HARNACK'S INEQUALITY IN SPACES OF HOMOGENEOUS TYPE

by

SHARAD DEEP SILWAL

B.A., Tribhuvan University, Nepal, 2000
M.A., Tribhuvan University, Nepal, 2002
M.S., Kansas State University, 2007
M.S., Kansas State University, 2009

AN ABSTRACT OF A DISSERTATION

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Abstract

Originally introduced in 1961 by Carl Gustav Axel Harnack [36] in the context of harmonic functions in $\mathbb{R}^2$, the so-called Harnack inequality has since been established for solutions to a wide variety of different partial differential equations (PDEs) by mathematicians at different times of its historical development. Among them, Moser's iterative scheme [47–49] and Krylov-Safonov's probabilistic method [43, 44] stand out as pioneering theories, both in terms of their originality and their impact on the study of regularity of solutions to PDEs. Caffarelli's work [12] in 1989 greatly simplified Krylov-Safonov's theory and established Harnack's inequality in the context of fully non-linear elliptic PDEs. In this scenario, Caffarelli and Gutiérrez's study of the linearized Monge-Ampère equation [15, 16] in 2002-2003 served as a motivation for axiomatizations of Krylov-Safonov-Caffarelli theory [3, 25, 57]. The main work in this dissertation is a new axiomatization of Krylov-Safonov-Caffarelli theory.

Our axiomatic approach to Harnack's inequality in spaces of homogeneous type has some distinctive features. It sheds more light onto the role of the so-called critical density property, a property which is at the heart of the techniques developed by Krylov and Safonov. Our structural assumptions become more natural, and thus, our theory better suited, in the context of variational PDEs. We base our method on the theory of Muckenhoupt's A_p weights. The dissertation also gives an application of our axiomatic approach to Harnack's inequality in the context of infinite graphs. We provide an alternate proof of Harnack's inequality for harmonic functions on graphs originally proved in [21].

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Our axiomatic approach to Harnack's inequality in spaces of homogeneous type has some distinctive features. It sheds more light onto the role of the so-called critical density property, a property which is at the heart of the techniques developed by Krylov and Safonov. Our structural assumptions become more natural, and thus, our theory better suited, in the context of variational PDEs. We base our method on the theory of Muckenhoupt's A_p weights. The dissertation also gives an application of our axiomatic approach to Harnack's inequality in the context of infinite graphs. We provide an alternate proof of Harnack's inequality for harmonic functions on graphs originally proved in [21].

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Acknowledgments

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Dedication

To Lord Saraswati and my parents

Chapter 1

Introduction

This chapter lays the foundation for the work in this dissertation by familiarizing the reader with some background knowledge necessary to understand the main work. The whole chapter is devoted to establishing basic definitions, notations and results that will be used in subsequent chapters. Section 1.1 introduces the thematic inequality, namely, Harnack's inequality and related concepts and contexts. Section 1.2 provides motivation to our work in this dissertation together with a historical perspective of previous related works. Lastly, Section 1.3 provides the layout of the entire dissertation.

1.1 Harnack's inequality

This section provides a review of Harnack's inequality and a necessary framework for its study in the context of spaces of homogeneous type.

Harnack's inequality is a property for a non-negative function on a domain, say, $\Omega \subset \mathbb{R}^n$. This is an intrinsic property of both the function and the domain and is defined in terms of the function's behavior over balls which are significantly interior to the domain.

Definition 1. A function $u \geq 0$ defined on a domain Ω is said to verify *Harnack's inequality* or have the *Harnack* property if there exists a $C_H > 1$ such that for every ball $B_r(x_0)$ with $B_{2r}(x_0) \subset \Omega$,

$$\sup_{B_r(x_0)} u \leq C_H \inf_{B_r(x_0)} u.$$

Although Harnack's inequality has been phrased and presented in various ways in different contexts at different times, Definition 1 is the version that is currently used in the theory of partial differential equations and was given by Kellogg in [40].

Often, the interest lies in verifying Harnack's inequality for a class of functions such as the solutions to a certain PDE rather than an individual function. Let us look at a simple example in $\mathbb{R}$.

Example 2. Consider the non-negative solutions to the following differential equation:

$$\frac{d^2u}{dx^2} = 0 \text{ on } \Omega := (0,1) \subset \mathbb{R}.$$

Its solution space $\mathbb{K}_\Omega$ is the set of all non-negative linear functions defined on Ω. Specifically, $\mathbb{K}_\Omega = A \cup B$, where

$$A = \{u(x) := mx + b \mid m \geq 0, b \geq 0\}$$

and

$$B = \{u(x) := m(1-x) + b \mid m \geq 0, b \geq 0\}.$$

Our claim is that $\mathbb{K}_\Omega$ has a Harnack property, i.e., there exists a constant $C_H > 1$ such that given any interval $I := (l, r)$ with $2I := \left(l - \left(\frac{r-l}{2}\right), r + \left(\frac{r-l}{2}\right)\right) \subset (0,1)$ and any $u \in \mathbb{K}_\Omega$,

$$\sup_I u \leq C_H \inf_I u.$$

Note that the worst possible admissible interval at the left end point is $I_0 := (\varepsilon, 3\varepsilon), \varepsilon \in (0, \frac{1}{4})$ and at the right end point is $I_1 := (1-3\varepsilon, 1-\varepsilon), \varepsilon \in (0, \frac{1}{4})$. The case $m = 0$ is trivial since any $C_H > 1$ works. Also, without loss of generality, we can assume $b = 0$ since for any $\tilde{m}, x_1, x_2 \in \mathbb{R}, b > 0$ and $C > 1$, we always have

$$\tilde{m}x_1 \leq C\tilde{m}x_2 \Rightarrow \tilde{m}x_1 + b \leq C(\tilde{m}x_2 + b).$$

Hence, with the assumption $m > 0$ and $b = 0$, if $u \in A$,

$$\sup_{I_0} u = 3m\varepsilon = 3\inf_{I_0} u$$

and, since $0 < \varepsilon < \frac{1}{4}$,

$$\frac{\sup_{I_1} u}{\inf_{I_1} u} = \frac{m(1-\varepsilon)}{m(1-3\varepsilon)} = \frac{(1-\varepsilon)}{(1-3\varepsilon)} \leq \frac{1-0}{1-3\left(\frac{1}{4}\right)} = 4.$$

Similar bounds can be established for the case $u \in B$. Thus, $C_H = 4$ works for $\Omega := (0,1)$. In fact, the Harnack property will still hold true if we consider any interval $\Omega := (L, R) \subset \mathbb{R}$, in which case C_H, now, will necessarily depend on L and R.

Remark 3. As discussed above, we will be concentrating our attention mostly to a family of functions throughout this dissertation. Hence, let us establish some notation at this point. Let X be a space and $\Omega \subset X$ be a domain. Then, we denote by $\mathbb{K}_\Omega$ a family of functions with domain contained in Ω, and if $u \in \mathbb{K}_\Omega$ and $A \subset \mathrm{dom}(u)$ then we write $u \in \mathbb{K}_\Omega(A)$.

Example 4. We saw earlier that linear functions in $\mathbb{R}$ make examples of functions having a Harnack property. Much richer examples of functions can be found in $\mathbb{R}^2$. The simplest ones are harmonic functions, which, by definition, are solutions to the PDE

$$\triangle u := \sum_{i=1}^{2} u_{ii} = 0.$$

For example, $u(x_1, x_2) = x_1^2 - x_2^2$ is a harmonic function. Thus, $v(x_1, x_2) = x_1^2 - x_2^2 + 1$ has a Harnack property in $(-1,1)^2 \subset \mathbb{R}^2$. Note that u is concave up in x_1 and concave down in x_2. The essence of harmonicity in several variables is that these two types of concavity in a sense cancel each other out. Typical examples of harmonic functions in $\mathbb{R}^2$ are provided by the real and imaginary parts of holomorphic functions which are smooth functions of a complex variable z. Polynomials, trig, log and exponential functions in a complex variable z are examples of holomorphic functions.

Example 5. The space of the non-negative solutions to the second-order uniformly elliptic operators in any of the three canonical forms as given in Definition 12 below has a Harnack property.

Next, we give some definitions in order to introduce the second-order uniformly elliptic operators, which are frequently referred to throughout the dissertation.

Definition 6. An $n \times n$ (real) matrix A is said to be *positive-definite* and we write $A > 0$ if $\langle A(x)\xi, \xi\rangle := \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j > 0$ for every $0 \neq \xi \in \mathbb{R}^n$. The quantity $\langle A(x)\xi, \xi\rangle$ is called the *quadratic form* of A. If we instead ask $\langle A(x)\xi, \xi\rangle \geq 0$ for every $0 \neq \xi \in \mathbb{R}^n$ (i.e., $\langle A(x)\xi, \xi\rangle = 0$ is possible for some $0 \neq \xi \in \mathbb{R}^n$), then A is called *positive-semidefinite* and we write $A \geq 0$. If A and B are two matrices, then we order $A \leq B$, if $B - A \geq 0$.

The following facts about positive-semidefinite matrices hold true:

(a) $A \geq 0 \Rightarrow a_{ii} \geq 0 \Rightarrow \mathrm{tr}(A) \geq 0$.

(b) $M \geq N > 0 \Rightarrow N^{-1} \geq M^{-1} > 0$.

(c) $r > 0, M > 0 \Rightarrow rM > 0$.

(d) $M, N > 0 \Rightarrow M + N, MNM, NMN > 0$. If $MN = NM$ then $MN > 0$.

(e) $M > N > 0 \Rightarrow M^{1/2} > N^{1/2} > 0$.

Definition 7. Let $\Omega \subset \mathbb{R}^n$ and $0 < \lambda < \Lambda < \infty$. A measurable, $n \times n$ matrix-valued function A defined on Ω is said to be *uniformly elliptic* in Ω with constants λ and Λ and we write $A \in \mathcal{A}(\lambda, \Lambda, \Omega)$ if

(a) $A(x) = A(x)^T, x \in \Omega$.

(b) $\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda|\xi|^2$, $x \in \Omega$, $\xi \in \mathbb{R}^n$.

Remark 8. By Definition 6 and Definition 7, $A \in \mathcal{A}(\lambda, \Lambda, \Omega)$ implies that for every $x \in \Omega$, $A(x)$ is a symmetric positive-definite matrix. Furthermore, we have $0 < \lambda I_n \leq A(x) \leq \Lambda I_n$. The consequence of the second assumption in Definition 7 is that we have, for every $x \in \Omega$,

$$\lambda \leq a_{ii}(x) \leq \Lambda, \quad |a_{ij}(x)| \leq \Lambda - \lambda,\ i \neq j.$$

This means $A(x)$ is, in fact, bounded.

Remark 9. Definition 7 is the analytic interpretation of uniform ellipticity. There are two other interpretations, namely, algebraic and geometric. Next, we provide the definitions of uniform ellipticity based on these interpretations without establishing the equivalence between them. The algebraic interpretation is provided by the following definition:

Definition 10 (Algebraic definition of uniform ellipticity)**.** Let $\Omega \subset \mathbb{R}^n$ be open and $0 < \lambda \leq \Lambda < \infty$. A symmetric real-valued matrix $A(x) = (a_{ij}(x))$ is said to be *uniformly elliptic* in Ω with constants λ and Λ if for every $\sigma > 0$, the following condition holds:

- If $A(x)\xi = \sigma\xi$ for every $x \in \Omega$ and for some $0 \neq \xi \in \mathbb{R}^n$, then $\sigma \in [\lambda, \Lambda]$.

In other words, $A \in \mathcal{A}(\lambda, \Lambda, \Omega)$ iff for every $x \in \Omega$ and an eigenvalue $\sigma \in \mathbb{R}$ of $A(x)$, $\lambda \leq \sigma \leq \Lambda$. Although, a matrix can have complex eigenvalues in general, it is to be noted that a symmetric real matrix has all real eigenvalues.

The geometric interpretation of uniform ellipticity with constants λ and Λ is that the matrix transforms the unit ball into an ellipse which lies inside of the annuli with inner and outer radii λ and Λ respectively. It is illustrated in Figure 1.1 and is given by the following definition:

Definition 11 (Geometric definition of uniform ellipticity)**.** Let $\Omega \subset \mathbb{R}^n$ be open and $0 < \lambda \leq \Lambda < \infty$. Let $U = \{\xi \in \mathbb{R}^n \mid |\xi| = 1\}$. A symmetric real-valued matrix $A(x) = (a_{ij}(x))$ is is said to be *uniformly elliptic* in Ω with constants λ and Λ if for every $x \in \Omega$ and for every $\xi \in U$, we have $\lambda \leq |A(x)\xi| \leq \Lambda$.

In other words, $A \in \mathcal{A}(\lambda, \Lambda, \Omega)$ iff for every $x \in \Omega$ and $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, we have $\lambda \leq |A(x)\xi| \leq \Lambda$.

Let u be a real-valued function defined in $\Omega \subset \mathbb{R}^n$ and $A \in \mathcal{A}(\lambda, \Lambda, \Omega)$ and denote $u_i := D_i u = \frac{\partial u}{\partial x_i}$. The most general form of second-order linear differential operator in a domain $\Omega \subset \mathbb{R}^n$ is

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u + b_i(x) D_i u + c(x) u, \; x \in \Omega,$$

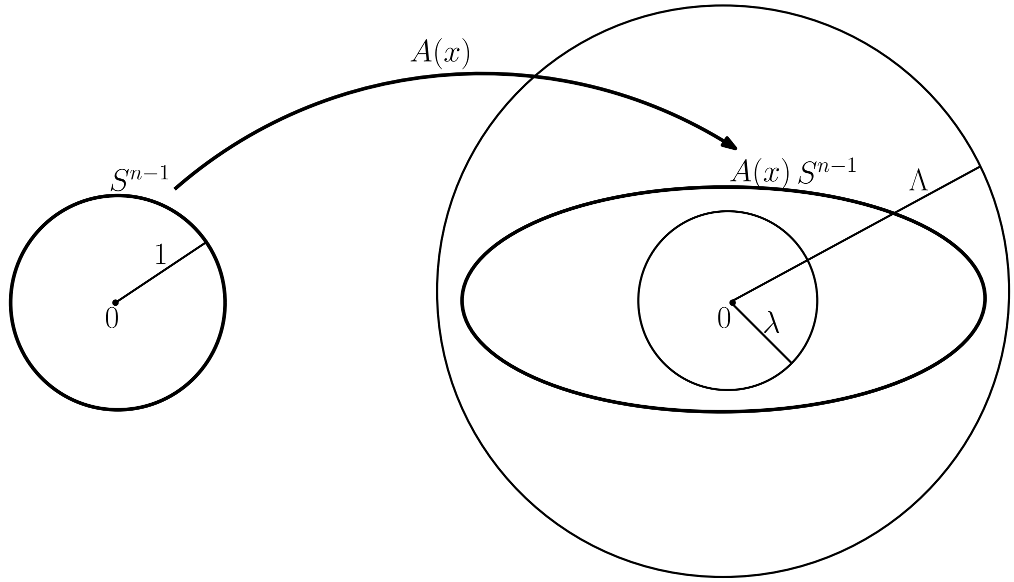


Figure 1.1: *Ellipticity of $A(x)$ with constants λ, Λ.*

where $a_{ij}, b_i, c \in L^\infty(\Omega)$. Since the highest-order terms control the qualitative behaviors of solutions to a PDE, we often restrict our attention to the second-order linear differential operator without lower-order terms:

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x) D_{ij} u,\ x \in \Omega,$$

where $a_{ij} \in L^\infty(\Omega)$.

Definition 12. Based on how they originate, uniformly elliptic operators can be categorized into the following three forms:

(1) Divergence form: $\mathcal{L}u := \displaystyle\sum_{i,j=1}^{n} (a_{ij}(x)u_i)_j = \operatorname{div}(A(x)\nabla u)$.

(2) Non-divergence form: $Lu := \displaystyle\sum_{i,j=1}^{n} a_{ij}(x)u_{ij} = \operatorname{tr}(A(x)D^2u)$.

(3) Adjoint form: $L^*u := \displaystyle\sum_{i,j=1}^{n} (a_{ij}(x)u)_{ij}$.

Notice the special case: $A(x) := (a_{ij}(x)) = I_n$ when all three forms yield the Laplacian operator $\triangle = \displaystyle\sum_{i=1}^{n} D_{ii}$.

Example 13. It is easy to see that any non-negative function which attains a zero in a domain cannot have a Harnack property there unless the function is identically zero. Hence, $u(x) = |x|$ cannot have a Harnack property in any interval $(-\varepsilon, \varepsilon) \subset \mathbb{R}$. Also, it is very easy to find individual functions that does have a Harnack property. Any function u that is bounded and bounded away from zero has a Harnack property. Indeed, if $0 < \lambda < u(x) < \Lambda$ for any $x \in \Omega$, then

$$\frac{\sup_I u}{\inf_I u} \leq \frac{\Lambda}{\lambda}, \forall I \subset \Omega.$$

However, in practical applications, the interest lies in establishing Harnack for some family of functions as described in Remark 3 above rather than for individual functions.

The Harnack property is a very strong property and leads to many deep consequences. We would like to mention some of them here.

Proposition 14. *Suppose that u defined in Ω has a Harnack property, i.e., there exists a structural constant $C_H > 1$ such that for every $B_r \subset \Omega$,*

$$\sup_{B_{r/2}} u \leq C_H \inf_{B_{r/2}} u.$$

Additionally, assume that given any two points $x, x_0 \in \Omega$, it is possible to find an $\varepsilon > 0$ and a finite number of balls $B_0, B_1, \ldots, B_k$, each with radius ε, having the following properties:

(i) $2B_i \subset \Omega$ *for every* i.

(ii) $B_0 = B_\varepsilon(x_0)$ *and* $B_k = B_\varepsilon(x)$.

(iii) $B_i \cap B_{i+1} \neq \emptyset$ *for* $i = 0, 1, \ldots, k-1$.

Then, if u has a zero in Ω then $u \equiv 0$ in Ω.

Proof. Assume $x_0 \in \Omega$ is a zero of u and take any point $x \in \Omega$. Next, obtain the balls $B_0, B_1, \ldots, B_k$ for these two points $x, x_0 \in \Omega$ as given by hypothesis. Then, by the non-negativity of u,

$$0 \leq \sup_{B_0} u \leq C_H \inf_{B_0} u = 0,$$

which implies $u \equiv 0$ in B_0. Next, we take another zero $x_1 \in B_0 \cap B_1$ and keep repeating this procedure until we eventually get $u \equiv 0$ in B_k. $\square$

The next definition and the discussions to follow that are intended to set the background to show one of the most important consequences of the Harnack property.

Definition 15. Let $0 < \alpha \leq 1$. We say a function u defined on a domain Ω is *α-Hölder continuous*, and write $u \in C^{0,\alpha}(\Omega)$, if and only if there exists a constant $C > 0$ such that

$$|u(x) - u(y)| \leq C|x - y|^{\alpha}, \tag{1.1.1}$$

for all $x, y \in \Omega$. In the case of $\alpha = 1$, u is more commonly said to be *Lipschitz continuous*. The constant C that makes the inequality (1.1.1) sharp is denoted by

$$|u|_{C^{0,\alpha}(\Omega)} := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left\{ \frac{|u(x) - u(y)|}{|x - y|^{\alpha}} \right\}.$$

In general, $u \in C^{k,\alpha}(\Omega)$, where k is a non-negative integer, means that $u \in C^k(\Omega)$ (i.e., u is differentiable up to order k) and $\partial^{\beta} u \in C^{0,\alpha}(\Omega)$ for every multi-index β with $|\beta| = k$ (i.e., all its k^{th} partial derivatives are α-Hölder continuous). A geometric illustration of a α-Hölder continuity is given by Figure 1.2. Assume first that $u(0) = 0$, then using $x = 0$ in (1.1.1), the α-Hölder continuity of u means that the graph of u lies below the graph of $v_p(y) = C|y|^{\alpha}$ and above the graph of $v_n(y) = -C|y|^{\alpha}$. Now, with any arbitrary x in (1.1.1), the α-Hölder continuity of u requires that the exact same condition holds with a change of coordinates where the axes are parallel to the rectangular axes and the origin is shifted from $(0, 0)$ to $(x, u(x))$.

Remark 16. It is clear by definition that Hölder continuity implies uniform continuity. The function

$$f(x) := \begin{cases} \frac{1}{\log x}, & \text{if } x \in (0, 10^{-2}] \\ 0, & \text{if } x = 0, \end{cases}$$

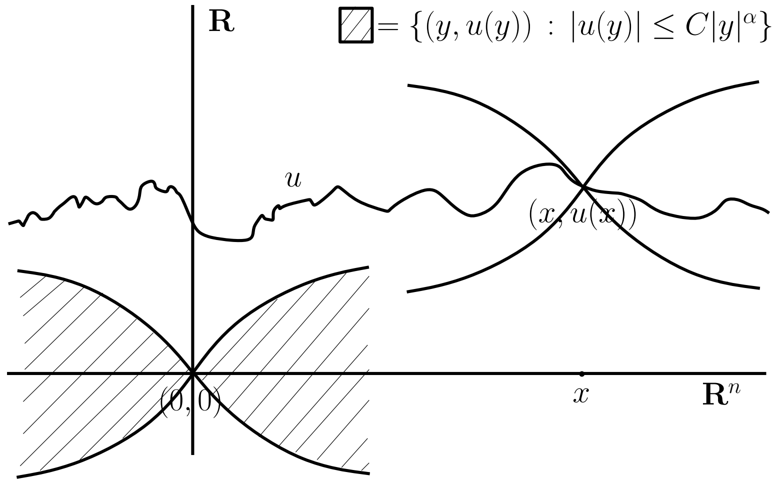


Figure 1.2: *α-Hölder continuity of u at x means that the graph of u lies encapsulated inside of the graphs of $v(y) = \pm C|y|^\alpha$ translated from $(0,0)$ to $(x, u(x))$.*

is uniformly continuous in $[0, 10^{-2}]$ but is not α-Hölder continuous at 0 for any $0 < \alpha \leq 1$. Indeed, otherwise, there would exist a constant $C > 0$ such that

$$\left|0 - \frac{1}{\log x}\right| \leq C|x|^\alpha, \forall x > 0,$$

which, in turn, implies

$$0 < \frac{1}{C} \leq |\log x||x|^\alpha, \forall x > 0,$$

which is not true since the right hand side goes to 0 as $x \to 0^+$. Also, uniform continuity implies continuity but the reverse implication is true only when the domain is compact. An example of a function having no uniform continuity in an unbounded domain is given by $u(x) = e^x$, $x \in \mathbb{R}^n$, and in a bounded domain is given by $u(x) = \tan x$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. If $\alpha > 1$, Definition 15 just yields constant functions because for any $x \in \Omega, \omega \in \mathbb{S}^{n-1}$, we have

$$\left|\frac{u(x + r\omega) - u(x)}{r}\right| \leq |u|_{C^{0,\alpha}(\Omega)}|r|^{\alpha-1} \to 0 \text{ as } r \to 0,$$

which shows that $\partial_\omega u(x) = 0$.

One of the virtues of Harnack's inequality is that, in the context of a family of functions (the space of non-negative solutions to some differential operator, in practice), it implies

Hölder continuity, which, in turn, leads to C^∞ regularity. In 1900, David Hilbert posed a celebrated list of 23 problems [37] to the mathematics community which, in large part, dominated the direction of mathematical research of the twentieth century. Hilbert's 19th problem asked whether or not the solutions to regular problems in the calculus of variations always are analytic. Ennio De Giorgi [31] in 1957 and John Nash [51] in 1958 through their independent results were able to completely resolve this problem by providing an affirmative answer. The way they accomplished this is through the establishment of Hölder regularity for solutions to the divergence-form PDE with uniform ellipticity (see Definition 12) followed by a bootstrapping argument. We leave out the details of the Hilbert's 19th problem such as its relation with the C^∞ regularity of the solutions to the above-mentioned PDE and the bootstrapping argument. These can be found in many books or papers, for instance, see [30, 35, 58]. The second part of the solution to Hilbert's 19th problem, namely, the bootstrapping argument, was a classic technique known to mathematicians long before the problem was resolved. The actual breakthrough, therefore, was the first part, namely, the establishment of Hölder regularity for the solutions to the above-mentioned PDE. Later in 1961, Moser [47] establishes Harnack's inequality for the solutions to the same PDE and obtains Hölder regularity as one of its consequences. Since then, Harnack's inequality in the current literature has emerged as the canonical method by which to arrive at Hölder regularity. We provide a very quick review of this method in the form of Proposition 18 below, very much in the spirit of [35].

Lemma 17 ([35]). *Let w be nondecreasing in $(0, R]$. Let $\gamma, \tau \in (0,1)$ and w be such that*

$$w(\tau r) \leq \gamma w(r). \tag{1.1.2}$$

Then, for every $\mu \in (0,1)$, we have

$$w(r) \leq C \left(\frac{r}{R}\right)^\alpha w(R), \tag{1.1.3}$$

where $C = \frac{1}{\gamma}$ and $\alpha = (1-\mu)\frac{\log \gamma}{\log \tau}$.

Proof. We only need to prove (1.1.3) for $r \in (0, R)$. Set $r_1 := r^\mu R^{1-\mu}$ so that

$$\frac{r}{r_1} = \left(\frac{r}{R}\right)^{1-\mu}. \tag{1.1.4}$$

Since $\frac{\log(r/r_1)}{\log \tau} > 0$, there exists an integer $k \geq 1$ such that

$$k - 1 \leq \frac{\log(r/r_1)}{\log \tau} \leq k,$$

which, upon multiplying by $\log \gamma < 0$, yields

$$\gamma^k \leq \left(\frac{r}{r_1}\right)^{\frac{\log \gamma}{\log \tau}}. \tag{1.1.5}$$

Iterating (1.1.2) and then using (1.1.4) and (1.1.5), we get

$$\begin{aligned} w(r) &\leq w(\tau^{k-1} r_1) \leq \gamma^{k-1} w(r_1) \leq \gamma^{k-1} w(R) \\ &\leq \frac{1}{\gamma}\left(\frac{r}{r_1}\right)^{\frac{\log \gamma}{\log \tau}} w(R) \\ &= \frac{1}{\gamma}\left(\frac{r}{R}\right)^{(1-\mu)\frac{\log \gamma}{\log \tau}} w(R), \end{aligned}$$

which is (1.1.3). □

Proposition 18 ([35]). *Let L be a differential operator in $\Omega \subset \mathbb{R}^n$ with the following properties:*

(i) $L(u + c) = Lu$, $c \in \mathbb{R}$.

(ii) There exists an integer $k \geq 1$ such that L is homogeneous of order k, *i.e.,* $L(cu) = c^k Lu$, *for $c > 0$.*

Suppose that non-negative solutions to L in $\Omega \subset \mathbb{R}^n$ has a Harnack property, i.e., there exists a structural constant $C_H = C_H(n, L, \Omega)$ such that if $u \geq 0$ solves $Lu = 0$ in $\Omega \subset \mathbb{R}^n$, then, for every $B_r \subset \Omega$,

$$\sup_{B_{r/2}} u \leq C_H \inf_{B_{r/2}} u. \tag{1.1.6}$$

Then, if $u \in L^p(\Omega), p > 0$ solves $Lu = 0$ in $\Omega \subset \mathbb{R}^n$, there exists some structural constant $\alpha \in (0,1)$ such that $u \in C^{0,\alpha}(\Omega)$. Moreover, there exists a structural constant $C > 0$ such that, for any $B_R(x_0) \subset \Omega$ and any $x, y \in B_{R/2}(x_0)$,

$$|u(x) - u(y)| \leq C \left(\frac{|x-y|}{R} \right)^{\alpha} \left(\frac{1}{R^n} \int_{B_R(x_0)} u(z)^p \, dz \right)^{\frac{1}{p}}. \tag{1.1.7}$$

Proof. We establish (1.1.7) by proving a Hölder modulus of continuity estimate. Let $r \in (0, R)$ and define the following:

$$B_R := B_R(x_0), \quad B_r := B_r(x_0), \quad M(r) := \max_{B_r} u, \quad m(r) := \min_{B_r} u, \quad w(r) := M(r) - m(r).$$

Then, it will suffice to show that, for every $r \in (0, \frac{R}{2}]$,

$$w(r) \leq C \left(\frac{r}{R} \right)^{\alpha} \left(\frac{1}{R^n} \int_{B_R} u(x)^p \, dx \right)^{\frac{1}{p}}. \tag{1.1.8}$$

Due to the assumptions (i) and (ii) in the hypothesis, $M(r) - u \geq 0$ is a solution to $Lu = 0$ in B_r. Hence, by Harnack's inequality (1.1.6), we have,

$$\sup_{B_{r/2}} (M(r) - u) \leq C_H \inf_{B_{r/2}} (M(r) - u),$$

which is

$$M(r) - m\left(\frac{r}{2}\right) \leq C_H \left(M(r) - M\left(\frac{r}{2}\right) \right). \tag{1.1.9}$$

Likewise, $u - m(r) \geq 0$ is a solution to $Lu = 0$ in B_r. So, by Harnack's inequality (1.1.6), we have

$$M\left(\frac{r}{2}\right) - m(r) \leq C_H \left(m\left(\frac{r}{2}\right) - m(r) \right). \tag{1.1.10}$$

Adding (1.1.9) and (1.1.10) together gives

$$w(r) + w\left(\frac{r}{2}\right) \leq C_H \left(w(r) - w\left(\frac{r}{2}\right) \right),$$

which is

$$w\left(\frac{r}{2}\right) \leq \gamma w(r), \quad \gamma = \frac{C_H - 1}{C_H + 1} \in (0,1). \tag{1.1.11}$$

Now, applying Lemma 17 to (1.1.11) and a choice of $\mu \in (0,1)$ sufficiently close to 1, we get a structural constant $C > 0$ and $\alpha = (1-\mu)\frac{\log \gamma}{\log \tau} \in (0,1)$ such that, for every $r \in (0, \frac{R}{2}]$, we have

$$w(r) \le C\left(\frac{r}{R}\right)^{\alpha} w\left(\frac{R}{2}\right) \le C\left(\frac{r}{R}\right)^{\alpha} M\left(\frac{R}{2}\right). \tag{1.1.12}$$

On the other hand, denoting by $|\cdot|$ the Lebesgue measure in $\mathbb{R}^n$, we have

$$M\left(\frac{R}{2}\right) := \sup_{B_{R/2}} u \le C_H \inf_{B_{R/2}} u \le C_H \left(\frac{1}{|B_{R/2}|}\int_{B_{R/2}} u(x)^p\, dx\right)^{\frac{1}{p}} \le C_H \left(\frac{1}{|B_R|}\int_{B_R} u(x)^p\, dx\right)^{\frac{1}{p}},$$

which, together with (1.1.12), yields (1.1.8). $\square$

Corollary 19 ([35], Liouville's theorem)**.** *Assume the hypotheses of Proposition 18. If $\Omega := \mathbb{R}^n$ and u is bounded, then u is constant.*

Proof. Let $B_r(x_0)$ be any arbitrary ball in $\mathbb{R}^n$. By (1.1.11), we have that there exists a structural constant $\gamma \in (0,1)$ such that

$$w\left(\frac{r}{2}\right) \le \gamma w(r),$$

which, upon iteration up to k times, yields

$$w(r) \le \gamma^k w(2^k r). \tag{1.1.13}$$

On the other hand, u is bounded if and only if w is bounded. Thus, since u is bounded, we have $w(2^k r) \le C$ for every $k \ge 1$. Hence, taking the limit $k \to \infty$ in (1.1.13) yields $w(r) = 0$. That is, $\sup_{B_r(x_0)} u = \inf_{B_r(x_0)} u$, which means u is constant in $B_r(x_0)$. Since $B_r(x_0) \subset \mathbb{R}^n$ is arbitrary, u is constant in all of $\mathbb{R}^n$. $\square$

Having discussed Harnack's inequality together with some examples and consequences, a short discussion of its historical development that motivates the work in this dissertation is in order. However, we would like to deal with that later in Section 1.2 and devote the rest of this section to reviewing some preliminary concepts relevant to the dissertation.

Definition 20. A *quasi-metric* space is a pair (X, d) where X is a non-empty set and d is a quasi-distance on X, that is, $d : X \times X \to [0, \infty)$ such that

(i) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(ii) $d(x, y) = 0$ if and only if $x = y$,

(iii) and there exists $K \geq 1$ (quasi-triangle constant), such that

$$d(x, y) \leq K(d(x, z) + d(z, y)), \quad x, y, z \in X.$$

On a quasi-metric space (X, d), we can define the d-ball with center $x \in X$ and radius $r > 0$ in (X, d) by

$$B_r(x) := \{y \in X : \; d(x, y) < r\}.$$

For $B = B_r(x)$ and $\lambda > 0$, λB denotes the ball $B_{\lambda r}(x)$.

Definition 21. Let (X, d) be a quasi-metric space with the quasi-triangle constant K. Let μ be a measure defined on the d-balls of X. We say that (X, d, μ) is a *space of homogeneous type* or *doubling quasi-metric space* if μ satisfies the *doubling property*, that is, if there exists a doubling constant $C_\mu > 1$ such that

$$0 < \mu(B_{2r}(x)) \leq C_\mu \, \mu(B_r(x)), \quad x \in X, r > 0. \tag{1.1.14}$$

Every constant depending only on K and C_μ is called a *geometric constant.*

Clearly, Euclidean spaces $\mathbb{R}^n$ equipped with the Euclidean distance d and the Lebesgue measure $\mathcal{L}$ are examples of spaces of homogeneous type. A more interesting example is given by the triad $(\mathbb{R}^n, \rho_\phi, \mu_\phi)$ introduced in [15] (see Section 3.2.2 for a detailed description), where $\phi \in C^2$ is a suitable convex function, $\mu_\phi(x) := \det D^2\phi(x)$ is the *Monge-Ampère measure* and

$$\rho_\phi(x, y) := \max\{\phi(y) - \phi(x) - \langle \nabla\phi(x), y - x\rangle, \phi(x) - \phi(y) - \langle \nabla\phi(y), x - y\rangle\}.$$

For $\phi \in C^2$ to be a suitable function, all that is required is $\mu_\phi(x) > 0$ for all $x \in \mathbb{R}^n$. For example, taking $\phi(x) := \frac{1}{2}|x|^2$, we get $\rho_\phi(x, y) = 2d(x, y)^2$, where d is the Euclidean distance and the measure $d\mu_\phi(x)$ reduces to the Lebesgue measure dx. A second interesting example of a space of homogeneous type is given by a connected graph G with a uniform bound on the number of neighbors of its vertices. The natural way to define a distance on a graph is as follows: the distance between vertices x and y is the length of the shortest path joining x and y. The natural measure on a graph is the counting measure #. Now, the only extra condition required for a graph to have the structure of a space of homogeneous type is the doubling property for the counting measure. k-regular trees are examples of graphs which admit the doubling property. These are graphs generated through an infinite iterative process starting with a finite graph with k edges and then generating in each step a new graph by replacing each edge with a copy of the initial graph.

Next, we record a remark due to Macías and Segovia [45, 46] which provides us with a very useful assertion that will endow a quasi-metric with additional properties.

Remark 22 ([45, 46], Remark 8.10 [56])**.** If (X, ρ) is a quasi-metric space, then there exists a quasi-distance ρ' on X and constants $C > 0$ and $\theta \in (0, 1)$ such that

(i) ρ' is equivalent to ρ,

(ii) for every $x, y, z \in X$ and $r > 0$,

$$|\rho'(x, z) - \rho'(y, z)| \leq C\rho'(x, y)^\theta \left(\rho'(x, z) + \rho'(y, z)\right)^{1-\theta}. \tag{1.1.15}$$

In fact, this will have the following consequences:

(a) $\rho'(x, y) = d(x, y)^q$, for some distance d in X and $q > 1$.

(b) If (X, ρ, μ) is a space of homogeneous type, then for each ρ'-ball B, B is an open set and the triple (B, ρ', μ) is a space of homogenous type with constants uniform in B, depending only on the constants for (X, ρ, μ)

Thus, for all intents and purposes of our dissertation, we can assume that the underlying quasi-distance ρ of our space of homogeneous type (X, ρ, μ) have the same properties as ρ'. In particular, for all practical purposes of this dissertation, we can assume that the quasi-distance ρ is essentially a power of a distance. In other words, every quasi-metric space is metrizable.

A small lemma which will be used extensively throughout the dissertation is established next. Lemma 23 below extracts some very handy properties of a quasi-distance through the quasi-triangle inequality (iii) and the doubling property (1.1.14).

