CUBICAL CATEGORIES, TQFTS AND POSSIBLE NEW REPRESENTATIONS FOR THE POINCARE GROUP

by

DANY MAJARD

M.Sc., Université de la Méditerranée, France, 2006

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
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KANSAS STATE UNIVERSITY
Manhattan, Kansas
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Abstract

In this thesis we explore the possibilities of obtaining Topological Quantum Field Theories using cobordisms with corners to break further down in the structure of manifolds of a given dimension. The algebraic data obtained is described in the language of higher category theory, more precisely in its cubical approach which we explore here as well. Interesting connections are proposed to some important objects in Physics: the representations of the Poincaré group. Finally we will describe in great details the topological tools needed to describe the categories of cobordisms with corners and give some conjectures on their nature.

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Abstract

In this thesis we explore the possibilities of obtaining Topological Quantum Field Theories using cobordisms with corners to break further down in the structure of manifolds of a given dimension. The algebraic data obtained is described in the language of higher category theory, more precisely in its cubical approach which we explore here as well. Interesting connections are proposed to some important objects in Physics: the representations of the Poincaré group. Finally we will describe in great details the topological tools needed to describe the categories of cobordisms with corners and give some conjectures on their nature.

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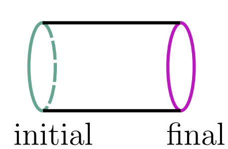
Chapter 1

Introduction

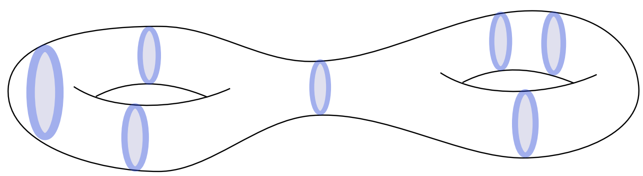
Eilenberg and MacLane's category theory [48] brought a considerable paradigm shift [2] in the mathematical community since its introduction in the early 1940s. The language of category theory has spread to other areas of science in these last 70 years, including computer science, physics [19], economics [27] and biology [38, 30] to cite a few. One of its most notable achievement has been to provide a better way to encode and understand processes and move away from a set theoretic mindset, which encodes the immutable. And as processes are prevalent in physics, it is not surprising to meet categories all throughout the field. Groups, which describe fundamental symmetries of our Universe, are first hand examples, and of extreme importance in classical mechanics and in relativity. Moreover their representations form categories whose structures give fundamental particles and their relationships in the standard model.

Our concern, in this thesis, is to build new models for **Topological Quantum Field Theories**. TQFTs emerged from physics as a fairly simple idea: a space or a system in a certain state, such as a slice of space time, is a topological space enriched with some algebraic data and whose dynamics "propagates" it to another state. The propagation, as a whole, is a topological space itself and the TQFT encodes the transformation on the associated algebraic data in a coherent manner. If, for example, the system stays in the same state, represented by a circle, the propagation would correspond a cylinder and the operation would just be the identity on the algebraic data associated to the circle.



As we mentioned previously, category theory seems to be language in which such processes are expressed best and the category of topological spaces and their "evolutions" is called the **cobordism category**. To get an algebraic description of the system, so that we can quantify, produce forecasts, et cetera, we need a **functor**: a rule to reproduce the content of one category, the topological one, into another category, the algebraic one that contains our data. In our case the functor is what we call a TQFT and the category of algebraic data is often the category of finite dimensional vector spaces and linear maps. These axioms were given in this language by Atiyah in 1988 [9], and states that a TQFT is a **monoidal functor**, which means that it preserves the notion of combination of isolated systems, or in topological terms the disjoint union.

Our original task was to use the construction of Crane [21] for oriented 3d TQFTs and to extend it to the unoriented case, just as Turaev did for the oriented 2d TQFTs in 2006 [55]. Crane's TQFT is a state sum, which means finding a decomposition of the manifold, labeling the pieces, summing over all possible labellings and then making sure that it does not depend on the choice of decomposition. In our case the decomposition uses Heegard splittings of 3-manifolds [52]. A Heegard splitting is the data of a handlebody, which is a manifold with borders obtained by gluing handles to balls, together with an isomorphism of its boundary such that gluing two copies of the handlebody via the given isomorphism results in a space isomorphic to the considered 3-manifold. The state sum is then found by considering a decomposition of the handlebody given by "slicing it",



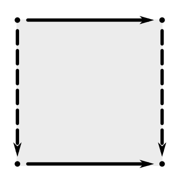
labeling the boundary disks of the pieces with irreducible representations of a quantum groups and the pieces with intertwiners between these representations. Tensoring all these together and summing over all possible labellings results in an intertwiner from the trivial representation to the representation V our labeling of the disks gave us, i.e. an invariant vector $v \in V$. Then by using a representation of the mapping class group of the handlebody we may get a linear map $\phi : V \to V$ corresponding to the isomorphism of the Heegard splitting and therefore a number for the 3-manifold by $v \cdot \phi(v)$. The invariance under all the choices made is what the subject of [21].
Our intuition was that there was an intrinsic way of formulating the model, i.e. without referring to the actual representations of the quantum group, and that once this formulation was found, new examples would come out and the generalization to unoriented 3-manifolds would be as straightforward as in the 2d case.
This led us to rediscover double categories in a very natural way from the topology standpoint : manifolds with borders glue like morphisms in a category while manifolds with corners glue like squares in a double category. Though double categories have actually been around since the mid 1960s, from the work of Ehresmann [26], they remain little known. Double categories have allowed some important breakthroughs in topology, such as the understanding of non commutative higher homotopy theory and its associated Van Kampen theorem by Brown and Higgins [16], and even in TQFTs where Kerler and Lyubashenko [40] found non semisimple 3D TQFTs for oriented manifolds using them, though in a different manner from that in this thesis.

The principle underlying higher category theory, to which double categories belong, is the understanding that if sets were to be thought of as “0-dimensional”, categories would be “1-dimensional” and that they therefore are but the first step in the widening of our toolbox for constructing TQFTs. Finding the right candidate for the “2-dimensional” entity, and more importantly the “n-dimensional” one, has been the source of a considerable amount of work during the last 30 years, c.f. [10] for a better historical perspective. We believe that the most qualified candidate is to be found in **cubical categories**, or **n-tuple categories**.

In the rest of the introduction, we will present the chapters and expose their material in a sketched, simplified way in order for the reader to get somewhat of a general view of the thesis. The reader may then read the sections of his choice and refer back to the introduction and the table of content to navigate his way through the thesis.

1.1 Double categories

With this vision in mind, sets are akin to points, all separated to each other, and categories are akin to a graph, with points and arrows between them. **Double categories** are akin to double graphs, with points, vertical and horizontal arrows between them, themselves the boundaries of squares. A general element of a double category can then be represented by a square



and such squares can be composed in two directions, horizontally and vertically. Each direction has identities, which forces the horizontal arrows and vertical arrows to be categories themselves, and satisfy the interchange law, which is a 2 dimensional notion of “commutativity”. If we denote the compositions by $\circ_h$ and $\circ_v$, the interchange reads :

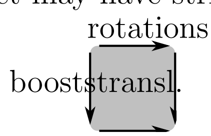
$$(A \circ_v B) \circ_h (C \circ_v D) = (A \circ_h C) \circ_v (B \circ_h D)$$

and it is true whenever both sides exist. Accordingly, a **double functor** is a rule reproducing the content of a double category in another double category in such a way that composition is respected.
Double categories have symmetries that may be expressed functorially, just as categories had with the **opposite category** construction, except that instead of sharing symmetries with the interval, double categories share symmetries with the square. There will then be horizontal opposite double categories, vertical opposite categories and more. Moreover there now exists two possible notions of “transverse inverses”, called either **connections**

and **adjunctions**, that is, for a square A, a square A' such that horizontal composition of the two horizontally yields a vertical identity,

$$A \circ_h A' = id_v$$

A very interesting family of examples of double categories stem from **double groupoids**: double categories where every square is invertible w.r.t. both compositions. There is a very interesting familiy of double groupoids we will be talking about, the exclusive double groupoids. In the case where they have a single object, they are **double groups** and we will show that these are in one-to-one correspondence with semi direct products of an abelian group with a bicrossed products of groups [3]. This family is already very interesting due to its relation to the expansion problem in group theory[4], but we make it even more appealing here by showing that the Poincaré group, absolutely central to modern physics, is very naturally an exclusive double group. Indeed the KAN decomposition of the Lorentz group, one of its subgroups, allows a presentation where boosts are confined to one boundary type, rotation in Euclidian 3-space are confined to the other boundary type while the translations are encoded in the squares. This fact may have striking consequences but they will not be



found until the theory of representations of double groups is constructed and sufficiently understood. The impact though may be glanced by considering the recent EPRL model [23, 22, 28], which is currently the most promising attempt to date to a quantum theory of gravity. In this model, the spacial part of the Poicaré group and its space-time components have distinct roles and we are hoping that this fact will be the example that stimulates more research on double groups.

Another interesting feature of double groups with potential interest for theoretical Physics is the existence of a certain diagram, called **core diagram** [18] or in other contexts butterfly[1], that describes it. To what extent the core diagram is a classifying invariant of double group is an open question but we already know it to be very strong. These diagrams have a very interesting structure and we make a connection to recent discoveries in particle physics by identifying subgroups of $SO(4)$, corresponding to 4D platonic solids, as giving such diagrams. These subgroups are of importance regarding the neutrino oscillations, as testifies the work of Ma [47, 46]. As neutrino oscillation seems to be the key to many mysteries in particle physics, we believe that this fact may inspire further research on double groups as well.
All in all, it seems too much of a coincidence for all these unrelated pieces, the group of relativity of the theory of Einstein, the neutrino oscillations of particle physics, and the TQFTs in dimension 3 and 4, to lead to double groups and more generally cubical categories without a connection to exist. We would then like to speculate that there will be TQFTs build out representations of double groups such as the ones we mentioned.

1.2 n-tuple categories

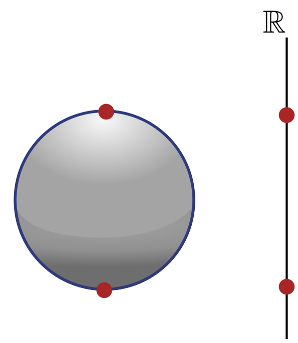
The question arising naturally is then : "What about dimension n categories?". The definition of a double category extends naturally to any dimension and we get **n-tuple categories** where a general element is an n-cube and compose associatively along n directions, each of them "commuting" with the others, i.e. respecting the interchange laws. As double categories are best described as **internal categories** in the categories of small categories, we will recall the definition as well as that of an **internal functor** and then extend the study to **internal natural transformations**, **internal natural cells** and **internal comparisons** that seem to appear nowhere in the literature. We will show that there is a recursive process at play that is different from the one found in the n-category program. As n-categories are funded on opeotopes and that opeotopes are collapsed n-cubes, it will appear that n-categories are a special case of n-tuple categories and will give the bestiary of degeneracies in dimension 2.

The definition of double groupoids extends naturally to n-tuple groupoids and we will show that so does the notion of exclusivity. It will allow us to generalize the results obtained in dimension 2 to any dimension, which confirms the conjecture of Brown [17] for dimension 3. The notion of core diagram needs to be extended though and this matter is not addressed in the present document.

Instead we will try to give a hint on how a theory of **weak internal categories** and **weak n-tuple categories** ought to be constructed harvesting the fruits of the analysis of the **strict** (regular) case. We will give in detail the example of a weak double category. A prime example, and the one that brought us to consider n-tuple categories in the first place, is the notion of **spans** (or **cospans**) in a category with pull-backs (respectively push-forwards), of which cobordisms are a subcategory.

1.3 Singularity theory

The real challenge in the description of TQFTs is the description of the cobordism category by generators and relations. It is a standard result [41] that in dimension 2 it is equivalent to a category generated by a Frobenius object for the oriented cobordisms and an extended Frobenius algebra for the unoriented case, but no such result is known for higher dimensions, though Lurie's approach claims such a result, via (∞, n)-categories [45]. But what we propose is to take lessons from the dimension 2 and adapt it properly to dimensions 3 and 4. This is made possible by the use of n-tuple categories. Indeed, in dimension two the result is solely based on Morse theory [49], which is the analysis of critical points of a certain class of smooth functions. For compact manifolds it boils down to smooth maps from a manifold with corners to an interval.



And since the oriented interval is the model upon which we build category theory, it is obvious that it is the correct tool to investigate the categories of cobordisms. The problem is that while the theory of critical points of Morse functions gets only slightly more complex as dimensions grow, it is not so of the preimages of opens around a critical value. We will give a brief description of some subcategory that is always present, the one generated by spheres, but in the general case, they become exponentially more intractable. This explains the extreme difficulty to find TQFTs, as we defined them, in higher dimensions. This situation gets back to manageable computations if we move away from the interval onto higher dimensional cubes, as we will explain in detail. The structures needed to do such work is then, very obviously, n-tuple categories of **n-tuple cospans**, modeling cobordisms with corners.
To find the generators and relations, we give some results of Singularity theory and show that for dimension 3 and 4 there is a finite number of generators and relations. We recall that in dimension 2, generators are given by singularities of the **fold** type and relations by folds and **cusps**. Then we show that in dimension 3, they are given by singularities of the following types : **folds**,**cusps**, and **swallowtails** for relations only. Then, depending on some combinatorial data that we will give, each of these types may have more than one corresponding generator or relation.

1.4 Prospects

This research is, as of the writing of this document, a programme of research in progress. In the last chapter we will describe the parts that still need work and give a few conjectures and expectations regarding higher category theory, higher group theory and the higher categories of cobordisms with corners. It is our hope that we will be able to carry it through and that it generates fruitful collaborations with like-minded mathematicians and physicists.
We would like to thank the encouragements and ideas of Prof. Louis Crane and Prof. David Yetter all throughout the birth of this programme. This thesis would not exist without them. Also we would like to particularly thank Prof. Crane for his fantastic insight.

Chapter 2

Double categories

As we mentioned in Chapter 1.1 double categories are the natural successor to categories in the climbing of dimensions. As our TQFT on cobordisms with corners will be based on double categories and double functors, we will now explore in details these entities and find very interesting examples of double groups that we think will provide some of the first tools to actually compute such a TQFT.

2.1 Double categories

2.1.1 Definitions

Definition 2.1.1. *A **double category** is a set* $(O, H, V, S, s_{1,h}, t_{1,h}, s_{1,v}, t_{1.v}, s_{2,h}, t_{2,h}, s_{2,v}, t_{2,v}, \imath_{1,h}, \imath_{1,v}, \imath_{2,h}, \imath_{2,v}, \circ_{1,h}, \circ_{1,v}, \circ_{2,h}, \circ_{2,v})$ *where :*

- *The sets O,H, V,S are respectively objects, horizontal arrows, vertical arrows and squares.*
- *The following maps of sets are the source and target maps :*

$$s_{1,h}, t_{1,h} : H \to O \qquad s_{2,h}, t_{2,h} : S \to V$$
$$s_{1,v}, t_{1.v} : V \to O \qquad s_{2,v}, t_{2,v} : S \to H$$

- *The following maps of sets are the identity maps :*

$$\imath_{1,h} : O \to H \qquad \imath_{2,h} : V \to S$$
$$\imath_{1,v} : O \to V \qquad \imath_{2,v} : H \to S$$

- *The following maps of sets are the composition maps.*

$$\circ_{1,h} : H \times_O H \to H \qquad \circ_{2,h} : S \times_V S \to S$$
$$\circ_{1,v} : V \times_O V \to V \qquad \circ_{2,v} : S \times_H S \to S$$

- *The following holds :*

– *Sources and targets are compatible the following way :*

$$s_{2,h}s_{1,v} = s_{2,v}s_{1,h} \qquad s_{2,h}t_{1,v} = t_{2,v}s_{1,h}$$
$$t_{2,h}s_{1,v} = s_{2,v}t_{1,h} \qquad t_{2,h}t_{1,v} = t_{2,v}t_{1,h}$$

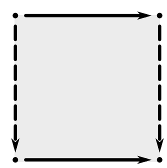
– $\imath_{i,\alpha}$ *is an identity for the composition* $\circ_{i,\alpha}$*.*

– *the compositions* $\circ_{i,\alpha}$ *are associative.*

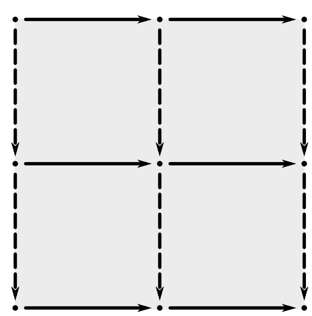
– $\circ_{2,v}\circ_{2,h} = \circ_{2,h}\circ_{2,v}$ *whenever it makes sense.*

We will drop the first index in the source, target and identity maps when obvious. For example, for a vertical arrow f, we will write its identity square $i_h(f)$.

A general element of a double category looks like a square:



It has vertical and horizontal sources and targets, the sides, which themselves have sources and targets, the corners. It also has horizontal and vertical compositions with their associated identities. More specifically, a double category is constituted of one kind of objects, two (possibly different) kinds of arrows and one kind of squares. The above picture depicted the two different types of arrows with plain and dotted lines. It should be noted that the vertical source of a square is a horizontal arrow and vice-versa. Accordingly, vertical identities are identities on horizontal arrows. It is imposed on double categories that the "interchange law" holds, i.e. any assortment of the sort :



yields the same square regardless of how it is composed. In a paper published in 1993, [51] R.Dawson & R.Pare gave general conditions for the existence of composition for general "tilings", or arrangements of squares and it is a good read to comprehend a little better what happens in double categories.

Definition 2.1.2. *A **double functor** between double categories is a map of sets sending objects to objects, arrows to arrows and squares to squares while respecting source, target, identities and composition.*

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, dashed, "g"'] \arrow[dr, phantom, "A"] & b \arrow[d, dashed, "g'"] \\ c \arrow[r, "f'"'] & d \end{tikzcd} \longrightarrow \begin{tikzcd} F(a) \arrow[r, "F(f)"] \arrow[d, dashed, "F(g)"'] \arrow[dr, phantom, "F(A)"] & F(b) \arrow[d, dashed, "F(g')"] \\ F(c) \arrow[r, "F(f')"'] & F(d) \end{tikzcd}$$

They are the two dimensional equivalent of functors of categories. Let's visualize what a functor F would do :

Definition 2.1.3. *Let F and G be double functors : $C \to D$, a* ***horizontal double natural transformation*** *from F to G is a map of sets that sends vertical arrows of C to squares in D in such a way that it intertwines these functors horizontally.*

Let's visualize what a horizontal double natural transformation $\omega : F \to G$ does :

$$\begin{tikzcd} a \arrow[d, dashed, "f"'] \\ b \end{tikzcd} \longrightarrow \begin{tikzcd} F(a) \arrow[r, "\omega(a)"] \arrow[d, dashed, "F(f)"'] \arrow[dr, phantom, "\omega(f)"] & G(a) \arrow[d, dashed, "G(f)"] \\ F(b) \arrow[r, "\omega(b)"'] & G(b) \end{tikzcd}$$

The intertwining is represented by the following identity :

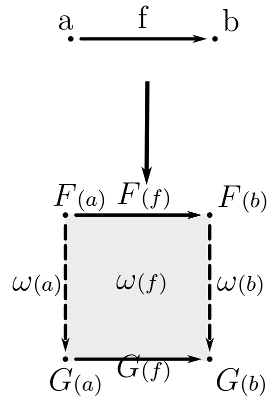
$$\begin{tikzcd} F(a) \arrow[r, "F(h)"] \arrow[d, dashed, "F(f)"'] \arrow[dr, phantom, "F(A)"] & F(c) \arrow[r, "\omega(c)"] \arrow[d, dashed, "F(g)"] \arrow[dr, phantom, "\omega(g)"] & G(c) \arrow[d, dashed, "G(g)"] \\ F(b) \arrow[r, "F(k)"'] & F(d) \arrow[r, "\omega(d)"'] & G(d) \end{tikzcd}$$

$$=$$

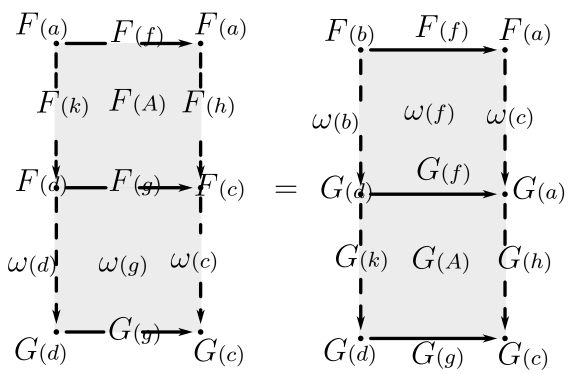
$$\begin{tikzcd} F(a) \arrow[r, "\omega(a)"] \arrow[d, dashed, "F(f)"'] \arrow[dr, phantom, "\omega(f)"] & G(a) \arrow[r, "G(h)"] \arrow[d, dashed, "G(f)"] \arrow[dr, phantom, "G(A)"] & G(c) \arrow[d, dashed, "G(g)"] \\ F(b) \arrow[r, "\omega(b)"'] & G(d) \arrow[r, "G(k)"'] & G(d) \end{tikzcd}$$

Definition 2.1.4. *Let F and G be double functors : $C \to D$, a* ***horizontal double natural transformation*** *from F to G is a map of sets that sends vertical arrows of C to squares in D in such a way that it intertwines these functors horizontally.*

Let's visualize what a vertical double natural transformation $\omega : F \to G$ does :



The intertwining this time being vertical, it is represented by the identity :



From their intertwining properties it is clear that these double natural transformations compose in two ways and that this composition is associative and unital. We will prove this later in a more abstract setting. Being given horizontal and vertical double natural transformations one could look for an entity intertwining them. Such an entity exists and has been given different names throughout the literature, the most common being comparison and modification. They are built on the same principle as natural transformations of functor of categories but assign a square instead of an arrow to objects.

Definition 2.1.5. *Given four double functors $F, G, H, J : C \to D$, two horizontal double n.t. $\omega : F \to G$ and $\Omega : H \to J$, and two vertical double n.t. $\delta : G \to H$ and $\Delta : F \to J$, a* ***double comparison*** *is a map of sets that sends objects of C to squares in D in such a way that it intertwines all the above entities in a plane.*

Let's visualize what such a double comparison does :

$$
\begin{tikzcd}
\bullet \, \mathrm{a} \arrow[r] & F_{(a)} \arrow[r, "\omega(a)"] \arrow[d, dashed, "\Delta(a)"'] \arrow[dr, phantom, "\diamond(a)"] & G_{(a)} \arrow[d, dashed, "\delta(a)"] \\
& J_{(a)} \arrow[r, "\Omega(a)"'] & H_{(a)}
\end{tikzcd}
$$

Intertwining on a plane means that for a given square in $\underline{\mathbf{C}}$ the four compositions below are equal :

$$
\begin{tikzcd}
F_{(a)} \arrow[r, "F(f)"] \arrow[d, dashed, "F(k)"'] \arrow[dr, phantom, "F(A)"] & F_{(b)} \arrow[r, "\omega(b)"] \arrow[d, dashed, "F(l)"] \arrow[dr, phantom, "\omega(l)"] & G_{(b)} \arrow[d, dashed, "G(l)"] \\
F_{(c)} \arrow[r, "F(g)"] \arrow[d, dashed, "\Delta(c)"'] \arrow[dr, phantom, "\Delta(g)"] & F_{(d)} \arrow[r, "\omega(d)"] \arrow[d, dashed, "\Delta(d)"] \arrow[dr, phantom, "\diamond(d)"] & G_{(d)} \arrow[d, dashed, "\delta(d)"] \\
J_{(c)} \arrow[r, "J(g)"] & J_{(d)} \arrow[r, "\Omega(d)"] & H_{(d)}
\end{tikzcd}
\qquad
\begin{tikzcd}
F_{(a)} \arrow[r, "\omega(a)"] \arrow[d, dashed, "F(k)"'] \arrow[dr, phantom, "\omega(k)"] & G_{(a)} \arrow[r, "G(f)"] \arrow[d, dashed, "G(k)"] \arrow[dr, phantom, "G(A)"] & G_{(b)} \arrow[d, dashed, "G(l)"] \\
F_{(c)} \arrow[r, "\omega(c)"] \arrow[d, dashed, "\Delta(c)"'] \arrow[dr, phantom, "\diamond(c)"] & G_{(c)} \arrow[r, "G(g)"] \arrow[d, dashed, "\delta(c)"] \arrow[dr, phantom, "\delta(g)"] & G_{(d)} \arrow[d, dashed, "\delta(d)"] \\
J_{(c)} \arrow[r, "\Omega(c)"] & H_{(c)} \arrow[r, "H(g)"] & H_{(d)}
\end{tikzcd}
$$

$$
\begin{tikzcd}
F_{(a)} \arrow[r, "F(f)"] \arrow[d, dashed, "\Delta(a)"'] \arrow[dr, phantom, "\Delta(f)"] & F_{(a)} \arrow[r, "\omega(b)"] \arrow[d, dashed, "\Delta(b)"] \arrow[dr, phantom, "\diamond(b)"] & G_{(b)} \arrow[d, dashed, "\delta(b)"] \\
J_{(a)} \arrow[r, "J(f)"] \arrow[d, dashed, "J(k)"'] \arrow[dr, phantom, "J(\beta)"] & J_{(b)} \arrow[r, "\Omega(b)"] \arrow[d, dashed, "J(l)"] \arrow[dr, phantom, "\Omega(l)"] & H_{(b)} \arrow[d, dashed, "H(l)"] \\
J_{(c)} \arrow[r, "J(g)"] & J_{(d)} \arrow[r, "\Omega(d)"] & H_{(d)}
\end{tikzcd}
\qquad
\begin{tikzcd}
F_{(a)} \arrow[r, "\omega(a)"] \arrow[d, dashed, "\Delta(a)"'] \arrow[dr, phantom, "\diamond(a)"] & G_{(a)} \arrow[r, "G(f)"] \arrow[d, dashed, "\delta(a)"] \arrow[dr, phantom, "\delta(f)"] & G_{(b)} \arrow[d, dashed, "\delta(l)"] \\
J_{(a)} \arrow[r, "\omega(c)"] \arrow[d, dashed, "J(k)"'] \arrow[dr, phantom, "\Omega(k)"] & H_{(a)} \arrow[r, "H(f)"] \arrow[d, dashed, "H(k)"] \arrow[dr, phantom, "H(A)"] & H_{(b)} \arrow[d, dashed, "H(l)"] \\
J_{(c)} \arrow[r, "\Omega(c)"] & H_{(c)} \arrow[r, "H(g)"] & H_{(d)}
\end{tikzcd}
$$

2.1.2 Examples

- Let $\underline{\mathbf{C}}$ be a category, then the set of all square diagrams in $\underline{\mathbf{C}}$ forms a double category.
  - objects are the objects in $\underline{\mathbf{C}}$.
  - horizontal arrows are the arrows of $\underline{\mathbf{C}}$
  - vertical arrows are the arrows of $\underline{\mathbf{C}}$
  - squares are square diagram in $\underline{\mathbf{C}}$

- Let $\underline{\mathbf{C}}$ be a category, then the set of all commutative square diagrams in $\underline{\mathbf{C}}$ forms a double category.
- Let $\underline{\mathbf{C}}$ be a 2-category. Then the **quintet double category** of $\underline{\mathbf{C}}$ is the double category whose :
  - objects are the objects in $\underline{\mathbf{C}}$.
  - horizontal arrows are the arrows of $\underline{\mathbf{C}}$
  - vertical arrows are the arrows of $\underline{\mathbf{C}}$
  - squares are 2-cells of $\underline{\mathbf{C}}$: $s_h(S)t_v(S) \to s_v(S)t_h(S)$ for a square S.

  A square in a quintet double category then looks like the following :

$$\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "h"'] & {} \arrow[d, "g"] \arrow[dl, Rightarrow, "\eta"'] \\ {} \arrow[r, "j"'] & {} \end{tikzcd}$$

  with η a natural transformation $fg \to hj$.

2.1.3 Symmetries

As we explained in Section 1.1, categories had symmetries originating from the ones of the interval. It is this symmetry that allowed the existence of covariant and contravariant functors. As we have not encountered the equivalent study for double categories, we will write down all their symmetries now and the associated types of functors. It is important to us as our categories of interest, the cobordism n-tuple categories, will possess most of these symmetries.

