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Stability of solutions to abstract evolution equations with delay

A.G. Ramm
Department of Mathematics
Kansas State University, Manhattan, KS 66506-2602, USA
ramm@math.ksu.edu

Abstract

An equation $\dot{u} = A(t)u + B(t)F(t, u(t-\tau))$, $u(t) = v(t), -\tau \le t \le 0$ is considered, where $A(t)$ and $B(t)$ are linear operators in a Hilbert space H, $\dot{u} = \frac{du}{dt}$, $F : H \to H$ is a non-linear operator, and $\tau > 0$ is a constant. Under some assumptions on $A(t), B(t)$ and $F(t, u)$ sufficient conditions are given for the solution $u(t)$ to exist globally, i.e, for all $t \ge 0$, to be globally bounded, and to tend to zero at a specified rate as $t \to \infty$.

MSC: 34G20, 34K20, 37L05, 47J35

Keywords: abstract evolution problems; delay; stability; differential inequality.

1 Introduction

Consider an abstract evolution problem

$$\dot{u} = A(t)u + B(t)F(t, u(t-\tau)), \tag{1}$$

$$u(t) = v(t), \quad -\tau \le t \le 0 \tag{2}$$

where $u(t) \in H$, H is a Hilbert space, $A(t)$ and $B(t)$ are linear operators in H, $F(t, u)$ is a nonlinear operator in H, and $\tau > 0$ is a constant.

Let us assume that $A(t)$ is a closed densely defined operator, $D(A(t)) = D(A)$, $D(A)$ is the domain of $A(t)$, independent of t, and

$$\mathrm{Re}(A(t)u, u) \le -\gamma(t)(u, u), \tag{3}$$

$$||B(t)|| \le b(t), \tag{4}$$

$$||F(t, u)|| \le \alpha(t, g), \qquad g := ||u(t)||. \tag{5}$$

We assume that problem (1)-(2) has a unique local solution. Sufficient conditions for this can be found in the literature, see, e.g., [1].

We assume that the function $\alpha(t,g) \geq 0$ satisfies a local Lipschitz condition with respect to g, is continuous with respect to t on $[-\tau, \infty)$ and is non-decreasing with respect to g, and that functions $b(t)$ and $\gamma(t)$ are continuous on $[-\tau, \infty)$.

Our aim is to give sufficient conditions for global existence, that is, existence for all $t \geq 0$, global boundedness, and stability of the solution to problem (1)-(2).

There is a large literature on functional differential equations, see [1]-[4], and references therein. The method that we propose is new. A version of our method was used in a study of the Dynamical Systems Method (DSM) for solving operator equations, see [5]-[8]. This method is generalized in [6] to the case of abstract differential equations without delay and with persistently acting perturbations, see also a recent paper [9].

Our approach is as follows: multiply equation (1) by $u(t)$ in H and take the real part to get

$$\mathrm{Re}(\dot{u}, u) = \mathrm{Re}(A(t)u(t), u(t)) + \mathrm{Re}(B(t)F(t, u(t-\tau)), u), \quad t \geq 0. \tag{6}$$

Let $g(t) := ||u(t)||$. Then equation (6) yields an inequality

$$g\dot{g} \leq -\gamma(t)g^2 + b(t)\alpha(t, g(t-\tau))g, \quad t \geq 0. \tag{7}$$

Since $g(t) \geq 0$, inequality (7) implies

$$\dot{g}(t) \leq l(g) := -\gamma(t)g(t) + b(t)\alpha(t, g(t-\tau)), \quad t \geq 0, \tag{8}$$

and $g(t) := ||v(t)||$, $-\tau \leq t \leq 0$. Indeed, at the points at which $g(t) > 0$, inequality (7) is equivalent to (8) and $\dot{g}(t) = \mathrm{Re}(\dot{u}, \frac{u(t)}{||u(t)||})$.

If $g(t) = 0$ on an open interval, $t \in (a,b)$, then $\dot{g}(t) = 0$, $t \in (a,b)$, and inequality (8) holds since $b(t) \geq 0$ and $\alpha(t,g) \geq 0$.

If $g(s) = 0$ but in any neighborhood $(s-\delta, s) \cup (s, s+\delta)$, $g(t) \neq 0$ provided that $\delta > 0$ is sufficiently small, then by $\dot{g}(s)$ we understand the derivative from the right:

$$\dot{g}(s) = \lim_{h \to +0} g(s+h)h^{-1} = ||\dot{u}(s)||. \tag{9}$$

Inequality (8) then follows from (7) by continuity as $t \to s + 0$.

The following lemma is key for our results.