Lemma 23. *Let (X, d, μ) be a space of homogeneous type with quasi-triangle and doubling constants K and C_μ respectively. Define a geometric constant $\zeta := \log_2 C_\mu$. Then*

$$\mu(B_{R_2}(x_0)) \le C_\mu \left(\frac{R_2}{R_1}\right)^\zeta \mu(B_{R_1}(x_0)), \quad \forall\, 0 < R_1 < R_2. \tag{1.1.16}$$

Furthermore, if either (i) $B_s(z) \subset B_{2Ks}(x_0)$ or (ii) $x_0 \in B_r(z)$ holds, then

$$\mu(B_r(x_0)) \ge \frac{1}{(2K)^\zeta C_\mu} \left(\frac{r}{s}\right)^\zeta \mu(B_s(z)), \quad \forall\, 0 < r < s. \tag{1.1.17}$$

Proof. Let us first prove (1.1.16), which is a consequence of the doubling property (1.1.14). Since $\frac{\log\left(\frac{R_2}{R_1}\right)}{\log 2} > 0$, there exists an integer $k \ge 1$ such that

$$(k-1) < \frac{\log\left(\frac{R_2}{R_1}\right)}{\log 2} < k. \tag{1.1.18}$$

On the other hand, using logarithm tricks ($a^{\log b} = b^{\log a}$ and $\log_a b = \frac{\log b}{\log a}$) and $\zeta := \log_2 C_\mu$, we obtain

$$C_\mu^{\frac{\log\left(\frac{R_2}{R_1}\right)}{\log 2}} = \left(\frac{R_2}{R_1}\right)^\zeta. \tag{1.1.19}$$

Next, using $R_2 = \frac{R_2}{R_1} R_1 \le 2^k R_1$ and the doubling property (1.1.14), we have

$$\mu(B_{R_2}(x_0)) \le \mu((B_{2^k R_1}(x_0)) \le C_\mu C_\mu^{k-1} \mu(B_{R_1}(x_0)),$$

which, together with (1.1.18) and (1.1.19), yields (1.1.16). Now, since $x_0 \in B_s(x_0)$, by the quasi-triangle inequality, $B_s(x_0) \subset B_{2Ks}(x_0)$. Since $r < s$, this implies $B_r(x_0) \subset B_{2Ks}(x_0)$. Using (1.1.16) to this inclusion, we get

$$C_\mu \left(\frac{2Ks}{r}\right)^\zeta \mu(B_r(x_0)) \geq \mu(B_{2Ks}(x_0)),$$

which upon using $B_s(z) \subset B_{2Ks}(x_0)$, yields

$$\mu(B_s(z)) \leq C_\mu (2K)^\zeta \left(\frac{s}{r}\right)^\zeta \mu(B_r(x_0)).$$

Finally, since $x_0 \in B_r(z)$ and $0 < r < s$ together imply $B_s(z) \subset B_{2Ks}(x_0)$, the lemma holds in that case too. □

1.2 Motivation and historical notes

Harnack's inequality whose study in recent times permeates most general types of spaces originated in 1887 in the context of Euclidean spaces. It was first introduced by Carl Gustav Axel Harnack in his book [36] on potential theory. He introduced this inequality as a property verified by non-negative harmonic functions in $\mathbb{R}^2$. This inequality was soon to be used to deduce several powerful consequences and deep results. Also, this inequality was found to be true for harmonic functions in any dimension and for more generalized class of functions thereafter.

A historical perspective of Haranck's inequality suggests that it is a natural property for elliptic operators. In 1961, Moser [47, 49] established Harnack's inequality for non-negative solutions to the divergence-form operator

$$\mathcal{L}u := \sum_{i,j=1}^{n} (a_{ij}(x)u_i)_j = \operatorname{div}(A(x)\nabla u) \text{ in } \Omega \subset \mathbb{R}^n.$$

The iteration techniques he used to accomplish that are one of the greatest contributions in the study of Harnack's inequality. In 1981, Krylov-Safonov [43, 44] proved Harnack's inequality for non-negative solutions to the non-divergence-form operator

$$Lu := \sum_{i,j=1}^{n} a_{ij}(x)u_{ij} = \operatorname{tr}(A(x)D^2u) \text{ in } \Omega \subset \mathbb{R}^n.$$

The authors had introduced completely new measure-theoretic tools in their approach to Harnack's inequality. Di Benedetto and Trudinger in 1984 proved Harnack's inequality in the context of divergence-form operators using the techniques of Krylov-Safonov. Both Moser's and Krylov-Safonov's techniques have been proven ground-breaking in the study of regularity problems of solutions to elliptic PDEs. We will illustrate these techniques and discuss them in greater detail in Chapter 3.1 and Chapter 3.2 respectively. Eventually, Harnack's inequality for the non-negative solutions to the adjoint-form operator

$$L^*u := \sum_{i,j=1}^{n} (a_{ij}(x)u)_{ij} \text{ in } \Omega \subset \mathbb{R}^n$$

was proved by Fabes and Stroock [24] in 1984. In 1989, Caffarelli proved Harnack's inequality in the context of fully non-linear elliptic operators of the form

$$F(D^2u) = 0 \text{ in } \Omega \subset \mathbb{R}^n.$$

The uniform ellipticity condition for F means that

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} \left(\frac{\partial F}{\partial x_{ij}}\right) \xi_i\xi_j \leq \Lambda|\xi|^2, \quad x \in \Omega,\, \xi \in \mathbb{R}^n.$$

Caffarelli [12] significantly simplified Krylov-Safonov's techniques in establishing Harnack's inequality for fully non-linear equations. Later in 1996 and 1997, Caffarelli and Gutiérrez [15, 16] employ these techniques to acquire Harnack's inequality for the solutions to the linearized Monge-Ampère operator. In fact, it is precisely the work of Caffarelli and Gutiérrez that inspired axiomatizations of Harnack's inequality in spaces of homogeneous type. Aimar, Forzani, and Toledano establish one axiomatization in 2001 and Di Fazio, Gutiérrez, and Lanconelli prove another in 2008. All these stories of historical development of Harnack's inequality have provided a direct motivation for the work in this dissertation whose main theme is also a novel axiomatization of Harnack's inequality.

The critical step toward Harnack in all approaches has been the following so-called local boundedness inequality:

$$\sup_{B_r(x_0)} u(x) \leq \frac{C}{|B_{2r}(x_0)|} \int_{B_{2r}(x_0)} u(x)\, dx. \tag{1.2.20}$$

Note that the ball on the right is two times dilation of the one on the left with respect to the center. A stronger version of this inequality has the same ball on both sides and this version is clearly implied by (1.2.20) as follows:

$$\begin{aligned}\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} u(x)\,dx &\le \frac{1}{|B_r(x_0)|}\int_{B_{2r}(x_0)} u(x)\,dx\\ &= \frac{2^n}{2^n|B_r(x_0)|}\int_{B_{2r}(x_0)} u(x)\,dx\\ &= \frac{2^n}{|B_{2r}(x_0)|}\int_{B_{2r}(x_0)} u(x)\,dx.\end{aligned}$$

Note that 2^n is a structural uniform constant independent of the ball or the function and keeps the essence of the inequality intact. Also, to be noted is that this has been possible only because the Lebesgue measure $|\cdot|$ is doubling. Hence, the same program can be pulled off as long as we have a measure with the doubling property (1.1.14).

In order to establish the Harnack result for solutions, most approaches first acquire (1.2.20) for supersolutions. For example, superharmonic functions, i.e., functions u verifying

$$\triangle u := \sum_{i=1}^{n} D_{ii}u \ge 0 \text{ in } \Omega \subset \mathbb{R}^n$$

possess the following property which also goes by the name of a mean value inequality:

$$u(x_0) \le \frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} u(x)\,dx. \tag{1.2.21}$$

Next, we claim that (1.2.21) implies (1.2.20) if $B_{2r}(x_0) \subset \Omega$. This requires that $\sup_{B_r(x_0)} u(x) = u(y)$ for some $y \in \Omega$, and $B_r(y) \subset \Omega$. Then, by the mean value inequality,

$$\begin{aligned}\sup_{B_r(x_0)} u(x) = u(y) &\le \frac{1}{|B_r(y)|}\int_{B_r(y)} u(x)\,dx\\ &= \frac{1}{|B_r(x_0)|}\int_{B_r(y)} u(x)\,dx\\ &\le \frac{1}{|B_r(x_0)|}\int_{B_{2r}(x_0)} u(x)\,dx\\ &= \frac{2^n}{|B_{2r}(x_0)|}\int_{B_{2r}(x_0)} u(x)\,dx,\end{aligned}$$

which is (1.2.20). Moser's idea of accomplishing the critical step of (1.2.20) is through his celebrated iterative scheme. Having established it for a supersolution u, he repeats the same procedure for a supersolution $\frac{1}{u}$ which paves the way to Harnack's inequality for a solution u. The inequality (1.2.20) is also the key step in Krylov-Safonov's approach to Harnack which replaces variational tools by probabilistic tools in their methodology. Their method identifies the so-called critical density property as the crucial property required to obtain (1.2.20). Krylov-Safonov's measure-theoretic approach to Harnack was greatly simplified and reinterpreted by Caffarelli. Our axiomatization to Harnack's inequality in spaces of homogeneous type is mainly based on Krylov-Safonov-Caffarelli's approach and also sheds more light onto the role of the critical density property in obtaining the crucial inequality (1.2.20).

1.3 Outline of dissertation

The main work in this dissertation, namely, a new axiomatic approach to Harnack's inequality in spaces of homogeneous type, is located in Chapter 4. The dissertation has 9 major chapters followed by a summary chapter. The dissertation from hereon is organized as follows:

Chapter 2 introduces reverse inequalities and reviews its theory. Harnack's inequality, as we have seen in previous sections, is the most extreme form of a reverse inequality as it reverses the two extreme values a function can attain over a ball, namely the infimum and the supremum. However, there are a whole range of other values that a function can attain in between and thereby a whole range of reverse inequalities is possible. The properties of various reverse inequalities, as it turns out, define many well-known classes in Analysis such as Muckenhoupt's A_p classes. This chapter also introduces a diagrammatic representation of these properties. We also reprove some classic results with the perspective of reverse inequalities. Also, this representation, in particular, has allowed us to illustrate various approaches to Haranck's inequalities in this dissertation.

Chapter 3 reviews two major approaches to Harnack's inequality, namely Moser's and Krylov-Safonov's. Section 3.1 is entirely devoted to a review of Moser's approach to Harnack's inequality. Moser's approach serves as a good motivation and provides a comparative study for other approaches to Harnack's inequality. Moser's iteration scheme is a very powerful tool and has been carried out in several contexts. This chapter will present Moser's iteration technique in the context of the divergence-form operator with no lower-order terms. Section 3.2 introduces the ground-breaking measure-theoretic approach to Harnack's inequality pioneered by Krylov and Safonov. Their probabilistic methods originally developed in the context of non-divergence-form elliptic operators have made a deep impact on the study of regularity properties of solutions to PDEs. As such, their techniques have been adapted to several more general types of PDEs, most notably to fully non-linear elliptic operators by Caffarelli. Caffarelli is credited with enriching and simplifying the Krylov-Safonov theory to a great extent. Krylov-Safonov and Caffarelli's work essentially paved the way for axiomatization of Harnack's inequality. This chapter briefly reviews some of the axiomatic approaches to Harnack's inequality in spaces of homogeneous type.

Chapter 4 is the main work of the dissertation. Section 4.1 introduces a property that is central to our novel approach and establishes some basic results relating various properties and Section 4.2 provides the statement and proof of the main theorem. The proof has been broken down into multiple steps which are arranged in several subsections leading to the final proof. The subsequent sections (Section 4.3 through Section 4.7) discuss the assumptions of the theorem or make comparison with other related results such as Bombieri's lemma. These sections contain remarks which provide more insight about our main result.

Chapter 5 further explores the role played by the critical density property. Originally invented by Krylov-Safonov, this property has been the central tool in all axiomatic approaches to Harnack's inequality. In this chapter, we give the definition of a new property we call the explosive critical density property and through it prove the so-called power-like decay property in metric spaces with a little bit of more structure.

Chapter 6 is an application of the theory in Chapter 4 to analysis on graphs. It provides an alternate proof of Harnack's inquality on graphs established originally by Delmotte

[21]. Having discussed the necessary structure for an analysis on graphs along with some examples, it gives the new proof of Harnack's inequality for harmonic functions on graphs.

Chapter 7 is the summary of the entire dissertation. The main purpose of this chapter is to provide a very brief overview of our main result in a self-contained way. Therefore, it provides relevant definitions and references, states the main theorem and lists a few pertinent remarks.

Chapter 2

Reverse inequalities

This chapter revisits some classic results in the theory of well-known classes of weights such as Muckenhoupt's A_p classes and reverse Hölder classes. It describes a new visually-flavored approach to represent these classes as special cases of a more general type of reverse classes. This idea of visual formalism was first introduced in our preprint [39] currently under development. As we shall see in this chapter, this new perspective gives us significantly simple proofs of many classic results in the theory of reverse inequalities. Moreover, its diagrammatic approach has potential applications in classroom pedagogy of this theory. In particular, it will come in useful in subsequent chapters of this dissertation, namely, in Chapter 3 to illustrate the two most important approaches to Harnack's inequality and in Chapter 4 to illustrate our new approach to Harnack's inequality.

The chapter begins with Section 2.1 stating some well-known definitions and basic results. Section 2.2 and Section 2.3 provide the definition of the reverse class introduced in this chapter and some of its properties respectively. Section 2.4 reproves self-improving properties of A_p classes recast in the theory of reverse classes propounded in this chapter. We record results related to weighted A_p and RH_s classes in Section 2.5. Section 2.6 establishes an interpolation result for reverse classes, which, in turn, gives the classic result $BMO = BLO - BLO$ as a corollary.

2.1 Some well-known classes

We begin by defining a weight and introducing some well-known classes of weights. For our purposes, we refer to as a weight any non-negative function $w \geq 0$ which is locally integrable to some power $p \in [-\infty, \infty]$. Note that this is at variance with the usual reference of a weight in the literature as a locally integrable non-negative function, i.e., $p = 1$ is held fixed. Also, throughout this chapter, (X, d, μ) is a space of homogeneous type, $\Omega \subset X$ is a domain and we only consider the balls in the collection

$$\mathcal{B} = \{B \subset \Omega \mid 2B \subset \Omega\}.$$

Definition 24 (p-mean)**.** Given a weight w and a Euclidean ball $B \subset \Omega$, we define the *p-mean of w over B* by

$$w(p, B) := \left(\frac{1}{\mu(B)} \int_B w^p \, d\mu \right)^{\frac{1}{p}}.$$

Some special cases of Definition 24 are noteworthy.

- The arithmetic mean, $p = 1$.
- The geometric mean, $p = 0$.

$$\lim_{p \to 0} \left(\frac{1}{\mu(B)} \int_B w^p \, d\mu \right)^{\frac{1}{p}} = \exp \left(\frac{1}{\mu(B)} \int_B \ln w \, d\mu \right).$$

Proof. This is proved easily by using the L'Hospital's rule as follows:

$$\begin{aligned}
\lim_{p \to 0} \frac{\ln \left(\frac{1}{\mu(B)} \int_B w^p \, d\mu \right)}{p} &= \lim_{p \to 0} \frac{1}{\left(\frac{1}{\mu(B)} \int_B w^p \, d\mu \right)} \left(\frac{1}{\mu(B)} \int_B w^p \ln w \, d\mu \right) \\
&= \frac{1}{\left(\frac{1}{\mu(B)} \int_B d\mu \right)} \left(\frac{1}{\mu(B)} \int_B \ln w \, d\mu \right) \\
&= \left(\frac{\mu(B)}{\mu(B)^2} \int_B \ln w \, d\mu \right).
\end{aligned}$$

□

- The harmonic mean, $p = -1$.

- Supremum, $p = \infty$.

$$\lim_{p\to\infty}\left(\frac{1}{\mu(B)}\int_B w^p\,d\mu\right)^{\frac{1}{p}} = \operatorname*{ess.sup}_B w.$$

Proof. To show this, first we prove the following claim:

Claim 25. *If either of the following two cases:*

(1) $p > 0$ *and* $w_0 := \operatorname*{ess.sup}_B w$

(2) $p < 0$ *and* $w_0 := \operatorname*{ess.inf}_B w$

holds, then we have

$$\lim_{p\to\pm\infty}\frac{\ln\left(\frac{1}{\mu(B)}\int_B\left(\frac{w}{w_0}\right)^p d\mu\right)}{p} = 0.$$

First, let us prove the claim. Note that in either case, we have $w_0^p \leq \int_B w^p\,d\mu$ and $w^p \leq w_0^p$. Hence, if $p > 0$ ($p < 0$), we have

$$\begin{aligned}
\frac{1}{p}\ln\left(\frac{1}{\mu(B)}\right) &= \frac{1}{p}\ln\left(\frac{1}{\mu(B)}\left(\frac{w_0}{w_0}\right)^p\right)\\
&\leq (\geq)\frac{1}{p}\ln\left(\frac{1}{\mu(B)}\int_B\left(\frac{w}{w_0}\right)^p d\mu\right)\\
&\leq (\geq)\frac{1}{p}\ln\left(\frac{1}{\mu(B)}\int_B\left(\frac{w_0}{w_0}\right)^p d\mu\right)\\
&= \frac{1}{p}\ln\left(\frac{\mu(B)}{\mu(B)}\right) = 0.
\end{aligned}$$

Hence, taking $p \to \pm\infty$, by the squeeze theorem, the claim is established. Now, using the first case of the claim,

$$\begin{aligned}
&\lim_{p\to\infty} \frac{\ln\left(\frac{1}{\mu(B)}\int_B w^p\,d\mu\right)}{p} \\
&= \lim_{p\to\infty} \frac{1}{p}\ln\left(\frac{1}{\mu(B)}\int_B \left(\frac{w}{w_0}\right)^p w_0^p\,d\mu\right) \\
&= \lim_{p\to\infty} \frac{1}{p}\ln w_0^p + \lim_{p\to\infty} \frac{1}{p}\ln\left(\frac{1}{\mu(B)}\int_B \left(\frac{w}{w_0}\right)^p w_0^p\,d\mu\right) \\
&= \ln w_0 = \ln\left(\operatorname*{ess.sup}_B w\right).
\end{aligned}$$

Now, taking exp on both sides, concludes the proof. □

- Infimum, $p = -\infty$.

$$\lim_{p\to-\infty}\left(\frac{1}{\mu(B)}\int_B w^p\,d\mu\right)^{\frac{1}{p}} = \operatorname*{ess.inf}_B w.$$

Either by Hölder's inequality:

$$\int_B uw\,d\mu \le \left(\int_B u^p\,d\mu\right)^{1/p}\left(\int_B w^{p'}\,d\mu\right)^{1/p'}, p>1,\ p' = \frac{1}{p-1}+1,$$

or, by Jensen's inequality:

$$\phi\left(\frac{1}{\mu(B)}\int_B w\,d\mu\right) \le \frac{1}{\mu(B)}\int_B \phi(w)\,d\mu, \quad \phi \text{ convex },$$

it follows that the p-mean is an increasing function of the exponent p. This means, for any $w > 0$ and a ball B, the inequality

$$\left(\frac{1}{\mu(B)}\int_B w^p\,d\mu\right)^{1/p} \le \left(\frac{1}{\mu(B)}\int_B w^q\,d\mu\right)^{1/q}, p<q,$$

holds naturally. Indeed, for the case $0 < p < q$, we have $q/p > 1$, or, $\phi(x) = x^{q/p}$ is convex, so either one of the above mentioned inequalities applies directly. For the case $p := -b < q := -a < 0$, we have $0 < a < b$, so that the previous case can be applied to w^{-1}. Finally, for the case $q < 0 < p$, we can take $q \nearrow 0$ and $p \searrow 0$ in the previous cases. Note that this works only because we have

$$\lim_{p\to 0}\left(\frac{1}{\mu(B)}\int_B w^p\,d\mu\right)^{\frac{1}{p}} = \exp\left(\frac{1}{\mu(B)}\int_B \ln w\,d\mu\right),$$

so that the two sided limits coincide.

The purpose of this chapter is to study the weights which satisfy the reverse inequality of (2.1) up to a constant that doesn't depend on the balls. The study of this kind has been pioneered by Muckenhoupt with the introduction of A_p weights in [50] in the context of weighted L^p-estimates for the Hardy maximal function. Since then, several authors (see, for example, [6, 19, 20, 28, 55]) have extensively studied weights verifying such reverse inequalities.

Next, we recall the definitions of some well-known spaces of non-negative functions, namely, Mckenhoupt's A_p weights reverse Hölder classes, BMO and BLO. In general, a weight is a locally integrable non-negative function, i.e., whose 1-means exist. However, in this chapter, the term weight is used rather loosely to refer to any non-negative function whose p-mean exists for some $p \in [-\infty, \infty]$.

Definition 26. Let $1 < p < \infty$. We write $w \in A_p$ and say that u is a Muckenhoupt A_p weight if the following inequality holds:

$$[w]_{A_p} := \sup_{B \in \mathcal{B}} \left(\frac{1}{\mu(B)} \int_B w \, d\mu \right) \left(\frac{1}{\mu(B)} \int_B w^{\frac{1}{1-p}} \, d\mu \right)^{p-1} < \infty. \tag{2.1.1}$$

Definition 27. $w \in A_1$ if and only if the following inequality holds:

$$[w]_{A_1} := \sup_{B \in \mathcal{B}} \left(\frac{1}{\mu(B)} \int_B w \, d\mu \right) \left(\operatorname*{ess.inf}_B w \right)^{-1} < \infty. \tag{2.1.2}$$

Definition 28. $w \in A_\infty$ if and only if the following inequality holds:

$$[w]_{A_\infty} := \sup_{B \in \mathcal{B}} \left(\frac{1}{\mu(B)} \int_B w \, d\mu \right) \exp \left(\frac{1}{\mu(B)} \int_B \ln w \, d\mu \right)^{-1} < \infty. \tag{2.1.3}$$

Definition 29. Let $1 < s \leq \infty$. We write $w \in RH_s$ and say that u is a reverse Hölder class of order s if the following inequality holds:

$$[w]_{RH_s} := \sup_{B \in \mathcal{B}} \left(\frac{1}{\mu(B)} \int_B w \, d\mu \right)^{-1} \left(\frac{1}{\mu(B)} \int_B w^s \, d\mu \right)^{\frac{1}{s}} < \infty. \tag{2.1.4}$$

Some well-known facts about A_p and reverse Hölder classes are listed in the following proposition. Their proofs can be found in many Fourier analysis books, for example, in [32].

Proposition 30. (1) *If* $1 < p_1 < p_2 \leq \infty$, $A_{p_1} \subset A_{p_2}$.

(2) $A_\infty = \bigcup_{p=1}^{\infty} A_p$.

(3) $w \in A_p$ *if and only if* $w = w_1 w_2^{1-p}$, *for some* $w_1, w_2 \in A_1$.

(4) *If* $w \in A_p$, $1 < p < \infty$, *then there exists* $\delta > 0$ *such that* $w \in RH_{1+\delta}$.

Definition 31. $w \in BMO$ if and only if the following inequality holds:

$$[w]_{BMO} := \sup_{B \in \mathcal{B}} \left(\frac{1}{\mu(B)} \int_B |w - w_B| \, d\mu \right) < \infty, \, w_B := \left(\frac{1}{\mu(B)} \int_B w \, d\mu \right). \tag{2.1.5}$$

Definition 32. $w \in BLO$ if and only if the following inequality holds:

$$[w]_{BLO} := \sup_{B \in \mathcal{B}} \left(\frac{1}{\mu(B)} \int_B \left| w - \operatorname*{ess.inf}_B w \right| \, d\mu \right) < \infty. \tag{2.1.6}$$

Proposition 33 below gives the characterizations of BMO and BLO in terms of A_p weights. The proof of (1) can be found in many Fourier analysis books, for example, in [32]. The proof of (2) can be found in [20].

Proposition 33. (1) $\log w \in BMO$ *if and only if* $w^\varepsilon \in A_p$, *for some* $p > 1$ *and* $\varepsilon > 0$.

(2) $\log w \in BLO$ *if and only if* $w^\varepsilon \in A_1$, *for some* $\varepsilon > 0$.

2.2 Reverse Classes (RC)

In this section, we give our definition of a more general type of a reverse class.

Definition 34. Let $-\infty \leq r < s \leq \infty$ and $C > 0$. We write $w \in RC(r, s, C)$, or simply, $w \in RC(r, s)$, and say that w is in the reverse class with exponents r and s if, for every ball $B \in \mathcal{B}$, the following inequality holds:

$$\left(\frac{1}{\mu(B)} \int_B w^s \, d\mu \right)^{1/s} \leq C \left(\frac{1}{\mu(B)} \int_B w^r \, d\mu \right)^{1/r}.$$

Figure 2.1 is a diagram to illustrate Definition 34 where points on the extended real line represent the possible values for the exponent r of the r-mean. The reverse class with exponents r and s, by definition and as specified in (34), requires the s-mean to be uniformly bounded by the r-mean. On the other hand, the r-mean is an increasing function of r and the reverse inequality of (34) always holds with $C = 1$. Hence, this implies that the reverse class with exponents r and s, in fact, is equivalent to the uniform comparability of the r-mean and the s-mean. Figure 2.1 shows a visual representation of a weight in the reverse class with exponents r and s where the two exponents are joined by a solid straight line with arrowheads pointing at them.

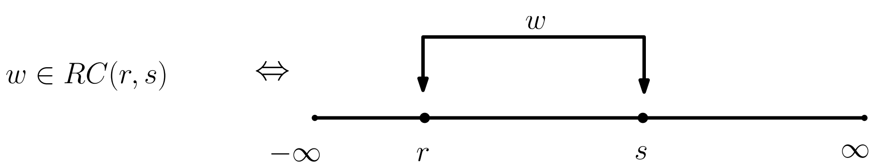


Figure 2.1: *The reverse class with exponents r and s.*

Definition 35. Let $-\infty \le r < s \le \infty$ and $C > 0$. We write $w \in RC^w(r, s, C)$, or simply, $w \in RC^w(r, s)$, and say that w is in the weak reverse class with exponents r and s if, for every ball $B \in \mathcal{B}$, the following inequality holds:

$$\left(\frac{1}{\mu(B)} \int_B w^s \, d\mu \right)^{1/s} \le C \left(\frac{1}{\mu(2B)} \int_{2B} w^r \, d\mu \right)^{1/r}.$$

Definition 35 is visually represented by joining the two exponents r and s by a dashed straight line with arrowheads pointing at them as illustrated in Figure 2.2.

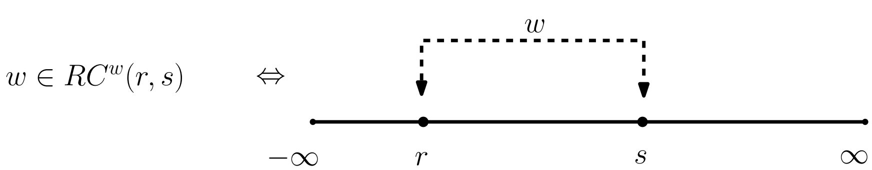


Figure 2.2: *The weak reverse class with exponents r and s.*

Theorem 36 below gives the characterizations for two well-known classes of weights, namely, Muckenhoupt's A_p weights and reverse Hölder classes in terms of reverse classes. Figure 2.3 and Figure 2.4 illustrate these characterizations.

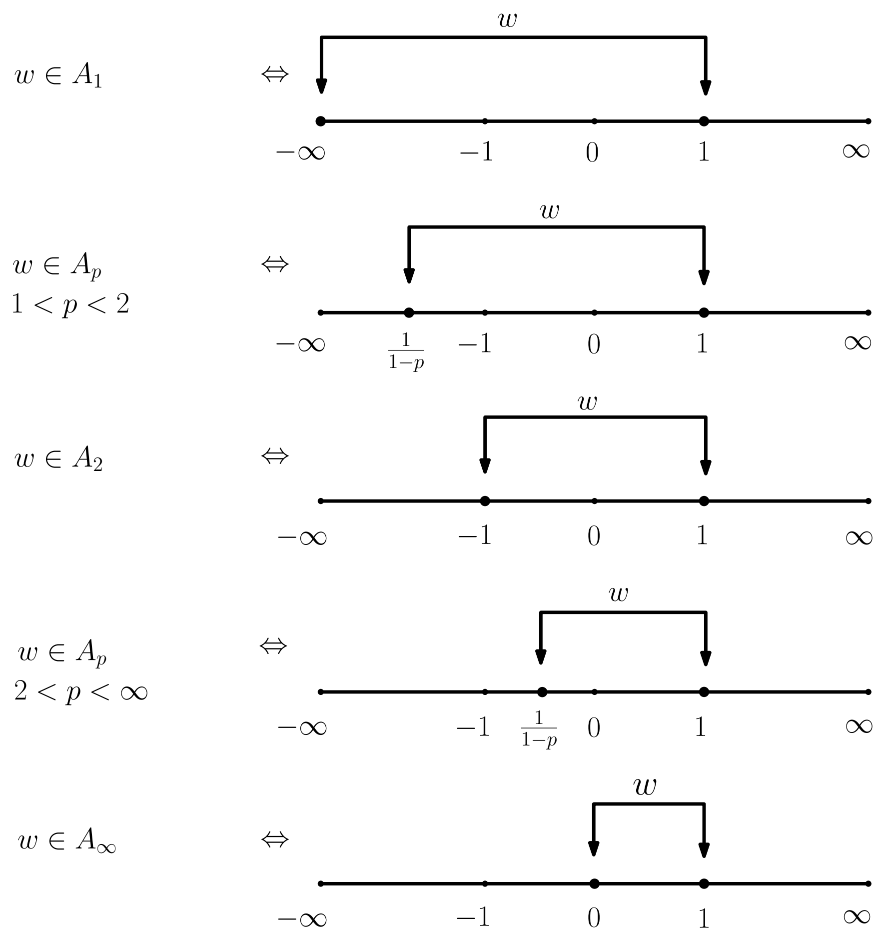


Figure 2.3: *The A_p classes.*

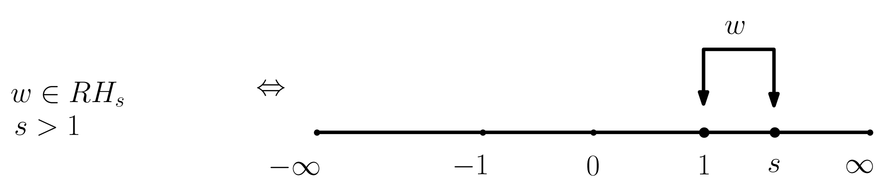


Figure 2.4: *The RH_s classes.*

Theorem 36. *Let $RC(a,b,C)$ be the reverse class defined in Definition 34. Then*

(1) $w \in A_1$ *if and only if* $w \in RC(-\infty, 1, [w]_{A_1})$.

(2) $w \in A_p$, $1 < p < \infty$, *if and only if* $w \in RC(\frac{1}{1-p}, 1, [w]_{A_p})$.

(3) $w \in A_\infty$ *if and only if* $w \in RC(0, 1, [w]_{A_\infty})$.

(4) $w \in RH_s$, $s > 1$, *if and only if* $w \in RC(1, s, [w]_{RH_s})$, $s > 1$.

2.3 Some properties of reverse classes

We begin this section by making two elementary observations verified by reverse classes stated in Lemma 37.

Lemma 37. *(a) If* $w \in RC(r, \tilde{r}, C_1) \cap RC(\tilde{r}, s, C_2)$ *then* $w \in RC(r, s, C_1C_2)$.

(b) If $w \in RC(r, s, C_3)$ *where* $r \le \tilde{r} < \tilde{s} \le s$ *then* $w \in RC(\tilde{r}, \tilde{s}, C_1)$.

Proof. In order to prove (a), first assume that $w \in RC(r, \tilde{r}, C_1) \cap RC(\tilde{r}, s, C_2)$. Then,

$$\left(\frac{1}{\mu(B)}\int_B w^s\,d\mu\right)^{1/s} \le C_2\left(\frac{1}{\mu(B)}\int_B w^{\tilde{r}}\,d\mu\right)^{1/\tilde{r}} \le C_1C_2\left(\frac{1}{\mu(B)}\int_B w^r\,d\mu\right)^{1/r}. \tag{2.3.7}$$

Next, let us prove (b). Since $r \le \tilde{r}$ and $\tilde{s} \le s$, we have natural inequalities

$$\left(\frac{1}{\mu(B)}\int_B w^r\,d\mu\right)^{1/r} \le \left(\frac{1}{\mu(B)}\int_B w^{\tilde{r}}\,d\mu\right)^{1/\tilde{r}}$$

and

$$\left(\frac{1}{\mu(B)}\int_B w^{\tilde{s}}\,d\mu\right)^{1/\tilde{s}} \le \left(\frac{1}{\mu(B)}\int_B w^s\,d\mu\right)^{1/s}.$$

Now, combining (2.3) and (2.3) with the defining inequality of $w \in RC(r, s, C_3)$, (b) follows at once. ∎

The visualization of Lemma 37 is given in Figure 2.5 and Figure 2.6.

Theorem 38 establishes a key fact that will be used frequently throughout this paper to obtain some new as well as well-known results.

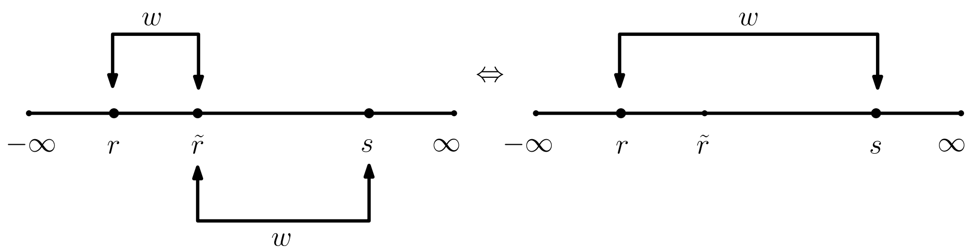


Figure 2.5: *Extension of reverse classes connect naturally.*

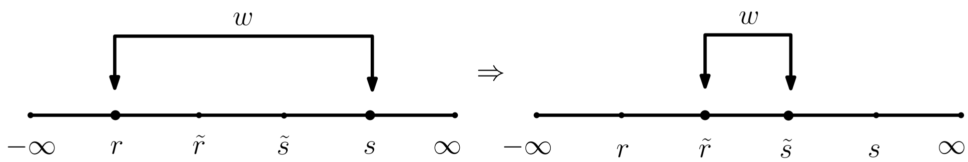


Figure 2.6: *Reverse classes shrink from either side naturally.*

Theorem 38. *If $w \in RC(r, s, C)$ then*

(1) $w^\theta \in RC(\frac{r}{\theta}, \frac{s}{\theta}, C^\theta)$ *for all* $\theta > 0$

(2) $w^\theta \in RC(\frac{s}{\theta}, \frac{r}{\theta}, C^{|\theta|})$ *for all* $\theta < 0$.

Proof. Let us prove (38). Since $w \in RC(r, s, C)$, it follows that

$$\left(\frac{1}{\mu(B)}\int_B w^s \, d\mu\right)^{1/s} = \left(\frac{1}{\mu(B)}\int_B w^{\theta\frac{s}{\theta}} \, d\mu\right)^{1/s} \le C\left(\frac{1}{\mu(B)}\int_B w^{\theta\frac{r}{\theta}} \, d\mu\right)^{1/r}.$$

For $\theta > 0$ we get

$$\left(\frac{1}{\mu(B)}\int_B w^{\theta\frac{s}{\theta}} \, d\mu\right)^{\theta/s} \le C^\theta\left(\frac{1}{\mu(B)}\int_B w^{\theta\frac{r}{\theta}} \, d\mu\right)^{\theta/r},$$

that is,$w^\theta \in RC(\frac{r}{\theta}, \frac{s}{\theta}, C^\theta)$. This proves (1). To prove (38), we use the same argument as above. From (2.3) we obtain

$$\left(\frac{1}{\mu(B)}\int_B w^{\theta\frac{s}{\theta}} \, d\mu\right)^{\theta/s} \ge C^\theta\left(\frac{1}{\mu(B)}\int_B w^{\theta\frac{r}{\theta}} \, d\mu\right)^{\theta/r}, \quad \forall\theta < 0.$$

Thus, $w^\theta \in RC(\frac{s}{\theta}, \frac{r}{\theta}, C^{|\theta|})$ $\forall\theta < 0$.

This proves the theorem. □

Corollary 39. *Let $RC(a, b, C)$ be the reverse class defined in Definition 34. Then*

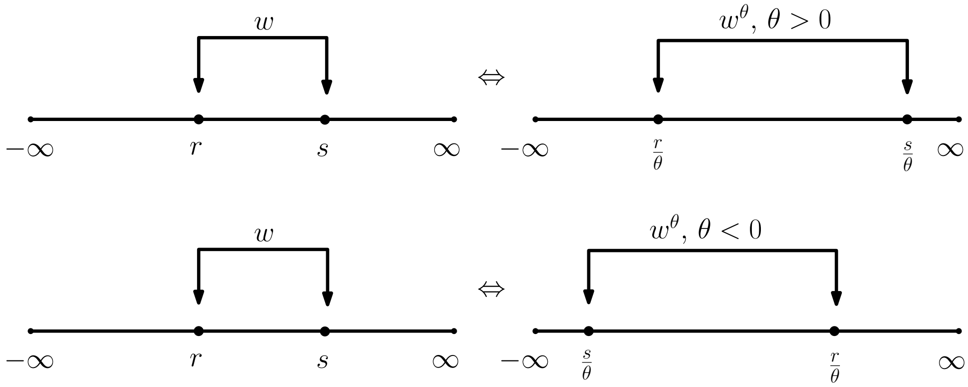


Figure 2.7: *The figure for the reverse class raised to some power.*

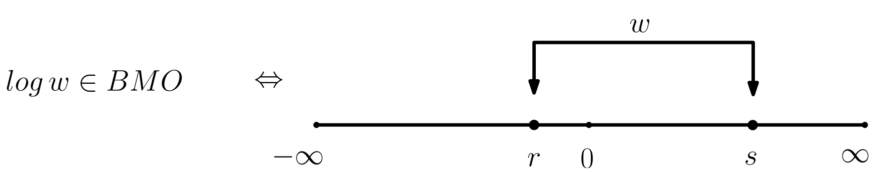


Figure 2.8: *The characterization of BMO.*

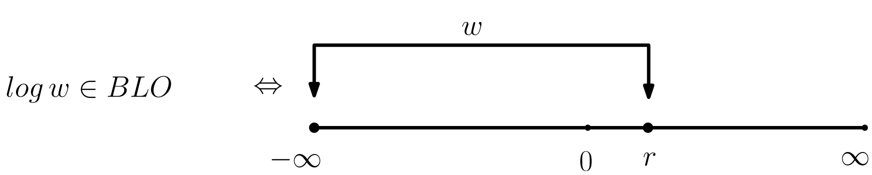


Figure 2.9: *The characterization of BLO.*

(1) $\log w \in BMO$ *if and only if* $w \in RC(r,s)$ *for some* $r < 0 < s$.