Definition 2.1.6. *Let $\underline{\boldsymbol{C}}$ be a double category.*

- *It is **edge symmetric** if there exist an equivalence $\phi : \underline{\boldsymbol{C}}_{1,h} \to \underline{\boldsymbol{C}}_{1,v}$ that is the identity on objects.*
- *Its **horizontally opposite** category $\underline{\boldsymbol{C}}^{hop}$ is the double category whose horizontal category is the opposite of the one of $\underline{\boldsymbol{C}}$, the vertical one is unchanged and $\underline{\boldsymbol{C}}_2^{hop}\big((f,g),(h,j)\big) = \underline{\boldsymbol{C}}_2\big((f^{op},g^{op}),(j,h)\big)$.*
- *Its **vertically opposite** category $\underline{\boldsymbol{C}}^{hop}$ is the double category whose vertical category is the opposite of the one of $\underline{\boldsymbol{C}}$, the horizontal one is unchanged and $\underline{\boldsymbol{C}}_2^{vop}\big((f,g),(h,j)\big) = \underline{\boldsymbol{C}}_2\big((g,f),(h^{op},j^{op})\big)$.*
- *Its **opposite** category $\underline{\boldsymbol{C}}^{op}$ is the double category whose vertical category is the opposite of the one of $\underline{\boldsymbol{C}}$, the horizontal one is the opposite of the one of $\underline{\boldsymbol{C}}$ and $\underline{\boldsymbol{C}}_2^{op}\big((f,g),(h,j)\big) = \underline{\boldsymbol{C}}_2\big((g^{op},f^{op}),(j^{op},h^{op})\big)$.*

- *Its **flip category** is the double category $\underline{\boldsymbol{C}}^{fl}$ such that $\underline{\boldsymbol{C}}^{fl}_0 = \underline{\boldsymbol{C}}_0$, $\underline{\boldsymbol{C}}^{fl}_{1,v} = \underline{\boldsymbol{C}}_{1,h}$, $\underline{\boldsymbol{C}}^{fl}_{1,h} = \underline{\boldsymbol{C}}_{1,v}$ and*

$$\underline{\boldsymbol{C}}^{fl}_2\big((f,g),(h,i)\big) = \underline{\boldsymbol{C}}_2\big((h,i),(f,g)\big)$$

Definition 2.1.7. *A **horizontally contravariant double functor** $\underline{\boldsymbol{C}} \to \underline{\boldsymbol{D}}$ is a functor $\underline{\boldsymbol{C}}^{hop} \to \underline{\boldsymbol{D}}$. A **vertically contravariant double functor** $\underline{\boldsymbol{C}} \to \underline{\boldsymbol{D}}$ is a functor $\underline{\boldsymbol{C}}^{vop} \to \underline{\boldsymbol{D}}$. A **contravariant double functor** $\underline{\boldsymbol{C}} \to \underline{\boldsymbol{D}}$ is a functor $\underline{\boldsymbol{C}}^{op} \to \underline{\boldsymbol{D}}$. A **flip double functor** $\underline{\boldsymbol{C}} \to \underline{\boldsymbol{D}}$ is a functor $\underline{\boldsymbol{C}}^{fl} \to \underline{\boldsymbol{D}}$.*

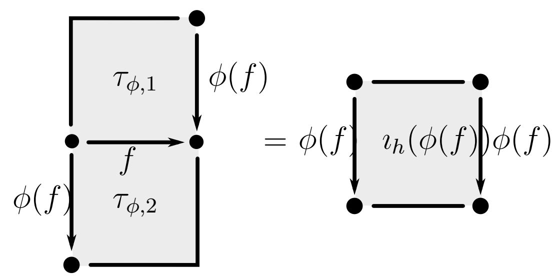
Lemma 2.1.1. *Let $(\_)^{hop}$ be a horizontally contravariant double functor, $(\_)^{vop}$ be a vertically contravariant double functor, etc... The following holds :*

$$\begin{aligned}
\big((_)_1 \circ_{2,h} (_)_2\big)^{hop} &= (_)^{hop}_2 \circ_{2,h} (_)^{hop}_1 \\
\big((_)_1 \circ_{2.v} (_)_2\big)^{hop} &= (_)^{hop}_1 \circ_{2,v} (_)^{hop}_2 \\
\big((_)_1 \circ_{2,h} (_)_2\big)^{vop} &= (_)^{hop}_1 \circ_{2,h} (_)^{hop}_2 \\
\big((_)_1 \circ_{2,v} (_)_2\big)^{vop} &= (_)^{hop}_2 \circ_{2,v} (_)^{hop}_1 \\
\big((_)_1 \circ_{2,h} (_)_2\big)^{op} &= (_)^{op}_2 \circ_{2,h} (_)^{op}_1 \\
\big((_)_1 \circ_{2,v} (_)_2\big)^{op} &= (_)^{op}_2 \circ_{2,v} (_)^{op}_1 \\
\big((_)_1 \circ^{h} (_)_2\big)^{fl} &= (_)^{fl}_1 \circ^{v} (_)^{fl}_2 \\
\big((_)_1 \circ^{v} (_)_2\big)^{fl} &= (_)^{fl}_1 \circ^{h} (_)^{fl}_2
\end{aligned}$$

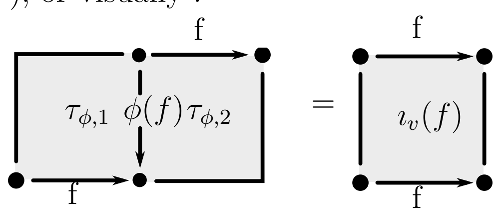
Definition 2.1.8. *A double category $\underline{\boldsymbol{C}}$ is **horizontally self-dual** if it is equivalent to $\underline{\boldsymbol{C}}^{hop}$. It is **vertically self-dual** is it is equivalent to $\underline{\boldsymbol{C}}^{hop}$. It is **self-dual** is it is equivalent to $\underline{\boldsymbol{C}}^{op}$. An edge symmetric double category is **reflexive** if it is equivalent to its flip category.*

2.1.4 Connections and adjoints

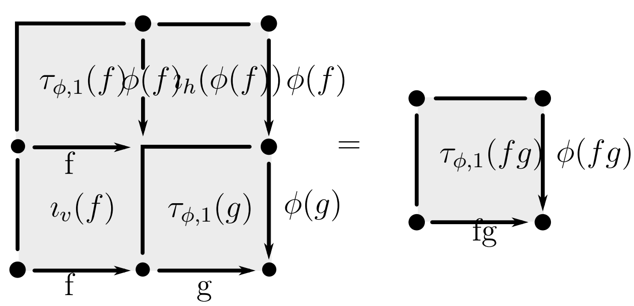
The first and strongest intuition when discovering double categories is to think of squares as commutative squares, assuming along the way that both types of arrows are identical and that a square has no other information than its boundaries. Though the notion of commutative squares in a category yields a double category, it is not something that we can make sense of in any double category. Moreover if are working in a double category, we have to ask what is a commutative cube whose faces are squares?
The notion that arises when considering this question is that of a connection, as proposed by Al-Alg and Brown [5]. A connection is a way to "rotate" faces of a kind to faces of another kind in an invertible way, while carrying the composition along. Let $(\underline{\mathbf{C}}, \phi)$ be an edge symmetric double category. Then a connection τ_ϕ is the assignment of a pair of squares $(\tau_{\phi,1}(f), \tau_{\phi,2}(f))$ for every $f \in C_{1,h}$ such that $\tau_{\phi,1}(f) \circ_v \tau_{\phi,2}(f) = \imath_h(\phi(f))$, or visually :



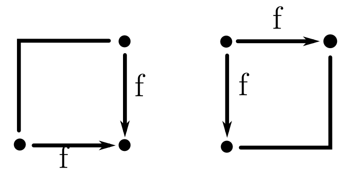
and $\tau_{\phi,1}(f) \circ_h \tau_{\phi,2}(f) = \imath_v(f)$, or visually :



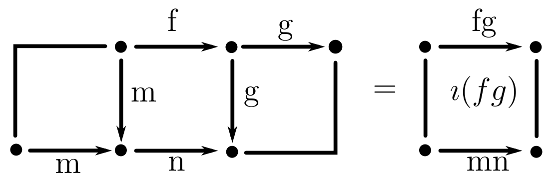
and finally $(\tau_{\phi,1}(f) \circ_h \imath_h(\phi(f))) \circ_2 (\imath_v(f) \circ_h \tau_{\phi,1}(g)) = \tau_{\phi,1}(fg)$



The double category with connection that is the most familiar is the double category of all possible square diagrams, commutative or not, in a given category $\underline{\mathbf{C}}$. It defines a functor $Q : \underline{\mathbf{Cat}} \to \underline{\mathbf{DbCat}}$. It has a canonical connection given by the identity isofunctor on $\underline{\mathbf{C}}$. Its elements have the form :
The connection "acts" on squares by vertical, or horizontal, "conjugation". It yields squares



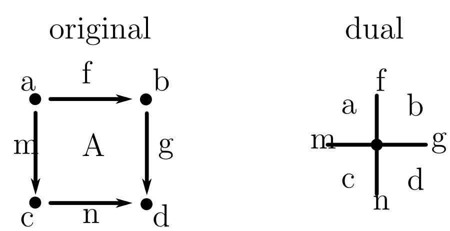
whose horizontal, resp. vertical, arrows are identities. The classical notion of commutative square is that of a square that gives an identity when conjugated by the canonical connection.

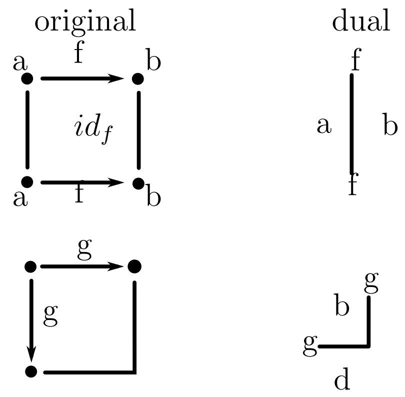


We will therefore define the notion of commutativity w.r.t. a connection in this fashion. The remaining part of the section will be devoted to showing that it is precisely defined and that it extends to all n-tuple categories.

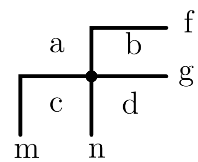
Lemma 2.1.2. *Let $\underline{\boldsymbol{D}}$ be a double category with connection. Then the set $\underline{\boldsymbol{D}}_c$ of all squares that are formed by composing identities and connections is a sub double category of $\underline{\boldsymbol{D}}$. Moreover if $A \in \underline{\boldsymbol{D}}$ and $d \in \underline{\boldsymbol{D}}_c$ such that $A \circ_h d$, $d \circ_h A$, $A \circ_v d$ or $d \circ_v A$ is in $\underline{\boldsymbol{D}}_c$, then $A \in \underline{\boldsymbol{D}}_c$*

Proof. The first claim is straightforward since it contains identities and composing squares made out of identities and connections are themselves made out of identities and connections.
To prove the second statement we will introduce the dual representation of a double category. In this representation we will draw the dual graph of the original one, i.e. we will draw squares as points in the plane, vertical arrows as horizontal segments, horizontal arrows as vertical segments and objects as areas. We will not draw points for identities and connections.

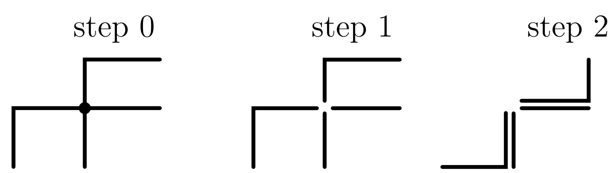




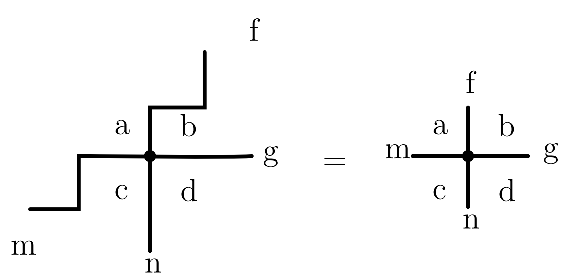
Then suppose that, for $A \in \underline{\mathbf{D}}$, there exists $\tilde{A} \in \underline{\mathbf{D}}_c$ that can be written as A composed with identities and connections. Then $\tilde{A}$ can be dually represented as a diagram with a single dot on a single crossing. For example :



We may now prove that there is a composite of elements of $\underline{\mathbf{D}}_c$ that is equal to A. Consider the previous graph to which you remove a ball at the origin. The result contains four non crossing paths. Rotating these four paths by $\pi 2$ about their respective endpoints farthest from the origin yields a new diagram. On the example,it gives



Then we may place the diagram for A back in the center of the diagram just obtained. What we get is then just A.



$\square$

Though there is a faster way of proving this theorem, this approach has the advantage to be easily generalized to higher dimensions. What we defined as commutative squares in a double category $\underline{\mathbf{D}}$ with connection are then all the elements of $\underline{\mathbf{D}}_c$. To define commutative cube, we will need to define connections for triple categories.

2.2 Double Groupoids

2.2.1 Structure of groupoids

In this section we present the structure of groupoids, as we developed it. Though it is original work, we believe that the results are standard. As the definitions and techniques extend to our study of higher groupoids, we believe that it is an integral part of the chapter. See Weinstein [58] and Brown [14] for a general introduction to groupoids that show their importance in different fields and Higgins [36] for a more thorough treatment of the subject.

Definition 2.2.1. *A **groupoid** is a category where all morphisms are invertible. It is **connected** if every homset is nonempty. It is **discrete** if all morphisms are identities.*

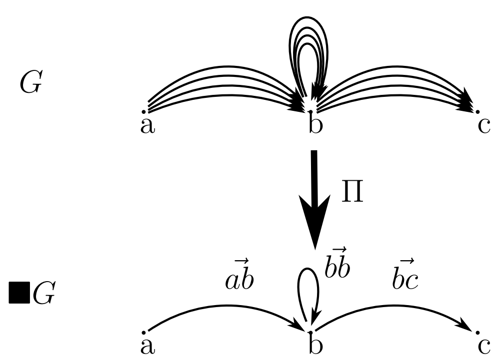
With regards to this definition a group is a groupoid with one object and a group bundle is a groupoid whose arrows never have a different source and target.

Lemma 2.2.1. *Let $\underline{\boldsymbol{G}}$ be a groupoid, then it is the disjoint union of connected groupoids*

Proof. Let $\underline{\mathbf{G}}$ be a groupoid and suppose that $\underline{\mathbf{G}}(a, z) = \emptyset$. Then $\underline{\mathbf{G}}(a, b) \neq \emptyset$ implies $\underline{\mathbf{G}}(b, z) = \emptyset$, otherwise composition gives an arrow $a \to z$, giving a contradiction. $\square$

Definition 2.2.2. *Let X be a set, then the **coarse** groupoid $\underline{\square X}$ on X is the groupoid with X as objects and $\underline{\square X}(a, b) = \{\vec{ab}\}$, i.e. every possible homset has a unique element.*
*Let G be a groupoid, and $\Pi : \underline{\boldsymbol{G}} \to \square \underline{\boldsymbol{G}}_0$, be defined by $\Pi(f) = \vec{ab}$, for $f \in \underline{\boldsymbol{G}}(a, b)$. Its image is called the **frame** $\blacksquare\underline{\boldsymbol{G}}$ of $\underline{\boldsymbol{G}}$. A groupoid is **slim** if it is isomorphic to its frame. The **core** $\underline{\boldsymbol{G}_\bullet}$ of G is the subgroupoid of $\underline{\boldsymbol{G}}$ consisting of arrows whose source and target are identical.*

In summary, the projection Π collapses Homsets to only keep source and target, in other words it is a bundle :



The core of G, on the other hand, retains the information on what the "fibers" look like.

Lemma 2.2.2. *Let $\underline{\boldsymbol{G}}$ be a connected groupoid, then its core is a principal bundle and its frame is coarse.*

Proof. Let $a, b \in \underline{\mathbf{G}}_0$, and pick $f \in \underline{\mathbf{G}}(a,b)$. Then $f^{-1} \circ (\_) \circ f : \underline{\mathbf{G}}(a) \to \underline{\mathbf{G}}(b)$ is an isomorphism of groups whose inverse is $f \circ (\_) \circ f^{-1}$. Since $\underline{\mathbf{G}}$ is connected, $\underline{\mathbf{G}}(a) \simeq \underline{\mathbf{G}}(b) \quad \forall a, b \in \underline{\mathbf{G}}_0$,oving that its core is a principal bundle. Moreover, since $\underline{\mathbf{G}}(a,b) \neq \emptyset \, \forall a, b \in \underline{\mathbf{G}}_0$, the projection is on its frame is onto and full, i.e $\blacksquare\underline{\mathbf{G}} = \square\underline{\mathbf{G}}$. $\square$

Theorem 2.2.1. *Let $\underline{\boldsymbol{G}}$ be a groupoid, then every section* [1] *of $\Pi : \underline{\boldsymbol{G}} \to \blacksquare\underline{\boldsymbol{G}}$ in* ***Graph*** *defines a unique isomorphism of graphs:*

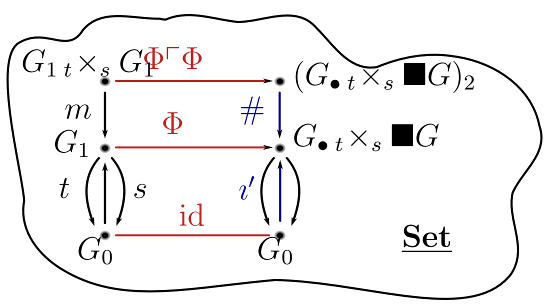
$$\Phi : G_{\bullet\, t}\times_s \blacksquare\underline{\boldsymbol{G}} \to \underline{\boldsymbol{G}}$$

Moreover there exists a unique groupoid structure on $G_{\bullet\, t} \times_s \blacksquare G$ that makes Φ an isomorphism of groupoids.

Proof. Assume a section $! : \blacksquare\underline{\mathbf{G}} \to \underline{\mathbf{G}}$ in $\underline{\mathbf{Graph}}$, then for $f \in \underline{\mathbf{G}}(a,b)$, there exists a unique $u_f \in \underline{\mathbf{G}}(a,a)$ defined by $u_f := f(!(\vec{ab}))^{-1}$ such that $f = u_f!(\vec{ab})$. This shows that

$$a \xrightarrow{f} b \quad = \quad a \xrightarrow{u_f} a \xrightarrow{!} b$$

$\Phi(f) := (u_f, \vec{ab})$ defines an isomorphism of graphs and since it is the identity on objects, it lifts to an isomorphism on the pullback $\underline{\mathbf{G}}_t \times_s \underline{\mathbf{G}}$ of the target and source map. Then the multiplication m on $\underline{\mathbf{G}}$ defines a multiplication $\#$ on $G_{\bullet\, t} \times_s \blacksquare G$ by $(\Phi^{\ulcorner}\Phi)^{-1} m \Phi$. Similarly it defines an identity $\imath'$ by $\imath\Phi$. The picture is the following :



The axioms for associativity and unit follow from Φ being an isomorphism of graphs. $\square$

Corrollary 2.2.1. *Let $\underline{\boldsymbol{G}}$ be a connected groupoid with a chosen fiber bundle structure on its core with fiber G. Then every section* [2] *of $\Pi : \underline{\boldsymbol{G}} \to \square\underline{\boldsymbol{G}}_0$ in* ***Graph*** *defines a unique isomorphism of graphs:*

$$\Phi : G \times \square\underline{\boldsymbol{G}}_0 \to \underline{\boldsymbol{G}}$$

Moreover there exists a unique groupoid structure on $G \times \square\underline{\boldsymbol{G}}_0$ that makes Φ an isomorphism of groupoids.

[1] graph morphism $! : \blacksquare G \to G$ that is the identity on objects

[2] graph morphism $! : \blacksquare G \to G$ that is the identity on objects

Define the following notation : $u_{\vec{ab}} := (u, \vec{ab}) \in G_{\bullet\; t}\!\times_s \blacksquare\underline{\mathbf{G}}$.

Lemma 2.2.3. *The above defined composition is given as :*

$$u_{\vec{ab}} \# v_{\vec{bc}} = \Big(u(\vec{ab}\neg v)\phi_{\vec{ab},\vec{bc}} \Big)_{\vec{ac}}$$

by two maps :

$$\phi : \blacksquare\underline{\boldsymbol{G}}_{0\; t}\!\times_s \blacksquare\underline{\boldsymbol{G}}_0 \to G_\bullet \qquad \neg : \blacksquare\underline{\boldsymbol{G}}_{0\; t}\!\times_s G_\bullet \to G_\bullet$$

such that $\vec{ab}\neg : \underline{\boldsymbol{G}}(b) \to \underline{\boldsymbol{G}}(a)$ is a group isomorphism and that the following identities are satisfied:

$$\begin{aligned} s(\phi_{\vec{ab},\vec{bc}}) &= a & \qquad \vec{ab}\neg(\vec{bc}\neg u) &= \phi_{\vec{ab},\vec{bc}}(\vec{ac}\neg u)(\phi_{\vec{ab},\vec{bc}})^{-1} \\ \phi_{\vec{aa},\vec{ab}} &= \phi_{\vec{aa},\vec{aa}} & (\vec{ab}\neg\phi_{\vec{bc},\vec{cd}}) &= \phi_{\vec{ab},\vec{bc}}\phi_{\vec{ac},\vec{cd}}(\phi_{\vec{ab},\vec{bd}})^{-1} \end{aligned}$$

Moreover the identity is given by $\imath'(a) = ((\phi_{\vec{aa},\vec{aa}})^{-1})_{\vec{aa}}$.

Proof. ϕ is defined by : $e_{\vec{ab}} \# e_{\vec{bc}} =: (\phi_{\vec{ab},\vec{bc}})_{\vec{ac}}$, or in pictures :

$$\begin{tikzcd} a \arrow[r, bend left, "!"] & b \arrow[r, bend left, "!"] & c & = & a \arrow[r, "\phi_{\vec{ab},\vec{bc}}"] & a \arrow[r, bend left, "!"] & c \end{tikzcd}$$

It encodes how much the section fails to be a section of groupoids. $\neg$ is defined by : $u_{\vec{ab}} \# v_{\vec{bc}} =: \big(u(\vec{ab}\neg v)\big)_{\vec{ab}} \# e_{\vec{bc}}$, or in pictures :

$$\begin{tikzcd} a \arrow[r, bend left, "!"] & b \arrow[r, "u"] & b & = & a \arrow[r, "\vec{ab}\neg u"] & a \arrow[r, bend left, "!"] & b \end{tikzcd}$$

or equivalently : $\vec{ab}\neg u := (!\vec{ab})\, u\, (!\vec{ab})^{-1}$. So that indeed, the above composition is given by Φ^{-1} of :

$$\begin{tikzcd} u_{a,b} \# v_{b,c} := & a \arrow[r, "u"] & a \arrow[r, bend left, "!"] & b \arrow[r, "v"] & b \arrow[r, bend left, "!"] & c \\ & a \arrow[r, "u"] & a \arrow[r, "a\neg v"] & a \arrow[r, bend left, "!"] & b \arrow[r, bend left, "!"] & c \\ & a \arrow[r, "u"] & a \arrow[r, "a\neg v"] & a \arrow[r, "\phi_{a,b,c}"] & a \arrow[r, bend left, "!"] & c \end{tikzcd}$$

Associativity in G imposes the claimed relations on $\triangleright$ and ϕ, via the following :

$$\begin{tikzcd}
\underset{\text{a}}{\cdot} \arrow[r, bend left, "!"] & \underset{\text{b}}{\cdot} \arrow[r, "u"] & \underset{\text{b}}{\cdot} \arrow[r, "v"] & \underset{\text{b}}{\cdot}
\end{tikzcd}$$

$$\begin{tikzcd}
\underset{\text{a}}{\cdot} \arrow[r, bend left, "!"] & \underset{\text{b}}{\cdot} \arrow[r, bend left, "!"] & \underset{\text{c}}{\cdot} \arrow[r, "u"] & \underset{\text{c}}{\cdot}
\end{tikzcd}$$

$$\begin{tikzcd}
\underset{\text{a}}{\cdot} \arrow[r, bend left, "!"] & \underset{\text{b}}{\cdot} \arrow[r, bend left, "!"] & \underset{\text{c}}{\cdot} \arrow[r, bend left, "!"] & \underset{\text{d}}{\cdot}
\end{tikzcd}$$

Moreover $!(\vec{aa})!(\vec{ab}) = \phi_{\vec{aa},\vec{ab}}!(\vec{ab})$ shows that $\phi_{\vec{aa},\vec{ab}} =!(\vec{aa})$ for all $b \in B$, hence the last condition. Now we can see that the identity $\imath'$ of $\#$ is given by $((\phi_{\vec{aa},\vec{aa}})^{-1})_{\vec{aa}} = \Phi^{-1}(id_a)$, via :

$$\begin{tikzcd}
\underset{\text{a}}{\cdot} \arrow[r, "\imath'(a)"] & \underset{\text{a}}{\cdot} \arrow[r, bend left, "!"] & \underset{\text{a}}{\cdot} \arrow[r, bend left, "!"] & \underset{\text{b}}{\cdot} & = & \underset{\text{a}}{\cdot} \arrow[r, dash, "id_a"] & \underset{\text{a}}{\cdot} \arrow[r, bend left, "!"] & \underset{\text{b}}{\cdot}
\end{tikzcd}$$

□

This shows that necessary and sufficient information needed to describe a groupoid is given by its frame, its core, and two special maps.

Corrollary 2.2.2. *Let G be a group and B a set. Then connected groupoid structures on B with fiber G are given by pairs of maps*

$$\phi : B \times B \times B \to G \qquad \rho : B \times B \times G \to G$$

such that $\rho(a, b, \_)$ is a group isomorphism and that

$$\begin{aligned}
\rho(a, b, \rho(b, c, g)) &= \phi(a, b, c)\rho(a, c, g)\phi(a, b, c)^{-1} \\
\rho(a, b, \phi(b, cd)) &= \phi(a, b, c)\phi(a, c, d)\phi(a, b, d)^{-1} \\
\phi(a, a, b) &= \phi(a, a, a)
\end{aligned}$$

Corrollary 2.2.3. *Let (B, G, ρ, ϕ) and $(B, G, \tilde{\rho}, \tilde{\phi})$ be two groupoids on a pair (B, G) as defined above. Then they are isomorphic if and only if there is a map $\Gamma : B \times B \to G$ satisfying :*

$$\Gamma(a, b)\tilde{\rho}(a, b, g) = \rho(a, b, g)\Gamma(a, b)$$
$$\Gamma(a, b)\rho(a, b, \Gamma(b, c))\phi(a, b, c) = \tilde{\phi}(a, b, c)\Gamma(a, c)$$

Proof. Suppose that you have two sections ! and ?, then define $\Gamma_{\vec{ab}} :=?(\vec{ab})(!(\vec{ab}))^{-1}$. The above relations come from transforming ? to ! in the following pictures :

$$\begin{tikzcd} a \arrow[r, bend left, "?"] & a \arrow[r, bend left, "?"] & b \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, bend left, "?"] & b \arrow[r, dash, "id_a"] & a \end{tikzcd}$$

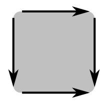
$\square$

Remark that being given one such groupoid, all equivalent groupoid structures on (B, G) can be constructed from it by any map $\Gamma : B \times B \to G$. Without loss of generality, one can then consider that $!\vec{aa} = id_a$ and the axiom $\phi_{\vec{aa},\vec{ab}} = \phi_{\vec{aa},\vec{ab}}$ becomes $\phi_{\vec{aa},\vec{ab}} = id_a$.

2.2.2 Definitions

Definition 2.2.3. *A **double groupoid** is a (strict) double category where every square has both horizontal and vertical inverses*[3]*. A **double group** is a double groupoid with a single object.*

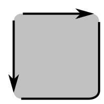
To save time and space, we will take the convention not to represent objects any more unless absolutely necessary, and we will let the orientation of the page determine what is horizontal and what is vertical. The picture then becomes :



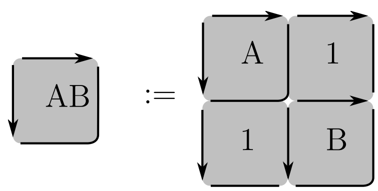
The boundary groups of a double group are far from determining it. Further studies of double groupoids will tell us what extra structure they embody.

Definition 2.2.4. *Let τ be a double groupoid, then:*

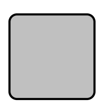
- *Its **core groupoid** $\tau_{\lrcorner}$ is the diagonal groupoid of elements of τ whose targets are identities. They are squares of the form :*

  whose multiplication is defined by :

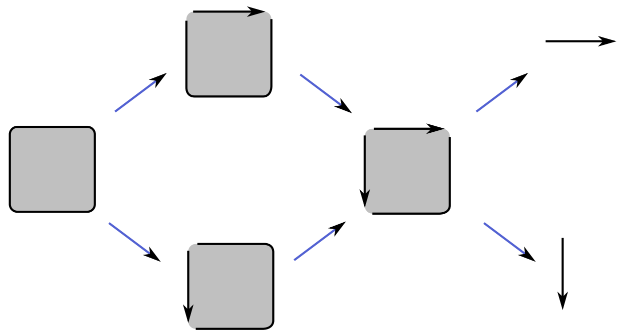
  

- *Its **core bundle** $\tau_{\bullet}$ is the sub groupoid of $\tau_{\lrcorner}$ whose boundaries are all identities. It is a group bundle over the objects whose elements are squares of the form :*

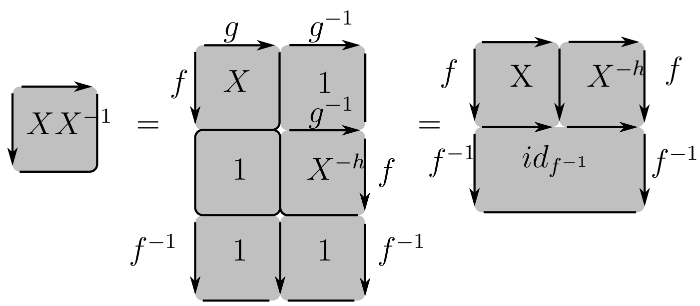
[3] Note that consequently vertical and horizontal arrows are invertible as well.

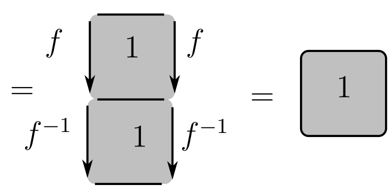
- *Its* ***core diagram*** *is the following diagram of groupoids:*

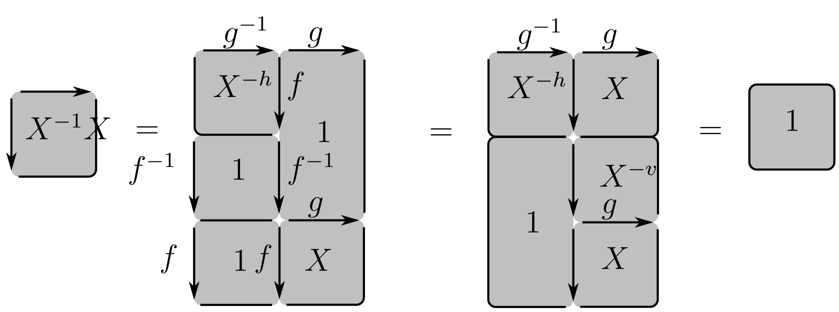


Lemma 2.2.4. *The core groupoid is a groupoid.*

Proof. Identity and associtivity follow from the double category axioms. Let $X \in \tau_\lrcorner$ such that $f := s_h(X)$, then $X^{-1} := X^{-h} \circ_v id_f$. Then





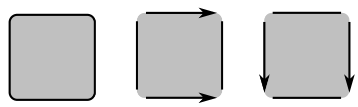


proves that X has both a right and a left inverse, and therefore is invertible. □

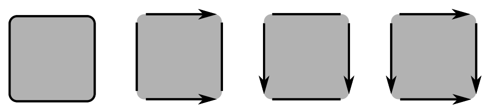
Core groupoids appeared in [18] by Brown and Mackenzie, where they show that a certain class of double groupoid is determined by their cores. Note that it is not the case in general, though it provides a great deal of information. Let's consider the simplest non trivial core diagram of a double group:

$$\begin{tikzcd}
 & \{e\} \arrow[dr] & & Z_2 \\
\{e\} \arrow[ur] \arrow[dr] & & \{e\} \arrow[ur] \arrow[dr] & \\
 & \{e\} \arrow[ur] & & Z_2
\end{tikzcd}$$

Then there are two double groupoids having it as a core groupoid. Since there is only one non-identity arrow in each direction, it will not be labelled. The smallest one has only identities :



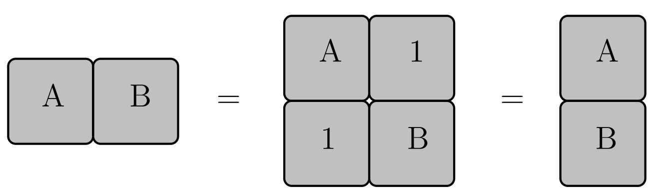
The other one has one more square that is its own inverse :



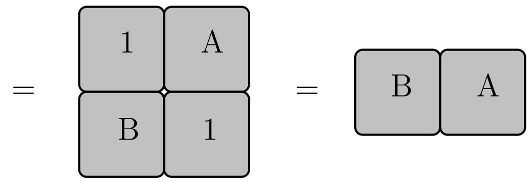
Remark that any other square would impact the core diagram. The situation becomes more complex as the core diagrams involve bigger groupoids but even such a simple example makes the point. Let's analyse the core diagram further.