Lemma 1. *If there exists a function $\mu(t) > 0$, defined for all $t \geq -\tau$, such that*

$$b(t)\alpha(t, \frac{1}{\mu(t-\tau)})\mu(t) \leq \gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)}, \quad t \geq 0, \tag{10}$$

and

$$\mu(t)g(t) \leq 1, \quad t \in [-\tau, 0], \tag{11}$$

then any solution $g(t) \geq 0$ to inequality (8) *exists for all $t \geq 0$ and satisfies the following inequality:*

$$0 \leq g(t) \leq \frac{1}{\mu(t)}, \qquad \forall t \geq 0. \tag{12}$$

Remark 1. *If one proves inequality* (12) *for any $t \geq 0$ for which g is defined, then, since $\mu(t)$ is defined on all of $\mathbb{R}_+ = [0, \infty)$, inequality* (12) *implies that $g(t) \geq 0$ is defined on all of $\mathbb{R}_+$. Moreover, if $\lim_{t\to\infty} \mu(t) = +\infty$, then* $\lim_{t\to\infty} g(t) = 0$.

In section 2, we show how to choose $\mu(t)$ and to use Lemma 1 in order to obtain estimates for the solution to problem (1)-(2).

Proof of Lemma 1. Let us use inequality (8)

$$\dot{g}(t) \leq l(g). \tag{13}$$

Then inequalities (10) and (11) can be written as

$$l(\frac{1}{\mu(t)}) \leq \frac{d\mu^{-1}(t)}{dt}, \quad t \geq 0; \qquad \mu^{-1}(t) \geq g(t) \quad t \in [-\tau, 0]. \tag{14}$$

Let w_n solve the problem

$$\dot{w}_n = l(w_n) - \frac{1}{n}, \quad t \geq 0; \quad w_n(t) = g(t), t \in [-\tau, 0], \tag{15}$$
$$n = 1, 2, 3, ...$$

Let us prove that

$$w_n(t) \leq \mu^{-1}(t), \qquad \forall t \geq 0. \tag{16}$$

If (16) is proved, then one takes into account that $\lim_{n\to\infty} w_n = w$, where

$$\dot{w} = l(w), \quad t \geq 0; \quad w(t) = g(t), \qquad t \in [-\tau, 0], \tag{17}$$

and concludes, passing to the limit $n \to \infty$, that

$$w(t) \leq \mu^{-1}(t), \qquad \forall t \geq 0. \tag{18}$$

An argument similar to the one that will lead to inequality (16) will also yield the inequality $g(t) \le w(t)$, $t \ge 0$. This inequality and (18) imply the desired conclusion (12).

Therefore, to complete the proof of (12), it is sufficient to prove (16). In order to prove (16), note that if $w_n(0) < \mu^{-1}(0)$, then there exists an interval $(0, t_1)$, $t_1 > 0$, such that $w_n(t) < \mu^{-1}(t)$ when $t \in [0, t_1)$. If $w_n(0) = \mu^{-1}(0)$, then

$$\dot{w}_n(0) = l(w_n)|_{t=0} - \frac{1}{n} < l(w_n)|_{t=0} = l(g)|_{t=0} \le \frac{d\mu^{-1}(t)}{dt}|_{t=0},$$

where we have used the assumption about non-decreasing of $a(t, g)$ with respect to g and the inequality $\mu^{-1}(-\tau) \ge g(-\tau)$, that follows from assumption (11). Consequently, one has

$$w_n(0) = \mu^{-1}(0), \qquad \dot{w}_n(0) < \frac{d\mu^{-1}(t)}{dt}|_{t=0}.$$

Therefore, in this case there exists a number $t_2 > 0$ such that on the interval $(0, t_2)$ one has

$$w_n(t) < \mu^{-1}(t), \qquad 0 < t < t_2. \tag{19}$$

Let $t_3 := \min(t_1, t_2)$. Let us prove that $t_3 = \infty$. Assume the contrary. Then there exists a (minimal) $s > 0$, $s = \sup t_3$, such that

$$w_n(t) < \mu^{-1}(t), \quad t < s; \qquad w_n(s) = \mu^{-1}(s). \tag{20}$$

At the point s the following inequalities hold:

$$\dot{w}_n(s) = l(w_n(s)) - \frac{1}{n} < l(w_n(s)) \le l(\mu^{-1}(s)) \le \frac{d\mu^{-1}(t)}{dt}|_{t=s}, \tag{21}$$

where the non-decreasing of $a(t, g)$ with respect to g was used, and the inequality $w_n(s-\tau) < \mu^{-1}(s-\tau)$, which is a consequence of inequality (20), was taken into account. Thus,

$$\dot{w}_n(s) < \frac{d\mu^{-1}(t)}{dt}|_{t=s}.$$

By continuity, one has

$$\dot{w}_n(t) < \frac{d\mu^{-1}(t)}{dt}, \qquad s - \delta \le t \le s, \tag{22}$$

for a sufficiently small $\delta > 0$.

Integrate (22) on the interval $[s-\delta, s]$ and get

$$w_n(s) - w_n(s-\delta) < \mu^{-1}(s) - \mu^{-1}(s-\delta). \tag{23}$$

Since $w_n(s) = \mu^{-1}(s)$, inequality (23) implies

$$\mu^{-1}(s-\delta) < w_n(s-\delta). \tag{24}$$

This inequality contradicts inequality (20). This contradiction proves that the assumption $t_3 < \infty$ is false, so $t_3 = \infty$. Consequently,

$$w_n(t) < \mu^{-1}(t), \qquad \forall t > 0. \tag{25}$$

Passing to the limit $n \to \infty$ in (25), one gets (18).