(2) $\log w \in BLO$ *if and only if* $w \in RC(-\infty, r)$ *for some* $r > 0$.

Proof. The proof is immediate by applying Theorem 38 together with Theorem 36 and Proposition 30. □

Corollary 39 gives the characterizations of BMO and BLO in terms of reverse classes, which are visualized by Figure 2.8 and Figure 2.9 respectively.

Corollary 40. $RH_\infty = \cap_{s>1} RH_s$.

Proof. We just need to prove that if $w \in RC(1,\infty,C_1)$, then $w \in \cap_{s>1} RC(1,s,C_s)$. Since $w \in RC(1,\infty,C_1)$, it follows that

$$w(x) \le \sup_B w \le C_1 \frac{1}{\mu(B)} \int_B w\, d\mu, \tag{2.3.8}$$

which implies

$$\left(\frac{1}{\mu(B)} \int_B w^s\, d\mu\right)^{1/s} \le C_1 \frac{1}{\mu(B)} \int_B w\, d\mu, \quad \forall s > 1, \tag{2.3.9}$$

that is, $w \in \cap_{s>1} RC(1,s,C_s)$. □

Corollary 41. *If* $w \in A_p$, $1 < p < \infty$ *then* $w^{1-q} \in A_q$, $\frac{1}{p} + \frac{1}{q} = 1$ *and* $[w^{1-q}]_{A_q} = [w]_{A_p}^{q-1}$.

Proof. By Theorem 36, we need to show that if $w \in RC(\frac{1}{1-p}, 1, C_p)$, $1 < p < \infty$ then $w^{1-q} \in RC\left(\frac{1}{1-q}, 1, C_p^{|1-q|}\right)$, $\frac{1}{p} + \frac{1}{q} = 1$. Then, the proof follows from the relation $(1-p)(1-q) = 1$ and Theorem 38 with $\theta = 1 - q < 0$. □

Corollary 42. *If* $w \in A_1$ *then, for every* $p > 1$, $w^{1-p} \in A_p$.

Proof. By Theorem 36, we need to show that if $w \in RC(-\infty, 1, C_1)$ then $w^{1-p} \in RC(\frac{1}{1-p}, \infty, C)$. Then, the proof follows from Theorem 38 with $\theta = 1 - p < 0$. □

Corollary 43. *If* $w \in A_1 \cap RH_s$, *then for every* $p > 1$, $w^{1-p} \in RC(\frac{1}{1-q}, \infty, C)$ *for every* $q > (p-1)/s + 1$.

Proof. Let $s, p > 1$ and $q > (p-1)/s + 1$. By Theorem 36, $w \in A_1 \cap RH_s$ is equivalent to $w \in RC(-\infty, 1, C_1) \cap RC(1, s, C_2)$. This, by Lemma 37, yields $w \in RC(-\infty, s, C_3)$. Then

$$w^{1-p} \in RC\left(\frac{s}{1-p}, \infty, C\right) = RC\left(\frac{1}{1-q}, \infty, C\right), \quad q > (p-1)/s + 1.$$

□

Corollary 44. *If $w \in RH_\infty \cap A_p$, then $w^{1-q} \in RC(-\infty, 1, C_3)$ where $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof. By Theorem 36, we have $w \in RC(1, \infty, C_1) \cap RC(\frac{1}{1-p}, 1, C_2)$, which, by Lemma 37, yields $w \in RC(\frac{1}{1-p}, \infty, C)$. Applying Theorem 38 with $\theta = (1-q) < 0$, we get $w^{1-q} \in RC(-\infty, 1, C_3)$, and Corollary 44 is proved. □

Corollary 45. *$w \in A_p \cap RH_s$ if and only if $w^s \in A_q$, where $q = s(p-1) + 1$.*

Proof. By Theorem 36, we need to show $w \in RC(\frac{1}{1-p}, 1, C_1) \cap RC(1, s, C_2)$ if and only if $w^s \in RC(\frac{1}{1-q}, 1, C)$. By Lemma 37, $w \in RC(\frac{1}{1-p}, 1, C_1) \cap RC(1, s, C_2)$ is equivalent to $w \in RC(\frac{1}{1-p}, s, C_3)$. This, by Theorem 38 with $\theta = s > 0$, is equivalent to

$$w^s \in RC\left(\frac{1}{(1-p)s}, 1, C\right) = RC\left(\frac{1}{1-q}, 1, C\right), \quad q = s(p-1) + 1,$$

and Corollary 45 is proved. □

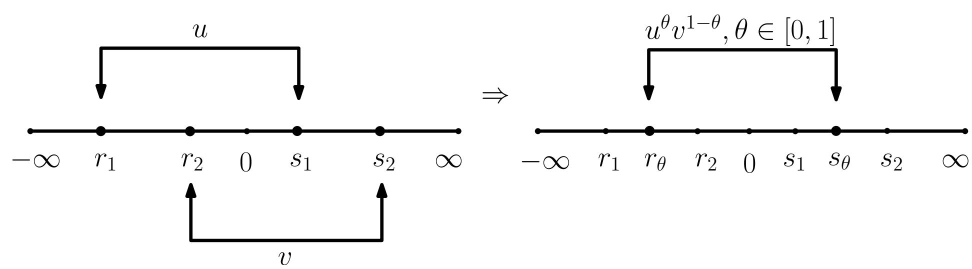


Figure 2.10: *Interpolation between two reverse classes.*

Corollary 46. *If $w \in A_p$ then $w^{1+\delta} \in A_q$ for some $\delta > 0$, where $q = (1+\delta)(p-1) + 1$.*

Proof. If $w \in A_p$, by (4) of Proposition 30, $w \in RC(1, 1+\delta, C)$ for some $\delta > 0$. On the other hand, by Theorem 36, $w \in A_p$ is equivalent to $w \in RC(\frac{1}{1-p}, 1, C_1)$. Hence $w \in RC(\frac{1}{1-p}, 1, C_1) \cap RC(1, 1+\delta, C)$ for some $\delta > 0$. This, together with Corollary 45, imply $w^{1+\delta} \in RC(\frac{1}{1-q}, 1, C_2)$, $q = (1+\delta)(p-1) + 1$, and Corollary 46 is proved. □

2.4 Self-improving properties

In this section we define two self-improving properties and show that $A_p, 1 < p < \infty$ has these properties.

Definition 47. The reverse class $RC(r, s, C_1)$ has the right self-improving property if for each $w \in RC(r, s, C_1)$, there exists $\varepsilon, C_2 > 0$ such that $w \in RC(r, s + \varepsilon, C_2)$.

Definition 48. The reverse class $RC(r, s, C_1)$ has the left self-improving property if for each $w \in RC(r, s, C_1)$, there exists $\varepsilon, C_2 > 0$ such that $w \in RC(r - \varepsilon, s, C_2)$.

The diagrams to illustrate the self-improving weights are provided in Figure 2.11. We provide a proof of the well-known fact that A_p classes possess self-improving properties in both directions in Theorem 49 and visualize it in Figure 2.12.

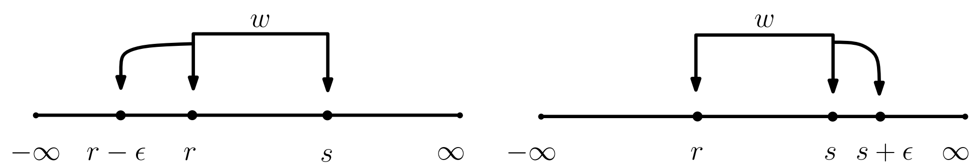


Figure 2.11: *Self-improving reverse classes are the ones that can be extended.*

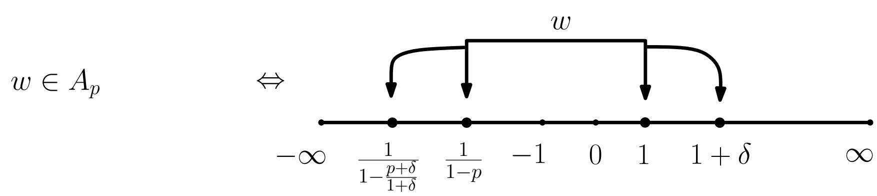


Figure 2.12: *The self-improving properties of the A_p classes.*

Theorem 49. *$A_p, 1 < p < \infty$, has both the right and left self-improving properties.*

Proof. Let $w \in A_p, 1 < p < \infty$. By Proposition 30, there exists $\varepsilon > 0$ such that $w \in RH_{1+\varepsilon}$. This, by virtue of Theorem 36, means $w \in RC(\frac{1}{1-p}, 1, C_1)$ implies $w \in RC(1, 1+\varepsilon)$. Hence, A_p has the right self-improving property. For the other direction, we intend to show that there exists $s \in (1,p)$ such that $w \in RC(\frac{1}{1-s}, 1, C_1)$, which would establish the left self-improving property since $1 < s < p$ implies $\frac{1}{1-s} < \frac{1}{1-p} < 0$. Let $\frac{1}{p} + \frac{1}{q} = 1$. Then, by Corollary 41, $w^{1-q} \in RC\left(\frac{1}{1-q}, 1, C_1^{|1-q|}\right)$. This together with (4) of Proposition 30 imply $w^{1-q} \in RC(1, 1+\delta, C_2)$ for some $\delta > 0$. Using Theorem 38 with $\theta = \frac{1}{1-q} < 0$, we obtain

$$w \in RC\left((1-q)(1+\delta), 1-q, C_2^{|\frac{1}{1-q}|}\right) = RC\left(\frac{1+\delta}{1-p}, \frac{1}{1-p}, C_2^{|\frac{1}{1-q}|}\right).$$

Since $w \in RC(\frac{1}{1-p}, 1, C_1) \cap RC\left(\frac{1+\delta}{1-p}, \frac{1}{1-p}, C_2^{|\frac{1}{1-q}|}\right)$, it follows from Lemma 37 that $w \in RC(\frac{1+\delta}{1-p}, 1, C_3)$, where $C_3 = C_1 C_2^{|\frac{1}{1-q}|}$. Indeed, for $s := \frac{p+\delta}{1-p}$, we have $s \in (1,p)$ and $\frac{1+\delta}{1-p} = \frac{1}{1-s}$, and Theorem 49 is proved. $\square$

2.5 Weighted A_p and RH_s classes

Thus far, we have only presented reverse classes with respect to the underlying measure μ. The reverse classes, A_p classes and reverse Hölder classes with respect to μ, are denoted $RC(r,s,C)$, A_p and RH_s respectively. In this section, we introduce a weighted p-mean and record some results about w-weighted reverse classes, A_p classes and reverse Hölder classes which are denoted by $RC(r,s,C; w\,d\mu)$, $A_p(w\,d\mu)$ and $RH_s(w\,d\mu)$ respectively.

Definition 50. Given a function $u > 0$, a weight $w > 0$, a ball $B \in \mathcal{B}$ and $p \in [-\infty, \infty]$, we define the expression

$$u(p, B; w\,d\mu) := \left(\frac{1}{w(B)} \int_B u^p w\,d\mu\right)^{1/p}, w(B) := \int_B w\,d\mu,$$

whenever it exists, to be the *w-weighted p-mean* of u over B.

Definition 50 easily yields the respective definitions of $RC(r,s; w\,d\mu)$, $A_p(w\,d\mu)$ and $RH_s(w\,d\mu)$. Since these are all straightforward, we avoid reiterating them explicitly. The main result in this section encoded in Theorem 51 states the property of reflection about the exponent 1 between a reverse class of w and the w-weighted reverse class of w^{-1}. It will deduce an important property relating weighted A_p and reverse Hölder classes in the form of Corollary 52 below.

Theorem 51. *The following hold true:*

(i) $w \in RC(1, q, C)$ *if and only if* $w^{-1} \in RC\left(1-q, 1, C^{\frac{q}{q-1}}; w\,d\mu\right)$.

(ii) $w \in RC(q, 1, C)$ *if and only if* $w^{-1} \in RC\left(1, 1-q, C^{\frac{q}{q-1}}; w\,d\mu\right)$.

Proof. Since the proofs of (i) and (ii) are similar, we only give the proof of (i). We do this explicitly by establishing a series of equivalent inequalities using only the definitions of these reverse classes as follows :

$$
\begin{aligned}
&w \in RC(1, q, C)\\
\frac{1}{\mu(B)}\int_B w\,d\mu &\le C\left(\frac{1}{\mu(B)}\int_B w^q\,d\mu\right)^{\frac{1}{q}}\\
\frac{w(B)}{\mu(B)} &\le C\left(\frac{1}{w(B)}\int_B w^q\,d\mu\right)^{\frac{1}{q}}\left(\frac{w(B)}{\mu(B)}\right)^{\frac{1}{q}}\\
\left(\frac{1}{\mu(B)}\int_B w^q\,d\mu\right)^{-\frac{1}{q}} &\le C\left(\frac{\mu(B)}{w(B)}\right)^{\frac{q-1}{q}} = C\left(\frac{1}{w(B)}\int_B d\mu\right)^{\frac{q-1}{q}}\\
\left[\left(\frac{1}{\mu(B)}\int_B (w^{-1})^{1-q} w\,d\mu\right)^{\frac{1}{1-q}}\right]^{\frac{q-1}{q}} &\le C\left(\frac{1}{w(B)}\int_B w^{-1}w\,d\mu\right)^{\frac{q-1}{q}}\\
\left(\frac{1}{\mu(B)}\int_B (w^{-1})^{1-q} w\,d\mu\right)^{\frac{1}{1-q}} &\le C^{\frac{q}{q-1}}\left(\frac{1}{w(B)}\int_B w^{-1}w\,d\mu\right)\\
&w^{-1} \in RC\left(1-q, 1, C^{\frac{q}{q-1}}; w\,d\mu\right)
\end{aligned}
$$

□

Corollary 52. *Let* $p, p' > 1$ *be Hölder conjugates, i.e.,* $\frac{1}{p} + \frac{1}{p'} = 1$. *The following hold true:*

(a) $w \in RH_p$ *if and only if* $w^{-1} \in A_{p'}(w\,d\mu)$.

(b) $w \in A_p$ *if and only if* $w^{-1} \in RH_{p'}(w\,d\mu)$.

Proof. (a) follows by taking $q = p$ in (i) and using $RC(1, p) = RH_p$ and $RC(1-p, 1; w\,d\mu) = A_{p'}(w\,d\mu)$. Similarly, (b) follows by taking $q = 1 - p'$ in (ii) and using $RC(1-p', 1) = A_p$ and $RC(1, p'; w\,d\mu) = RH_{p'}(w\,d\mu)$. □

2.6 Interpolation and Factorization

In this section, we give an interpolation result between two reverse classes encoded in Theorem 53 and illustrated in Figure 2.13.

Theorem 53. *Let* $r_1 \le r_2 < 0 < s_1 \le s_2$, *and* $u \in RC(r_1, s_1, C_1)$ *and* $v \in RC(r_2, s_2, C_2)$. *Then* $u^\theta v^{1-\theta} \in RC(r_\theta, s_\theta, C_3)$ *where* $\theta \in [0,1]$, $r_\theta = \frac{r_1 r_2}{\theta r_2 + (1-\theta) r_1} < 0$, $s_\theta = \frac{s_1 s_2}{\theta s_2 + (1-\theta) s_1}$.

Proof. It is trivial for $\theta = 0$ and $\theta = 1$. So, assume $\theta \in (0,1)$. Let us prove that

$$\left(\frac{1}{\mu(B)} \int_B u^{s_\theta \theta} v^{s_\theta (1-\theta)}\, d\mu\right)^{1/s_\theta} \left(\frac{1}{\mu(B)} \int_B u^{r_\theta \theta} v^{(1-\theta) r_\theta}\, d\mu\right)^{-1/r_\theta} \le C,$$

where C is a constant. By Hölder's inequality with the exponents $q = \frac{r_1}{\theta r_\theta}$ and $q' = \frac{r_2}{(1-\theta) r_\theta}$, one gets

$$\begin{aligned}\left(\frac{1}{\mu(B)} \int_B u^{r_\theta \theta} v^{(1-\theta) r_\theta}\, d\mu\right)^{-1/r_\theta} &\le \left(\frac{1}{\mu(B)} \int_B u^{r_1}\, d\mu\right)^{-\frac{\theta}{r_1}} \left(\frac{1}{\mu(B)} \int_B v^{r_2}\, d\mu\right)^{-(1-\theta)/r_2} \\ &=: m^{-\theta} n^{-(1-\theta)}.\end{aligned}$$

Again, applying Hölder's inequality with the exponents $r = \frac{s_1}{\theta s_\theta}$ and $r' = \frac{s_2}{(1-\theta) s_\theta}$, and using the assumptions $u \in RC(r_1, s_1, C_1)$ and $v \in RC(r_2, s_2, C_2)$,

$$\begin{aligned}\left(\frac{1}{\mu(B)} \int_B u^{s_\theta \theta} v^{s_\theta (1-\theta)}\, d\mu\right)^{1/s_\theta} &\le \left(\frac{1}{\mu(B)} \int_B u^{s_1}\, d\mu\right)^{\frac{\theta}{s_1}} \left(\frac{1}{\mu(B)} \int_B v^{s_2}\, d\mu\right)^{\frac{1-\theta}{s_2}} \\ &\le C_1 C_2 m^\theta n^{1-\theta}.\end{aligned}$$

Therefore, (2.6) follows from (2.6) and (2.6). $\square$

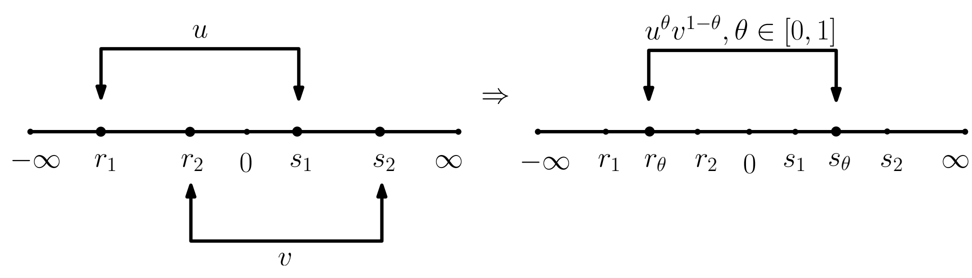


Figure 2.13: *Interpolation between two reverse classes.*

Theorem 55 is a factorization result for reverse classes and is illustrated in Figure 2.14. The 'if' part of Theorem 55 lays out a condition for two intersecting diagrams of reverse classes to produce a new diagram of a reverse class. Note the requirement that the intersecting diagrams need to cross the zero and touch either ∞ or $-\infty$. Conversely, the 'only if' part of Theorem 55 says that a weight in a reverse class crossing the zero can be factorized as a product of two weights such that one extends the reverse class to the extreme right and the other to the extreme left.

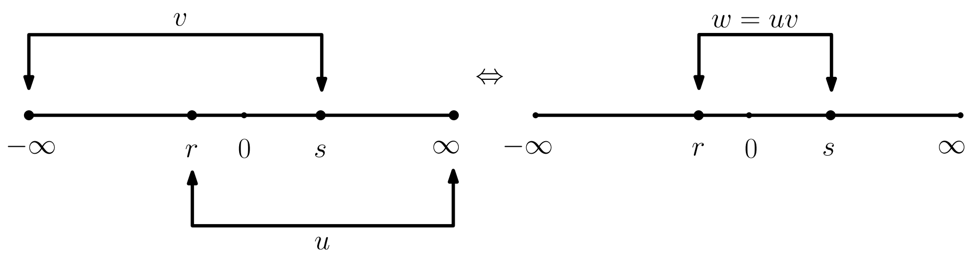


Figure 2.14: *Factorization of a weight in a reverse class crossing the zero*

Corollary 54. *Let $w_j \in A_{p_j}$, $j = 1, 2$, where $1 \le p_1 < p_2 < \infty$, then $w_1^\theta w_2^{1-\theta} \in A_{p_3}$ where $p_3 = \theta p_1 + (1-\theta)p_2$ and $\theta \in (0, 1)$.*

Proof. The proof follows immediately from Theorem 53 after the statement is recast in terms of reverse classes as follows: if $w_j \in RC(\frac{1}{1-p_j}, 1, C_j)$, where $j = 1, 2$, and $1 \le p_1 < p_2 < \infty$, then $w_1^\theta w_2^{1-\theta} \in RC(\frac{1}{1-[\theta p_1+(1-\theta)p_2]}, 1, C_3)$, $\theta \in (0, 1)$. $\square$

Theorem 55. *Let $r < 0 < s$. Then $w \in RC(r, s, C_3)$ if and only if $w = uv$ for some $u \in RC(r, \infty, C_2)$ and $v \in RC(-\infty, s, C_1)$.*

Proof. For the 'if' part, we have $u \in RC(r, \infty, C_2)$ and $v \in RC(-\infty, s, C_1)$ and we need to prove $uv \in RC(r, s, C_3)$, which amounts to proving

$$\left(\frac{1}{\mu(B)} \int_B (uv)^s \, d\mu\right)^{1/s} \left(\frac{1}{\mu(B)} \int_B (uv)^r \, d\mu\right)^{-1/r} \le C_3.$$

Using the assumptions $u \in RC(r, \infty, C_2)$ and $v \in RC(-\infty, s, C_1)$, we have

$$\begin{aligned}\left(\frac{1}{\mu(B)}\int_B (uv)^s \, d\mu\right)^{1/s} &\leq \sup_B u \left(\frac{1}{\mu(B)}\int_B v^s \, d\mu\right)^{1/s} \\ &\leq C_2 \left(\frac{1}{\mu(B)}\int_B u^r \, d\mu\right)^{1/r} \left(\frac{1}{\mu(B)}\int_B v^s \, d\mu\right)^{1/s} \\ &=: C_2 m n\end{aligned}$$

and

$$\begin{aligned}\left(\frac{1}{\mu(B)}\int_B (uv)^r \, d\mu\right)^{-1/r} &= \left(\frac{1}{\mu(B)}\int_B \frac{u^r}{v^{-r}} \, d\mu\right)^{-1/r} \\ &\leq (\inf_B v)^{-1} \left(\frac{1}{\mu(B)}\int_B u^r \, d\mu\right)^{-1/r} \\ &\leq C_1 \left(\frac{1}{\mu(B)}\int_B v^s \, d\mu\right)^{-1/s} \left(\frac{1}{\mu(B)}\int_B u^r \, d\mu\right)^{-1/r} \\ &= C_1 n^{-1} m^{-1}.\end{aligned}$$

Hence, the estimate (2.6) follows from (2.6) and (2.6). For the 'only if' part, we have $w \in RC(r, s, C_3)$ and we need to show that there exist $u \in RC(r, \infty, C_2)$ and $v \in RC(-\infty, s, C_1)$ such that $w = uv$. By Theorem 38 and Theorem 36, $w^s \in A_p, p = (1 - \frac{s}{r})$. Then, by (3) of Proposition 30, there exist $\tilde{u}, \tilde{v} \in A_1$ such that $w^s \in \tilde{u}\tilde{v}^{1-p}$. Defining $u := \tilde{u}^{\frac{1}{s}}$ and $v := \tilde{v}^{\frac{1-p}{s}}$, we get $w^s = uv$. Now, since $\tilde{u}, \tilde{v} \in A_1$, by Theorem 36, $\tilde{u}, \tilde{v} \in RC(-\infty, 1)$. Hence, by Theorem 38, since $s > 0$, $u := \tilde{u}^{\frac{1}{s}} \in RC(-\infty, s)$ and since $\frac{1-p}{s} < 0$, $v := \tilde{v}^{\frac{1-p}{s}} \in RC(\frac{s}{1-p}, \infty) = RC(r, \infty)$, where $r := \frac{s}{1-p} < 0$. □

Corollary 56. *$BMO = BLO - BLO$.*

Proof. In light of Corollary 39, we only need to show that $w \in RC(r, s), r < 0 < s$ if and only if $w = w_1 w_2^{-1}$ for some $w_1 \in (-\infty, r_1), r_1 > 0$ and $w_2 \in (-\infty, r_2), r_2 > 0$. Let $r < 0 < s$ and $w \in RC(r, s)$. Then, taking $w_1 \in (-\infty, s)$ and $w_2 \in (-\infty, -r)$, by Theorem 55, we get $w = w_1 w_2^{-1}$. Conversely, let $w_1 \in (-\infty, r_1), r_1 > 0$ and $w_2 \in (-\infty, r_2), r_2 > 0$. Then, by Theorem 55, $w = w_1 w_2^{-1} \in RC(-r_2, r_1)$. Since $-r_2 < 0 < r_1$, this proves Corollary 56. □

Remark 57. Theorem 55 is about the reverse classes of two factors which intersect into a reverse class crossing the zero. It specifies the criterion so that the product of the two factors lies in the intersecting reverse class. For this to hold, it is absolutely essential for both factors to touch opposite infinities. This fact has been substantiated by Examples 58 and 59 where Theorem 55 fails. These examples are constructed with the help of the following recipe (see [32], p. 286): $|x|^a \in A_p(\mathbb{R}^n)$ if and only if $-n < a < n(p-1)$.

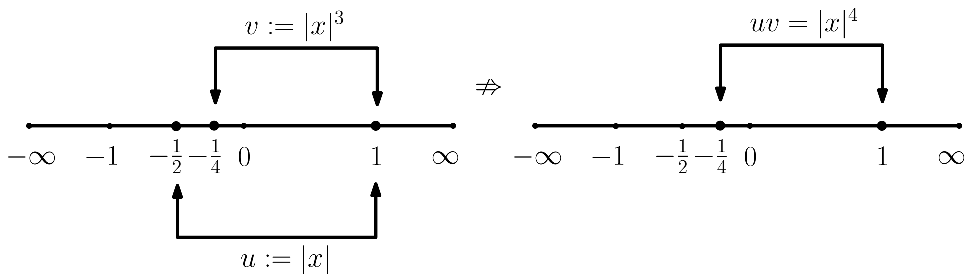


Figure 2.15: *The product will not be in the intersecting reverse class if the reverse classes of both factors do not touch an infinity.*

Example 58. In this example, the reverse classes of both the factors do not touch an infinity. Consequently, by the failure of Theorem 55, the product do not lie in the intersecting reverse class. Take $u := |x| \in A_3(\mathbb{R}) = RC(-\frac{1}{2}, 1)$ and $v := |x|^3 \in A_5(\mathbb{R}) = RC(-\frac{1}{4}, 1)$. Clearly, we have

$$RC\left(-\frac{1}{2}, 1\right) \cap RC\left(-\frac{1}{4}, 1\right) = RC\left(-\frac{1}{4}, 1\right).$$

But, $uv = |x|^4 \notin A_5(\mathbb{R})$. This example is illustrated in Figure 2.15. $\square$

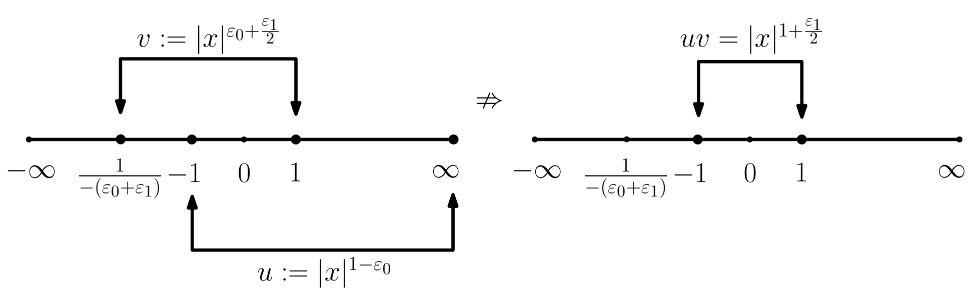


Figure 2.16: *The product will not be in the intersecting reverse class if the reverse class of one of the factors fail to touch an infinity.*

Example 59. In the following example illustrated in Figure 2.16, only one of the factors touch an infinity, which, by Theorem 55, is not sufficient for the product to lie in the intersecting reverse class. Take $\varepsilon_0, \varepsilon_1 > 0$ so that $0 < (\varepsilon_0 + \varepsilon_1) \ll 1$. Then, since $-1 < -1 + \varepsilon_0 < 0$, $|x|^{-1+\varepsilon_0} \in A_1(\mathbb{R}) = RC(-\infty, 1)$. Or, equivalently, $u := |x|^{1-\varepsilon_0} \in RC(-1, \infty)$. Again, since $-1 < \varepsilon_0 + \frac{\varepsilon_1}{2} < \varepsilon_0 + \varepsilon_1$, $v := |x|^{\varepsilon_0 + \frac{\varepsilon_1}{2}} \in A_{1+\varepsilon_0+\varepsilon_1}(\mathbb{R}) = RC(\frac{1}{-(\varepsilon_0+\varepsilon_1)}, 1)$. Now, since $\frac{1}{-(\varepsilon_0+\varepsilon_1)} \ll -1$, $RC(\frac{1}{-(\varepsilon_0+\varepsilon_1)}, 1) \subset\subset RC(-1, 1)$. Hence, $v := |x|^{\varepsilon_0 + \frac{\varepsilon_1}{2}} \in RC(-1, 1) = A_2(\mathbb{R})$. Clearly,

$$RC(-1, \infty) \cap RC(-1, 1) = RC(-1, 1).$$

But, $uv = |x|^{1+\frac{\varepsilon_1}{2}} \notin A_2(\mathbb{R})$ since $1 + \frac{\varepsilon_1}{2} > 1$. $\square$

Chapter 3

Two major approaches to Harnack

This chapter reviews two major approaches to Harnack's inequality, namely, Moser's approach [49] in Section 3.1 and Krylov-Safonov's approach [43, 44] in Section 3.2. Moser's and Krylov-Safonov's theories stand as cornerstones in the study of regularity properties of solutions to elliptic PDEs. Their rich and flexible techniques are still being exploited by many researchers to obtain new results or generalizations in many different contexts (for example, see [7, 21] for Moser's and [25, 26] for Krylov-Safonov's).

This chapter is intended to set the stage before we introduce our original approach to Harnack's inequality in the next chapter. As such, this chapter provides motivation as well as background knowledge and notation for our own work. In fact, our approach to Harnack's inequality is going to be a modified version of Krylov-Safonov's approach.

3.1 Moser's approach to Harnack

Harnack's inequality in the context of divergence-form elliptic operators was first verified by Moser [47]. In this section, we provide a brief but explicit review of Moser's Harnack theory. In other words, this is going to be an expository section dedicated to illustrating Moser's famous iteration techniques [49].

Let $\Omega \subset \mathbb{R}^n$ be a domain and let $\mathcal{B}_\Omega$ denote the collection of Euclidean balls B such that $4B \subset \Omega$. By means of his celebrated iterative procedure J. Moser proved in [49] that positive subsolutions to homogeneous divergence-form uniformly elliptic PDEs satisfy weak reverse-Hölder inequalities of the form

$$\sup_B u \le C \left(\frac{1}{|2B|} \int_{2B} u(x)^q \, dx \right)^{\frac{1}{q}}, \quad B \in \mathcal{B}_\Omega, q > 0, \tag{3.1.1}$$

where C depends on n, q, and the ellipticity constants and $|B|$ stands for Lebesgue measure of B. Now, if u is a positive solution, the previous result applied to $1/u$ yields, in addition, the inequalities

$$\left(\frac{1}{|2B|} \int_{2B} u(x)^{-q} \, dx \right)^{-\frac{1}{q}} \le C \inf_B u, \quad B \in \mathcal{B}_\Omega, q > 0. \tag{3.1.2}$$

Then, Moser's Harnack inequality for u,

$$\sup_B u \le C \inf_B u, \quad B \in \mathcal{B}_\Omega, \tag{3.1.3}$$

follows after proving that $\log u \in BMO(\Omega)$, whenever u is a positive supersolution. Indeed, the John-Nirenberg inequality renders the equivalence between $\log u \in BMO(\Omega)$ and the existence of $q_0 > 0$ such that $u^{q_0} \in A_2(\mathcal{B}_\Omega)$. Finally, the choice $q := q_0$ in (3.1.1) and (3.1.2) yields (3.1.3). Moser's approach remains a cornerstone in the study of regularity properties of solutions to PDEs. It is flexible enough to be carried out in the context of other divergence-form PDEs in doubling quasi-metric spaces that sustain a Poincaré-type inequality (for instance, see [1] for the $p(x)$-Laplacian, [7] for quasi-minimizers of p-Dirichlet integrals, [17] for degenerate elliptic PDEs, [21] for infinite graphs, [29] for Dirichlet forms in homogeneous spaces, [34] for X-elliptic operators, etc.)

Next, we review one of Moser's key results as stated above in (3.1.1), also known as a local-boundedness estimate, for positive subsolutions to homogeneous divergence-form uniformly elliptic PDEs encoded in Theorem 61 below. The proof of Theorem 61 given here has been taken from the book [35] and perfectly serves the purpose of elucidating Moser's celebrated iteration scheme. Also, for the sake of brevity, we just write $\int_B u$ to mean $\int_B u(x)\, dx$.

First, let us state and prove a very handy technical lemma. This lemma should remind the reader of another similar lemma, namely, Lemma 17 given in Section 1.1. Lemma 60 will also be used later in the proof of Corollary 90 in Section 4.2.2.

Lemma 60 (Lemma 4.3 in [35]). *Let $\vartheta \in [0,1), \alpha > 0$ and $A \in \mathbb{R}$. If $f \geq 0$ is bounded in $[\tau_0, \tau_1] \subset [0, \infty)$, then for every $\tau_0 \leq t < s \leq \tau_1$, we have*

$$f(t) \leq \vartheta f(s) + \frac{A}{(s-t)^\alpha} \Rightarrow f(t) \leq C(\alpha, \vartheta) \frac{A}{(s-t)^\alpha}.$$

Proof. Choose $\tau \in (0,1)$ such that $\vartheta \tau^{-\alpha} < 1$. With $t_0 = t$, let $t_{i+1} = t_i + (1-\tau)\tau^i (s-t)$ so that $t_i \nearrow s$.

$$\begin{aligned} f(t) &\leq \vartheta \left[\vartheta f(t_2) + \frac{A}{[(1-\tau)\tau(s-t)]^\alpha} \right] + \frac{A}{[(1-\tau)(s-t)]^\alpha} \\ &= \vartheta^2 f(t_2) + \frac{A}{(1-\tau)^\alpha (s-t)^\alpha} \left(\frac{\vartheta}{\tau^\alpha} + 1 \right) \\ &\leq \vartheta^k f(t_k) + \frac{A}{(1-\tau)^\alpha (s-t)^\alpha} \sum_{i=0}^{k-1} \left(\frac{\vartheta}{\tau^\alpha} \right)^i. \end{aligned}$$

Letting $k \to \infty$, proves the lemma. $\square$

Theorem 61 (Moser, 1961). *Let A be a uniformly elliptic matrix in $\Omega \subset \mathbb{R}^n$ with constants λ and Λ. There exists a constant $C = C(n, \lambda, \Lambda)$ such that if $u \in H^1(\Omega)$ is a subsolution to $\mathcal{L}u := (a_{ij}(x) u_i)_j$, $x \in \Omega$, i.e.,*

$$\int_\Omega a_{ij} u_i \varphi_j \leq 0, \ \forall \varphi \in H_0^1(\Omega), \ \varphi \geq 0, \tag{3.1.4}$$

then for any ball $B_r(x_0) \subset \Omega$, any $\theta \in (0,1)$ and any $p > 0$,

$$\sup_{B_{\theta r}(x_0)} u^+ \leq \frac{C}{(1-\theta)^{n/p}} \|u^+\|_{L^p(B_r(x_0))}.$$

Proof. As the PDE is translation and scale-invariant, without loss of generality, we assume, $B_r(x_0) = B_1(0) =: B_1$. Let us, for simplicity, break the proof into a few sections.

