z_1^n(x_1^{n-1}+\theta_1^n) + z_2^n[x_2^{n-1}+\sum_{i=1}^{2} F_i^n(\theta_i^n)], \quad n = 1, 2, \ldots, 4 \tag{3-4}$$

and

$$z_1^{n-1} = \frac{\partial H^n}{\partial x_1^{n-1}} = z_1^n \quad , \qquad (3\text{-}5)$$

$$n = 1,\ 2,\ 3,\ 4$$

$$z_2^{n-1} = \frac{\partial H^n}{\partial x_2^{n-1}} = z_2^n \quad , \qquad (3\text{-}6)$$

since

$$z_2^N = c_2 = 1 \quad ,$$

thus,

$$z_2^n = 1 \quad , \quad n = 1,\ 2,\ 3,\ 4 \quad .$$

Since z_1^n and x_1^{n-1} are considered to be constants in searching for a stationary or minimum point (with respect to θ^n), of the Hamiltonian function given by equation (3-4), it is convenient to define the variable part of the Hamiltonian function as

$$H_v^n = z_i^n \theta_i^n + \sum_{i=1}^{2} F_i^n(\theta_i^n) \quad . \qquad (3\text{-}7)$$

Substituting the value of $F_i^n(\theta_i^n)$ into the above equation yields

$$H_v^n = (z_1^n + a_1^n - a_2^n - 2b_2^n D^n)\theta_1^n + (b_1^n + b_2^n)(\theta_1^n)^2 \quad .$$

Stage 1:

The variable part of the Hamiltonian equation for the first demand point (stage) is

$$H_v^1 = (z_1^1 - 2.1)\theta_1^1 \quad , \quad \bar{z}_1^1 = 2.1$$

At this stage the Hamiltonian function H_v^1 is a linear function with respect to θ_1^1. Thus the optimal decision occurs at one point of the boundary. The minimum conditions at which H_v^1 = min. are then evaluated according to the sign of the value of $(z_1^1-2.1)$. The conditions are

(a) H_v^1 = min. at $\theta_1^1 = 0$ when $z_1^1 > 2.1$,

(b) H_v^1 = min. at $0 \leq \theta_1^1 \leq 25$ when $z_1^1 = 2.1$,

(c) H_v^1 = min. at $\theta_1^1 = 25$ when $z_1^1 < 2.1$.

Stage 2:

The variable part of the Hamiltonian equation for the second demand point (stage) is

$$H_v^2 = (z_1^2 + 0.9)\theta_1^2 + 0.01(\theta_1^2)^2 \quad .$$

According to the maximum principle algorithm, the optimal decision will be found where

$$\frac{\partial H_v^n}{\partial \theta_1^n} = 0 \quad .$$

Then,

$$\frac{\partial H_v^2}{\partial \theta_1^2} = (z_1^2 + 0.9) + 0.02(\theta_1^2) = 0$$

and

$$\frac{\partial^2 H_v^2}{\partial (\theta_1^2)^2} = 0.02 > 0 \quad .$$

Therefore, H_v^2 is minimum at the following conditions

$$\theta_1^2 = 0 \qquad \text{when} \qquad z_1^2 = -0.9 \quad ,$$

$$\theta_1^2 = 45 \qquad \text{when} \qquad z_1^2 = -1.8 \quad ,$$

$$\theta_1^2 = -50z_1^2-45 \qquad \text{when} \qquad -1.8 \leq z_1^2 \leq -0.9 \quad .$$

The conditions for H_v^2 to be minimum are shown in figure 3-1. This graph shows that the optimal decision is at a boundary of the admissible set. Having this condition and knowing that the performance equation is linear in the state variable x_1^{n-1} , the following condition of the discrete maximum principle is satisfied:

$$S \text{ minimum} \rightleftarrows H^n \text{ minimum}$$

Stage 3:

The variable part of the Hamiltonian equation for the third demand point (stage) is

$$H_v^3 = (z^3 - 2.6)\theta_1^3 + 0.1(\theta_1^3)^2 \quad .$$

At this stage the Hamiltonian function H_v^3 has the same form as that in the second stage. Consequently, the optimal decision can be obtained in the same way. Then,

$$\frac{\partial H_v^3}{\partial \theta_1^3} = 0 = (z_1^3 - 2.6) + 0.2(\theta_1^3) \quad ,$$