Lemma 2.2.5. *The core bundle of a double groupoid is an abelian group bundle over its objects.*

Proof. This is the celebrated Eckmann-Hilton argument :



This first step shows that the horizontal and vertical compositions are identical and correspond to the composition in the core bundle. If we continue to exploit the existence and uniqueness of identity squares on objects and the interchange law, we get the result announced.



Note that we are literally rotating A and B around each other. □

In [6], Andruskiewitsch and Natale show that a special kind of double groupoid called vacant, corresponds exactly to factorizations of groupoids, which in the group case can also be found in literature as matched pair of groups, bicrossed products, knit products or Zappa-Szep products of groups [3, 4]. A matched pair of groups is a pair of subgroups H,K of a group G such that each element of G can be uniquely written $g = hk$ for $h \in H$ and $k \in K$. A matched pair of groups is isomorphic to a bicrossed product of the same groups, written $H \bowtie K$. The rest of the section will extend the result to a bigger class of double groupoids. Let's recall the definition of vacancy.

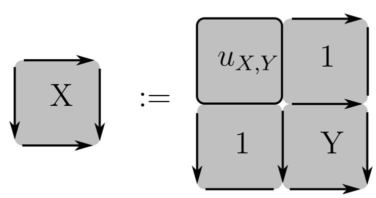
Definition 2.2.5. *A double groupoid is* ***vacant*** *if any pair of possible horizontal sources and vertical targets is the boundary of a unique square.*

The following definitions will help refine the notion:

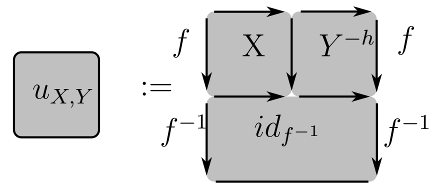
Definition 2.2.6. *A double groupoid is* ***slim*** *if its core bundle is the trivial bundle. It is* ***exclusive*** *if its core groupoid and its core bundle are identical.*

Remember that the core bundle is a totally disconnected subgroupoid of the core groupoids, the one whose square have all boundaries identity arrows. Slim double groupoids have at most one cell per boundary, as the next lemma shows, so the only data they contain is which boundaries correspond to a square, in other words a slim double groupoid is a special subset of $H \times H \times V \times V$.

Lemma 2.2.6. *Let τ be a double groupoid, $X, Y \in \tau$ with the same boundary. Then there exists a unique element $u_{X,Y}$ in the core of τ such that :*



Proof. Defining $u_{X,Y}$ by :

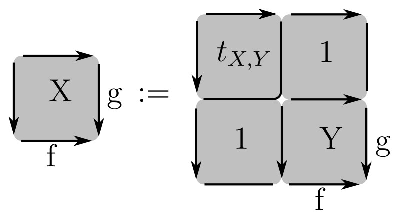


and using inverses to isolate X yields the claim. □

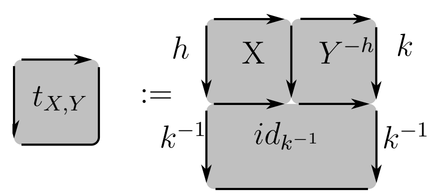
Corrollary 2.2.4. *A double groupoid is slim if and only if there is at most one square for a given set of boundaries.*

Exclusive double groupoids may have many squares for a given boundary condition but this boundary is completely determined by the knowledge of two of its boundaries, one of each type, as the following lemma shows.

Lemma 2.2.7. *Let τ be a double groupoid, $X, Y \in \tau$ with the same targets. Then there exists a unique element $t_{X,Y}$ in the core groupoid of τ such that :*



Proof. Just as in the previous lemma, defining $t_{X,Y}$ by :



and using inverses to isolate X yields the claim. □

Corrollary 2.2.5. *A double groupoid is exclusive if and only if the boundary of its squares is determined by one of their boundaries of each type.*

Proof. Suppose two squares share one boundary of each type, then one of their inverses (horizontal, vertical or both) will share both their targets. By the above lemma they also share their sources. Using the same inverse again yields the result. □

Definition 2.2.7. *A double groupoid is* ***maximal*** *if any pair of vertical-horizontal arrows is the target pair of a square.*

In the example we gave earlier, both double groupoids were exclusive but only the second was maximal.

Lemma 2.2.8. *A double groupoid is vacant if and only if it is slim, maximal and exclusive.*

Proof. By the above lemma it is slim and exclusive. A simple use of vertical inverses show that it is maximal as well. □

A better definition for maximality would be the following:

"A double groupoid is maximal if it is maximal in the poset of double groupoids sharing a given core diagram."

But it is yet to be proven that these definitions are equivalent in the case of exclusive double groupoids, which brings us to the problem of determining how many different exclusive double groupoids exist for a given core bundle. It is an open question and is reserved for further research.
Exclusive double groups are interesting in their own right and the next section will provide a prime example of these. The following theorem is original and motivated the example. The definitions needed to understand its statement are to be found in the proof.

Theorem 2.2.2. *Maximal exclusive double groupoids equipped with a section over their frame are in one to one correspondence with fibered semi-direct products of an abelian group bundle with a matched pair of groupoids.*

Proof.

Definition 2.2.8. *Let* ***2-sub*** *be the category whose objects are ordered triples* (G, H, K) *consisting of a groupoid* G *and two of its subgroupoids* H *and* K*, and whose arrows* $f : (G, H, K) \to (G', H', K')$ *are functors* $f : G \to G'$ *such that* $f(H) \subset H'$ *and* $f(K) \subset K'$*.*

Definition 2.2.9. *Let* $(G, H, K) \in$ ***2-Sub****, and define* $\Gamma(G, H, K)$ *as the double groupoid defined by :*

- *The horizontal arrow groupoid is* H*.*
- *The vertical arrow groupoid is* K*.*
- *There exist a square with sources* (h, k) *and targets* (h', k') *if anf only if* $hk' = kh'$

We will denote a square by its defining identity. Also, for $f : G \to G' \in$ ***Gpd*** *a functor we define* $\Gamma(f) : \Gamma((G, H, K)) \to \Gamma((G', H', K'))$ *by* $\Gamma(f)(hk' = kh') = \big(f(h)f(k') = f(k)f(h')\big)$

Lemma 2.2.9. Γ *is a functor* ***2-Sub*** $\to$ ***slimDblGpd****.*

Proof. First let's prove that $\Gamma(G, H, K)$ is a double category. Consider the squares $hk' = kh'$ and $jk'' = k'j'$, then the horizontal composition of the two is the square $(hj)k'' = k(h'j')$, which is well defined since $hjk'' = hk'j' = kh'j'$. The vertical composition is defined similarly by $(hk' = kh') \circ_v (h'k''' = k''h'') = h(k'k''') = (kk'')h''$ and associativity follows from associativity in G. Identities are given by $he = eh$ and $ek = ke$ where e is the identity of G. Now since the targets of Γ are slim, functors between them are defined by their values on the boundaries, which are given by functors. Γ then sends a functor to its restrictions on the given subgroupoids, which is done functorially. $\square$

Lemma 2.2.10. *The following are true :*

- *$\Gamma(G, H, K)$ is exclusive if and only if $H \cap K$ is discrete.*

- *$\Gamma(G, H, K)$ is maximal if and only if $HK = KH$*

Proof. The core groupoid of $\Gamma(G, H, K)$ is composed of squares $he = ke$, hence it is $H \cap K$. When the latter is discrete, the double groupoid is exclusive and vice versa, which proves the first claim. Now squares exist if they correspond to elements of $HK \cap KH$, then $hk \notin KH$ if ands only if there exist no square $hk = \cdots$. If there exist such a square for every pair (h, k), the double groupoid is maximal. $\square$

Definition 2.2.10. *Let* $\underline{\textbf{\textit{VacDblGpd}}}$ *be the category of vacant double groupoids and* $\underline{\textbf{\textit{2-matchSub}}}$ *the subcategory of* $\underline{\textbf{\textit{2-Sub}}}$ *that satisfy the above conditions. Then* $\Gamma : \underline{\textbf{\textit{2-matchSub}}} \to \underline{\textbf{\textit{VacDblGrp}}}$ *has a left adjoint* Λ

Proof. Let τ be a vacant double groupoid, X, Y be squares in τ. Define $\tilde{\tau}$ to be the groupoid whose objects are objects of τ and arrows $X : a \to b$ are squares X with $s_h s_v(X) = a$ and $t_h t_v(X) = b$. Composition is given by the following : if composable, XY is be the unique square in τ having a barycentric subdivision with X in the upper left and Y in the lower right. The existence of such a square is guaranteed by maximality and its uniqueness by exclusivity and slimness. The same argument guarantees associativity and identities are given by $\imath_a := \imath_h(\imath_v(a))$.
$\Lambda(\tau) := (\tilde{\tau}, \imath_v(\tau_h), \imath_h(\tau_v))$ defines the functor on objects and $\Lambda(\tau) \in$ **2-matchSub** since the following is true in τ :

$$\begin{bmatrix} \imath & X \\ \imath & \imath \end{bmatrix} = \begin{bmatrix} \imath & \imath \\ X & \imath \end{bmatrix}$$

For a double functor $f : \tau \to \tau'$, the corresponding functor $\Lambda(f)$ is given by $\Lambda(f)(X) = f(X)$. Since double functors preserve identities, $\Lambda(f) \in$ **2-matchSub**.
The adjunction is then given by the isomorphism

$$\phi : \underline{\textbf{VacDblGpd}}(\tau, \Gamma(G)) \to \underline{\textbf{2-matchSub}}(\Lambda(\tau), (G, H, K))$$

such that if $F(X) = (hk' = kh')$ then $\phi(F)(X) = hk'$. To see that it is indeed invertible, recall that if $(G, H, K) \in$ **2-matchSub**, then any element of HK can be uniquely written as an element of KH and therefore correspond to a unique square in ΓG. $\square$

Lemma 2.2.11. *Let* $\underline{\textit{\textbf{2-Match}}}$ *be the full subcategory of* $\underline{\textit{\textbf{2-matchSub}}}$ *such that* $(G, H, K) \in \underline{\textit{\textbf{2-Match}}} \iff G = HK$*. Then* Λ*'s image is in* $\underline{\textit{\textbf{2-Match}}}$ *and the above adjunction is an equivalence of categories.*

Proof. We have already shown that the image is what is claimed, but to show that the adjunction is an equivalence, we need to show that its unit and counit are isomorphisms. Let $\nu : \tau \to \Gamma(\Lambda(\tau))$ and $\eta : \Lambda(\Gamma(G)) \to G$ be the adjunction's unit and counit. Then they are given by

$$\nu(X) = (\begin{bmatrix} \imath & X \\ \imath & \imath \end{bmatrix} = \begin{bmatrix} \imath & \imath \\ X & \imath \end{bmatrix})$$
$$\eta(hk = kh) = hk$$

and are both iso. □

This proof can also be found is [7], and it shows that vacant double groupoids are in correspondence with factorizations of groupoids. In the group case it means that vacant double groups are in correspondence with bi-crossed products of groups, matched pairs of groups or Zappa-Szep products of groups, as all appear in the literature. We now need to remove the slimness condition.

Definition 2.2.11. *Let* $\underline{\textit{\textbf{MaxExcl}}}$ *be the category whose objects are sections of maximal exclusive double groupoids over their frame and whose arows are section preserving functors. Let* $\underline{\textit{\textbf{2-semi}}}$ *be the following category :*

- *objects are quadruples* (G, H, K, A)*, where* $(G, H, K) \in \underline{\textit{\textbf{2-matchSub}}}$ *and* A *a totally disconnected abelian subgroupoid of* G *such that* $hah^{-1} \in A$*,* $kak^{-1} \in A$ *and* $A \cap H = A \cap K$ *is discrete.*
- *arrows are functors sending the subgroupoids into their counterparts.*

Given such a quadruple, one can build a double groupoid with a canonical section the following way : squares are pairs $(a, hk' = kh')$ such that $s(a) = s(h)$ and the section is $(hk' = kh') \to (e, hk' = kh')$. Composition is given by

$$\begin{aligned}(a, hk' = kh') \circ_h (b, h''k'' = k'h''') &= (ahbh^{-1}, (hh'')k'' = k(h'k''')) \\ (a, hk' = kh') \circ_v (b, h'k''' = k''h'') &= (akbk^{-1}, h(k'k''') = (kk'')h''))\end{aligned}$$

Identities are $(\imath(\imath a), h\imath(b) = \imath(a)h)$ and $(\imath(\imath a), \imath(a)k = k\imath(b))$ and associativity boils down to

$$\begin{aligned}hbh^{-1}(hh'c(hh')^{-1}) &= hbh'ch'^{-1}h^{-1} \\ &= h(bh'ch'^{-1})h^{-1}\end{aligned}$$

and inverses are given by

$$\begin{aligned}(a, hk' = kh')^{-h} &= (h^{-1}a^{-1}h, h^{-1}k = k'h'^{-1}) \\ (a, hk' = kh')^{-v} &= (k^{-1}a^{-1}k, h'k'^{-1} = k^{-1}h)\end{aligned}$$

An arrow $f \in \underline{\textbf{2-semi}}$ then defines a double functor by $f(a, hk' = kh') = (f(a), f(h)f(k') = f(k)f(h'))$.

Lemma 2.2.12. *Let $\Gamma : \underline{\textbf{2-semi}} \to \underline{\textbf{MaxExcl}}$ be the functor defined above, then there exists a functor Λ and an isomorphism ϕ such that (Λ, Γ, ϕ) is an adjunction. Moreover, restricted to the full subcategory of* $\underline{\textbf{2-semi}}$ *whose objects are such that $G = AKH$, the adjunction is an equivalence of categories.*

Proof. Given a section $! : \blacksquare\tau \to \tau$ of a double groupoid over its frame, maximality and exclusivity ensure that, for given squares X and Y, there exist a unique square with subdivision :

$$\begin{bmatrix} X & ! \\ ! & Y \end{bmatrix}$$

Just as in the case of the core groupoid, this defines a composition on the set of squares of τ. When ! is a double functor this composition is associative and unital. Moreover it then has inverses given by

$$X^{-1} := \imath_v(t_h(X)^{-v})X^{-h}\imath_v(s_h(X)^{-v}) = \imath_h(t_v(X)^{-h})X^{-v}\imath_h(s_v(X)^{-h})$$

and hence defines a groupoid.
Since arrows in **MaxExcl** preserve sections, it preserves composition of the above defined groupoid and the process defines a functor Λ as claimed. Now

$$\phi : \underline{\textbf{MaxExcl}}(\tau, \Gamma(G)) \to \underline{\textbf{2-semi}}(\Lambda(\tau), G)$$

is given by $\phi(F)(X) = ahk'$ if $F(X) = (a, hk' = kh')$. To see that it is invertible, we need to recall a previous lemma that shows that $X = a(!(\Pi(X)))$ in $\Lambda(\tau)$ since X and $!(\Pi(X))$ have the same boundary.Therefore $f \in \underline{\textbf{2-semi}}(\Lambda(\tau), G) \iff f(X) \in AHK \subset G$, in which case corresond to a unique square in $\Gamma(G)$, proving the adjunction claim. Now suppose that $G = AHK$, then maps in $\underline{\textbf{2-semi}}(G, \Lambda(\tau))$ give uniquely maps in $\underline{\textbf{MaxExcl}}(\Gamma(G), \tau)$ by $(ahk') \to X$ being sent to $(a, hk' = kh') \to X$. □

This finishes the proof. □

2.2.3 The Poincare Group

To get to interesting examples, we chose to reduce our attention to double groups, in which case, the last chapter taught us that the structure of a vacant double group is a bicrossed product of groups. A very special case of such a decomposition is given by the Iwasawa decomposition of semisimple Lie groups, also called K(AN) decomposition for the subgroups appearing in the decomposition are usually denoted K, A and N. For more information on Lie double groups, see Brown and Mackenzie [18].
There is a Lie group that is very special to any theoretical physicist : the Lorentz group, SO(3,1). It is the group of symmetries of Minkowski spacetime in special relativity, i.e. the linear transformations of $\mathbb{R}^4$ preserving the diagonal matrix diag(-1 1 1 1) while fixing the origin. It contains rotations of euclidian space and "boosts", together with combinations of them, and while rotations form a subgroup, the boosts do not. Its K(AN) decomposition decomposes SO(3,1) accordingly, as shows the following lemma :

Lemma 2.2.13. *The Iwasawa decomposition of SO(3,1) is given by the following subgroups:*

$$K := exp\left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & -a & 0 & c \\ 0 & -b & -c & 0 \end{bmatrix} \quad |a, b, c \in \mathbb{R} \right\} \simeq SO(3)$$

$$A := exp\left\{ \begin{bmatrix} 0 & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad |a \in \mathbb{R} \right\} \simeq SO(1,1)$$

$$N := exp\left\{ \begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & a & b \\ a & -a & 0 & 0 \\ b & -b & 0 & 0 \end{bmatrix} \quad |a, b \in \mathbb{R} \right\}$$

This result can be found in the book of J.Gallier [31] amongst others. Keeping AN as a single subgroup gives a bicrossed product presentation of the Lorentz group, which corresponds to a unique vacant double group. This vacant double group has boundary groups $SO(3)$ and $SO(1,1) \bowtie N$.
But just as the isometries of Euclidian space consist of rotations and translations, the Lorentz transformations form only a subgroup of the isometries of Minkowski spacetime: the Poincaré group. This group has then a canonical presentation as a semi-direct product between the Lorentz group and the group of translations in $\mathbb{R}^4$, which is abelian. We can then infer :

Corrollary 2.2.6. *The Poincaré group has a decomposition of the form :*

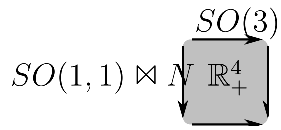
$$Poinc \simeq (K \bowtie (AN)) \ltimes \mathbb{R}^4_+$$

This decomposition is therefore represented by a maximal exclusive double group whose core is $(R^4, +)$ and boundary groups are $SO(3)$ and $SO(1,1) \bowtie N$

The corresponding maximal exclusive double group under the equivalence of the theorem has the following core diagram :

$$\begin{tikzcd}
 & \mathbb{R}^4_+ \arrow[dr] & & SO(3) \\
\mathbb{R}^4_+ \arrow[ur] \arrow[dr] & & \mathbb{R}^4_+ \arrow[ur] \arrow[dr] & \\
 & \mathbb{R}^4_+ \arrow[ur] & & SO(1,1) \bowtie N
\end{tikzcd}$$

Schematically the roles of the different groups are summarized in the following picture :



As claimed in the introduction, the presentation of the Poincare group as a double group has the very attractive property that the translations, the boosts and the rotations are kept distinct. Moreover, it is expected that maximal exclusive triple groups would allow to separate $SO(1,1) \bowtie N$. Of course, other Lie group decompositions of the Lorentz group can be considered and would give other double groups.

2.2.4 Hyperplatonic solids

In our previous examples, we have considered very simple core groupoids, namely ones that were no bigger than the core bundle. But what about the remaining cases? Remark that, in the case of double groups, the core groupoid is, up to isomorphism, fully determined by the span of groups :

$$\begin{tikzcd} & \tau_{\lrcorner} \arrow[dl] \arrow[dr] & \\ \tau_h & & \tau_v \end{tikzcd}$$

Indeed, taking the kernels of the projections and then the pullback of the obtained cospan reconstructs the core diagram up to isomorphism. This brings us to the following characterization :

Lemma 2.2.14. *Core diagrams of double groups are in one to one correspondance with exact sequences*

$$\{e\} \to A \to G \to H \times K$$

where $G, H, K \in \underline{\boldsymbol{Grp}}$ *and* $A \in \underline{\boldsymbol{AbGrp}}$.

Proof. By universality of the pull-back that is given by the product, we have a unique arrow $\tau_{\lrcorner} \to \tau_h \times \tau_v$, as shown in the picture :

$$\begin{tikzcd} \tau_{\lrcorner} \arrow[rr, dotted, "!"] \arrow[dr] \arrow[drrr] & & \tau_h \times \tau_v \arrow[dl] \arrow[dr] & \\ & \tau_h & & \tau_v \end{tikzcd}$$

But the kernel of this map is the group of squares with horizontal and vertical sources trivial, i.e. the core bundle. This gives the exact sequence $\{e\} \to \tau_\bullet \to \tau_{\lrcorner} \to \tau_h \times \tau_v$ and proves the equivalence one way. For the other way, the projections of the product recover the span, from which we get the cospan on G by taking kernels. The span on A is recovered using the universality of the fibered product $P \times_G Q$, as shows the picture :

$$\begin{tikzcd}
& & Q \arrow[dr] & & & H \\
P \times_G Q \arrow[urr] \arrow[drr] & A \arrow[l, dotted, "!"'] \arrow[ur] \arrow[dr] \arrow[rr] & & G \arrow[r] \arrow[urr] \arrow[drr] & H \times V \arrow[ur] \arrow[dr] & \\
& & P \arrow[ur] & & & V
\end{tikzcd}$$

□

Corrollary 2.2.7. *Core groups of slim double groups are in one to one correspondence with exact sequences*

$$\{e\} \to \tau_{\lrcorner} \to \tau_h \times \tau_v$$

In other words slim core diagrams correspond to subgroups of the product of the horizontal and vertical groups. This is particularly interesting in the light of the recent work of Ma [47]. It seems that discrete symmetries may play an important role in upcoming theoretical physics.
In fact, regular and semi-regular polytopes in n dimensional Euclidian space can be studied from the finite subgroups of $SO(n)$, as shown in Coxeter's book [20]. For $n = 4$, it exhibits a behavior that doesn't exist in other dimensions. Indeed, the following lemma can be found in the literature, for example in [56] .

Lemma 2.2.15. *There exist a short exact sequence :*

$$\{e\} \to \mathbb{Z}_2 \to Spin(n) \to SO(n) \to \{e\}$$

Moreover, when $n = 4$, there is an isomorphism $Spin(4) \simeq SU(2) \times SU(2)$.

This shows that subgroups of $SO(4)$ correspond to some subgroups of $SU(2) \times SU(2)$ by considering the preimage. The study of such subgroups was done by P.DuVal in [25] where he shows that they are classified by group isomorphisms $A/A_0 \to B/B_0$ where $A_0 \lhd A \subset SU(2)$ and $B_0 \lhd B \subset SU(2)$. Putting the pieces together we understand that regular polytopes in dimension 4 give us, through their groups of symmetry, core diagrams of double groups. It is an important question to determine what or how many double groups share the same core diagram. We have managed to dodge this difficulty until now thanks to the availability of maximal double groups in the exclusive case but the question gets more complicated for arbitrary core diagrams. This done, we will be able to understand which ones are of significant importance for the regular polytopes. Moreover it would classify double groupoids and is seen by the author as constituting a possible program for future research.

Chapter 3

Higher Cubical Categories

As chapter 2 gave a precise example of what we will call higher cubical categories, the dimension two case, the present chapter will develop the theory in all generality. Remark that though there are examples of double categories in the literature, it is very hard to find references to triple categories. Yet the notion was introduced by Ehresmann in 1968 in an article called "Categories structurees généralisées" [26]. To show the theory in all its grace, we believe that it is more elegant to use the notion of internalization, contrary to what we did in the first chapter. Weak triple categories will be needed to describe cobordisms with corners before we may quotient to obtain a double category just as for the usual cobordism category we start with a weak double category of cospans and isomorphisms and quotient it to get a category, which we will explain in Chapter 4.

3.1 Internalization

Families of algebraic objects such as monoids, groups, rings, modules, algebras et cetera are defined as sets with extra structure. The axioms of these structures can be expressed diagramatically, with arrows being maps of sets. But thought of as abstract objects, these structures can be defined using the same diagrams in other categories, not just $\underline{\mathbf{Set}}$. This process is called **internalization** and it is the topic of this section. Let's start with a "philosophical lemma"

Note that the construction of internal structures has a reflexive property :

Lemma 3.1.1 (Philosophical lemma)**.** *Let $\underline{\mathbf{C}}$ be a small category. Let $\underline{\mathbf{Object\ 1}}$ be be the category of internal (Structure 1) in $\underline{\mathbf{C}}$, $\underline{\mathbf{Object\ 2}}$ the category of internal (Structure 2) in $\underline{\mathbf{C}}$. Then an internal (Structure 2) in $\underline{\mathbf{Object\ 1}}$ is equivalent to an internal (Structure 1) in $\underline{\mathbf{Object\ 2}}$.*

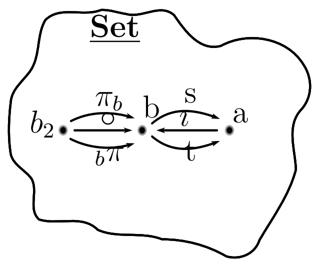
Proof. Some internal object can be viewed as a functor from a certain category, representing the diagram. In this case the category $\underline{\mathbf{Object\ 1}}$ of (Structure 1) in $\underline{\mathbf{C}}$ can therefore be assimilated to $\underline{\mathbf{C}}^A$ for some category A. Similarly the category $\underline{\mathbf{Object\ 1}}$ of (Structure 1) in $\underline{\mathbf{C}}$ is can be assimilated to $\underline{\mathbf{C}}^B$ for some category B. Then the category of (Structure 1) in $\underline{\textbf{Object 2}}$ is $(\underline{\mathbf{C}}^B)^A$ and the category of (Structure 2) in $\underline{\textbf{Object 1}}$ is $(\underline{\mathbf{C}}^A)^B$. Or since $\underline{\textbf{Cat}}$ is cartesian closed $(\underline{\mathbf{C}}^B)^A \simeq (\underline{\mathbf{C}}^B)^A$, as long as A, B and $\underline{\mathbf{C}}$ are small. □

Already, from this analysis, some objects are getting our immediate attention. What is an internal monoid in the category of monoids ? What is an internal category in the category of monoids, or equivalently an internal monoid in the category of categories ? What is an internal category in the category of categories ? What kind of special properties have internal ($Structure1$) in $\underline{\textbf{Object 1}}$? Understanding the last two questions will give us answers to the previous ones.

3.1.1 Internal categories

Definition 3.1.1. *An **internal category** in a category $\underline{\boldsymbol{C}}$ with pullbacks is a sextuple $(a, b, s, t, \imath, \circ)$, where :*

- *a and b are objects of $\underline{\boldsymbol{C}}$.*
- *$(s, t, \circ, \imath)$ are arrows in $\underline{\boldsymbol{C}}$ respectively called source, target, composition and identity as in the picture :*



where b_2, ${}_b\pi$ and π_b are the pull-back of t and s.

$$\begin{tikzcd} b_2 \arrow[r, "\pi_b"] \arrow[d, "{}_b\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & b \arrow[d, "s"] \\ b \arrow[r, "t"'] & a \end{tikzcd}$$

More generally b_n is a chosen limit of :

$$\underbrace{b \xrightarrow{t} a \xleftarrow{s} b \xrightarrow{t} \cdots \xleftarrow{s} b \xrightarrow{t} a \xleftarrow{s} b}_{\text{n-1 times}}$$

- *some relations are satisfied:*

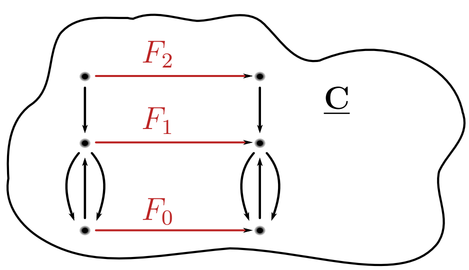
$$\begin{tikzcd} a \arrow[r, "\imath"] \arrow[d, dash, "id_a"'] & b \arrow[d, "s"] \\ a \arrow[r, dash, "ib_a"'] & a \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\imath"] \arrow[d, dash, "id_a"'] & b \arrow[d, "t"] \\ a \arrow[r, dash, "ib_a"'] & a \end{tikzcd} \qquad \begin{tikzcd} b_2 \arrow[r, "\pi_b"] \arrow[d, "\circ"'] & b \arrow[d, "t"] \\ b \arrow[r, "t"'] & a \end{tikzcd} \qquad \begin{tikzcd} b_2 \arrow[r, "\circ"] \arrow[d, "{}_b\pi"'] & b \arrow[d, "s"] \\ b \arrow[r, "s"'] & a \end{tikzcd}$$

$$
\begin{tikzcd}
b \arrow[r, "id_b \ulcorner t\imath"] \arrow[d, "id_b"'] & b_2 \arrow[d, "\circ"] \\
b \arrow[r, "ib_b"'] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
b \arrow[r, "s\imath \ulcorner id_b"] \arrow[d, "id_b"'] & b_2 \arrow[d, "\circ"] \\
b \arrow[r, "ib_b"'] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
b_3 \arrow[r, "\circ \ulcorner \pi_b"] \arrow[d, "{}_b\pi \ulcorner \circ"'] & b_2 \arrow[d, "\circ"] \\
b_2 \arrow[r, "circ"'] & b
\end{tikzcd}
$$

This diagrammatic definition taken in <u>**Set**</u> yields the usual notion of a small category. Let's take a few lines to show why this is the case. The set a is the set of objects of a category, the set b its set of morphisms. The maps s and t, sometimes denoted d and c for domain and codomain, associate respectively a source and a target to the elements of b, the morphisms. The set b_2, pullback of the pair (t, s), contains pairs of composable morphisms. The projections ${}_b\pi$ and π_b give respectively the first and second elements of the pairs and the map $\circ$ is the composition. The map $\imath$ selects an identity amongst morphisms with identical source and targets for every object. The axioms are then in order, the source and target axioms for identities, the source and target axioms for composition, the right and left unit axioms and finally the associativity axiom. Remark that the source and target axioms are a necessary condition for the unique maps used in the other axioms to exist.

3.1.2 Internal functors

Definition 3.1.2. *An **internal functor** between two internal categories is a triple of morphisms (F_0, F_1, F_2):*

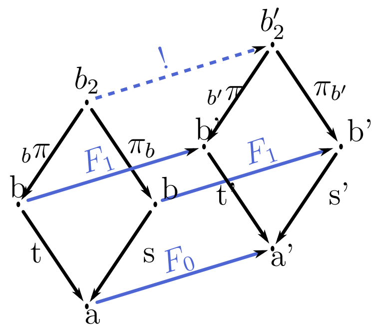


such that all possible diagrams commute, i.e.:

$$
\begin{tikzcd}
b \arrow[r, "F_1"] \arrow[d, "s"'] & b' \arrow[d, "s'"] \\
a \arrow[r, "F_0"'] & a'
\end{tikzcd}
\quad
\begin{tikzcd}
b \arrow[r, "F_1"] \arrow[d, "t"'] & b' \arrow[d, "t'"] \\
a \arrow[r, "F_0"'] & a'
\end{tikzcd}
\quad
\begin{tikzcd}
b_2 \arrow[r, "F_2"] \arrow[d, "\pi_b"'] & b'_2 \arrow[d, "\pi_{b'}"] \\
b \arrow[r, "F_1"'] & b'
\end{tikzcd}
\quad
\begin{tikzcd}
b_2 \arrow[r, "F_2"] \arrow[d, "{}_b\pi"'] & b'_2 \arrow[d, "{}_{b'}\pi"] \\
b \arrow[r, "F_1"'] & b'
\end{tikzcd}
$$

$$
\begin{tikzcd}
b \arrow[r, "F_1"] \arrow[d, "\imath"'] & b' \arrow[d, "\imath'"] \\
a \arrow[r, "F_0"'] & a'
\end{tikzcd}
\qquad
\begin{tikzcd}
b_2 \arrow[r, "F_2"] \arrow[d, "\circ"'] & b_2' \arrow[d, "\circ'"] \\
b \arrow[r, "F_1"'] & b'
\end{tikzcd}
$$

Note that since b_2 is a pullback, the first four conditions force F_2 to be the map factoring ${}_b\pi F_1$ and $\pi_b F_1$, as shown in the following picture :



Actually the definition should have a map between the chosen limits for any n but the relations would impose that they all are the unique map given by the pullback. It will be understood from now on that the induced unique arrow is written $F_2 := {}_b\pi F_1 \ulcorner \pi_b F_1$, or more generally $F_n := {}_b\pi F_1 \ulcorner \ldots \ulcorner \pi_b F_1$ n times, and we will give a functor by a pair of maps only $F := (F_0, F_1)$.

