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Summary: Assume that $D \subset \mathbb{R}^2$ is a bounded domain, diffeomorphic to a disc, star-shaped, with a $C^{1,\lambda}$ boundary $C$, $\lambda > 0$, which can be represented in polar coordinates as $r = f(\phi)$, where $f > 0$ is a smooth $2\pi-$periodic function. Let $\psi_{\pm n} := \psi_{\pm n}(\phi) := e^{\pm in\phi}f^n(\phi)$.

Theorem. Assume that
$$\int_0^{2\pi} \psi_{\pm n} f^2(\phi) d\phi = 0 \quad n = 1, 2, \ldots$$
Then $f = \text{const}$.

1 Formulation of the result

Assume that $D \subset \mathbb{R}^2$ is a bounded domain, diffeomorphic to a disc, star-shaped, with a $C^{1,\lambda}$ boundary $C$, $\lambda > 0$, which can be represented in polar coordinates as $r = f(\phi)$, where $f > 0$ is a smooth $2\pi-$periodic function. Let $\psi_{\pm n} := \psi_{\pm n}(\phi) := e^{\pm in\phi}f^n(\phi)$.

Theorem 1.1 Assume that
$$\int_0^{2\pi} \psi_{\pm n} f^2(\phi) d\phi = 0 \quad n = 1, 2, \ldots \quad (1.1)$$
Then $f = \text{const}$.

Remark 1.2 A similar result is true for $D \subset \mathbb{R}^m$, $m > 2$. Its proof is essentially the same.

Remark 1.3 The author raised the question, answered in Theorem 1.1, while thinking about the Pompeiu problem, see Chapter 11 in [1]. This question is of interest regardless of its relation to the Pompeiu problem since it gives an unusual result concerning completeness of a set of functions.

In Section 2 a proof is given.

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2 Proof

Assumption (1.1) implies that

\[ \int_D h_n dx = 0 \quad n = 1, 2, \ldots, \]  

(2.1)

where \( h_n := r^{|n|} e^{\pm in\phi} \) are harmonic functions regular at the origin, \( x \in \mathbb{R}^2 \), \( x = (r, \phi), \)

where \((r, \phi)\) are polar coordinates. To see that (1.1) is equivalent to (2.1), write the left-hand side of (2.1) in polar coordinates, integrate over \( r \) from 0 to \( f(\phi) \), and get (1.1).

Let \( y \in \mathbb{R}^2 \), \( B_R \) be a ball (disc), centered at the origin and containing \( D \) inside, \( B_R' \) be its complement in \( \mathbb{R}^2 \), and \( G(x, y) = \frac{1}{2\pi} \ln \frac{1}{|x-y|} \) be the fundamental solution of the Laplace equation in \( \mathbb{R}^2 \). Let

\[ r := |x|, \quad r' := |y|, \quad x \cdot y = rr' \cos \theta. \]

Then, for \( r > r' \), one has

\[ 2\pi G(x, y) = -\left[ \ln r + \frac{1}{2} \left( \ln \left( 1 - \frac{r'}{r} e^{i\theta} \right) + \ln \left( 1 - \frac{r'}{r} e^{-i\theta} \right) \right) \right], \quad r > r'. \]

(2.2)

Expanding \( \ln(1 - \frac{r'}{r} e^{\pm i\theta}) \) in Taylor series, which is possible since \( \frac{r'}{r} < 1 \), one gets

\[ \ln \left( 1 - \frac{r'}{r} e^{i\theta} \right) = -\sum_{n=1}^{\infty} \frac{h_n}{n r^{n+1}}, \quad r > r', \quad h_n = (r')^n e^{\pm in\theta}. \]

(2.3)

We conclude from the assumption (2.1) and from (2.2)–(2.3) that

\[ \int_D G(x, y) dy = -\frac{1}{2\pi} |D| \ln r, \quad r > R, \]

(2.4)

where \( |D| \) denotes area of \( D \).

Using the method from [2] (see also [3]) we derive from (2.4) that \( D \) is a disc.

It follows from (2.4) that the harmonic in \( D' = \mathbb{R}^2 \setminus D \) function

\[ u(x) := \int_D G(x, y) dy = -\frac{1}{2\pi} |D| \ln r, \quad r > R, \]

(2.5)

solves the equation

\[ \Delta u(x) = -\eta |D|, \]

(2.6)

where \( \eta \) is the characteristic function of \( D \), that is, \( \eta = 1 \) in \( D \), and \( \eta = 0 \) in \( D' \). Let \( C_R \) be the boundary of \( B_R \). A harmonic in \( B_R \) function \( h \) satisfies the conditions

\[ \int_{C_R} h_N ds = 0, \quad \int_{C_R} h ds = 2\pi h(0). \]

(2.7)

It follows from (2.5) that the functions \( u(x) \) and \( u_N(x) \) are constant on \( C_R \), since the normal \( N \) on \( C_R \) is directed along the radius. Multiply (2.6) by an arbitrary regular at the origin harmonic function \( h = h_n \), integrate over a disc \( B_R \), and use (2.7) to get

\[ \int_D h dx = \int_{C_R} (u h_N - u_N h) ds = c h(0), \quad c = const. \]

(2.8)
If $h$ is harmonic in $B_R$, then so is $h(gx)$, where $g$ is a rotation by an arbitrary angle $\alpha$ around $z$-axis, the axis perpendicular to $D$. Since $h(g0) = h(0)$, one can replace $h(x)$ by $h(gx)$ in (2.8), differentiate with respect to $\alpha$ and then set $\alpha = 0$. This yields
\[
\int_{D} \nabla h(x) \cdot [e_3, x] dx = 0, \tag{2.9}
\]
where $e_3$ is a unit vector along $z$-axis, $\cdot$ stands for the scalar product, $[e_3, x]$ is the vector product in $\mathbb{R}^3$, and $h$ is an arbitrary harmonic function in $B_R$, regular at the origin. One has
\[
\nabla h(x) \cdot [e_3, x] = \nabla \cdot (h[e_3, x]), \tag{2.10}
\]
because $\nabla \cdot [e_3, x] = 0$. Thus, integrating by parts in (2.9), one gets
\[
\int_{C} (-N_1s_2 + N_2s_1) h ds = 0, \tag{2.11}
\]
where $N_j, j = 1, 2$, are the components of the outer unit normal $N$ to $C$. It is proved in [2] that the set of restrictions of all harmonic functions in $B_R$, regular at the origin, onto a closed curve $C \subset B_R$, diffeomorphic to a circle, is dense in $L^2(C)$. Therefore, (2.11) implies
\[
-N_1s_2 + N_2s_1 = 0 \quad \forall s \in C. \tag{2.12}
\]
Let us derive from equation (2.12) that $C$ is a circle. Geometrically equation (2.12) means that the radius-vector $r := s_1e_1 + s_2e_2$ of the boundary $C$ is parallel to the normal $N$ to $C$, namely, $[r, N] = 0$. The unit tangential vector to $C$ is $t = dr/ds$, where $s$ is the arclength of $C$, and the normal $N$ is directed along $dt/ds$.

Since the normal $N$ is orthogonal to $t$, and $N$ is parallel to $r$ according to (2.12), it follows that $t \cdot r = 0$. Thus,
\[
dr/ds \cdot r = 0 \quad \forall s \in C. \tag{2.13}
\]
Consequently,
\[
r \cdot r = \text{const} \quad \forall s \in C. \tag{2.14}
\]
Therefore, $C$ is a circle, and $D$ is a disc.

Thus, Theorem 1.1 is proved. \hfill \Box

References


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