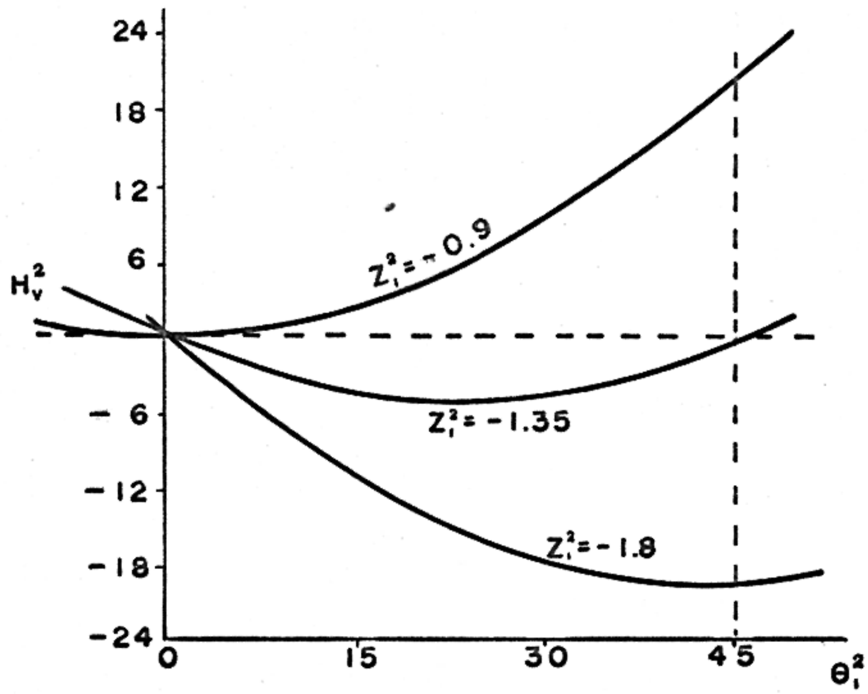

Fig. 3-1. H_v^2 vs θ_1^2

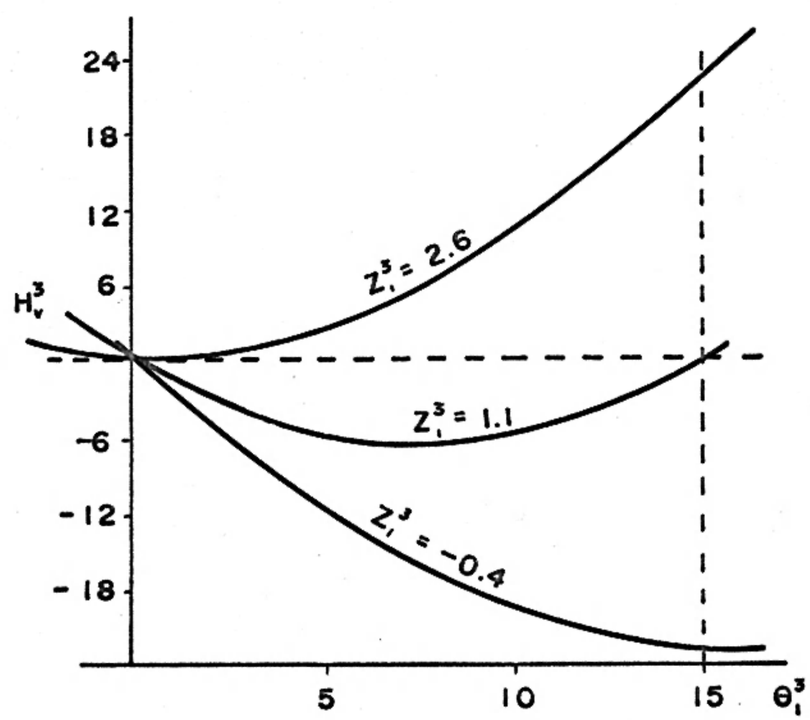

Fig. 3-2. H_v^3 vs θ_1^3

$$\frac{\partial^2 H_v^3}{\partial(\theta_1^3)^2} = 0.2 > 0 \quad .$$

Therefore, H_v^3 is a minimum at the following conditions:

$$\theta_1^3 = 0 \qquad \text{if} \qquad z_1^3 = 2.6 \quad ,$$

$$\theta_1^3 = 15 \qquad \text{if} \qquad z_1^3 = -0.4 \quad ,$$

$$\theta_1^3 = 13 - 5\, z_1^3 \qquad \text{if} \qquad -0.4 \leq z_1^3 \leq 2.6 \quad .$$

The conditions for H_v^3 to be a minimum are shown in Fig. 3-2. As in the second stage the condition

$$S \text{ minimum} \rightleftarrows H^n \text{ minimum}$$

is satisfied.

Stage 4:

The variable part of the Hamiltonian equation for the fourth demand point (stage) is

$$H_v^4 = (z_1^4 + 4)\theta_1^4 - 0.05(\theta_1^4)^2 \quad .$$

At this stage the Hamiltonian function H_v^4 has a form similar to those in stages (2) and (3). However, the second derivative of H^4 with respect to θ_1^4 is negative as shown below.

$$\frac{\partial H^4}{\partial \theta_1^4} = 0 = (z_1^4 + 4) - 0.1(\theta_1^4) \quad ,$$

$$\frac{\partial^2 H^4}{\partial(\theta_1^4)^2} = -\ 0.1 < 0 \quad .$$

Therefore, from the condition $\frac{\partial H_v^4}{\partial\theta_1^4} = 0$, we cannot obtain θ_1^4 which yields the minimum of H_v^4. The minimum of H_v^4 occurs at the boundary of the constraint of θ_1^4, as shown in Fig. 3-3. Therefore, the conditions for H_v^4 to be a minimum are

$$H_v^4 = \text{minima at:} \quad \theta_1^4 = 0 \quad \text{if} \quad z^4 \geq -3.5 \quad ,$$

$$\theta_1^4 = 10 \quad \text{if} \quad z^4 \leq -3.5 \quad .$$

These conditions are shown in Fig. 3-3. The conditions for all H_v^n to be minima are sumarized in Table 3-2. Fig. 3-4 shows the breaking values of z_1^n. From equation (3-5) the following values are obtained:

$$z_1^1 = z_1^2 = z_1^3 = z_1^4 \quad .$$

The value of z_1 can now be determined by the condition

$$\sum_{n=1}^{4} \theta_1^n = 55 \quad .$$

The systematic search for the value of z_1 which satisfies this condition should give rise to an optimal solution.

The condition, $-0.4 \leq z_1^n \leq 2.6$, gives no feasible solution because $\theta_1^2 - 0$, that is, $\theta_2^2 = 45$ and $W_2 = 40$. This situation is illustrated in Table 3-3.