Lemma 3.1.2. *Let $A := (a, b, \cdots)$, $A' := (a', b', \cdots)$ and $A'' := (a'', b'', \cdots)$ be internal categories in a fixed category $\underline{\boldsymbol{C}}$. Let $F := (F_0, F_1) : A \to A'$ and $G := (G_0, G_1) : A' \to A''$ be internal functors. Then the functor $FG := (F_0\, G_0, F_1\, G_1) : A \to A''$ defines an associative composition with identity $(id_a, id_b) : A \to A$*

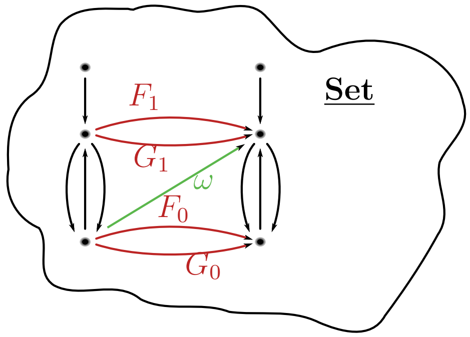
Proof. Since a functor is a pair of arrows, associativity comes for free as long as the composition is well defined. But since pasting commutative diagrams yields commutative diagrams, the above defined pair of arrows is indeed an internal functor. Now the pair (id_a, id_b) is an internal functor since all diagrams commute trivially. Moreover $(id_a, id_b)F = (id_a F_0,\, id_b F_1) = F$, showing that it is an identity for $(a, b, \cdots)$ with respect to this composition. $\square$

Then, given a category $\underline{\mathbf{C}}$, one can build a new category $\underline{\mathbf{IntCat(C)}}$ whose objects are internal categories and arrows internal functors. $\underline{\mathbf{IntCat(Set)}}$= $\underline{\mathbf{Cat}}$, the category of all small categories.

3.1.3 Internal natural transformations

As the real appeal for categories came with natural transformations, internal categories should have little to say were we to stop here. The construction of natural transformations on categories and functors allows us to generalize and define the notion internally.

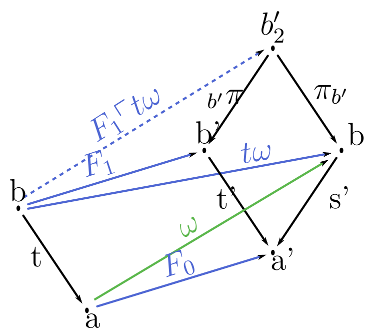
Definition 3.1.3. *Given two internal functors F and G, an* ***internal natural transformation*** *is a morphism ω :*



such that the following diagrams commute:

$$\begin{tikzcd} a \arrow[r, "\omega"] \arrow[dr, "F_0"'] & b' \arrow[d, "s"] \\ & a' \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "\omega"] \arrow[dr, "G_0"'] & b' \arrow[d, "t"] \\ & a' \end{tikzcd} \qquad \begin{tikzcd} b \arrow[r, "F_1\ulcorner t\omega"] \arrow[d, "s\omega\ulcorner G_1"] & b'_2 \arrow[d, "\circ'"] \\ b'_2 \arrow[r, "\circ"] & b' \end{tikzcd}$$

Once again the first two conditions allow the unique maps used in the third one to exist. They are mere prerequisites for the naturality axiom to be stated, and once again they concern the graph structure. For clarity, let's draw the pullback diagram inducing $F_1\ulcorner t\omega$: Note that $F_0\imath$ is a natural transformation between F and itself. Natural transformations



compose two different ways, usually called horizontal and vertical for reasons that should soon become obvious.

Lemma 3.1.3. *Let $\omega : F \to G$ and $\gamma : G \to H$ be internal natural transformations, then $\omega \bullet \gamma := (\omega\ulcorner\gamma)\circ'$ is an internal natural transformation $F \to H$. This composition is associative and has identities given by $F \to F_0\imath$.*

Proof. First let's show that it is a natural transformation from F to H :

$$
\begin{aligned}
(\omega \bullet \gamma)s' &= \big((\omega\ulcorner\gamma)\circ'\big)s' \\
&\big((\omega\ulcorner\gamma)_{b'}\pi\big)s' \\
&= \omega s' = F_0 \\
(\omega \bullet \gamma)t' &= \big((\omega\ulcorner\gamma)\circ'\big)t' \\
&\big((\omega\ulcorner\gamma)\pi_{b'}\big)t' \\
&= \gamma t' = H_0 \\
(F_1\ulcorner(t(\omega \bullet \gamma)))\circ' &= (F_1\ulcorner(t(\omega\ulcorner\gamma)\circ'))\circ' \\
&= (F_1\ulcorner(t(\omega\ulcorner\gamma)))(1\ulcorner\circ')\circ' \\
&= (F_1\ulcorner t\omega\ulcorner t\gamma)(1\ulcorner\circ')\circ' \\
&= (F_1\ulcorner t\omega\ulcorner t\gamma)(\circ'\ulcorner 1)\circ' \\
&= (((F_1\ulcorner t\omega)\circ')\ulcorner t\gamma)\circ' \\
&= (((s\omega\ulcorner G_1)\circ')\ulcorner t\gamma)\circ' \\
&= (s\omega\ulcorner G_1\ulcorner t\gamma)(\circ'\ulcorner 1)\circ' \\
&= (s\omega\ulcorner G_1\ulcorner t\gamma)(1\ulcorner\circ')\circ' \\
&= (s\omega\ulcorner((G_1\ulcorner t\gamma)\circ'))\circ' \\
&= (s\omega\ulcorner((s\gamma\ulcorner H_1)\circ'))\circ' \\
&= (s\omega\ulcorner s\gamma\ulcorner H_1)(1\ulcorner\circ')\circ' \\
&= ((s(\omega\ulcorner\gamma))\ulcorner H_1)(\circ'\ulcorner 1)\circ' \\
&= (s(\omega \bullet \gamma)\ulcorner H_1)\circ'
\end{aligned}
$$

From the uniqueness of the pullback, we find that $(F\ulcorner G\ulcorner H)(({}_{b'_2}\pi\circ')\ulcorner\pi_{b'}) = ((F\ulcorner G)\circ')\ulcorner H$, and then :

$$
\begin{aligned}
(\omega \bullet \gamma) \bullet \lambda &= (((\omega\ulcorner\gamma)\circ')\ulcorner\lambda)\circ' \\
&= (\omega\ulcorner\gamma\ulcorner\lambda)(({}_{b'_2}\pi\circ')\ulcorner\pi_{b'})\circ' \\
&= (\omega\ulcorner\gamma\ulcorner\lambda)({}_{b'}\pi\ulcorner(\pi_{b'_2}\circ'))\circ' \\
&= (\omega\ulcorner((\gamma\ulcorner\lambda)\circ'))\circ' \\
&= \omega \bullet (\gamma \bullet \lambda)
\end{aligned}
$$

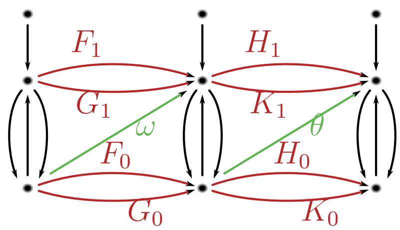
Therefore the uniqueness of the pullback and the associativity of composition insure the associativity of $\bullet$. Finally,

$$
\begin{aligned}
(F_0\imath) \bullet \omega &= (F_0\imath\ulcorner\omega)\circ' \\
&= ((\omega s'i')\ulcorner\omega)\circ' \\
&= \omega(s'i'\ulcorner id_{b'})\circ' \\
&= \omega
\end{aligned}
$$

$$
\begin{aligned}
\omega \bullet (G_0\imath) &= (\omega{}^\ulcorner G_0\imath)\circ' \\
&= \omega(id_{b'}{}^\ulcorner t' i')\circ' \\
&= \omega
\end{aligned}
$$

So that as claimed $F_0\imath$ is the identity on F. ☐

Suppose now that you have the following situation :



then one can create $\omega_H := \omega H_1 : FH \to GH$ and ${}_G\theta := G_0\theta : GH \to GK$[1], which are natural transformations as well, and compose them with the previous defined composition, so that $\omega \diamond \theta := \omega_H \bullet {}_G\theta$.

Lemma 3.1.4. *The above constructed composition is associative and unital.*

Proof. Let $\omega : F \to G$, $\theta : H \to K$ and $\eta : L \to M$, then :

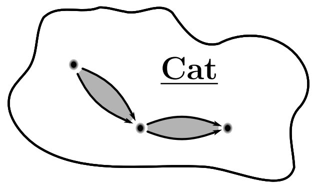
$$
\begin{aligned}
(\omega \diamond \theta) \diamond \eta &= (\omega_H \bullet {}_G\theta)_L \bullet {}_{GK}\eta \\
&= (((\omega H_1{}^\urcorner G_0\theta) \circ' L_1){}^\ulcorner G_0K_0\eta)\circ'' \\
&= (((\omega H_1L_1{}^\urcorner G_0\theta L_1)\circ''){}^\ulcorner G_0L_0\eta)\circ'' \\
&= (\omega H_1L_1{}^\urcorner G_0\theta L_1{}^\ulcorner G_0K_0\eta)(1{}^\ulcorner\circ'')\circ'' \\
&= (\omega H_1L_1{}^\urcorner G_0\theta L_1{}^\ulcorner G_0K_0\eta)(\circ''{}^\ulcorner 1)\circ'' \\
&= (\omega H_1L_1{}^\urcorner G_0((\theta L_1{}^\ulcorner K_0\eta)\circ''))\circ'' \\
&= \omega_{HL} \bullet {}_G(\theta_L \bullet {}_K\eta) \\
&= \omega \diamond (\theta \diamond \eta) \\
\omega \diamond \imath' &= \omega_{id} \bullet {}_G\imath' \\
&= (\omega{}^\ulcorner G_0\imath')\circ' \\
&= (\omega{}^\ulcorner \omega t\imath')\circ' \\
&= \omega(id{}^\ulcorner t\imath')\circ' \\
&= \omega
\end{aligned}
$$

The left unit proof follows the same pattern as the right unit one. ☐

[1] this is the whiskering

3.1.4 Emerging recursive phenomenons

Out of internal categories, functors and natural transformations emerges the picture of an entity called a 2-category. It is a category-enriched category, i.e. homsets are actually hom-categories and composition is functorial. Therefore **Cat** looks like the following:



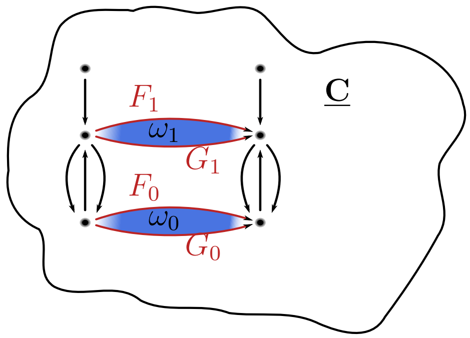
Which is to say objects, arrows and arrows between arrows. This seems to give a hint of a recursive process to define higher dimensional versions of categories as entities with objects, arrows, 2-arrows between arrows, 3-arrows between 2-arrows etc... This is called the globular approach and the weak case has been worked on notably by Ross Street, John Baez and his students. For more information of the history of n-categories check [54, 10]. Higher categories of dimension n are then called n-categories. It has been shown by Ronald Brown and Al-Alg that the approach we will be taking, the cubical approach, is in some very precise cases equivalent to the globular approach. Although its name has everything to do with the kind of pictures one draws when dealing with such objects, the recursive process at play is to take internal categories in (_), starting with **Set**. The categories obtained are called n-tuple (or multiple) categories, which are the same as n-categories for $n = 1$ but become more and more general as n grows, always including n-categories as special cases.

3.1.5 Internal natural cells

The previously defined double categories, double functors and horizontal double natural transformations are respectively internal categories, internal functors and internal natural transformations in **Cat**. To recover vertical double natural transformations and comparisons, we need to use the fact that **Cat** is a 2-category. Let's define them internally and investigate their compositions. We will need the notion of pullbacks in a 2-category, which is defined in [42]. For an arrow f in $\underline{\mathbf{C}}$, we will write id_f for its identity 2-cell.

Definition 3.1.4. *Given two parallel internal functors F and G between internal categories in a 2-category $\underline{\mathbf{C}}$, an **internal natural cell** $\omega : F \to G$ is a pair $(\omega_0 : F_0 \to G_0, \omega_1 : F_1 \to G_1)$ of 2-cells such that :*

$$(\omega_1) \circ id_s = id_{s'} \circ \omega_0$$
$$(\omega_1) \circ id_t = id_{t'} \circ \omega_0$$
$$id_\imath \circ (\omega_1) = \omega_0 \circ id_{\imath'}$$
$$(\omega_2) \circ id_m = id_{m'} \circ \omega_1$$



It is represented by the following picture where all possible diagrams commute. and the previous notation $\omega_2 := {}_b\pi\omega_1{}^\ulcorner\pi_b\omega_1$ has been used. Remark that abstractly internal natural cells look like higher dimensional functors.

Lemma 3.1.5. *The compositions of 2-cells lifts to compositions on internal natural cells.*

Proof. Let $\circ$ be the horizontal composition in the 2-category, $\circ_v$ its vertical composition. Let F,G and H,K two pairs of parallel functors such that $F \circ G$ is defined. Then for two internal natural cells $\beta : F \to G$ and $\gamma : H \to K$, the composite $\beta \circ \gamma := (\beta_0 \circ \gamma_0, \beta_1 \circ \gamma_1)$ is an internal natural cell.

$$\begin{aligned}(\beta \circ \gamma)_1 \circ s'' &= (\beta_1 \circ \gamma_1) \circ s'' = \beta_1 \circ (\gamma_1 \circ s'') = \beta_1 \circ (s' \circ \gamma_0)\\ &= (\beta_1 \circ s') \circ \gamma_0 = (s \circ \beta_0) \circ \gamma_0 = s \circ (\beta_0 \circ \gamma_0)\\ &= s \circ (\beta \circ \gamma)_0\end{aligned}$$

the proof for t and $\imath$ follow the same principle, similarly for m :

$$\begin{aligned}(\beta \circ \gamma)_2 \circ m'' &= (\beta_2 \circ \gamma_2) \circ m'' = \beta_2 \circ (\gamma_2 \circ m'') = \beta_2 \circ (m' \circ \gamma_1)\\ &= (\beta_2 \circ m') \circ \gamma_1 = (m \circ \beta_2) \circ \gamma_1 = m \circ (\beta_1 \circ \gamma_1)\\ &= m \circ (\beta \circ \gamma)_1\end{aligned}$$

associativity and units are then inherited from the composition in the 2-category. Now let $\bullet$ be the vertical composition and $\kappa : G \to L$, if the composite $\beta \bullet \kappa := (\beta_0 \bullet \kappa_0, \beta_1 \bullet \kappa_1)$ is again a natural cell, associativity and unit of the composition will follow from the axion of a 2-category. Or :

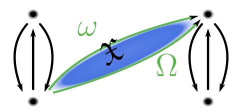
$$\begin{aligned}(\beta \bullet \kappa)_1 \circ s' &= (\beta_1 \bullet \kappa_1) \circ s'\\ &= (\beta_1 \circ s') \bullet (\kappa_1 \circ s')\\ &= (s \circ \beta_0) \bullet (s \circ \kappa_0)\\ &= s \circ (\beta \bullet \kappa)_0\end{aligned}$$

the proof for t,m and $\imath$ follow the same principle. $\square$

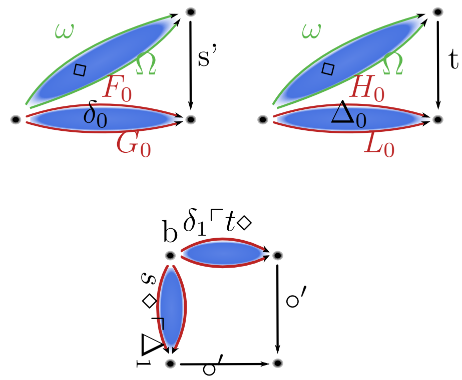
Note that internal natural cells share the same axioms as functors except that it involves 2-cells instead of 1-cells (arrows).

3.1.6 Internal comparisons

Definition 3.1.5. *Given four internal functors $F, G, H, J : C \to D$, two internal n.t. $\omega : F \to G$ and $\Omega : H \to J$, and two internal natural cells $\delta : G \to J$ and $\Delta : F \to H$, an **internal comparison** $\mathfrak{X} : (\omega, \delta) \to (\Omega, \Delta)$ is a 2-cell $\mathfrak{X} : \omega \to \Omega$, i.e.*



such that the following diagrams commute :



In the literature, comparisons may be found as well under the name modifications. Note that internal comparisons share the same axioms as internal natural transformations. Internal comparisons have 3 different compositions, all associative and unital that reduce on the boundary to the three kind of composition that appeared for natural transformations and natural cells.

Lemma 3.1.6. *Let $\omega_1 : F \to G$, $\omega_2 : H \to J$ and $\omega_3 : K \to L$ be internal natural transformations, $\delta_1 : F \to J$, $\delta_2 : H \to L$, $\Delta_1 : G \to H$ and $\Delta_2 : J \to K$ be internal natural cells, and $\mathfrak{X} : (\omega_1, \delta_1) \to (\omega_2, \Delta_1)$ and $\mathfrak{Y} : (\omega_2, \delta_2) \to (\omega_3, \Delta_2)$ internal comparisons. Then composing $\mathfrak{X}$ to $\mathfrak{Y}$ in their homcategory yields a new internal comparison $\mathfrak{X} \star \mathfrak{Y} : (\omega_1, \delta_1 \star \delta_2) \to (\omega_3, \Delta_1 \star \Delta_2)$.*
This composition is associative and has a unit.

Proof. Let $\star$ denote the composition of 2-cells along their boundary 1-cells. Then the axioms are guaranteed by the following distributivity axiom on $\star$. Let's prove the commutativity of the first diagram on $\mathfrak{X} \star \mathfrak{Y}$.

$$\begin{aligned}(\mathfrak{X} \star \mathfrak{Y})s' &= (\mathfrak{X}s') \star (\mathfrak{Y}s') \\ &= \delta_1 \star \delta_2\end{aligned}$$

The other two identities are proved the same way. Associativity and unit follows from the associativity and unit of $\star$ as a 2-cell. $\square$

Lemma 3.1.7. *Let $\omega_1 : F \to G$, $\omega_2 : K \to L$, $\gamma_1 : G \to H$ and $\gamma_2 : L \to M$ be internal natural transformations, $\Delta_1 : F \to K$, $\Delta_2 : G \to L$ and $\Delta_3 : H \to M$ be natural cells, and $\mathfrak{X} : (\omega_1, \Delta_1) \to (\omega_2, \Delta_2)$ and $\mathfrak{Y} : (\gamma_1, \Delta_2) \to (\gamma_2, \Delta_3)$ internal comparisons. Then $\mathfrak{X} \bullet \mathfrak{Y} := (\mathfrak{X}^\ulcorner \mathfrak{Y})\circ'$ is an internal comparison $(\omega_1 \bullet \gamma_1, \Delta_1) \to (\omega_2 \bullet \gamma_2, \Delta_3)$. This composition is associative and has identities given by $\delta \to \delta_0 \imath$.*

Proof. First let's show that it is a comparison $(\omega_1 \bullet \gamma_1, \Delta_1) \to (\omega_2 \bullet \gamma_2, \Delta_3)$:

$$\begin{aligned}
(\mathfrak{X} \bullet \mathfrak{Y})s' &= \big((\mathfrak{X}^\ulcorner \mathfrak{Y})\circ'\big)s' \\
&\big((\mathfrak{X}^\ulcorner \mathfrak{Y})_{b'}\pi\big)s' \\
&= \mathfrak{X}s' = (\Delta_1)_0 \\
(\mathfrak{X} \bullet \mathfrak{Y})t' &= \big((\mathfrak{X}^\ulcorner \mathfrak{Y})\circ'\big)t' \\
&\big((\mathfrak{X}^\ulcorner \mathfrak{Y})\pi_{b'}\big)t' \\
&= \mathfrak{Y}t' = (\Delta_3)_0 \\
((\Delta_1)_1{}^\ulcorner(t(\mathfrak{X} \bullet \mathfrak{Y})))\circ' &= ((\Delta_1)_1{}^\ulcorner(t(\mathfrak{X}^\ulcorner \mathfrak{Y})\circ'))\circ' \\
&= ((\Delta_1)_1{}^\ulcorner(t(\mathfrak{X}^\ulcorner \mathfrak{Y})))(1^\ulcorner \circ')\circ' \\
&= ((\Delta_1)_1{}^\ulcorner t\mathfrak{X}^\ulcorner t\mathfrak{Y})(1^\ulcorner \circ')\circ' \\
&= ((\Delta_1)_1{}^\ulcorner t\mathfrak{X}^\ulcorner t\mathfrak{Y})(\circ'^\ulcorner 1)\circ' \\
&= ((((\Delta_1)_1{}^\ulcorner t\mathfrak{X})\circ')^\ulcorner t\mathfrak{Y})\circ' \\
&= (((s\mathfrak{X}^\ulcorner(\Delta_2)_1)\circ')^\ulcorner t\mathfrak{Y})\circ' \\
&= (s\mathfrak{X}^\ulcorner(\Delta_2)_1{}^\ulcorner t\mathfrak{Y})(\circ'^\ulcorner 1)\circ' \\
&= (s\mathfrak{X}^\ulcorner(\Delta_2)_1{}^\ulcorner t\mathfrak{Y})(1^\ulcorner \circ')\circ' \\
&= (s\mathfrak{X}^\ulcorner(((\Delta_2)_1{}^\ulcorner t\mathfrak{Y})\circ'))\circ' \\
&= (s\mathfrak{X}^\ulcorner((s\mathfrak{Y}^\ulcorner(\Delta_3)_1)\circ'))\circ' \\
&= (s\mathfrak{X}^\ulcorner s\mathfrak{Y}^\ulcorner(\Delta_3)_1)(1^\ulcorner \circ')\circ' \\
&= ((s(\mathfrak{X}^\ulcorner \mathfrak{Y}))^\ulcorner(\Delta_3)_1)(\circ'^\ulcorner 1)\circ' \\
&= (s(\mathfrak{X} \bullet \mathfrak{Y})^\ulcorner(\Delta_3)_1)\circ'
\end{aligned}$$

From the uniqueness of the pullback, we find that $(F^\ulcorner G^\ulcorner H)(({}_{b'_2}\pi\circ')^\ulcorner \pi_{b'}) = ((F^\ulcorner G)\circ')^\ulcorner H$, and then :

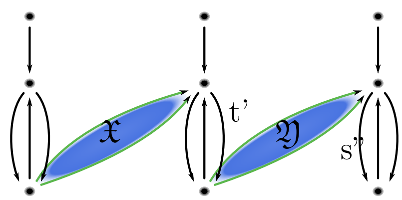
$$\begin{aligned}
(\mathfrak{X} \bullet \mathfrak{Y}) \bullet \mathfrak{Z} &= (((\mathfrak{X}^\ulcorner \mathfrak{Y})\circ')^\ulcorner \mathfrak{Z})\circ' \\
&= (\mathfrak{X}^\ulcorner \mathfrak{Y}^\ulcorner \mathfrak{Z})(({}_{b'_2}\pi\circ')^\ulcorner \pi_{b'})\circ' \\
&= (\mathfrak{X}^\ulcorner \mathfrak{Y}^\ulcorner \mathfrak{Z})({}_{b'}\pi^\ulcorner(\pi_{b'_2}\circ'))\circ' \\
&= (\mathfrak{X}^\ulcorner((\mathfrak{Y}^\ulcorner \mathfrak{Z})\circ'))\circ' \\
&= \mathfrak{X} \bullet (\mathfrak{Y} \bullet \mathfrak{Z})
\end{aligned}$$

Therefore the uniqueness of the pullback and the associativity of composition ensure the associativity of $\bullet$. Finally

$$\begin{aligned}
(\delta_0 \imath) \bullet \mathfrak{X} &= (\delta_0 \imath^\ulcorner \mathfrak{X}) \circ' \\
&= ((\mathfrak{X} s' i')^\ulcorner \mathfrak{X}) \circ' \\
&= \mathfrak{X}(s' i'^\ulcorner id_{b'}) \circ' \\
&= \mathfrak{X} \\
\mathfrak{X} \bullet (\Delta_0 \imath) &= (\mathfrak{X}^\ulcorner \Delta_0 \imath) \circ' \\
&= \mathfrak{X}(id_{b'}{}^\ulcorner t' i') \circ' \\
&= \mathfrak{X}
\end{aligned}$$

So that as claimed $\delta_0 \imath$ is the identity on δ. $\square$

Suppose now that you have the following situation :



where $\mathfrak{X} t' = \gamma_0$ and $\mathfrak{Y} s'' = \delta_0$ then one can create $\mathfrak{X}_\delta := \mathfrak{X}\delta_1$ and ${}_\gamma\mathfrak{Y} := \gamma_0 \mathfrak{Y}$, which are comparisons as well, and compose them with the previous defined composition, so that $\mathfrak{X} \diamond \mathfrak{Y} := \mathfrak{X}_H \bullet {}_G\mathfrak{Y}$.

Lemma 3.1.8. *The above constructed composition is associative and unital.*

Proof. Let $\mathfrak{X} : (\omega, \delta) \to (\Omega, \Delta)$, $\mathfrak{Y} : (\omega', \delta') \to (\Omega', \Delta')$ and $\mathfrak{Z} : (\omega'', \delta'') \to (\Omega'', \Delta'')$, then :

$$\begin{aligned}
(\mathfrak{X} \diamond \mathfrak{Y}) \diamond \mathfrak{Z} &= (\mathfrak{X}'_\delta \bullet {}_\Delta \mathfrak{Y})''_\delta \bullet {}_{\Delta\Delta'} \mathfrak{Z} \\
&= (((\mathfrak{X}\delta'_1{}^\urcorner \Delta_0 \mathfrak{Y}) \circ' \delta''_1)^\ulcorner \Delta_0 \Delta'_0 \mathfrak{Z}) \circ'' \\
&= (((\mathfrak{X}\delta'_1 \delta''_1{}^\urcorner \Delta_0 \mathfrak{Y} \delta''_1) \circ'')^\ulcorner \Delta_0 \Delta'_0 \mathfrak{Z}) \circ'' \\
&= (\mathfrak{X}\delta'_1 \delta''_1{}^\urcorner \Delta_0 \mathfrak{Y} \delta''_1{}^\ulcorner \Delta_0 \Delta'_0 \mathfrak{Z})(1^\ulcorner \circ'') \circ'' \\
&= (\mathfrak{X}\delta'_1 \delta''_1{}^\urcorner \Delta_0 \mathfrak{Y} \delta''_1{}^\ulcorner \Delta_0 \Delta'_0 \mathfrak{Z})(\circ''^\ulcorner 1) \circ'' \\
&= (\mathfrak{X}\delta'_1 \delta''_1{}^\urcorner \Delta_0 ((\mathfrak{Y} \delta''_1{}^\ulcorner \Delta'_0 \mathfrak{Z}) \circ'')) \circ'' \\
&= \mathfrak{X}_{\delta'\delta''} \bullet {}_\Delta (\mathfrak{Y}_K \bullet {}_{\Delta'} \mathfrak{Z}) \\
&= \mathfrak{X} \diamond (\mathfrak{Y} \diamond \mathfrak{Z}) \\
\mathfrak{X} \diamond \imath' &= \mathfrak{X}_{id} \bullet {}_\Delta \imath' \\
&= (\mathfrak{X}^\ulcorner \Delta_0 \imath') \circ' \\
&= (\mathfrak{X}^\ulcorner \mathfrak{X} t \imath') \circ' \\
&= \mathfrak{X}(id^\ulcorner t \imath') \circ' \\
&= \mathfrak{X}
\end{aligned}$$

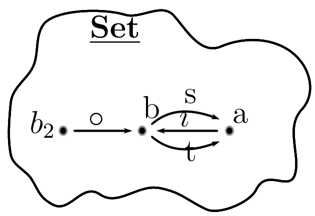
The left unit proof follows the same pattern as the right unit one. $\square$

Internal comparisons have therefore 3 compositions, all associative with unit. Moreover, these compositions intertwine each other.

When starting with a 2-category in place of a mere category, internal functors, internal natural transformations, internal natural cells and internal comparisons are available. What emerges out of this description is a "category" where Homsets are double categories and composition is functorial, which is to say that **dbCat** is a **dbCat**-enriched category !

3.2 Degeneracies

From now on the projections from the pullback will be omitted in the picture, and hence we draw an internal category in a category with pullbacks as :



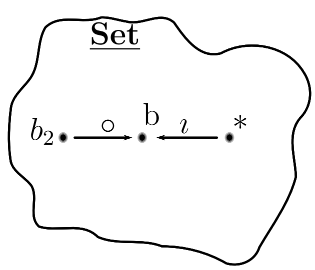
There are two "degenerate" versions of a category. The first one happens when the source or target morphism (s or t) is taken to be the identity.