A similar argument proves that

$$g(t) \le w(t), \qquad \forall t \ge 0. \tag{26}$$

Combining inequalities (18) and (26), one obtains (12).

Lemma 1 is proved. $\Box$

2 Estimates of solutions to evolution problem

Let us apply Lemma 1 to the solution of problem (1) - (2).

In order to choose $\mu(t)$, *let us assume that*

$$\gamma(t) = \gamma = const > 0, \qquad b(t) \le \frac{\gamma}{2}, \qquad \alpha(t,g) \le c_0 g^p, \tag{27}$$

where $c_0 > 0$ and $p > 1$ are constants, and $b(t) \ge 0$, $\alpha(t,g) \ge 0$ and $\alpha(t,g)$ is non-decreasing with respect to g.

Let us choose

$$\mu(t) = \lambda e^{\nu t},$$

where λ and ν are positive constants. Then $\frac{\dot{\mu}}{\mu} = \nu$. Choose $\nu = 0.5\gamma$. Inequality (10) holds if

$$c_0 \lambda^{1-p} e^{0.5\gamma(p-1)\tau} \le 1. \tag{28}$$

Define $\Gamma := \max_{t\in[-\tau,0]} |g(t)|$. Then inequality (11) holds if

$$\Gamma \le \lambda^{-1}. \tag{29}$$

Choose $\lambda = \Gamma^{-1}$. Then inequality (29) holds. Inequality (28) holds if

$$c_0\Gamma^{p-1}e^{0.5(p-1)\gamma\tau} \le 1. \tag{30}$$

Inequality (30) holds if c_0 is sufficiently small, or if Γ is sufficiently small. The last conclusion is based on the assumption $p > 1$.

We have proved the following theorem.

Theorem 1. *Assume that* (3) *holds with* $\gamma(t) = \gamma = const > 0$, (4) *holds with* $b(t) \le \frac{\gamma}{2}$, (27) *and* (30) *hold. Then the solution to problem* (1)-(2) *exists for all* $t \ge 0$ *and satisfies the following inequality*

$$||u(t)|| \le \Gamma^{p-1}e^{-0.5\gamma t}, \qquad \forall t \ge 0. \tag{31}$$

Estimate (31) of Theorem 1 implies *exponential stability* of the solution to problem (1)-(2). One could assume that γ depends on t. This will be done in the next example.

Consider now the case when $\gamma = \gamma(t)$ tends to zero as $t \to \infty$.

Assume that

$$\gamma(t) = \frac{c_1}{(1+t)^{m_1}}, \quad b(t) \le \frac{c_2}{(1+t)^{m_2}}, \quad \alpha(t,g) \le \frac{c_3}{(1+t)^{m_3}}g^p, \tag{32}$$

where $c_j, m_j > 0$, $j = 1,2,3$, and $p > 1$ are constants, and $\alpha(t,g)$ is non-decreasing with respect to g.

Choose $\mu(t)$ of the form

$$\mu(t) = \lambda(1+t+\tau)^\nu, \qquad \lambda, \nu > 0, \tag{33}$$

where λ and ν are positive constants. Then

$$\frac{\dot{\mu}(t)}{\mu(t)} = \frac{\nu}{1+t+\tau} \le \frac{\nu}{1+\tau}, \qquad t \ge 0.$$

Denote, as above, $\Gamma := \max_{t\in[-\tau,0]} |g(t)|$. Then inequalities (10) and (11) hold if

$$\frac{c_2c_3\Gamma^{p-1}}{(1+t)^{m_2+m_3+(p-1)\nu}} \le \frac{c_1}{(1+t)^{m_1}} - \frac{\nu}{1+t}, \qquad t \ge 0, \tag{34}$$

$$\lambda\Gamma \le 1. \tag{35}$$

Inequality (35) holds if $\lambda = \Gamma^{-1}$.

Assume that

$$m_2 + m_3 + (p-1)\nu \ge 1, \qquad m_1 \le 1. \tag{36}$$

Then inequality (34) holds for all $t \geq 0$ provided that

$$c_2 c_3 \Gamma^{p-1} \leq c_1 - \nu. \tag{37}$$

Inequality (37) holds if $\nu < c_1$ and $c_2 c_3$ is sufficiently small. If these conditions are satisfied then, by Lemma 1, one gets

$$||u(t)|| \leq \frac{\Gamma}{(1+t+\tau)^{\nu}}, \qquad \forall t \geq 0. \tag{38}$$

We have proved the following theorem

Theorem 2. *Assume that* (32) *and* (36) *hold,* $\lambda = \frac{1}{\Gamma}$, $\nu < c_1$, *and* $c_2 c_3$ *is sufficiently small that* (37) *holds. Then the solution to problem* (1)-(2) *exists for all* $t \geq 0$, *and estimate* (38) *holds.*

Our method, based on lemma 1, is very flexible and applicable to many other problems, see, for example, [6]-[8]. If the delay is absent in the abstract differential equation (1), then the assumption that $\alpha(t, g)$ is non-decreasing with respect to g can be dropped, see [6].

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