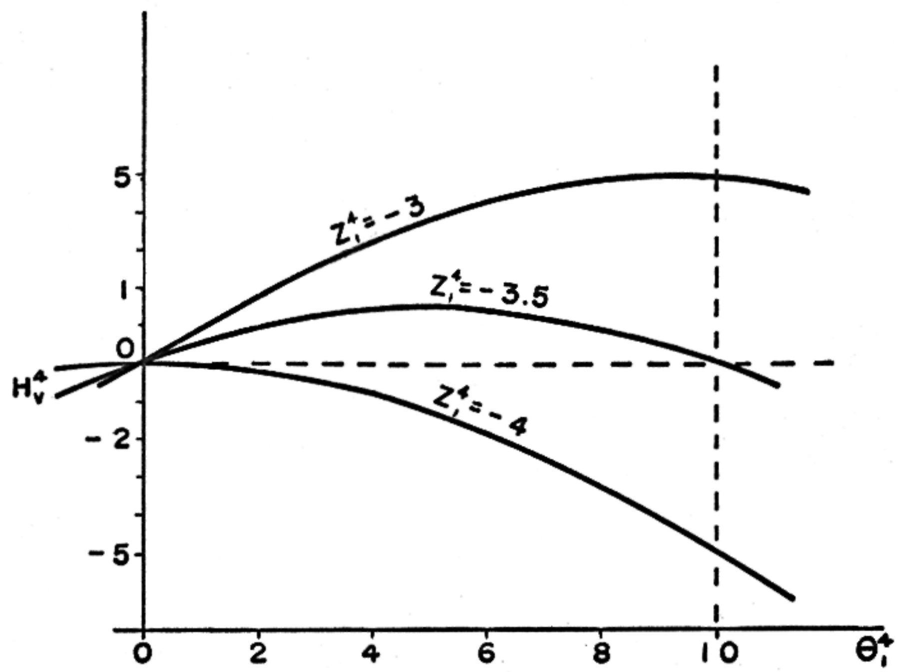

Fig. 3-3. H_v^4 vs θ_1^4

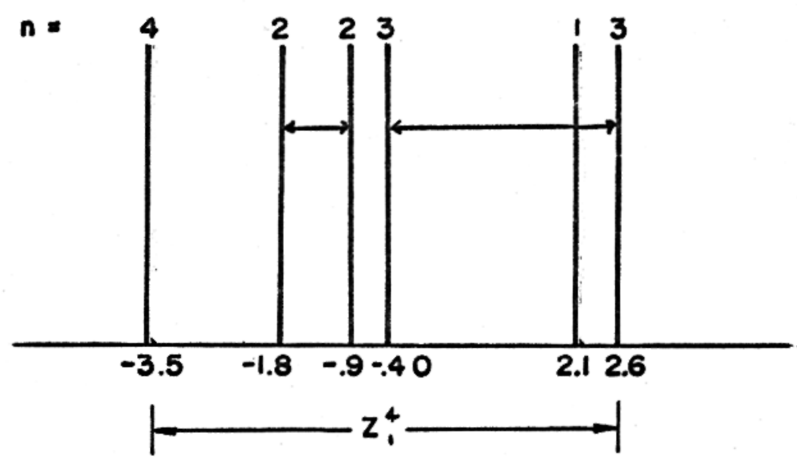

Fig. 3-4. Breaking values of adjoint variable Z_1^n

Table 3-2. Conditions necessary for H_v to be minimum

n	Minima occur at	
	θ_1^n	z_1^n
1	0	> 2.1
	$0 \leq \theta_1^n \leq 25$	$= 2.1$
	25	< 2.1
2	0	≥ -0.9
	$-45 -50\ z_1^2$	$-1.8 = z_1^2 = -0.9$
	45	≤ -1.8
3	0	≥ 2.6
	$13 - 5\ z_1^3$	$-0.4 \leq z_1^3 \leq 2.6$
	15	≤ -0.4
4	0	$z_1^4 \geq -3.5$
	10	$z_1^4 \leq -3.5$

Table 3-3. θ_i^n corresponding to the values of $-0.4 \leq z_1^n \leq 2.6$

i / n	1	2	D^n
1			25
2	0	45	45
3	$13 - 5z_1^3$		15
4	0	10	10
W_i	55	40	95

Table 3-4. θ_i^n corresponding to the values of $-1.8 \leq z_1^n \leq -0.9$

i / n	1	2	D^n
1	25 $\longrightarrow$	0	25
2	$-45 - 50z_1^2$	$45 - \theta_1^2$	45
3	15 $\longrightarrow$	0	15
4	0 $\longrightarrow$	10	10
W_i	55	40	95

For the condition, $-1.8 \leq z_1^n \leq -0.9$, Table 3-4 is obtained. From the constraint $W_1 = 55$, we obtain

$$25 - 45 - 50\ z_1^2 + 15 = 55 \quad ,$$

or

$$z_1^2 = \frac{-60}{50} = -1.2 \quad .$$

This satisfies the condition, $-1.8 \leq z_1 \leq -0.9$, which of course automatically satisfies the constraint $\sum_{n=1}^{4} \theta_1^4 = W_1 = 55$. The corresponding value of θ_i^2 obtained by the substitution of $z_1 = -1.2$, is 15. Although this θ_1^2 value is an integer, it is worth mentioning that there are cases in which it may not be an integer. Then it must be rounded off to the nearest integer.

The total cost for the above solution is

$$\sum_{n=1}^{4} \sum_{i=1}^{2} F_i^n(0_i^n) = \$\ 202.75 \quad .$$

Since the value of z_1^n in the region $-3.5 < z_1^n < -1.8$ does not yield a feasible solution, corresponding to the value of z_1 in the region is $\theta_i^1 = 25$, $\theta_i^2 = 45$; therefore, $\theta_i^1 + \theta_i^2 = 70 > W_1 = 55$ which does not satisfy constraint (iii). Therefore, Table 3-5 shows the only feasible solution. Numerical simulation of the problem is shown in Table 3-6. The total cost for this solution is

$$\sum_{n=1}^{4} \sum_{i-1}^{2} F_i^n(0_i^n) = \$\ 207.20 \quad .$$

This indicates that the solution given by Table 3-5 is indeed the optimal solution.

Table 3-5. θ_i^n corresponding to the values of $z_1^2 = -1.2$

n \ i	1	2	D^n
1	25	0	25
2	15	30	45
3	15	0	15
4	0	10	10
W_i	55	40	95

Table 3-6. Perturbation

n \ i	1	2	D^n
1	25	0	25
2	15	30	45
3	14	1	15
4	1	9	10
W_i	55	40	95

CASE 4. A PROBLEM ON BREEDING-MARKETING POLICY OF CATTLE WITH LINEAR COST FUNCTION

PROBLEM [2].

You manage a herd of cattle of size I, where I is an arbitrary but given number. You have the perogative, at the end of each year, of sending one part of the herd to market and retaining the other part for breeding purposes. Assume that the total dollar value of X cattle sent to market is given by $R(X) = 200X$ dollars and that Y cattle retained for breeding purposes yield 1.5Y at the beginning of the next year. Assume that breeding costs are \$80 per animal per year, i.e., total breeding costs per period are 80Y dollars. Assume that there are no marketing, feeding, or other costs of any kind.

1. Formulate the basic recurrence relation to use in determining a breeding marketing policy which maximizes the total return over an N-year period, at the end of which all cattle must be sold or given away.
2. Find the optimal policies and profits for the process.

FORMULATION AND SOLUTION BY THE DISCRETE MAXIMUM PRINCIPLE

Let

θ^n = amount of cattle sold during the n^{th} period,

x_1^{n-1} = cattle available at the beginning of the n^{th} period,

x_2^n = sum of returns up to and including the n^{th} period.

Since the decision for the last period has already been taken we will not consider that period as stage. In order to be consistent with the notation of the algorithm, we shall define the total number of periods as N + 1. Thus,

N + 1 = total number of periods

where N = number of stages.

Performance Equations:

$$x_1^n = 1.5(x_1^{n-1} - \theta^n) = T_1^n(x^{n-1};\theta^n), \quad n = 1, 2, \ldots, N \tag{4-1}$$

$$x_1^o = I \text{ (given)} , \tag{4-1a}$$

$$x_2^n = x_2^{n-1} + 200(\theta^n) - 80(x_1^{n-1} - \theta^n) = T_2^n(x^{n-1};\theta^n) , \quad n = 1, 2, \ldots, N \tag{4-2}$$

$$x_2^o = 0 \quad . \tag{4-2a}$$

Objective Function:

$$\text{Maximize } S = \sum_{i=1}^{2} c_i x_i^N = c_1 x_1^N + c_2 x_2^N = 200x_1^N + x_2^N \quad . \tag{4-3}$$

Therefore,

$$c_1 = 200 \tag{4-3a}$$

and

$$c_2 = 1 \quad . \tag{4-3b}$$

Hamiltonian Function:

$$H^n = z_1^n[1.5(x_1^{n-1} - \theta^n)] + z_2^n[x_2^{n-1} + 200(\theta^n) - 80(x_1^{n-1} - \theta^n)] , \quad n = 1, 2, \ldots, N \quad . \tag{4-4}$$