Lemma 3.2.1. *If any of $s, t, \imath$ are identities then they all are and so is the composition. The relations are then trivially satisfied.*

Proof. If $s = id$, then condition 1 forces $\imath = id$ and $t = id$ follows from condition 2. The projections from the pullback are then identities as well, which forces the composition to be the identity. □

Definition 3.2.1. *An internal category whose defining morphisms are identities is called **discrete**.*

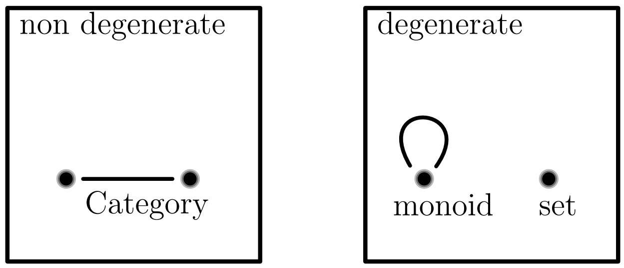
A discrete internal category in $\underline{\mathbf{C}}$ is equivalent to an object of $\underline{\mathbf{C}}$. The second degenerate case is the case where the object of objects, "a" in our definition, is a terminal object. The source and target morphisms are then unique and the structure becomes :



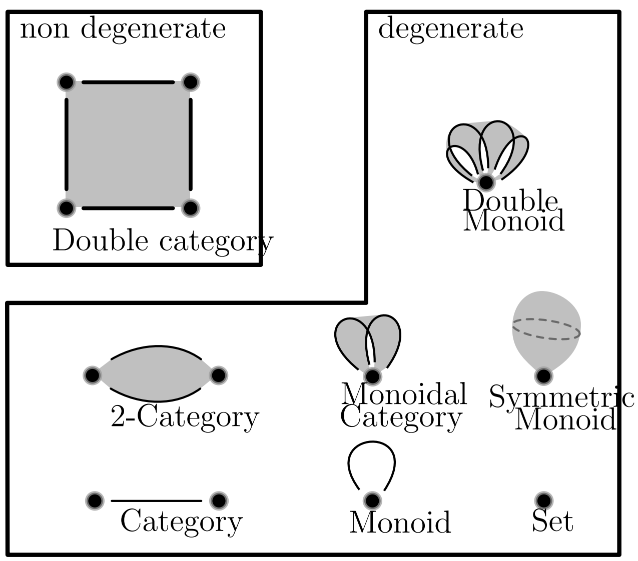
Definition 3.2.2. *An internal category whose vertices object is terminal is called an **internal monoid**.*

Indeed we find that $b_2 = b \times b$ and that $\circ$ is therefore an associative multiplication on b. The identity $\imath$ is equivalent to an element of b, its image, that acts as a unit with respect to the multiplication and the whole structure is a monoid in the usual sense.

So, a category whose arrows are identities only is a set, whereas a category with only one object is a monoid. These correspond to the two degenerate versions of a segment :



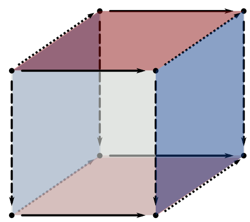
In the first one the vertices (objects) were collapsed whereas the second one has its edges (arrows) collapsed. Similarly double categories will have degenerate versions corresponding to the collapses of vertices, edges of same kind and surfaces. The list in pictures is :



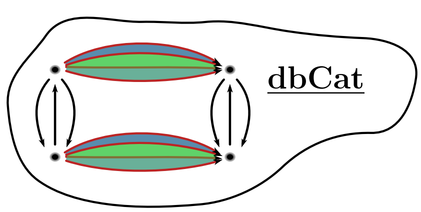
As promised, useful and usual category theory entities appear as degeneracies, such as 2-categories, monoidal categories and symmetric monoids. The case of triple categories has many more degeneracies , including all of the preceeding and amongst the rest some familiar entities like 3-categories, monoidal 2-categories and symmetric monoidal categories. The symmetrization appearing is the mechanism whose shadow is known as the Eckmann-Hilton argument, as presented in Lemma 1.2. If one has a double category structure (or substructure) where all edges are identities, then Eckmann-Hilton argument applies. Now consider cubes whose only non trivial elements are two parallel faces and the volume, all the rest being identities, then the above pictorial argument holds in a plane parallel to those non trivial faces. It shows that a triple category whose only non trivial elements are squares of one kind and cubes is in fact a symmetric monoidal category. Any structure we find in dimension n, will appear with a monoidal structure in dimension n+1 and with a symmetric monoidal structure in dimension n+2.

3.3 n-tuple Categories

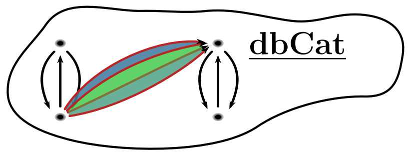
The next step in the inductive process is to take internal categories in **dbCat**. The resulting structure is called a **triple category**. It contains one kind of objects, three kinds of arrows, three kinds of squares and 1 kind of cubes. General cubes look like :



and can be composed in three directions. The functoriality of the internal composition ensures that the extra two interchange laws one can write are satisfied and that therefore the subdivision of a cube into eight cubes gives a unique composition. As before, cubes have a group of symmetries that acts upon the category **TrCat** of triple categories. Under this action triple functors don't change presentation. Triple natural transformations are sent to triple natural cells, which are of two types since **dbCat** has double natural transformations and double natural cells as 2-cells. This same fact gives us two choices for triple comparisons though a third type appears : its diagrammatic description is identical to the one of internal natural cells and internal functors but with "footballs" instead of 2-cells and arrows. It is represented by the picture :



It is therefore some kind of 3-dimensional functor, just as internal natural cells were 2-dimensional functors. Following the same scheme a new entity appears, it can be thought of a a 3-dimensional natural transformation, just as comparisons were 2-dimensional natural transformations. It has the form :



It is defined using 8 triple functors, 4 triple natural transformations, 4 triple natural cells of each type, and 2 comparisons of each type forming a cube. It can be composed in 4 different ways. Again we find that **TrCat**, the category of triple categories, is naturally a triple category enriched category !

Lemma 3.3.1. *The category of n-tuple categories is naturally enriched over n-tuple categories.*

Proof. this was first proved by Ehresmann himself in [13] but can be found in more modern language in [43]. □

It may be hard for the reader to deduce a direct definition of n-tuple categories from the recursion of internalization alone. We therefore will give one now. Let [n] be the set of integers from 1 to n, and for a nonempty subset I of $[n]$ define $\hat{I} := [n] \setminus I$. By abuse of notation braces will be omitted in subscripts, for example $n_{ij} := n_{\{ij\}}$ and $n_{\hat{i}} = n_{[n]\setminus\{i\}}$.

Definition 3.3.1. *A n**-tuple category** $\mathfrak{C}$ is a set*

$$\left\{\mathfrak{C}_\emptyset, \{\mathfrak{C}_I\}, \{s_I\}, \{t_I\}, \{\imath_I\}\, \forall I \subseteq [n] \text{ nonempty}, \{\circ_i\}\, \forall i \in [n]\right\}$$

where :

- *The following maps of sets are the source and target maps:*

$$s_I, t_I\ : \mathfrak{C}_J \to \mathfrak{C}_{J\setminus I} \quad \forall I \subseteq J \subseteq [n]$$

  The pullback of the pair $(t_i,\, s_i)$ is denoted by $\underset{i}{\times}$.

- *The following maps of sets are the identity maps:*

$$\imath_I\ : \mathfrak{C}_J \to \mathfrak{C}_{J\cup I} \quad \forall J \subseteq [n] \text{ s.t. } I \cap J = \emptyset$$

- *The following maps of sets are the composition maps:*

$$\circ_i :\ \mathfrak{C}_I \underset{i}{\times} \mathfrak{C}_I \to \mathfrak{C}_I \quad \forall I \text{ containing } i$$

- *Sources, targets and compositions are compatible the following way:*

$$s_I s_J = s_{J\cup I} \qquad t_I t_J = t_{J\cup I}$$
$$(s_I \underset{j}{\times} s_I)\circ_j = (\circ_j \underset{i}{\times} \circ_j) s_I \qquad (t_I \underset{j}{\times} t_I)\circ_j = (\circ_j \underset{i}{\times} \circ_j) t_I$$

- *Compositions are associative and satisfy the interchange laws:*

$$(\circ_i \underset{j}{\times} \circ_i)\circ_j = (\circ_j \underset{i}{\times} \circ_j) \circ_i \quad \forall i \neq j \in [n]$$

- $\imath_i$ *is an identity for the composition* $\circ_i$ *and* $\circ_j(\imath_I) = (\imath_I \underset{j}{\times} \imath_I)\circ_j$ *for all* $j \notin I$.

The elements of $\mathfrak{C}_\emptyset$, $\{\mathfrak{C}_I\}$ *for* $I \subset [n]$ *and* $\mathfrak{C}_{[n]}$ *are respectively called objects, faces and n-cubes.*

Lemma 3.3.2. *The interchange laws impose* $\imath_i\imath_j = \imath_j\imath_i$, *and therefore* $\imath_I\imath_J = \imath_{J\cup I}$.

3.4 n-tuple groupoids

In this section, we develop tools to extend the results that we obtained for double groupoids to higher dimensions. It is entirely original as there is no literature on the structure of n-tuple groupoids. For this reason $n \geq 2$ for the rest of theis section.

Definition 3.4.1. *An n-**tuple groupoid** is an n-tuple category whose n-cubes are invertible in all directions.*

The inverses will be denoted as follows, suppose that X is an n-cube, then its inverse with respect to $\circ_i$ will be denoted X^{-i}. The interchange laws then ensure that $(X^{-i})^{-j} = (X^{-j})^{-i}$. We can then define the unique inverse $X^{-ijk\dots}$ of X in the combined directions $i, j, k\dots$.

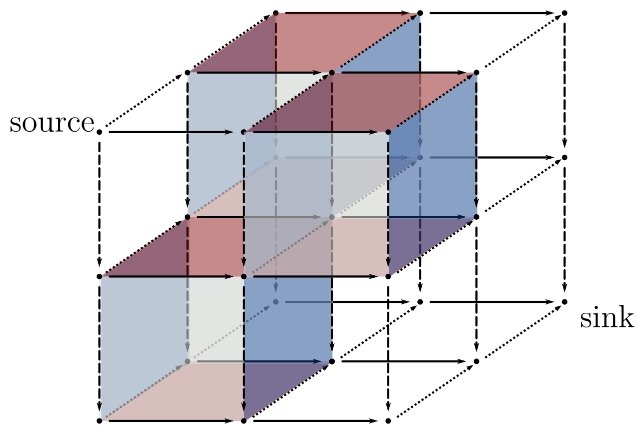
3.4.1 Barycentric subdivisions

Definition 3.4.2. *An arrangement of n-cubes combinatorially equivalent to the one given by excluding the subspaces* $x_i = \frac{1}{2}$ *for all* $i \in [n]$ *from* $[0,1]^n \subset \mathbb{R}^n$ *is called a **barycentric subdivision** of the n-cube. Arrangements combinatorially equivalent to one found by excluding the subspaces* $x_i = 1/3$ *and* $x_i = 2/3$ *for all* $i \in [n]$ *is called the **division in thirds**.*

With this definition in mind we can see the interchange law as ensuring that barycentric subdivisions have a uniquely defined composition.

Definition 3.4.3. *In the barycentric subdivision of the n-cube, the **partition** of a sub n-cube is an ordered pair* (A, B) *of complementary subsets of* $[n]$ *such that* $i \in A$ *if and only if the* i^{th} *source of the sub n-cube is part of the barycentric division of the* i^{th} *boundary of the original n-cube. The **depth** of a sub n-cube is the cardinality of B.*

Then the n-cube adjacent to the source corner has depth 0 and the one adjacent to the sink corner has depth n. Here are for example the 3-subcubes of depth 1:



Every sub n-cube has n external boundaries and n internal boundaries and the i^{th} target of a sub n-cube with partition (A, B) is internal to the subdivision if and only if $i \in A$. Therefore an n-cube has $2n$ boundaries, n of which are internal.

Lemma 3.4.1. *A sub n-cube of depth i has a common boundary (n-1)-cube with i sub n-cubes of depth (i-1) and with $(n-i)$ sub n-cubes of depth $(i+1)$.*

Proof. Consider a sub n-cube Q, then $t_J(Q)$ is internal to the decomposition if and only if $J \in A_Q$. Similarly, for another sub n-cube Q', $s_J(Q')$ is internal to the decomposition if and only if $J \in B_Q$. Then defining $R := A_Q \cap B_{Q'}$ we can conclude that $s_R(Q') = t_R(Q)$ and that for $R \subset S \subset [n]$, $s_S(Q') \neq t_S(Q)$. Therefore a sub n-cube Q of depth $i-1$ shares a boundary (n-1)-cube with Q' of depth i if and only if $|A_Q \cap B_{Q'}| = 1$, or equivalently $B_Q \subset B_{Q'}$. Since $|B_{Q'}| = |B_Q| + 1 = i$ there are i such n-cubes Q for a given Q'. □

Lemma 3.4.2. *The intersection of an n-cube of depth i and the n-cube of depth 0 is an $(n-i)$-cube. Its intersection with the n-cube of depth n is an i-cube. Together, these two intersection contain edges of all n directions.*

Proof. The sub n-cube α of depth 0 satisfies $A_\alpha = [n]$, so from the previous proof we can conclude that $Q \cap \alpha = s_{B_Q}$ and has codimension $|B_Q| = i$. Similarly if Ω is the sub n-cube of depth n, $B_\Omega = [n]$, so the intersection $Q \cap \Omega = t_{A_Q}$ and has dimension $|B_Q| = i$. □

Let X be an n-cube of τ and consider a barycentric subdivision where X has depth n. Place identities in all positions of positive depth. Then the squares that can be placed with depth 0 must have all targets the identity on the source of X, i.e. :

$$t_i = \imath_{\hat{i}}(s_{[n]}(X))$$

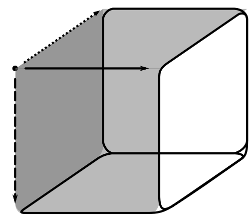
3.4.2 Core groupoids

Definition 3.4.4. *Let τ be an n-tuple groupoid and define*

$$\begin{aligned} \tau_{\lrcorner} :&= \{\textit{n-cubes whose recursive targets are identities}\} \\ &= \{X \in \tau | t_i(X) = \imath_{\hat{i}}(t_{[n]}(X))\} \end{aligned}$$

*For $u \in \tau_\lrcorner$ and $X \in \tau$ such that $t_{[n]}(u) = s_{[n]}(X)$ define the **transmutation of** X **by** u, denoted $u \cdot X$, to be the n-cube accepting a barycentric subdivision with u of depth 0, X of depth n and all others identities, as defined above.*

For example, in dimension 3 a element of $\tau_\lrcorner$ looks like the following, where arrows without heads and non colored squares are identities.



Lemma 3.4.3. *Let $u, v \in \tau_\lrcorner$, then $u \cdot v \in \tau_\lrcorner$. Moreover $(\tau_\lrcorner, \cdot, \imath)$ is a groupoid.*

Proof. Let the sources and targets of arrows in said groupoid be the sources and sinks of the n-cubes of $\tau_\lrcorner$. First note that $t_i(u \cdot X) = t_i(X)$, so that $v \in \tau_\lrcorner$ implies $u \cdot v \in \tau_\lrcorner$. This defines a composition on $\tau_\lrcorner$. The associativity of this composition is a direct consequence of the interchange law and the uniqueness of identities. Indeed, for $u, v, w, \in \tau_\lrcorner$, $(u \cdot v) \cdot w$ and $u \cdot (v \cdot w)$ are given by two different order of composition of the same 3x3x...x3 (n times) array of n-cubes with u,v and w in order on the diagonal and the rest filled by identities. But as the interchange laws insures that the two compositions are equal, associativity is proved. The identities given by identity n-cubes on objects. It remains to prove that inverses exist and uniqueness will follow.
To do so, consider an equation $b(X) = Y$ where $b(X)$ is a barycentric subdivision with one indeterminate sub-cube X, then one can solve for X in τ. For example, in dimension 2, for some fixed $A, B, C, Y \in \tau$:

$$\begin{aligned} (A \circ_h B) \circ_v (C \circ_h X) &= Y \\ C \circ_h X &= (A \circ_h B)^{-v} \circ_v Y \\ X &= C^{-h} \circ_h \left((A \circ_h B)^{-v} \circ_v Y\right) \end{aligned}$$

This proves the existence of left inverses in $\tau_\lrcorner$ since we can solve $X \cdot u = id_{t_{[n]}(u)}$ as all n-cubes of non zero depth in $X \cdot u$ are uniquely determined by u. It is not the case for $u \cdot X$, so a little more work is needed to find a right inverse. We will give a proof by induction. Note that in dimension 1 the core diagram of a groupoid is the groupoid itself, so the lemma is true. Now for any other dimension if a right inverse u^{-1} of u exists in $\tau_\lrcorner$, then its i^{th} source is the right inverse to the i^{th} source of u in the core of the i^{th} boundary $(n-1)$-groupoid of τ and the boundaries of $u \cdot u^{-1}$ are their product in it. Now all n-cubes of depth smaller than n in $u \cdot u^{-1}$ are either u itself or identity cubes uniquely determined by these boundaries inverses. Hence supposing the existence of right inverses on the boundaries, the only indeterminate in $u \cdot u^{-1} = id_{s_{[n]}(u)}$ is the inverse, which proves the existence, by induction, of the right inverse of u in τ. □

Definition 3.4.5. *The groupoid $\tau_\lrcorner$ is called the **core groupoid** of τ*

Let $u \in \tau_\lrcorner$, then the assignment $u \to u\cdot$ defines an action of groupoids on $\tau_{[n]}$, the set of n-cubes of τ. The next lemma shows that it is transitive on n-cubes with common targets.

Lemma 3.4.4. *Let $X, Y \in \tau$ such that $t_i(X) = t_i(Y)$, $\forall i$, then there exists a unique element $u_{XY} \in \tau_\lrcorner$ such that*

$$X = u_{XY} \cdot Y$$

Proof. We prove this as we did in 2.2.7. Indeed since $t_i(X \circ_1 Y^{-1}) = \imath(t_i s_1(X))$ for all $i \neq 1$, $\big(((X \circ_1 Y^{-1}) \circ_2 \imath(s_1(Y)^{-}2)) \circ_3 \imath(s_1 2(Y)^{-}3) \cdots\big) \in \tau_\lrcorner$. Using inverses to solve for X proves the lemma. $\square$

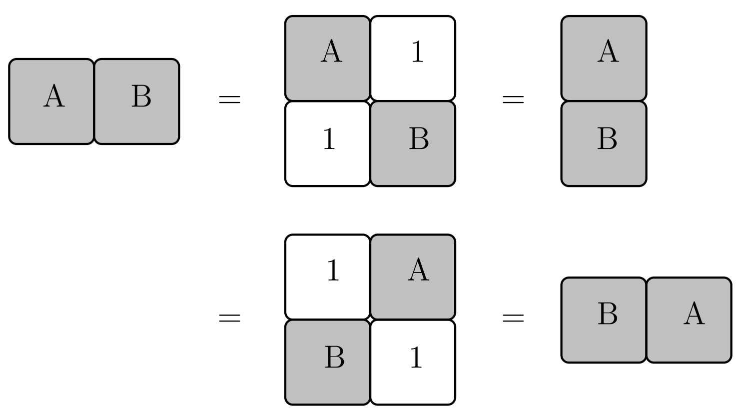
Definition 3.4.6. *Let $\tau_\bullet$ be the sub groupoid of $\tau_\lrcorner$ composed of n-cubes whose boundaries are all identities. It is called the **core bundle**.*

Lemma 3.4.5. *The core bundle is an abelian group bundle over τ_0, the objects of τ.*

Proof. This is a generalized Eckmann-Hilton argument. N-cubes whose boundaries are identities "slide" along each other. First note that for $u, v \in \tau_\bullet$:

$$u \cdot v = u \circ_i v \qquad \forall i \in [n]$$

To see this, pick $i \in [n]$, $u, v \in \tau_\bullet$ and compose all n-cubes intersecting the i^{th} source of the big n-cube in the barycentric decomposition defining $u \cdot v$. This will give you u since all other subcubes are identities on the source of u. The rest of the subcubes compose to v and therefore, by the interchange law, $u \cdot v = u \circ_i v$. Now pick two directions i, j and apply the following two dimensional argument in the plane ij :



which shows that $u \cdot v = u \circ_i v = v \circ_i u = v \cdot u$. $\square$

Corrollary 3.4.1. *Let $X, Y \in \tau$, then X and Y share all boundaries iff $u_{XY} \in \tau_\bullet$.*

Definition 3.4.7. *A n-tuple groupoid is **slim** if its core bundle is trivial*

Corrollary 3.4.2. *A n-tuple groupoid is* ***slim*** *iff there is at most one n-cube per boundary condition.*

Definition 3.4.8. *A n-tuple groupoid is* ***exclusive*** *if $\tau_{\lrcorner} = \tau_{\bullet}$.*

A n-cube then belongs to the core groupoid if and only if all its faces are identities.

Corrollary 3.4.3. *A n-tuple groupoid is exclusive if and only if the boundary of its n-cubes are determined by one of their boundaries of each type.*

Proof. Suppose that $X, Y \in \tau$ share a boundary of each type and define

$$I := \{i \in [n] | t_i(X) \neq t_i(Y)\}$$

Then $s_i(X) = s_i(Y) \, \forall i \in I$. But in this case $t_j(X^{-I}) = t_j(Y^{-I}) \, \forall j \in [n]$. But since τ is exclusive these inverses have the same boundary, proving that X and Y have the same boundaries as well. $\square$

Lemma 3.4.6. *Let τ be a n-tuple groupoid. If all boundary i-tuple groupoids are slim and all boundary double groupoids are exclusive then τ is exclusive.*
Moreover in this case the following is true :

$$t_{\hat{i}}(X) = t_{\hat{i}}(Y) \quad \forall i \in [n] \iff X = u \cdot Y \, for \, u \in \tau_{\bullet}$$

Proof. Suppose that τ_{ij} is exclusive for all $i, j \in \mathbb{Z}_n$, then for $X \in \tau_{\lrcorner}$:

$$\begin{aligned} & t_{\hat{i}}(X) = \imath_i(s_{[n]}(X)) \, \forall i \in [n] \\ \Rightarrow \quad & s_j t_{\hat{i}\hat{j}}(X) = \imath_i(s_{[n]}(X)) \, \forall i \neq j \in [n] \\ \Rightarrow \quad & s_{jk} t_{\hat{i}\hat{j}\hat{k}}(X) = \imath_i(s_{[n]}(X)) \, \forall i \neq j \neq k \in [n] \\ & \Rightarrow \cdots \\ \Rightarrow \quad & s_{\hat{i}}(X) = \imath_i(s_{[n]}(X)) \, \forall i \in [n] \end{aligned}$$

Which shows that all 1-arrows of X are identities. Now since the boundary double groupoids of τ are slim, it shows that all sub 2-cubes of X are identities. Since all boundary triple groupoids of τ are slim, all sub 3-cubes of X are identities. Repeat the argument to dimension n-1 to prove that τ is exclusive.
In this case, the above argument shows that all boundary i-tuple groupoids are exclusive and slim. Therefore the boundary of an n-cube $X \in \tau$ is fixed by $\{t_{\hat{i}}(X)\}$, so if X and Y share these arrows, they share their whole boundary and by a previous lemma differ by an element of $\tau_{\lrcorner}$, which in this situation is equal to $\tau_{\bullet}$ $\square$

Definition 3.4.9. *An n-tuple groupoid τ is* ***maximal*** *if for any $(f_1, f_2, \cdots, f_n)$ s.t. $f_i \in \tau_i$ and $t_i(f_i) = t_j(f_j) \quad \forall i, j \in \mathbb{Z}_n$ there exists $X \in \tau$ s.t. $t_{\hat{i}}(X) = f_i$*

Definition 3.4.10. *An n-tuple groupoid is* ***maximally exclusive*** *if*

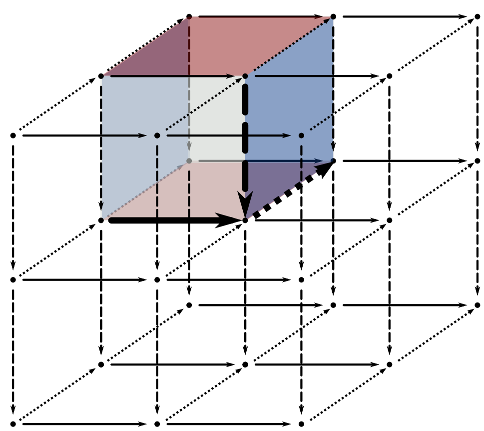
- *all boundary i-tuple groupoids are slim for $i > 1$*
- *all boundary double groupoids are exclusive*
- *it is maximal*

A n-tuple groupoid is ***vacant*** *if it is slim and maximally exclusive.*

Let X be an n-cube in a vacant groupoid then $\{t_{\hat{i}}\}_{[n]}$, the set of all boundary 1-arrows targeted at the sink, determines X uniquely and such an X exists for any possible such combination.

Lemma 3.4.7. *Let τ be a vacant n-tuple groupoid, $X, Y \in \tau$, then there exist a unique n-cube $X \cdot Y$ in τ that has a barycentric subdivision with X of depth 0 and Y of depth n. Moreover $(\tau, \cdot)$ is a groupoid.*

Proof. As mentioned above, all positions of any depth different from 0 or n have an internal edge of each index shared with either X or Y. For example, in the case $n = 3$ a 3-cube of depth 1 shares internal edges of two index with the depth 0 3-cube and of the third index with the depth n 3-cube, as in the picture :



Since the n-tuple groupoid is vacant, these positions have a unique filler which proves that the composition is well defined. Its associativity is guaranteed by uniqueness of fillers and the interchange laws. Identities are identity n-cubes on source or sink.
Now to show that inverses exist, consider an n-cube X and place it in position 0. As depth 1 cubes' intersections with X are (n-1) cubes, they only need one arrow in the direction not contained in the shared boundary to be determined. But for X to have an inverse, the n-cube Q of depth 1 whose yet undetermined arrows in direction i needs to satisfy $s_{\hat{i}}(Q) = s_{\hat{i}}(X)^{-1}$. That determines uniquely all depth 1 n-cubes. From a previous lemma depth i cubes share at least two boundary $(n-1)$-cubes with depth $(i-1)$ sub n-cubes, for $i > 1$. This fact determines inductively all subcubes of depth greater than 1. Their composition is a n-cube with a boundary arrow of each type being an identity and is therefore $\imath_{[n]}(s_{[n]}(X))$, showing that X has a right inverse.
But since $(X \cdot Y)^{-[n]} = X^{-[n]} \cdot Y^{-[n]}$, a right inverse to $X^{-[n]}$ is a left inverse to X, proving that X is invertible. $\square$

3.4.3 Equivalence with factorizations of subgroupoids

Let **nGpd** be the category of n-tuple groupoids and n-tuple functors and let **nSub** be the category defined by :

- Objects are $(n+1)$ tuples $(G, H_1, H_2, \cdots, H_n)$ where G is a groupoid, $\{H_i\}_{i\in[n]}$ are subgroupoids.
- Arrows are functors $f : G \to G'$ such that $f(H_i) \subset H'_i \quad \forall i \in [n]$

In this subsection we will build an adjunction between the two categories and find subcategories on which the adjunction restricts to an equivalence of categories. First we build a functor Γ : **nSub** $\to$ **nGpd**
For a $(n+1)$ tuples $(G, H_1, H_2, \cdots, H_n)$, let $\Gamma(G, H_1, H_2, \cdots, H_n)$ be the n-tuple groupoid defined by:

- Objects of $\Gamma(G, H_1, H_2, \cdots, H_n)$ are objects of G.
- $\Gamma(G, H_1, H_2, \cdots, H_n)_i = H_i$.
- n-cubes are commutative cubes, to be defined below.

Let K be the set of all possible cubes that would comply with the first two conditions. Each path from the source object to the sink object of an n-cube gives a sequence of composable arrows in G, with each arrow in a different subgroupoid. Each path then defines a map of sets $K \to G$ by composing the sequences. The n-cubes of K that have a constant value under all paths are the cubes of $\Gamma(G, H_1, H_2, \cdots, H_n)$.

Lemma 3.4.8. $\Gamma(G, H_1, H_2, \cdots, H_n)$ *is a n-tuple groupoid.*

Proof. The idea behind this proof is that pasting commutative diagrams along common boundaries produces other commutative diagrams in an associative way. $\square$

Suppose that $F : (G, H_1, H_2, \cdots, H_n) \to (G', H'_1, H'_2, \cdots, H'_n)$ is a subgroupoid preserving functor, then we can define $\Gamma(F)$ as the n-tuple functor such that $\Gamma(F)_i = F|_{H_i}$.

Theorem 3.4.1. *Let $I \subset [n]$ have at least two elements, then*

- $\Gamma(G, H_1, \cdots, H_n)_I$ *is slim*
- $\Gamma(G, H_1, \cdots, H_n)_I$ *is exclusive if and only if* $\bigcap_{i\in I} H_i$ *is discrete*
- *If* $\Gamma(G, H_1, \cdots, H_n)$ *is maximal then*

$$H_{\sigma(1)}H_{\sigma(2)}\cdots H_{\sigma(n)} = H_1H_2\cdots H_n \quad \forall\sigma \in S_n$$

Proof. By definition a n-cube exists in $\Gamma(G, H_1, \cdots, H_n)$ if and only if it is a commutative diagram in G. It is therefore uniquely defined by its boundary arrows. With the help of identities this proves the first part.
For the second, note that by definition:

$$X \in \big(\Gamma(G, H_1, \cdots, H_n)_I\big)_\lrcorner \iff s_{\hat{i}}(X) = s_{\hat{j}}(X)\, \forall i, j \in I$$

which if $\big(\Gamma(G, H_1, \cdots, H_n)_I\big)_\lrcorner$ is exclusive are all forced to be identities and vice-versa, proving the second part.
For the third part, note that by use of inverses, maximality shows that for any composable n-tuples $h_1 h_2 \cdots h_n$ of $H_1 H_2 \cdots H_n$, there is an n-cube X with a path source-sink whose i^{th} arrow is h_i. But as cube of $\Gamma(G, H_1, \cdots, H_n)$ correspond to elements of $\bigcap_{\sigma \in S_n} H_{\sigma(1)} H_{\sigma(2)} \cdots H_{\sigma(n)}$ the third part is proven. □

Definition 3.4.11. *Let* ***<u>nMatchSub</u>*** *be the full subcategory of* ***<u>nSub</u>*** *whose objects are n-tuples* $(G, H_1, H_2, \cdots, H_n)$ *such that* $H_i \cap H_j$ *is discrete for all* $i \neq j$*, and* ***<u>nMatch</u>*** *the full subcategory of* ***<u>nMatchsub</u>*** *whose objects satisfy* $G = H_1 H_2 \cdots H_n$ *and* $H_i H_j = H_j H_i$ *for all* $i, j \in [n]$.

