WHAT CALCULUS DO STUDENTS LEARN AFTER CALCULUS?

by

TODD MOORE

B.S., Kansas State University, 2006
M.S., Kansas State University, 2009

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas

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Abstract

Engineering majors and Mathematics Education majors are two groups that take the basic, core Mathematics classes. Whereas Engineering majors go on to apply this mathematics to real world situations, Mathematics Education majors apply this mathematics to deeper, abstract mathematics. Senior students from each group were interviewed about “function” and “accumulation” to examine any differences in learning between the two groups that may be tied to the use of mathematics in these different contexts. Variation between individuals was found to be greater than variation between the two groups; however, several differences between the two groups were evident. Among these were higher levels of conceptual understanding in Engineering majors as well as higher levels of confidence and willingness to try problems even when they did not necessarily know how to work them.

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Dedication

I would like to dedicate this dissertation to my father, Col. Hylan C. Moore, M.D., who, despite passing away over twenty years ago, continued to motivate me every day to get my doctorate and whose hard work made sure that I did not face the same obstacles he did in getting his, and to my mother, Mary Sue Moore, who continued to push me every day so that it actually happened and whose hard work made sure that I did not squander the opportunities I was given.

Chapter 1 - Introduction

Overview of the Study

Folklore tells us that we often do not learn concepts from a mathematics course until the next class. At least one study suggests that this “appears to be not quite accurate, rather several courses may be necessary" (Selden, Selden, Hauk, & Mason, 1999). So, do students learn Calculus during the Calculus sequence? Obviously, they must learn some; otherwise they would not be successful in moving through the sequence. However, many studies, as well as every professor’s experience, show that Calculus students often have significant trouble learning the concepts involved in their classes (Selden, Mason, & Selden, 1989; White & Mitchelmore, 1996). Since some students presumably learn Calculus at some point, this raises the question: what Calculus do students learn after Calculus?

This study aims to begin to answer that question. More specifically, it aims to answer the question of how students’ level of understanding of “function” and “accumulation” (in terms of integration) change after the students have finished the Calculus sequence. These are two of the most important concepts in Calculus, and indeed in mathematics. The ability to understand “function” deeply is necessary for a strong understanding of mathematics (Sfard, 1991). Similarly, in order to understand integration, it is important that students have an understanding of the integral as accumulation (Thompson & Silverman, 2008). Despite these concepts’ importance and Calculus class’s stressing of them, teaching and learning of concepts like “function” and “accumulation” are still a problem (Tzur & Simon, 2004).

Students’ conceptual problems with functions are well documented (Markovits, Eylon, & Bruckheimer, 1986; Vinner & Dreyfus, 1989; Even, 1993). This includes problems with algebraic representations of functions (Clement, Lochhead, & Monk, 1981; Markovitz, Eylon, & Bruckheimer 1986), understanding functions from the definition (Vinner & Dreyfus 1989), and appealing to algebra to solve calculus problems even when the calculus concepts are known (Selden, Selden, Hauk, & Mason, 1999). These problems persist even as students advance through the Calculus sequence with the concept of “function” developing slowly throughout a student’s undergraduate career (Carlson 1998). Students’ understanding increases over time, though, and the number of misconceptions decrease from Calculus I to Differential Equations

(Selden, Selden, Hauk, & Mason, 1999). However, even the best students with a fair amount of math still do not totally understand the concept of "function" (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Carlson 1998).

These conceptual difficulties are common with accumulation as well. Unsurprisingly, students near the beginning of the Calculus sequence do not have a strong understanding of integration or accumulation (Selden, Mason, & Selden, 1989; Rasslan & Tall, 2002; Mahir, 2006). Difficulties persisted all the way up through Differential Equations (Selden, Selden, Hauk, & Mason, 1999; Bennett, Moore, & Nguyen, 2011). Specifically, these problems included notions of infinity and limits (Tall, 1993; Thompson & Silverman, 2008; Maharaj, 2010), notation (Orton, 1980; Mundy 1984; Tall, 1993; Cui, Rebello, & Bennett, 2007), and Riemann sums and accumulation (Orton, 1983; Cui, Rebello, & Bennett, 2007; Thompson & Silverman, 2008; Nguyen, 2011). Some of these problems persisted to the point where, in one study, engineers in the study "had almost completed their formal mathematical educations… leaving them limited opportunity in future mathematics courses to improve their non-routine problem solving abilities" (Selden, Selden, Hauk, & Mason, 1999).

A full understanding of "function" and "accumulation" requires the ability to apply it in other fields and within different contexts in mathematics (Markovitz, Eylon, & Bruckheimer 1986). Many of our Calculus students are taking Calculus to do just this. Since many Calculus students are taking the courses to become engineers or mathematics teachers, this study will focus on senior-level Engineering majors and Mathematics Education majors. Many Engineering majors have difficulties applying their mathematics understanding to their Physics and Engineering classes, however. "They tended to use oversimplified algebraic relationships to avoid using calculus because they do not understand the underlying assumptions of the relationships" (Cui, Rebello, & Bennett, 2005). Their difficulty using integrals in physics problems came mainly from their lack of understanding of what was being accumulated (Nguyen, 2011). And overall, many successful Engineering majors have less conceptual understanding of calculus concepts than we would hope (Selden, Selden, Hauk, & Mason, 1999). These problems are not limited to Engineering majors; Even (1993) and others have noted that many future teachers also have many misconceptions about functions

Since Engineering majors and Mathematics Education majors eventually gain an understanding of these concepts and they do not seem to do it by the end of the Calculus sequence, it is possible that they are gaining it as a result of backward transfer from their later classes in which they see the mathematics used in context. Backward transfer would happen when a student applies old knowledge to a new context and this application strengthens their understanding of the original concept, i.e., understanding is transferring from the new situation backward to the old.

There seem to be very few studies that research whether this backward transfer happens in the context of mathematics despite its possible importance in the way our students gain their expertise (Hohensee, 2011). More generally, there are very few studies that even address the transfer of understanding in post-Calculus mathematics and physics courses (Karakök, 2009). This study will attempt to fill both of these holes. There also appears to be some conflicting studies, with some finding poor understanding of mathematics teachers (Even, 1993) and some finding stronger performance (Vinner & Dreyfus 1989). This study will aim to evaluate just what levels of understanding Mathematics Education and Engineering majors attain.

Conflicting results of studies of mathematics students' understanding are unsurprising given the range of understandings of students even in Differential Equations that was demonstrated in the previous longitudinal study that this study will build upon (Bennett, Moore, & Nguyen, 2011). That study showed that in Differential Equations, students range from the lowest levels of conceptual understanding up into the higher levels of understanding. This study will build upon the longitudinal study by measuring the understanding of senior-level Mathematics Education and Engineering majors and compare them against the levels of understanding possessed by Differential Equations students.

Research Questions

The goal of this study is to attempt to measure any amount of back transfer among Mathematics Education majors and Engineering majors from their upper level major courses back to Calculus that occurs between the time they finish the Calculus sequence and graduation. As such, there are two main questions:

1. Does back transfer occur?
2. If so, does the application of mathematics in different contexts or majors cause different levels of back transfer?

Brief Description of the Methodology

This study will use quantitative and qualitative methods to answer the research questions. To begin, Mathematics Education and Engineering majors will be selected on a volunteer basis to participate in an hour-long, one-on-one, conversation-style interview that will cover calculus based questions aimed at discovering students' understanding of "function" and "accumulation". The students will be rated on a modified APOS scale and the results will be compared against the levels of understanding of Differential Equations students as rated by the previous longitudinal study (Bennett, Moore, & Nguyen, 2011). Any major change in the distribution will be taken as evidence of what happens to students' understanding after they finish the Calculus sequence. A decrease in understanding could indicate that students forget what they are not explicitly learning anymore; an increase should indicate back transfer.

The differences between the movements of the Engineering majors versus the Mathematics Education majors will also be studied to determine if the different contexts the students used their calculus knowledge in led to different levels of back transfer. Roughly speaking, the Engineering majors apply their calculus knowledge to real-world situations, whereas the Mathematics Education majors apply their calculus knowledge to higher-level, abstract mathematics classes.

The quantitative portion of this study will consist of a non-negative matrix factorization made from a vocabulary matrix. This matrix will be made up of how many times each student said each word from the list of words used in the interviews. The goal of the non-negative matrix factorization is to see if it can isolate groups of students based on common vocabularies. For example, if Engineering majors use a different vocabulary than Mathematics Education majors, or if students with strong conceptual understanding differ from those with weak understanding, then the matrix factorization should show that.

Limitations of the Study

All studies have their limitations. The limitations of this study fall into five categories: self-selection issues, demographic differences between the two groups, different sizes of the two groups, comparison with the previous longitudinal study, and the limited scope of the study.

Since this study will be run on a volunteer basis, there is the always-present issue with who chooses to participate. There are also the larger self-selection issues with who chooses to become Engineering majors versus Mathematics Education majors. Perhaps naturally stronger students choose one major over the other.

Related to self-selection problems are the demographic issues involved with the differences between the two majors. Engineering tends to have a higher percentage of males versus females compared to Mathematics Education. While differences in the numbers of each gender interviewed will be attempted to be kept to a minimum, the fact that participants are chosen on a volunteer basis will complicate this.

The third limitation of this study is the difference sizes of the groups. The university where this study will take place has a large Engineering program while the numbers of Mathematics Education majors are much smaller.

The fourth limitation of this study is actually a limitation of the previous longitudinal study upon which this study draws. The previous longitudinal study did not differentiate between majors; it lumped all students together in each level of understanding. Therefore, the baseline that this study compares the two majors against is an imperfect baseline as it consists of both majors and is being used to compare against each major individually. This, likely, will not cause that much trouble, since the students will not have had vastly different college experiences by the time they are in Differential Equations.

The fifth limitation of this study is the limited scope that the researcher will be able to take on with this study. The goal is to measure whether back transfer is happening and, if so, whether context affected the amount of back transfer. Therefore this study aims to measure possibly different levels of understanding possessed by seniors compared to students in Differential Equations as well as compare Engineering seniors with Mathematics Education seniors. The goal is not to explain how this back transfer occurs or to explain why different contexts might correspond to different levels of back transfer. Answers to those questions would require much more extensive and directed research than will be possible in this study and would constitute a possible area of future research.

Chapter 2 - Literature Review

The purpose of this study, in general, is to examine what happens to students' calculus knowledge after they have left the Calculus sequence. More specifically, this research attempts to measure senior-level students' conceptual understanding of function and accumulation in terms of APOS theory. This is done with the aim of seeing if the group as a whole has forgotten or retained their understanding, or if, on the other hand, using the mathematics in their higher-level classes has actually transferred backwards to increase the students' conceptual understanding.

This chapter will give an overview of research relevant to this study. This includes a review of the research indicating students' difficulties with the concepts of function and accumulation. To explain the tool used to rate the students' conceptual understanding, a review of APOS theory is provided along with some of its applications. Finally, a sampling of some of the various theories of transfer is given, as well as what research has been done investigating backward transfer.

Difficulties in Learning Function and Accumulation

"Function" and "accumulation" are two of the most important concepts in any Calculus class. However, these are two concepts that students have a great deal of trouble learning (Tzur & Simon, 2004; Tall, 1993; White & Mitchelmore, 1996). Even when students do learn these topics, they learn them procedural so that they can pass the test rather than learning them conceptually and deeply (Tall, 1993).

Functions

There has been a significant amount of research on students' understanding of function. Most have come to the conclusion that students do not understand functions in the way that mathematics professors would hope (Even, 1993; Markovits, Eylon, & Bruckheimer, 1986; Vinner & Dreyfus, 1989). Carlson (1998) studied College Algebra students among others in a cross-sectional study and came to the conclusion that College Algebra students had a narrow view of functions that included the view that all functions should each be definable by a single, algebraic formula. This would exclude most complicated functions, piecewise functions, many discontinuous functions, etc.

This adherence to algebraic formulae is troublesome because students have more problems gaining conceptual understanding when working with algebraic representations of functions and transferring from graphical representation to algebraic is even harder than the reverse (Markovitz, Eylon, & Bruckheimer 1986). Clement found that “many students have difficulty expressing relationships algebraically” (Clement, Lochhead, & Monk, 1981).

Students’ problems with functions are not limited to algebraic representations. Carlson (1998) studied, in part, students who had passed College Algebra and concluded that many of the students have a low level of conceptual understanding, possess a pointwise view of functions, and see the evaluation of functions simply as substitution and as a series of memorized procedures. These problems extend to when teachers introduce functions by emphasizing the formal definition. There is evidence that even after studying the formal definition of “function”, and even when students could give the formal definition of “function”, they still did not understand it and, further, that a reliance on emphasizing the formal definition may be to blame (Even, 1993; Vinner & Dreyfus, 1989).

Conceptual problems in understanding “function” are not limited to those who have limited mathematical backgrounds. Clement found that "even after taking a semester or more of Calculus, many students have difficulty expressing relationships algebraically" (Clement, Lochhead, & Monk, 1981). This aligns with Carlson’s (1998) findings that even the best College Algebra and second semester Calculus students have conceptual difficulty with many of the topics in their respective classes. She goes on to say that while second semester Calculus students have a stronger, more general view of functions they still encounter difficulties with covariation and the dynamic behavior of functions.

Even students with stronger mathematics backgrounds do not understand “function” as strongly as professors would hope (Breidenbach, Dubinsky, Hawks, & Nichols, 1992). In fact, as students advance through their college careers, even the basic concept of “function” develops slowly (Carlson, 1998). This extends to those that will teach mathematics in the future. Many future teachers still retain numerous misconceptions about functions including the views that functions are equations and that functions are "known", simply-defined relationships (Even 1993). One might hope that these misconceptions disappear before the end of a students’ college career, however, in the same study, Even said that even secondary teachers do not necessarily have a “modern concept of function”. This is not just a shortcoming of future teachers; many successful engineering majors have lower conceptual understanding of calculus topics than professors would hope (Selden, Selden, Hauk, & Mason, 1999). This is also not a shortcoming of non-mathematicians or undergraduates. Carlson (1998) found that while graduate students had stronger covariational understanding of functions as well as a better handling of multiple representations, even the best students do not completely understand "function" to the extent that one would like.