Adjoint Vector:

$$z_1^{n-1} = \frac{\partial H^n}{\partial x_1^{n-1}} = 1.5z_1^n - 80z_2^n \quad , \quad n = 1, 2, \ldots, N \tag{4-5}$$

$$z_2^{n-1} = \frac{\partial H^n}{\partial x_2^{n-1}} = z_2^n \quad , \quad n = 1, 2, \ldots, N \quad . \tag{4-6}$$

Application of the condition $c_i = z_i^N$ yields

$$z_1^N = 200 \quad , \tag{4-7}$$

and

$$z_2^N = 1 \quad . \tag{4-8}$$

Combination of equations (4-6) and (4-8) yields

$$z_2^n = 1, \quad n = 1, 2, \ldots, N \quad . \tag{4-9}$$

Substituting equations (4-7) and (4-9) into equations (4-5) and (4-6) we can determine the value of z_1^n.

Inspection of equation (4-4) indicates that the Hamiltonian function H^n is a linear function with respect to θ^n. Thus the optimal decision at each stage occurs at the boundary and consequently cannot be found by setting $\frac{\partial H^n}{\partial \theta^n} = 0$. But it can be obtained by numerical search. The conditions to be satisfied will be:

$$H^n = \text{maximum}, \quad n = 1, 2, \ldots, N \quad .$$

Having obtained the general N-stage solution we shall solve first for two periods as indicated in the second question and then for five periods assuming that the initial amount of cattle is 200 head and the selling price of the cattle is \$200.

(1). $N + 1 = 2 \quad , \quad N = 1$

The variable portion of the Hamiltonian function H_v^n obtained from equation (4-4) is

$$H_v^n = -1.5z_1^n(\theta^n) + 280(\theta^n)$$

or

$$H_v^n = \theta^n(280 - 1.5z_1^n) \quad .$$

From (4-7) we obtain $z_1^n = 200$.

Therefore,

$$H_v^1 = \theta^n(280 - 1.5(200)) = \theta^1(-20), \quad 0 \leq \theta^1 \leq x_1^{n-1} \quad .$$

The maximum of H_v^1 occurs at $\theta^1 = 0$.

Then,

$$x_1^1 = 1.5I \quad ,$$

$$x_2^1 = -80I \quad ,$$

$$S = 200x_1^1 + x_2^1 = 200(1.5\ I) - 80\ I$$

$$= 220\ I = 220\ x_1^o \quad .$$

(2). Extending to five years and assuming the initial amount of cattle equal to 200 head we obtain:

$$N + 1 = 5 \quad , \quad N = 4 \quad .$$

From equation (4-4) we obtain

$$H_v^n = \theta^n(280 - 1.5z_1^n) \quad , \quad n = 1, 2, 3, 4 \quad .$$

From equation (4-7) we obtain $z_1^N = z_1^4 = 200$. Substitution of this value and equation (4-9) into equation (4-5) gives

$$z_1^3 = 220 \quad ,$$

$$z_1^2 = 250 \quad ,$$

$$z_1^1 = 295 \quad .$$

Substituting these values into the Hamiltonian function yields

Stage 1:

$$H_v = \theta^1(280 - 1.5(295)) = \theta^1(-162.5), \quad 0 \leq \theta^1 \leq x_1^o$$

The maximum of H_v^1 occurs at $\theta^1 = 0$.

Stage 2:

$$H_v^2 = \theta^2(280 - 1.5(250)) = \theta^2(-95) \quad , \ 0 \leq \theta^2 \leq x_1^1$$

The maximum of H_v^2 occurs at $\theta^2 = 0$

Stage 3:

$$H_v^3 = \theta^3(280 - 1.5(220)) = \theta^3(-60) \quad , \quad 0 \leq \theta^3 \leq x_1^2$$

The maximum of H_v^3 occurs at $\theta^3 = 0$.

Stage 4:

$$H_v^4 = \theta^4(280 - 1.5(200)) = \theta^4(-20) \quad , \quad 0 \leq \theta^4 \leq x_1^3$$

The maximum of H_V^4 occurs at $\theta^4 = 0$.

Knowing the optimal sequence of decisions we can evaluate the values of x_1^N and x_2^N using equations (4-1) and (4-2). Thus,

$$x_1^1 = 300 \quad , \qquad x_2^1 = -80(200) \quad ,$$

$$x_1^2 = 450 \quad , \qquad x_2^2 = -80(500) \quad ,$$

$$x_1^3 = 675 \quad , \qquad x_2^3 = -80(950) \quad ,$$

$$x_1^4 = 1072.5 \ , \qquad x_2^4 = -80(1625) \quad .$$

$$\text{Maximum return} = S = 200x_1^4 + x_2^4$$

$$= \$72{,}500.$$

It can be observed that the best policy is to keep the cattle and sell them at the end of the last stage. (In other words the cattle should be retained for four years and sold during the fifth period.)

Now we shall consider the case where the selling price is \$150 per head. The rest of the information remains the same as in the previous example. Determine the breeding marketing policy which maximizes the total return over a five year period.

(3). Extending to five years and assuming the initial amount of cattle equal to 200 head and the selling price of the cattle equal to \$150 a head we would have:

$$N + 1 = 5 \quad , \qquad N = 4 \quad .$$

From equation (4-4) we obtain the variable portion of the Hamiltonian function as

$$H_v^n = \theta^n(230 - 1.5z_1^n) \quad .$$

From equation (4-7) we obtain $z_1^4 = 150$.

Substitution of the above value and equation (4-9) into equation (4-5) gives

$$z_1^3 = 145 \quad ,$$

$$z_1^2 = 137.5 \quad ,$$

$$z_1^1 = 206.2 \quad ,$$

Substituting these values into the Hamiltonian function yields

Stage 1:

$$H_v^1 = \theta^1(230 - 1.5(206.25)) = \theta^1(-79.4) \quad , \quad 0 \leq \theta^1 \leq x_1^o \ .$$

The maximum of H_v^1 occurs at $\theta^1 = 0$.

Stage 2:

$$H_v^2 = \theta^1(230 - 1.5(137.5)) = \theta^1(23.75) \quad , \quad 0 \leq \theta^1 = x^1 \ .$$

The maximum of H_v^2 occurs at $\theta^2 = x_1^1$.

Stage 3:

$$H_v^3 = \theta^3(230 - 1.5(145)) = \theta^3(12.5) \quad , \quad 0 \leq \theta^3 \leq x_1^2 \ .$$

The maximum of H_v^3 occurs at $\theta^3 = x_1^2$.