Theorem 3.4.2. Γ : ***<u>nMatchSub</u>*** $\to$ ***<u>nVacant</u>*** *has a left adjoint* Λ *given by* $\Lambda(\tau) = ((\tau, \cdot), \tau_1, \tau_2, \cdots, \tau_n)$*. Moreover* $(\Gamma|_{\underline{\textbf{\textit{nMatch}}}}, \Lambda)$ *is an equivalence of categories.*

Proof. Let $(G, H_1, \cdots, H_n) \in$ **<u>nMatchSub</u>** and $\tau \in$ **<u>nVacant</u>**. Then since every element of $(\tau, \cdot)$ has a decomposition as composition of elements of τ_i,

$$\underline{\textbf{nMatchSub}}\big(\Lambda(\tau),\, (G, H_1, \cdots, H_2)\big) \simeq \{(f_1, \cdots, f_n) | f_i : \tau_i \to H_i \text{ and } f_i = f_j \text{ on objects}\}$$

moreover τ has at most one n-cube per acceptable 1-boundary, so an n-tuple functor is fixed by its values on arrows, i.e.

$$\underline{\textbf{nVacant}}\big(\tau,\, \Gamma(G, H_1, \cdots, H_n)\big) \simeq \{(f_1, \cdots, f_n) | f_i : \tau_i \to H_i \text{ and } f_i = f_j \text{ on objects}\}$$

proving the adjunction.
Now if $(G, H_1, H_2, \cdots, H_n) \in$ **<u>nMatch</u>**, every arrow of G can be written as a composition of arrows of $\{H_i\}$, hence :

$$\underline{\textbf{nMatchSub}}\big((G, H_1, \cdots, H_2),\, \Lambda(\tau)\big) \simeq \{(f_1, \cdots, f_n) | f_i : \tau_i \to H_i \text{ and } f_i = f_j \text{ on objects}\}$$

$$\underline{\textbf{nVacant}}\big(\Gamma(G, H_1, \cdots, H_n),\, \tau\big) \simeq \{(f_1, \cdots, f_n) | f_i : \tau_i \to H_i \text{ and } f_i = f_j \text{ on objects}\}$$

which proves the equivalence of categories between **<u>nMatch</u>** and **<u>nVacant</u>**. □

This theorem allows us to see some decompositions of groups as higher dimensional groups, where dimension is taken in a very categorical sense.

Corrollary 3.4.4. *Vacant n-tuple groups are in functorial correspondence with matched n-tuples of subgroups.*

3.4.4 Maximally exclusive n-tuple groupoids

Definition 3.4.12. *Let $(\tau_{\hat{1}}, \cdots, \tau_{\hat{n}})$ be boundary $(n-1)$-tuple groupoids of some n-tuple groupoid τ. Then the **coarse n-tuple groupoid** $\square(\tau_{\hat{1}}, \cdots, \tau_{\hat{n}})$ is the slim n-tuple groupoid such that $\square(\tau_{\hat{1}}, \cdots, \tau_{\hat{n}})_i = \tau_{\hat{i}}$ and an n-cube exists iff its boundary is admissible, i.e. if its boundary $(n-1)$ cubes agree on their common boundaries. The **frame** $\blacksquare\tau$ of τ is then the image of the functor :*

$$\Pi : \tau \to \square(\tau_{\hat{1}}, \cdots, \tau_{\hat{i}}, \cdots, \tau_{\hat{n}})$$

such that $\Pi s_i = s_i \Pi$ and $\Pi t_i = t_i \Pi$ for all $i \in [n]$.

In this light, a maximally exclusive n-tuple groupoid is one whose frame is vacant. When trying to define a diagonal composition for n-cubes, we previously used the uniqueness of fillers for n-cubes of depth 1 to (n-1) in the barycentric division of the n-cube that vacancy provides. In the present case this uniqueness disappears, though the boundaries of such cubes are fixed. We therefore need to make a consistent choice of fillers to define a diagonal groupoid out of a maximally exclusive n-tuple groupoid.
Let $! : \blacksquare\tau \to \tau$ be a section of Π as n-tuple graphs and $X, Y \in \tau$, then one can use the section to fill the barycentric subdivision of the n-cube with X of depth 0 and Y of depth n. Denote the composite of the subdivision by $X \cdot_! Y$, and the graph defined by $(\tau_{[n]}, s_{[n]}, t_{[n]})$ with the above product by $(\tau, \cdot_!)$

Lemma 3.4.9. *The following is true :*

$$! \in \underline{\textbf{\textit{n-plGpd}}}(\square\tau, \tau) \Rightarrow (\tau, \cdot_!) \in \underline{\textbf{\textit{Gpd}}}$$

Proof. Let $X, Y, Z \in \tau$ such that $t_{[n]}(Z) = s_{[n]}(Y)$ and $t_{[n]}(Y) = s_{[n]}(Z)$. Then $(X \cdot Y) \cdot Z$ and $X \cdot (Y \cdot Z)$ are both equal to the cube obtained by composing the "subdivision in thirds" of the n-cube where X, Y, Z are placed on the diagonal source-sink and all other n-subcubes are filled with elements of the section. Since the frame is vacant, such a filling exists and the composition is associative. Identities on objects are the identities of the groupoid and inverses are given by the following argument:
Let X be placed in position of depth 0. From a previous theorem, there exists a filling of the barycentric subdivision with section n-cubes such that all boundaries of the composition are identities on $s_{[n]}(X)$. In other words, $\exists Y \in \blacksquare\tau$ such that $X \cdot !(Y) = u_X \in \tau_\bullet$. Then:

$$\begin{aligned} X \cdot !(Y) \cdot (u_X)^{-i} &= u_X \cdot (u_X)^{-i} \\ &= u_X \circ_i (u_X)^{-i} \\ &= \imath(s_{[n]}(X)) \end{aligned}$$

Which shows that X has a right inverse. The same procedure shows that it has a left inverse and therefore that $(\tau, \cdot_!)$ is a groupoid. □

As promised in the introduction, we can extend this result to further decompositions of groupoids. Let **nSemiSub** be the category defined by :

- Objects are $(n+2)$ tuples $(G, A, H_1, H_2, \cdots, H_n)$ where $(G, H_1, \cdots, H_n) \in$ **nMatchSub**, $A \in G$ is an abelian group bundle on the objects of G and $ha(h)^{-1} \in A$ for all $h \in H_i$ and all $i \in [n]$.

- Arrows are functors $f : G \to G'$ such that $f(H_i) \subset H_i'$ for all $i \in [n]$ and $f(A) \in A'$.

Let **nSemi** be the full subcategory of **nSemiSub** where objects $(G, A, H_1, \cdots, H_n)$ satisfy $G = AH_1H_2 \cdots H_n$.
Let **nMaxExcl** be the category whose objects are pairs $(\tau, !)$ and arrows are section preserving n-tuple functors. Then we can build a functor

$$\tilde{\Gamma} : \underline{\mathbf{nSemi}} \to \underline{\mathbf{nMaxExcl}}$$

by building $\tilde{\Gamma}(G, A, H_1, \cdots, H_n)$, the n-tuple groupoid whose n-cubes are pairs (X, a) with $X \in \Gamma(G, H_1, \cdots, H_n)$ and $a \in A$ and whose compositions are given by :

$$(X, a) \circ_i (Y, b) = \big(X \circ_i Y,\ as_{\hat{i}}(X)b(s_{\hat{i}}(X)^{-1})\big)$$

A direct computation shows that these compositions define an n-tuple groupoid. together with the section $! : \Gamma(G, H_1 \cdots, H_n) \to \tilde{\Gamma}(G, H_1 \cdots, H_n)$ given by $!(X) = \big(X, \imath(s_{[n]}(X))\big)$. On arrows of **nSemi**, $\tilde{\Gamma}$ is given by:

$$\tilde{\Gamma}(F)(X, a) = \big(\Gamma(F)(X), F(a)\big)$$

Once again a direct computation shows that this defines a functor. We are now ready to state the theorem.

Theorem 3.4.3. *The functor $\tilde{\Gamma}$: **nSemiSub** $\to$ **nMaxExcl** has a left adjoint $\tilde{\Lambda}$ defined by:*

$$\tilde{\Lambda}(\tau, !) = \big((\tau, \cdot_!), \tau_\bullet, \tau_1, \cdots, \tau_n\big)$$
$$\tilde{\Lambda}(F) = F$$

Moreover $(\tilde{\Gamma}|_{\underline{\mathbf{nSemi}}}, \tilde{\Lambda})$ is an equivalence of categories

Proof. □

Let τ be maximally exclusive, then :

$$\underline{\mathbf{nMaxExcl}}(\omega, \tau) \simeq \{(F_0, F_1, \cdots, F_n) | F_0 : \omega_\bullet \to \tau_\bullet \text{ and } F_i : \omega_i \to \tau_i\}$$

Moreover if $(G, A, H_1, \cdots, H_n) \in$ **nSemi**, then:

$$\underline{\mathbf{nSemi}}\big((G, H_1, \cdots, H_n),\ (K, B, L_1, \cdots, L_n)\big) \simeq$$
$$\{(F_0, F_1, \cdots, F_n) | F_0 : A \to B \text{ and } F_i : H_i \to L_i\}$$

Considering that the image of $\tilde{\Lambda}$ is by definition in **nSemi**, it is enough to prove the two statements.

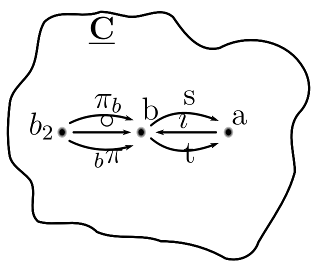
3.5 Weakening

This section is dedicated to building the weak versions of our previous n-tuple categories. The starting point of the weakening is <u>**Cat**</u> since the principal ingredient to the weakening is a non trivial 2-categorical structure. With it one can use the 2-cells as fillers for the previously commutative squares.

3.5.1 weak internal categories

Definition 3.5.1. *A **weak internal category** in a 2-category with pullbacks is constituted of :*

- *A set of arrows :*



- *A set of 2-cells :*

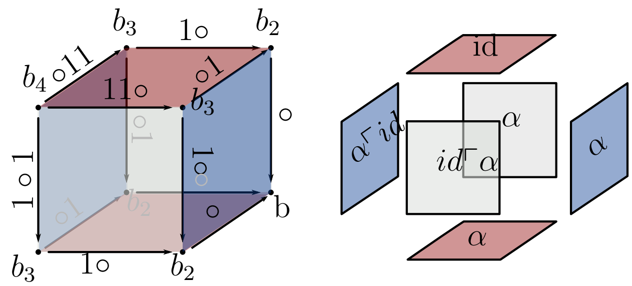
$$
\begin{tikzcd}
b \arrow[r, "id_b \ulcorner t\imath"] \arrow[d, "id_b"'] \arrow[dr, phantom, "\rho \Swarrow"] & b_2 \arrow[d, "\circ"] \\
b \arrow[r, "id_b"'] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
b \arrow[r, "s\imath \ulcorner id_b"] \arrow[d, "id_b"'] \arrow[dr, phantom, "\lambda \Swarrow"] & b \arrow[d, "\circ"] \\
b \arrow[r, "id_b"'] & b
\end{tikzcd}
\qquad
\begin{tikzcd}
b_3 \arrow[r, "\circ \ulcorner \pi_b"] \arrow[d, "{}_b\pi \ulcorner \circ"'] \arrow[dr, phantom, "\alpha \Swarrow"] & b_2 \arrow[d, "\circ"] \\
b_2 \arrow[r, "\circ"'] & b
\end{tikzcd}
$$

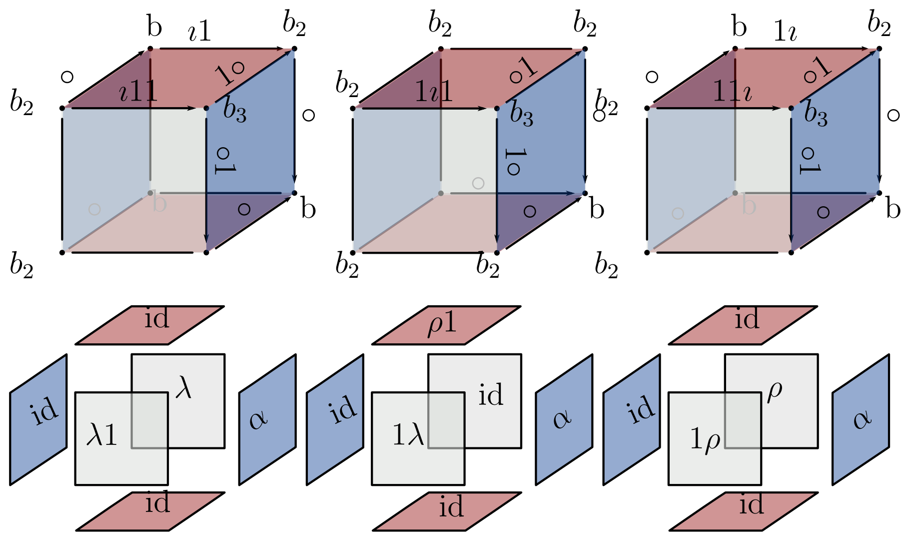
- *Commutative diagrams for arrows :*

$$
\begin{tikzcd}
b_2 \arrow[r, "\pi_b"] \arrow[d, "{}_b\pi"'] \arrow[dr, phantom, very near start, "\lrcorner"] & b \arrow[d, "s"] \\
b \arrow[r, "t"'] & a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "\imath"] \arrow[d, "id_a"'] & b \arrow[d, "s"] \\
a \arrow[r, "id_a"'] & a
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "\imath"] \arrow[d, "id_a"'] & b \arrow[d, "t"] \\
a \arrow[r, "id_a"'] & a
\end{tikzcd}
$$

$$
\begin{tikzcd}
b_2 \arrow[r, "\pi_b"] \arrow[d, "\circ"'] & b \arrow[d, "t"] \\
b \arrow[r, "t"'] & a
\end{tikzcd}
\qquad
\begin{tikzcd}
b_2 \arrow[r, "\circ"] \arrow[d, "{}_b\pi"'] & b \arrow[d, "s"] \\
b \arrow[r, "s"'] & a
\end{tikzcd}
$$

- *Commutative diagrams for cells :*





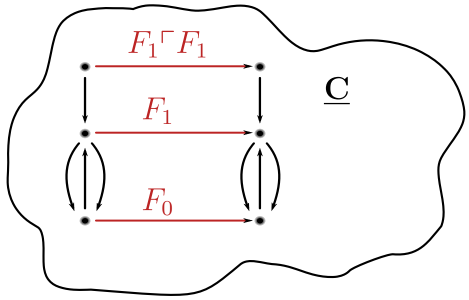
- $\alpha t = \pi_b 1_t$ *and* $\alpha s = {}_b\pi 1_s$

The previous cubes for associativity and units are usually called the pentagon condition and the triangle conditions. Remark that on top of the cube for associativity there is a trivial 2-cell : $(\circ^\ulcorner id^\ulcorner id)(id^\ulcorner\circ) \to (id^\ulcorner id^\ulcorner\circ)(\circ^\ulcorner id)$. The choice of a non trivial cell for it results in even weaker forms whose degenerate case can be found in the literature under the name of pre-monoidal categories [57, 37]. This point will not be further discussed in this paper though. Note that all the diagrams concerning the underlying graph structure, i.e. involving source and target, have not been weakened. The definition of weak internal functor will give hints that such a structure exists but as we fail to see the relevance we will not address it.

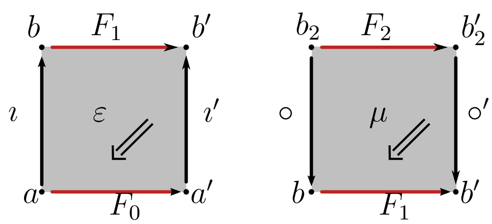
3.5.2 weak internal functor

Definition 3.5.2. *A* ***weak internal functor*** *between two weak internal categories is constituted of :*

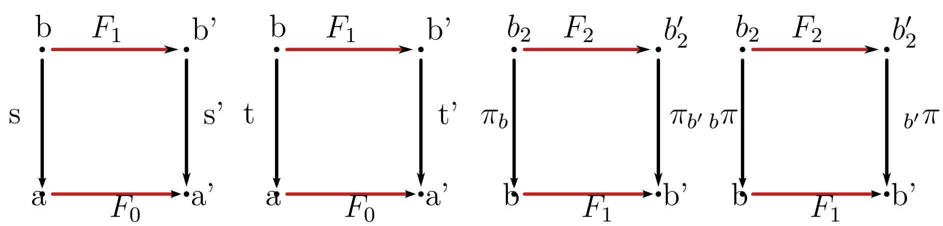
- *A pair of arrows :*



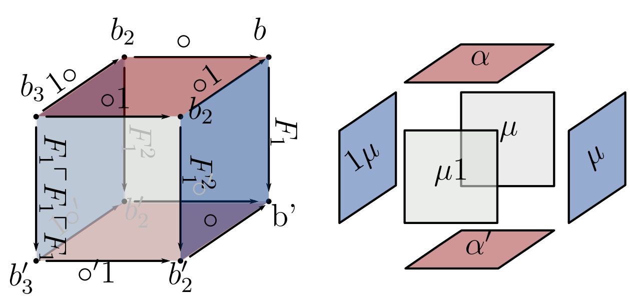
- *A pair of 2-cells :*

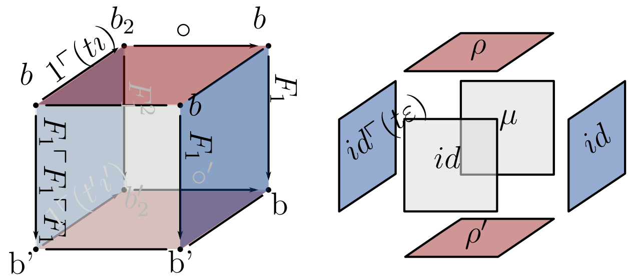


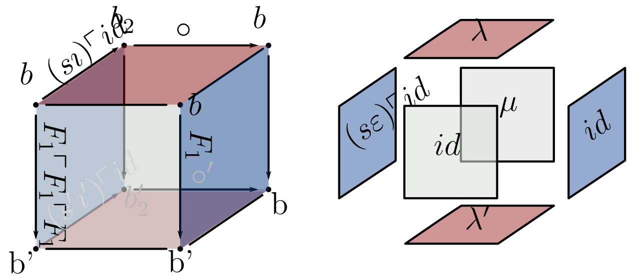
- *A set of commutative diagrams for arrows:*

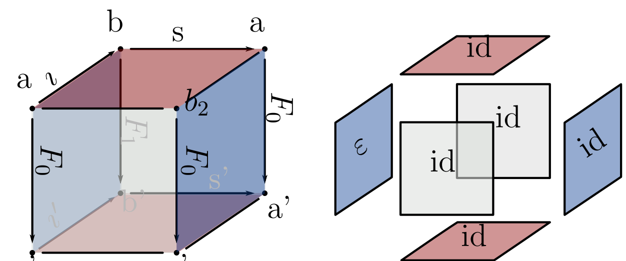


- *A set of commutative diagrams for cells :*

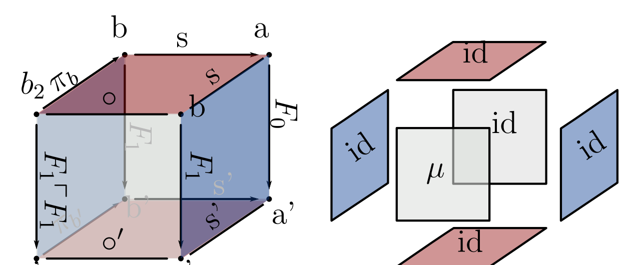








+ same with targets instead of sources



+ same with targets instead of sources

The first three cubes are showing the intertwining of the 2-cells of the weak internal category structure whereas the last four are necessary to the definition of the pullback used in the intertwining cubes. Showing that functors compose associatively is now straightforward as it gets down to composing commutative cubes and the trivial functor consists in identity arrows, making the cubes trivially commutative.

3.6 The category of weak double categories

Since **Set** can be seen as a 2-category and therefore one can consider weak internal categories in it. But since all its 2-cells are identities these will all be strict categories. Therefore our first example of weak internal categories is to be found with **Cat** as a base category. The obtained structures are called weak double categories. Remark that the previous symmetry interchanging horizontal and vertical arrows ,the flip, is not present any more. The fact that we are starting with **Set**, a trivial 2-category, prevents the existence of a double category weak in both direction. Let's now unpack the definitions.

Weak double categories

According to 3.5.1 a weak double category has a horizontal composition that is associative with unit only up to some natural vertical squares that we will call associators and unitors, and a vertical composition that is strictly associative and unital. Which means the following :

- Every triple of composable horizontal arrows (f, g, h) have a corresponding associator square $\alpha_{f,g,h}$ such that for composable squares A, B, C:

$$\begin{tikzcd}
\bullet \arrow[rr, "f \circ_h g"] \arrow[d, "l"'] \arrow[drr, phantom, "A \circ_h B"] & & \bullet \arrow[r, "h"] \arrow[d, "m"] \arrow[dr, phantom, "C"] & \bullet \arrow[d, "n"] \\
\bullet \arrow[rr, "i \circ_h j"'] \arrow[d, no head] \arrow[drrr, phantom, "\alpha_{i,j,k}"] & & \bullet \arrow[r, "k"'] & \bullet \arrow[d, no head] \\
\bullet \arrow[r, "i"'] & \bullet \arrow[rr, "j \circ_h k"'] & & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[rr, "f \circ_h g"] \arrow[d, no head] \arrow[drrr, phantom, "\alpha_{f,g,h}"] & & \bullet \arrow[r, "h"] & \bullet \arrow[d, no head] \\
\bullet \arrow[r, "f"] \arrow[d, "l"'] \arrow[dr, phantom, "A"] & \bullet \arrow[rr, "g \circ_h h"] \arrow[d] \arrow[drr, phantom, "B \circ_h C"] & & \bullet \arrow[d, "n"] \\
\bullet \arrow[r, "i"'] & \bullet \arrow[rr, "j \circ_h k"'] & & \bullet
\end{tikzcd}$$

- Every horizontal arrow f has two corresponding unitor squares ρ_f and λ_f such that, for any square A:

$$\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d, "l"'] \arrow[dr, phantom, "A"] & \bullet \arrow[r, no head] \arrow[d, "k"] \arrow[dr, phantom, "\imath_h(k)"] & \bullet \arrow[d, "k"] \\
\bullet \arrow[r, "i"'] \arrow[d, no head] \arrow[drr, phantom, "\rho_i"] & \bullet \arrow[r, no head] & \bullet \arrow[d, no head] \\
\bullet \arrow[rr, "i"'] & & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, "f"] \arrow[d, no head] \arrow[drr, phantom, "\rho_f"] & \bullet \arrow[r, no head] & \bullet \arrow[d, no head] \\
\bullet \arrow[rr, "f"] \arrow[d, "l"'] \arrow[drr, phantom, "A"] & & \bullet \arrow[d, "k"] \\
\bullet \arrow[rr, "i"'] & & \bullet
\end{tikzcd}$$

$$
\begin{tikzcd}
\bullet \arrow[r, dash] \arrow[d, "h"'] \arrow[dr, phantom, "\imath_2(h)"] & \bullet \arrow[r, "f"] \arrow[d, "h"'] \arrow[dr, phantom, "A"] & \bullet \arrow[d] \\
\bullet \arrow[r, dash] \arrow[d, dash] \arrow[drr, phantom, "\lambda_g"] & \bullet \arrow[r, "g"'] & \bullet \arrow[d, dash] \\
\bullet \arrow[rr, "f"'] & & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, dash] \arrow[d, dash] \arrow[drr, phantom, "\lambda_f"] & \bullet \arrow[r, "f"] & \bullet \arrow[d, dash] \\
\bullet \arrow[d, "h"'] \arrow[drr, phantom, "A"] & & \bullet \arrow[ll, "f"'] \arrow[d] \\
\bullet & & \bullet \arrow[ll, "g"]
\end{tikzcd}
$$

The "middle four" or interchange law still holds. The degenerate versions of the weak double category give the usual notions of bicategory and (weak) monoidal categories and the choice present in representing a 2-category as a vertical or horizontal double category disappears.

3.6.1 weak double functors

A weak double functor is going to send squares to squares in such a way that the boundaries and the vertical composition are respected, but will respect horizontal composition and unit up to vertical squares. It means the following:

- for every pair of composable horizontal arrows (f, g), there is an intertwining square:

$$
\begin{tikzcd}
\bullet \arrow[r, "F(f)"] \arrow[d, "F(h)"'] \arrow[dr, phantom, "F(A)"] & \bullet \arrow[r, "F(m)"] \arrow[d, "F(k)"'] \arrow[dr, phantom, "F(B)"] & \bullet \arrow[d, "F(n)"] \\
\bullet \arrow[r, "F(g)"'] \arrow[d, dash] \arrow[drr, phantom, "\mu_{g,p}"] & \bullet \arrow[r, "F(p)"'] & \bullet \arrow[d, dash] \\
\bullet \arrow[rr, "F(g \circ_2 p)"'] & & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, "F(f)"] \arrow[d, dash] \arrow[drr, phantom, "\mu_{f,m}"] & \bullet \arrow[r, "F(m)"] & \bullet \arrow[d, dash] \\
\bullet \arrow[rr, "F(f \circ_2 m)"] \arrow[d, "F(h)"'] \arrow[drr, phantom, "F(A \circ_2 B)"] & & \bullet \arrow[d, "F(k)"] \\
\bullet \arrow[rr, "g \circ_2 p"'] & & \bullet
\end{tikzcd}
$$

- for every horizontal identity, there is an intertwining square :

$$
\begin{tikzcd}
F(a) \arrow[r, no head] \arrow[d, "F(f)"'] \arrow[dr, phantom, "\imath'_2(F(f))"] & F(a) \arrow[d, "F(f)"] \\
F(b) \arrow[r, no head] \arrow[d, no head] \arrow[dr, phantom, "\varepsilon_b"] & F(b) \arrow[d, no head] \\
F(b) \arrow[r, "F(\imath_2(b))"'] & F(b)
\end{tikzcd}
=
\begin{tikzcd}
F(a) \arrow[r, no head] \arrow[d, no head] \arrow[dr, phantom, "\varepsilon_a"] & F(a) \arrow[d, no head] \\
F(a) \arrow[r, "F(\imath_2(a))"] \arrow[d, "F(f)"'] \arrow[dr, phantom, "F(\imath_2(f))"] & F(a) \arrow[d, "F(f)"] \\
F(b) \arrow[r, "F(\imath_2(b))"'] & F(b)
\end{tikzcd}
$$

- the following holds for any triple of composable horizontal arrows (f, g, h)

$$
\begin{tikzcd}
\bullet \arrow[rr, "F(f) \circ'_2 F(g)"] \arrow[d, no head] \arrow[drrr, phantom, "\alpha_{F(f),F(g),F(h)}"] & & \bullet \arrow[r, "F(h)"] & \bullet \arrow[d, no head] \\
\bullet \arrow[r, "F(f)"] \arrow[d, no head] \arrow[dr, phantom, "\imath'_1(F(f))"] & \bullet \arrow[rr, "F(g) \circ'_2 F(h)"] \arrow[d, no head] \arrow[drr, phantom, "\mu_{g,h}"] & & \bullet \arrow[d, no head] \\
\bullet \arrow[r, no head, "F(f)"] \arrow[d, no head] \arrow[drrr, phantom, "\mu_{f,g \circ_2 h}"] & \bullet \arrow[rr, "F(g \circ_2 h)"] & & \bullet \arrow[d, no head] \\
\bullet \arrow[rrr, "F(f \circ_2 (g \circ_2 h))"'] & & & \bullet
\end{tikzcd}
=
\begin{tikzcd}
\bullet \arrow[r, "F(f) \circ'_2 F(g)"] \arrow[d, no head] \arrow[dr, phantom, "\mu_{F(f),F(g)}"] & \bullet \arrow[r, "F(h)"] \arrow[d, no head] \arrow[dr, phantom, "\imath'_1(F(h))"] & \bullet \arrow[d, no head] \\
\bullet \arrow[r, "F(f \circ_2 g)"] \arrow[d, no head] \arrow[drr, phantom, "\mu_{f \circ_2 g,h}"] & \bullet \arrow[r, "F(h)"] & \bullet \arrow[d, no head] \\
\bullet \arrow[rr, "F((f \circ_2 g) \circ_2 h)"] \arrow[d, no head] \arrow[drr, phantom, "F(\alpha_{f,g,h})"] & & \bullet \arrow[d, no head] \\
\bullet \arrow[rr, "F(f \circ_2 (g \circ_2 h))"'] & & \bullet
\end{tikzcd}
$$

- the following holds for any horizontal arrows f :

$$
\begin{tikzcd}
F(a) \arrow[r, "F(f)"] \arrow[d, no head] \arrow[dr, phantom, "\imath'_1(F(F))"] & F(b) \arrow[r, no head] \arrow[d, no head] \arrow[dr, phantom, "\varepsilon_b"] & F(b) \arrow[d, no head] \\
F(a) \arrow[r, "F(f)"] \arrow[d, no head] \arrow[drr, phantom, "\mu_{f,\imath_2(b)}"] & F(b) \arrow[r, "F(\imath_2(b))"] & F(b) \arrow[d, no head] \\
F(a) \arrow[rr, "F(f \circ_2 \imath_2(b))"] \arrow[d, no head] \arrow[drr, phantom, "F(\rho_f)"] & & F(b) \arrow[d, no head] \\
F(a) \arrow[rr, "F(f)"'] & & F(b)
\end{tikzcd}
=
\begin{tikzcd}
F(a) \arrow[r, "F(f) \circ_2 \imath'_2(F(b))"] \arrow[d, no head] \arrow[dr, phantom, "\rho'_{F(f)}"] & F(b) \arrow[d, no head] \\
F(a) \arrow[r, "F(f)"'] & F(b)
\end{tikzcd}
$$

+ same for λ

These functors restrict to the usual definition of functors for bicategories and monoidal categories when considering the degenerate cases.

3.6.2 weak double natural transformations

A weak double natural transformation will consist of squares intertwining two weak functors vertically. It means the following :

- for every horizontal arrow f there is an intertwining square :

$$
\begin{tikzcd}
\mathrm{F(a)} \arrow[r, "\mathrm{F(f)}"] \arrow[d, dash, "\mathrm{F(h)}"'] \arrow[dr, phantom, "\omega_f)"] & \mathrm{F(b)} \arrow[d, dash] \\
\mathrm{G(a)} \arrow[r, "\mathrm{G(f)}"] \arrow[d] \arrow[dr, phantom, "\mathrm{G(A)}"] & \mathrm{G(b)} \arrow[d, "\mathrm{g(k)}"] \\
\mathrm{G(c)} \arrow[r, "\mathrm{G(g)}"] & \mathrm{G(d)}
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{F(a)} \arrow[r, "\mathrm{F(f)}"] \arrow[d, "\mathrm{F(h)}"'] \arrow[dr, phantom, "\mathrm{F(A)}"] & \mathrm{F(b)} \arrow[d, "\mathrm{F(k)}"] \\
\mathrm{F(c)} \arrow[r, "\mathrm{F(g)}"] \arrow[d, dash] \arrow[dr, phantom, "\omega_g"] & \mathrm{F(d)} \arrow[d, dash] \\
\mathrm{G(c)} \arrow[r, "\mathrm{G(g)}"] & \mathrm{G(d)}
\end{tikzcd}
$$

- the following holds for any pairs of composable horizontal arrows :

$$
\begin{tikzcd}
\mathrm{F(a)} \arrow[r, "\mathrm{F(f)}"] \arrow[d, dash] \arrow[dr, phantom, "\omega_f"] & \mathrm{F(b)} \arrow[r, "\mathrm{F(g)}"] \arrow[d, dash] \arrow[dr, phantom, "\omega_g"] & \mathrm{F(b)} \arrow[d, dash] \\
\mathrm{G(a)} \arrow[r, "\mathrm{G(f)}"] \arrow[d, dash] \arrow[drr, phantom, "\mu_{G,(f,g)}"] & \mathrm{G(b)} \arrow[r, "\mathrm{G(g)}"] & \mathrm{G(b)} \arrow[d, dash] \\
\mathrm{G(a)} \arrow[rr, "G(f \circ_2 g)"] & & \mathrm{G(b)}
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{F(a)} \arrow[r, "F(f) \circ_2' (F(g))"] \arrow[d, dash] \arrow[dr, phantom, "mu_{F,(f,g)}"] & \mathrm{F(b)} \arrow[d, dash] \\
\mathrm{F(a)} \arrow[r, "F(f \circ_2 g)"'] \arrow[d, dash] \arrow[dr, phantom, "\omega_{f \circ_2 g}"] & \mathrm{F(b)} \arrow[d, dash] \\
\mathrm{G(a)} \arrow[r, "F(f)"'] & \mathrm{G(b)}
\end{tikzcd}
$$

Since they require vertical intertwining arrows, weak natural transformations are available for the degenerate cases of bicategories and monoidal categories only when the two intertwined functors are identical on the level of objects.