So why is conceptual understanding of "function" really that important if most of our students, including those who will become teachers of mathematics, do not fully understand function? "The ability of seeing a function or a number both as a process and an object is indispensible for a deep understanding of mathematics" (Sfard, 1991). Beyond learning mathematics for the sake of learning mathematics, a conceptual understanding of function requires, and is required for, the use in fields other than mathematics along with different contexts within mathematics (Markovitz, Eylon, & Bruckheimer, 1986). In particular, many of our students are taking mathematics classes to apply them in the future to the context of Physics or Engineering. However, students often have trouble interpreting their mathematical understanding in a physics context (Tuminaro, 2004). This does not necessarily reflect that students do not understand the mathematics; even after students have demonstrated in a first course in Calculus that they know the concepts, many still cannot solve non-routine problems (Selden, Mason, & Selden, 1989). However, competency in procedural skill without conceptual understanding compounds problems when applying mathematics to Physics because it prevents students from bringing their procedural competency to bear. "They tended to use oversimplified algebraic relationships to avoid using calculus because they do not understand the underlying assumptions of the relationships" (Cui, Rebello, & Bennett, 2005). So it is important that students have a conceptual understanding of functions if they hope to pursue a field that requires the use of functions in problems solving.

Accumulation

Students' conceptual difficulties are not limited to the topic of "function"; these show up in the topic of "accumulation" as well. Accumulation in this study is thought of in terms of integration, i.e. the accumulation of a measured quantity when using integration to solve problems. This is a crucial concept, since a student cannot fully understand integration without the ability to view integration as accumulation (Thompson & Silverman, 2008). This is compounded by students' lack of understanding of function because in order to understand accumulation, students need a covariational view of function (Thompson & Silverman, 2008).

Unsurprisingly, students' conceptual problems with accumulation and integration start early. Few beginning students have a conceptual understanding of integration (Rasslan & Tall, 2002). This is not limited to beginning students, however. Students who have passed a first year in Calculus do not have a proper conceptual understanding of integration (Mahir, 2006), and many cannot solve non-routine problems even when they know the concepts involved (Selden, Mason, & Selden, 1989). One might expect this from those with limited Calculus experience; however, while Differential Equations students showed progress over first year Calculus students, most were still unable to solve non-routine Calculus problems (Selden, Selden, Hauk, & Mason, 1999). When confronted with these non-routine problems, students preferred to use algebra over calculus even when they knew how to do the calculus (Selden, Selden, Hauk, & Mason, 1999; Cui, Rebello, & Bennett, 2005).

So, general problems working with and understanding accumulation are not limited to the first courses in college. There are many specific problems students have with these concepts as well. One of these is that students have trouble with infinity and limits (Tall, 1993; Maharaj, 2010; Orton, 1983; Thompson & Silverman, 2008). Students have trouble with integral notation (Tall, 1993), evaluating integrals when the function is negative or the upper limit is less than the lower limit (Orton, 1980), and evaluating integrals when the function involves an absolute value (Mundy, 1984). Students also seem to have trouble understanding what each element means in an integral problem (Cui, Rebello, & Bennett, 2007).

Nguyen (2011) studied students in an introductory, calculus-based engineering physics course to investigate their difficulties in applying integration in a physics context as well as what hints proved helpful in helping them solve the problems. He found that students encountered significant problems in doing this, including failing to view an integral as a Riemann sum and, further, that students' difficulties using integrals in physics largely originated from their lack of regard or their lack of understanding of what quantities were being accumulated. It was not that the students could not "do the math"; he said that students are usually "very fluent" while computing a mathematical problem but have little to no conceptual understanding of the ideas behind that computation. More specifically, many students could find the accumulation of charge by setting up and carrying out an integral, but they did it without considering, conceptually, what was being accumulated. More generally, most students did not think about the integral as an accumulation at all when solving the problems. Even more troubling, Nguyen found that students might not understand the concept that the integral of a function calculates the area under the curve of that function.

Orton (1983) had some similar findings to Nguyen, saying that a majority of students, despite a procedural competency with Riemann sums, viewed the limit of a Riemann sum as an approximation to an integral instead of being equal to it. Along with the importance of Riemann sums to integration (Orton 1983; Thompson & Silverman, 2008), these problems point to a larger problem with the concept of accumulation. Thompson and Silverman (2008) took this a step further in saying the problem with "the idea of accumulation functions is that it is rarely taught with the intention that students actually understand it."

Despite the possible truth of that last statement, these problems are not permanent roadblocks, however. As students move through the Calculus sequence, from Calculus to Differential Equations, the number of these Calculus misconceptions decreases (Selden, Selden, Hauk, & Mason, 1999). However, Engineering majors in this study still exhibited worrisome abilities to solve non-routine problems and since they "had almost completed their formal mathematical educations", this leaves them "limited opportunity in future mathematics courses to improve their non-routine problem solving abilities". So, even many successful students exhibit conceptual difficulties all the way through their college careers.

APOS Theory

After so much talk about what conceptual understanding the students possess on particular Calculus topics, one might ask how to measure conceptual understanding. The conceptual framework chosen for this research is APOS Theory. This was done because it was designed for just this type of conceptual research (Dubinsky & MacDonald, 2001; Weyer, 2010; Tall, 1999), APOS is used by numerous other studies that this study study can compare to (Tall, 1999; Asiala, et al., 1996; Weyer, 2010; Maharaj, 2010; Kabael, 2011), and it was necessary to line up with the previous longitudinal study that this one builds on (Bennett, Moore, & Nguyen, 2011).

APOS stands for Action-Process-Object-Schema for the levels of understanding that students move through as they move toward mastery of a subject. APOS theory was proposed by Dubinsky and was based as an extension to the collegiate level of the last stage of the cognitive construction developed by Piaget for children up to about 16 to describe how actions become generalized into processes, reified into mental objects, and finally become placed in an overall schema (Dubinsky & MacDonald, 2001; Tall, 1999; Weyer, 2010). APOS theory was designed not just as a model to measure the levels of students' learning and conceptual understanding, but also as an attempt to understand "what an educational program can do to help in this learning" (Dubinsky & MacDonald, 2001). APOS theory is suited well as a framework for analyzing a wide array of mathematical learning pertaining specifically to more complex concepts (Tall, 1999; Weyer 2010). Tall (1999) discussed biological and neurological behaviors and concluded that there may be biological evidence that backs up the APOS approach, however also neurologic reasons to believe it may be imperfect when applied to some specific other fields.

The first level of the APOS framework is the Action level. "An action is a transformation of objects perceived by the individual as essentially external and as requiring, either explicitly or from memory, step-by-step instructions on how to perform the operation" (Dubinsky & MacDonald, 2001). Action does not require a deep level of thought and little to no understanding; if a student is at the Action level, they are just carrying out procedures. Specifically, if a student is unable to understand a function, or to understand that a situation involves a function, without a specific formula or equation, then that student is likely at Action (Asiala, et al., 1996). At this level, the student views functions as an equation or formula to be evaluated (plug-and-chug) and any manipulation of the function is limited to manipulation of a graph or equation.

"When an action is repeated and the individual reflects upon it, he or she can make an internal mental construction called a process which the individual can think of as performing the same kind of action, but no longer with the need of external stimuli" (Dubinsky & MacDonald, 2001). Contrasted with the need for an external control at the Action level, at Process processes are internal and under one's control; therefore, students at Process can describe or reverse the steps without performing them (Asiala, et al., 1996). When working with functions, evidence that a student is at Process include not needing domain and range to be restricted to numbers, not needing specific, defined functions to be able to imagine operations with functions (Breidenbach, Dubinsky, Hawks & Nichols, 1992), viewing a function as transforming inputs to outputs rather than one input going to one output, and the ability to compose or invert functions (Asiala, et al., 1996).

"An object is constructed from a process when the individual becomes aware of the process as a totality and realizes that transformations can act on it" (Dubinsky & MacDonald, 2001). A process becomes an object when it is viewed as a total thing that can be manipulated (Asiala, et al., 1996), and only when a student can think of the process as a fully-fledged object, can one say that the concept has been reified (Sfard, 1991). For example, being able to think of a set of all positive functions requires an object view of function because one first has to be able to think of a process that only outputs positive values and then be able to think of those infinitely many different and not specifically defined processes as objects that can be collected into a set. Because so much upper level mathematics requires the manipulation of functions as objects, Object level understanding of "function" is important to be able to understand mathematics (Sfard, 1991). However, the shift from process to object is difficult and not many teaching strategies have been successful in helping students make this transition (Asiala, et al., 1996).

"A schema for a certain mathematical concept is an individual's collection of actions, processes, objects, and other schemas which are linked by some general principles to form a framework in the individual's mind that may be brought to bear upon a problem situation involving that concept" (Dubinsky & MacDonald, 2001). For example, functions and operations on them can be made into function spaces and applied to things like dual spaces, spaces of linear mappings, and function algebras (Asiala, et al., 1996). While the final goal is for students' understanding to reach the level of Schema (Weyer, 2010), the levels aren't necessarily learned linearly in this order, but instead are more of a "partially ordered sequence" (Dubinsky & MacDonald, 2001; Weyer, 2010). Unlike the other three levels that have stronger definitions and plenty of supporting research, the idea of the Schema level has not been as strongly formulated (Asiala, et al., 1996).

APOS theory is not without its limitations. Research into where students rate in terms of APOS theory can be difficult because APOS only measures to which level a student is capable of understanding a concept, it does not offer any information to which level a student will respond since a student will not always respond to their highest level of understanding (Dubinsky, 2000). "The fact that an individual possesses a certain mental structure does not mean that he or she will necessarily apply it in a given situation" (Maharaj, 2010). One reason this happens is that, when solving a problem, students tend to fall back on earlier, more familiar mathematical techniques with which they are more comfortable (Selden, Selden, Hauk, & Mason, 1999). Another is that regardless of comfort level, a student “can have the knowledge, but not think of using it" (Selden, Selden, Hauk, & Mason, 1999). In general, quantitative results in APOS will not be entirely reliable because of this inherent mobility of students through the levels as they work a problem, but qualitative results in APOS are time consuming, necessitating smaller sample size (Dubinsky, 2000).

Quite a few researchers have looked at APOS in mathematics; however most have looked at the movement from Action to Process. Selden, et al. (1999) studied Differential Equations students and found that they have a tendency to fall back on lower-level techniques including relying more often on arithmetic and algebraic techniques rather than calculus ones. This may have been less to do with a lack of understanding and more to do with students having the proper knowledge, but not thinking of using it (Selden, Selden, Hauk, & Mason, 1999).

There appears to be less research on movement in the higher levels, specifically little research of the Schema stage. Weyer (2010) studies 22 undergraduate students in a Discrete Mathematics course; most started at Action with some at Process. She said "some conclusions can be drawn from the findings of this study. The findings show that even after years in math classes, education is not getting people past the Process stage.” She also observed that Mathematics majors are usually the ones likely to achieve Object and Schema stage.

Kabael (2011) looked at Mathematics Education students’ understanding of single and two-variable functions in a two-variable calculus class. That study found that students range all the way from Action up to Object level but found that there were probably none in Schema. Further than that, Sfard (1991) found that the Object level is difficult to reach and many do not do so.

Dubinsky (2000) clarified Schema in his discussion of cosets: Schema "will be coherent in the sense that the individual will have some means (explicit or implicit), perhaps the formal definition, of determining, for any phenomenon encountered, what relationship it has to her or his conception of cosets." So in this way, Schema is what allows students to effectively evaluate new information and new situations and know how they fit into their current understanding of the concept.

Theories of Transfer

Much of the research on the transfer of knowledge between contexts began with Thorndike's study of identical elements between problems (Thorndike & Woodworth, 1901; Thorndike, 1906). He took a limited view on transfer, saying it occurs by applying knowledge from one problem to identical elements in another problem. This is the view that became the traditional view of transfer. Traditional transfer studies "privilege the perspective of the observer and rely on models of expert performance, accepting as evidence of transfer only specific correspondences defined a priori as being the "right" mappings" (Lobato, 2006). Gick and Holyoak (1980) described transfer, saying it compares situations the expert perceives to be analogous and discusses how students will discover these identical elements to solve the new problem. Unfortunately, this view led to the opinion that transfer does not happen in a broad way because traditional transfer is difficult for students (Bransford & Schwartz, 1999; Clement, Lochhead, & Monk, 1981).

In order to have transfer, a student must be able to understand the concept in the first context (Bransford & Schwartz, 1999). Nguyen (2011) found that student's problems working with integrals in physics problems came from their lack of understanding or their disregard for what was being accumulated. Transfer does not have to be limited to moving to a new field, it can involve transferring within a field; many students who have passed a first Calculus course cannot solve non-routine problems even when those problems involved known ideas (Selden, Mason, & Selden, 1989). Rebello, et al. (2007) looked at transfer from a structured context (an undergraduate Mathematics course) to an unstructured one (approximated by an undergraduate Physics course). They reasoned that if students cannot transfer their mathematical understanding to a Physics class, they are unlikely to be able to transfer it to real-world situations. They found that "the main difficulty that students appear to have does not lie in their lack of understanding of mathematics per se, rather it lies in their inability to see how mathematics is appropriately applied to physics problems", i.e. lack of transfer in the traditional sense. In fact, the common complaint among Physics and Engineering professors that they receive their students out of mathematics classes without sufficient mathematics preparation likely is due to the students' inability to transfer their mathematics knowledge to a Physics context in the traditional view (Cui, 2006; Rebello, et al., 2007).

Cui (2006) studied students taking a second semester of an Engineering Physics course and found that Calculus students retained their mathematics knowledge well and solved calculus problems easily; however, they had difficulties applying their knowledge to physics problems despite having seen similar problems previously. Cui found that one of the problems was that students could not distinguish the important factors in a problem and so they resorted to novice-level strategies to solve them. For example, the students could not decide the bounds and variable of integration despite competency with integration in Calculus. Tuminaro (2004) made a similar finding, saying a major cause for students' failure in applying mathematical knowledge to physics problem solving was not the lack of the necessary knowledge but the inability to apply that knowledge in a physics context.