Some preliminary observations:

Set $\bar{u} = u^+$. For an arbitrary $m > 0$, define

$$\bar{u}_m = \min\{\bar{u}, m\} = \begin{cases} \bar{u} & \text{if} \quad u < m \\ m & \text{if} \quad u \geq m. \end{cases}$$

Note

$$\bar{u}_m \nearrow \bar{u} \qquad \text{as } m \to \infty.$$

Given $\eta \in C_0^1(B_1), \eta \geq 0$ to be fixed later, define, for an arbitrary $\beta \geq 0$, the test function

$$\varphi := \eta^2 \bar{u}_m^\beta \bar{u} \in H_0^1(B_1).$$

Note that we need only consider (3.1.4) over the domain $\{u > 0\}$ since $\varphi = 0$ if $u \leq 0$ and also note that

$$u = \bar{u} \quad \text{ over } \{u > 0\} \tag{3.1.5}$$

Likewise, since either $\bar{u}_m = \bar{u}$ or $\bar{u}_m$ is a constant, note that

$$\begin{aligned} D\bar{u} &= D\bar{u}_m \quad \text{ over } \{|D\bar{u}_m| > 0\} \\ \bar{u}D\bar{u}_m &= \bar{u}_m D\bar{u}_m. \end{aligned} \tag{3.1.6}$$

Claim 62. *The subsolution condition (3.1.4) with $\varphi := \eta^2 \bar{u}_m^\beta \bar{u}$ yields*

$$\lambda\beta \int_{B_1} \eta^2 \bar{u}_m^\beta |D\bar{u}_m|^2 + \frac{\lambda}{2} \int_{B_1} \eta^2 \bar{u}_m^\beta |D\bar{u}|^2 \leq \frac{2\Lambda^2}{\lambda} \int_{B_1} |D\eta|^2 \bar{u}_m^\beta \bar{u}^2. \tag{3.1.7}$$

Since $ab = (\varepsilon a)(\frac{b}{\varepsilon}) \leq \frac{\varepsilon^2}{2}a^2 + \frac{1}{2\varepsilon^2}b^2$, with $\varepsilon^2 = \frac{\lambda}{2\Lambda}$, we have

$$-2\Lambda \int ab \geq -\frac{\lambda}{2} \int a^2 - \frac{2\Lambda^2}{\lambda} \int b^2. \tag{3.1.8}$$

Now, (3.1.4), ellipticity of (a_{ij}), (3.1.5), (3.1.6) and (3.1.8) all together yield

$$\begin{aligned} 0 &\geq \int_{B_1} a_{ij} D_i u (\eta^2 \bar{u}_m^\beta D_j u + \beta \eta^2 \bar{u} \bar{u}_m^{\beta-1} D_j \bar{u}_m + 2\eta D_j \eta \bar{u}_m^\beta \bar{u}) \\ &\geq \lambda\beta \int_{B_1} |D\bar{u}_m|^2 \eta^2 \bar{u}_m^\beta + \lambda \int_{B_1} \eta^2 \bar{u}_m^\beta |Du|^2 - 2\Lambda \int_{B_1} |D\bar{u}||D\eta| \bar{u}_m^\beta \bar{u}\eta \\ &\geq \lambda\beta \int_{B_1} |D\bar{u}_m|^2 \eta^2 \bar{u}_m^\beta + \frac{\lambda}{2} \int_{B_1} \eta^2 \bar{u}_m^\beta |Du|^2 - \frac{2\Lambda^2}{\lambda} \int_{B_1} |D\bar{u}||D\eta| \bar{u}_m^\beta \bar{u}\eta. \end{aligned}$$

<u>*Using the claim:*</u>

Define $w := \bar{u}_m^{\beta/2} \bar{u}$. Then, by Leibnitz and (3.1.6),

$$|Dw|^2 \leq 2(1+\beta) \left[\beta \bar{u}_m^\beta |D\bar{u}_m|^2 + \bar{u}_m^\beta |D\bar{u}|^2 \right], \tag{3.1.9}$$

using $(aA + bB)^2 \leq 2\max\{a^2, b^2\}(A^2 + B^2)$. Now, (3.1.7) and (3.1.9) together yield the following Cacciopoli estimate

$$\int_{B_1} |Dw|^2 \eta^2 \leq C(n, \lambda, \Lambda)(1+\beta) \int_{B_1} |D\eta|^2 w^2. \tag{3.1.10}$$

Leibinitz's rule and (3.1.10) gives

$$\int_{B_1} |D(w\eta)|^2 \leq C(1+\beta) \int_{B_1} |D\eta|^2 w^2. \tag{3.1.11}$$

Setting $\rho := \frac{n}{n-2}$ so that $2\rho = 2^*$, Sobolev and (3.1.11) yield

$$\left(\frac{1}{C}\int_{B_1} |w\eta|^{2\rho}\right)^{1/\rho} \leq \int_{B_1} |D(w\eta)|^2 \leq C(1+\beta) \int_{B_1} |D\eta|^2 w^2. \tag{3.1.12}$$

For an arbitrary $0 < r < R \leq 1$, choosing $\eta \in C_0^1(B_R), \eta \equiv 1$ in B_r and $|D\eta| \leq \frac{2}{R-r}$,

$$\left(\int_{B_r} w^{2\rho}\right)^{1/\rho} \leq \frac{C(1+\beta)}{(R-r)^2} \int_{B_R} w^2. \tag{3.1.13}$$

Using $w := \bar{u}_m^{\beta/2}\bar{u}$ and Monotone convergence theorem as $m \to \infty$,

$$\left(\int_{B_r} \bar{u}^{(\beta+2)\rho}\right)^{1/\rho} \leq \frac{C(1+\beta)}{(R-r)^2} \int_{B_R} \bar{u}^{\beta+2},\ \beta \geq 0. \tag{3.1.14}$$

Iterations for the case $p = 2$:

By (3.1.14), for every $\gamma := \beta + 2 \geq 2$ and every $0 < r < R \leq 1$,

$$\|\bar{u}\|_{L^{\gamma\rho}(B_r)} \leq \left(\frac{C(\gamma-1)}{(R-r)^2}\right)^{1/\gamma} \|\bar{u}\|_{L^\gamma(B_R)}. \tag{3.1.15}$$

With $\gamma_0 = 2$ and $r_0 = 1$, let $\gamma_i = \gamma_{i-1}\rho$ and $r_i = \frac{1}{2} + \frac{1}{2^{i+1}}$ so that $\gamma_i \nearrow \infty$ and $r_i \searrow \frac{1}{2}$. Then, since

$$\begin{aligned}\left(\frac{C(\gamma_i - 1)}{(r_{i+1}-r_i)^2}\right)^{\frac{1}{\gamma_i}} &\leq C(2\rho^i - 1)^{\frac{1}{2\rho^i}} 2^{\frac{2(i+2)}{2\rho^i}} \leq C(2\rho^i)^{\frac{1}{2\rho^i}} 2^{\frac{i}{\rho^i}} 2^{\frac{2}{\rho^i}} \\ &\leq C(2\rho^{\frac{1}{2}})^{\frac{i}{\rho^i}} 2^{\frac{i}{\rho^i}} (2^2)^{\frac{i}{\rho^i}} = C(n,\lambda,\Lambda)^{\frac{i}{\rho^i}},\end{aligned}$$

$$\begin{aligned}\|\bar{u}\|_{L^{\gamma_i}(B_{r_i})} &\leq C(n,\lambda,\Lambda)^{\frac{i}{\rho^i}} \|\bar{u}\|_{L^{\gamma_{i-1}}(B_{r_{i-1}})} \\ &\leq C^{\left(\sum_{k=0}^{i} \frac{k}{\rho^k}\right)} \|\bar{u}\|_{L^2(B_1)}.\end{aligned} \tag{3.1.16}$$

Letting $i \to \infty$,

$$\|\bar{u}\|_{L^\infty(B_{1/2})} \equiv \sup_{B_{1/2}} \bar{u} \leq C\|\bar{u}\|_{L^2(B_1)}. \tag{3.1.17}$$

Some observations for the case $p \geq 2$:

Running the iterations in (3.1.16) with $\gamma_0 = p \geq 2$ and making a dilation substitution $\tilde{u}(y) = u(Ry), y \in B_1$, (3.1.17) becomes

$$\|\bar{u}\|_{L^\infty(B_{R/2})} \equiv \sup_{B_{R/2}} \bar{u} \leq \frac{C}{R^{n/p}} \|\bar{u}\|_{L^p(B_R)}. \tag{3.1.18}$$

Let $y \in B_{\theta R}$, $\theta \in (0,1)$, $0 < R \leq 1$. We wish to apply (3.1.18) to $B_{(1-\theta)R}(y)$ instead of B_R.

Note that for every $y \in B_{\theta R}$, $\theta \in (0,1)$, $0 < R \leq 1$,

$$\|\bar{u}\|_{L^p(B_{(1-\theta)R}(y))} \leq \|\bar{u}\|_{L^p(B_R)}. \tag{3.1.19}$$

Also, note that there exists a $y \in B_{\theta R}$, $\theta \in (0,1)$, $0 < R \leq 1$, such that

$$\|\bar{u}\|_{L^\infty(B_{\theta R})} \leq \|\bar{u}\|_{L^\infty(B_{\frac{(1-\theta)R}{2}}(y))}, \tag{3.1.20}$$

because it is possible to choose $y \in B_{\theta R}$ in such a way that the ball $B_{\frac{(1-\theta)R}{2}}(y)$ contains the point (or, the boundary) where $\bar{u}$ attains its max (or, sup) in $B_{\theta R}$.

Iterations for the case $p \geq 2$:

Now, for $p \geq 2$ and some $y \in B_{\theta R}$, $\theta \in (0,1)$, $0 < R \leq 1$, we have by (3.1.18),

$$\|\bar{u}\|_{L^\infty(B_{R/2})} \equiv \sup_{B_{R/2}} \bar{u} \leq \frac{C}{R^{n/p}} \|\bar{u}\|_{L^p(B_R)}.$$

By (3.1.19),

$$\|\bar{u}\|_{L^p(B_{(1-\theta)R}(y))} \leq \|\bar{u}\|_{L^p(B_R)}.$$

By (3.1.20),

$$\|\bar{u}\|_{L^\infty(B_{\theta R})} \leq \|\bar{u}\|_{L^\infty(B_{\frac{(1-\theta)R}{2}}(y))}.$$

By (3.1.20), (3.1.18) applied to $B_{(1-\theta)R}(y)$ instead of B_R, and (3.1.19),

$$\|\bar{u}\|_{L^\infty(B_{\theta R})} \leq \frac{C}{[(1-\theta)R]^{n/p}} \|\bar{u}\|_{L^p(B_R)}. \tag{3.1.21}$$

Now, take $R = 1$ in (3.1.21) and we are done for the case $p \geq 2$.

Some observations for the case $p \in (0,2)$:

Recall (3.1.21): for every $\theta \in (0,1)$ and every $0 < R \le 1$,

$$\|\bar{u}\|_{L^\infty(B_{\theta R})} \le \frac{C(n,\lambda,\Lambda)}{[(1-\theta)R]^{n/2}}\|\bar{u}\|_{L^2(B_R)}. \tag{3.1.22}$$

For $p \in (0,2)$,

$$\|\bar{u}\|^2_{L^2(B_R)} := \int_{B_R} \bar{u}^2 \le \|\bar{u}\|^{2-p}_{L^\infty(B_R)} \int_{B_R} \bar{u}^p. \tag{3.1.23}$$

Since $ab = (\varepsilon a)(\frac{b}{\varepsilon}) \le \frac{\varepsilon^{\tilde{p}}}{\tilde{p}}a^{\tilde{p}} + \frac{1}{\tilde{p}'\varepsilon^{\tilde{p}'}}b^{\tilde{p}'}$, with $\tilde{p} = \frac{2}{2-p}$ (so that $\tilde{p}' = \frac{2}{p}$) and ε such that $\frac{\varepsilon^{\tilde{p}}}{\tilde{p}} = \frac{1}{2}$, we have

$$a^{\frac{1}{\tilde{p}}}b^{\frac{1}{2}} \le \frac{1}{2}a + C(p)b^{\frac{2}{p}}. \tag{3.1.24}$$

Then, (3.1.22), (3.1.23) raised to $\frac{1}{2}$, and (3.1.24) yield

$$\|\bar{u}\|_{L^\infty(B_{\theta R})} \le \frac{1}{2}\|\bar{u}\|_{L^\infty(B_R)} + \frac{C(n,\lambda,\Lambda,p)}{[(1-\theta)R]^{n/p}}\left(\int_{B_R} \bar{u}^p\right)^{\frac{1}{p}}. \tag{3.1.25}$$

Proof for the case $p \in (0,2)$:

Using the function $f(t) := \|\bar{u}\|_{L^\infty(B_t)}$, $t \in (0,1]$, (3.1.25) becomes

$$f(\theta R) \le \frac{1}{2}f(R) + \frac{C(n,\lambda,\Lambda,p)}{[(1-\theta)R]^{n/p}}\|\bar{u}\|_{L^p(B_1)}. \tag{3.1.26}$$

Using Lemma 60, (3.1.26) implies

$$f(\theta R) \le \frac{C(n,\lambda,\Lambda,p)}{[(1-\theta)R]^{n/p}}\|\bar{u}\|_{L^p(B_1)}.$$

Letting $R \to 1^-$,

$$\|\bar{u}\|_{L^\infty(B_\theta)} \le \frac{C(n,\lambda,\Lambda,p)}{[(1-\theta)]^{n/p}}\|\bar{u}\|_{L^p(B_1)}.$$

This concludes the proof of the theorem for the case $p \in (0,2)$.

$\square$

As an application of the visual formalism of reverse inequalities introduced in Chapter 2, we elucidate Moser's approach to Harnack in Figure 3.1 via a step-by-step illustration. Recalling Theorem 38 and the illustration of an A_2 weight in Figure 2.3, note that what we have in Step 5 of this illustration is exactly, for some uniform constants $\delta, p_0 > 0$, $u^{\delta p_0} \in A_2(\Omega)$. Hence, it is quite apt to call Moser's approach an A_2 approach to Harnack.

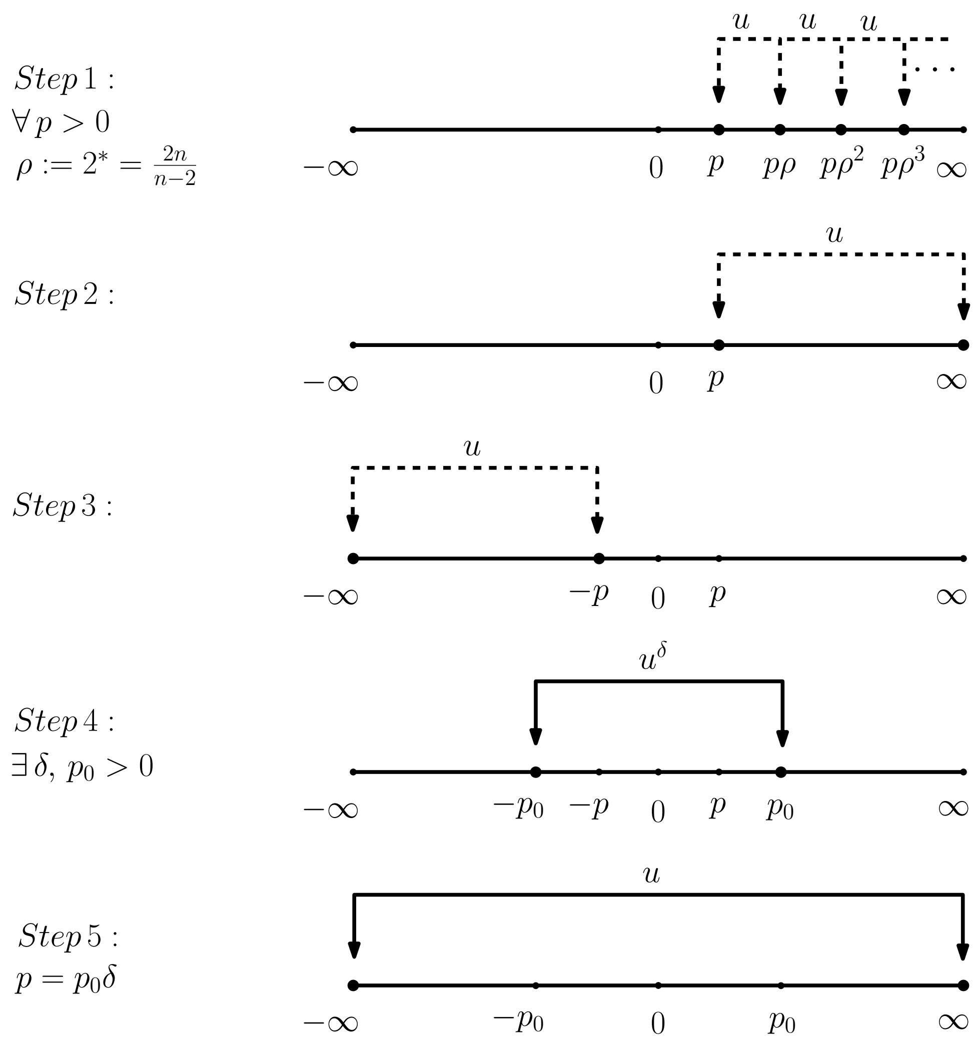


Figure 3.1: *Steps of Moser's approach (A_2 approach) to Harnack.*

3.2 Kylov-Safonov's approach to Harnack

In this section, we will review the theory propounded by Krylov and Safonov [43, 44] while studying regularity properties of solutions to elliptic operators in non-divergence form:

$$Lu := \operatorname{tr}(A(x)D^2u) = 0,\ x \in \Omega \subset \mathbb{R}^n. \tag{3.2.27}$$

Using probabilistic tools which were completely unconventional at that time, they were able to prove Harnack's inequality for solutions to (3.2.27). The review of Krylov-Safonov's Harnack theory will be divided into two subsections. Section 3.2.1 introduces their ground-breaking measure-theoretic tools and Section 3.2.2 introduces some more properties and describes some axiomatic adaptations of Krylov-Safonov's theory in spaces of homogeneous type.

3.2.1 Measure-theoretic tools

Before Krylov and Safonov, most of the tools used in the study of regularity properties of solutions to elliptic operators were variational ones such as energy estimates. Under such circumstances, Krylov and Safonov's measure-theoretic tools genuinely came as a boon, especially in the study of non-variational operators. Next, we give the definitions of some of the properties introduced by Krylov and Safonov. All these definitions are set in a space of homogeneous type (X, d, μ) and are in the spirit of [3], [12], [14], and [25].

First, let us recall the notation we introduced in Remark 3, which will be frequently used hereafter. Let (X, d, μ) be a space of homogeneous type (see Definition 20) and $\Omega \subset X$ be a domain. Then, we denote by $\mathbb{K}_\Omega$ a family of μ-measurable functions with domain contained in Ω. If $u \in \mathbb{K}_\Omega$ and $A \subset \operatorname{dom}(u)$ then we write $u \in \mathbb{K}_\Omega(A)$.

Definition 63. Given $\varepsilon \in (0, 1)$ and $M \geq 1$, $\mathbb{K}_\Omega$ is said to satisfy the *critical density property* with constants M and ε if for every ball $B_{2R}(x_0) \subset \Omega$ and for every $u \in \mathbb{K}_\Omega(B_{2R}(x_0))$ with

$$\mu(\{x \in B_R(x_0) \mid u(x) \geq M\} \geq \varepsilon\mu(B_R(x_0)),$$

we have

$$\inf_{B_{R/2}(x_0)} u > 1.$$

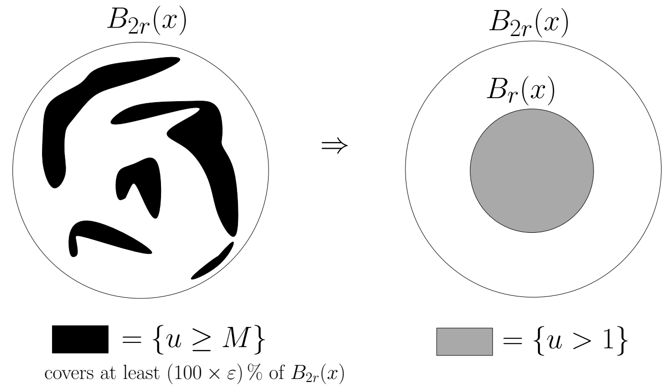


Figure 3.2: *Critical density with constants $M \geq 1$ and $\varepsilon \in (0, 1)$.*

The critical density property lies at the heart of the techniques developed by Krylov and Safonov. This property is illustrated in Figure 3.2 and is interpreted in probabilistic terms as follows: u is said to have the critical density property with constants $\varepsilon \in (0, 1)$ and $M \geq 1$ if the proportion of the region where $u \geq M$ in a ball B_R being at least $\varepsilon\%$ guarantees that u is at least 1 in the ball $B_{R/2}$. Note that the definition only states that $u \geq M$ over a significant area of the ball and does not specify its exact location of this region. Interpreted again in terms of temperature, for a function with a critical density property, if the portion of a ball where it is very hot passes a certain critical point, then it must be at least warm over half that ball.

Definition 64. Let $\gamma \in (0, 1)$. $\mathbb{K}_\Omega$ is said to satisfy the *double-ball property* with constant γ if for every ball $B_{2R}(x_0) \subset \Omega$ and for every $u \in \mathbb{K}_\Omega(B_{2R}(x_0))$ with

$$\inf_{B_{R/2}(x_0)} u \geq 1,$$

it follows that

$$\inf_{B_R(x_0)} u \geq \gamma.$$

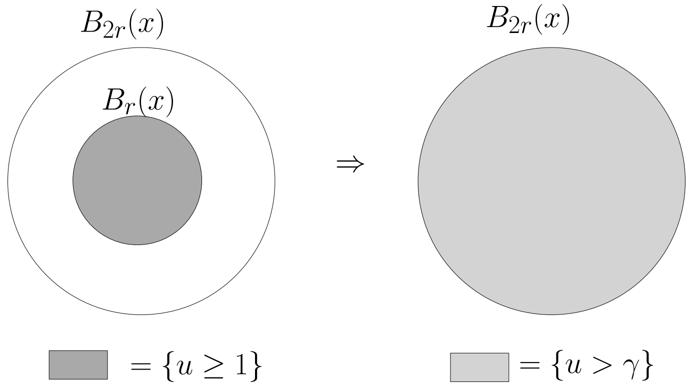


Figure 3.3: *Double-ball property with constant $\gamma \in (0,1)$.*

The double-ball property is illustrated in Figure 3.3. A function u is said to have the double-ball property with constant $\gamma \in (0,1)$ if $u \geq 1$ in all of $B_{R/2}$ guarantees $u > \gamma$ in all of B_R. Interpreted again in terms of temperature, for a function with a double-ball property, if it is warm over a ball, then it must be at least mild over twice that ball.

Definition 65. Let $\varrho \in (0,1)$ and $N > 1$. $\mathbb{K}_\Omega$ is said to satisfy the *power-like decay property* with constants N and ϱ, if for every $u \in \mathbb{K}_\Omega(B_{2R}(x_0))$ with

$$\inf_{B_R(x_0)} u \leq 1,$$

it follows that

$$\mu(\{x \in B_{R/2}(x_0) : u(x) > N^k\}) \leq \varrho^k \mu(B_{R/2}(x_0)), \quad k \in \mathbb{N}.$$

A function u is said to have the power-like decay property with constants $\varrho \in (0,1)$ and $N > 1$ if the proportion of region where $u > N^k$ in a ball decays in the order of ϱ^k as k increases. Referring to the illustration in Figure 3.4, we can visualize this property as the area of the part of the line $u = N^k$ (note that, in general, this is a hyperplane) lying below the graph of u being decreased like ϱ^k as k increases.

Definition 66. $\mathbb{K}_\Omega$ is said to satisfy the *weak Harnack property* if there exist constants $C'_H > 1$ and $\delta > 0$ such that for every $B_{2R}(x_0) \subset \Omega$ and for every $u \in \mathbb{K}_\Omega(B_{2R}(x_0))$,

$$\frac{1}{\mu(B_{R/2}(x_0))} \int_{B_{R/2}(x_0)} u^\delta d\mu \leq C'_H \inf_{B_{R/2}(x_0)} u^\delta.$$

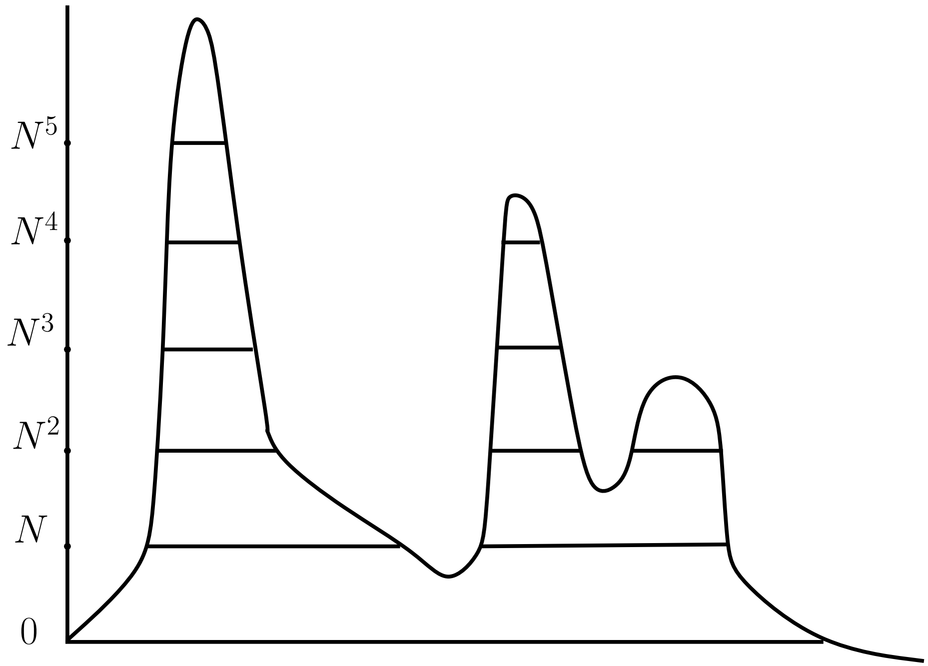


Figure 3.4: *A figure to illustrate the power-like decay property.*

Let us, for the moment, consider the family of subsolutions (or, supersolutions) to the non-divergence-form operator (3.2.27). It is easily checked that this functional set is closed under multiplication by positive constants, which motivates the following definition:

Definition 67. Let $\Omega \subset X$ be a domain. $\mathbb{K}_\Omega$ denotes a family of μ-measurable functions with domain contained in Ω, and if $u \in \mathbb{K}_\Omega$ and $A \subset \mathrm{dom}(u)$ then we write $u \in \mathbb{K}_\Omega(A)$. Here $\mathrm{dom}(u)$ stands for the domain of the function u. We say that $\mathbb{K}_\Omega$ is closed under *multiplication by positive constants* if whenever $u \in \mathbb{K}_\Omega$ and $\alpha > 0$, then $\alpha u \in \mathbb{K}_\Omega$, and we say $\mathbb{K}_\Omega$ is closed under *multiplication by small constants* if whenever $u \in K_\Omega$ and $\tau \in (0, 1)$, then $\tau u \in \mathbb{K}_\Omega$. Also, we say $\mathbb{K}_\Omega$ has a property to mean that the property holds uniformly for every $u \in \mathbb{K}_\Omega$, i.e., all the constants associated to the property are purely structural and do not depend on individual functions in $\mathbb{K}_\Omega$.

Remark 68. The assumption of closedness under multiplication by positive constants applies to many families of functions and, in practice, does not constrain the theory. For instance, the supersolutions (or subsolutions) of the second-order uniformly elliptic operators introduced in Definition 12 are all closed under multiplication by positive constants. In fact, even the solution class to the Monge-Ampère operator $M(u) := \det(D^2u) = 0$, clearly not linear, has the property of being closed under multiplication by positive constants. Also, it turns out in the case when $\mathbb{K}_\Omega$ is closed under multiplication by positive constants, the power-like decay property is equivalent to the weak Harnack property. This will be explicitly proved later in Section 4.1 as Proposition 81.

3.2.2 Some axiomatic approaches to Harnack

We begin this section by describing the linearized Monge-Ampère equation that motivated axiomatic approaches toward Harnack's inequality.

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a convex C^2 function whose associated Monge-Ampère measure $\mu_\varphi(x) := \det D^2\varphi(x)$ satisfies $\mu_\varphi(x) > 0$ for all $x \in \mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$ be an open set and fix $0 < \Lambda_1 \leq \Lambda_2 < \infty$. For each $x \in \Omega$ let $A(x)$ be a symmetric matrix with continuous coefficients in Ω. We write $A \in E(\Lambda_1, \Lambda_2, D^2\varphi, \Omega)$ if

$$\Lambda_1 \leq \langle (D^2\varphi(x))^{1/2}A(x)(D^2\varphi(x))^{1/2}\xi, \xi\rangle \leq \Lambda_2, \quad x \in \Omega, \xi \in \mathbb{S}^{n-1}.$$

In other words,

$$\Lambda_1 I \leq (D^2\varphi)^{1/2}A(D^2\varphi)^{1/2} \leq \Lambda_2 I \quad \text{in } \Omega.$$

For $A \in E(\Lambda_1, \Lambda_2, D^2\varphi, \Omega)$, consider the non-divergence form elliptic operator L_A defined as

$$L_A u(x) := \operatorname{tr}(A(x)D^2u(x)) + c(x)u(x) \quad x \in \Omega,$$

where the potential $c(x)$ satisfies $c \in L^\infty(\Omega)$ and $c \leq 0$ in Ω. We say that L_A *is adapted to* φ since the geometric and measure theoretic objects to study L_A are determined by the convex function φ. Notice that for arbitrary φ the eigenvalues of $A(x)$ are comparable to those of $(D^2\varphi(x))^{-1}$ which, although positive, might get close to 0 or ∞. Hence, the apriori degeneracy of A is circumvented by conveniently adopting the geometry and the measure theory associated to φ. Indeed, following L. Caffarelli in [13], given $x \in \mathbb{R}^n$ and $r > 0$, a *section* of φ centered at x with height r is the open bounded convex set

$$S_\varphi(x,r) := \{y \in \mathbb{R}^n : \varphi(y) < \varphi(x) + \langle \nabla\varphi(x), y - x\rangle + r\}.$$

Sections of a convex function is illustrated in Figure 3.5. It should be noted that the center x of the section $S_\varphi(x,r)$ need not be the center of mass denoted by x^* in Figure 3.5.

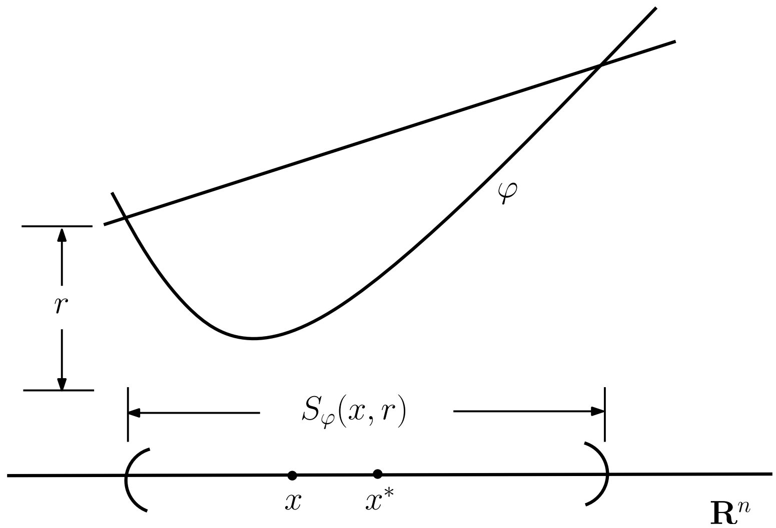


Figure 3.5: *Section of a convex function φ.*

Also, it can be noticed that in the case $\varphi(x) = \varphi_2(x) := \frac{1}{2}|x|^2$ we have $A \in E(\Lambda_1, \Lambda_2, D^2\varphi_2, \Omega)$ if and only if A is uniformly elliptic. In this special case, the measure $d\mu_{\varphi_2}(x) := \det D^2\varphi_2(x)dx$ reduces to the Lebesgue measure dx. Indeed, since

$$\frac{\partial \varphi_2(x)}{\partial x_i} = \frac{1}{2}2|x|\frac{x_i}{|x|} = x_i,$$

we have the gradient $\nabla\varphi_2(x) = x$ and the Hessian $D^2\varphi_2(x) = I_n$, which implies $\det D^2\varphi_2(x) =$

1. Also, the sections turn out to be Euclidean balls. Indeed, for every $x \in \mathbb{R}^n$ and $r > 0$,

$$\begin{aligned} S_{\varphi_2}(x, r/2) &= \left\{ y : \varphi_2(y) - \varphi_2(x) - \langle \nabla\varphi_2(x), y - x \rangle < \frac{r}{2} \right\} \\ &= \left\{ y : \frac{1}{2}|y|^2 - \frac{1}{2}|x|^2 - \langle x, y - x \rangle < \frac{r}{2} \right\} \\ &= \left\{ y : \frac{1}{2}|x|^2 - x \cdot y + \frac{1}{2}|y|^2 < \frac{r}{2} \right\} \\ &= \left\{ y : \frac{|x - y|^2}{2} < \frac{r}{2} \right\} \\ &= \{ y : |x - y| < \sqrt{r} \} \\ &= B(x, \sqrt{r}). \end{aligned}$$

The class $E(\Lambda_1, \Lambda_2, D^2\varphi, \Omega)$ was originally introduced by L. Caffarelli and C. Gutiérrez [15, 16] in their pioneering work on the linearized Monge-Ampère equation.

Next, we introduce some axiomatic approaches to Harnack's inequality in spaces of homogeneous type. This makes for a convenient comparison with our new axiomatic approach found in Chapter 4. We give some old and new definitions in order to fully state the main results of these axiomatic approaches. Proofs of these results found in their original works are not reproduced here. However, some insight on these earlier axiomatic approaches can be found in Chapter 4 in terms of discussions on the structural assumptions of these various approaches.

Krylov-Safonov's measure-theoretic techniques, greatly simplified and enriched by L. Caffarelli in [12], appear to capture the essence of ellipticity in non-variational settings including fully non-linear elliptic PDEs [12] and degenerate elliptic PDEs such as the linearized Monge-Ampère equation [15, 16] as well as variational ones including divergence-form elliptic PDEs with *a priori* energy estimates (see [4] in the Euclidean case and [42] for metric spaces with a calculus of order 1) and adjoint solutions to non-divergence elliptic operators [24], just to name a few. In all rigor, the linearized Monge-Ampère equation possesses a double nature, both variational and non-variational. It was precisely the pioneering work of Caffarelli and Gutiérrez on the linearized Monge-Ampère equation [15, 16], where convex functions prescribe the relevant geometric and measure-theoretic framework, that led to the axiomatization of the Krylov-Safanov-Caffarelli approach in the context of spaces of homogeneous type [3, 25, 57].

Next, we introduce some definitions that have been used in some of the axiomatic approaches to Harnack in spaces of homogeneous type.

Definition 69. A function u defined on Ω is said to be *upper semi-continuous* at x_0 if there exists a ball $B_\varepsilon(x_0) \subset \Omega$ such that, for every $x \in B_\varepsilon(x_0)$, we have

$$\limsup_{x \to x_0} f(x) \leq f(x_0).$$

Definition 70. A doubling quasi-metric space (X, d, μ) is said to satisfy the *ring condition* if there exists a non-negative function ω such that $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ and, for every ball $B_r(x)$ and every $\varepsilon \in (0, 1)$, we have

$$\mu(B_r(x)) \setminus B_{(1-\varepsilon)r}(x) \leq \omega(\varepsilon)\mu(B_r(x)).$$

Definition 71. A doubling quasi-metric space (X, d, μ) is said to satisfy the *non-empty annulus condition* if for every $x \in X$ and every $r > 0$, we have

$$\mathrm{Ann}_{\frac{r}{2},r}(x) := \{y \in X \mid r/2 < |x - y| < r\} = \emptyset.$$

Under different sets of assumptions on the doubling quasi-metric space (X, d, μ) (e.g., ring condition, non-empty annulus, unboundedness, etc.), Toledano [57], Aimar, Forzani, and Toledano [3] (assuming also that $\mathbb{K}_\Omega$ contains only upper semi-continuous functions), and Di Fazio, Gutiérrez, and Lanconelli [25] (with no semi-continuity assumptions on $\mathbb{K}_\Omega$), proved the following

Theorem 72. ([3, Theorem 3.1(d)], [25, Theorem 4.7], [57, Theorem 3.10]) *If $\mathbb{K}_\Omega$ is closed under multiplication by positive constants, and possesses the critical density property with constants M and ε and the doubling-ball property with constant γ for some $\varepsilon, \gamma \in (0, 1)$ and $M > 1$, then $\mathbb{K}_\Omega$ satisfies the power-like decay property with some constants $N > 1$ and $\varrho \in (0, 1)$ depending only on ε, γ, M, and geometric constants.*

Theorem 73. ([3, Theorem 3.10(e)], [57, Theorem 3.10]) *If $\mathbb{K}_\Omega$ is closed under multiplication by positive constants, and possesses the critical density and doubling-ball properties with some $\varepsilon, \gamma \in (0,1)$ and $M > 1$, then $\mathbb{K}_\Omega$ has the Harnack property with constants C_H and η_0 depending only on ε, γ, and geometric constants.*

Theorem 74. ([25, Theorem 5.1]) *If $\mathbb{K}_\Omega$ is closed under multiplication by positive constants, and possesses the power-like decay property with constants N and ϱ and whenever $u \in \mathbb{K}_\Omega(B_R(x_0))$ and $\lambda \geq u$ in $B_R(x_0)$ then $\lambda - u \in \mathbb{K}_\Omega(B_R(x_0))$, then $\mathbb{K}_\Omega$ satisfies the Harnack property with constant C_H depending only on N, ϱ, and geometric constants.*

All the above-mentioned approaches to Harnack share the techniques pioneered by Krylov-Safonov. These approaches, via covering lemmas, are ultimately based on the power-like decay of the distribution functions of non-negative sub and supersolutions. As mentioned in Remark 68 above (this will be proved later in Proposition 81), the power-like decay property for u is equivalent to its weak Harnack property, which, in turn, amounts to the existence of $\delta \in (0,1)$ such that $u^\delta \in A_1(\mathcal{B}_\Omega)$. Thus, in contrast to Moser's approach to Harnack which is an A_2 approach illustrated in Figure 3.1, Krylov-Safonov's approach to Harnack illustrated in Figure 3.6 is conspicuously an A_1 approach.