Stage 4:

$$H_v^4 = \theta^4(230 - 15(150)) = \theta^4(5) \quad , \quad 0 \leq \theta^4 \leq x_1^3 \ .$$

The maximum of H_v^4 occurs at $\theta^4 = x_1^3$.

The values of x_1^n and x_2^n are obtained by using equations (4-1) and (4-2). Thus

$$x_1^1 = 300 \quad , \qquad x_2^1 = -80(200)$$

$$x_1^2 = 0 \quad , \qquad x_2^2 = 0 \quad ,$$

$$x_1^3 = 0 \quad , \qquad x_2^3 = 0 \quad ,$$

$$x_1^4 = 0 \quad , \qquad x_2^4 = 0 \quad .$$

Maximum return over the five years period would be:

$$S = 150(300) \; -80(200)$$

$$= \$29{,}500.$$

The sequential decisions obtained in this example are quite different from those obtained in the previous one.

CASE 5. A PERSONNEL AND PRODUCTION SCHEDULING PROBLEM (A TWO DECISION VARIABLE PROBLEM)

PROBLEM [1]

It is necessary to plan operations in a situation in which initial conditions, cost, and requirements are given as follows:

P^o = 2,000, production rate in previous period,

W^o = 600, work force in previous period,

I^o = 300, inventory at the end of previous period,

K = 3, production units per worker per period in regular time.

Costs:

$\$200(W^n-W^{n-1})^2$ = cost due to change in work force,

$\$50\ P^n$ = production cost,

$\$25(P^n-KW^n)^2$ = overtime cost,

$\$20(500-I^n)^2$ = inventory cost.

Requirements:

S^1 = 3,000, sales in period No. 1,

S^2 = 1,800, sales in period No. 2,

S^3 = 2,400, sales in period No. 3.

Back order is permitted.

The management desires to make those plans which will result in the lowest operating cost in meeting the above requirements.

(a) Express the system objective function mathematically.

(b) Solve for an optimum schedule.

FORMULATION AND SOLUTION BY THE DISCRETE MAXIMUM PRINCIPLE

Let

$\theta_1^n = p^n - p^{n-1}$ = difference in production between the n^{th} period and the previous period,

$x_1^n = p^n$ = production rate during the n^{th} period,

$\theta_2^n = w^n - w^{n-1}$ = difference in work force between the n^{th} period and the previous period,

$x_2^n = w^n$ = work force during the n^{th} period,

$x_3^n = I^3$ = inventory at the end of the n^{th} period,

x_4^n = cost up to and including the n^{th} period.

Performance Equations:

Production rate:

$$x_1^n = x_1^{n-1} + \theta_1^n = T_1(x^{n-1};\ \theta^n) \quad , \tag{5-1}$$

$$x_1^o = P^o = 2{,}000 \quad . \tag{5-1a}$$

Work force:

$$x_2^n = x_2^{n-1} + \theta_2^n = T_2(x^{n-1};\theta^n) \quad , \tag{5-2}$$

$$x_2^o = W^o = 600 \quad . \tag{5-2a}$$

Inventory:

$$x_3^n = x_3^{n-1} + x_1^n - S^n$$

$$= x_3^{n-1} + x_1^{n-1} + \theta_1^n - S^n = T_3(x^{n-1};\ \theta^n) \quad , \qquad (5\text{-}3)$$

$$x_3^o = I^o = 300 \quad . \qquad (5\text{-}3a)$$

Costs:

$$x_4^n = x_4^{n-1} + 200(\theta_2^n)^2 + 50x_1^n + 25(x_1^n - Kx_2^n)^2 + 20(500 - x_3^n)^2 , \qquad (5\text{-}4)$$

$$x_4^o = 0 \quad . \qquad (5\text{-}4a)$$

Substituting the values of x_1^n, x_2^n, and x_3^n into equation (5-4) gives

$$x_4^n = x_4^{n-1} + 200(\theta_2^n)^2 + 50(x_1^{n-1} + \theta_1^n) + 25(x_1^{n-1} + \theta_1^n - Kx_2^{n-1} - K\theta_2^n)^2$$

$$+ 20(500 - x_3^{n-1} - x_1^{n-1} - \theta_1^n + S^n)^2 \quad , \quad n = 1, 2, 3 \qquad (5\text{-}5)$$

$$= T_4(x^{n-1};\ \theta^n) \quad .$$

Objective function:

$$\text{Minimize } S = \sum_{i=1}^{4} c_i x_i^N = c_1 x_1^N + c_2 x_2^N + c_3 x_3^N + c_4 x_4^N = x_4^N \quad . \qquad (5\text{-}6)$$

Therefore,

$$c_1 = 0 \quad , \qquad (5\text{-}6a)$$

$$c_2 = 0 \quad , \qquad (5\text{-}6b)$$

$$c_3 = 0 \quad , \tag{5-6c}$$

$$c_4 = 1 \quad . \tag{5-6d}$$

The Hamiltonian Function is

$$H^n = z_1^n(x_1^{n-1}+\theta_1^n) + z_2^n(x_2^{n-1}+\theta_2^n) + z_3^n(x_3^{n-1}+x_1^{n-1}+\theta_1^n-S^n)^2$$
$$+ z_4^n[x_4^{n-1}+200(\theta_2^n)^2 + 50(x_1^{n-1}+\theta_1^n) + 25(x_1^{n-1}+\theta_1^n-Kx_2^{n-1}-K\theta_2^n)^2$$
$$+ 20(500-x_3^{n-1}-x_1^{n-1}-\theta_1^n+S^n)^2] \quad , \quad n = 1, 2, 3 . \tag{5-7}$$

The four adjoint variables z_1, z_2, z_3 and z_4 are

$$z_1^{n-1} = \frac{\partial H^n}{\partial x_1^{n-1}} = z_1^n + z_3^n + z_4^n\ [50 + 50(x_1^{n-1} + \theta_1^n - Kx_2^{n-1} - K\theta_2^n) -$$
$$40(500 - x_3^{n-1} - x_1^{n-1} - \theta_1^n + S^n)] \quad , \tag{5-8}$$

$$z_1^N = c_1 = 0 \quad , \tag{5-8a}$$

$$z_2^{n-1} = \frac{\partial H^n}{\partial x_2^{n-1}} = z_2^n - 50Kz_4^n(x^{n-1} + \theta_1^n - Kx_2^{n-1} - K\theta_2^n) \quad , \tag{5-9}$$

$$z_2^N = c_2 = 0 \quad , \tag{5-9a}$$

$$z_3^{n-1} = \frac{\partial H^n}{\partial x_3^{n-1}} = z_3^n - 40z_4^n(500 - x_3^{n-1} - x_1^{n-1} - \theta_1^n + S^n) , \tag{5-10}$$

$$z_3^N = c_3 = 0 \quad , \tag{5-10a}$$

$$z_4^{n-1} = \frac{\partial H^n}{\partial x_4^{n-1}} = z_4^n \quad , \tag{5-11}$$

$$z_4^N = 1 \quad . \tag{5-11a}$$

The combination of equations (5-11) and (5-11a) gives

$$z_4^n = 1 \quad , \quad n = 1, 2, \ldots, N \quad . \tag{5-11b}$$

According to the algorithm of the discrete maximum principle, the optimal sequences of θ_1^n and θ_2^n are determined by the condition

$$\frac{\partial H^n}{\partial \theta_i^n} = 0 \quad , \quad n = 1, 2, \ldots, N \quad ; \quad i = 1, 2 \quad .$$