In another study involving traditional transfer, students who had reviewed fractions using paper strip manipulatives did not draw on this experience the next day when comparing similar fraction problems (Tzur, 2004). Tzur blamed this on students' learning with these manipulatives as being participatory whereas answering the questions required an anticipatory strategy. But regardless the reason, the end result was another failure of students to transfer their knowledge in the traditional sense. This is unsurprising since "transferring from analogous examples can be challenging even for sophisticated and motivated learners" (Gentner, Loewenstein, & Thompson, 2004).

Nguyen (2011) blamed this phenomenon in his study on an unbalance between conceptual and procedural knowledge leading to students' to have difficulties in applying their mathematical knowledge to physics. For example, many students could not recognize work as the integral of force despite having been taught that, and those that could do so did not necessarily have a conceptual understanding of what they were doing.

Gick & Holyoak (1980) found that traditional transfer is not likely even if students remember previous example because they do not necessarily see the usefulness of the previous example. They presented students with the following Attack-Dispersion story:

> "A general wishes to capture a fortress located in the center of a country. There are many roads radiating outward from the fortress. All have been mined so that while small groups of men can pass over the roads safely, any large force will detonate the mines. A full-scale direct attack is therefore impossible. The general's solution is to divide his army

> into small groups, send each group to the head of a different road, and have the groups converge simultaneously on the fortress."

They then presented the students with the following Radiation problem:

> "Suppose you are a doctor faced with a patient who has a malignant tumor in his stomach. It is impossible to operate on the patient, but unless the tumor is destroyed the patient will die. There is a kind of ray that can be used to destroy the tumor. If the rays reach the tumor all at once at a sufficiently high intensity, the tumor will be destroyed. Unfortunately, at this intensity the healthy tissue that the rays pass through on the way to the tumor will also be destroyed. At lower intensities the rays are harmless to healthy tissue, but they will not affect the tumor either. What type of procedure might be used to destroy the tumor with the rays, and at the same time avoid destroying the healthy tissue?"

The solution to the Radiation problem is analogous to the Attack-Dispersion story: simultaneously fire several low-intensity rays at the tumor from many different directions. Doing this will prevent dangerous amounts of rays from passing through any healthy tissue leaving them unharmed, and still have the effect of concentrating the rays on the tumor in sufficient intensity to destroy the tumor. Students were presented with the Attack-Dispersion story and asked to solve the Radiation problem. Despite the nearly identically analogous situations, the study found that students still failed to transfer their knowledge without hints in order to be able to solve the problem.

Selden, et al. (1999) also recorded a failure in traditional transfer in their study of Differential Equation students solving non-routine problems (Selden, Selden, Hauk, & Mason, 1999). They found that Differential Equations students were still mostly unable to solve non-routine calculus problems; they tended to use algebra over calculus when solving non-routine problems despite a sufficient understanding of calculus. However, they did show progress over first year calculus students. This last observation, along with the many studies failing to find traditional transfer, indicates that perhaps a new view of transfer is needed.

It would seem that the traditional definition of transfer would be the common sense definition: because a student has seen something before, they should be able to solve the same problem in a different context. That would seem to be what learning is all about; however, this view of transfer is hard to measure and appears to indicate that students do not transfer their knowledge. Traditional transfer does not seem to happen even though we have all experienced it in our own lives, so this must be a failure in the way we have defined transfer. We must have to define it in a different way, not least of all because backward transfer (the purpose of this study) does not make sense with the view of traditional transfer. In traditional transfer, you learn something and then transfer it over to a new context by means of analogous elements. Any effect of this new situation on the old one would represent new learning and would not take effect until one looked back at the old situation, in which case it would just be more transfer. Luckily, many researchers have come to the conclusion that the traditional view of transfer ought to be more broadly defined (Bransford & Schwartz, 1999; Lobato & Seibert, 2002; Lobato, 2006; Rebello, et al., 2007; Hohensee, 2011).

Alternative Views of Transfer

The traditional view of transfer is oriented from the view of an expert; this is the view that has dominated earlier research (Hohensee, 2011). Part of the problem with the traditional view is that what experts designate as a surface feature can give students more difficulties than one might expect (Lobato & Seibert, 2002). Lobato (2003) saw this problem and shifted from an "expert" point of view to an "actor's" point of view.

Actor oriented transfer (AOT) looks at learning and transfer more broadly (Lobato, 2006). Traditional transfer relies on students solving problems that experts have decided have similar structural forms but different surface features (Lobato, 2003, 2006). Instead of seeing if the students can view the situation in the way an expert sees it as similar, AOT looks at how a student views situations as similar (Lobato, 2003). It considers any influence of prior experiences on students' handling of new problems (Lobato, 2006). An actor's view of transfer gives another difference from traditional transfer; the traditional view only counts "correct" strategies as evidence of transfer, whereas in AOT even incorrect strategies can show evidence of earlier learning (Lobato, 2003).

Lobato and Seibert (2002) conducted a study of 8th – 10th graders where they examined slopes in terms of making ramps of certain heights. They performed two studies: one from a traditional view and another from the view of AOT. They failed to find transfer from a traditional point of view; students recalled the definition of slope but rejected it as irrelevant to the problem. The students viewed the similarities between the presented problems in a different way than the experts expected them to. However, in terms of AOT, the students were more successful. The AOT study found substantial influence of previous learning on student's understanding of proportional reasoning and over several sessions the students greatly improved their proportional reasoning.

In a similar paper, Lobato (2003) described her study where students learned to find slopes of staircases and then applied this to slopes of playground slides in an attempt to measure transfer. From a traditional point of view, the students should have looked past the surface differences of staircases versus slides, noticed the underlying similarities, and been able to transfer their understanding to the new problem. However, the student fared poorly from the traditional point of view; only 40% of the students showed evidence of transfer. The students viewed the similarities and differences between the two problems differently than the experts did. From the point of view of AOT, the study revealed significant evidence of transfer, with every student showing some amount of transfer.

Cui (2006) studied students in a second semester Engineering Physics course. The students had a sufficient mathematical skill and solved calculus problems easily, but had difficulties applying this knowledge to physics problems despite seeing similar problems previously. From a traditional point of view, this would represent a failure to transfer their mathematical skill. The students could not pick out the important features in the problem and, as a result, resorted to novice-level strategies to solve them. For example, students could not decide on the bounds and variable of integration despite being able to do so in a mathematical context. However, a strong correlation between students' calculus and physics performances on a physics exam indicates AOT despite the poor traditional transfer.

Karakök (2009) conducted a study of junior year Physics students which he says is one of the few, or the only, study of transfer in a post-Calculus mathematics or physics course to that point. Karakök looked at transfer of eigenvalues and eigenvectors to physics from the point of view of AOT. Again, traditional transfer was elusive with almost none of the participants exhibiting any, but once again, the students exhibited signs of AOT.

Bransford and Schwartz (1999) were of the same view that evidence of transfer emerges when we adapt what we are looking for to the way students view problems. They described the traditional point of view of transfer as relying on "sequestered problem solving" (SPS). That is, after a student learns a concept, the traditional point of view then has students directly apply this knowledge to solve a new problem with no context, no discussion, and no repeated measurement beyond the first attempts to solve the problem. Bransford and Schwartz theorized that SPS and the related "direct application" theory of transfer was responsible for the belief that transfer does not happen.

They, instead, take a more actor oriented approach and broaden transfer to include "preparation for future learning" (PFL), which is more akin to practicing how to learn and looks at transfer as what effects it has on how people learn new information. Rather than focus on how far off novices are from the answers experts expect, PFL focuses instead on how these people are more prepared than those without the previous experiences. Instead of measuring to what degree students can answer new problems, PFL focuses on "evidence for useful learning trajectories".

They give the example of a recently graduated mathematics educator. The educators do not enter the classroom for the first time as experts, but instead with the preparation to learn to be experts. College has not taught them exactly the skills they will need to directly transfer to the classroom as much as it has taught them how to assess what skills they need and how to achieve them. They say transfer does not reveal itself easily in the form of spontaneously being able to solve new problems, but instead in how students use strategies to learn the correct answers and how they "critically evaluate new information".

According to Bransford and Schwartz, PFL focuses on assumptions students make and the sophistication of their solution strategies. In this way, PFL looks for active conceptual change rather than the passive persistence of knowledge and behaviors that SPS does. This broader view is why PFL allows for the discovery of transfer that would be missed by a SPS view of transfer and so PFL makes evidence of transfer more visible.

Singley & Anderson (1989) looked at transfer in the context of text editors. Their research had similarities to PFL. They asked how learning a text editor affected the learning of a second text editor. They found that the benefits of previous experience with the first text editor did not reveal themselves immediately, i.e. that the study failed to find transfer from the traditional point of view. However, the benefits of the experience were greater on the second day than the first. The experience with the first text editor affected the way the students viewed the learning and what strategies they used, allowing them to learn the second editor more easily. This is evidence of transfer from the actor's point of view as well as evidence of PFL.

Another view of transfer from the view of the actor is the concept of Horizontal Transfer versus Vertical Transfer (Rebello, et al., 2007). They define Horizontal Transfer as the ability to directly apply knowledge to a problem that is structured similarly to the way knowledge was learned, i.e., the ability to do the plug-and-chug problems at the end of the section. This corresponds roughly to the view of transfer as SPS as well as the traditional, expert view of transfer. Horizontal Transfer often just relies on pattern matching.

Vertical Transfer is defined as an ability to apply intuition obtained from learning to a non-structured problem to create a solution strategy. Rather than aligning a predetermined knowledge structure as in Horizontal Transfer, the student makes one up on the spot through repeated constructions and deconstructions of strategies and associations. Vertical Transfer involves choosing an appropriate representation from several available ones or making a new one. Vertical Transfer is similar to PFL and AOT.

So in terms of research, Vertical Transfer would likely be the more fruitful, since it seems to align with AOT and PFL. However, for the same reason, this makes Vertical Transfer harder to test for than Horizontal Transfer since it requires more in depth investigation and measurement of longer term effects. Horizontal Transfer is easier to test for because it only requires testing students with problems, such as the exercises at the end of the chapter in a textbook. In the end, though, few book problems, but most real-world problems require Vertical Transfer rather than Horizontal Transfer (Rebello, et al., 2007).

A similar, but possibly unknown to Rebello, et al., construction was one of Lateral and Vertical Transfer (Gagné, 1966). Lateral Transfer is transfer of knowledge spread broadly over situations at the same level of complexity, such as transfer between similar problems or between contexts. Gagné's Vertical Transfer is the transfer of knowledge from lower-level to higher-level skills that are in a prerequisite relationship. So Gagné's Lateral and Vertical Transfer are similar to the Vertical and Horizontal Transfer of Rebello, et al., respectively.

Regardless of exactly which flavor of transfer is most correct or most convenient, for the purposes of this study, it is only important that transfer will be viewed from an actor's perspective. Hohensee (2011), who also conducted a study of backward transfer, adopted an actor's point of view of transfer "because the primary interest was to investigate all changes in reasoning, not just changes that result in expert-like performance."

Back Transfer

Anecdotally, many students have experienced the situation where they have successfully taken a multivariable Calculus course, but they did not really conceptually understand it until they took an Electromagnetism Physics course and saw the mathematics in context. This would mean that transferring the knowledge to a new context actually strengthened the previous knowledge, i.e. from an AOT view, transfer happened backward. This is what this paper will refer to as "back transfer" or "backward transfer".

Beyond anecdotal evidence, there is some logical reason to believe that this phenomenon happens. We know students eventually get to Schema since all experts were once students who did not start at Schema, and we also know that students do not reach this level by the end of the Calculus sequence (Carlson 1998; Kabael, 2011). Since many experts, including many Engineers, do not take further mathematics classes after the Calculus sequence and since they do not reach Schema in the Calculus series, they likely do it by back transfer driven by using the mathematics in the context of their field.

Selden, et al. (Selden, Selden, Hauk, & Mason, 1999) presented a similar sentiment, saying, "Our three studies suggest the folklore that one only really learns a course's material in the next course appears to be not quite accurate, rather several courses may be necessary." They also stated that Engineers in this study "had almost completed their formal mathematical educations… leaving them limited opportunity in future mathematics courses to improve their non-routine problem solving abilities". Since Engineers presumably learn to solve non-routine problems, it seems that they are gaining this mathematics competency after they have left the mathematics courses. These observations seem to be evidence of back transfer in mathematics.

Linguistics

Most of the previous research on back transfer has been conducted in the field of Linguistics (Hohensee, 2011). This probably should not be too surprising given that the old adage that learning a second language makes learning the third much easier indicates Linguistics is a fruitful field for transfer research. Several studies have found evidence of back transfer (Tsimpli, Sorace, Heycock, & Filiaci, 2004; Camarata, Nelson, Gillum, & Camarata, 2009; Su, 2001; Marton, 2006). This includes both studies that found productive and unproductive back transfer (Tsimpli, Sorace, Heycock, & Filiaci, 2004), as well as studies that found forward and back transfer happening in the same students (Su, 2001).

Camarata, et al. (Camarata, Nelson, Gillum, & Camarata, 2009) studied children with specific language impairments. The treatment group was given expressive language intervention with no auditory processing training activities and they found that this group made significantly greater gains in receptive language skill than did the control group. They concluded that this was evidence of productive backward transfer from expressive language skills to receptive language skills.

Su (2001) compared bilingual Chinese speakers (whose second language was English) and bilingual English speakers (whose second language was Chinese) with monolingual Chinese and English controls. Su found that intermediate and advanced EFL (English as a Foreign Language) speakers (whose first language is Chinese) had different word order strategies when processing their first language (Chinese) than monolingual native Chinese speakers. This indicates that bilingual speakers apply experience from their new language to process their old language, resulting in back transfer.