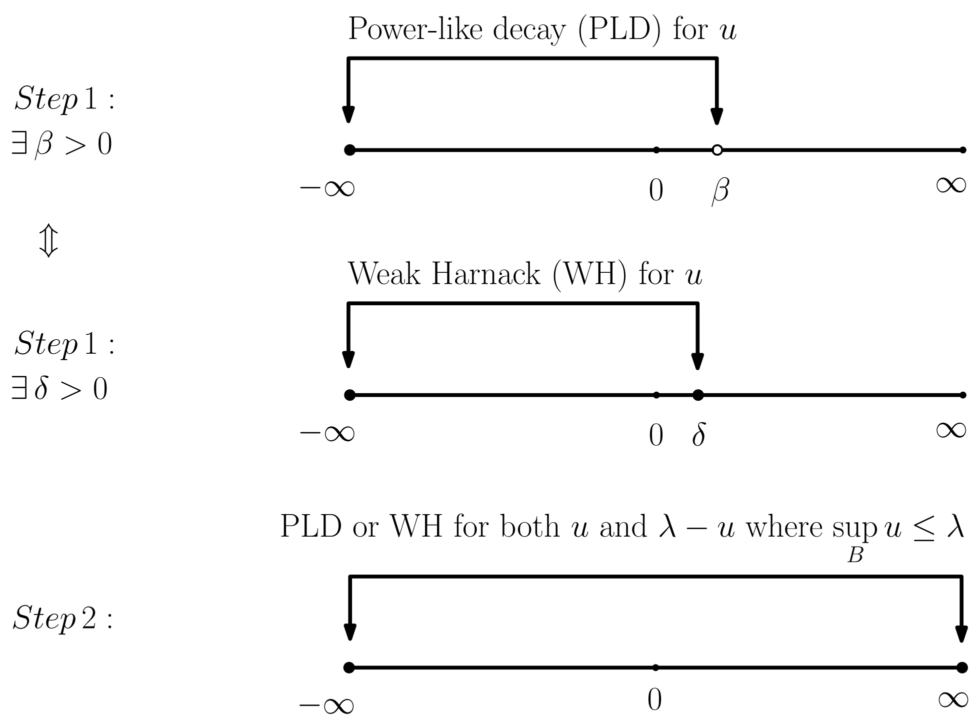


Figure 3.6: *Krylov-Safonov's approach (A_1 approach) to Harnack.*

Chapter 4

A novel approach to Harnack

This chapter contains the theory of the main work in this dissertation. The theme is a novel axiomatic approach to Harnack's inequality in spaces of homogeneous type. This work was first introduced in our collaborative work [38]. The main result is stated in Theorem 82 and is found in Section 4.2.3.

The novel approach to Harnack's inequality introduced in this chapter is a modified version of Krylov-Safonov's approach reviewed in Section 3.2 of Chapter 3. Krylov-Safonov's approach was based on the two properties: the critical density and the double-ball properties. Our approach replaces the double-ball property by the doubling property (as a weight). The doubling property of a measure has already been introduced in Chapter 1 as Definition 21. The doubling property of a function u (as a weight) is simply the doubling property of a measure whose density is u. Section 4.1 reintroduces this definition and some of its consequences in relation to other properties. Section 4.2 is devoted to the statement and proof of Harnack's inequality via our approach encoded in Theorem 82. Finally, sections Section 4.3 through 4.7 provide discussions about the main result and its assumptions vis-á-vis other related results.

The proof of Theorem 82 is broken down into three parts which are framed in three subsections. Section 4.2.1 establishes a key lemma stating a classic result in analysis in the setting of spaces of homogeneous type. Although it is a very well-known result, we have been unable to find its proof in its entirety, which is why we provide its proof here.

Section 4.2.2 establishes another key intermediate step, namely, Corollary 90, toward the proof of Theorem 82. It is one of the main results of our theory and establishes a key local-boundedness type result. A result similar to this serves as a critical step toward Harnack's inequality in both Moser's and Krylov-Safonov's approaches. Finally, Section 4.2.3 concludes the proof of Theorem 82.

After the main theorem has been established, five short sections have been devoted to providing more insight about the main result. Section 4.3 makes a remark about an assumption of Theorem 82, namely, that of the density of continuous functions in L^1 space. Section 4.4 justifies the novel idea in our approach, that is, the replacement of the double-ball property by the doubling property as our assumption on the structure. It does so by showing that these two properties are not related to one another. Section 4.5 identifies a natural context of our approach by showing that the doubling property appears naturally for variational operators just like the way the double-ball property is natural for non-variational operators. Section 4.6 seeks to make sure that all the structure assumed in our approach is indeed necessary and that there is no redundancy of assumptions. Finally, Section 4.7 makes a comparison of our approach to a classic result due to Bombieri [8, Theorem 4].

4.1 The doubling property (as a weight)

We have already described the doubling property for a measure ((1.1.14) in Section 1). In this section, we will introduce the doubling property (as a weight) for a function. This property in our axiomatic approach to Harnack is going to replace the double-ball property in earlier axiomatic approaches. In order to put this replacement in perspective, first we recall the definition of the double-ball property (Definition 64) introduced in Section 3.2:

Let $\gamma \in (0,1)$. $\mathbb{K}_\Omega$ is said to satisfy the *double-ball property* with constant γ if for every ball $B_{2R}(x_0) \subset \Omega$ and for every $u \in \mathbb{K}_\Omega(B_{2R}(x_0))$,

$$\inf_{B_{R/2}(x_0)} u \geq 1 \Rightarrow \inf_{B_R(x_0)} u \geq \gamma.$$

The property defined below is the main novelty in our approach to Harnack's inequality:

Definition 75. Let $C_D > 1$. $\mathbb{K}_\Omega$ is said to satisfy the *doubling property* with constant C_D if for every ball $B_{2R}(x_0) \subset \Omega$ and for every $u \in \mathbb{K}_\Omega(B_{2R}(x_0))$

$$\int_{B_R(x_0)} u\,d\mu \le C_D \int_{B_{R/2}(x_0)} u\,d\mu. \tag{4.1.1}$$

The doubling property of a function u is also referred to as the doubling of u as a weight. Comparing the doubling property of a measure given by (1.1.14) in Chapter 1, the doubling property of a function u given by (4.1.1) in Definition 75 is clearly seen to be the doubling property of a measure ν such that $u = \frac{d\nu}{d\mu}$. Our goal in this chapter is to establish an alternative path to the Harnack inequality by replacing the double-ball property with the doubling property of u as a weight. That is, we replace the pointwise doubling condition (76) by the integral doubling condition (4.1.1). We shall see later in Section 4.4 that the double-ball property and the doubling property are independent of one another.

Our axiomatic approach to Harnack's inequality is set for a family of functions closed under multiplication by positive constants (see Definition 67). If $\mathbb{K}_\Omega$ is closed under multiplication by positive constants, the following assertions hold true:

(a) The weak Harnack property (see Definition 66) is equivalent to the power-like decay property (see Definition 65).

(b) Further assuming that $\lambda - u \in \mathbb{K}_\Omega$ whenever $u \in \mathbb{K}_\Omega$ and $\sup_B u \le \lambda$, the weak Harnack property for $\mathbb{K}_\Omega$ yields the Harnack property for $\mathbb{K}_\Omega$.

The remainder of this section is devoted to establishing the equivalence between the power-like decay property and the weak Harnack property. The Harnack property will be dealt with in the next section.

Proposition 76. *If $\mathbb{K}_\Omega$ is closed under multiplication by positive constants, $u \in \mathbb{K}_\Omega$ has the double-ball property with constant γ if and only if the infimum of u has the doubling property with constant $\frac{1}{\gamma}$, that is,*

$$\inf_{B_{R/2}(x_0)} u \le \frac{1}{\gamma} \inf_{B_R(x_0)} u.$$

Proof. First, assume the doubling property for $\inf u$ and the hypothesis that $\inf\limits_{B_{R/2}(x_0)} u \geq 1$. Then,

$$1 \leq \inf_{B_{R/2}(x_0)} u \leq \frac{1}{\gamma} \inf_{B_R(x_0)} u,$$

which yields $\inf\limits_{B_R(x_0)} u \geq \gamma$, and we have the double-ball property. Conversely, let us assume the double-ball property for $\mathbb{K}_\Omega$ and consider the function $\frac{u}{\inf\limits_{B_{R/2}(x_0)} u}$, which, in particular, has this property. Hence,

$$\inf_{B_R(x_0)} \left(\frac{u}{\inf\limits_{B_{R/2}(x_0)} u} \right) \geq \gamma,$$

or, $\inf\limits_{B_{R/2}(x_0)} u \leq \frac{1}{\gamma} \inf\limits_{B_R(x_0)} u$, which is the double-ball property for $\inf u$. $\square$

Corollary 77. *Let $\mathbb{K}_\Omega$, closed under multiplication by positive constants, have the double-ball property. If $0 \leq u \in \mathbb{K}_\Omega$ and $u \not\equiv 0$, then $u > 0$.*

Proof. This is clear since if there existed an x_0 such that $u(x_0) = 0$, then

$$\inf_{B_{R/2}(x_0)} u = \inf_{B_R(x_0)} u = 0,$$

which means u cannot have the double-ball property. $\square$

Corollary 78. *Suppose $\mathbb{K}_\Omega$ has the weak Harnack property. That is, there exists a $\delta > 0$ and $C'_H > 1$ such that, for every $u \in \mathbb{K}_\Omega$,*

$$\frac{1}{\mu(B)} \int_{B_R} u^\delta \, d\mu \leq C'_H \inf_{B_R} u^\delta, \tag{4.1.2}$$

Then, $\mathbb{K}_\Omega$ has the double-ball property with constant $1/\left(C'_H C_D\right)^{\frac{1}{\delta}}$.

Proof. Using natural inequalities, doubling property of μ, and the hypothesis, we obtain

$$\begin{aligned}
\inf_{B_R} u^\delta &\leq \frac{1}{\mu(B_R)} \int_{B_R} u^\delta \\
&\leq C_D \frac{1}{\mu(B_{2R})} \int_{B_{2R}} u^\delta \\
&\leq C_D \, C'_H \inf_{B_{2R}} u^\delta = C_D \, C'_H \left(\inf_{B_{2R}} u \right)^\delta,
\end{aligned}$$

which yields

$$\inf_{B_R} u \leq (C'_H C_D)^{\frac{1}{\delta}} \inf_{B_{2R}} u.$$

□

Recall that the power-like decay property (see Definition 65) with constants $N > 1$ and $\varrho \in (0,1)$ for $\mathbb{K}_\Omega$, a family of functions closed under multiplication by positive constants, means that, for every $u \in \mathbb{K}_\Omega(B_{2R}(x_0))$,

$$\frac{\mu(\{x \in B_{R/2}(x_0) \mid u(x) > N^k\})}{\mu(B_{R/2}(x_0))} \leq \varrho^k \inf_{B_R(x_0)} u, \quad k \in \mathbb{N} \tag{4.1.3}$$

Lemma 79. *Let $\mathbb{K}_\Omega$ be closed under multiplication by positive constants. The inequality (4.1.3) is equivalent to the following inequality:*

$$\sup_{t>0} t \left(\frac{\mu(\{x \in B_{R/2}(x_0) : u(x) > t\})}{\mu(B_{R/2}(x_0))} \right)^{1/\beta} \leq C_0 \left(\inf_{B_R(x_0)} u \right)^{1/\beta}, \tag{4.1.4}$$

for some constants $C_0 > 0$ and $\beta > 0$. Moreover, the pairs of constants (N, ϱ) and (C_0, β) depend only on each other.

Proof. First, we assume (4.1.3) and prove (4.1.4). Having had $N > 1$ and $\varrho \in (0,1)$ in our hands, we choose $C_0 = N > 0$ and $\beta = \frac{\ln(1/\varrho)}{2 \ln N} > 0$. That is,

$$C_0 = N, \quad \varrho = \frac{1}{N^{2\beta}}. \tag{4.1.5}$$

Now, given $t \geq N$, there exists $k_0 \in \mathbb{N}$ such that, upon using (4.1.5), we obtain

$$N^{k_0} \leq t \leq \frac{C_0}{\varrho^{k_0/\beta}} = N^{2k_0+1}. \tag{4.1.6}$$

Hence, by (4.1.3) and (4.1.6), we get

$$\begin{aligned}
\mu(\{x \in B_{R/2}(x_0) \mid u(x) > t\}) &\leq \mu(\{x \in B_{R/2}(x_0) \mid u(x) > N^{k_0}\}) \\
&\leq \varrho^{k_0} \mu(B_{R/2}(x_0)) \inf_{B_R(x_0)} u \\
&\leq \frac{C_0^\beta}{t^\beta} \mu(B_{R/2}(x_0)) \inf_{B_R(x_0)} u,
\end{aligned}$$

which yields

$$\sup_{t\geq N} t\left(\frac{\mu(\{x\in B_{R/2}(x_0) : u(x)>t\})}{\mu(B_{R/2}(x_0))}\right)^{1/\beta} < C_0\left(\inf_{B_R(x_0)} u\right)^{1/\beta}. \tag{4.1.7}$$

On the other hand, we always have

$$\sup_{0<t<N} t\left(\frac{\mu(\{x\in B_{R/2}(x_0) : u(x)>t\})}{\mu(B_{R/2}(x_0))}\right)^{1/\beta} < N(1)^{1/\beta} = C_0,$$

which, upon replacing u by $\frac{u}{\left(\inf_{B_R(x_0)} u\right)^{1/\beta}}$ and t by $\frac{t}{\left(\inf_{B_R(x_0)} u\right)^{1/\beta}}$, yields

$$\sup_{0<t<N} t\left(\frac{\mu(\{x\in B_{R/2}(x_0) : u(x)>t\})}{\mu(B_{R/2}(x_0))}\right)^{1/\beta} < C_0\left(\inf_{B_R(x_0)} u\right)^{1/\beta}. \tag{4.1.8}$$

Finally, combining (4.1.7) and (4.1.8) yields (4.1.4). Conversely, let us prove that (4.1.4) implies (4.1.3). Having had $C_0>0$ and $\beta>0$ in our hands, we choose $N>1$ and $\varrho\in(0,1)$ in such a way that, for every $k\in\mathbb{N}$, we have

$$\frac{C_0}{\varrho^{k/\beta}} \leq N^k. \tag{4.1.9}$$

Indeed, (4.1.9) can be easily accomplished as follows in the two different cases:
If $C_0\in(0,1)$, we set $\varrho := C_0$ and $N := \frac{1}{\varrho^{1/\beta}}$. In this case, (4.1.9) is equivalent to $C_0N^k\leq N^k$, which is certainly true for every $k\in\mathbb{N}$ since $C_0\in(0,1)$. And, if $C_0>1$, then set $N:=C_0^2$ and $\varrho := \frac{1}{N^{\beta/2}}$. In this case, (4.1.9) is equivalent to $C_0^{k+1}\leq C_0^{2k}$, which again is true for every $k\in\mathbb{N}$. Now, by (4.1.9), given any $k\in\mathbb{N}$, there exists $t_0>0$ such that

$$\frac{C_0}{\varrho^{k/\beta}} \leq t_0 \leq N^k. \tag{4.1.10}$$

Now, by (4.1.10) and (4.1.4), we obtain

$$\begin{aligned}\frac{\mu(\{x\in B_{R/2}(x_0) \mid u(x)>N^k\})}{\mu(B_{R/2}(x_0))} &\leq \frac{\mu(\{x\in B_{R/2}(x_0) \mid u(x)>t_0\})}{\mu(B_{R/2}(x_0))}\\ &\leq \frac{C_0^\beta}{t_0^\beta}\inf_{B_R(x_0)} u\\ &\leq \varrho^k \inf_{B_R(x_0)} u,\end{aligned}$$

which is exactly (4.1.3). □

Lemma 80. *Let $\mathbb{K}_\Omega$ be closed under multiplication by positive constants. If*

$$\sup_{t>0} t \left(\frac{\mu(\{x \in B_{R/2}(x_0) : u(x) > t\})}{\mu(B_{R/2}(x_0))} \right)^{1/\beta} \leq C_0 \left(\inf_{B_{R/2}(x_0)} u \right)^{1/\beta}. \tag{4.1.11}$$

then, for every $\delta \in (0, \beta)$, $u^\delta \in A_1(\mathcal{B}_\Omega)$.

Proof. First, we consider the case when $\inf\limits_{B_{R/2}(x_0)} u \leq 1$. Then, (4.1.11) implies, for every $t > 0$

$$\frac{\mu(\{x \in B_{R/2}(x_0) : u(x) > t\})}{\mu(B_{R/2}(x_0))} \leq C_0^\beta t^{-\beta},$$

which holds, in particular, for every $t \geq 1$. Also, for every $t > 0$,

$$\frac{\mu(\{x \in B_{R/2}(x_0) : u(x) > t\})}{\mu(B_{R/2}(x_0))} \leq 1,$$

which holds, in particular, for every $0 < t < 1$. Using these inequalities,

$$\begin{aligned} \frac{1}{\mu(B_{R/2}(x_0))} \int_{B_{R/2}(x_0)} u^\delta \, d\mu &= \frac{\delta}{\mu(B_{R/2}(x_0))} \int_0^\infty t^{\delta-1} \mu(\{x \in B_{R/2}(x_0) : u(x) > t\}) \, dt \\ &\leq \delta C_0^\beta \int_1^\infty t^{\delta-1-\beta} \, dt + \delta C_0^\beta \int_0^1 t^{\delta-1} \, dt, \end{aligned}$$

which, upon choosing $\delta \in (0, \beta)$, yields

$$\frac{1}{\mu(B_{R/2}(x_0))} \int_{B_{R/2}(x_0)} u^\delta \, d\mu \leq C_0'(\delta, C_0, \beta). \tag{4.1.12}$$

Finally, replacing u by $\frac{u}{\inf_{B_{R/2}(x_0)} u}$ in (4.1.12), we obtain, for any u,

$$\frac{1}{\mu(B_{R/2}(x_0))} \int_{B_{R/2}(x_0)} u^\delta \, d\mu \leq C_0' \inf_{B_{R/2}(x_0)} u^\delta,$$

which, by definition, means $u^\delta \in A_1(\mathcal{B}_\Omega)$. ∎

Proposition 81. *Let $\mathbb{K}_\Omega$ be closed under multiplication by positive constants. The power-like decay property is equivalent to the weak-Harnack property for $\mathbb{K}_\Omega$.*

Proof. By Lemma 79, the alternative definition of the power-like decay property is (4.1.4), which trivially implies (4.1.11). Then, by Lemma 80, there exists a $\delta > 0$ such that for every $u \in \mathbb{K}_\Omega$, $u^\delta \in A_1(\mathcal{B}_\Omega)$, which is a weak Harnack property. Conversely, assume the weak Harnack property (4.1.2). Then, using the doubling property for μ, we have

$$\begin{aligned}
C'_H \inf_{B_R} u^\delta &\geq \frac{C'_H}{\mu(B_R)} \int_{B_R} u^\delta \, d\mu \\
&\geq \frac{C'_H}{C_D \mu(B_{R/2})} \int_{B_{R/2}} u^\delta \, d\mu \\
&\geq \frac{C'_H}{C_D \mu(B_{R/2})} \int_{\{x \in B_{R/2} | u(x) > t\}} u^\delta \, d\mu \\
&\geq \frac{C'_H t^\delta}{C_D \mu(B_{R/2})} \mu(\{x \in B_{R/2} \mid u(x) > t\}) \\
&= \frac{C'_H t^\delta}{C_0^\delta} \frac{\mu(\{x \in B_{R/2} \mid u(x) > t\})}{\mu(B_{R/2})},
\end{aligned}$$

which yields (4.1.4), an equivalent definition of the power-like decay property due to Lemma 79.

□

4.2 A modified Krylov-Safonov's approach to Harnack

The most important portion of the dissertation, this section contains the statement and the proof of our novel axiomatic approach to Harnack's inequality.

Our main result is encoded in the following theorem:

Theorem 82. *Let (X, d, μ) be a doubling quasi-metric space such that continuous functions are dense in $L^1(X, d\mu)$. Suppose that $\mathbb{K}_\Omega$ is closed under multiplication by positive constants, and possesses the critical density property with constants M and ε. Also, assume that whenever $u \in \mathbb{K}_\Omega(B_R(x_0))$ and $\lambda \geq u$ in $B_R(x_0)$ then $\lambda - u \in \mathbb{K}_\Omega(B_R(x_0))$. If, in addition, there exists $\varrho > 0$ such that whenever $u \in \mathbb{K}_\Omega$, u^ϱ is a doubling weight, that is, there exists a constant $C_D \geq 1$ such that*

$$\int_{B_{2r}(x)} u^\varrho \, d\mu \leq C_D \int_{B_r(x)} u^\varrho \, d\mu, \tag{4.2.13}$$

for all $B_r(x)$ with $B_{8Kr}(x) \subset\subset \Omega$, then $\mathbb{K}_\Omega$ satisfies the Harnack property with constant C_H depending only on ε, M, C_D, and the doubling quasi-metric constants K and C_μ.

The proof of Theorem 82 has been divided into three parts given in three subsections.

4.2.1 A classical result in the setting of metric spaces: $\cup_{s>1} RH_s \subset \cup_{p>1} A_p$

In this section, we will state and proof a result we require in proving Theorem 82 in the form of a lemma. This lemma belongs to the folklore of real analysis in spaces of homogeneous type. However, we have been unable to locate a proof. For the sake of completeness we sketch a proof that involves classical A_p techniques (see, for instance, [32], p. 269, or [56], p. 212) as well as some modern ideas on dyadic real analysis in spaces of homogeneous type developed by Aimar, Bernardis, and Iaffei in [2].

In a space of homogeneous type (X, d, μ), there exists a family of M. Christ's dyadic cubes $\mathcal{D} = \cup_{n\in\mathbb{Z}}\{Q_k^n : k \in \mathfrak{I}_n\}$, where $\mathfrak{I}_n$ is a countable set of indices, satisfying, in particular, the following properties (see [2] for the proofs of these as well as other more we have omitted here):

(i) For every $n \in \mathbb{Z}$, $\{Q_k^n : k \in \mathfrak{I}_n\}$ is a disjoint collection of n-generation cubes.

(ii) For every $m, n \in \mathbb{Z}, k \in \mathfrak{I}_m$ and $l \in \mathfrak{I}_n$, if $m < n$ then either $Q_l^n \subset Q_k^m$ or $Q_l^n \cap Q_k^m = \emptyset$.

(iii) For every $n \in \mathbb{Z}$, $\mu(X \setminus \cup_{k\in\mathfrak{I}_n} Q_k^n) = 0$.

(iv) $\mu(X) < \infty$ if and only if there exist $n \in \mathbb{Z}$ and $k \in \mathfrak{I}_n$ such that $X = Q_k^n$.

(v) There exist geometric constants $0 < \xi < 1, C_1 > 0$ and $a > 0$ such that for every $Q := Q_k^n \in \mathcal{D}$, there exists a ball $B_{a\xi^n}^Q =: B^Q$ of radius $a\xi^n$ such that $B^Q \subset Q \subset C_1 B^Q$. Note the definition $cB_r(x_0) := \{y \in X \mid d(y, x_0) < cr\}, c \geq 0$.

Lemma 83. *Let (Y, d, μ) be a space of homogeneous type. Define the weighted dyadic maximal function*

$$\mathcal{M}^{dyad}(f)(x) = \sup_{\substack{Q \in \mathcal{D} \\ x \in Q}} \frac{1}{w(Q)} \int_Q |f(y)| w(y)\, d\mu,$$

for $x \in \cup_{Q \in \mathcal{D}} Q$ and $\mathcal{M}^{dyad}(f)(x) = 0$ for $x \in Y \setminus \cup_{Q \in \mathcal{D}} Q$. Then for every $\lambda > 0$ and for every $f \in L^1(Y, w\, d\mu)$, the following hold:

(a) There exists a disjoint family $\mathcal{F} \subset \mathcal{D}$ such that

$$\{x \in Y \mid \mathcal{M}^{dyad}(f)(x) > \lambda\} = \cup_{Q \in \mathcal{F}} Q.$$

Moreover, if m is an integer such that for every $Q \in \mathcal{D}$ and its parent $\tilde{Q} \in \mathcal{D}, w(\tilde{Q}) \leq 2^m w(Q)$, then for every $Q \in \mathcal{F}$,

$$\lambda < \frac{1}{w(Q)} \int_Q |f(y)| w(y)\, d\mu \leq 2^m \lambda.$$

(b) The weak-type $(1, 1)$ inequality:

$$w(\{x \in Y \mid \mathcal{M}^{dyad}(f)(x) > \lambda\}) \leq \frac{1}{\lambda} \int_Y |f(y)| w(y)\, d\mu.$$

(c) If, in addition, continuous functions are dense in $L^1(Y, w\, d\mu)$, then

$$|f(x)| \leq \mathcal{M}^{dyad}(f)(x), w\text{-a.e.} x \in Y.$$

Proof. If $\lambda < \frac{1}{w(Y)} \int_Y |f(y)| w(y)\, d\mu$, then $w(Y) < \infty$, which, by property (iv) of M. Christ's dyadic cubes, is true if and only if there exists $Q' \in \mathcal{D}$ such that $Q' = Y$. In this case, we set $\mathcal{F} = \{Q'\}$. Suppose the other case, i.e., $\frac{1}{w(Q)} \int_Q |f(y)| w(y)\, d\mu \leq \lambda$. Set $\mathcal{H} = \{Q \in \mathcal{D} \mid \lambda < \frac{1}{w(Q)} \int_Q |f(y)| w(y)\, d\mu\}$. If $\mathcal{H} = \emptyset$, then $\mathcal{F} = \emptyset$. Otherwise, we claim there exists a maximal $Q \in \mathcal{D}$ such that $Q \in \mathcal{H}$. If $w(Y) < \infty$, this is obvious since every $Q \in \mathcal{D}$ is bounded and there exists a maximal diameter of these cubes. In the unbounded case, this is true because as $\operatorname{diam} Q \to \infty$, the expression

$$\frac{1}{w(Q)} \int_Q |f(y)| w(y)\, d\mu \leq \frac{\|f\|_{L^1(Y, w\, d\mu)}}{w(Q)} \to 0$$

which is not possible for $Q \in \mathcal{H}$, as it cannot go lower than $\lambda > 0$. Now, part (a) follows by setting $\mathcal{F} = \{Q \in \mathcal{H} \mid Q \text{ is maximal}\}$. Indeed, for every $Q \in \mathcal{D}$ and its parent $\tilde{Q} \in \mathcal{D}$, by maximality,

$$\lambda < \frac{1}{w(Q)} \int_Q |f(y)|w(y)\, d\mu \leq \frac{w(\tilde{Q})}{w(Q)} \frac{1}{w(\tilde{Q})} \int_{\tilde{Q}} |f(y)|w(y)\, d\mu \leq 2^m \lambda.$$

Using part (a), there exists a disjoint family $\mathcal{F} \subset \mathcal{D}$ such that

$$w(\{x \in Y \mid \mathcal{M}^{dyad}(f)(x) > \lambda\}) = \sum_{Q \in \mathcal{F}} w(Q) \leq \sum_{Q \in \mathcal{F}} \frac{1}{\lambda} \int_Q |f(y)|w(y)\, d\mu,$$

which is part (b).

Finally, for part (c), observe that $T_Q(f) = \frac{1}{w(Q)} \int_Q f(y)w(y)\, d\mu$ is a linear operator on $L^1(Y, w\, d\mu)$ and $\mathcal{M}^{dyad}$ defined by $\mathcal{M}^{dyad}(f)(x) = \sup\limits_{x \in Q \in \mathcal{D}} T_Q(|f|)(x)$ is weak $(1,1)$. Then, by Theorem 2.2, p. 27 in [23], the set

$$\mathcal{T} = \{f \in L^1(Y, w\, d\mu) \mid \lim_{\substack{\operatorname{diam} Q \to 0 \\ x \in Q}} T_Q(|f|)(x) = |f(x)| \text{ } w\text{-}a.e.\}$$

is closed in $L^1(Y, w\, d\mu)$. But, every continuous function is clearly in $\mathcal{T}$. So, the assumption that continuous functions are dense in $L^1(Y, w\, d\mu)$ implies, for every $f \in L^1(Y, w\, d\mu)$ and for w-a.e. $x \in Y$,

$$|f(x)| = \lim_{\substack{\operatorname{diam} Q \to 0 \\ x \in Q}} T_Q(|f|)(x) \leq \mathcal{M}^{dyad}(f)(x).$$

$\square$

Lemma 84. *Let (Y, d, μ) be a space of homogeneous type and $w \in RH_s$, for some $s > 1$. Then the following hold:*

(i) There exist $\gamma, \delta \in (0,1)$ such that for every $Q \in \mathcal{D}$ and μ-measurable $E \subset Q$,

$$w(E) \leq \gamma w(Q) \Rightarrow \mu(E) \leq \delta \mu(Q). \tag{4.2.14}$$

(ii) w is doubling, i.e., there exists a geometric constant $C_2 > 0$ such that for every $Q \in \mathcal{D}$ and its parent $\tilde{Q}, w(\tilde{Q}) \leq C_2 w(Q)$.

Proof. First we assert that it suffices to prove the above results for every ball B instead of every cube $Q \in \mathcal{D}$ because given any $Q \in \mathcal{D}$, say, of generation $j \in \mathbb{Z}$, and its parent $\tilde{Q}$, by property (v) of M. Christ's dyadic cubes, there exist balls B^Q and $B^{\tilde{Q}}$ of radii $a\xi^j$ and $a\xi^{j-1}$ respectively such that $B^Q \subset Q \subset C_1 B^Q$ and $B^Q \subset Q \subset \tilde{Q} \subset C_1 B^{\tilde{Q}}$. Indeed, the assertion for part (i) is straightforward and the one for part (ii) follows by virtue of Lemma 23 since the ratio of the radii of B^Q and $C_1 B^{\tilde{Q}}$ is a geometric constant independent of Q.

Using Hölder's inequality and reverse Hölder's inequality, we have

$$\frac{w(E)}{w(B)} \leq [w]_{RH_s} \left(\frac{\mu(E)}{\mu(B)} \right)^{1/s'},$$

which implies there exists a pair $\tilde{\gamma}, \tilde{\delta} \in (0,1)$ such that for every ball B and μ-measurable $E \subset B$,

$$\mu(E) \leq \tilde{\delta}\mu(B) \Rightarrow w(E) \leq \tilde{\gamma} w(B). \tag{4.2.15}$$

Choosing $B \setminus E$ as E in (4.2.15), this becomes

$$(1-\tilde{\delta})\mu(B) \leq \mu(E) \Rightarrow (1-\tilde{\gamma})w(B) \leq w(E),$$

which is equivalent to

$$(1-\tilde{\delta})\mu(B) < \mu(E) \Rightarrow (1-\tilde{\gamma})w(B) < w(E), \tag{4.2.16}$$

for possibly a new pair of constants $\tilde{\gamma}, \tilde{\delta} \in (0,1)$. This implies (4.2.14) with constants $\gamma := (1-\tilde{\gamma}), \delta := (1-\tilde{\delta}) \in (0,1)$ for balls B instead of cubes Q and part (i) is proved.

Since μ is doubling, using Lemma 23, we can choose a $k \in \mathbb{N}$ such that for every ball B, we have $\mu\left(\left(\frac{1}{2}\right)^{1/k} B\right) \geq C_\mu \left(\frac{1}{2}\right)^{\zeta/k} \mu(B) > (1-\tilde{\delta})\mu(B)$. Then, (4.2.16) yields $w\left(\left(\frac{1}{2}\right)^{1/k} B\right) > (1-\tilde{\gamma})w(B)$. Thus, it is possible to choose a constant $C_w > 1$ such that for every ball B,

$$w\left(\frac{1}{2}B\right) = w\left(\left(\frac{1}{2}\right)^{k/k} B\right) \geq (1-\tilde{\gamma})^k w(B) \geq \frac{1}{C_w} w(B),$$

which is the doubling condition for w with respect to balls. □

Lemma 85. *Let (Y, d, μ) be a space of homogeneous type such that continuous functions are dense in $L^1(Y, d\mu)$. Suppose that $w \in RH_s$ for some $s > 1$. Then, there exists $p > 1$, depending only on the RH_s-characteristic of w and geometric constants, such that $w \in A_p$.*

Proof. Fix $Q_0 \in \mathcal{D}$, and set $d\mu_0 := \frac{1}{\mu(Q_0)} d\mu$ and $w_0(x) := \frac{1}{w(Q_0)} w(x)$, so that $\mu_0(Q_0) = w_0(Q_0) = 1$. Notice that $d\mu_0$ has doubling constant uniform in Q_0 and $w_0 \in RH_s$ also with constant uniform in Q_0.

We will prove $w_0 \in A_p$ for some $p > 1$ by showing the existence of a $r = p' > 1$ and a constant C_3 uniform in Q_0 such that

$$\int_{Q_0} \mu(Q_0)^{1-r} w_0^{1-r} \, d\mu_0 \leq C_3. \tag{4.2.17}$$

Indeed, using the definitions of μ_0 and w_0, for every dyadic cube Q_0, (4.2.17) yields

$$\frac{1}{\mu(Q_0)} \int_{Q_0} w \, d\mu \left(\frac{1}{\mu(Q_0)} \int_{Q_0} w^{1-r} \, d\mu \right)^{\frac{1}{r-1}} \leq C_3,$$

which, by definition, says that w belongs to the dyadic A_p class with $p = r'$. Since w is doubling, by Theorem 4.1 in [2], we finally obtain $w \in A_p$, as desired. Next, we proceed to establish (4.2.17).

Define the dyadic maximal function

$$\mathcal{M}_0^{dyad}(f)(x) = \sup_{\substack{Q \in \mathcal{D} \\ x \in Q}} \frac{1}{w_0(Q)} \int_Q |f(y)| w_0(y) \, d\mu_0,$$

for $x \in \cup_{Q \in \mathcal{D}} Q$ and $\mathcal{M}_0^{dyad}(f)(x) = 0$ for $x \in Y \setminus \cup_{Q \in \mathcal{D}} Q$. Lemma 83(c) implies

$$|f(x)| \leq \mathcal{M}_0^{dyad}(f)(x), \quad w_0\text{-a.e.}\, x \in Y. \tag{4.2.18}$$

For $k \in \mathbb{N}_0$, set

$$E^k := \{x \in Q_0 : \mathcal{M}_0^{dyad}(f)(x) > 2^{kN}\},$$

where $N \in \mathbb{N}_0$ is to be fixed later. By Lemma 83(a) with $\lambda = 2^{Nk}$, there exists a disjoint family $\mathcal{F}_k \subset \mathcal{D}$ such that $E^k = \cup_{Q' \in \mathcal{F}_k} Q'$ and for every $Q' \in \mathcal{F}_k$,

$$2^{Nk} < \frac{1}{w_0(Q')} \int_{Q'} |f(y)| w_0(y) \, d\mu_0. \tag{4.2.19}$$

Again, by Lemma 84(ii) and Lemma 83(a), there exists a disjoint family $\mathcal{F}_{k-1} \subset \mathcal{D}$ such that $E^{k-1} = \cup_{Q \in \mathcal{F}_{k-1}} Q$ and for every $Q \in \mathcal{F}_{k-1}$,

$$\frac{1}{w_0(Q)} \int_Q |f(y)| w_0(y)\, d\mu_0 \leq 2^m 2^{N(k-1)}, \tag{4.2.20}$$

for some $m \in \mathbb{N}$ satisfying $C_2 \leq 2^m$. Fix an arbitrary $Q \in \mathcal{F}_{k-1}$. Note, from property (ii) of M. Christ's dyadic cubes, that for every $Q' \in \mathcal{F}_k$, either $Q' \cap Q = Q'$ or $Q' \cap Q = \emptyset$. If $Q' \cap Q = \emptyset$, then we trivially have

$$2^{Nk} w_0(Q' \cap Q) \leq \int_Q |f(y)| w_0(y)\, d\mu_0. \tag{4.2.21}$$

In the case $Q' \cap Q = Q'$, from (4.2.19),

$$2^{Nk} w_0(Q' \cap Q) < \int_{Q' \cap Q} |f(y)| w_0(y)\, d\mu_0 \leq \int_Q |f(y)| w_0(y)\, d\mu_0. \tag{4.2.22}$$

Summing over $Q' \in \mathcal{F}_k$, from (4.2.21) and (4.2.22), we have, for every $Q \in \mathcal{F}_{k-1}$,

$$2^{Nk} w_0(E^k \cap Q) \leq \int_Q |f(y)| w_0(y)\, d\mu_0,$$

which, upon using (4.2.20), yields

$$w_0(E^k \cap Q) \leq \gamma w_0(Q) \tag{4.2.23}$$

after choosing $N \in \mathbb{N}_0$ so that $\frac{2^m}{\gamma} \leq 2^N$, where $\gamma \in (0,1)$ is the constant in Lemma 84 (i). As a consequence of this lemma, there exists a $\delta \in (0,1)$ such that $\mu_0(E^k) \leq \delta \mu_0(E^{k-1}) \leq \delta^k \mu_0(Q_0) = \delta^k$ for each $k \in \mathbb{N}_0$.