Therefore,

$$\frac{\partial H^n}{\partial \theta_1^n} = 0 = z_1^n + z_3^n + [50(x_1^{n-1}+\theta_1^n-Kx_2^{n-1}-K\theta_2^n) + 50 -$$

$$40(500 - x_3^{n-1}+x_1^{n-1}-\theta_1^n+S^n)] \quad , \tag{5-12}$$

$$\frac{\partial H^n}{\partial \theta_2^n} = 0 = z_2^n + [400(\theta_2^n) - 50K(x_1^{n-1}+\theta_1^n-Kx_2^{n-1}-K\theta_2^n)] \quad . \tag{5-13}$$

Combination of equations (5-9), (5-11b) and (5-13) gives

$$\theta_1^n = \frac{8}{K}(\theta_2^n) - \frac{8}{K}(\theta_2^{n+1}) - (x_1^{n-1}-Kx_2^{n-1}-K\theta_2^n) \quad . \tag{5-14}$$

Equation (5-14) gives one of the optimality conditions for the multistage process under consideration. Another recurrence relation for the other optimality condition can be found by combining equations (5-8), (5-10), and (5-12), and substituting (5-14) and the value of K into the resulting equation yielding

$$0 = 2000 - 4x_3^{n-1} + 4S^n - 12x_2^{n-1} - 36\theta_2^n + \frac{112\ \theta_2^{n+1}}{3} - \frac{40\ \theta_2^{n+2}}{3} \quad . \tag{5-15}$$

Thus the optimization problem can be solved by the following procedure utilizing the optimality conditions, equations (5-14) and (5-15), together with the set of performance equations (5-1) through (5-4).

CALCULATION PROCEDURE

STEP 1. Choosing n = 1 and substituting the known values into (5-14) gives

$$\theta_1^1 = \frac{17}{3}(\theta_2^1) - \frac{8}{3}(\theta_2^2) - 200 \quad . \tag{A}$$

STEP 2. Choosing n = 1 and substituting the known values into (5-15) gives

$$27(\theta_2^1) - 28(\theta_2^2) + 10(\theta_2^3) = 4200 \quad . \tag{B}$$

STEP 3. Assume a value for θ_2^1.

STEP 4. Assume a value for θ_2^2.

STEP 5. Calculate θ_1^1 and θ_2^3 from equations (A) and (B) respectively.

STEP 6. Calculate x_1^1, x_2^1, and x_3^1 from equations (5-1), (5-2), and (5-3) respectively.

STEP 7. Choose n = 2 and substitute the known values into (5-14) to obtain

$$\theta_1^2 = \frac{17}{3}(\theta_2^2) - \frac{8}{3}(\theta_2^3) - (x_1^1 - 3x_2^1) \quad . \tag{C}$$

STEP 8. Calculate x_1^2, x_2^2, and x_3^2 from equations (5-1), (5-2) and (5-3) respectively.

STEP 9. Choose n = 2 and substitute the known values into (5-15) to obtain

$$\theta_2^4 = [2000 - 4x_3^1 + 4S^2 - 12x_2^1 - 36(\theta_2^2) + 112(\theta_2^3)]\frac{3}{40} \quad .$$

This value will be used in the next step. Although this problem is a three-stage problem we can evaluate θ_2^4 if it is not specified because it represents the work force. It could be considered zero in cases where the work force is utilized for other purposes after running production.

STEP 10. Choose n = 3 and substitute the known values into (5-14). Hence

$$\theta_1^3 = \frac{17}{3}(\theta_2^3) - \frac{8}{3}(\theta_2^4) - (x^2 - 3x^2) \quad . \tag{D}$$

Compute θ_1^3 from this equation.

STEP 11. Calculate x_1^3, x_2^3, and x_3^3 from equations (5-1), (5-2), and (5-3).

STEP 12. If the θ_2^4 value was preassigned, it is compared with the calculated value in step 9. In the event that these values are equal the problem is solved. Otherwise, steps (4) through (9) are repeated. If the θ_2^4 value was not preassigned then we have to fix the desired inventory at the end of the last period under consideration. In this case, steps (3) through (11) are repeated until the computed value of x_3^3 is equal to x_3^3 given.

STEP 13. Substitute the values of θ_1^n, θ_2^n, x_1^n, x_2^n, and x_3^n into the equation (5-4) to obtain the total minimum cost.

The numerical answer is not presented here. As can be observed the recurrence relations obtained for both optimality conditions are general forms to solve a N-stage process.

CASE 6. A WAREHOUSING DECISION PROBLEM

PROBLEM

Given a warehouse of fixed capacity and an initial stock of certain products, which is subject to known seasonal fluctuation in selling price and cost, and given a delay between the purchasing and the receiving of the product, what is the optimal pattern of purchasing, storage, and sales over a given period of time?

A numerical example is as follows [12]: A man is engaged in buying and selling identical items, each of which requires considerable storage space. He operates from a warehouse which has capacity of 500 items. He can order on the 15th of each month, at the prices shown below, for delivery on the first of the following month. During a month he can sell any amount up to his total stock on hand, at the market prices given below. If he starts the year with 200 items in stock, how much should he plan to purchase and sell each month in order to maximize his profits (cash receipt and cash expenditures) for the year?

Cost [C_i]		Sales prices [P_i]	
January 15	150	January	165
February 15	155	February	165
March 15	165	March	185
April 15	160	April	175
May 15	160	May	170
June 15	160	June	155
July 15	155	July	155

August 15	150	August	155
September 15	155	September	160
October 15	155	October	170
November 15	150	November	175
December 15	150	December	170

FORMULATION AND SOLUTION BY THE DISCRETE MAXIMUM PRINCIPLE

Considering each month as a stage, let us define the following notation at the n^{th} stage (month):

C^n = cost per unit,

P^n = selling price per unit,

θ^n = amount of items sold at n^{th} stage (month),

x_1^n = amount of items to be ordered in the n^{th} stage, for delivery on the first of the following month (stage),

x_2^n = sum of the cash receipts minus cash expenditures up to and including the n^{th} stage,

I^n = storage at the first of the month (stage),

W = capacity of the warehouse = 500 items.