Marton (2006) gives an example that resembles the findings of Su. He gives an example of a Cantonese speaker hearing a word for the first time who cannot determine the vocal aspects (sound and tone) of the word. The speaker then hears a second word with the same sound but different tone and immediately knows the tone of the second word and that of the first. In this way, there is immediate forward and backward transfer as each word reinforces the other.

Other Research

Gentner, Loewenstein, and Thompson (2004) conducted a study with professional management consultants. The consultants were split into two groups: a comparison group and a separate case group; both groups were given two analogous situations. The comparison group was given both situations and asked to explain the situations and explore the similarities between the two. The separate case group was given the two situations one at a time and after each were asked to explain the situations. After these two situations were discussed, both groups were asked to recall an example from their experience that resembled the situations. Despite both groups having similar amounts of experience, the comparison group was able to retrieve better and more appropriate experiences from their memory than was the separate case group. They also ran a similar experiment where the separate case group was given the analogy between the situations, but the group that was given the comparison still did not fare as well as the comparison group. These results indicate that the comparison group is applying this new information to reinterpret their old experiences in light of the new comparison. In this way, having students compare two partially understood situations themselves facilitates backward transfer as well as forward transfer more effectively than when they are told the comparison.

Research in Mathematics

Seemingly, the only major study of back transfer in Mathematics was conducted by Hohensee (2011). He defined backward transfer as the influence of new knowledge on prior knowledge by gaining and generalizing the new knowledge. Hohensee gave a pre-interview on linear functions, then instructed students on quadratic functions, and followed up with a post-interview on linear functions. His first iteration found unproductive effects of back transfer where instruction on quadratic functions actually decreased students' effectiveness evaluating linear functions. While this is not the type of back transfer that educators hope will happen, it is still effective evidence that back transfer is happening.

Hohensee then revised his instruction to try to correct the problems and attempt to use quadratic functions to deepen student understanding of linear functions. This time it was successful: after instruction on quadratic functions, students reasoned more productively about linear functions including proportional reasoning, drawing diagrams, and having meaningful explanations of division. Students reasoned more productively with changes in quantity during post-interview with linear and quadratic functions despite not receiving any more instruction on linear functions. In particular, the average students and below average students benefitted the most in their understanding of linear functions just by studying quadratic functions. Especially in light of the first, less than successful iteration, this shows not only that backward transfer occurs, but also the important effect of proper instruction on eliciting it.

Previous Longitudinal Study

This study aims to build upon a previous longitudinal study (Bennett, Moore, and Nguyen, 2011) that rated students' understanding of "function" and "accumulation" on the APOS scale as they moved through the Calculus sequence. That study began with a group of volunteers selected from Calculus I each of whom were brought in for an hour-long, semi-structured conversation-style, one-on-one interviews that were designed to uncover the student's conceptual understanding to allow for rating on a modified APOS scale. The standard APOS scale was modified in order to refine it by adding the intermediate stages of Action/Process and Process/Object for students on the border of the respective APOS levels. The study tracked the students as they moved through Calculus II, III, and Differential Equations and reinterviewed them once or twice in each class to rerate them on the modified APOS scale. The following graph shows the results of the students' ratings throughout the Calculus sequence:

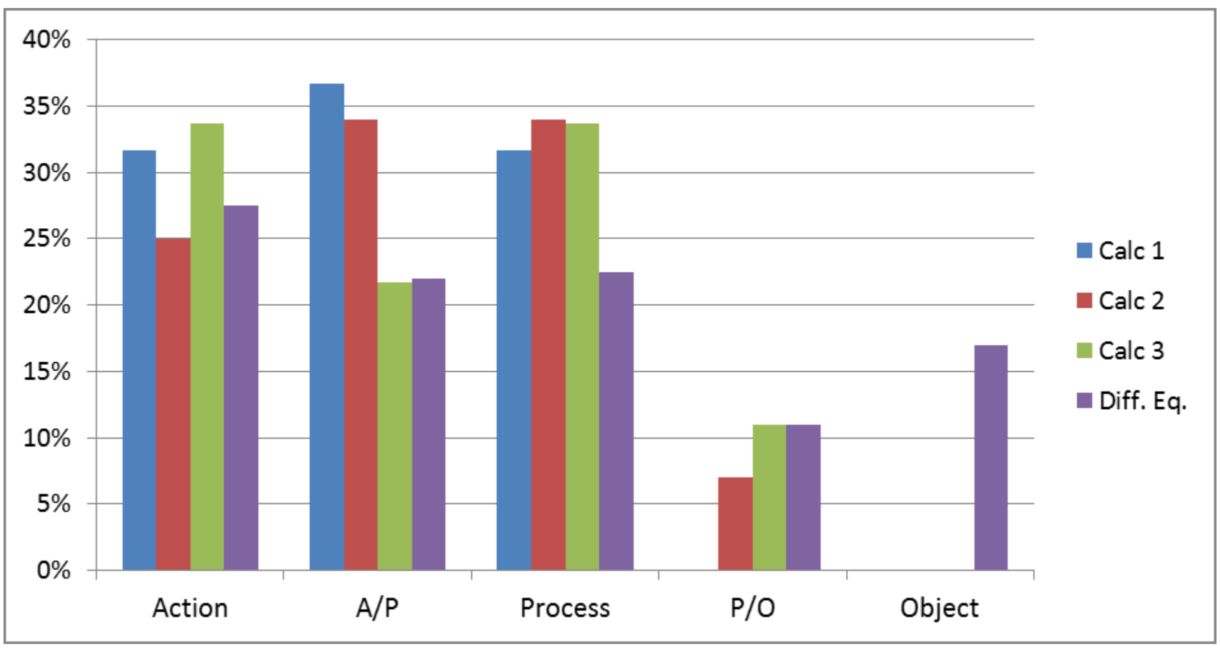

Figure 2.1 APOS distribution of students in previous longitudinal study

One can pull out many observations from the above graph. We can see that in Calculus I, students only show up in the Action, Action/Process, and Process levels; there are none in Process/Object or Object. This makes sense; one would expect Calculus I students to start off in the lower levels of conceptual understanding. Once they gain more experience, we see some of the students in Calculus II begin to move into the Process/Object level; however, most of the Calculus II students are still in Process or below. Others have noted the difficulties of Calculus students when it comes to conceptually understanding the mathematics (Clement, Lochhead, & Monk, 1981; White & Mitchelmore, 1996).

There is not much conceptual improvement of the students as a whole between Calculus II and Calculus III. As a result, there are a fair number of students who have still not even advanced into the Process stage. This aligns somewhat with what Breidenbach, et al. (Breidenbach, Dubinsky, Hawks, & Nichols, 1992) found when they said that even students with a fair amount of mathematics do not understand the function concept. However, this longitudinal study found that this cannot be said as a blanket statement; almost half of the students in Calculus III are at least at the Process level. This indicates that many Calculus III students have at least a fair conceptual understanding of "function" and "accumulation".

Finally, in Differential Equations students begin to gain an Object level understanding. This is not unexpected because Differential Equations is the class in which students must heavily use functions as objects as inputs to equations and as solutions to problems. This study does seem, however, slightly at odds with some previous studies. Selden, et al. (Selden, Selden, Hauk, & Mason, 1999) concluded that Differential Equations students were still mostly unable to solve non-routine Calculus problems. This would seem to indicate that they do not have a strong conceptual understanding, but over half of the Differential Equations students were measured at Process or above. These two findings are not necessarily contradictions, though. For one, almost half of the Differential Equations students are not even at Process. Second, solving non-routine Calculus problems may be sufficiently different enough that problems with transfer begin to become evident. Selden, et al. did make the observation, though, that they showed progress over first year Calculus students, which is in agreement with the results of the longitudinal study.

This study also seems to disagree with the statement made by Cui, Rebello, and Bennett (2005) when they said that Calculus students "tended to use oversimplified algebraic relationships to avoid using calculus because they do not understand the underlying assumptions". Many of the students in Differential Equations do have a strong conceptual understanding of the mathematics. However, this is probably not as different as it appears. Their study was a small study looking to measure transfer from Calculus to Physics. Almost half of Differential Equations students still do not have a strong understanding of mathematics and we have already seen that transfer, in the traditional sense, is extremely hard to measure. The only real difference was that the longitudinal study was sufficiently large enough and long enough to notice the subtleties and differences in the conceptual understanding of mathematics students.

Students in Differential Equations are far from a homogenous group; they range from the lowest level of understanding, Action, all the way up to Object. This variability makes statements about "average" Calculus students difficult. This general distribution of students in the Calculus sequence, including the lack of students at the Schema level, aligns with the findings of Kabael (2011) who found that students reach from Action up to Object but that there were probably none in Schema.

Carlson (1998) did a cross-sectional study that had a similar purpose to this longitudinal study: to see what level students understanding is at in various stages of college. However, she did not use the framework of APOS theory. Carlson studied three different groups: 30 post-College Algebra students, 16 second-semester Calculus students, and 14 graduate students. She concluded that many students just out of College Algebra have a narrow view of "function" where all functions should be definable by a single algebraic formula, they possess a pointwise view of functions, and they view evaluation of function as algebraic substitution and as a memorized procedure. This would be the equivalent of saying that they are mainly at Action. The longitudinal study agrees that this is true of many of them, and is not true of all of them.

Carlson found that second semester (Calculus II) students have a more general view of functions, but still have difficulties with covariation and dynamic representations. She said that even the best students in the first two groups have trouble with some concepts in their respective classes, but high performing second semester Calculus students viewed functions as processes. This is similar to saying that Calculus II students have deeper conceptual understanding than Calculus I students and the highest are around the Process level, but not all of them are yet at Process. The longitudinal study agrees with this finding also.

Carlson found that the concept of "function" develops slowly as students move through their undergraduate career and that even the best students do not totally understand "function". By the time they become graduate students, however, they have stronger understanding of covariation and multiple representations. The first sentence aligns with the finding that the students, as a whole, in the longitudinal study slowly increase through the levels but do not reach the level of Schema by the end of the Calculus sequence. The second sentence essentially says that graduate students have likely ascended at least to the levels of Process and Object. This is also supported by the longitudinal study that found that many students do get into the Object level by the end of Differential Equation.

Summary

The problems students encounter with "function" and "accumulation" have been laid out in this chapter as well a description of APOS theory and studies of forward and backward transfer. The previous longitudinal study, of which this will be an extension, was also described. This study will attempt to extend the longitudinal one by measuring, on the same modified APOS scale, where senior-level students are in their understanding of "function" and "accumulation". It will use this to attempt to measure any amount of back transfer from Engineering and Mathematics Education majors that might be observed.

Since the traditional view of transfer is difficult to measure and seemingly limited in scope, and since backward transfer does not make sense under the traditional view, this research will adopt an actor's point of view of transfer in line with AOT, PFL, or Vertical Transfer. However, exactly which specific view of transfer is not too important since evidence of back transfer will be inferred by any increase in students' understanding of the Calculus concepts despite having no more Calculus instruction.

Studying the understanding of senior-level Engineering and Mathematics Education majors is especially of interest in light of Even's (1993) conclusion that many prospective teachers still have many conceptual problems with functions and that many secondary teachers do not have a "modern concept of a function". On the other hand, Vinner & Dreyfus (1989) found that the performance of junior high teachers was similar to mathematics majors. About Engineers, Cui, Rebello, and Bennett (2005) and Selden, Selden, Hauk, & Mason (1999), among others, have documented their conceptual problems including misconceptions, tending to use algebra instead of calculus in calculus problems, and generally not having the levels of understanding that one would hope.

Chapter 3 - Methodology

Purpose of the Study

This study aims to measure growth of students' understanding of calculus concepts as a result of back transfer from classes taken after they have advanced beyond the Calculus sequence. Student interviews were used to assess conceptual understanding and the distribution of students among the levels was then compared against a previous longitudinal study to determine if back transfer was occurring. Other qualitative methods as well as some quantitative methods were also used to analyze students' understanding.

Research Questions

The questions that this research will attempt to answer about student understanding and back transfer are the following:

1. Does back transfer occur?
2. If so, does the application of math in different contexts or majors cause different levels of back transfer?

Participants and Setting

This study took place in a Midwestern university of about 24,000 undergraduate and graduate students located in a community of about 52,000 people. The participants consisted of 20 senior-level engineering majors and nine senior-level mathematics education majors. Of the nine Mathematics Education seniors interviewed, roughly half were male (5 out of 9) compared to the 20 Engineering seniors of which all 20 were male. Each group was selected from their classes in the Spring semester of 2011. The engineering majors were recruited from ECE512: Linear Systems in the College of Engineering. The mathematics education majors were recruited from MATH 570: History of Mathematics in the Mathematics Department.

All participants cooperated voluntarily and signed the University's Informed Consent Form after being told about the research and their role in it as well as having any question answered that they might have regarding the interview process. Students were informed that their anonymity would be maintained through the use of pseudonyms.

Data Collection

Longitudinal Study

This study was developed from a previous longitudinal study that was conducted from Fall 2009 to Spring 2011. The purpose of that study was to understand how students' conceptual understanding grew within the Calculus sequence. This was accomplished by interviewing students once or twice a semester to assess their conceptual understanding of certain calculus topics including function and accumulation in regards to integration.

Initial recruiting started with students from Math 220: Analytical Geometry and Calculus I during Fall 2009. These students participated in hour-long, one-on-one interviews each semester as they progressed through the Calculus sequence: from Math 220 to Math 240: Elementary Differential Equations. The students' conceptual understanding was rated each semester and the breakdown for each class was graphed (see Results). This graph was used as the basis for the measure of conceptual growth for this study.

Cross Sectional Study

This study is a cross sectional study of senior-level students in engineering and mathematics education students that compares these students with the results of the previous longitudinal study to determine if any back transfer occurred.

The senior level students described in the Participants and Setting section were recruited for hour-long, one-on-one interviews. The researcher visited the Spring 2011 ECE512: Linear Systems and MATH 570: History of Mathematics classes to recruit students for the interviews in exchange for 10 dollars. Volunteers were asked to indicate to the researcher what times they would be available to be interviewed and then the students were each emailed to set up an interview with them based on these times.