Now, for $r > 1$ to be fixed later, choosing $f = \mu(Q_0)^{-1} w_0^{-1} \chi_{Q_0}$ and using (4.2.18), we can write

$$\begin{aligned}
&\int_{Q_0} \mu(Q_0)^{1-r} w_0^{1-r}\, d\mu_0 = \int_{Q_0} f^{r-1}\, d\mu_0 \leq \int_{Q_0} \mathcal{M}_0^{dyad}(f)^{r-1}\, d\mu_0 \\
&\leq \int_{Q_0 \cap \{\mathcal{M}_0^{dyad}(f) \leq 1\}} \mathcal{M}_0^{dyad}(f)^{r-1}\, d\mu_0 + \sum_{k \in \mathbb{N}_0} \int_{E^k \setminus E^{k+1}} \mathcal{M}_0^{dyad}(f)^{r-1}\, d\mu_0 \\
&\leq \mu_0(Q_0) + \sum_{k \in \mathbb{N}_0} 2^{N(k+1)(r-1)} \mu_0(E^k) \leq 1 + \sum_{k \in \mathbb{N}_0} 2^{N(k+1)(r-1)} \delta^k =: C_3 < \infty,
\end{aligned}$$

provided that r is close enough to 1, which proves (4.2.17) and we are done. $\square$

4.2.2 The critical density property implies a RH_∞^{weak} property

The proof of Theorem 82 will follow from the interaction between the reverse class RH_∞ and the critical density property. As a first step toward's Harnack's inequality, in this section, we establish a local-boundedness type result also known as a RH_∞^{weak} property via the critical density property. As most of the recent literature on the critical-density approach to Harnack's inequality, this section heavily relies on the techniques developed by L. Caffarelli in [12] (see, in particular, Lemma 5 in [12]). The idea of replacing infima with averages can be traced back to [26], p. 263.

Theorem 86. *Let (X, d, μ) be a doubling quasi-metric space and $\Omega \subset X$ an open subset. Assume that $\mathbb{K}_\Omega$ has the following properties:*

(i) It is closed under multiplication by small constants (see Definition 67).

(ii) It possesses the critical density property with constants $M > 1$ and $\varepsilon \in (0,1)$.

(iii) Whenever $u \leq \lambda$ in $B_R(x_0)$, then $\lambda - u \in \mathbb{K}_\Omega(B_R(x_0))$.

Let $u \geq 0$ in $B_{8KR}(z_0)$ be such that

$$\frac{1}{\mu(B_{4KR}(z_0))} \int_{B_{4KR}(z_0)} u \, d\mu \leq 1. \tag{4.2.24}$$

Let ν be the structural constant defined by

$$\nu := \frac{2M}{2M-1} > 1.$$

If $x_0 \in B_R(z_0)$ and $\rho < R$ satisfy

$$u(x_0) > \nu^{j-1} M, \tag{4.2.25}$$

with,

$$\mu(B_\rho(x_0)) \geq \frac{2C_\mu (4K)^\zeta}{\nu^j (1-\varepsilon)} \mu(B_R(z_0)) \tag{4.2.26}$$

for some $j \in \mathbb{N}$ such that

$$\nu^j M - \nu^{j-1} M > 1, \tag{4.2.27}$$

where K is the quasi-triangle constant and $\zeta := \log_2 C_\mu$ is as in Lemma 23, then

$$\sup_{B_\rho(x_0)} u > \nu^j M.$$

Proof. Assume, by contradiction,

$$\sup_{B_\rho(x_0)} u \leq \nu^j M. \tag{4.2.28}$$

Set

$$S_1 := \{x \in B_{2KR}(z_0) \mid u(x) \geq \frac{\nu^j M}{2}\}$$

and

$$S_2 := \{x \in B_\rho(x_0) \mid w(x) \geq M\},$$

where

$$w(x) := \frac{\nu^j M - u(x)}{\nu^j M - \nu^{j-1} M}.$$

Claim 87. *The following assertions finish off the proof of Theorem 86:*

(1) $B_\rho(x_0) \subset S_1 \cup S_2$.

(2) $\mu(S_1) \leq \frac{2C_\mu (4K)^\zeta}{\nu^j} \mu(B_R(z_0))$.

(3) $\mu(S_2) < \varepsilon \mu(B_\rho(x_0))$.

Indeed, Claim 87 yields

$$\mu(B_\rho(x_0)) < \frac{2C_\mu (4K)^\zeta}{\nu^j} \mu(B_R(z_0)) + \varepsilon \mu(B_\rho(x_0)),$$

which contradicts (4.2.26), and the theorem is proved.

Proof of (1) of Claim 87:

Let $x \in B_\rho(x_0)$. Since $\rho < R$ and $x_0 \in B_R(z_0)$,

$$B_\rho(x_0) \subset B_{2KR}(z_0).$$

- If $u(x) \geq \frac{\nu^j M}{2}$, $x \in S_1 := \{x \in B_{2KR}(z_0) \mid u(x) \geq \frac{\nu^j M}{2}\}$.

- If $u(x) < \frac{\nu^j M}{2}$, then

$$w(x) := \frac{\nu^j M - u(x)}{\nu^j M - \nu^{j-1} M} > \frac{\nu}{2(\nu-1)} =: M,$$

which implies

$$x \in S_2 := \{x \in B_\rho(x_0) \mid w(x) \geq M\}.$$

Proof of (2) of Claim 87:

Choose k such that $M^{k-1} \leq \frac{\nu^j}{2} < M^k$. Then

$$\begin{aligned} S_1 &:= \{x \in B_{2KR}(z_0) \mid u(x) \geq \frac{\nu^j M}{2}\} \\ &\subset \{x \in B_{2KR}(z_0) \mid u(x) \geq M^k\} =: E. \end{aligned}$$

From (4.2.24),

$$1 \geq \frac{1}{\mu(B_{4KR}(z_0))} \int_E u(x)\, dx \geq M^k \frac{\mu(E)}{\mu(B_{4KR}(z_0))}.$$

Hence, using Lemma 23,

$$\begin{aligned} \mu(S_1) \leq \mu(E) \leq \frac{1}{M^k} \mu(B_{4KR}(z_0)) &\leq \frac{C_\mu (4K)^\zeta}{M^k} \mu(B_R(z_0)) \\ &\leq \frac{2C_\mu (4K)^\zeta}{\nu^j} \mu(B_R(z_0)). \end{aligned}$$

Proof of (3) of Claim 87:

By (ii), (iii) and (4.2.28), $\nu^j M - u(x)$ has the critical density property in $B_\rho(x_0)$. Then, by (i) and (4.2.27),

$$w(x) := \frac{\nu^j M - u(x)}{\nu^j M - \nu^{j-1} M}$$

also has the critical density property in $B_\rho(x_0)$. On the other hand, by (4.2.25), $w(x_0) \leq 1$, which yields

$$\inf_{B_{\rho/2}(x_0)} w \leq 1.$$

This means w did not reach the critical density in $B_\rho(x_0)$, i.e.,

$$\mu(S_2) := \mu(\{x \in B_\rho(x_0) \mid w(x) \geq M\}) < \varepsilon \mu(B_\rho(x_0)).$$

$\square$

The following lemma will be needed in the proof of Theorem 89.

Lemma 88. *Suppose $R > 0$, $\alpha \in (0,1)$, $\{d_n\}_{n\in\mathbb{N}_0} \subset (0,\infty)$, $\{\theta_n\}_{n\in\mathbb{N}_0} \subset (0,R)$ such that*

$$d_{n+1} \le d_n + \theta_n^\alpha (d_n + d_{n+1})^{1-\alpha}, \ d_0 \le R,$$

and

$$\theta_{n+1} < \theta_n < R, \, n \in \mathbb{N}_0,$$

then

$$d_{n+1} \le R \prod_{j=0}^{n} \left[1 + c_0 \left(\frac{\theta_j}{R}\right)^\alpha\right], \tag{4.2.29}$$

where

$$c_0 := \frac{2}{\left[1 - \left(\frac{\theta_0}{R}\right)^\alpha\right]} > 0.$$

Moreover, it follows from (4.2.29) that

$$d_{n+1} \le R \exp\left[\sum_{j=0}^{\infty} c_0 \left(\frac{\theta_j}{R}\right)^\alpha\right]. \tag{4.2.30}$$

Proof. We proceed by induction in n. For $n = 0$ we have

$$d_1 \le d_0 + \theta_0^\alpha (d_0 + d_1)^{1-\alpha}.$$

Dividing the above inequality by R and using the estimate $d_0 \le R$ yields

$$\begin{aligned} \frac{d_1}{R} &\le \frac{d_0}{R} + \left(\frac{\theta_0}{R}\right)^\alpha \left(\frac{d_0}{R} + \frac{d_1}{R}\right)^{1-\alpha} \le 1 + \left(\frac{\theta_0}{R}\right)^\alpha \left(1 + \frac{d_1}{R}\right)^{1-\alpha} \\ &\le 1 + \left(\frac{\theta_0}{R}\right)^\alpha \left(1 + \frac{d_1}{R}\right) = 1 + \left(\frac{\theta_0}{R}\right)^\alpha + \left(\frac{\theta_0}{R}\right)^\alpha \frac{d_1}{R}. \end{aligned} \tag{4.2.31}$$

Since $\left(\frac{\theta_0}{R}\right)^\alpha \in (0,1)$, using (4.2.31), we obtain

$$\begin{aligned} \frac{d_1}{R} &\le \frac{1 + \left(\frac{\theta_0}{R}\right)^\alpha}{1 - \left(\frac{\theta_0}{R}\right)^\alpha} = \frac{R^\alpha + \theta_0^\alpha}{R^\alpha - \theta_0^\alpha} = \frac{(R^\alpha - \theta_0^\alpha) + 2\theta_0^\alpha}{R^\alpha - \theta_0^\alpha} \\ &= 1 + \frac{2\theta_0^\alpha}{R^\alpha - \theta_0^\alpha} = 1 + \frac{2\theta_0^\alpha / R^\alpha}{(R^\alpha - \theta_0^\alpha)/R^\alpha} \\ &= 1 + \frac{2\left(\frac{\theta_0}{R}\right)^\alpha}{\left[1 - \left(\frac{\theta_0}{R}\right)^\alpha\right]} = 1 + c_0 \left(\frac{\theta_0}{R}\right)^\alpha. \end{aligned}$$

Thus, inequality (4.2.29) holds for $n = 0$. Assume now that it holds for all $n \leq m$. By inductive hypothesis,

$$\begin{aligned} d_{m+1} &\leq d_m + \theta_m^\alpha (d_m + d_{m+1})^{1-\alpha} \\ &\leq R \prod_{j=0}^{m-1} \left[1 + c_0 \left(\frac{\theta_j}{R}\right)^\alpha\right] + \theta_m^\alpha \left\{R \prod_{j=0}^{m-1} \left[1 + c_0 \left(\frac{\theta_j}{R}\right)^\alpha\right] + d_{m+1}\right\}^{1-\alpha}. \end{aligned}$$

Dividing the above inequality by R yields

$$\begin{aligned} \frac{d_{m+1}}{R} &\leq \prod_{j=0}^{m-1} \left[1 + c_0 \left(\frac{\theta_j}{R}\right)^\alpha\right] + \left(\frac{\theta_m}{R}\right)^\alpha \left\{\prod_{j=0}^{m-1} \left[1 + c_0 \left(\frac{\theta_j}{R}\right)^\alpha\right] + \frac{d_{m+1}}{R}\right\}^{1-\alpha} \\ &\leq \prod_{j=0}^{m-1} \left[1 + c_0 \left(\frac{\theta_j}{R}\right)^\alpha\right] + \left(\frac{\theta_m}{R}\right)^\alpha \left\{\prod_{j=0}^{m-1} \left[1 + c_0 \left(\frac{\theta_j}{R}\right)^\alpha\right] + \frac{d_{m+1}}{R}\right\}. \end{aligned} \tag{4.2.32}$$

Note that if $\varepsilon \in (0, 1)$, then $a \leq b + \varepsilon(b + a)$ implies $a \leq \left(\frac{1+\varepsilon}{1-\varepsilon}\right) b$. Applying this to (4.2.32) with $\left(\frac{\theta_m}{R}\right)^\alpha \in (0, 1)$ and using $\frac{1}{-\theta_m} \leq \frac{1}{-\theta_0}$, we obtain

$$\begin{aligned} \frac{d_{m+1}}{R} &\leq \left(\frac{1 + \left(\frac{\theta_m}{R}\right)^\alpha}{1 - \left(\frac{\theta_m}{R}\right)^\alpha}\right) \prod_{j=0}^{m-1} \left[1 + c_0 \left(\frac{\theta_j}{R}\right)^\alpha\right] = \left(\frac{R^\alpha + \theta_m^\alpha}{R^\alpha - \theta_m^\alpha}\right) \prod_{j=0}^{m-1} \left[1 + c_0 \left(\frac{\theta_j}{R}\right)^\alpha\right] \\ &= \left(1 + \frac{2\theta_m^\alpha}{R^\alpha - \theta_m^\alpha}\right) \prod_{j=0}^{m-1} \left[1 + c_0 \left(\frac{\theta_j}{R}\right)^\alpha\right] \leq \left(1 + \frac{2\theta_m^\alpha}{R^\alpha - \theta_0^\alpha}\right) \prod_{j=0}^{m-1} \left[1 + c_0 \left(\frac{\theta_j}{R}\right)^\alpha\right] \\ &\leq \left(1 + c_0 \frac{\theta_m^\alpha}{R^\alpha}\right) \prod_{j=0}^{m-1} \left[1 + c_0 \left(\frac{\theta_j}{R}\right)^\alpha\right] = \prod_{j=0}^{m} \left[1 + c_0 \left(\frac{\theta_j}{R}\right)^\alpha\right], \end{aligned}$$

which proves (4.2.29). Now, using the inequality $\ln(1 + x) \leq x$, we get

$$\prod_{j=0}^{n} \left[1 + c_0 \left(\frac{\theta_j}{R}\right)^\alpha\right] = \exp\left\{\sum_{j=0}^{n} \ln\left[1 + c_0 \left(\frac{\theta_j}{R}\right)^\alpha\right]\right\} \leq \exp\left[\sum_{j=0}^{n} c_0 \left(\frac{\theta_j}{R}\right)^\alpha\right],$$

upon using which in (4.2.29) yields (4.2.30). □

Theorem 89. *Let (X, d, μ) be a doubling quasi-metric space and $\Omega \subset X$ an open subset. Assume that $\mathbb{K}_\Omega$ has the critical density property with constants $M > 1$ and $\varepsilon \in (0, 1)$; and whenever $u \leq \lambda$ in $B_R(x_0)$, then $\lambda - u \in \mathbb{K}_\Omega(B_R(x_0))$. There exists a positive constant C_4, depending only on M, ε, and geometric constants, such that if $u \geq 0$ in $B_{8K\eta R}(x_0)$, for some $\eta > 1$, is locally bounded, then the inequality*

$$\frac{1}{\mu(B_{8KR}(x_0))} \int_{B_{8KR}(x_0)} u \, d\mu \leq 1 \tag{4.2.33}$$

implies

$$\sup_{B_R(x_0)} u \leq C_4. \tag{4.2.34}$$

Proof. We will establish the theorem by proving that

$$\sup_{B_R(x_0)} u \leq \nu^{m-1} M$$

for sufficiently large m (depending only on the structure), where

$$\nu := \frac{M}{M - \frac{1}{2}} > 1. \tag{4.2.35}$$

Let us assume by contradiction that

$$\sup_{B_R(x_0)} u > \nu^{m-1} M. \tag{4.2.36}$$

The idea of the proof now is that with this sufficiently large m, (4.2.36) implies that u is unbounded in the ball $B_{2R}(x_0)$, i.e., there exists a sequence $\{x_{m+j}\}_{j\in\mathbb{N}_0}$ in $B_{2R}(x_0) \subset \Omega$ such that the following two conditions hold:

$$x_{m+j} \in B_{2R}(x_0), \;\; \forall j \in \mathbb{N}_0 \tag{4.2.37}$$

and

$$u(x_{m+j}) > \nu^{m+j-1} M, \;\; \forall j \in \mathbb{N}_0, \tag{4.2.38}$$

which imply

$$\sup_{B_{2R}(x_0)} u \geq u(x_{m+j}) > \nu^{m+j-1} M, \, \forall j \in \mathbb{N}_0.$$

This contradicts the local-boundedness of u and the theorem is proved.

Define a decreasing sequence $\{\rho_{m+j}\}_{j\in\mathbb{N}_0}$ by

$$\rho_{m+j} := C_1 \nu^{-(\frac{m+j}{\zeta})} R, \;\; j \in \mathbb{N}_0, \tag{4.2.39}$$

where $\zeta > 0$ is the geometric constant of Lemma 23, $\nu > 1$ is the constant defined in (4.2.35) and $C_1 > 0$ is a structural constant to be chosen shortly. By assumption (4.2.36), there exists

$$x_m \in B_R(x_0) \tag{4.2.40}$$

such that

$$u(x_m) > \nu^{m-1}M. \tag{4.2.41}$$

By (4.2.39), since $\nu^{1/\zeta} > 1$,

$$\frac{\rho_{m+j}}{R} = C_1\nu^{-(m+j)/\zeta} \leq C_1\nu^{-m/\zeta}, \tag{4.2.42}$$

one may choose m large enough so that

$$C_1\nu^{-(\frac{m}{\zeta})} < 1, \tag{4.2.43}$$

which yields

$$\rho_{m+j} < R < 2R, \, \forall j \in \mathbb{N}_0. \tag{4.2.44}$$

On the other hand, by (4.2.40),

$$x_m \in B_{2R}(x_0), \tag{4.2.45}$$

which, in turn, implies that

$$B_{2R}(x_0) \subset B_{4R}(x_m) \subset B_{2K2R}(x_m). \tag{4.2.46}$$

Then, by Lemma 23 with (4.2.44) and (4.2.46), we have

$$\mu(B_{\rho_m}(x_m)) \geq \frac{1}{C_\mu(2K)^\zeta}\left(\frac{\rho_m}{2R}\right)^\zeta \mu(B_{2R}(x_0)) = \frac{C_1^\zeta}{(4K)^\zeta C_\mu}\nu^{-m}\mu(B_{2R}(x_0)),$$

where, now, using (4.2.42) and choosing C_1 so that

$$\frac{C_1^\zeta}{(4K)^\zeta C_\mu} \geq \frac{2C_\mu(4K)^\zeta}{1-\epsilon},$$

we obtain

$$\mu(B_{\rho_m}(x_m)) \geq \frac{2C_\mu(4K)^\zeta}{(1-\epsilon)\nu^m}\mu(B_{2R}(x_0)). \tag{4.2.47}$$

Next, we improve the $m \in \mathbb{N}$ chosen earlier in (4.2.43) by making it even bigger so that

$$\nu^m M - \nu^{m-1}M > 1, \tag{4.2.48}$$

which, in fact, due to $\nu > 1$, implies

$$\nu^{m+j} M - \nu^{m+j-1} M > 1,\ \forall j \in \mathbb{N}_0. \tag{4.2.49}$$

Thus, (4.2.33), (4.2.45), (4.2.44), (4.2.41), (4.2.47), (4.2.48), and Theorem 86 (with x_m, m, ρ_m, x_0, and $2R$ in this proof playing the roles of x_0, j, ρ, z_0, and R, respectively, in the statement of Theorem 86) imply

$$\sup_{B_{\rho_m}(x_m)} u > \nu^m M,$$

which, in turn, implies there exists $x_{m+1} \in B_{\rho_m}(x_m)$, or, equivalently,

$$d(x_{m+1}, x_m) < \rho_m, \tag{4.2.50}$$

such that

$$u(x_{m+1}) > \nu^m M. \tag{4.2.51}$$

Now, the idea is to iterate the use of Theorem 86 to generate the sequence $\{x_{m+j}\}_{j \in \mathbb{N}_0}$. We shall do it only once by replacing m by $m+1$ in the previous use to obtain x_{m+2}. Now, to this end, (4.2.33), (4.2.44), and (4.2.49) (generic version of (4.2.48)) can be used as is as before. The inequality (4.2.41) has already been updated to (4.2.51). Our next goal is to update (4.2.45) by showing

$$x_{m+1} \in B_{2R}(x_0), \tag{4.2.52}$$

where $m \in \mathbb{N}$, chosen above to verify (4.2.43) and (4.2.48), is updated once again. Then applying Theorem 86 one more time yields

$$\sup_{B_{\rho_{m+1}}(x_{m+1})} u > \nu^{m+1} M,$$

which, in turn, implies there exists $x_{m+2} \in B_{\rho_{m+1}}(x_{m+1})$, or, equivalently,

$$d(x_{m+2}, x_{m+1}) < \rho_{m+1},$$

such that

$$u(x_{m+1}) > \nu^m M,$$

from where the next iteration of the use of Theorem 86 can be started.

Now, let us prove (4.2.52). We begin by setting

$$d_j := d(x_{m+j}, x_0), \quad j \in \mathbb{N}_0. \tag{4.2.53}$$

By Remark 22, there exist geometric constants $\beta > 0$ and $\alpha \in (0,1)$ such that

$$d_1 \leq d_0 + \beta(d(x_{m+1}, x_m))^{\alpha}(d_0 + d_1)^{1-\alpha},$$

which, upon using (4.2.50) and the decreasing sequence $\{\theta_j\}_{j\in\mathbb{N}_0}$ defined as

$$\theta_j := \beta^{1/\alpha}\rho_{m+j},\ j \in \mathbb{N}_0,$$

becomes

$$d_1 \leq d_0 + \theta_0^{\alpha}(d_0 + d_1)^{1-\alpha}. \tag{4.2.54}$$

Next, we update $m \in \mathbb{N}$ one final time so that, in addition to (4.2.43) and (4.2.48), it also verifies

$$\theta_j := \beta^{1/\alpha}\rho_{m+j} \leq \theta_0 := \beta^{1/\alpha}\rho_m < R,\ \forall j \in \mathbb{N}_0 \tag{4.2.55}$$

and

$$\exp\left[\frac{c_0\beta C_1^{\alpha}}{\nu^{(m-1)\alpha/\zeta}}\frac{1}{(\nu-1)^{\alpha/\zeta}}\right] < 2. \tag{4.2.56}$$

Now, using Lemma 88 along with (4.2.55) and (4.2.56), we have

$$\begin{aligned} d_1 &\leq \left[\sum_{j=0}^{\infty} c_0\left(\frac{\theta_j}{R}\right)^{\alpha}\right]R = \left[\sum_{j=0}^{\infty} c_0\frac{\beta C_1^{\alpha}}{\nu^{\left(\frac{m+j}{\zeta}\right)^{\alpha}}}\right]R = \left[\frac{c_0\beta C_1^{\alpha}}{\nu^{\frac{j\alpha}{\zeta}}}\sum_{j=0}^{\infty}\frac{1}{\nu^{\frac{j\alpha}{\zeta}}}\right]R \\ &= \exp\left[\frac{c_0\beta C_1^{\alpha}}{\nu^{m\alpha/\zeta}}\frac{1}{\left(1-(\frac{1}{\nu})^{\alpha/\zeta}\right)}\right] = \exp\left[\frac{c_0\beta C_1^{\alpha}}{\nu^{(m-1)\alpha/\zeta}}\frac{1}{(\nu-1)^{\alpha/\zeta}}\right] < 2R, \end{aligned}$$

which, by (4.2.53), means that

$$x_{m+1} \in B_{2R}(x_0). \tag{4.2.57}$$

In order for us to be able to apply Theorem 86 iteratively, notice that we do have

$$x_{m+j} \in B_{2R}(x_0),\ \forall j \in \mathbb{N}_0. \tag{4.2.58}$$

Indeed, very similar to what we did before, using (4.2.57), we can update (4.2.2) and (4.2.47) to

$$d_2 \leq d_1 + \theta_1^\alpha (d_1 + d_2)^{1-\alpha}$$

and

$$\mu(B_{\rho_{m+1}}(x_{m+1})) \geq \frac{2C_\mu (4K)^\zeta}{(1-\epsilon)\nu^{m+1}} \mu(B_{2R}(x_0)).$$

respectively. Iteration of the above process renders a sequence $\{x_{m+j}\}_{j=1}^\infty$ such that conditions (4.2.37) and (4.2.38) hold. □

The next corollary in the special case of (X, d) as a metric space follows from Remark 4.4 in [42]. Here, we provide its proof for the case when (X, d) is a quasi-metric space.

Corollary 90. *Suppose that (X, d) is a quasi-metric space and all the assumptions of Theorem 89 hold. Then, for every $\sigma > 0$ there exist constants $\alpha > 0$ and $c > 0$ such that for every ball $B_R(x_0) \subset \Omega$ and for all $0 < s < t < 1$*

$$\sup_{B_{sR}(x_0)} u \leq \frac{c}{(t-s)^{\alpha/\sigma}} \left(\frac{1}{\mu(B_{tR}(x_0))} \int_{B_{tR}(x_0)} u^\sigma d\mu \right)^{\frac{1}{\sigma}}. \tag{4.2.59}$$

Proof. Let ϵ be any positive constant. Then, by the definition of supremum, there exists $y \in B_{sR}(x_0)$ such that

$$\sup_{B_{sR}(x_0)} u - \epsilon \leq u(y).$$

This, together with (4.2.34), implies

$$\sup_{B_{sR}(x_0)} u \leq u(y) + \epsilon \leq \sup_{B_{\frac{(t-s)R}{16K}}(y)} u + \epsilon \leq c \fint_{B_{(t-s)R/2}(y)} u + \epsilon. \tag{4.2.60}$$

We have

$$B_{(t-s)R/2}(y) \subset B_{2tKR}(x_0). \tag{4.2.61}$$

Indeed, let $x \in B_{(t-s)R/2}(y)$. Then, by the quasi triangle inequality, one gets

$$d(x, x_0) \leq K[d(x, y) + d(y, x_0)] \leq K \left[\frac{(t-s)R}{2} + sR \right] < 2tKR,$$

which proves (4.2.61). Therefore, using inequality (1.1.17), we obtain

$$\mu(B_{(t-s)R/2}(y)) \geq C_Q \left(\frac{t-s}{t}\right)^Q \mu(B_{2tKR}(x_0)) \geq C_Q \left(\frac{t-s}{t}\right)^Q \mu(B_{tR}(x_0)),$$

where C_Q and Q are constants depending on the doubling constant of the measure μ. This, together with (4.2.60), yields

$$\begin{aligned}\sup_{B_{sR}(x_0)} u &\leq c\frac{\mu(B_{tR}(x_0))}{\mu(B_{(t-s)R/2}(y))}\fint_{B_{tR}(x_0)} u + \epsilon \leq c\frac{t^Q}{C_Q(t-s)^Q}\fint_{B_{tR}(x_0)} u + \epsilon \\ &\leq \frac{c}{C_Q}\frac{1}{(t-s)^Q}\fint_{B_{tR}(x_0)} u + \epsilon.\end{aligned}$$

Thus,

$$\sup_{B_{sR}(x_0)} u \leq \frac{c}{(t-s)^\alpha}\fint_{B_{tR}(x_0)} u, \tag{4.2.62}$$

where c and α are constants depending on the doubling constant C_Q.

Let us prove inequality (4.2.59). For $\sigma > 1$ the inequality follows from (4.2.62) and Hölder's inequality. For $0 < \sigma < 1$ we have

$$\begin{aligned}\sup_{B_{sR}(x_0)} u &\leq \frac{c}{(t-s)^\alpha}\fint_{B_{tR}(x_0)} u^{1-\sigma}u^\sigma \leq (\sup_{B_{tR}(x_0)} u)^{1-\sigma}\frac{c}{(t-s)^\alpha}\fint_{B_{tR}(x_0)} u^\sigma \\ &\leq \epsilon \sup_{B_{tR}(x_0)} u + \frac{c(\epsilon)}{(t-s)^{\alpha/\sigma}}\left(\fint_{B_{tR}(x_0)} u^\sigma\right)^{1/\sigma},\end{aligned} \tag{4.2.63}$$

where $\epsilon \in (0,1)$ and the Young's inequality was used. Now (4.2.59) is proved upon using Lemma 60 and (4.2.63). This proves the corollary.

□

4.2.3 Proof of Theorem 82: The critical density and RH_∞ properties imply Harnack's inequality

This section will finish the proof of Theorem 82 using the results established in Section 4.2.1 and Section 4.2.2. Using the visual formalism introduced in Chapter 2, the steps of the proof of Theorem 82 are illustrated in Figure 4.1 through Figure 4.4.

Step 1: By Corollary 90, $u^\sigma \in RH_\infty^{weak}$ for every $\sigma > 0$ (with structural constants).

Step 2: Choosing $\sigma := \varrho$, where ϱ is as in (7.0.3) we obtain that $u^\varrho \in RH_\infty = \cap_{s>1} RH_s$.

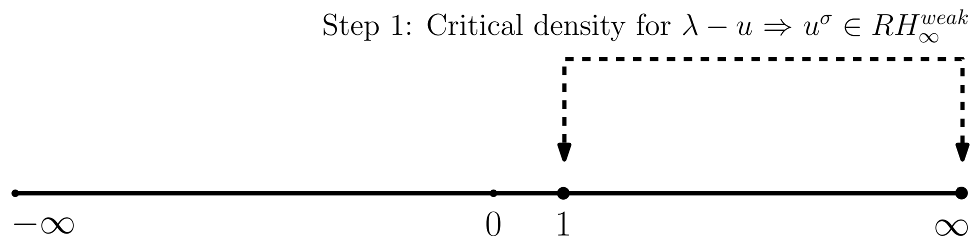


Figure 4.1: *Step 1 of modified A_1 approach.*

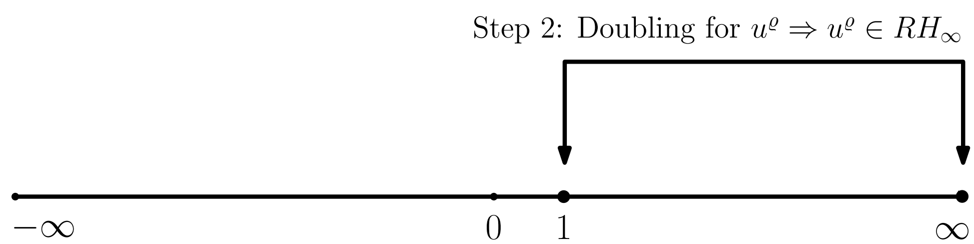


Figure 4.2: *Step 2 of modified A_1 approach.*

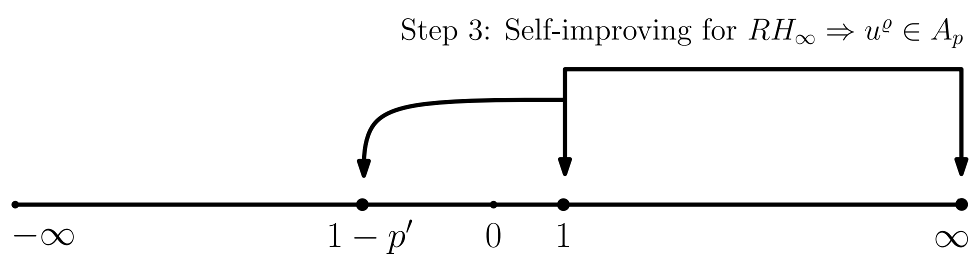


Figure 4.3: *Step 3 of modified A_1 approach.*

Step 3: By Lemma 85, due to the assumption that continuous functions are dense in $L^1(X, d\mu)$, we have $\cup_{s>1} RH_s \subset \cup_{p>1} A_p$.

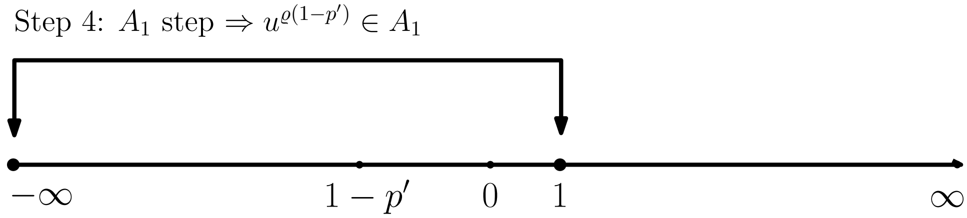


Figure 4.4: *Step 4 of modified A_1 approach.*

Step 4: $\cup_{p>1}(RH_\infty \cap A_p)^{1-p'} \subset A_1$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Hence, $u^{\varrho(1-p')} \in A_1$.

Finally, let us assume $\inf\limits_{B_{R/2}} u = 1$. Then, by the critical density property for u with constants M and ε, it follows by contraposition that u did not reach the critical density in B_R. That is,

$$\begin{aligned}
(1-\varepsilon) &\leq \frac{\mu(\{x \in B_R : u(x) < M\})}{\mu(B_R)} \\
&= \frac{\mu(\{x \in B_R : M^{\varrho(1-p')} < u(x)^{\varrho(1-p')}\})}{\mu(B_R)} \\
&\leq \frac{M^{\varrho(p'-1)}}{\mu(B_R)} \int_{B_R} u^{\varrho(1-p')}\, d\mu \\
&\leq M^{\varrho(p'-1)} [u^{\varrho(1-p')}]_{A_1} \inf_{B_R} \left(u^{\varrho(1-p')}\right) \\
&\leq M^{\varrho(p'-1)} [u^{\varrho(1-p')}]_{A_1} \inf_{B_R} \left(\frac{1}{u^{\varrho(p'-1)}}\right) \\
&\leq M^{\varrho(p'-1)} [u^{\varrho(1-p')}]_{A_1} \frac{1}{\sup\limits_{B_R} \left(u^{\varrho(p'-1)}\right)} \\
&\leq M^{\varrho(p'-1)} [u^{\varrho(1-p')}]_{A_1} \frac{1}{\left(\sup\limits_{B_R} u\right)^{\varrho(p'-1)}}
\end{aligned}$$

Hence, raising both sides to $\frac{1}{\varrho(1-p')} > 0$,

$$\sup_{B_R} u \leq M \left(\frac{[u^{\varrho(1-p')}]_{A_1}}{(1-\varepsilon)}\right)^{\frac{1}{\varrho(p'-1)}},$$

and the desired Harnack inequality is obtained. $\square$

4.3 Density of continuous functions in L^1

In our axiomatization of Harnack's inequality, the hypothesis of the density of continuous functions in $L^1(X, d\mu)$ to the structure of space of homogeneous type is something that comes as a replacement of covering lemmas or BMO properties seen in earlier axiomatic approaches to Harnack's inequality. The only place where we make use of this hypothesis is to ensure the validity of the Lebesgue Differentiation Theorem for $L^1_{loc}(X, d\mu)$-functions invoked in Section 4.2.1. The hypothesis together with the weak (1,1)-type of the maximal function [18] guarantees this fact. In this section, we explain why the assumption of the density of $C(X)$ in $L^1(X)$ is a mild hypothesis.

We have made a note in Section 1 that a consequence of Remark 22 is that a space of homogeneous type (X, d, μ) is metrizable. Then, by [52, Theorem 3.14], in locally compact metric spaces, $C_c(X)$, the set of all continuous functions on X with bounded supports, is dense in $L^p(X)$ for all $p \in [1, \infty)$.

Alternatively, a measure-theoretic criterion for density of continuous functions is given by Theorem 1.8 in [57]: if for every $\varepsilon > 0$ and every μ-measurable set $A \subset X$ with $\mu(A) < \infty$ there exists an open set $E \subset X$ such that $\mu(E \Delta A) < \varepsilon$, then the continuous functions with finite μ-measure support are dense in $L^p(X, d\mu)$ for all $p \in [1, \infty)$. Here Δ denotes the symmetric difference.

4.4 Double-ball property vs. doubling property

In a nutshell, the diagram to bear in mind for methods to prove Harnack's inequality has been the following:

$$\text{Critical density} + \text{Double-ball} = \text{ Harnack}$$
$$\text{Critical density} + \text{Doubling} = \text{ Harnack}$$

The novelty of our approach has been the replacement of the double-ball property adopted by earlier approaches with the doubling property. Two natural questions arise: Are the the double-ball property and doubling property related? In practice, when proving Harnack's inequality, should one of them be preferred over the other? In this section we show that the double-ball property and the doubling property of u as a weight are unrelated. Thus, when combined with the critical density property, either property can be used to non-trivially prove Harnack's inequality.

Let us first see that the doubling property of u as a weight does not imply the double-ball property. Let u be a continuous weight in, say, $\mathbb{R}^n$ with Lebesgue measure. Then, if u has the double-ball property we must have $u > 0$, unless $u \equiv 0$. Indeed, if there is a point $x_0 \in \mathbb{R}^n$ with $u(x_0) = 0$, by considering arbitrarily small balls B with $x_0 \in 2B \setminus B$, the conclusion follows. Hence, any (non-zero) doubling weight u that vanishes somewhere will provide an example that doubling of u does not imply double-ball property. Examples of such weights can be constructed, for instance, as $u(x) := |p(x)|$, $x \in \mathbb{R}^n$, where p is polynomial that vanishes somewhere. Since $u(x) = |p(x)|$ and p is a polynomial, we have $u \in A_\infty$, and consequently u will be a doubling weight. However, since p has a zero, by Corollary 77, it cannot have the double-ball property.