3.7 The span construction

Definition 3.7.1. *Let $\underline{\boldsymbol{C}}$ be a category. A **span** in $\underline{\boldsymbol{C}}$ is a pair of arrows in $\underline{\boldsymbol{C}}$ having the same source. A **cospan** is a pair of arrows sharing the same target.*

Spans and cospans may be pasted to each other to give new ones. For example, in the situation

$$
\begin{tikzcd}
& M \arrow[dl] \arrow[dr] & & N \arrow[dl] \arrow[dr] & \\
a & & b & & c
\end{tikzcd}
$$

supposing that we have a canonical way of "finishing the middle square",

$$\begin{tikzcd}
 & & K \arrow[dl] \arrow[dr] & & \\
 & M \arrow[dl] \arrow[dr] & & N \arrow[dl] \arrow[dr] & \\
a & & b & & c
\end{tikzcd}$$

we may get a new span

$$\begin{tikzcd}
 & K \arrow[dl] \arrow[dr] & \\
a & & c
\end{tikzcd}$$

by composing the outer arrows together. Which means that, given a choice of pull-back for any cospan in the category, spans may be composed. Similarly, for a choice of push forward for any span in the category, cospans may be composed. Let's denote the composition $M \circ N$. It is generally not associative, nor unital. Luckily though the universal properties of the pull-back, resp. push-forward, give unique isomorphisms that make the composition both associative and unital up to isomorphisms. Which , in the case of spans, means that there is a unique isomorphism α such that the following diagram commutes :

$$\begin{tikzcd}
 & (M \circ N) \circ P \arrow[dl] \arrow[dr] \arrow[dd, "\alpha"'] & \\
a & & d \\
 & M \circ (N \circ P) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

This all fits in a weak double category $\underline{\mathbf{Span(C)}}$ whose :

- objects are the objects in $\underline{\mathbf{C}}$.
- horizontal arrows are spans in $\underline{\mathbf{C}}$
- vertical arrows are the arrows of $\underline{\mathbf{C}}$
- squares are commutative diagrams of the form:

$$\begin{tikzcd}
a \arrow[d] & M \arrow[l] \arrow[r] \arrow[d] & b \arrow[d] \\
c & N \arrow[l] \arrow[r] & d
\end{tikzcd}$$

with an obvious identification of the boundaries.

- Composition is done by pullback.

Indeed, its universal properties ensure that when two squares are composable, i.e.

$$\begin{tikzcd}
a \arrow[d] & M \arrow[l] \arrow[r] \arrow[d] & b \arrow[d] & P \arrow[l] \arrow[r] \arrow[d] & e \arrow[d] \\
c & N \arrow[l] \arrow[r] & d & Q \arrow[l] \arrow[r] & f
\end{tikzcd}$$

there is a unique arrow from $M \circ P$ to $N \circ Q$ making the following diagram commutative:

$$\begin{tikzcd}
 & & M \circ P \arrow[dll] \arrow[drr] \arrow[d, dotted] & & \\
M \arrow[d] \arrow[drr] & & N \circ Q \arrow[dll] \arrow[drr] & & P \arrow[d] \arrow[dll] \\
N \arrow[drr] & & b \arrow[d] & & Q \arrow[dll] \\
 & & d & &
\end{tikzcd}$$

so that the composite is:

$$\begin{tikzcd}
a \arrow[d] & M \circ P \arrow[l] \arrow[r] \arrow[d, dotted] & e \arrow[d] \\
c & N \circ Q \arrow[l] \arrow[r] & f
\end{tikzcd}$$

The unique morphisms for associativity and unit are then squares that have identities as vertical arrows and satisfy the conditions given previously. when weak double categories were defined. This weak double category is fairly big but it contains very interesting sub double categories. First, remark that the subset of all squares that are vertically invertible contains all identities, contains all structural squares[2] and is closed under both compositions. It is therefore a sub double category $\underline{\mathbf{Span}[\mathbf{C}]}$ of $\underline{\mathbf{Span}(\mathbf{C})}$. Due to the invertibility condition,$\underline{\mathbf{Span}[\mathbf{C}]}$ may be represented as a weak internal double category in the 2-category of groupoids.
There is a way to obtain a category out of this construction, which is to compose the functor[3] $\underline{\mathbf{Span}[]}$ with the forgetful functor $\underline{\mathbf{wIntCat(Gpd)}} \to \underline{\mathbf{IntCat(Set)}}$ obtained from the functor $\underline{\mathbf{Gpd}} \to \underline{\mathbf{Set}}$ that sends a groupoid to its set of connected components and functors to maps of sets on the connected components.
The result is an internal category in $\underline{\mathbf{Set}}$, i.e. a category as claimed. Note that the construction is similar for cospans. To get a bigger category we can even start with a smaller sub double category of $\underline{\mathbf{Span}[\mathbf{C}]}$, the one composed of all squares that have identities as

[2]the associativity and unit squares
[3]which is actually a triple functor

vertical arrows. The connected components of the groupoid of objects in the weak internal category presentation is then discrete and therefore the category of "quotients" has many more objects.

Chapter 4

The Cobordism Categories

We are now getting to the topological part of the thesis. Its category theory side is mainly motivated by the fact that cobordisms are special cospans, as described in chapter 3.7. As we mentioned in Chapter 1.3, the way to find generators and relations for the categories of cobordisms is to use singularity theory. As Morse theory is the only tool classically considered we will review it and take lessons from its failure. We will then consider higher dimensional equivalents to Morse theory, which in the 2d case were called 2-Morse by Gay and Kirby [32]. These will allow a simpler description of the cobordisms as cobordisms with corners from which a usual TQFT may be extracted by considering cobordisms with empty boundaries.
For the whole chapter, M will denote a closed compact smooth manifold of dimension m, N a smooth manifold of dimension n. The material on Morse theory is widely spread but we recommend Milnor's book [49] and Guillemin and Golubitsky's book [33].

4.1 Morse theory and Cobordisms

The theory of Morse functions is the corner stone of 2D-TQFT since it allows a description of the category of 2 dimensional cobordisms up to equivalence. Let us recall how.

Definition 4.1.1. *A smooth map $f : M \to N$ between manifolds is said to be **critical** at $p \in M$ if $df|_p$ is neither surjective not injective. Otherwise, it is said to be **regular** at p. The set of points where f is critical is called the **critical points** of f and their images the **critical values** of f.*

For most functions, the critical set is actually a submanifold of M. In the case where $n = 1$, this critical manifold is of dimension 0, or in other words is composed of isolated points.

Definition 4.1.2. *A smooth map $f : M \to \mathbb{R}$ is called **Morse** if its critical points are isolated. It is also called **simple** if no two critical points share the same critical value.*

Lemma 4.1.1. *f is Morse iff its Hessian is non-degenerate.*

Lemma 4.1.2. *Morse functions are stable and dense in the space of smooth functions.*

We will only consider simple Morse functions for the rest of the section.
The non degeneracy of the Hessian shows us how to classify the critical points of a Morse function. Indeed its eigenvalues are then, up to local diffeomorphism either $+1$ or -1. A Morse function is then locally equivalent to a function of the type:

- local surjection : $(x_1, \cdots, x_m) \to x_1$
- "fold": $(x_1, \cdots, x_m) \to \sum_{i=1}^{\lceil \frac{m}{2} \rceil} (x_i)^2 + \sum_{i=\lceil \frac{m}{2} \rceil + 1}^{m} \pm (x_j)^2$

So suppose that we are given a function on M, then the following lemmas will help us find the structure of M thanks to the critical points of f.

Lemma 4.1.3. *Let $f : M \to \mathbb{R}$ be a smooth function with no critical value on an interval $[a, b]$, then $f^{-1}([a, b]) \simeq f^{-1}(a) \times [a, b]$.*

Let's consider the case of a critical point p where all eigenvalues of the Hessian are of the same sign. Then in a neighborhood of p, the function is equivalent to :

$$(x_1, \cdots, x_m) \to \sum_{i=1}^{m} (x_i)^2$$

Lemma 4.1.4. *Let $[a, b]$ be an interval with $f(p)$ as unique critical value for f. Then:*

$$\begin{aligned} f^{-1}([a, b]) &\simeq \left(f^{-1}(a) \times [a, b]\right) \coprod B_n \\ \textit{or} \qquad\qquad &\simeq \left(f^{-1}(b) \times [a, b]\right) \coprod B_n \end{aligned}$$

Proof. Let K be the connected component to the critical point p of $f^{-1}([a, b])$. From Lemma ... we know that $f^{-1}([a, b]) \setminus K$ is a cylinder, i.e. $\left(f^{-1}(a) \setminus K\right) \times [0, 1]$.
By assumption, there is a neighborhood U of p where the function is equivalent to the one given above. Then w.l.o.g. we may consider that U intersects $f^{-1}(a)$ or $f^{-1}(b)$ non trivially or in other words, that there exists $c \in \mathbb{R}$ such that the set of solutions to

$$\sum_{i=1}^{n} (x_i)^2 \leq c$$

is diffeomorphic to a submanifold of $f^{-1}([a, b]) \setminus K$, since $f(p) = 0$. Yet these solutions are diffeormorphic to balls of dimension m, hence already closed connected. We therefore obtain the claimed result. $\square$

For an interval containing a unique critical value corresponding to such a critical point, the only information required to know the diffeomorphism class of its preimage under f is the preimage of its boundaries. It is not given that it is always the case.
For now, let us investigate fully the $m = 2$ case. The only other type of critical point is a point where f is equivalent to $(x_1, x_2) \to x_1^2 - x_2^2$.

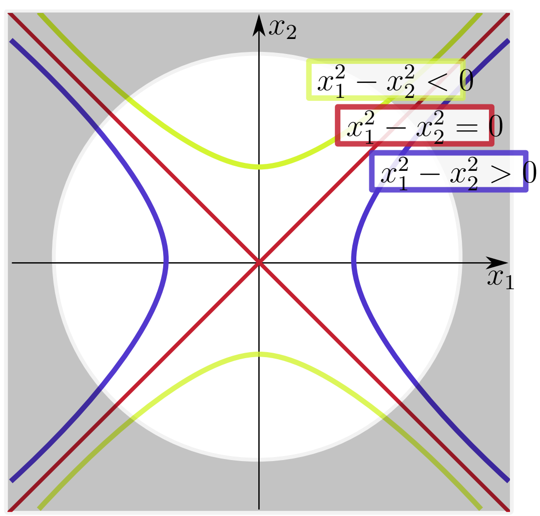
Lemma 4.1.5. *Let $[a, b]$ be an interval with $f(p)$ as unique critical value for f. Then:*

$$
\begin{aligned}
& f^{-1}([a,b]) \simeq \left(f^{-1}(a) \times [a,b]\right) \coprod \left(S^2 \setminus (B^2 \coprod B^2 \coprod B^2)\right) \\
\textit{or} \quad & \phantom{f^{-1}([a,b])} \simeq \left(f^{-1}(a) \times [a,b]\right) \coprod \left(\mathbb{R}P^2 \setminus (B^2 \coprod B^2)\right)
\end{aligned}
$$

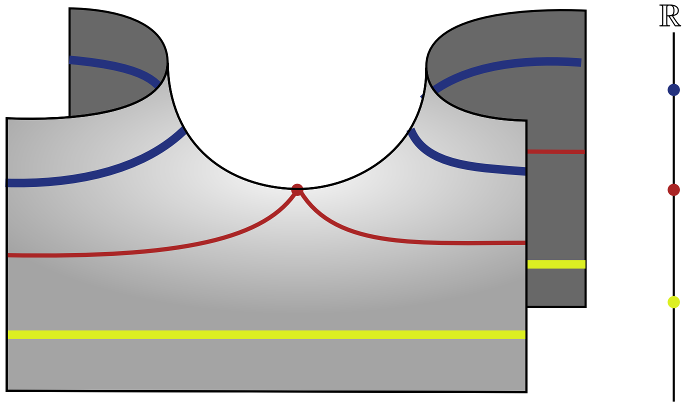
Proof. Let K be the connected component to the critical point p of $f^{-1}([a, b])$. From Lemma 4.1.3 we know that $f^{-1}([a, b]) \setminus K$ is a cylinder, i.e. $\left(f^{-1}(a) \setminus K\right) \times [0, 1]$. By assumption, there is a neighborhood U of p where the function is equivalent to the one given above. Then w.l.o.g. we may consider that U intersects $f^{-1}(a)$ or $f^{-1}(b)$ non trivially or in other words, that there exists $c, d \in \mathbb{R}$ such that the set of solutions to

$$c \leq x_1^2 - x_2^2 \leq d$$

is diffeomorphic to a submanifold of $f^{-1}([a, b]) \setminus K$, since $f(p) = 0$. Yet it is not compact, as shows the following picture:

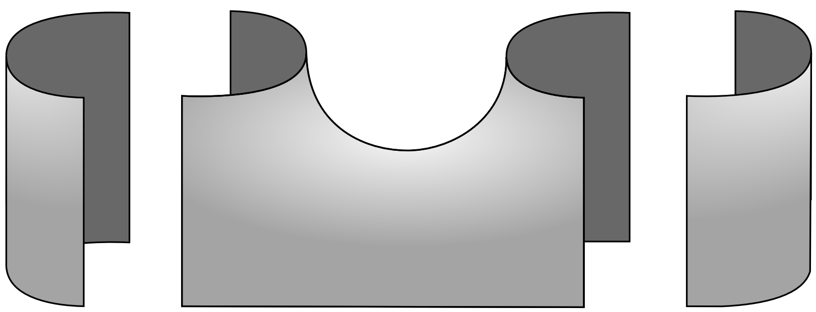


Therefore $f^{-1}([a,b])$ cannot be fully contained in U, however we decide to pick a and b. The critical point is called a saddle point, due to this other way of visualizing the neighborhood U : But since $f^{-1}(a) \cap \overline{U} \simeq [0,1] \coprod [0,1]$ and $K \setminus \overline{U}$ does not contain critical

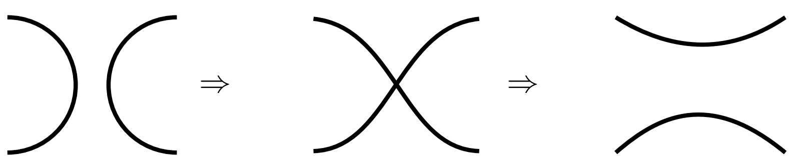


$(x_1, x_2) \to x_1^2 - x_2^2$

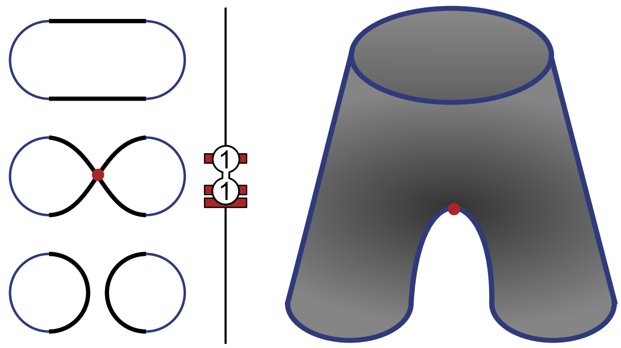
points, this last one is diffeomorphic to two squares and we can determine the diffeomorphism class of $f^{-1}([a,b])$ by knowing how $f^{-1}(a) \cap \overline{U}$ glues with $f^{-1}(a) \cap (K \setminus U)$.



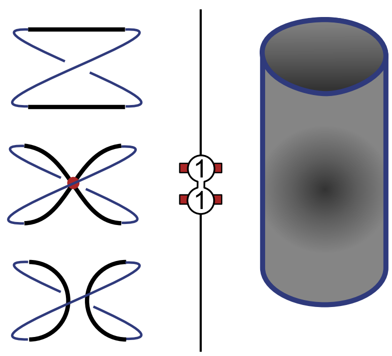
Note that under this light, we see that the saddle point corresponds to the resolutions of a crossing of a knot, and there are two non equivalent ways of completing the crossing. One



gives the trinion, or pair of pants, which is actually a punctured sphere :



while the second is a punctured $\mathbb{R}P^2$: $\square$



In the pictures above we introduced a notation on the image of f. At a critical value we indicated in white bubbles the number of folds going in each direction. For example if f is locally equivalent to $(x_1, x_2) \to x_1^2 + x_2^2$ there will be two folds going in the same direction, either up or down on $\mathbb{R}$ at the critical value. We will then find a 2 and a 0 in the bubbles. The red rectangles of each parts of a critical value encode the number of S_1 in corresponding resolving of the crossing.
We have now been able to classify the simplest types of 2d cobordisms that we may encounter, up to isomorphism. Now it remains to find the relations between these. It is clear for example that gluing two cylinders together yields another cylinder, but how can we find the rest ?

4.1.1 Cerf theory

To answer this question, we need to investigate the space of functions more closely. This is the aim of this section. The space $C_\infty(M, N)$ of functions between two smooth manifolds M and N can be given a topology, called the Whitney topology. A thorough description of this topology can be found in [33]. A path in this space is a homotopy between two functions $M \to \mathbb{R}$.

Lemma 4.1.6. *The set of Morse functions is open and dense in $C_\infty(M,\mathbb{R})$. The set of simple Morse functions is a residual set in $C_\infty(M,\mathbb{R})$.*

Proof. cf [33] $\square$

Lemma 4.1.7. *The space $C_\infty(M,\mathbb{R})$ is path connected.*

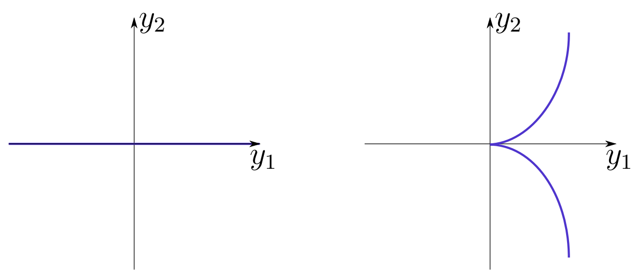
Proof. The proof lies in the fact that for$f \in C_\infty(M,\mathbb{R})$, there exists a continuous path α to the constant function 0 given by $\alpha(t) := (1-t)f$. $\square$

In other words, two Morse functions f and f' are always connected by a homotopy $h : M \times I \to \mathbb{R}$, to which we can associate a map $\tilde{h} : M \times I \to \mathbb{R} \times I$ such that $\tilde{h}(x,t) = (h(x,t),t)$. Moreover, being a path, the composition $M \times I \xrightarrow{h} \mathbb{R} \times I \xrightarrow{\pi_I} I$ has no critical points. Therefore locally, a generic homotopy is a generic map $\mathbb{R}^3 \to \mathbb{R}^2$ such that the map $\mathbb{R}^3 \to \mathbb{R}$ given by the projection on the first component is non singular.

Lemma 4.1.8. *Let p be a critical point of a generic smooth map $f : M \to N$ where $dim(M) = 3$ and $dim(N) = 2$. Then f is locally equivalent to one of the following form at p:*

$$(x_1, x_2, x_3) \to (x_1, x_2^2 \pm x_3^2)$$
$$(x_1, x_2, x_3) \to (x_1, x_2^3 + x_1x_2 + x_3^2)$$

The critical set for the first type is $\{(x_1, 0, 0)\}$ and its image is the set $\{(x_1, 0)\}$ whereas the critical set for the second type is $\{(-3x_2^2, x_2, 0)\}$ with image $\{-3x_2^2, -2x_2^3)\}$. Now when



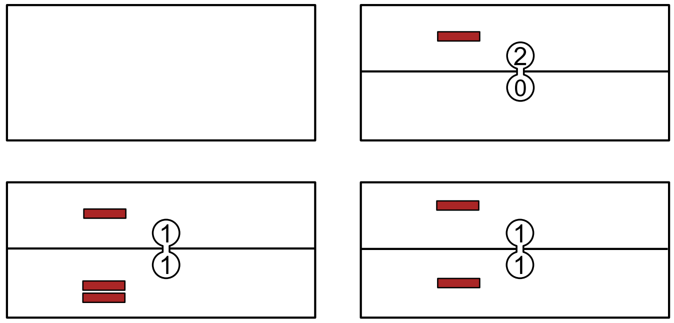
type 1 type 2

we consider cobordisms, the functions are required to be constant on the boundaries and their homotopies to be homotopies rel boundary, i.e. $h(\delta M, t) = h(\delta M, 0)$ for all $t \in [0,1]$. The singularities of maps $M \to \mathbb{R}^2$ that correspond to homotopies must therefore respect two conditions:

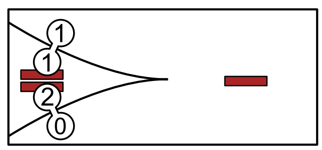
- There is no critical points on $(\delta M) \times [0,1]$
- The map $h_{\mathbb{R}}\pi$ is non singular.

In terms of critical values, it implies that the possible singularities of the homotopy, up to isomorphisms in $\mathbb{R}^2$ and local isotopies, can therefore only be:

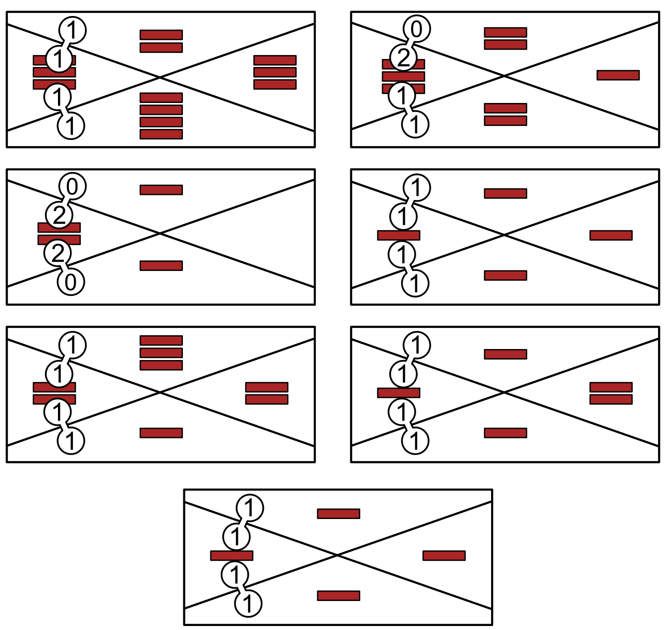
- of type I_0 which, as relations, are tautologies,



- of type I_1, which as relations are birth/death of critical points,



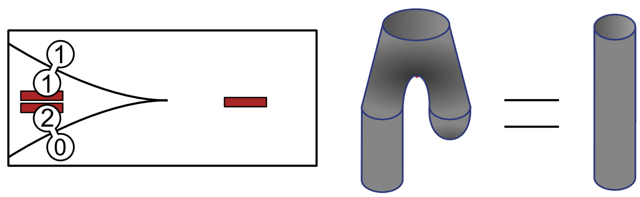
- or be singularities where the map is non simple.



Proof. Suppose that we have critical values of type I_0 only. Then, as the restriction of the map to the singular manifold is a local embedding of a compact 1-manifold, we either have an intersection point or it is an embedding. Assuming the latter we need only to describe the intersection of the critical values with $h(\delta(M \times [0,1]))$, which is composed of two points, assuming that the starting function has a critical point. From the first condition this intersection cannot happen on $(\delta M) \times [0,1]$. It therefore has to be on $h(M \times \{0,1\})$. Yet if it is on $h(M \times \{0\})$ or $h(M \times \{1\})$ only, we can show that the second condition is violated. We are therefore left with the given possibilities. The same analysis in the non simple case give the same conclusions. Of course, the information contained in the diagrams concern only the restriction of the homotopy to the connected components to the singularities. The actual diagrams of the function may have more red squares, corresponding to the number of S^1 contained in the preimage of a point, in which case the number of red squares added is the same in all quadrants. Concerning the I_1 type, we can conclude that the intersection has to happen on $h(M \times \{0\})$ or $h(M \times \{1\})$ only, hence the claimed diagrams. □

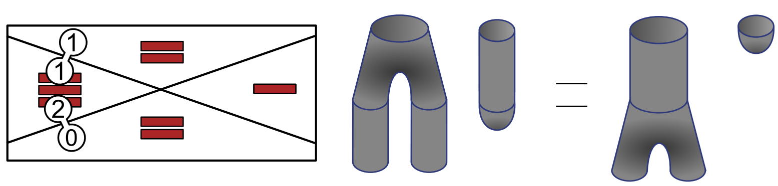
The analysis is standard and can be found in more details in [41]. Let us now give an explicit description of each of the above in terms of relations on cobordisms.

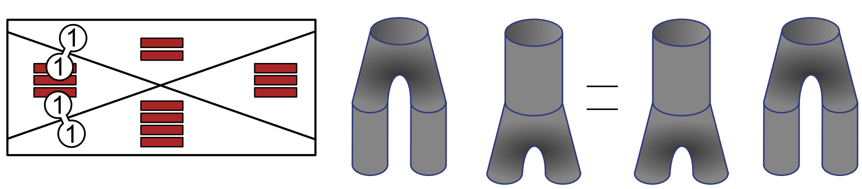
- The singulatiry of type I_1 correspond to the following relations :

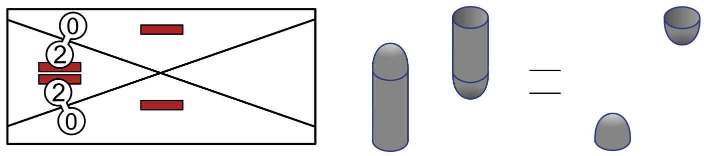


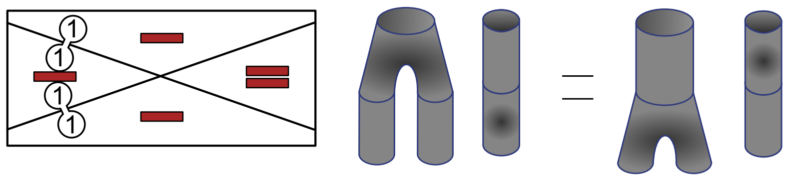
  which, with its symmetries correspond to a left unit, right unit, left counit and right counit axioms.

- The crossings of I_0 singularities correspond to the following relations:

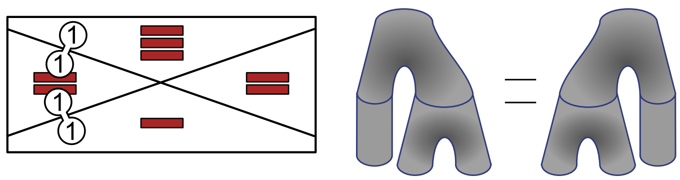




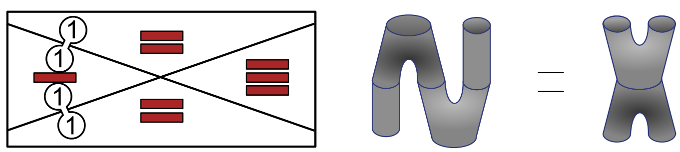




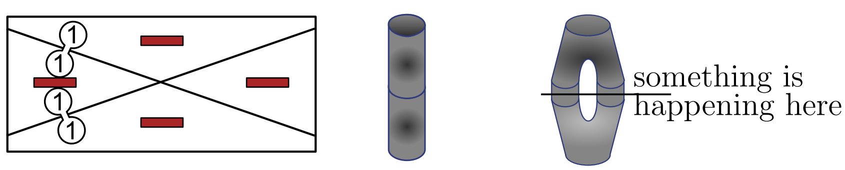
which are functoriality axioms for the disjoint union,



which corresponds to the associativity and coassociativity axioms,

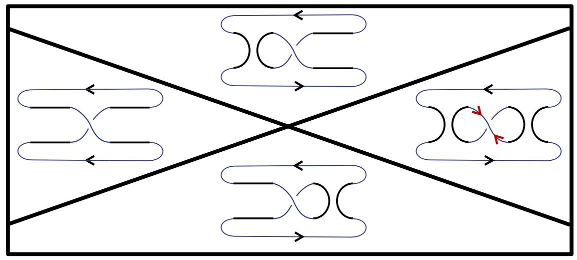


which is the frobenius axion, and



which needs further investigation.

This last axiom corresponds topologically to the fact that the connected sum of two copies of $\mathbb{R}P^2$ is isomorphic to a Klein bottle.To understand better what is happening in the last diagram we need to consider the resolutions of intersections at play. The picture is the following :



Remark that while orienting the links, we get two different orientations on the inner right circle to make the transitions to the links above and below coherent. Therefore a "twist" needs to be introduced in the gluing of the trinions, which would indeed create a punctured Klein bottle. We will see how the "twist" fits in the model in the section called Cylinders.

The obtained structure is akin to a frobenius algebra object in a monoidal category, except that it has a few more axioms. In [55], Turaev defined this entity as an **extended Frobenius algebra**. If we remove the resolution that yields unoriented manifolds, we recover the classical result for 2d TQFTs, i.e. the category of 2d cobordisms is equivalent to a category generated by a frobenius algebra object.

4.2 Towards higher dimensions

Regardless of the dimension, there is a sub-category of $\underline{\mathbf{Cob}}_n$ that is always present. It is the one generated by the spheres. We will describe it in the first part of this section. Moreover, they share the presence of cylinders representing mapping class groups of their objects, which is the object of the second part.