Each interview was conducted by the researcher and took place in a conference room in the mathematics building at the time agreed upon with the student. Before the interview began, the student was informed of the purpose of the research as well as given an informed consent form to sign if they chose and asked for their permission to be recorded. If permission was granted, a digital recording device was used throughout the interview to capture the audio for later analysis.

Interviews were conversation-style following a protocol (see Appendix B - Interview Protocol) with predetermined questions along with spontaneous, related follow-up questions determined by the course of the interview to get at understanding. Questions were taken mostly from the level of Calculus I and II with two questions coming from Differential Equations. The questions were chosen based on whether each would demonstrate a student's understanding of various calculus concepts including function, differentiation, and integration.

The following is an excerpt from one of the interviews that illustrates the conversational style of the interviews that starts with a protocol question and continues with spontaneous, unscripted questions to further reveal how the student understood the concept:

Researcher: So, what is […] a function?

[…]

Interviewee: It's a formula using ...uh I think of a function as you've got like your what it's called and then like equal and then you've got your formula of the… the rest of the...

Researcher: So a function is an equation then?

Interviewee: Yeah, an equation. um...

Researcher: So if we think about... um... So is any equation a function?

Interviewee: Off the top of my head, I want to say yeah, but it seems like a trick question. I mean, I think there's got to be some…

[…]

Researcher: So, y equals x squared: that's a function, right? It's a parabola.

Interviewee: Doesn't it have to be like differentiable at points. I don't know. Something. Maybe not. Or continuous. It has to be continuous doesn't it? Maybe not. Because you can have to sum of... yeah. So this would look like...

Researcher: [Drawing a discontinuous function] So is that a function?

Interviewee: Uh... yeah. I think it is.

Researcher: Ok. So it doesn't have to be continuous?

Interviewee: I think that's right.

Each thread of questioning continued in this way until the researcher was satisfied that the student's level of understanding had been revealed. At the end of the interview, students were given 10 dollars and had any questions they might have answered.

The interview data was analyzed in three passes in order to discover trends in learning and understanding. The first pass was qualitative: rating students based on the modified APOS rating. The second pass was a quantitative attempt to classify different groups based on vocabularies. This quantitative pass prompted a third pass that was a qualitative classification of students based on confidence and willingness to participate in answering questions.

Data Analysis

After all interviews had been completed, each interview was transcribed personally by the researcher by listening to digital media files of the interview and typing with the aid of transcribing software. The level of understanding of function and accumulation was assessed for each student based on a modified APOS scale.

1st Pass – Rating on a Modified APOS Scale

All students were rated on a modified APOS scale. This scale was created by modifying the standard APOS scale by adding the intermediate levels Action/Process and Process/Object between the standard Action and Process levels and the Process and Object levels respectively. A student falling in one of these two levels indicated that they could sometimes operate at the higher level but they could not operate fully at this level and still relied heavily on the lower level of understanding. This modification allowed for a finer scale that was more suited to semester by semester comparisons.

A student was rated by the researcher by analyzing how they answered the series of questions in the interview and deciding to what level a student could think about a particular topic. A student would be rated at a particular level if the researcher felt that the student could think about a topic and operate within a problem at that level. The rating was not an indication to which level a student would naturally operate when answering a question but rather an indication of the highest level a student could reasonably operate when answering a question (see Appendix A - Modified APOS Rubric).

The overall rating for each student was assigned based on a composite of the student's individual ratings on each question. Once all students had been rated, the percent of engineering majors and the percent of mathematics education majors at each level were graphed (see Results). The distributions among the levels for each major was then compared against those for the Differential Equations students to observe any trends in the distribution that would indicate back transfer occurring after the Calculus sequence.

2nd Pass – Non-Negative Matrix Factorization

Data mining techniques were next used with two goals in mind. First was the hope to develop an automated model for differentiating students based on conceptual understanding or any other significant distinction. This could allow the researcher to rate students in an automated fashion, thus lowering the amount of time required to do so, and allowing for an expanded number of students to be studied. Non-negative matrix factorizations are a standard tool in developing such models. A second goal of the factorization was to use computer techniques to identify themes that might have been overlooked initially.

To carry out the non-negative matrix factorization, the transcriptions were used to create a matrix where each entry was the total number of times each student said each word. So the sixth entry in the eighth row would be the number of times the eighth students said the sixth most commonly used word from the interviews. Different variations of this matrix can be created by including either all of the words spoken in the interviews by students or some subset of words. For example, the most commonly spoken word in the English language is "I". Therefore, this word will likely not add helpful information.

There were 29 students and a collection of approximately 4,000 words. Different numbers of word were used in different runs depending on which common words were included. Then the matrix, V, which would be formed, would be a 29x4000 matrix. The non-negative matrix factorization (NMF) creates two matrices, W and H, whose entries are non-negative and whose product, WH, approximates V. W and H were chosen to a local optimization in the Frobenius norm. Factorization was handled by the computer program, R, using the NMF package which implements the algorithms from Lee & Seung, 2001.

The dimensions of the two matrices are chosen by the researcher. For example, the researcher can choose matrix H to be a 4x4000 matrix, thereby forcing W to be a 29x4 matrix. The idea behind the NMF is that matrix H can then by interpreted to have discovered four different vocabularies that students use and the relative frequency of each word in each vocabulary. Matrix W could then be interpreted to represent the relative amount of each vocabulary that each student used in hopes of discovering that students could be classified based on what vocabularies they used. This factorization can be run many times with different restrictions produce various NMFs. Some of the restrictions included using different dimensions (different numbers of vocabularies), and throwing out common or uncommon words. Each NMF is then reviewed by the researcher in attempt to discover different groups of students.

Runs were chosen with 3, 4, and 5 vocabularies, and results were similar. Unfortunately, none of these factorizations was successful at grouping based on major or level of understanding. Some of the factorizations, however, did create groups based on words linked to confidence or willingness to participate in answering questions such as “don’t” and “know”, “understand”, etc. These groupings prompted coding the transcriptions based on each student’s confidence and willingness to participate.

3rd Pass – Coding

Sentences, phrases or other segments of information in each interview were individually coded using a proprietary coding scheme. Each code consists of a 5-tuple with the entries consisting of Topic, Level of Understanding, Current Representation or Action, Confidence, and Willingness to Participate. Topic was rated with a number from 1 to 5 to indicate what the general topic that was being discussed. Level of Understanding was given a rating from 1 to 3 to indicate what level the student was working at with 1 being Action, 2 being Process, and 3 being Object.

Since this was a measure of the level at which the student was currently working, the intermediate levels Action/Process and Process/Object were not given a number as they describe a level of understanding overall and do not lend themselves well to rating individual moments of work. In other words, at a specific point in time, a student will be working at Action, Process, or Object, but cannot be working at Action/Process or Process/Object. Overall, a student’s level of understanding may fall between Action and Process or Process and Object however, so these are overall ratings.

The third entry, Current Representation, was coded with a single letter to indicate what specific representation of a concept the student was currently discussing: graph, equation, rate of change, etc. Confidence and Willingness to Participate each were coded with a number from 1 to 3 where 1 indicated below average, 2 indicated average, and 3 was above average. The following gives a description each entry:

Entry #	**Entry**	**Description of Levels**
1	Topic	1: Function
		2: Accumulation
		3: Rate of Change
		4: Student Not Contributing
		5: Miscellaneous Discussion
2	Lev. of Und.	1: Action
		2: Process
		3: Object
3	Current Rep.	a: Accumulation
		c: Rate of change
		d: Derivative
		e: Equation
		f: Formula
		g: Graph
		i: Integral
		o: Operator/operation
		r: Relationship
		t: Tangent
		u: Area under the curve
4	Confidence	1: The student was not confident in their work.
		2: The student had average confidence in their work.
		3: The student was very confident in their work.

5	Will. to Part.	1: The student was not forthcoming with answers. 2: The student answered what was asked. 3: The student supplied information and progressed discussion on their own without much prompting.

For example, one phrase might get coded 21u21. This would indicate that the researcher and student were discussing or working with accumulation at an Action level specifically talking about integration as area under the curve. The student was acting with average confidence but an unwillingness to participate in the discussion: short answers and not contributing new information, for example. Each phrase or idea was coded in this way.

These codes were then used by the researcher to rate each interview with an average level of confidence and willingness from 1 to 3 using half steps so that they would be more comparable to the rating of the student's level of understanding on the modified APOS scale. The data was then represented in graphical form (see Results).

Reliability and Validity

In the previous longitudinal study, in order to ensure reliability of the results, two colleagues independently rated the students and any differences in rating were discussed until all three raters were satisfied with a common rating. A rubric was not made that could strictly rate each student as each interview was uniquely guided by each student's level of understanding and answers. However, each rating was guided by the general rubric as well as the researcher's years of experience with teaching and grading student understanding on exams and homework. This cross-sectional study utilizes the same rating system and therefore the reliability derives from that of the previous study.

The purpose of this study is to see if there is any growth of student understanding of calculus concepts after the student leaves the Calculus sequence as well as to discover any differences in such growth between engineers and mathematics education majors. The levels of student understanding throughout the Calculus sequence were measured in the previous longitudinal study. So, the levels of understanding of students as they leave the Calculus sequence are known. Any significant, measured growth beyond this must necessarily happen after the Calculus sequence. Likewise, any differences in understanding between the two different majors can be attributed to the inherent differences between the majors.

Chapter 4 - Results

Introduction

The student interview data was analyzed in three passes in order to discover trends in learning and understanding. The first pass was qualitative: rating students based on the modified APOS rating. The second pass was a quantitative attempt to classify different groups based on vocabularies. This quantitative pass prompted a third pass that was a qualitative classification of students based on confidence and willingness to participate in answering questions.

1st Pass – Qualitative

Each student was rated on the modified APOS scale as Action, Action/Process, Process, Process/Object, or Object based on an overall, composite understanding of the concepts of function and accumulation. Rather than reflecting an average of which level the student worked when solving problems, each rating represents the level at which the student could reasonably understand the material. Since research has indicated that students do not reach the Schema level by this stage (Carlson 1998; Kabael, 2011), this level was left off of the modified APOS framework. Indeed, none of the students interviewed were at the Schema stage.

Excerpts from various interviews are presented in this section. These excerpts are not intended to demonstrate all the information that was considered when rating each student. They are presented to provide some examples of statements that might be helpful in rating students, especially statements or ideas that are not explicitly laid out in the rating rubric. They also are intended to illustrate certain points of interest in the interviews.

Action Level

Of the 20 Engineering Seniors, only one was rated as having an Action level understanding. Two of the nine Education Seniors were given an Action rating. All the students worked at an Action level at some point since that is usually the easiest way to solve a familiar problem. When a student tries to find the derivative of a polynomial, for example, they do not need to think about tangent lines, rates of change, slopes, or derivative operators, they only need to think about the exponential and linearity rules of derivatives. That is, they only need to remember the procedure of multiplying the coefficient by the power and then lowering the exponent by one. The student would be working at the Action level regardless of whether they have a higher understanding of the material because they question does not require a higher level of thinking.

Therefore an Action level rating is demonstrated by a lack of working above action rather than any particular thing they say. However, certain themes were more commonly associated with students working at an action level. These included comments such as “that’s the way my teacher told me to do it”, “I am/am not a visual learner”, or more directly, “I’m bad at math”.

The following is an excerpt from the eighth Mathematics Education student interview which was rated as Action and exhibits many of these themes:

ED8: Ok, let's see... flower... I'm trying to remember.

Int: What do you mean?

ED8: Well, because we put the... ok. Because when you have like even or odd, then you can do the flowers.

Int: Oh, ok.

ED8: Is this a circle? I think it is. If I remember correctly. This is from Calc II.

Int: Yeah.

ED8: I hate parameterizing things. That was one of my least favorite things. I can't do it... I can't... I can't graph... I can't see things; I'm not a visual learner.

ED8: Um... Yes, it would be.

Int: Would be what?

ED8: [mumbling]

Int: Oh, you’re doing…

ED8: Um... in for t, to be in... one's for it. I tutor every once in... I tutored Calc III last semester, so they used... these every once in a while, so I was trying to transform back into something that my mind actually, like... I did, um... When you tutor enough, it gets stuck in your head. Um...

ED8: I don't actually remember. I think it's a circle, I have a feeling it's a circle. It's not a flower and that's about all I got for you.

Int: So it’s either a circle or a flower?

ED8: I don't know, I don't remember. Um, and then you have, like, the heart thing too. I just don't remember what shapes because they were ridiculous to me.

Int: How would we decide if that is a circle or a flower?

ED8: Um... We were taught to plug in... you pick your values t you can plug in from both of those and then plot it on your x... coordinate system.

The student exhibited many of the traits related to working at an Action level. When first presented with the problem, she attempted to guess the shape based on experience matching the shapes with the equations rather than plugging in numbers or thinking about behavior of cosine and sine. After attempting to solve the problem by calling on memorized facts, she tries to think about her tutoring experience saying “When you tutor enough, it gets stuck in your head,” indicating that she had learned the material through rote memorization. Later, she says, “I just don't remember what shapes because they were ridiculous to me.”

Another common theme is that she says she hates parameterization and says, “I can't see things, I'm not a visual learner.” This indicates she did not have a deep understanding of the concepts in question; she cannot “see” what is going on when graphing a function. When finally pushed enough, she says “We were taught to plug in... you pick your values t you can plug in from both of those and then plot it on your x... coordinate system.”

The same student exhibited similar methods when answering Question C (see CHAPTER 1 - Chapter 1 -Appendix B - Interview Protocol) about finding area under a curve. This student rated at an Action level of accumulation. The following is an excerpt from the student interview:

Int: Can we figure out where to put that line “c” to maximize the total amount of shaded area?

ED8: You should be able to, yeah.

Int: Alright. How?

ED8: I don't know. That’s a good question.

Well, because... that line is just... is just an equation. Um... Are you already at a maximum point?