Performance equations:

$$x_1^n = 500 - (I^n - \theta^n) \quad , \quad x_1^o = 0 \qquad (6\text{-}1)$$

$$I^n = x_1^{n-1} \quad , \quad I^o = I^1 = 200 \quad , \quad n = 2, 3, \ldots N \qquad (6\text{-}1a)$$

$$x_2^n = x_2^{n-1} + P^n(\theta^n) - C^n(x_1^n) \quad , \qquad (6\text{-}2)$$

$$x_2^0 = 0 \quad . \qquad (6\text{-}2a)$$

Constraints:

$$x_1^n \geq 0 \quad ,$$

$\theta^n \leq I^n$ (stock on hand at the beginning of the n^{th} stage). (6-3)

Objective function:

$$\text{Maximize} \quad S = \sum_{i=1}^{2} c_i x_i^N = x_2^N \quad . \qquad (6\text{-}4)$$

Substituting equation (6-1) into equation (6-2) yields

$$x_2^n = x_2^{n-1} + [P^n(\theta^n) - C^n(500 - (x_1^{n-1} - \theta^n)] \quad . \qquad (6\text{-}5)$$

The Hamiltonian function and the adjoint vector are:

$$H^n = z_1^n x_1^n + z_2^n x_2^n$$

$$= z_1^n[500 - (x_1^{n-1} - \theta^n)] + z_2^n[x_2^{n-1} + P^n(\theta^n) - C^n$$

$$(500 - (x_1^{n-1} - \theta^n)] \quad , \qquad (6\text{-}6)$$

$$z_1^{n-1} = \frac{\partial H^n}{\partial x_1^{n-1}} = -z_1^n + C^n z_2^n \quad , \qquad (6\text{-}7)$$

$$z_1^N = 0 \quad , \qquad (6\text{-}8)$$

$$z_2^{n-1} = \frac{\partial H^n}{\partial x_2^{n-1}} = z_2^n \quad , \tag{6-9}$$

$$z_2^N = 1 \quad . \tag{6-10}$$

From equations (6-9) and (6-10), we obtain:

$$z_2^n = 1 \quad , \qquad n = 1, 2, \ldots, N. \tag{6-11}$$

Then the Hamiltonian function becomes

$$H^n = z_1^n[500 - (x_1^{n-1} - \theta^n)] + [x_2^{n-1} + P^n(\theta^n) - 500C^n +$$

$$C^n x_1^{n-1} - C^n(\theta^n)] \quad . \tag{6-12}$$

From equations (6-7) and (6-8) we calculate the values for z_1^n, $n = 1, 2, \ldots, 12$.

$z_1^{12} = 0$, $z_1^6 = 5$,

$z_1^{11} = +150$, $z_1^5 = 155$,

$z_1^{10} = 0$, $z_1^4 = 5$,

$z_1^9 = 155$, $z_1^3 = 155$,

$z_1^8 = 0$, $z_1^2 = 10$,

$z_1^7 = 150$, $z_1^1 = 145$.

The variable portion of the Hamiltonian function given in equation (6-12) is

$$H_v^n = (z_1^n + P^n - C^n)\theta^n \qquad , \qquad n = 1, 2, \ldots, N \quad .$$

The substitution of the z_1^n values into this equation yields

$$H_v^1 = (z_1^1 + P^1 - C^1)\theta^1$$

$$= (160)\theta^1 \qquad , \qquad 0 \le \theta^1 \le I^1 \quad .$$

H_v^1 is maximum at $\theta^1 = I^1$.

$$H_v^2 = (z_1^2 + P^2 - C^2)\theta^2$$

$$= (20)\theta^2 \qquad , \qquad 0 \le \theta^2 \le I^2 \quad .$$

H_v^2 is maximum at $\theta^2 = I^2$.

$$H_v^3 = (z_1^3 + P^3 - C^3)\theta^3$$

$$= (175)\theta^3 \qquad , \qquad 0 \le \theta^3 \le I^3 \quad .$$

H_v^3 is maximum at $\theta^3 = I^3$.

$$H_v^4 = (z_1^4 + P^4 - C^4)\theta^4$$

$$= (20)\theta^4 \qquad , \qquad 0 \le \theta^4 \le I^4 \quad .$$

H_v^4 is maximum at $\theta^4 = I^4$.

$$H_v^5 = (z_1^5 + P^5 - C^5)\theta^5$$

$$= (165)\theta^5 \qquad , \qquad 0 \le \theta^5 \le I^5 \quad .$$

H_v^5 is maximum at $\theta^5 = I^5$.

$$H_v^6 = (z_1^6 + P^6 - C^6)\theta^6$$

$$= 0 \quad , \qquad 0 \leq \theta^6 \leq I^6 \quad .$$

H_v^6 is maximum at $H_v^6 = \theta^6$. At this stage there will be no sales.

$$H_v^7 = (z_1^7 + P^7 - C^7)\theta^7$$

$$= (150)\theta^7 \quad , \qquad 0 \leq \theta^7 \leq I^7 \quad .$$

H_v^7 is maximum at $\theta^7 = I^7$.

$$H_v^8 = (z_1^8 + P^8 - C^8)\theta^8$$

$$= 5\theta^8 \quad , \qquad 0 \leq \theta^8 \leq I^8 \quad .$$

H_v^8 is maximum at $\theta^8 = I^8$.

$$H_v^9 = (z_1^9 + P^9 - C^9)\theta^9$$

$$= (160)\theta^9 \quad , \qquad 0 \leq \theta^9 \leq I^9 \quad .$$

H_v^9 is maximum at $\theta^9 = I^9$.

$$H_v^{10} = (z_1^{10} + P^{10} - C^{10})\theta^{10} \quad ,$$

$$= (15)\theta^{10} \quad , \qquad 0 \leq \theta^{10} \leq I^{10} \quad .$$

H_v^{10} is maximum at $\theta^{10} = I^{10}$.

$$H_v^{11} = (z_1^{11} + P^{11} - C^{11})\theta^{11}$$

$$= (175)\theta^{11} \quad , \qquad 0 \leq \theta^{11} \leq I^{11} \quad .$$

H_v^{11} is maximum at $\theta^{11} = I^{11}$.

$$H_v^{12} = (z_1^{12} + P^{12} - C^{12})\theta^{12}$$

$$= (20)\theta^{12} \quad , \qquad 0 \leq \theta^{12} \leq I^{12}.$$

H_v^{12} is maximum at $\theta^{12} = I^{12}$.

Substituting the θ^n values into equation (6-1) we obtain:

$x_1^1 = 500 - (I^1 - \theta^1)$, $I^1 = 200$, $\theta^1 = I^1 = 200$,

$x_1^1 = 500$.