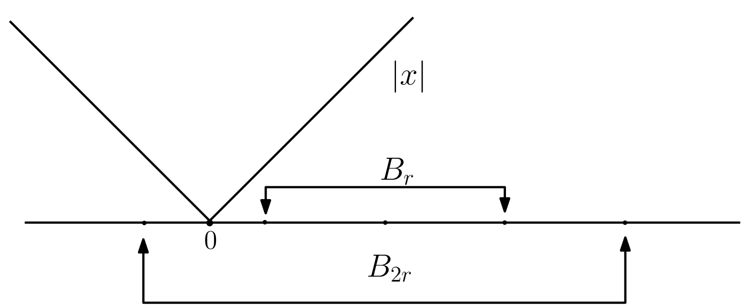


Figure 4.5: *$u(x) = |x|$ is doubling but not double-ball in a domain containing the origin.*

On the other hand, an example of a function u on the real line possessing the double-ball property but which is not doubling is provided by $u(t) := \frac{1}{1+t^2}$. Indeed, notice that for every interval (a, b) we have $\inf_{(a,b)} u = \inf_{(-m,m)} u = u(m)$, where $m := \max\{|a|, |b|\}$, the double-ball property follows from the fact that $u(2z) \simeq u(z)$ for every $z \in \mathbb{R}$. Indeed,

$$u(2z) = \frac{1}{1+4z^2} \leq \frac{1}{1+z^2} = u(z) = 4\left(\frac{1}{4+4z^2}\right) \leq 4\left(\frac{1}{1+4z^2}\right) = 4u(2z).$$

However, the weight u cannot be doubling on $\mathbb{R}$ since $u(\mathbb{R}) = \int_{-\infty}^{\infty} u(t)\,dt = \pi < \infty$ and a doubling measure in $\mathbb{R}$ cannot be both non-zero and finite (see [56] p. 39, Remark 8.6(a)).

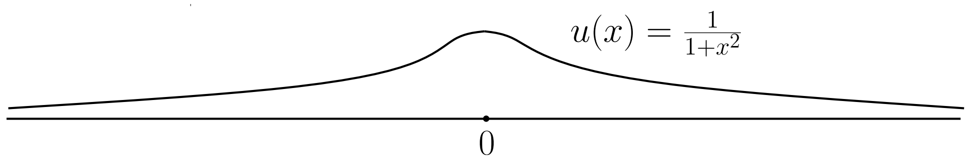


Figure 4.6: *u is double-ball in $\mathbb{R}$ with constant $\frac{1}{4}$ but not doubling since $u(\mathbb{R}) = \pi < \infty$.*

4.5 Variational vs. non-variational contexts

In Section 1, Definition 12 gave three forms of second-order elliptic operators, namely, divergence, non-divergence and adjoint based on their origin. However, based on the way their solutions are defined, operators can be divided into two broad categories: variational and non-variational. Divergence and adjoint form operators are variational operators since their solutions are defined in terms of test functions, making an explicit use of the property of integration by parts. Non-divergence form operators whose solutions are defined pointwise are non-variational operators. In this section, we analyze the two axiomatic approaches to Harnack's inequality from the viewpoint of variational and non-variational contexts.

The doubling property appears naturally as a property of non-negative supersolutions in the context of variational elliptic PDEs. For instance, in the case of divergence-form elliptic PDEs, the doubling property of (a positive power of) a non-negative weak subsolution u follows quite easily from the Poincaré inequality, an energy estimate for $\log u$, and the John-Nirenberg theorem which imply $\log u \in BMO$ and therefore $u^{q_0} \in A_2$ (for some $q_0 > 0$). Such short argument applies in wide variety of contexts, including quasi-minimizers in metric spaces, see, for instance, [7, Theorem 9.2], infinite graphs [21, Section 3], X-elliptic operators [34, Section 4, Step 2], $p(x)$-Laplacian [1, Section 4], Dirichlet forms [29, Lemma 2.3], etc. In addition, in the context of adjoint elliptic operators (which is also a variational one), the doubling property for non-negative weak supersolutions also has a very short proof, see

Lemma 2.0 in [24]. Being an integral condition, the doubling property appears to be better suited to these variational contexts.

On the other hand, in the case of non-divergence-form elliptic operators, where the variational tools are replaced by the maximum principle, the double-ball property (instead of the doubling property) seems to appear more naturally, see, for instance, Theorem 2.1.2 in [33] (or the shorter proof below) for non-divergence-form uniformly elliptic operators, and Theorem 2 in [16] for the case of the linearized Monge-Ampère operator which enjoys both a divergence and a non-divergence structure. Indeed, being a pointwise condition, the double-ball property is better suited to the maximum-principle-based contexts.

For the sake of illustration, we include proofs of the doubling property for divergence and adjoint form operators and of the double-ball property for non-divergence form operators.

Let $\Omega \subset \mathbb{R}^n$ be an open subset and that, for $x \in \Omega$, let $A(x)$ be a uniformly elliptic matrix with ellipticity constants $0 < \lambda \leq \Lambda$.

Theorem 91 (Moser, 1961)**.** *Let $A \in \mathcal{A}(\lambda, \Lambda, \Omega)$. and $\mathcal{L}u := \sum_{i,j=1}^{n} (a_{ij}(x)u_i)_j$, $x \in \Omega$. There exists a $\delta = \delta(n, \lambda, \Lambda) > 0$ such that if $u \geq 0$ is a supersolution to $\mathcal{L}u = 0$, i.e.,*

$$\sum_{i,j=1}^{n} \int_{\Omega} a_{ij} u_i \varphi_j \geq 0, \, \forall \varphi \in H_0^1(\Omega), \, \varphi \geq 0, \tag{4.5.64}$$

then u^δ is doubling, or equivalently, $w := \log u \in BMO(\Omega)$.

Proof. That u^δ is doubling follows if $u^\delta \in A_2(\Omega)$, and to prove $u^\delta \in A_2(\Omega)$, we only need to show that there exists a $C = C(n, \lambda, \Lambda) > 0$ such that for any ball $B_{2r} \subset\subset \Omega$,

$$\frac{1}{|B_r|} \int_{B_r} |w - w_{B_r}|^2 \leq C, \, w_{B_r} := \frac{1}{|B_r|} \int_{B_r} w. \tag{4.5.65}$$

By Poincaré's inequality for $H_0^1(\Omega)$, there exists a constant $C = C(n)$ such that

$$\frac{1}{|B_r|} \int_{B_r} |w - w_{B_r}|^2 \leq C \, r^2 \frac{1}{|B_r|} \int_{B_r} |\nabla w|^2. \tag{4.5.66}$$

Now, we finish off the proof of the theorem by using the following claim:

Claim 92. *Using $\varphi := \phi^2 u^{-1}$ for any $\phi \in L^\infty(B_{2r}) \cap H_0^1(B_{2r})$ in (4.5.64) yields*

$$\int_\Omega \phi^2 |\nabla w|^2 \leq 4\left(\frac{\Lambda}{\lambda}\right) \int_\Omega |\nabla \phi|^2. \tag{4.5.67}$$

Indeed, choosing a $\phi \in C_0^\infty$ with its support in B_{2r}, $\phi \equiv 1$ in B_r and $|\nabla \phi| \leq \frac{C_n}{r}$, (4.5.67) yields

$$\int_{B_r} |\nabla w|^2 \leq C_n \left(\frac{\Lambda}{\lambda}\right) r^{n-2},$$

which, in turn, together with (4.5.66), yields (4.5.65).

Next, let us return to the claim assumed earlier and prove (4.5.67). Choosing $\varphi = \psi u^{-1}$ in (4.5.64) where $\psi = \phi^2$ and $\phi \in L^\infty(B_{2r}) \cap H_0^1(B_{2r})$, we get

$$\sum_{i,j=1}^n \int_\Omega (a_{ij} w_i \psi_j - a_{ij} w_i w_j \psi) \geq 0.$$

$$\int_\Omega \langle A(x)\nabla w, \nabla \psi\rangle \geq \int_\Omega \langle A(x)\nabla w, \nabla w\rangle \psi.$$

Using the Cauchy-Schwartz inequality:

$$|\langle \xi, \zeta\rangle| \leq \|\xi\|\|\zeta\|,$$

for the inner products $\langle \xi, \zeta\rangle_A := \langle A(x)\xi, \zeta\rangle = a_{ij}\xi_i\zeta_j$ and $\langle f, g\rangle_{L^2} := \int_\Omega fg$ respectively,

$$\begin{aligned}
\int_\Omega \langle A(x)\nabla w, \nabla w\rangle \phi^2 &\leq \int_\Omega \langle A(x)\nabla w, \nabla \phi\rangle 2\phi \\
&\leq \int_\Omega 2\phi \langle A(x)\nabla w, \nabla w\rangle^{1/2} \langle A(x)\nabla \phi, \nabla \phi\rangle^{1/2} \\
&\leq 2\left(\int_\Omega \phi^2 \langle A(x)\nabla w, \nabla w\rangle\right)^{1/2} \left(\int_\Omega \langle A(x)\nabla \phi, \nabla \phi\rangle\right)^{1/2}.
\end{aligned}$$

Combining like integrals and squaring both sides we get,

$$\int_\Omega \langle A(x)\nabla w, \nabla w\rangle \phi^2 \leq 4 \int_\Omega \langle A(x)\nabla \phi, \nabla \phi\rangle,$$

which, upon using the ellipticity inequalities, namely,

$$\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Lambda|\xi|^2,$$

yields (4.5.67). □

The next theorem establishes the double-ball property for supersolutions of non-divergence-form uniformly elliptic operators. The proof here is based on Lemma 2.2 in [53].

Theorem 93 (Krylov-Safonov, 1981)**.** *Let $A \in \mathcal{A}(\lambda, \Lambda, \Omega)$. and $Lu := \sum_{i,j=1}^{n} a_{ij}(x)u_{ij}$, $x \in \Omega$. There exists a constant $\gamma = \gamma(n, \lambda, \Lambda) \in (0,1)$ such that if $u \geq 0$ and $Lu \geq 0$ then u is double-ball, i.e., given any ball $B_{4r} \subset\subset \Omega$, $r > 0$, we have*

$$\inf_{B_r} u \geq 1 \Rightarrow \inf_{B_{2r}} u \geq \gamma.$$

Proof. Assume center of B_r is the origin and $u > 0$ in $\bar{B}_{4r}$. Then it suffices to prove

$$\inf_{B_r} u \geq 1 \Rightarrow \inf_{B_{2r}} u \geq \left(1 - 2^{-p}\right) 2^{-p} =: \gamma, \tag{4.5.68}$$

for any $p \geq \frac{n\Lambda}{\lambda} - 2$. Next, we first finish off the proof of the theorem by assuming, for the moment, the following claim:

Claim 94. *For $v(x) := |\frac{x}{r}|^{-p} - 4^{-p}$, we have*

$$\begin{aligned} L(v-u) \geq 0 \quad &in \quad B_{4r} \setminus B_r, \\ v - u \leq 0 \quad &on \quad \partial\left(B_{4r} \setminus B_r\right). \end{aligned} \tag{4.5.69}$$

By the maximum principle, $v - u \leq 0$ in $B_{4r} \setminus B_r$. In particular, for $x \in B_{2r} \setminus B_r$,

$$\begin{aligned} u(x) \geq v(x) &= |\tfrac{x}{r}|^{-p} - 4^{-p} \\ &\geq 2^{-p} - 4^{-p} = 2^{-p}(1 - 2^{-p}) =: \gamma, \end{aligned}$$

which proves (4.5.68).

Now, we return to establishing the claim. For $x \in B_{4r} \setminus B_r$,

$$\begin{aligned} L(v) = L(|\tfrac{x}{r}|^{-p}) &= (-pr^p)|x|^{-p-4}\left[|x|^2 a_{ij}\delta_{ij} - (p+2)a_{ij}x_ix_j\right] \\ &\geq (-pr^p)|x|^{-p-4}\left[n\Lambda|x|^2 - (p+2)\lambda|x|^2\right] \\ &\geq (-pr^p)|x|^{-p-2}\left[n\Lambda - (p+2)\lambda\right] \\ &\geq 0 \geq L(u), \end{aligned} \tag{4.5.70}$$

For $x \in \partial B_{4r}$,

$$u(x) > 0 = v(x), \tag{4.5.71}$$

and for $x \in \partial B_r$,

$$u(x) \geq 1 > 1 - 4^{-p} = v(x). \tag{4.5.72}$$

Now, (4.5.70), (4.5.71) and (4.5.72) together imply (4.5.69). $\square$

The next theorem establishes the doubling property in the context of the adjoint-form operators.

Theorem 95 (Fabes-Stroock, 1984)**.** *Let $A \in \mathcal{A}(\lambda, \Lambda, \Omega)$. and $L^* u := \sum_{i,j=1}^{n} (a_{ij}(x)u)_{ij}$, $x \in \Omega$. There exists a constant $C = C(n, \lambda, \Lambda) > 0$ such that if $u \geq 0$ is a supersolution to $L^* u = 0$, i.e.,*

$$\int_\Omega u L(\varphi) := \sum_{i,j=1}^{n} \int_\Omega u a_{ij} \varphi_{ij} \leq 0, \, \forall \varphi \in C_0^2(\Omega), \, \varphi \geq 0, \tag{4.5.73}$$

then u is doubling, i.e., given any ball $B_{2r} \subset \Omega$, we have

$$\int_{B_r} u \leq C \int_{B_{r/2}} u. \tag{4.5.74}$$

Proof. To prove (4.5.74), it suffices to show

$$\int_{B_r} u \leq C \int_{B_{(1-\delta)r}} u, \tag{4.5.75}$$

for some $\delta \in (0,1)$ depending only on n, λ and Λ. In order to establish (4.5.75), for the time being, we assume, without proof, the following claim:

Claim 96. *There exists $h \in C_0^2(\Omega)$, $h \geq 0$ and structural constants $c_1, c_2 > 0$ and $\delta \in (0,1)$ such that*

$$\begin{aligned} L(h) &\geq 0 \text{ in } B_{(1+\delta)r} \setminus B_{(1-\delta)r}, \\ L(h) &\geq c_1 r^2 \text{ in } B_r \setminus B_{(1-\delta)r}, \\ |L(h)| &\leq c_2 r^2 \text{ in } B_r. \end{aligned} \tag{4.5.76}$$

$$\begin{aligned}\int_{B_r\setminus B_{(1-\delta)r}} u &\leq \frac{1}{c_1}\int_{B_{(1+\delta)r}\setminus B_{(1-\delta)r}} u\, L(h/r^2)\\ &= \frac{1}{c_1}\int_{B_{(1+\delta)r}} u\, L(h/r^2) - \frac{1}{c_1}\int_{B_{(1-\delta)r}} u\, L(h/r^2)\\ &\leq \frac{1}{c_1}\int_{B_{(1+\delta)r}} u\, L(h/r^2) + \frac{c_2}{c_1}\int_{B_{(1-\delta)r}} u \leq \tilde{c_2}\int_{B_{(1-\delta)r}} u,\end{aligned}$$

since $\int_{B_{(1+\delta)r}} u\, L(h/r^2) \leq 0$ from (4.5.73). This yields (4.5.75) with $C = 1 + \tilde{c_2}$.

Now, all that is left is to prove the claim assumed earlier. To construct the function h satisfying (4.5.76) as in the claim, assume, center of B_r is the origin and for $\delta \in (0,1)$ define $h(x) = [(1+\delta)^2 r^2 - |x|^2]^2$ in $B_{(1+\delta)r}$ and $h(x) = 0$ elsewhere.

$$h_i(x) = -4[(1+\delta)^2 r^2 - |x|^2]x_i,$$

$$h_{ij}(x) = -4(1+\delta)^2 r^2 \delta_{ij} + 4|x|^2\delta_{ij} + 8x_i x_j.$$

For $x \in B_r$,

$$\begin{aligned}|L(h)| = |a_{ij}h_{ij}| &\leq 4n\Lambda(1+\delta)^2 r^2 + 4n\Lambda|x|^2 + 8\Lambda|x|^2\\ &\leq c_2(n,\lambda,\Lambda,\delta)\, r^2.\end{aligned}$$

For $x \in B_{(1+\delta)r} \setminus B_{(1-\delta)r}$,

$$\begin{aligned}L(h) = a_{ij}h_{ij} &\geq 4n\Lambda[-(1+\delta)^2 r^2 + |x|^2] + 8\lambda|x|^2\\ &\geq 4n\Lambda r^2[-(1+\delta)^2 + (1-\delta)^2] + 8\lambda(1-\delta)^2 r^2\\ &\geq 4n\Lambda r^2[-4\delta] + 8\lambda(1-\delta)^2 r^2\\ &= r^2[-16n\Lambda\delta + 8\lambda(1-\delta)^2] \geq c_1(\lambda,\Lambda,n) r^2,\end{aligned}$$

upon choosing a $\delta = \delta(\lambda,\Lambda,n) \approx 0$. □

4.6 Insufficiency of the critical density property

In Proposition 4.3 of [25], it is shown that the double-ball property follows whenever the critical density property is sensitive enough, more precisely, whenever it holds true for

$0 < \varepsilon < 1/C_\mu^2$. This motivates the following definition:

Definition 97. Let $M \geq 1$ and $\varepsilon \in (0,1)$. A function u defined on Ω is said to have a *sensitive critical density (SCD)* property with constants M and ε if the following hold:

(1) u has the critical density property with constants M and ε.

(2) $0 < \varepsilon < 1/C_\mu^2$ where C_μ is the doubling constant of μ.

One of the axiomatic approaches we have mentioned is that critical density and double-ball together imply Harnack. The question then arises: is the critical density property alone is sufficient for Harnack? The answer is not in full generality, and only in the case of the sensitive critical density property. This section is devoted to the detailed discussion of this very fact.

In many practical contexts, it is proved that non-negative supersolutions to elliptic PDEs possess the critical density property with "arbitrary sensitivity", that is, for every $\varepsilon > 0$ in Definition 63. See, for instance, Theorem 7.5 in [25], Lemma 6.2 in [42], Proposition 3.1 in [4], Lemma 3.3 in [24], Theorem 4.9 in [35], etc. However, suppose, the critical density property is not as sensitive? What can we say in this case? The answer is: we can assert the conclusion in Theorem 89, but not much more. For instance, in $\mathbb{R}$ consider the function $u(x) := 2\chi_{[0,\infty)}(x)$. It is easy to verify that u has the critical density property with any $M \geq 1$ and $\varepsilon = .75$. Indeed, it is clearly observed in Figure 4.7 that if $u \geq 1$ in at least three-fourth of a ball B_r, then $u > 1$ in all of $B_{r/2}$. Moreover, if $\lambda \geq u$ in an interval I then $\lambda - u$ also has the critical density property with $\varepsilon = .75$ (contrast this with the statement of Theorem 74). However, u does not satisfy a Harnack's inequality, simply because there is no way $\sup u = 2$ can be comparable to $\inf u = 0$. Also, it is noteworthy that the critical density property of u is not sensitive enough in this example since $\varepsilon = .75 \geq \frac{1}{4} = \frac{1}{C_\mu^2}$, where μ is the Lebesgue measure in $\mathbb{R}$.

In the light of the just mentioned "arbitrarily-sensitive" critical density property and Proposition 4.3 in [25], it might seem as if the double-ball property, instead of the doubling

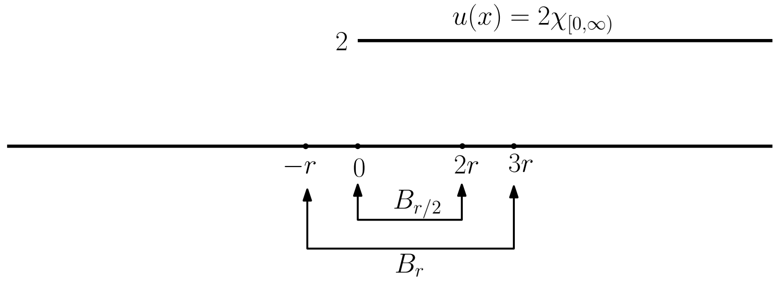


Figure 4.7: *Both u and $(2-u)$ have the critical density property with constants $M = 1$ and $\varepsilon = .75$ but u is not double-ball.*

property, plays an inherent role in the proof of Harnack's inequality. However, we point out that the proofs of the critical density property with arbitrary sensitivity cited above systematically involve, among other things, an estimate of the form

$$\int_{B_R} |\nabla \log u|^2 \, dx \leq CR^{n-2}. \tag{4.6.77}$$

Notice that, through Poincaré's inequality, (4.6.77) implies that $\log u \in BMO$ and, by the John-Nirenberg inequality, $u^{q_0} \in A_2$, for some $q_0 > 0$. In particular, u^{q_0} is a doubling weight. Thus, the doubling property, whether explicitly stated or not, is intrinsic to the approaches towards Harnack's inequality that involve estimates such as (4.6.77). For John-Nirenberg-type inequalities in doubling spaces, see [9].

4.7 A comparison with Bombieri's lemma

In this section, we present a view to regard our approach to Harnack's inequality proposed in this article (encoded in Theorem 82) as an alternative to a classical result due to E. Bombieri, known as Bombieri's lemma, see, for instance, [8, Theorem 4], [17, Lemma 3.14], [49, Lemma 3], and [54, Lemma 2.2.6]. Namely,

Lemma 98. (Bombieri's lemma) *Let $\tau > 0$, μ a doubling measure in $\mathbb{R}^n$ and u a non-negative bounded function on a ball $B \subset \mathbb{R}^n$. Assume the following two conditions:*

(B1) There are positive constants C_1 and δ such that

$$\operatorname*{ess.sup}_{sB} u^p \leq \frac{C_1}{(t-s)^\delta \mu(tB)} \int_{tB} u^p \, d\mu$$

for all $0 < p < 1/\tau$ *and* $1/2 \leq s \leq t \leq 1$.

(B2) For all $\lambda > 0$ *it holds*

$$\mu(\{x \in B : \log u(x) > \lambda\}) \leq \frac{C_1 \tau}{\lambda} \mu(B).$$

Then, there are positive constants C_2 *and* D *so that*

$$\operatorname*{ess.sup}_{\alpha B} u \leq e^{\frac{C_2 \tau}{(1-\alpha)^D}}, \quad 0 < \alpha < 1.$$

Indeed, in relation to Theorem 82, the role of condition (B1) in Bombieri's lemma is played by the critical density property (which, by Corollary 90, is stronger than (B1)) and condition (B2), which is a weaker version of (4.6.77) (more precisely, it is a weaker version of $\log u \in BMO$), is replaced by the (even weaker) doubling property for a power of u. As Bombieri's lemma, Theorem 82 also avoids the explicit use of the exponential integrability of BMO functions, i.e., John-Nirenberg's inequality.

As mentioned in the introduction, Moser's iterations yield the inequalities (7) (which, in turn, yield (4.2.59), i.e. (B1) in Bombieri's lemma). In the same PDE context, inequalities (7) can also be deduced by means of De Giorgi's truncations, see, for instance, [35, Chapter 4]. Moser's iterative procedure and De Giorgi's truncation method are both based on the interaction between a Sobolev inequality and an energy estimate (i.e., Caccioppoli's inequality). Corollary 90 now says that such interaction is built into the critical density property. In Section 4.2.3 we saw how the critical density and the doubling properties imply Harnack, in a novel, alternative way that avoids BMO, special covering lemmas, condition (B2), and Poincaré inequalities. Part of the virtue of Bombieri's lemma is that it significantly simplified Moser's proof of Harnack's inequality (see [54, Sections 2.2.3-2.3.2]). Along the same lines, all the proofs of the Harnack inequalities cited in this dissertation can be simplified through the use of Theorem 82, very much in the spirit of [25, Section 7]. However, we shall pursue this only in one context, namely, the context of graphs in Chapter 6.

Chapter 5

Power-like decay property in certain types of metric spaces

Quite often in the literature on elliptic (and parabolic) PDEs in a variety of contexts, the power-like decay property (Definition 65) is obtained for the distribution function of supersolutions via a property of the following type:

$$\inf_{B_{10s}} w \leq 1 \leq N \Rightarrow \frac{\mu\left(B_s \cap \{w \leq M\}\right)}{\mu(B_{Ns})} \geq \vartheta(N) > 0. \tag{5.0.1}$$

See, for instance, [14, Lemma 4.5] and [35, Lemma 5.13] for viscosity solutions of fully non-linear elliptic PDEs, [11, Lemma 5.1] and [41, Lemma 3.1] for elliptic equations on Riemannian manifolds, and [25, Theorem 4.7] and [3, Theorem 3.1] in abstract metric spaces.

The goal of this section is to unify all above-mentioned proofs into a single context, namely, the context of metric spaces which satisfy the so-called segment and segment-prolongation properties. Section 5.1 motivates a new definition for the property (5.0.1), Section 5.2 defines the new context of metric spaces and gives the statement of the main result of the chapter, and Section 5.3 contains the detailed proof of the main result.

5.1 The explosive critical density property

In Section 4.6, we defined the sensitive critical density property. In this section, we give the definition of the explosive critical density property and see how it yields the property (5.0.1), so commonly found in the literature.

The critical density property, that is, the implication

$$\frac{\mu\left(B_s \cap \{w \geq M\}\right)}{\mu(B_s)} \geq \varepsilon \Rightarrow \inf_{B_{s/2}} w \geq 1.$$

is an imploding property, in the sense that the information about the size of w is transferred inwards from B_s to $B_{s/2}$, boosting up w in $B_{s/2}$ from the outside. On the other hand, the double-ball property

$$\inf_{B_{s/2}} w \geq 1 \Rightarrow \inf_{B_s} w \geq \gamma,$$

is an explosive property which boosts w outwards. The combination of these two properties allows for an expansive wave with geometric attenuation. Namely,

$$\frac{\mu\left(B_s \cap \{w \geq M\}\right)}{\mu(B_s)} \geq \varepsilon \Rightarrow \inf_{B_{2s}} w \geq \gamma^2.$$

This combination motivates the following definition: a function w satisfies the *explosive critical density property* with constants $M \geq 1$ and $\eta \in (0,1)$, in an open set $\Omega \subset X$, if for all B_s with $B_{10s} \subset \Omega$ the following implication holds true

$$\inf_{B_{10s}} w \leq 1 \Rightarrow \frac{\mu\left(B_s \cap \{w \leq M\}\right)}{\mu(B_s)} \geq \eta. \tag{5.1.2}$$

The important feature about the implication (5.1.2) is that the ball B_s on the numerator is quantitatively smaller than the ball on which the infimum is taken. The fact that the ball B_s also appears on the denominator is essentially artificial. Indeed, we can replace it with either $B_{\tau s}, \tau \in (0,1)$ or $B_{Ns}, N \geq 1$. Replacing B_s with a smaller ball $B_{\tau s}$ incurs no cost at all since

$$\eta \leq \inf_{B_{10s}} w \leq 1 \Rightarrow \frac{\mu\left(B_s \cap \{w \leq M\}\right)}{\mu(B_s)} \leq \frac{\mu\left(B_s \cap \{w \leq M\}\right)}{\mu(B_{\tau s})}.$$

On the other hand, with $N \geq 1$, we have $B_s \subset B_{Ns}$ and and by Lemma 23 we get

$$\frac{\mu(B_s)}{\mu(B_{Ns})} \geq \frac{N^{-\zeta}}{C_\mu}$$

so that (5.1.2) implies

$$\inf_{B_{10s}} w \leq 1 \Rightarrow \frac{\mu\left(B_s \cap \{w \leq M\}\right)}{\mu(B_{Ns})} = \frac{\mu\left(B_s \cap \{w \leq M\}\right)}{\mu(B_{Ns})} \frac{\mu(B_s)}{\mu(B_s)} \geq \frac{\eta N^{-\zeta}}{C_\mu} =: \vartheta(N) > 0,$$

which, upon setting $\frac{\eta N^{-\zeta}}{C_\mu} =: \vartheta(N)$, takes the form of the ubiquitous property (5.0.1).

5.2 Segment and segment-prolongation properties

In this section, the main result of this chapter, i.e., the power-like decay property (Definition 65) in metric spaces with some additional structure, is stated. We begin by defining two new properties which gives us this additional structure in metric spaces.

Definition 99. A metric space (X, d) is said to possess the *segment property* if for every $x, y \in X$ and any r with $0 \leq r \leq d(x, y)$, there exist $z \in X$ with $d(x, z) = r$ and $d(x, y) = d(x, z) + d(z, y)$.
A metric space (X, d) is said to possess the *segment-prolongation property* if for every $x, y \in X$ and any r with $0 \leq d(x, y) \leq r \leq \operatorname{diam} X$, there exist $z \in X$ with $d(x, z) = r$ and $d(x, z) = d(x, y) + d(y, z)$.

As examples of metric spaces with the segment and segment-prolongation properties we mention that complete Riemannian manifolds equipped with the metric associated to a family of Lipschitz continuous vector fields has the segment property (locally). In particular, the Carnot-Caratheodory structure possesses the segment property (locally), see Remark 2.6 in [27] and references therein. Another rich family of metric spaces, which seems to dominate the relevant examples in differential geometry, with both the segment and segment-prolongation properties is the class of Busemann's G-spaces (see [10, pp. 37-38]). Yet another example of metric spaces with both the segment and segment-prolongation properties is the class of infinite graphs with the geodesic distance; in this case, the variable r in Definition 99 must be regarded as a natural number.

The main result in this section is

Theorem 100. *Let (X, d, μ) be a doubling metric space with the segment and segment-prolongation properties. Let $\Omega \subset X$ be an open subset and suppose that a functional set $\mathbb{K}_\Omega$ possesses property (5.1.2) with some constants $M > 1 > \eta > 0$. Also, assume that $\mathbb{K}_\Omega$ is closed under multiplication by small constants. Then, there exist geometric constants $\epsilon, C > 0$, depending also on M and η, such that for every $u \in \mathbb{K}_\Omega(B_R)$ and B_R with*

$B_{2R} \subset\subset \Omega$ *we have*

$$\mu(\{x \in B_R : u(x) > t\}) \leq C\mu(B_R)t^{-\epsilon}, \quad t > 0. \tag{5.2.3}$$

Theorem 100 comes as a complement to Theorem 4.7 in [25] and Theorem 3.1 in [3] where a similar conclusion is obtained under certain hypotheses that involve both the metric and the measure in (X, d, μ) or extra regularity on the function u. For example, the ring condition (see Definition 70) in [25] prescribes a bound for the rate of convergence to zero of $\mu(B_R \setminus B_{R(1-\varepsilon)})/\mu(B_R)$ as $\varepsilon \to 0$. An example of a metric space verifying the segment and segment-prolongation properties but not the ring condition is the case of an infinite graph with the counting measure, where $\mu(B_R \setminus B_{R(1-\varepsilon)}) \geq 1$ for all $\varepsilon > 0$ and $R \geq 1$.

5.3 Proof of the power-like decay via the explosive critical density

In this section, we will provide the proof of Theorem 100. Our proof uses the fact that spaces of homogeneous type always admit Vitali covering lemmas (see Theorem 1.2 in [18]); namely,

Lemma 101. (Vitali's covering lemma) *Let (X, d, μ) be a space of homogeneous type. There exists a geometric constant $K_0 \geq 1$ such that for any bounded subset $E \subset X$ and any covering $\{B(x, r(x))\}$ of E, there exists a collection of disjoint balls $\{B(x_j, r(x_j))\}_{j \in \mathbb{N}}$ so that the family $\{B(x_j, K_0 r(x_j))\}_{j \in \mathbb{N}}$ forms a covering of E.*

Lemma 102. *Let $\mathbb{K}_\Omega$ be as in Theorem 100. Fix $u \in \mathbb{K}_\Omega$ and $B_R = B_R(z)$. For $k \in \mathbb{N}_0$ define*

$$D_k := \{x \in B_R : u(x) \leq M^k\} \subset B_R.$$

Then, if $\inf\limits_{B_R} u \leq 1$, it follows that

$$\mu(B_R \setminus D_k) \leq \frac{1}{\eta_0}\mu(D_{k+1} \setminus D_k) \quad k \in \mathbb{N}_0, \tag{5.3.4}$$

with $\eta_0 := \vartheta(4K_0) \in (0,1)$*, where* ϑ *is as in* (5.0.1) *and* K_0 *is the geometric constant in Vitali's covering lemma.*

Proof. The assumption $\inf_{B_R} u \leq 1$ implies that $D_k \neq \emptyset$ for all $k \in \mathbb{N}_0$. In particular, given $x \in B_R = B_R(z)$ and $x' \in D_k$, we obtain

$$d(x, D_k) \leq d(x, x') \leq d(x, z) + d(z, x') < 2R.$$

For $k \in \mathbb{N}_0$ and $x \in B_R \setminus D_k$ (if $B_R \setminus D_k = \emptyset$, then (5.3.4) follows trivially) set

$$r_x := \frac{K_0}{2} d(x, D_k) < K_0 R. \tag{5.3.5}$$

Next, we consider two cases: Case 1: $r_x/(2K_0) \leq d(x,z)$ and Case 2: $d(x,z) < r_x/(2K_0)$. If the first case holds true, we use the segment property to find $y \in X$ such that

$$d(x,y) = \frac{r_x}{2K_0} \quad \text{and} \quad d(x,z) = d(x,y) + d(y,z). \tag{5.3.6}$$

If the second case holds true, we use the segment-prolongation property to find $y \in X$ such that

$$d(x,y) = \frac{r_x}{2K_0} \quad \text{and} \quad d(x,y) = d(x,z) + d(y,z). \tag{5.3.7}$$

Notice that, in both cases we have $d(x,y) = r_x/(2K_0) < R$ (where the last inequality is due to (5.3.5)). With this choice of $y \in X$ we claim that, in either case, we have

$$B_{\frac{r_x}{2K_0}}(y) \subset B_{\frac{r_x}{K_0}}(x) \cap B_R(z). \tag{5.3.8}$$

In order to check (5.3.8), we start with $B_{\frac{r_x}{2K_0}}(y) \subset B_{\frac{r_x}{K_0}}(x)$. In either Case 1 or Case 2 we proceed as follows: given $x_1 \in B_{\frac{r_x}{2K_0}}(y)$ we have

$$d(x, x_1) \leq d(x,y) + d(x_1, y) < \frac{r_x}{2K_0} + \frac{r_x}{2K_0} = \frac{r_x}{K_0}.$$

To prove the inclusion $B_{\frac{r_x}{2K_0}}(y) \subset B_R(z)$, in the first case, given $x_1 \in B_{\frac{r_x}{2K_0}}(y)$, (5.3.6) yields

$$\begin{aligned} d(x_1, z) &\leq d(x_1, y) + d(y,z) = d(x_1, y) + d(x,z) - d(x,y) \\ &\leq \frac{r_x}{2K_0} + d(x,z) - \frac{r_x}{2K_0} = d(x,z) < R. \end{aligned}$$

In the second case, we use (5.3.7) to write

$$\begin{aligned} d(x_1,z) &\le d(x_1,y)+d(y,z) < \frac{r_x}{2K_0} + d(y,z) = \frac{r_x}{2K_0} + d(x,y) - d(z,x) \\ &\le \frac{r_x}{2K_0} + d(x,y) = \frac{r_x}{K_0} < R. \end{aligned}$$

The rest of the proof just hinges upon (5.3.8) and it is independent of Case 1 or Case 2. From the inclusion (5.3.8) it follows that

$$B_{\frac{r_x}{2K_0}}(y) \cap D_{k+1} \subset B_{\frac{r_x}{K_0}}(x) \cap D_{k+1}. \tag{5.3.9}$$

Next, set $t := 2r_x$ so that

$$d(y, D_k) \le d(x,y) + d(x, D_k) = \frac{r_x}{2K_0} + \frac{2r_x}{K_0} = \frac{5r_x}{2K_0} = \frac{5t}{4K_0}.$$

In particular, $5t/(2K_0) \ge 2d(y, D_k)$. Hence, setting $w := u/M^k$, we have

$$\inf_{B_{\frac{5t}{2K_0}}(y)} w \le 1.$$

Hence, from the explosive critical density property (5.0.1) applied to w (notice that $M^k > 1$) with $s := \frac{t}{4K_0}$, $N := 4K_0$, and $\eta_0 := \vartheta(N)$, we have

$$\mu\left(B_{\frac{t}{4K_0}}(y) \cap \{w \le M\}\right) \ge \eta_0 \mu(B_t(y)). \tag{5.3.10}$$

Next we claim that

$$B_{r_x}(x) \subset B_t(y). \tag{5.3.11}$$

Indeed, given $x_2 \in B_{r_x}(x)$,

$$d(x_2, y) \le d(x_2, x) + d(x,y) \le r_x + \frac{r_x}{2K_0} < 2r_x = t.$$

Therefore, from (5.3.11), (5.3.10), and (5.3.9),

$$\begin{aligned} \eta_0 \mu(B_{r_x}(x)) &\le \eta_0 \mu(B_t(y)) \le \mu(B_{\frac{t}{4K_0}}(y) \cap \{w \le M\}) \\ &= \mu(B_{\frac{r_x}{2K_0}}(y) \cap \{w \le M\}) \\ &\le \mu(B_{r_x/K_0}(x) \cap D_{k+1}). \end{aligned}$$

That is, for every $x \in B_R \setminus D_k$, we have

$$\eta_0 \mu(B_{r_x}(x)) \leq \mu\left(B_{r_x/K_0}(x) \cap D_{k+1}\right), \tag{5.3.12}$$

where $r_x = \frac{K_0}{2} d(x, D_k)$. We now cover $B_R \setminus D_k$ with the balls $B_{r_x/K_0}(x)$, $x \in B_R \setminus D_k$, and use Vitali's covering lemma to extract a disjoint collection $\{B_j := B_{r_{x_j}/K_0}(x_j)\}_{j\in\mathbb{N}}$ so that $\{B_{r_{x_j}}(x_j)\}_{j\in\mathbb{N}}$ still covers $B_R \setminus D_k$. From the definition of r_x in (5.3.5) we have

$$B_j \cap D_k = \emptyset, \quad j \in \mathbb{N}, k \in \mathbb{N}_0.$$

In particular, for every $k \in \mathbb{N}_0$,

$$\bigcup_{j\in\mathbb{N}} (B_j \cap B_R) \subset B_R \setminus D_k,$$

and, consequently, always with disjoint union,

$$\bigcup_{j\in\mathbb{N}} (B_j \cap D_{k+1}) \subset D_{k+1} \setminus D_k. \tag{5.3.13}$$

From the covering property of $\{B_{r_{x_j}}(x_j)\}_{j\in\mathbb{N}}$, (5.3.12), and (5.3.13), we finally obtain

$$\begin{aligned}\mu(B_R \setminus D_k) &\leq \sum_{j\in\mathbb{N}} \mu(B_{r_{x_j}}(x_j)) \leq \frac{1}{\eta_0} \sum_{j\in\mathbb{N}} \mu\left(B_j \cap D_{k+1}\right) \\ &= \frac{1}{\eta_0} \mu\left(\bigcup_{j\in\mathbb{N}} (B_j \cap D_{k+1})\right) \leq \frac{1}{\eta_0} \mu(D_{k+1} \setminus D_k).\end{aligned}$$

□

Proof of Theorem 100. The proof of the theorem now quickly follows from (5.3.4). Indeed,

$$\begin{array}{rll} \mu(B_R \setminus D_k) & \leq \frac{1}{\eta_0} \mu(D_{k+1} \setminus D_k) & \Leftrightarrow \\ \mu(D_{k+1}) & \geq \eta_0 \mu(B_R) + (1-\eta_0)\mu(D_k) & \Leftrightarrow \\ \mu(B_R \setminus D_{k+1}) & \leq (1-\eta_0)\mu(B_R \setminus D_k) & \end{array}$$

which imply

$$\mu(\{x \in B_R : u(x) > M^{k+1}\}) \leq (1-\eta_0)^k \mu(B_R) \quad k \in \mathbb{N}_0,$$

and (5.2.3) follows. □

Chapter 6

Harnack's inequality on infinite graphs

This chapter exemplifies the practical usefulness of our Harnack theory presented in Chapter 4. It provides an application of our main result in the context of analysis on graphs. Specifically, using Theorem 82, we reprove Harnack's inequality for harmonic functions on graphs originally proved by [21] in 1997. Section 6.1 introduces the necessary structure on infinite graphs in order to make the reader familiar with analysis on graphs and Section 6.2 states and proves the main result of this chapter, namely, an alternative proof of Harnack's inequality for harmonic functions on infinite graphs.