4.2.1 Balls and pants

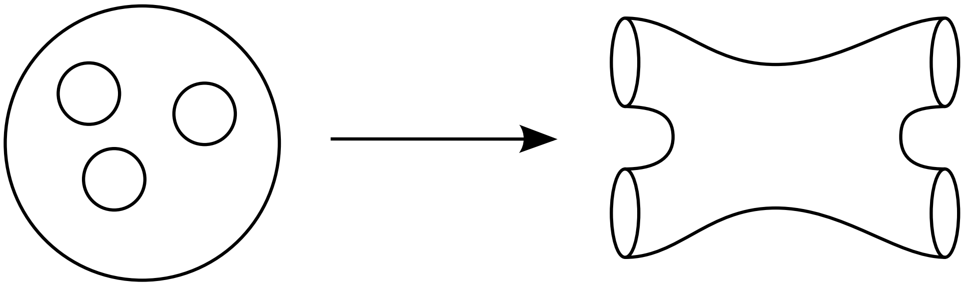
Let's consider the k-punctured m-sphere :

$$S^m_k := S^m \setminus \big(\coprod_{j=1}^{k} \imath_j(B^m)\big)$$

from which we can make a plethora of cobordisms :

$$P_m(a,b) = (S^m_{a+b}, \imath(\coprod_a S^{m-1} \coprod \imath(\coprod_b S^{m-1}))$$

Fixing a dimension m, we will drop it and simply write $P_{a,b}$ for $P_m(a,b)$. For example here is pictured $S_{2,4}$ giving $P_{2,2}$:



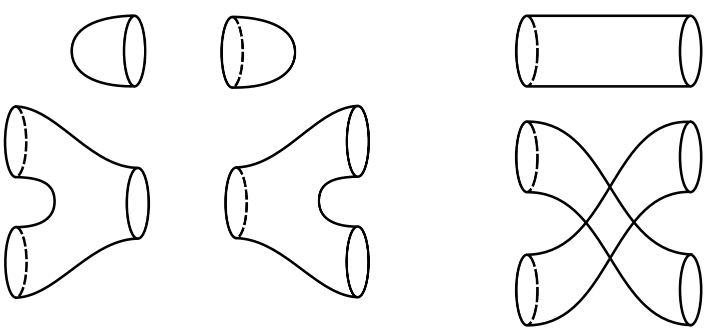
These cobordisms satisfy the following identities :

$$P_{a,b} \circ (P_{0,1} \coprod_{a-1} P_{1,1}) = P_{a-1,})$$
$$P_{a,b} \circ (P_{2,1} \coprod_{a-1} P_{1,1}) = P_{a+1,b}$$
$$(P_{1,0} \coprod_{b-1} P_{1,1}) \circ P_{a,b} = P_{a,b-1}$$
$$(P_{1,2} \coprod_{a-1} P_{1,1}) \circ P_{a,b} = P_{a,b+1}$$

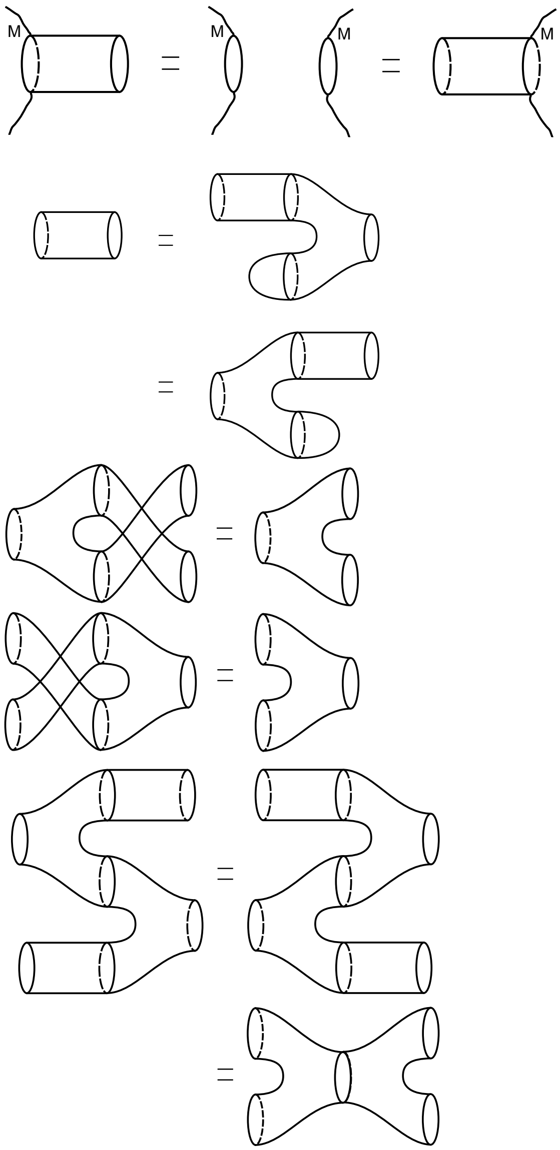
Moreover if M and N are connected manifolds we can get their connected sum by gluing $P_{1,0} \circ P_{2,1}$ to their punctured copies. Now $P_{1,1}$ plays a special role in our cobordism category. Our above definition was flawed, it requires a choice for $\imath$. Let's say that the choice has been made for all $P_{a,b}$. Take a puncture A, then:

$$\imath : S^n \to A \qquad \text{is the chosen embedding}$$

to get another choice, we want to glue a cylinder on A such that the glued boundary parts are both $\imath$ and the newly created one is given by that other choice. The next section will deal with that.
Therefore, from the above identities, the building blocks for all $P_{a,b}$ are :



That satisfy some properties :



4.2.2 Cylinders

Given two isomorphisms $f : B \to A$ and $g : C \to A$ one can define a cobordism ${}_fI_g$:

$$ {}_fI_g := (A \times I, f \coprod g) : B \to C $$

Lemma 4.2.1. *The following is true :* ${}_fI_g = {}_{id_B}I_{f^{-1}\circ g} = {}_{g^{-1}\circ f}I_{id_C}$

Proof. The following diagram clearly commutes :

$$\begin{tikzcd}
 & \partial(B\times I) \arrow[d, "\partial(f\times id_I)"] \\
B\coprod C \arrow[ur, "id_B\coprod(f^{-1}\circ g)"] \arrow[r, "f\coprod g"'] \arrow[dr, "(g^{-1}\circ f)\coprod id_C"'] & \partial(A\times I) \\
 & \partial(C\times I) \arrow[u, "\partial(g\times id_I)"']
\end{tikzcd}$$

□

Lemma 4.2.2. *The following is true :* ${}_{id}I_f = {}_{id}I_g \Leftrightarrow f \sim g$

Proof. $\underline{\Rightarrow}$

$$\begin{tikzcd}
 & \partial(A\times I) \arrow[dd, "\partial w"] \\
A\coprod B \arrow[ur, "id\coprod f"] \arrow[dr, "id\coprod g"'] & \\
 & \partial(A\times I)
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
B \arrow[r, "\imath_0"] \arrow[d, "f"] & B\times I \arrow[d, "w\circ(f\times id)"] & B \arrow[l, "\imath_1"'] \arrow[d, "g"] \\
A \arrow[r, "\imath_0"] \arrow[dr, "id"'] & A\times I \arrow[d, "quotient"] & A \arrow[l, "\imath_1"'] \arrow[dl, "id"] \\
 & A &
\end{tikzcd}$$

hence $f \sim g$

$\underline{\Leftarrow}$

if $h :\ B\times I \to\ A$ gives the isotopy then $\Phi := (h(x,t),t)$ gives the isomorphism of cobordisms.

□

Corrollary 4.2.1. *Let $f :\ B \to A$ and $g :\ C \to B$ be iso, then the gluing of their mapping cilynders is :*

$${}_{id}I_g \circ {}_{id}I_f = {}_{f^{-1}}I_g = {}_{id}I_{f\circ g}$$

Proof. ${}_{id}I_g \circ {}_{id}I_f = {}_{id}I_g \circ {}_{f^{-1}}I_{id} = {}_{f^{-1}}I_g$ □

Corrollary 4.2.2. *If $f :\ A \to A$ is iso, then $({}_{id}I_f)^{\circ n} = {}_{id}T_{f^{\circ n}}$*

Theorem 4.2.1. *Let$(M, \emptyset \coprod f)$ be a cobordism, then it is equivalent to :*

$${}_{id_{\partial M}}I_f \circ (M, \emptyset \coprod id_{\partial M})$$

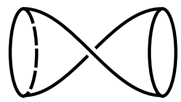
Proof. by the tubular neighborhood theorem, one can find a neighborhood of ∂M isomorphic to $\partial M \times I$, the rest follows. □

So that ${}_{id_{\partial M}}I_g \circ (M, \emptyset \coprod f) = (M, \emptyset \coprod g \circ f)$.
The conclusion is that for $a = simeqB$, there are at least as many cobordisms $A \to B$ as isotopic classes of maps $A \to B$.

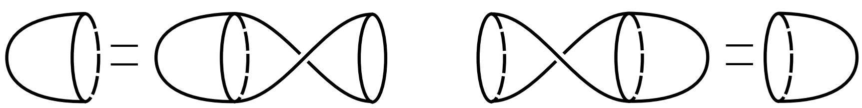
4.2.3 Twisting pairs of pants

Applying the previous result to $P_{(1,1)}$, we see that we can "twist" it by isotopy classes of isomorphisms of S_m. These are in a one-to-one correspondance with the invertible elements of $\pi^m(S^m)$. There may be lots of them but for every dimension we have at least two, the identity and the antipodal map f. We therefore have two cylinders :

 $= \quad {}_{id}I_{id}$

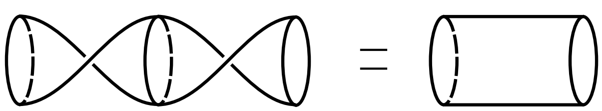
 $= \quad {}_{id}I_{f}$

Lemma 4.2.3. *The following are true:*



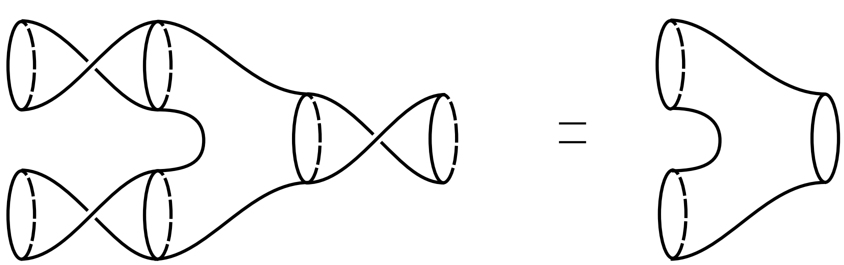
Proof. $f \in O_2(\mathbb{R}) \Rightarrow f$ extends to an isometry of $\mathbb{R}^2$, therefore an isomorphism of D_2. □

Lemma 4.2.4. *The following is true :*



Proof. The connected components of O_m are determined by the determinant, or $det(f^2) = (det(f))^2$, so any cylinder on S_m is involutive. □

Lemma 4.2.5. *The following is true:*



Proof. Once again the argument is that t is an isomorphism of D^m, so, up to isotopies on the punctures sphere, applying t on D^m is applying it on the boundaries. $\square$

These cobordisms make a full subcategory of any category of cobordisms. For the (1+1)-Cobordisms though, it fails to be the whole category by a single object, the punctured projective space.

4.2.4 The limitations of Morse theory

The problem encountered while trying to use Morse theory to describe the category of cobordisms in dimension greater than 2 stems from the following observation. While in dimension 1 a ball belonging to a compact 1-manifold can only belong to a sphere, it is not the case any more in higher dimensions. What used to just be a combinatorial problem about resolutions of crossings is gaining in complexity. The problem of finding the preimage of an interval containing a critical value gets much more complicated and in dimension 3 and 4 gives rise to the Kirby calculus [34]. We will see that by keeping the difference of dimensions of source and target of the smooth maps studied equal to 1, we may contain our studies to a combinatorial exercise, just as the 2d case. This point of view requires us to study singularity theory past Morse functions, which is what we will present in the next chapter.

4.2.5 N-tuple spans

If we consider functions from a 3 dimensional manifold to $\mathbb{R}^2$, we will obtain preimages of point in $\mathbb{R}^2$ that are 1-dimensional. The theory we are constructing is therefore not of manifolds with borders as cobordisms but manifolds with corners as cobordisms. The basic definition was given by Morton in [50] and extended by Feshbach and Voronov in [29]. Instead of special cospans in $\underline{\mathbf{Top}}$, we now have double cospans in $\underline{\mathbf{Top}}$.

Definition 4.2.1. *A **double cospan** is a commutative diagram of the sort:*

$$\begin{tikzcd}
L_1 \arrow[r] \arrow[d] & M_1 \arrow[d] & L_3 \arrow[l] \arrow[d] \\
M_2 \arrow[r] & N & M_4 \arrow[l] \\
L_2 \arrow[r] \arrow[u] & M_3 \arrow[u] & L_4 \arrow[l] \arrow[u]
\end{tikzcd}$$

They may be composed in two ways since from

$$
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & M \arrow[d] & \bullet \arrow[l] \arrow[r] \arrow[d] & N \arrow[d] & \bullet \arrow[l] \arrow[d] \\
\bullet \arrow[r] & D & \bullet \arrow[l] \arrow[r] & D' & \bullet \arrow[l] \\
\bullet \arrow[r] \arrow[u] & O \arrow[u] & \bullet \arrow[l] \arrow[r] \arrow[u] & P \arrow[u] & \bullet \arrow[l] \arrow[u]
\end{tikzcd}
$$

we obtain, using pull backs again,

$$
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M \circ N \arrow[d, dotted] & N \arrow[l] \arrow[d] & \bullet \arrow[l] \arrow[d] \\
\bullet \arrow[r] & D \arrow[r] & D \circ D' & D' \arrow[l] & \bullet \arrow[l] \\
\bullet \arrow[r] \arrow[u] & O \arrow[r] \arrow[u] & O \circ P \arrow[u, dotted] & P \arrow[l] \arrow[u] & \bullet \arrow[l] \arrow[u]
\end{tikzcd}
$$

which gives a composite:

$$
\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & M \circ N \arrow[d, dotted] & \bullet \arrow[l] \arrow[d] \\
\bullet \arrow[r] & D \circ D' & \bullet \arrow[l] \\
\bullet \arrow[r] \arrow[u] & O \circ P \arrow[u, dotted] & \bullet \arrow[l] \arrow[u]
\end{tikzcd}
$$

The second composition is obtained similarly when the diagrams paste vertically. But none of these compositions are associative and unital. But as happened for spans and cospans they are both associative and unital up to isomorphisms. We may then define a weak triple category of double cospans and morphisms. In this triple category there is a subset of cubes that are invertible along the direction bearing morphisms in $\underline{\mathbf{C}}$ and whose said morhism are all identities along the boundary. It is in our programme to show that this set is the "double bicategory" proposed by Morton [50], that it may be seen as a weak internal double category in the category of groupoids and that we may then obtain a double category of quotients using the same forgetful functor as we used in Chapter 2.

4.3 Singularity theory

The main sources for the content of this section are the books of Guillemin and Golubitsky [33], the book of Arnold [8] and Schommer-Pries' PhD thesis [53]. The definitions and proofs of theorems can be found there.

4.3.1 Thom-Boardman theory

Definition 4.3.1. *Let M be a smooth manifold. Then for $p \in M$ a* ***germ*** *at p is an equivalence class of smooth maps $M \to \mathbb{R}$ given by the following relation:*

$$f \sim g \iff \exists U \in M \text{ open around } p \text{ s.t. } f|_U = g|_U$$

Throughout this section $J_k(M, N)$ will denote the k^{th}-jet bundle over the product of manifolds $M \times N$. It is the space of equivalence classes of germs of functions agreeing up to k^{th} derivative. For a smooth map $f : M \to N$, its k^{th}-jet is denoted $j_k(f)$ and $j_k() : C^\infty(M, N) \to J_k(M, N)$ is defined as such is a smooth map. Let E^i be the sub-bundle of $J_1(M, N)$ of germs whose first derivative drops rank by i, and $\Sigma^i(f)$, its preimage by $j^1(f)$.

Lemma 4.3.1. *Let $f : M \to N$ be a smooth map, then $\Sigma^i(f)$ is a manifold if and only if $j^1(f) \pitchfork E^i$. In which case the codimension of $\Sigma^i(f)$ is given by:*

$$codim(\Sigma^i(f)) = i(|m - n| + i)$$

For low dimension, the table of codimensions is the following :

$m-n$ \ i	0	1	2	3	4
0	0	1	4	9	16
1	0	2	6	12	20
2	0	3	8	15	24
3	0	4	10	18	28
4	0	5	12	21	32

Keeping $m \leq 5$, we are limited to :

$m-n$ \ i	0	1	2	3	4
0	0	1	4		
1	0	2			
2	0	3			
3	0	4			
4	0	5			

which, for the cases where $m - n = 0$ or 1 reduces to :

$m-n$ \ i	0	1	2
0	0	1	4
1	0	2	

By the jet transversality theorem functions that are transversal to each E^i are dense in $C^\infty(M, N)$. We will call such a map **generic** for the remainder of the chapter.

Definition 4.3.2. *Let $f : M \to N$ be a generic smooth function, p be a critical point of f. Its **Thom Boardman symbol** is a finite sequence of integers $(i_0, \cdots, i_k)$ where $i_a \geq i_{a+1}$ for all $a < k$ and $i_k = 0$ such that :*

- *$df(p)$ drops rank by i_0.*
- *for $g := f|_{\Sigma(f)}$, $dg(p)$ drops rank by i_1.*
- *for $h := g|_{\Sigma(g)}$, $dh(p)$ drops rank by i_2.*
- *et cetera*

Though the Thom-Boardman symbol does not classify singularities entirely, they give precious information on the behavior of the function, especially at the critical manifold.

By an argument on dimensions, we can then conclude that as long as $m \leq 5$ and $m - n = 1$, the only singularities that appear in a generic function have Thom-Boardmann symbol I_k for $0 \leq k \leq (m-2)$ containing $(k+1)$ ones before the zero. But since the Thom Boardmann symbol is not a complete invariant of the singularity, we need to investigate further. Yet in our case there are not so much more to be found and we can classify the singularity types as Morin singularities.

4.3.2 Morin Singularities

Up to local diffeomorphism on M and N, a singularity with Boardman symbol I_k is of the following form :

$$f(x_1, \cdots, x_m) = \big(x_1, \cdots, x_{m-2}, x_m^{k+2} + \sum_{i=1}^{k} x_i x_m^i \pm x_{m-1}^2\big)$$

They are therefore determined by their Boardmann symbol and by the sign in front of the term x_{m-1}^2. In dimension 2 and 3, it corresponds to the singularities previously given. In dimension 4 we get the following :

- Folds : $(x_1, x_2, x_3, x_4) \to (x_1, x_2, x_4^2 \pm x_3^2)$.

- Cusps : $(x_1, x_2, x_3, x_4) \to (x_1, x_2, x_4^3 + x_1x_4 \pm x_3^2)$.
- Swallowtails : $(x_1, x_2, x_3, x_4) \to (x_1, x_2, x_4^4 + x_2x_4^2 + x_1x_4 \pm x_3^2)$.

In dimension 2 the only generators were folds and non simple critical values. Their relations were given by folds, cusps and non simple critical values. In dimension 3, as we will start laying down in the next chapter, the generators will be given by folds, cusps and non simple critical values. The relations will then be given by folds, cusps, swallowtails and non simple critical values.

Chapter 5

Prospects

I produced during my PhD the foundations of a programme to construct new TQFTs in dimension 3 and 4. Its bases we laid in the previous chapters and in this chapter we present the parts that need more work before being a definitive text can be produced. There is work to be done to solidify our understanding of weak higher category theory that are necessary to define the categories of cobordism with corners before quotienting to obtain strict ones. This will not be possible without a deeper understanding of commutative diagrams. There is also work to define a theory of representation for the double groups that we mentioned. We believe that once the last part of the programme is completed, i.e. the singularity theory part, the representation theory of double groups will provide the first examples of such higher TQFTs. Let us finish the thesis by explicitly giving the remaining part of the programme to be completed.

5.1 Higher Commutativity

As the definitions of the weak internal category and weak internal functor have shown, the definition of a commutative n-cube for any n whose faces are taken in $n-1$-tuple categories is central to the problem of defining weaker and weaker instances of higher categories. The notion of n-tuple categories with connection developed by Brown and Al'Alg [5] is too strong for anything interesting to happen. For example, it forces commutative cubes in a triple category to exist if and only if all their faces are invertible squares. It is not surprising though since they prove in their paper that such categories are equivalent to n-categories, showing how restrictive their condition for the existence of a connection to exist is.
Though we do not have a definitive theory of commutative cubes for any dimension, we know enough to define weak internal categories. To complete this part of the program, we will build upon Brown's and Higgins[15] and introduce the necessary modifications to make it general enough to apply to our needs.

5.2 Representations

As important to us as double groups are their representations. In dimension 3, the structure of the category of representations of certain algebraic entities, called quantum groups, gives us the basic ingredient to build a TQFT through state-sums. In the same fashion, some invariants of links and knots are given by categories of representations. Understanding the representations of our double groups will be important if we want to build TQFTs out of them. But first, let's recall what a representation of a group is. The basic notion of representation, called permutation representation, is a functor from the group G, viewed as a category with one object, to the category of sets :

$$G \to \underline{\mathbf{Set}}$$

or in more classical terms a group morphism from G to $\underline{\mathbf{Set}}$[S], the group of automorphisms of S. A linear representation, on the other hand, is a functor to the category of vector spaces over a field $\mathbb{K}$, most often chosen as $\mathbb{R}$ or $\mathbb{C}$:

$$G \to \underline{\mathbb{K}\text{-}\mathbf{Mod}}$$

or in more classical terms it is a group morphism to the group of invertible linear maps on a vector space V, $\underline{\mathbb{K}\text{-}\mathbf{Mod}}$[V]. The study of representations of 2-groups has pushed people to consider specific higher dimensional analogs of vector spaces, such as **2-Vect** [39, 12, 11], or **Meas** [24], while the basic notion of permutation representation was unsurprizingly thought of as functors to **Cat**, the prime example of a 2-category. Since 2-categories are special cases of double categories, representations of 2-groups should be special cases of representations of double groups and it therefore gives us suggestions to consider.

It seems correct to think of permutation representations of double groups as functors from a double group G to the "quintet" double category on the 2-category **Cat**, since it is the first double category at hand. The quintet double category contains a quintet double groupoid just as a category contains its groupoid of invertible maps. A representation of a double group would then be a double functor to it.

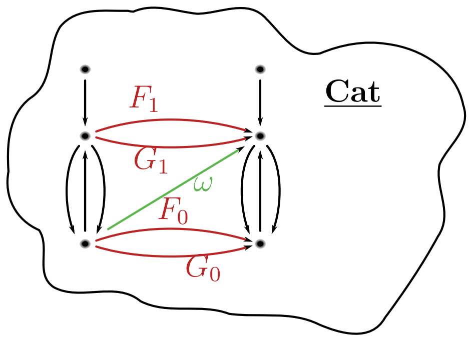
Another possibility for a target of the representation functor is given by the fact that the homsets of the category of double categories are themselves double categories. Such representations of a double group would then contain a representation of the horizontal group as horizontal double natural transformations, a representation of the vertical group as vertical double natural transformations and squares as comparisons.
Both examples above have equivalents for $\mathbb{K}$-linear categories, k-linear functors etc, which would give notions of linear representations. A third type of double category we may want to consider is the double category of functors and profunctors, cf [35]. Simple examples must be taken to see the relevance of the above representation theories.

sectionWeak internal structure

Weak internal natural transformations require a little bit more work than weak functors to define. The weakness of the functors they intertwine threatens the existence of pullbacks that are needed to naively extend the definition of internal natural transformation and though their degenerate counterparts are known, cf Leinster's book [44]. The definition should start the following way :

Definition 5.2.1. *A* ***weak internal natural transformation*** *between two weak internal functors consists of :*

- *An arrow :*



- *A cell :*

$$\begin{tikzcd}
b \arrow[r, "F_1 \ulcorner t\omega"] \arrow[d, "s\omega \ulcorner G_1"'] \arrow[dr, phantom, "\eta"] & b'_2 \arrow[d, "\circ'"] \\
b_2 \arrow[r, "\circ'"'] & \text{b'}
\end{tikzcd}$$

- *A pair of commutative diagrams :*

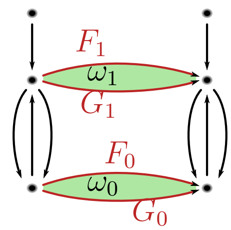
$$\begin{tikzcd}
\text{b} \arrow[r, "\omega"] \arrow[d, no head] & \text{b'} \arrow[d, "\text{s'}"] \\
\text{a} \arrow[r, "F_0"'] & \text{a'}
\end{tikzcd}
\qquad
\begin{tikzcd}
\text{b} \arrow[r, "\omega"] \arrow[d, no head] & \text{b'} \arrow[d, "\text{t'}"] \\
\text{a} \arrow[r, "G_0"'] & \text{a'}
\end{tikzcd}$$

- *A set of commutative cubes :*

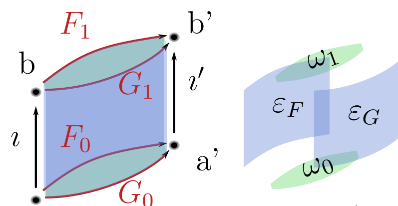
Weak internal natural cells may look simpler but its definition may require a little more work. We put it here to give an intuition of the phenomenon at play.

Definition 5.2.2. *A **weak internal natural cell** between two weak internal functors consists of :*

- *two 2-cells :*



- *and four commutative cubes :*



+ same for $(\circ, \circ')$, (s, s') and (t, t')

5.2.1 The weak conjecture

The mantra that seems to reign over the world of weak categories, from our standpoint is the following :

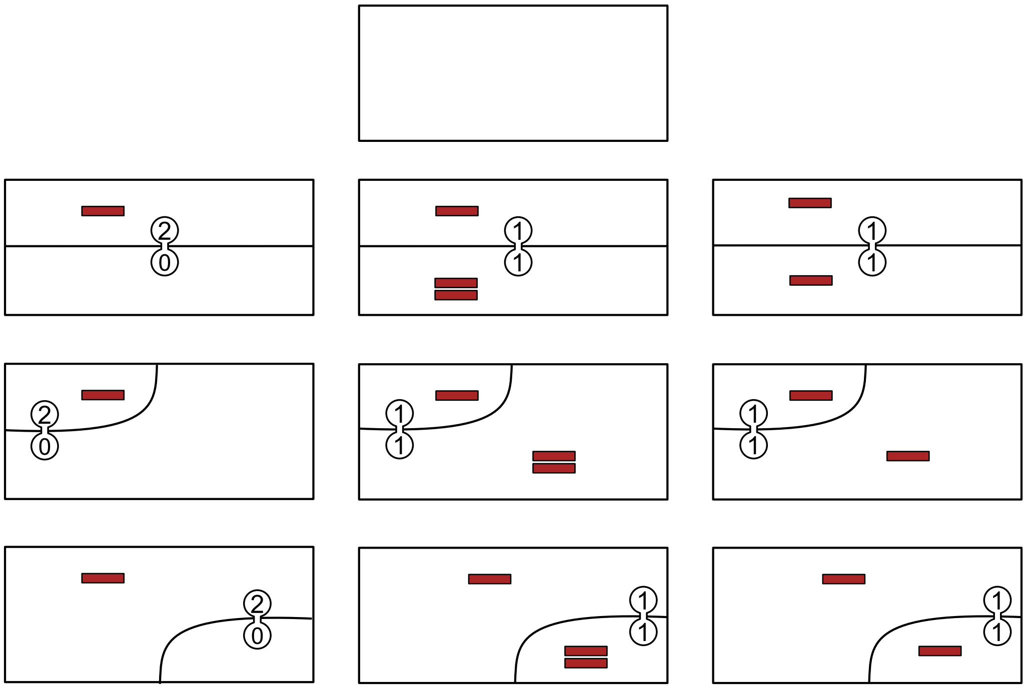
> ”At the beginning was strictness.
>
> The first weakness came only supported by it.
>
> From there on, any weakness came supported by a lesser one.”

In other words, any new weak composition can only be associative up to morphisms in another direction. It may be strict or weak itself, in which case it is supported by yet another direction. There is then a one-to-one correspondence between the different versions of weak n-tuple categories and planar rooted trees with n nodes.

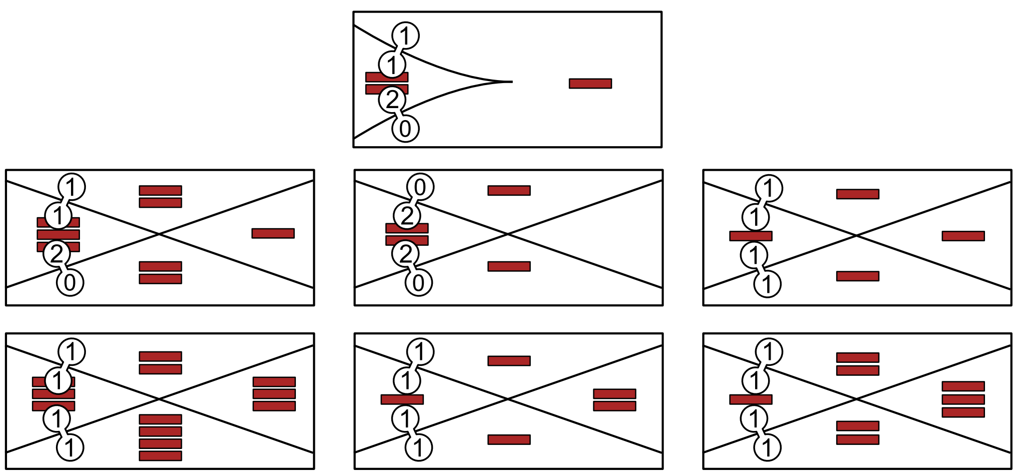
5.3 2-Morse theory and Cobordisms

Once the purely categorical part of the work on weak triple categories has been done, and that it has been showed that the triple category of double cospans and morphisms quotients to a double category, we will have to translate the quotienting to the 2-Morse setting, in terms of certain homotopies rel. corners.

We present here, based on the sole study of singularity theory presented previously, the expected generators for the double category that the programme should produce:

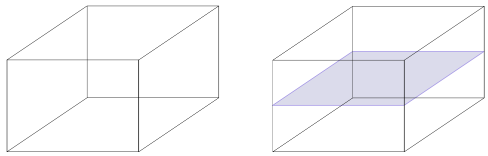


with their horizontal and vertical symmetric square: the identities and connections of the double category, and

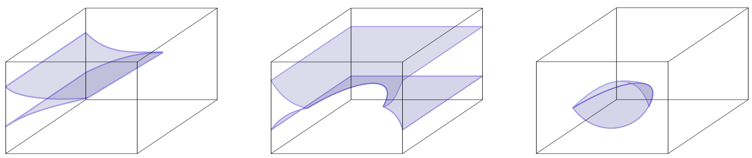


Their relations will be given by the corresponding Cerf theory, that we may call 2-Cerf, and that we present without the labellings. All the following diagrams should then have white bubbles and red rectangles in them, or better the corresponding resolutions of intersecting circles.

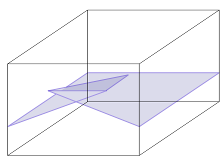
- no singularities or simple I_0 singularities:



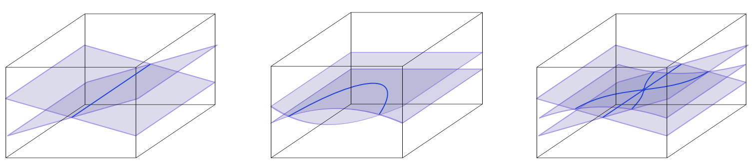
- I_1 singularities:



- I_2 singularities:



- non simple singularities:



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