$I^2 = x^1 = 500$, $\theta^2 = I^2 = 500$,

$x_1^2 = 500$,

$I^3 = 500$, $\theta^3 = I^3 = 500$,

$x_1^3 = 500$,

$I^4 = 500$, $\theta^4 = I^4 = 500$,

$x_1^4 = 500$,

$I^5 = 500$, $\theta^5 = I^5 = 500$,

$x_1^5 =$ Since during the 6th stage there will be no sales, there is no need to buy items in the previous stage. In other words, during the 5th stage there will be no purchase of goods. Therefore,

$I^6 = 0$, $\theta^6 = I^6 = 0$

x_1^6 Since $H_v^6 = 0$ is maximum, neither sale or purchase must be made at this stage.

$I^7 = 0$, $\theta^7 = I^7 = 0$,

$x_1^7 = 500$

$I^8 = 500$, $\theta^8 = I^8 = 500$,

$x_1^8 = 500$

$I^9 = 500$, $\theta^9 = I^9 = 500$,

$x_1^9 = 500$

$I^{10} = 500$, $\theta^{10} = I^{10} = 500$,

$x_1^{10} = 500$

$I^{11} = 500$, $\theta^{11} = I^{11} = 500$,

$x_1^{11} = 500$

$I^{12} = 500$, $\theta^{12} = I^{12} = 500$,

By substituting the values of θ^n and x_1^n into equation (6-2) we obtain:

$$x_2^1 = 150(200) - 150(500) \quad ,$$

$$x_2^2 = x_2^1 + 500(165 - 155) \quad ,$$

$$x_2^3 = x_2^2 + 500(185 - 165) \quad ,$$

$$x_2^4 = x_2^3 + 500(175 - 160) \quad ,$$

$$x_2^5 = x_2^4 + 500(170) \quad ,$$

$$x_2^6 = x_2^5 + 0 \quad ,$$

$$x_2^7 = x_2^6 - 500(155) \quad ,$$

$$x_2^8 = x_2^7 + 500(155 - 150) \quad ,$$

$$x_2^9 = x_2^8 + 500(160 - 155) \quad ,$$

$$x_2^{10} = x_2^9 + 500(170 - 155) \quad ,$$

$$x_2^{11} = x_2^{10} + 500(175 - 150) \quad ,$$

$$x_2^{12} = x_2^{11} + 500(170) \quad ,$$

$$\therefore \quad x_2^{12} = 200(165) + 130(500) = 98{,}800 \quad .$$

ANALYTICAL SOLUTION

Let

W = 500 units = warehousing capacity

P_i = selling price per unit

C_i = cost per unit

x_i = amount sold during the i^{th} period

Y_i = amount bought the 15th of each period

v = initial stock = 200 units

Specifically our buying capacity is constrained by the fact that the stock on hand during the n^{th} period cannot exceed the warehousing capacity. Since the amount ordered in the i^{th} period is received in the $(n + 1)^{th}$ period and the sales can be any amount up to the stock on hand, the inventory at the end of each period will be zero units. The amount ordered on the 15th of any period will be equal to the warehousing capacity and we will place an order only if the selling price corresponding to the next period is equal to or greater than the cost price. Thus, if $[P^n-C^{n-1}] > 0$, $n = 1,\ldots,N-1$ there will be profit and it will be proportional to the number of units sold during the $(n + 1)^{th}$ period which is equal to the number of units ordered in the n^{th} period. It is obvious that the initial stock is sold during the 1st period and its value must be added to the return obtained during the N-stage periods in order to obtain the total profit.

Numerical Results

Month	Cost Prices	Sales Prices	P^n-C^{n-1}	Buy	Sale
January	150	165	P^1	500	200
February	155	165	15	500	500
March	165	185	30	500	500
April	160	175	10	500	500
May	160	170	10	0	500
June	160	155	-5	0	0
July	155	155	-5	0	0
August	150	155	0	500	0
September	155	160	10	500	500
October	155	170	15	500	500
November	150	175	20	500	500
December	150	170	20		500

According to the P^n-C^{n-1} values obtained above we may decide whether we order in period n or not. The negative values indicate that there will not be any profit if we buy in May and sell in June. By the same reasoning we cannot order in June to sell in July. The zero value tells us that neither profit nor loss will occur. We multiply the positive differences by the constant value of the warehouse capacity, except for the P^1 value that is multiplied by the initial stock. Then adding all these products we get the maximum achievable return over the N periods.

$$\text{Profit} = 165 \times 200 + 130 \times 500 = \$98{,}000$$

CONCLUSION

The results obtained by the discrete maximum principle are the same given by dynamic programing and the analytical approach. However, in cases like this the discrete maximum principle and dynamic programing techniques are impractical if we compare them with the analytical approach, which is simple and saves time on calculation.

The analytical approach shows that the optimal policy at any stage is independent of initial stock at that stage and that it depends on the cost and selling price.

FURTHER DISCUSSION

The formulation and solution of this problem by the discrete maximum principle seem to be correct. However, the decisions taken in the seventh and eighth stages (months of June and July) were not correctly reasoned by the discrete maximum principle. A correct formulation in which all the stages are satisfied may exist. It is felt that this is a good example for testing students' degree of understanding of the rigorous reasoning of the discrete maximum principle. The above comments were made by the major advisor of this report.

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APPLICATIONS OF THE DISCRETE MAXIMUM PRINCIPLE TO SOME PROBLEMS IN INDUSTRIAL MANAGEMENT

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ABSTRACT

The objective of this report is to demonstrate the wide applicability of the discrete maximum principle in solving different industrial management problems. Six well known problems familiar to students in industrial engineering were formulated and solved. The standard way of solving problems by the discrete maximum principle is shown. The first case illustrates a single stage optimization problem, the economical lot size for inventory systems where the replenishment of stock is instantaneous and is determined over a period of time. The second case illustrates a multistage decision problem with memory in decision. One extension of the algorithms is applied to transform this problem into the standard form of the algorithm. The third case shows the quivalence of the objective function and the Hamiltonian function by one example of a transportation problem with non-linear cost function. The fourth case illustrates a fixed end point problem in a cattle breeding marketing problem. The fifth case presents a two decision variable per stage problem. Finally, a warehousing decision problem was set up and partially solved by the discrete maximum principle. This problem illustrates the use of this principle in linear functions subject to constraints.

Some comparisons with other methods are mentioned; however, the efficiency of the discrete maximum principle for solving these problems is not compared with that of the other methods.

By solving these problems by the maximum principle it is concluded that the algorithm does not only assist in solving them,

but in several cases such as the personnel scheduling problem (case 2) it provides a general recurrence equation which may provide an improvement in computational schemes heretofore proposed for more complex versions.