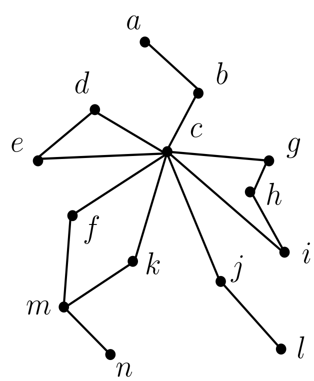
6.1 Harmonic functions on graphs

We begin with some definitions, notations and examples which will all be in the spirit of [21].

Definition 103. A *graph* G is a set of points or objects called *vertices* where some pairs of objects are connected by a (non-oriented) link called an *edge*. Two vertices x and y joined by an edge are called *neighbors* of one another and is denoted by $x \sim y$. The number of vertices of a vertex x is called the *degree* of x and is denoted by $\deg(x)$. A series of sequentially connected edges is called a *path* whose *length* is defined to be the number of edges in it. G is called a connected graph if given any two vertices, there always exists a path joining one to the other. A connected graph is a metric space where the *distance* between two vertices is defined to be the length of the shortest path joining them. For $n \geq 0$, a *ball* and its *boundary* are defined as follows:

$$B(x,n) := \{y \in G \mid d(x,y) \leq n-1\}, \quad \partial B := \{y \in G \mid d(x,y) = \lceil n-1 \rceil\},$$

where $\lceil n-1 \rceil$ is the smallest integer greater than or equal to $n-1$. Note that if $n \in (0,1]$, then $B(x,n) = \partial B(x,n) = \{x\}$, for every $x \in G$. Consistent with our notation thus far, if $B = B(x,n)$ and $k > 0$, we denote $kB = B(x,kn)$.



$B(c,1) = \{c\}$

$B(c,2) = \{c,b,d,e,f,g,k,j,i,g\}$

$B(c,3) = \{c,b,d,e,f,g,k,j,i,g,a,m,l,h\}$ (length ≤ 2 from c)

$\partial B(c,3) = \{c,a,d,e,m,h,l\}$(length $= 2$ from c)

$\partial B(c,R) \supset \partial B(c,R-2)$

$B(c,R) \setminus \partial B(c,R) \subset B(c,R-1)$

Figure 6.1: *Balls on a finite graph.*

Our work in this section is set in a connected graph, which, as noted above, is a metric space. Figure 6.1 illustrates a finite connected graph along with balls and boundaries in it. However, in order to do PDEs on a graph, we need yet more assumptions on our structure, which is what we will describe and define next.

As our first structural assumption, we require the doubling property of the counting measure # on our graph: there exists a $C_1 > 1$ such that for every ball B,

$$\#(2B) \le C_1 \#(B).$$

A graph G with this assumption is called a *doubling graph*, which clearly, is a doubling metric space.

Definition 104. A graph G is said to be *Ahlfors α-regular* if there exists a constant $C > 0$ such that for every $x \in G$,

$$\frac{1}{C} n^{\alpha} \le \#(B(x,n)) \le C n^{\alpha},\ \forall n > 0, n \notin \mathbb{N},$$

and

$$\frac{1}{C}(n+1)^{\alpha} \le \#(B(x,n)) \le C(n+1)^{\alpha},\ \forall n > 0, n \in \mathbb{N}.$$

More succinctly, a graph G is said to be *Ahlfors α-regular* if there exists a constant $C > 0$ such that for every $x \in G$,

$$\frac{1}{C} n^{\alpha} \le \#(\mathbf{B}(x,n)) \le C n^{\alpha},\ \forall n > 0,$$

where $\mathbf{B}(x,n) := \{y \in G \mid d(x,y) \le n\}$. Note that the new balls $\mathbf{B}(x,n)$ differ from the old balls $B(x,n)$ only when $n \in \mathbb{N}$. Clearly, Ahlfors α-regularity implies doubling and the doubling constant will necessarily depend on the structural constant α.

Example 105 ([22],[5])**.** A k-regular tree is an example of a doubling graph. A k-regular tree T_∞ is constructed as a limiting case $\lim\limits_{n\to\infty} T_n$, starting with a finite graph T_1 having k number of edges and $\operatorname{diam}(T_1) = \triangle$. The iterative process then is to construct $T_n, n = 2, 3, \ldots,$ by replacing each edge in T_1 by a copy of T_{n-1}, or, equivalently replacing each edge in T_{n-1} by a copy of T_1. Then $\operatorname{diam}(T_n) = \triangle^n$ and $\#(T_n) \approx k^n$. Furthermore, T_∞ is Ahlfors α-regular with $\alpha = \frac{\ln k}{\ln \triangle}$ and is thus doubling. Figure 6.2 illustrates a 3-regular tree which is Ahlfors α-regular with $\alpha = \frac{\ln 3}{\ln 2}$.

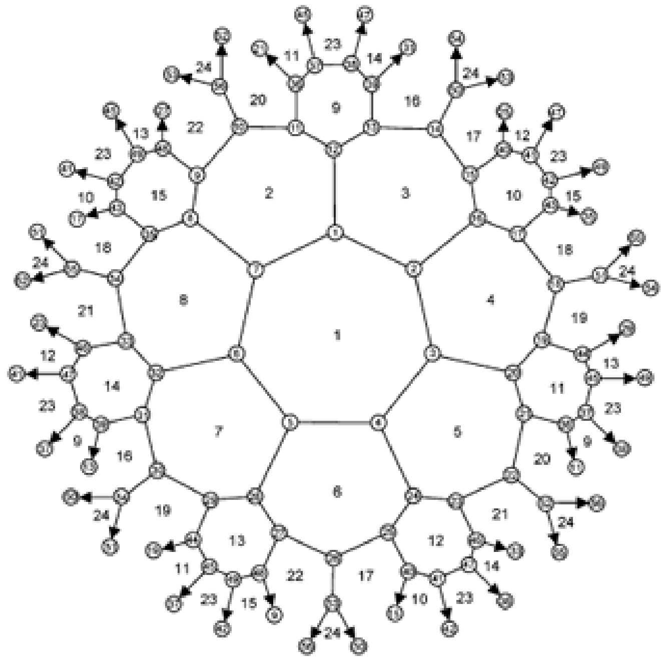


Figure 6.2: *A 3-regular tree is an infinite doubling graph.*

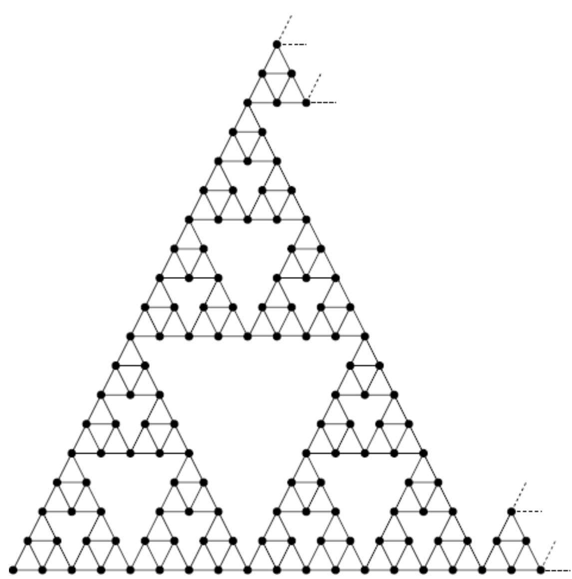


Figure 6.3: *The Sierpinski gasket is a doubling graph which does not admit the Poincaré inequality.*

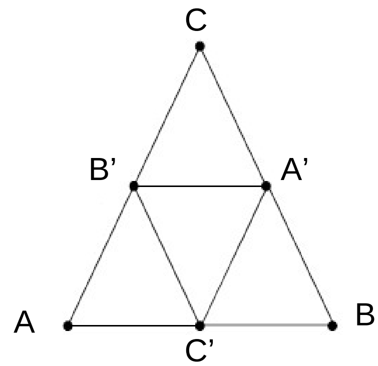


Figure 6.4: T_1*: The first step in the construction of the Sierpinski gasket.*

Example 106. ([22, Proposition 3.2]) The Sierpinski gasket illustrated in Figure 6.3 is Ahlfors α-regular with $\alpha = \frac{\ln 3}{\ln 2}$ and thus is a doubling graph. The Sierpinski gasket can be constructed through an iterative procedure as the limit $\lim\limits_{n\to\infty} T_n$, where T_n is the finite graph obtained at the n^{th} step for every $n \in \mathbb{N}$. Let T_0 be an equilateral triangle and T_1 as shown in Figure 6.4 be the first step construction of the Sierpinski gasket. Then for $n \geq 2$, the n-th step construction T_n is obtained by replacing each edge of T_0 by a copy of T_{n-1}.

Since, for every $x \in G$, $\#(B(x,1)) = 1$, there exists a uniform bound on the number of neighbors a vertex can have in a doubling graph as follows:

$$\#(B(x,2) \leq C_1 \, \#(B(x,1) = C_1.$$

This motivates the following definition:

Definition 107. A graph G is *locally uniformly finite* if there exists an $N \in \mathbb{N}$ such that for every $x \in G$,

$$\deg(x) := \#(\{y \in G \mid y \sim x\}) \leq N.$$

Definition 108. Given a real-valued function f defined on B, we denote

$$f_B := \frac{1}{\#(B)} \sum_{x \in B} f(x).$$

As our second structural assumption, we require that our graph admits the following inequality called the *Poincaré inequality*: there exists a $C_2 > 0$ such that for every ball $B^0 := B(x_0, n)$,

$$\sum_{x \in B^0} |f(x) - f_B|^2 \leq C_2 \, n^2 \sum_{\substack{x,y \in B^0 \\ x \sim y}} |f(x) - f(y)|^2.$$

Example 109 ([22])**.** Referring the reader to the original paper for more details, we provide two examples of graphs. Figure 6.5 is an example of a graph that admits the Poincaré inequality. The Poincaré inequality holds in this graph because it is possible to find a family of well-chosen paths without too many overlappings. To this end, one has to choose paths that first goes radially (changing i and j) then transversally (changing only k). The Sierpinski gasket illustrated in Figure 6.3 is an example of a graph where the poincaré inequality fails to hold because there are two many narrow parts at all scales in the geometry.

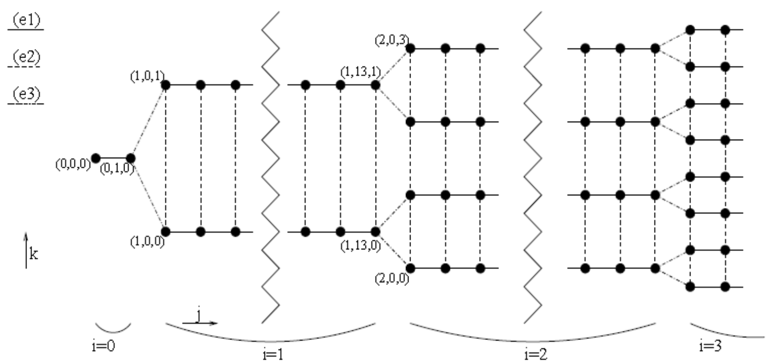


Figure 6.5: *A graph* $G = \{(i, j, k) \in \mathbb{N}^3 \mid j \leq 2^{\delta i}, k < 2^i\}$, $\delta := 3.802$, *as shown here admits the Poincaré inequality.*

Definition 110. A graph G is called a *weighted graph* if to every edge $x \sim y$ there is associated a weight $C_{xy} > 0$ with a default value of $C_{xy} = 1$. Given a real-valued function u defined on a weighted graph G, we define $\mathcal{L}$ or the *weighted discrete Laplacian* $\triangle$ as follows:

$$\mathcal{L}u(x) \equiv -\triangle u(x) := \sum_{y \sim x} C_{xy}(u(x) - u(y)).$$

The ellipticity of $\mathcal{L}$ is assumed in the following sense: there exists a $C_3 > 0$ such that

$$\frac{1}{C_3} \leq C_{xy} \leq C_3.$$

We say that u is *harmonic (subharmonic, superharmonic)* on B if $\mathcal{L}u \equiv (\leq, \geq)0$ on $B \setminus \partial B$.

Example 111 ([22])**.** Next, we give an example of a harmonic function u defined on the Sierpinski gasket illustrated in Figure 6.3 whose construction algorithm is given in Example 106. We begin with u defined on $T_0 = \{A, B, C\}$ which is a subset of T_1 shown in Figure 6.4. Then u is defined on all of T_1 as follows:

$$u(A') = \frac{u(A) + 2u(B) + 2u(C)}{5},$$
$$u(B') = \frac{2u(A) + u(B) + 2u(C)}{5},$$
$$u(C') = \frac{2u(A) + 2u(B) + u(C)}{5}.$$

Iterating the same procedure on each smaller triangle ad infinitum, we get a harmonic function u on the Sierpinski gasket with the given initial boundary value defined on $T_0 = \{A, B, C\}$.

6.2 A new approach to Harnack's inequality on graphs

The main result of this chapter is an alternative proof of Theorem 112 establishing Harnack's inequality for harmonic functions on graphs first proved by Delmotte in 1997. Delmotte's proof is based on Moser's approach we reviewed in Section 3.1 whereas our proof is going to be an application of our approach to Harnack dealt with in Chapter 4.

Theorem 112 (Delmotte, 1997)**.** *There exists a $C > 1$ depending only on the structural constants C_1, C_2 and C_3 associated to the graph such that if $u > 0$ is harmonic on $2B$, then*

$$\max_B u \leq C \min_B u.$$

Proof. In most practical applications, the doubling property for u is obtained by showing that $\ln u \in \mathrm{BMO}$ because this is equivalent to $u^\delta \in A_2$ for some $\delta > 0$ and it is a well-known fact that A_2 weights are doubling. Once the doubling property (Theorem 113) and the critical density property (Theorem 114) have been established for a solution u, then Theorem 112 follows as an immediate consequence of Theorem 82. $\square$

Theorem 113 (Energy estimate for $\ln u$)**.** *There exists a $C > 0$ depending only on C_1, C_2 and C_3 such that if $\mathcal{L}u \geq 0, u > 0$ on $77B^0, B^0 := B(x_0, n_0)$ and $n \leq n_0$, then*

$$\sum_{\substack{x,y\in 2B^0\\ x\sim y}} \left(\ln\frac{u(x)}{u(y)}\right)^2 \leq C\,\frac{1}{n^2}\#(B^0).$$

Consequently,

$$\ln u \in \mathrm{BMO}(11B^0).$$

Theorem 114 (Critical density for all $0 < \varepsilon < 1$)**.** *Given $0 < \varepsilon < 1$, there exists a $C > 0$ depending only on C_1, C_2, C_3 and ε such that if $\mathcal{L}u \geq 0, u > 0$ on $4B^0$, $B^0 := B(x_0, n_0)$, then*

$$\#(\{x \in 2B^0 \mid u(x) \geq 1\}) \geq \varepsilon\#(2B^0)$$

implies

$$\min_{B^0} u \geq C.$$

The proof of Theorem 113 closely follows Delmotte [21]. We also require a proposition (Proposition 115) and its corollary (Corollary 116) whose proofs can be found in [21].

Proposition 115. *Suppose $\eta \geq 0, \eta \equiv 0$ on ∂B^0. There exists a $C > 0$ depending only on C_1, C_2 and C_3 such that if $\mathcal{L}u \geq 0, u > 0$ on $B^0 := B(x_0, n_0)$, then*

$$\sum_{\substack{x,y\in B^0\\ x\sim y}} \left(\ln\frac{u(x)}{u(y)}\right)^2 \eta(x)^2 \leq C \sum_{\substack{x,y\in B^0\\ x\sim y}} |\eta(x)-\eta(y)|\eta(x)\left|\ln\frac{u(x)}{u(y)}\right|.$$

Corollary 116.

$$\sum_{\substack{x,y\in B^0\\ x\sim y}} \left(\ln\frac{u(x)}{u(y)}\right)^2 \eta(x)^2 \leq C \sum_{\substack{x,y\in B^0\\ x\sim y}} |\eta(x)-\eta(y)|^2.$$

Proof of Theorem 113: For $B := B(z,n) \subset 11B^0$, we apply Corollary 116 with the cut-off function η defined in $4B \subset (2\times 4-1)\times 11B^0$ as:

$$\begin{aligned}
\eta &\equiv 1 \quad \text{ on } 2B,\\
\eta(x) &= \frac{4n-1-d(z,x)}{2n} \quad \text{ on } 4B\setminus 2B,\\
|\eta(x)-\eta(y)|^2 &\le \frac{1}{4n^2},\ x\sim y,
\end{aligned}$$

to get

$$\begin{aligned}
\sum_{\substack{x,y\in 2B\\ x\sim y}} \left(\ln\frac{u(x)}{u(y)}\right)^2 &\le \sum_{\substack{x,y\in 4B\\ x\sim y}} \left(\ln\frac{u(x)}{u(y)}\right)^2 \eta(x)^2\\
&\le C \sum_{\substack{x,y\in 4B\\ x\sim y}} |\eta(x)-\eta(y)|^2\\
&\le C\frac{1}{n^2}\#(B).
\end{aligned}$$

Next, using Hölder, Poincaré and (6.2), we obtain

$$\begin{aligned}
\frac{1}{\#(B)}\sum_{x\in B}|\ln u(x)-(\ln u)_B| &\le \frac{1}{\#(B)^{1/2}}\left(\sum_{x\in B}|\ln u(x)-(\ln u)_B|^2\right)^{1/2}\\
&\le \frac{1}{\#(B)^{1/2}}\left(C_2\, n^2\sum_{\substack{x,y\in 2B\\ x\sim y}}\left(\ln\frac{u(x)}{u(y)}\right)^2\right)^{1/2}\\
&\le C,
\end{aligned}$$

which yields $\|\ln u\|_{\mathrm{BMO}(11B^0)} \le C$. $\square$

For the proof of Theorem 114, we need a proposition and a lemma first. The proof of Proposition 117 can be found in [21].

Proposition 117 (Local boundedness)**.** *There exists a $C>0$ depending only on C_1, C_2 and C_3 such that if $\mathcal{L}u \ge 0, u>0$ on $2B^0, B^0 := B(x_0,n_0)$, then*

$$\max_{B^0} u \le \frac{C}{\#(2B^0)}\left(\sum_{x\in 2B^0} u(x)^2\right)^{1/2}.$$

We also need the following lemma which is a discrete version of Fabes Lemma in [25].

Lemma 118 (Fabes Lemma)**.** *Let v be a real-valued function defined on B^0 and $0 < \varepsilon \leq 1$. There exists a $C > 0$ depending only on C_1, C_2, C_3 and ε such that if*

$$\#(E) := \#(\{x \in B^0 \mid v(x) = 0\}) \geq \varepsilon \#(B^0),$$

then

$$\frac{1}{\#(B^0)} \sum_{x \in B^0} v(x)^2 \leq C\, n^2 \frac{1}{\#(B^0)} \sum_{\substack{x,y \in B^0 \\ x \sim y}} (v(x) - v(y))^2 .$$

Proof.

$$\begin{aligned} |v(x)| = |v(x) - v_E| &\leq |v(x) - v_{B^0}| + |v_{B^0} - v_E| \\ &\leq |v(x) - v_{B^0}| + \frac{1}{\#(E)} \sum_{x \in B^0} |v(x) - v_{B^0}| \\ &\leq |v(x) - v_{B^0}| + \frac{1}{\varepsilon \#(B^0)} \sum_{x \in B^0} |v(x) - v_{B^0}|. \end{aligned}$$

Now, squaring, averaging over B^0, and using Poincaré,

$$\begin{aligned} \frac{1}{\#(B^0)} \sum_{x \in B^0} v(x)^2 &\leq \left(1 + \frac{1}{\varepsilon^2}\right) \frac{1}{\#(B^0)} \sum_{x \in B^0} (v(x) - v_{B^0})^2 \\ &\leq \left(1 + \frac{1}{\varepsilon^2}\right) \frac{C_2 n^2}{\#(B^0)} \sum_{\substack{x,y \in B^0 \\ x \sim y}} (v(x) - v(y))^2 . \end{aligned}$$

□

Proof of Theorem 114: Define $g(t) = \max\{-\ln t, 0\}$, $t > 0$ and note that by hypothesis on u, $v := g(u)$ verifies $\mathcal{L}v \leq 0$ on $4B^0$. Furthermore, note

$$\#(\{x \in 2B^0 \mid v(x) = 0\}) = \#(\{x \in 2B^0 \mid u(x) \geq 1\}) \geq \varepsilon \#(2B^0).$$

Now, using Proposition 117, Lemma 118 and Theorem 113 respectively,

$$\begin{aligned}
\max_{B^0} v &\leq C \left(\frac{1}{\#(2B^0)} \sum_{x \in 2B^0} v(x)^2 \right)^{1/2} \\
&\leq C \left(\frac{n^2}{\#(2B^0)} \sum_{\substack{x,y \in 2B^0 \\ x \sim y}} (v(x) - v(y))^2 \right)^{1/2} \\
&\leq C \frac{n}{\#(2B^0)^{1/2}} \left(\sum_{\substack{x,y \in 2B^0 \\ x \sim y}} \left(\ln \frac{u(x)}{u(y)} \right)^2 \right)^{1/2} \\
&\leq C \frac{n}{\#(2B^0)^{1/2}} \left(\frac{1}{n^2} \#(B^0) \right)^{1/2} = C,
\end{aligned}$$

which, in terms of u, becomes

$$\min_{B^0} u \geq e^{-C}.$$

$\square$

Chapter 7

Summary

In these final pages, we do a brief review of the dissertation by recapitulating some of its major ideas or contents. We include definitions, references, statement of the main result, and some pertinent remarks in as much a self-contained way as possible.

In 1887, Carl Gustav Axel Harnack in his book [36] on potential theory, introduced Harnack's inequality as a property verified by non-negative harmonic functions in the case of $n = 2$. A version of this inequality in the form it is currently used in the theory of partial differential equations reads: (first presented by Kellogg in [40]):

Theorem 119. *Let $\Omega \subset \mathbb{R}^n$ be a domain. There exists a constant $C_H = C_H(n, \Omega) > 1$ such that for any non-negative harmonic function u in Ω and every ball $B_{2R}(x_0) \subset \Omega$, we have*

$$\sup_{B_R(x_0)} u \leq C_H \inf_{B_R(x_0)} u.$$

Moser's and Krylov-Safonov's techniques in the context of divergence form PDEs [47, 49] and non-divergence form PDEs [43, 44] respectively are regarded as ground-breaking in the study of regularity problems of solutions to elliptic PDEs. Let $\Omega \subset \mathbb{R}^n$ be a domain, u a real-valued function on Ω, and A a matrix-valued function on Ω having a uniform ellipticity, i.e., it verifies for some $0 < \lambda < \Lambda$,

$$\lambda|\xi|^2 \leq \langle A(x)\xi, \xi\rangle := \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2,\ x \in \Omega,\ \xi \in \mathbb{R}^n.$$

In [47, 49], Moser verified Harnack's inequality for non-negative solutions to divergence-form elliptic operators defined as

$$\mathcal{L}u := \sum_{i,j=1}^{n} (a_{ij}(x)u_i)_j = \operatorname{div}(A(x)\nabla u),\ x \in \Omega. \tag{7.0.1}$$

The key step in Moser's approach is to get the following estimate:

$$\sup_B u \le C(n,q,\lambda,\Lambda) \left(\frac{1}{|B|} \int_{2B} u(x)^q \, dx \right)^{\frac{1}{q}},\ 2B \subset \Omega, \forall q > 0,$$

for a supersolution u which is accomplished through Moser's celebrated iterative scheme. Another breakthrough in this direction came along when Krylov and Safonov [43, 44] introduced new measure-theoretic and probabilistic tools to establish Harnack's inequality for non-negative solutions to non-divergence-form elliptic operators defined as

$$Lu := \sum_{i,j=1}^{n} a_{ij}(x)u_{ij} = \operatorname{tr}(A(x)D^2u),\ x \in \Omega. \tag{7.0.2}$$

The central idea in Krylov-Safonov's technique is a measure-theoretic property about a control on the size of a function in terms of a prescribed probability. Given $M > 1$ and $\varepsilon \in (0,1)$, a function $u \ge 0$ defined on Ω is said to have the *critical density property* with constants M and ε if for every $B_{2r} \subset \Omega$, the following holds true:

$$|\{x \in B_{2r} : u(x) > M\}| \ge \varepsilon |B_{2r}| \quad \Rightarrow \quad \inf_{B_r} u > 1.$$

Since the introduction of Moser's and Krylov-Safonov's techniques, several authors have been able to adapt, extend and generalize these techniques in several contexts. My research in this topic has been toward an axiomatization of Krylov-Safonov's approach to Harnack's inequality in spaces of homogeneous type. The main theme of this dissertation is the axiomatization of Harnack's inequality in the context of doubling quasi-metric spaces. This context is defined next.

A triad (X, d, μ) is said to be a *space of homogeneous type* if X is a non-empty set; d is a *quasi-metric* on X, i.e., it relaxes the triangle inequality of a metric: there exists a constant

$K > 1$ ($K = 1$ if d is a metric) such that

$$d(x, y) \leq K(d(x, z) + d(z, y)) \text{ for every } x, y, z \in X;$$

and μ is a measure compatible with the quasi-metric d in the sense that μ is doubling with respect to the d-balls: there exists a constant $C_\mu > 1$ such that for every d-ball $B_{2r} \subset X$, we have

$$\mu(B_{2r}) \leq C_\mu \mu(B_r).$$

Examples of spaces of homogeneous type include: (1) Euclidean spaces with a non-isotropic metric, $(\mathbb{R}^n, d, \mathcal{L})$, where d is the Euclidean metric and $\mathcal{L}$ is the Lebesgue measure; (2) $(\mathbb{R}^n, \rho_\phi, \mu_\phi)$, where $\phi \in C^2$ is a suitable convex function, $\mu_\phi(x) := \det D^2\phi(x)$ is the *Monge-Ampère measure* and

$$\rho_\phi(x, y) := \max\{\phi(y) - \phi(x) - \langle \nabla\phi(x), y - x\rangle, \phi(x) - \phi(y) - \langle \nabla\phi(y), x - y\rangle\};$$

(3) Boundaries of Lipchitz domains endowed with harmonic measure, (4) Connected Riemannian manifolds with a non-negative Ricci curvature, (5) Real connected Lie groups with polynomial volume growth, and (6) Graphs with a uniform bound on the number of neighbors of vertices.

Next, our axiomatization can be best understood by looking at one specific example of our result. Consider the second-order uniformly elliptic linear operator $\mathcal{L}$ in divergence form defined in (7.0.1). Using his celebrated technique of iterations, Moser [47, 49] established that

(a) the non-negative supersolutions of $\mathcal{L}$ possess the *property that these functions, raised to some positive power, are doubling as weights*: there exists constants $C_D > 1$ and $\varrho > 0$ such that if $u \geq 0$ is a supersolution of $\mathcal{L}$, then for every ball $B_{2R}(x_0) \subset \Omega$, u^ϱ is doubling as weight with constant C_D, i.e.,

$$\int_{B_{2R}(x_0)} u(x)^\varrho \, dx \leq C_D \int_{B_R(x_0)} u(x)^\varrho \, dx. \tag{7.0.3}$$

(b) the non-negative solutions of $\mathcal{L}$ verify *Harnack's inequality:* there exists a constant $C_H > 1$ such that if $u \geq 0$ is a solution of $\mathcal{L}$, then for every ball $B_{2R}(x_0) \subset \Omega$, we have

$$\sup_{B_R(x_0)} u \leq C_H \inf_{B_R(x_0)} u. \tag{7.0.4}$$

Next, we explain the goal of an axiomatization with reference to the above results. In the above example, the *underlying space* is the Euclidean space $\mathbb{R}^n$ together with the usual Euclidean metric and the usual Lebesgue measure and the *functional set* is the set of all non-negative supersolutions for (7.0.3) and the set of all non-negative solutions for (7.0.4) to the operator $\mathcal{L}$. In the case of solutions, the functional set is, in fact, a vector space over $\mathbb{R}$ and is called an $\mathbb{R}$-*functional space*. In the case of supersolutions, the functional set is called a *functional cone* since it is closed under multiplication by non-negative constants only. The premise of our axiomatic setting is to lay out sufficient assumptions on the *structure* so that Harnack's inequality can be established for a *functional set* defined on a domain of an *underlying space*. The structure refers to the domain, the underlying space and the functional set. The goal of the axiomatic setting, then, is to establish a property under bare minimum assumptions on the structure so that it can have the widest possible range of applicability. Krylov-Safonov [43, 44] introduced the following probabilistic properties in the context of second-order uniformly elliptic operators in non-divergence form:
For $M > 1$ and $\varepsilon \in (0,1)$, a function $u \geq 0$ defined on Ω is said to have the *critical density property* with constants M and ε if for every $B_{2r}(x_0) \subset \Omega$, the following holds true:

$$\mu\left(\{x \in B_{2r}(x_0) : u(x) > M\}\right) \geq \varepsilon \mu\left(B_{2r}(x_0)\right) \quad \Rightarrow \quad \inf_{B_r}(x_0) u > 1.$$

In addition to the critical density property, a few other properties were also used in these approaches which are defined below:

Let $\gamma \in (0,1)$. A function $u \geq 0$ defined on Ω is said to satisfy the *double-ball property* with constant γ if for every ball $B_{2R}(x_0) \subset \Omega$,

$$\inf_{B_{R/2}(x_0)} u \geq 1 \quad \Rightarrow \quad \inf_{B_R(x_0)} u \geq \gamma.$$

Let $\varrho \in (0,1)$ and $N > 1$. A function $u \geq 0$ defined on Ω is said to satisfy the *power-like decay property* with constants N and ϱ, if for every ball $B_{2R}(x_0) \subset \Omega$,

$$\inf_{B_R(x_0)} u \leq 1 \quad \Rightarrow \quad \mu(\{x \in B_{R/2}(x_0) : u(x) > N^k\}) \leq \varrho^k \mu(B_{R/2}(x_0)),\ k \in \mathbb{N}.$$

In order to axiomatize Harnack's inequality in a doubling quasi-metric space (X, d, μ), let Ω be a domain in X and $\mathbb{K}_\Omega$ denote a class of functions with domain contained in Ω which is closed under multiplication by positive constants. It helps to have in mind the class of subsolutions or supersolutions to homogeneous PDEs such as (7.0.1) and (7.0.2) as examples of $\mathbb{K}_\Omega$.

We say $\mathbb{K}_\Omega$ has a property P if every function in $\mathbb{K}_\Omega$ has the property P and the constants of the property are structural and do not depend on individual functions in $\mathbb{K}_\Omega$. Further denote $\mathbb{K}_\Omega(B_R(x_0)) := \{u \in \mathbb{K}_\Omega \mid B_R(x_0) \subset \mathrm{dom}(u)\}$.

Theorem 120 below gives an axiomatic approach to Harnack's inequality in spaces of homogeneous type, the proof of which is based on Krylov-Safonov's techniques.

Theorem 120. *Let (X, d, μ) be a doubling quasi-metric space. Let $\Omega \subset X$ be open. Let $\mathbb{K}_\Omega$ be a set of non-negative functions defined on X which is closed under multiplication by positive constants. Further denote $\mathbb{K}_\Omega(B_R(x_0)) := \{u \in \mathbb{K}_\Omega \mid B_R(x_0) \subset \mathrm{dom}(u)\}$. Assume the following:*

(i) The continuous functions are dense in $L^1(X, d\mu)$.

(ii) $\mathbb{K}_\Omega$ satisfies the critical density property with constants M and ε.

(iii) There exist constants $\varrho > 0$ and $C_D \geq 1$ such that whenever $u \in \mathbb{K}_\Omega$, u^ϱ is a doubling weight with constant C_D, that is,

$$\int_{B_{2r}(x)} u^\varrho \, d\mu \leq C_D \int_{B_r(x)} u^\varrho \, d\mu,$$

for every $B_r(x)$ with $B_{8Kr}(x) \subset\subset \Omega$.

(iv) Whenever $u \in \mathbb{K}_\Omega(B_R(x_0))$ and $\lambda \geq u$ in $B_R(x_0)$ then $\lambda - u \in \mathbb{K}_\Omega(B_R(x_0))$.

Then $\mathbb{K}_\Omega$ satisfies the Harnack property with constant C_H depending only on ε, M, C_D, and the doubling quasi-metric constants K and C_μ.

Our axiomatic approach to Harnack's inequality in doubling quasi-metric spaces provides an alternative to earlier such approaches [3, 25, 57]. These approaches were established under different sets of assumptions on the structure. Detailed discussions of the assumptions in our as well as others' approaches are done in the main body of the dissertation. Here, we only summarize them with a few remarks:

(1) Other axiomatic approaches assume the double-ball property instead of the doubling property as a weight, which is a novel idea in our approach.

(2) The double-ball property and the doubling property as a weight are unrelated to each other. More details are in Section 4.4.

(3) Aimar, Forzani and Toledano [3] had an additional assumption on the functional set $\mathbb{K}_\Omega$ which restricts it to consist of only upper semi-continuous functions.

(4) DiFazio, Gutiérrez and Lanconelli [25] do not make such an a priori assumption on the functional set but their method requires covering lemmas.

(5) Our approach bypasses the use of covering lemmas and BMO properties.

(6) Our approach is better-suited to variational operators than non-variational operators. More details are in Section 4.5.

(7) The density of continuity functions in L^1 is a mild assumption since every locally compact metric space has this property and every quasi-metric space is metrizable. More details are in Section 4.3.

(8) Our approach can be viewed as an alternative approach to Bombieri's result [8], which significantly simplified Moser's approach to Harnack. More details are in Section 4.7.

The dissertation also presents an application of our axiomatic approach to Harnack's inequality described in Chapter 4. Chapter 6 applies the the main result of the dissertation found in Section 4.2.3 to harmonic functions on graphs. Besides the axiomatization results and their applications, the dissertation also presents some new as well as expository results. Chapter 2 introduces a novel diagrammatic approach of looking at reverse inequalities which facilitates the illustration of various ideas throughout the dissertation. Chapter 5 axiomatizes the power-like decay property, a weaker version of the Harnack property (see Section 81), in the setting of metric spaces having the segment and segment-prolongation properties.

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