Just put a variable number in for it… You can change it back into an equation because that line is just... an equation.

I don't remember what it is, I've seen it before and I've had to learn it for a test, but I... can't recall it at all.

You could do that and like plug in different inputs and outputs to see what you get for your total shaded region. Like a guess and check method, that's all I can come up with right now.

Int: Ok. So you want to reverse engineer the equation for the line?

ED8: Um, we can.

Int: Once we have the equation for it, you have to then take the…

ED8: …different… oh... Uh, maximum shaded under the curve would be the integral. Um... we put that… right into that...

I think you'd have to reverse engineer the equation to take that... Sorry I... keep going into my monologue and thinking about stuff.

Int: That's fine.

ED8: The math stuff you're asking about is…

Int: So, if we have the equation, you'd plug in… you'd put in different points and see what the different shaded areas were?

ED8: mm hmm... so you see like the maximum amount.

Int: So how do you know what the shaded areas are? How do you find the shaded areas?

ED8: How do I find the shaded areas?

Int: You said put in different points and find the shaded areas and see which one is the biggest…

ED8: In this equation... See this is why I don't think you should... if you're going to be a math teacher you shouldn't go over um... [mumbles] for math because then you forget things that you've done before. Um...

You could also just simplistically move the height up and down and then count all your... all the squares that build it... the... concrete way of doing it, um...

Int: Ok, so estimate the area that gets the most squares?

ED8: And guesstimate, yeah.

Int: So you move "c" around, then you'd estimate the little squares and you see which one estimates the most little squares?

ED8: Yeah, that's all there is of it.

Um... I'm sure I cut off the, like, the top part. or does it just continue on forever and ever and ever... oh, ok. I'm just going to make sure...

Int: Ok. So, earlier you used a… Well, let me ask this… You want to move that line around…

ED8: But there's other possible way of doing it, right? But... like guess and check way, which most... people would... first go to if they didn't know how to change it. And then you could change it into the equation.

And then there's like... totally forgot, but there is a way to figure out... Um... it's... the boundaries of it and it just switches at this point.

I don't remember what they're called. But there's something you can do with both of those, I think. You could figure out with an equation, also. You need different outputs, then. Like different numbers and then you get the different outputs and then you'd have your maximum.

The student's first reaction to solving the problem was to try to come up with the equation for the line. This was coded as the student working at an Action level. It is not enough by itself to rate the student as having an Action level understanding, but it is enforced by further discussion. The student continued by saying, "I don't remember what it is, I've seen it before and I've had to learn it for a test, but I... can't recall it at all,'' indicating that the student was just trying to recall memorized procedures rather than thinking deeply about the concepts involved in the problem.

She then decides that she can move the line to different locations and count the number of "squares" that build it: "the concrete way of doing it". This gets at the basic idea behind solving the problem, but she is still doing it in a procedural way. This "guess and check" method is the discrete version of the correct method; however, it can only hope to estimate the answer. More importantly, it helps to rate the student as only having an Action level understanding of integration. Even after stumbling on the correct method, she is still doing it by plugging in inputs one at a time rather than viewing integration dynamically and moving the line around, continuously evaluating the change of area.

Action/Process Level

Two Engineering majors and one Mathematics Education major were rated in the intermediate Action/Process level. These students showed some signs that they could understand function and accumulation as being processes, but they had not yet achieved the level of Process. These students might exhibit some amounts of understanding of the concepts as processes, but trying to work with a Process view pushes their understanding too far, and so they primarily fall back to an Action level understanding to answer questions.

The following is an excerpt from the first Mathematics Education student interview illustrating some examples of why this student was rated as Action/Process. This excerpt begins after the student was asked the first part of Question B about graphing the parametric equations.

ED1: Um... cosine... that's, that's, uh... like that... then it goes from zero to two pi, so... That's one complete... revolution. So that would be... x of t. And then y of t would just be the sine. Which... would be until it makes one revolution, so x one...
Ok, then a period... one period over two pi.

Int: Ok, so you graphed x versus t and you graphed y versus t. What if I wanted to graph x versus y, so x and y are on the same graph.

ED1: On the same graph... um... Um... Maybe you could substitute this equation in for t here.
Or is that not such a good idea?

Int: So you're going to substitute y in for t?

ED1: Right. I'm not going to get any graphs of them... that would give you... but...

Int: Cosine of sine of t; do you know what that looks like?

ED1: So, yeah, that's probably...

Int: So what happens if I plug in 0 in for t? What x and y do I get?

ED1: mm... one, zero. So you could plot out points? So that would be...

[Student plots points and decides the graph is the unit circle.]

ED1: So it would be a circle. I guess if we kept plotting points... so it would be... three pi over... two... Yeah, it would be a circle.

Int: So would it be the whole circle? Would it go around once or would go around twice or whatever?

ED1: It would... uh, between... two pi... yeah it would just go around once.

Here the student began with some evidence that he was thinking about functions as dynamic processes when he graphed the sine and cosine graphs and indicated that they each made one full revolution as t ran between 0 and 2 pi. This could show some indication of the student viewing functions dynamically, at least with respect to trigonometric functions.

This alone is not enough to classify the student in a particular level. As further evidence, the student was later asked for a second time what a function is. The following is the excerpt from this discussion.

Int: So if "function" is different than "graph" and it is different than "equation", what is it?

ED1: A function is just... I don't know, I'd... I'd still put it as like the process of putting in inputs and getting different outputs out.

Int: So it's this process of changing things?

ED1: Right.

Int: So, can we put functions into functions if it is just this abstract way of changing things?

ED1: No. Um, because you have to have a concrete equation to put into another equation. So a function is an equation.

Int: So a function is an equation?

ED1: Yeah.

Int: Didn't you decide that the process of taking a derivative was a function?

ED1: Sure, why not. Um... Yeah. But... I mean, the derivative... I get it; it's a process of what you can do to something.

Int: And you said a function is a process?

ED1: Ok. Yeah.

Int: So in that sense, a derivative is a function?

ED1: Yeah.

Int: But you also said a function is this tangible thing that you can do something with and so that made you say it was an equation again.

ED1: Yeah, I guess this isn't... It can be an equation but doesn't necessarily have to be. But you could have a... I mean, you could have a function of just, like, a plot of points. Like if you have your x and your y and whatever it is... I mean, that could be a function. And that's not an equation. But you could put it into an equation.

Int: So it seems like you're saying that there's two different kinds of function. There are the ones that are equations and then there are the processes. So can you put the second kind into other functions? Like can I take the process of taking a derivative into a function?

ED1: Yeah. Like can you... can you put a derivative into... Yeah, you could put a process onto a function. But you couldn't, like, substitute in derivative for x.

Int: So there are some functions that I can put in other functions and some functions I can't put in other functions?

ED1: Yeah.

Here, the student began to describe a function as the process of taking inputs to outputs, indicating that the student had some understanding of functions as processes. When pushed further on the topic of composing functions, he decided that functions are the same as equations, because one needs an equation to put into another equation. This indicates that the student may not be comfortable thinking about functions as processes and it further demonstrates that the student is not thinking about functions as objects beyond having to rely on equation representations. This is strong evidence that the student should not be rated in the Process/Object or Object levels.

When given the example of the derivative operator taking inputs to outputs in the form of taking differentiable functions to their derivatives, the student again changed his mind about what functions are to allow derivatives to be functions. This both demonstrates that the student has some ability to view functions as being processes independent of equations as well as showing that the student may be wrestling with this view and not yet ready to abandon an Action view of functions.

Finally, the student tries to rectify these two competing views by saying that there are two different types of functions: those that are equations that can be put into other functions, and those that are processes independent of equations. This bolsters the view that the student is operating in the intermediate level between Action and Process.

This same student also exhibited some of these same difficulties when discussing accumulation in Question C. The following is the same student's discussion after being asked to maximize the shaded area in Question C.

ED1: The line c? To maximize the shaded area? Well, we'd put it at the top.

Int: So at the top will maximize the shaded area?

ED1: Um... well the shaded area is beneath the function and beneath c... and the shaded area is above the function and above c. But we want the greatest area, which... I mean, it bends this way so the greatest is obviously going to be below the function. So we want c as high as we can.

Int: So if we put it up here, we'll gain this, but we'll lose this because it will no longer be under the function.

ED1: Oh. Well, maybe we should maybe put it at the uh, the turning point. The... from where it starts... Uh, what would it be? The second derivative or something like that.

Int: What about the second derivative?

ED1: Um... That's the... what was it called? The... inflection point. Yeah, so... Right.

Int: So where is an inflection point?

ED1: Um, where the tangent gets to... starts... going negative, I guess. So where... the tangent...

Int: Do we have negative tangent lines here?

ED1: No, these tangent points are always positive. Or zero up here.

Int: So you don't want to put it all the way at the top anymore?

ED1: We... Um, this area is obviously, I mean, looks greater than this area. So if we scoot it up to the top we would lose that and gain that which is not what we want to do.

Int: So you're saying that if we put it all the way up, we get more area than if we put it all the way down.

ED1: Right.

Int: So somewhere in here, there's a maximum.

ED1: Right. Um... So, the maximum would be... uh... this distance from here... from the graph to this axis equals this distance.

Int: Ok, how did you decide that?

ED1: Um... Picked a number, I guess. It would be... Just from looking at the graph, basically. Um, you can see that um... the two distances equal each other. That's the widest point, I guess. The... the widest point for the bottom of the shaded region to the top of the shaded region. Like the... the point where... they can both be equal to each other.

Int: Ok, so why would we want them both to be equal to each other? How would we know that maximizes the area?

ED1: Um... I guess, basically, by looking at it. I'm not sure why it's maximum. It just... is. Um... Yeah. I don't really know how to... describe that process, I guess. It's...

Int: Well, tell me what you were thinking.

ED1: Well... You... I mean, I'm just... decided before that you want some of this and some of that. So... You're going to have to find the greatest area of this added to the greatest area of that. And... It's kind of like when you... have a rectangle... and the greatest area of it is going to be, like, the... the... Well, if you have a given area... and then that rectangle... and the greatest area is going to be when... one side and the other side are closest to each other.

Int: So when it comes out to be a square?

ED1: Right. That's the same premise as saying that it'd be this side and... well...

Int: So you tried to make it like a square?

ED1: Right, I mean, you want to maximize it by... Yeah, making it... as close to a square as possible, basically.

The student started out operating at an Action level. For example, he stated that the line should be put at the inflection point of the graph to maximize the amount of shaded area and that this was the point where the tangent line stated to go negative. The student may have been repeating procedures he had learned to find maxima – i.e. looking for when the tangent line went from positive to negative and using second derivatives to check for extrema. Using rote procedures without understanding is a defining feature of Action level work.

It was clear that the student did not understand the procedures that he was trying to use since, his description was incorrect and, more importantly, there are no inflection points or negative derivatives in the graph. He clearly was not using a Process understanding to reason through the problem.

After being pushed in the right direction, the student was able to come up with the correct answer: that the line should be place so that the graph intersects it at the half-way point. That he was able to come up with the answer without being led to it may indicate that he could visualize the line moving and changing the area. When compared to some of the other things that this student said in other discussions reinforces that this student was beginning to visualize the Process view of accumulation. However, the difficulty he had in trying to describe how he got the answer is indicative of not yet being able to fully understand the Process level view.

Process Level

Seven of the Engineering majors but only one of the Mathematics Education majors was rated at the Process level. Students working at the Process level with functions can view functions dynamically as something that transforms a set of inputs into the set of outputs. They can comfortably understand covariation and are able to visualize changes in the outputs based on changes in the inputs.

Aside from these standard features, another common, but not universal, feature was students who understand functions at a Process level, but because of the way the material was originally presented to them or whatever other reason, when asked about functions that take other functions as inputs and outputs, they resist calling these “functions” and prefer to call them “operators” or “transformations”. As long as they can understand these concepts at a Process level, the different vocabulary does not affect their rating since it only manifests as a difference in vocabulary and is not conceptually incorrect.

The following is an excerpt from the 17th Engineering major's interview illustrating a typical student distinguishing between "functions" and "transformations". This student was rated as process due to the things said in the following as well as other parts of the interview.

Int: What if we think about the process of taking a derivative? We take f to f'. Is the process of taking a derivative a function?

EN17: Um... I would say no. Because, um... You could say maybe y equals... d/dx that's f, but then that would be... but if you're saying y is a function of, um... you know, x and f is a function of x. You could say that. But, um... just the process of taking a derivate is... I think of it as more of a... um... transformation... than anything else.

Int: So what's the difference?

EN17: Um... Well, when you... when you take a derivative of something, you're um... you're altering it or transforming it. Um... So for example, if you just take... x and you take three times x, that's not necessarily a function, it's just... uh, an operation. Um... you know, on x.

Int: So a function takes a number and assigns to it a number and a transformation takes something and changes it into something else?

EN17: Right.

Int: So if a transformation can change one function into another function, can I transform a transformation?

EN17: Yeah, sure.

Int: So a transformation can take a transformation and change it?

EN17: Um... The result of a transformation is something you can do something with, but the... I mean that's probably not... correct terminology, but... Um... uh... I'd say it's uh, more... process... than anything else, I mean... You can uh... take the transformation of something and uh... I don't know.

Int: Can we add two transformations together, for example?

EN17: Um... I think it... makes more sense to say that you can add the results of two transformations together.

The student starts by resisting d/dx as a function because a function relates two variables, whereas d/dx as written did not indicate what the independent variable was. He follows this by saying derivative is a transformation because it alters functions and relates this to the process of multiplying by 3, which he calls an operation. So here he is distinguishing between functions as a formally written equation that takes numbers as inputs and outputs with operations that alter an input. This distinction was common and when rating these types of students, this kind of answer was taken as evidence that the student had a Process level understanding.

When asked if one can add two transformations, he responds that it "makes more sense to say that you can add the results of two transformations" indicating that he is viewing these particular functions no deeper than a Process view. Were he viewing the functions in an Object view, he should have been comfortable adding two operations as objects themselves instead of adding the outputs as objects.

A Process view of accumulation should manifest itself by the ability to visualize the dynamic change in area as the parameters change rather than having to plug in different parameters one at a time and comparing the resulting areas. The following is an excerpt from the first Engineering interview that demonstrates this ability while discussing Question C.

EN1: You could like start zeroing in on it that way, maybe, if you got close and you could move the other direction if you started to get, you know, started to get bigger or something.

Int: So you're thinking about moving the line and seeing what's happening to the area?

EN1: Yeah.

Int: What happens if I move this, um, line up just a little bit? What happens to the, uh, shaded area?

EN1: The shaded area will get... it looks like it's bigger because this section here is, is uh, wider than this side here.

Int: Ok.

EN1: So this, this area here is going to be increasing. The area under the curve is going to be increasing faster than this will be decreasing. Basically.

Int: So you thought about this little bit of area that I just increased by.

EN1: mm hmm.

Int: And you've said I've gained that but I've lost this.

EN1: Yeah.

Int: So, I gained more than I lost, so the integral went up?

EN1: Yeah.

Int: Will it keep moving up if I move it all the way up? Will it be maximized up here?

EN1: Um... Not entirely, there a point... I think there's going to be a point right about... right around there where you're going to be losing more... uh...

Int: Ok, so you're thinking about moving this up now.

EN1: Where this is basically half way...

Int: Ok, so you want this to be halfway across so, whatever this line is, so that it's halfway across. How did you decide it's halfway?

EN1: Well... I guess that it's not exactly a straight line, though, so that's not right.

Int: You said that the function is not a straight line?

EN1: Right, yeah, well I'm saying it's not a, uh... it curves, it's not a...

Int: It's not linear.

EN1: Yeah, it's not linear.

Int: Right, so um, where would this go then? How does that affect that? Because why did you say it was... why did you originally say it was halfway?

EN1: Well, I was... I originally said it was halfway because it's just width-wise you'd be... So the amount I would have just added... If your chunks going up are small enough, it's, it's going to be, uh... It's going to be just strictly based on... how, uh... wide each chunk is.

Int: Alright, so I've added this much, but I've lost this much?

EN1: Yeah, you're adding the same amount as you're losing at the maximum.

To solve this problem, the student thought about moving the line and said that, if the area started to get bigger, you would have to move the line the other direction. This indicates that he may have been thinking about the covariation of the area and the position of the line. This was reinforced when he was asked what happens to the area if the line is moved up a little. He compared the length of the line left of the graph with the length to the right because moving up increases the area to the left and decreases the area to the right. He also talked about the rates the area was increasing on the left versus the rate it was decreasing on the right. This implies that he was not taking individual locations and comparing the areas, but was rather thinking about moving the line and changing the area continuously.

When asked that, since the area increases as you move the line up, whether the area would be maxed by moving the line all the way up, he responded that there was a point where you start to lose faster than you gain and that this point should be when the function intersects the midpoint of the line "c". This, again, reinforces the Process rating this student was given.

Process/Object Level

Four Engineering seniors and three Mathematics Education seniors were rated as being at the Process/Object level for function and accumulation. These students are in the transition from the Process level of understanding to the higher Object level but cannot yet comfortably or sufficiently operate at Object enough to be rated as such. These students can sometimes view functions as objects themselves that can be manipulated and that have certain properties, but when pushed too far, they cannot delve deeper into this understanding.

The following is the 5th Mathematics Education senior’s interview in which she was rated as Process/Object. This student showed some signs of being able to understand functions at the Object level, but still had troubles and mostly was able only to demonstrate a Process level understanding of functions.

Int: What is a function?

ED5: Uh... As far as a formal answer goes... I couldn't uh... I couldn't give you other... much other than... it's a relationship between two... two um... variables, where you input one... you put... you have an input for one variable and that gives you the output for another.

Int: Alright, how do you think about “function”?

ED5: Um... Whenever I think about a function, I usually think about, like, the... the one to one test... when you're looking at the graph... So, if you have... it's a function if, for every x value, you have one y value.

Int: So when you say the “one to one test”, do you mean that it has to pass the Horizontal Line Test?

ED5: Uh... vertical.

Int: Ok, so you mean one x doesn’t go to two different y’s?

ED5: Yeah, exactly.

Int: Do those inputs and outputs have to be numbers?

ED5: Yeah. I guess, I mean... then just doing that, I would think they would, but I couldn't tell you formally.

Int: So if a function is a process of relating inputs to outputs, I can think of the process of taking the derivative as relating f to f’. So is the process of taking a derivative a function?

ED5: Is the process of taking a derivative... a function? Uh... it appears to me, the way it's drawn up, it looks like a function, yes.

Int: So the process of taking the derivative is a function?

ED5: It would probably be a... Yeah. Yes, I would say it is.

Int: So what would be its domain and its range?

ED5: The domain and range of the function or the domain and the range... of this? Um... Let's see... the domain would be... Let's see... every... let's see... I'm trying to think what domain and range... and then relate it to this.

Int: So what do domain and range usually mean?

ED5: Uh... The domain is every... let's see... every value that you can get... I believe... and the range is every value that you can... input. Maybe.

Int: You have the idea right, but you flipped them.

ED5: Did I flip them around? Ok. The input... I would say that the domain would be... your function and the output would be your... the uh... derivative of the function.

Int: Just this function?

ED5: A bunch of different functions. It just... yeah, it can be any function that you... wanted to input there and then you could find the derivative which would be your range.

This student started out by trying to give the "formal answer". This is kind of answer was observed at all levels of understanding where the student sees a disconnect between how they operationally understand a concept and how they think the instructor/interviewer wants them to understand the concept. The student then gave an answer somewhere between the Action and Process levels calling it a relationship between inputs and outputs but then indicating that she thinks about a function as a graph that passes the Vertical Line Test. Later, the student was comfortable saying that the process of taking a derivative was a function because it took f to f'. This usually indicates at least a Process view of function because the student is comfortable with functions as processes that relate inputs to outputs independent of whether they involve plugging inputs into equations.

After being reminded what "domain" and "range" mean, the student was easily able to find the domain and range of the derivative as a function, which may indicate that the student is beginning to view functions as objects at least as far as they are something that can be inputs and outputs of a function that does something to them.

The interviewer then asked if the domain was "just this function" in order to see if the student was just applying the procedures of blindly plugging in inputs at an Action level, but the student was easily able to see that it would be "a bunch of different functions". This indicates that the student is starting to view functions as objects that can be manipulated like numbers. However, overall, other questions in the interview indicated the student was still mostly operating at a Process level of understanding.

Accumulation is a subtler concept to try to evaluate directly at the Object level since one does not normally manipulate it in the same way one would a function. So the Object level will be rated more indirectly for accumulation. It is indicated by how a student handles multiple representations of accumulation and integration, how comfortably they answer conceptually harder questions and if they seem to view accumulation as more deeply than the single process of covariation between area and integration parameters. As a result, it is more difficult to illustrate this level for accumulation with an excerpt; the whole interview needs to be taken into account.

Students rated as Process/Object are in between these two levels, operating primarily at the lower level while showing some sign of understanding at the deeper level. The following excerpt is from the interview of the second Engineering major interview, which was rated as Process/Object. This excerpt starts well into the discussion about Question C.

Int: So what does that tell us about the area?

EN2: Um... Well... Over here it's steeper. As you move up... the... amount of area that you're... adding is... going... or what you're losing is normally decreasing, but it's... uh... decreasing at a... slower rate.

Int: So you're thinking about moving this line, “c”, around and seeing what happens to the area?

EN2: Yeah.

Int: And you started at the bottom and moved it up and said that, because your derivatives were big, you're adding a lot of area, but losing just a tiny amount over because you'd be adding this much and be losing this much?

EN2: Yeah, you're adding more area than you're losing... I'm not quite sure. Yeah, I think that's right because you're... Yeah. Because you're... yeah, because you're adding more area and then you shade to the point where you're losing it again. So that'd be some... point in the middle... where... Yeah, because it will be... adding more and you get a point where you'd start losing it again.

Int: Can you tell me where that point would be?

EN2: When the rate of change is zero... the... we gain as much as you lose. So... it would have to be like a maximum.

Int: Alright, where does that happen?

EN2: Um... Right at the point where both side... or the... where c and this were... like one half, one half. Like when the... the midpoint of the line.

Int: So that's where we reach a maximum?

EN2: Yeah. Um... If you take a... very small piece across, then it's going to have the same dimensions.

Int: So what does have to do with “integral”?

EN2: Um... Well... the integral is just adding up a bunch of... infinitely small pieces.

The student starts off by talking about continuously moving the line "c" and comparing the different rates at which the area change. This implies that the student is at least working at a Process level. As further evidence, he explains that when rate of change is 0, this means that it gains as much as it loses, so it must be at a maximum and that this happens at the midpoint. So this student seems to have at least a Process level understanding.

To explain why this would happen, he switches representations and describes the situation in terms more akin to Riemann sums by talking about "adding up a bunch of infinitely small pieces". This ability to switch between representations indicates that the student is beginning to view accumulation as one idea that manifests itself in multiple ways. When pushed further, however, he was not adept at this enough to be rated as Object.

Object Level

Six Engineering major seniors and two Mathematics Education seniors were rated as having achieved Object level understanding. When discussing functions, these students can view the concept of "function" inherently as an object itself that can be manipulated, put into other functions, and has different properties and multiple representations that are just different views on a single whole.

The following is an excerpt from the sixth Engineering major's interview in which the student was rated at an Object level of understanding of function. In this excerpt, the student exhibits some use of multiple representations as well as the ability to view functions as objects. He also displayed the common theme of distinguishing between "function" and "operator".

Int: What is a function?

EN6: Ok, it's like a mapping between some domain and some range.

Int: Ok, what do you mean by "a mapping"?

EN6: I mean whenever you're given an input, you associate, in whatever way you want, some output.

Int: So it's some sort of association between inputs and outputs. Is everything that associates inputs and outputs a function?

EN6: Uh... well... Um... I would say yes.

Int: Do these inputs and outputs have to be numbers?

EN6: No. Anything you want.

Int: Can you give me an example of an equation that doesn't use numbers?

EN6: Well, if you're talking about equations, then... you'd have to use numbers.

Int: Oh, sorry, I meant can you give me an example of a function that doesn't use numbers?

EN6: Um... Sure, I mean, if you don't... if you don't force me to use equations, then I can make a function that maps like the erasers in the room to the markers in the room. Uh, take the erasers and if it's uh... you know has some area then it can turn into a black marker. Something like that. And if it has a different area it turns into a red marker.

Int: So if I assign each function to its derivative… is the act of taking the derivative a function?

EN6: Uh, no it's an operator.

Int: Ok, what's the difference?

EN6: I... an operator is something that you apply to functions.

Int: Ok, so the only difference between functions and operators is that operators apply to functions?

EN6: Sure, like when you apply a function to... you apply functions to values and get out single values and you apply an operator to a function to get out another function.

Int: But didn't you just say I could have a function applying to erasers to give me markers?

EN6: Sure, like, but you apply an operator to the group... like... you know, so you apply the function to... a single eraser to get out another single thing. But an operator takes entire functions.

Int: Ok, but if my inputs and outputs can be anything can't I input functions into functions?

EN6: Um... well, I mean, ok, well no, because that's just a way of defining it, right? So I'm going to mean, like, if it's available to use as a function, then it can't be an input to a function. Well, you can have functions of functions; too, it doesn't matter really.

Int: Ok, so a function can take a function to another function?

EN6: Well, uh, sure.

The student seems very comfortable with functions as processes as well as the difference between function and equation, to the point of catching the interviewer on the mistake of saying "equation" instead of "function". He also easily came up with a function that takes inputs and outputs that were not numbers. These all indicate that the student has at least a strong Process level understanding of functions.

Despite this understanding, he still made the distinction between “operators” which apply to functions and “functions” which can apply to seemingly anything else. This might indicate that he does not view functions as objects that can be put into functions, but is could also merely be a result of the way that the Calculus sequence teaches about derivative "operators" and makes a distinction in the terms. The latter possibility is reinforced when the student is pushed on this incongruity; he quickly says that this is "just a way of defining it" and makes a distinction of vocabulary. Overall, his description of “operators” is consistent with rating him as having an Object level understanding of functions.

The same student displayed the same level of understanding when discussing Question C. As stated earlier, accumulation is more subtle to rate, resting mostly on indirect effects, like multiple representations and comfort with harder concepts. In the following excerpt, the student exhibits these traits, so because of this, and other parts of the interview, he was rated at Object.

Int: Can we figure out where to put line “c” to maximize the shaded area?

EN6: To maximize the shaded area... Um... and this is also... a line that restricts…

Int: Yeah, it’s restricted on the sides and top.

EN6: Well I can write some equations and figure it out, probably, but it looks like you... Um... Let's see. um... Maybe um... put it right... where it intersect the half-way point right here. It's right there.

Int: Ok, how did you decide that?

EN6: Um, I'm just thinking about how, if you take the line and move it up just a little bit, it's going to be favorable to move it up as long as you're adding more area here than you are in there. So you just got to put it, yeah, right on the middle.

Int: Ok, if you knew what the equation of this line was, how would you do it?

EN6: Um, you could write... I'm just going to use our area, so you can write an integral. So you write an expression for this area as a function of c... and that as a function of c. And it's simple enough that you can just take the derivative with respect to c and set it equal to zero.

Int: Ok, so you take the integral and then you take the derivative. How does that relate with what you did?

EN6: Um, when you do that, you're talking about the rate of change of an integral. Which says, if I vary my parameter, c, how much is my integral changing. And I was already talking about how if you move it up, as long as you're increasing more area here than there, then it's going to be favorable. So have... like positive derivative, but then right when you get here, if you move it down a little bit, in some sense, like, you're going to be increasing... that way and the other way. The signs work out somehow.

Int: Ok, you're changing your integral, so you're watching the rate of change of your integral and then you decided that when it stops increasing, you'd be at a maximum?

EN6: Or a minimum. Yeah.

Int: So when you did it without equations, were you taking the derivative of an integral?

EN6: Uh, I... I suppose, yeah. I mean there's different ways to think about it. So... Well, I mean, call it what you want... I mean I was looking at the area. And areas are integrals, so I was looking at integrals. And then I was moving the parameter to see how it changed. So I was looking at the change of an area. So, yeah, so I was looking at the derivative of an integral.

First, the student indicates that you could find the equation for the line and do the problem at an Action level, but then he immediately arrives at the correct answer. He explains his conclusion in terms that indicate he is at least viewing accumulation as a process.

When asked how he would solve the problem using the equation for the line, he immediately relates the method he used with an integral which yields a function of "c". He stated that he would then take the derivative of this function with respect to "c" to find when the derivative was zero, therefore indicating an extremum. When asked to explain how this was the same as his previous answer, he talked about watching how much his integral is changing and how an increase relates to a positive derivative. This use of multiple representations of accumulation, as well as the ease with which he switched from graphical accumulation to derivatives of integrals, indicate that he is viewing accumulation more deeply than a student with a Process level understanding.

This Object rating is reinforced by his comfort with taking the derivative of the integral and seeing that this was the rate of change of the integral and therefore the accumulation. Many students were only able to discover it was the derivative of an integral when explicitly led there, and then their reaction was usually that the Fundamental Theorem of Calculus implied that the derivative of the integral was the original function.

Total APOS Distribution

The following table and graph show the total number of students in each of the five levels of the modified APOS scale.

Table 4.1 APOS distribution of Engineering and Mathematics Education Majors

	A	**A/P**	**P**	**P/O**	**O**
Engineers	1	2	7	4	6
Math Ed	2	1	1	3	2

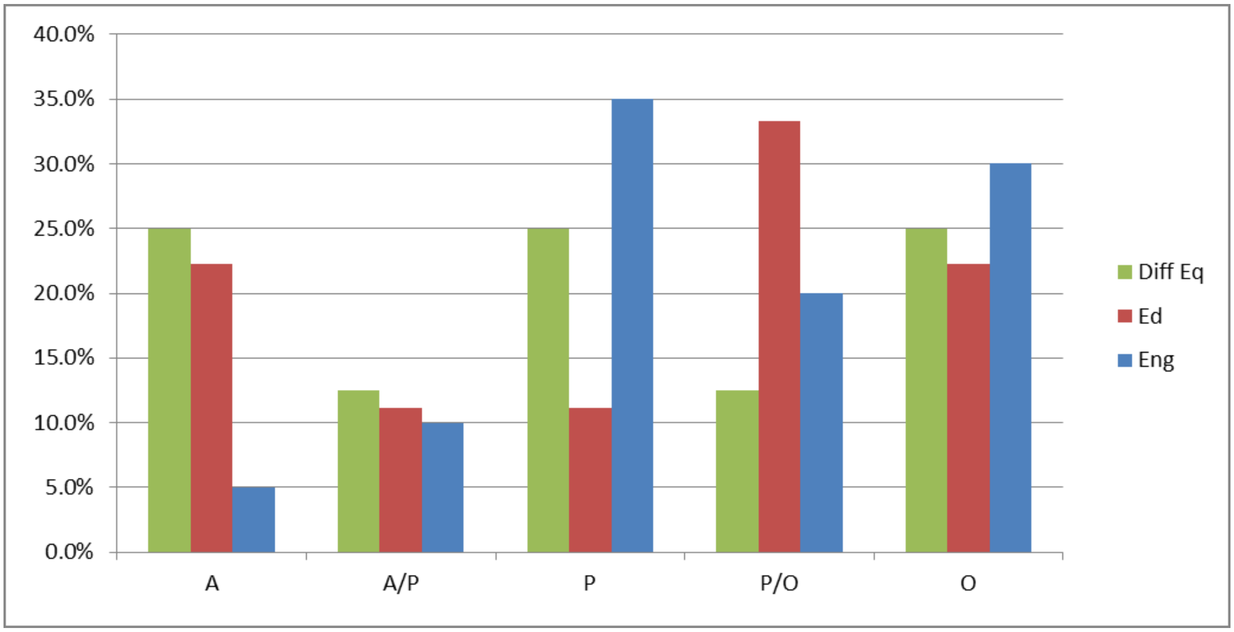

Figure 4.1 The two majors compared with Diff Eq students from the longitudinal study

2nd Pass – Quantitative

As stated in the Methodology section, a matrix was formed using the word frequency for each student and then several non-negative matrix factorizations were run with different parameters. None of these factorizations yielded any significant results with regard to grouping students based on major or level of understanding.

The only factorizations that were even marginally successful at grouping in a discernible way grouped based on words linked to confidence or willingness to participate in the conversation. These words included such words as "don't" and "know", "understand", etc. These groupings prompted another qualitative pass that consisted of coding the transcriptions based on each student's confidence and willingness to participate.

3rd Pass – Qualitative

Each sentence or group of sentences in each interview was rated using the coding scheme describe in the Methodology. This led to each student being rated overall for Confidence and Willingness out of five levels: 1, 1.5, 2, 2.5, and 3 where 1 represented the lowest level of confidence and willingness to participate in the discussion, 2 represents average confidence or willingness and 3 represents the highest levels, with the two intermediate levels put in to refine the scale and make it match up better with the five level modified APOS scale used to measure understanding.

Unlike in the 1st pass, interview excerpts do not easily demonstrate why a particular student was rated at a particular level for Confidence or Willingness. In rating these, it was not as important what the student said, it was more important how the student said what they said or how they responded over the whole interview. Therefore, short excerpts are not presented here, as they would be unhelpful at best. However, two longer excerpts are presented in the Appendices: Appendix D - Interview Excerpt 1, Appendix E - Interview Excerpt 2. Both excerpts are the full text of the conversations of Questions A and B of the protocol. The first excerpt is from the fourth Engineering major to be interviewed; he was rated as having an Object level understanding, with a Confidence and Willingness of 3 each. The second excerpt is from the twentieth Engineering major; he was rated as having a Process level understanding with a 1.5 for both of the Confidence and Willingness ratings.

Confidence still proves difficult, but not impossible, to pick out of the text, since the way things were said has been lost in transcribing it; however, Willingness is more discernible from the text. In the first excerpt, it is clear that the fourth Engineering student needs very little prompting to discuss what he is thinking, with him often offering monologues of his thoughts for extended periods of time. He seemed eager to discuss the topics at hand and therefore was rated at the highest level of Willingness. In the second excerpt, the twentieth Engineering student engages in an average amount of discussion with the researcher, with both contributing about equally to the conversation. However, as the interview passes, the student contributes less and less, often resorting to one sentence or even one word responses with the researching carrying most of the discussion. For this reason, the student was rated as having a 1.5 Willingness, i.e., a bit below average.

The total for each major in each of the levels of Confidence are shown in the next table followed by a graph of the same. The percent of students from that major in each level is in parentheses after the number in the table.

Table 4.2 Distribution of students in Confidence rating

	1	**1.5**	**2**	**2.5**	**3**
Engineers	2 (10%)	6 (30%)	4 (20%)	5 (25%)	3 (15%)

Math Ed	3 (33.3%)	3 (33.3%)	2 (22.2%)	1 (11.1%)	0 (0%)

Figure 4.2 Distribution of students in Confidence rating

As shown in the table and the graph, both groups were spread across most of the levels. However, no Mathematics Education students were rated as having level 3 confidence, whereas some Engineers were. The average Confidence rating of the Mathematics Education students was 1.56, and the average rating of the Engineering students was 2.03.

The total for each major in each of the levels of Willingness to Participate are shown in the next table followed by a graph of the same. The percent of students from that major in each level is in parentheses after the number in the table.

Table 4.3 Distribution of students in Willingness rating

	1	**1.5**	**2**	**2.5**	**3**
Engineers	1 (5%)	1 (5%)	7 (35%)	7 (35%)	4 (20%)
Math Ed	0 (0%)	0 (0%)	6 (66.7%)	3 (33.3%)	0 (0%)

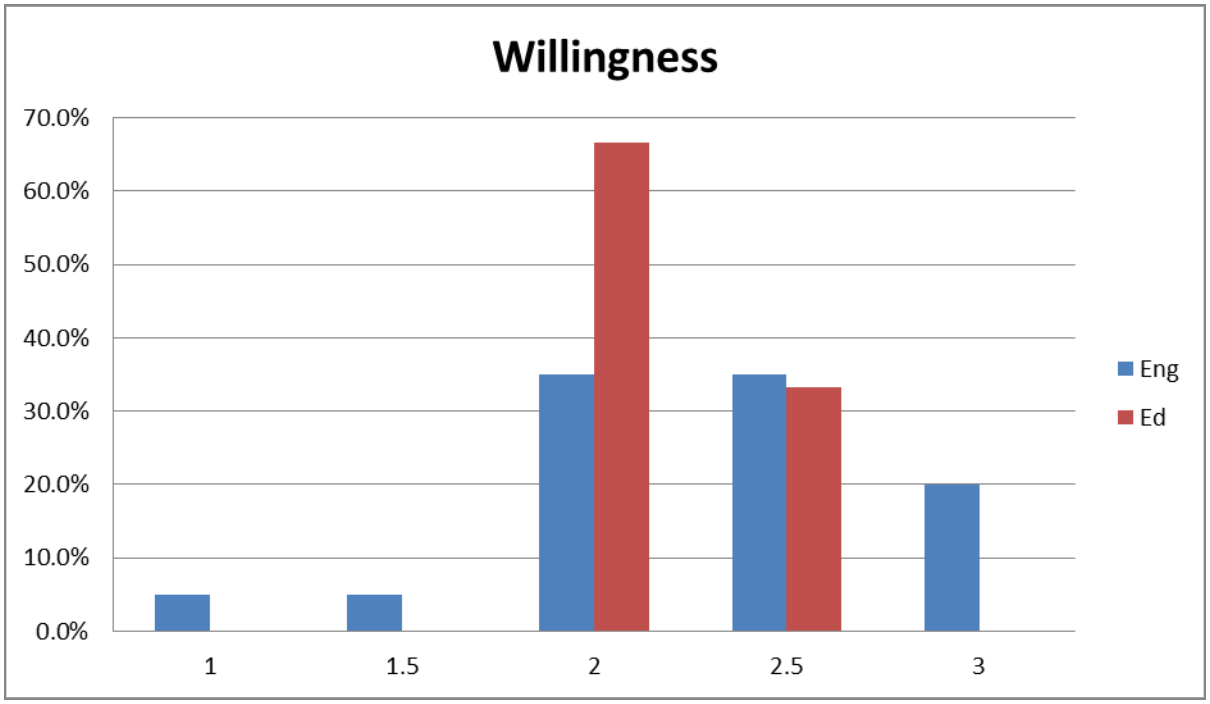

Figure 4.3 Distribution of students in Willingness rating

The table and graph show that the Engineering majors as a group exhibit more variability in their willingness to participate in the discussion than do the Mathematics Education majors. In fact, all Mathematics Education majors were rated as being either level 2 or 2.5, whereas Engineering majors can be found in all five levels. Overall, the average level of Willingness of the two groups did not differ much, with both averaging slightly over 2. The average of the Mathematics Education majors was 2.2 and the average level of Willingness of the Engineering majors was 2.3.

Chapter 5 - Conclusions

Restatement of the Problem

The overall aim of this study was to gain some sense of what happens to students' conceptual understanding of Calculus after they finish the Calculus sequence. There are three possibilities that can happen to a student's conceptual understanding as time separates them from when they first learned that concept. The first possibility is that conceptual understanding decreases over time, i.e., they forget. The second is that conceptual understanding does not change after they finish learning the concept in class, i.e., they have learned what they are going to learn. The last possibility is that the student's conceptual understanding of Calculus increases even after they have finished the Calculus sequence.

This last possibility indicates that learning was facilitated by events that took place after the student left the original class. This is referred to as back transfer: using a learned concept in a later context strengthens that previous knowledge. This study aims to measure whether this back transfer occurs and, if so, whether the application of math into different contexts causes different levels of back transfer.

Individual student interviews with senior Mathematics Education and Engineering majors were conducted and rated on a modified APOS scale to assess levels of conceptual understanding of "function" and "accumulation". These levels were then compared with a previous longitudinal study that tracked the growth of conceptual understanding of students as they progressed through the Calculus sequence. By comparing the levels of understanding of the senior students with the levels of understanding of the previous students at the end of the Calculus sequence, it can be determined how the students' understanding changed after they finished the Calculus sequence.

Summary of Research Methods

Twenty Engineering seniors and eleven Mathematics Education seniors were interviewed individually in hour-long, one-on-one sessions in semi-directed, conversation-style interviews to assess each student's conceptual understanding of certain Calculus topics. These levels of understanding were then compared against similar ratings of students who were just at the end of the Calculus sequence to compare how understanding changed over time after leaving the sequence.

Along with these ratings, a vocabulary matrix was made that consisted of how many times each student said each word from a word bank consisting of every different word said during the interviews. From this, non-negative matrix factorizations were created with varying parameter in an attempt to classify different groups of students based on different vocabularies. These matrix factorizations prompted the researcher to code each phrase from each student interview for relevant attributes. These attributes were the following: topic being discussed, level of understanding that the student is currently working with, specific representations or actions being discussed, confidence being demonstrated by the student, and the willingness of the student to participate in the current discussion.

These two majors were chosen for the interviews because they both take the Calculus sequence, Calculus I through Differential Equations. The Mathematics Education majors then go on to apply this knowledge to higher level, abstract mathematics classes while taking little to no mathematical science classes. Engineering majors, on the other hand, apply this knowledge to higher level mathematical science classes while taking little to no higher level, abstract mathematics classes.

Once interviewed, each student was rated on a modified APOS scale. Along with the standard APOS levels of Action, Process, Object, and Schema, the researcher added the intermediate levels of Action/Process and Process/Object between those respective levels. This modification is important in two ways: first, it was designed for the previous longitudinal study and so it was designed to be a finer scale that could more adequately measure changes in understanding between semesters, and second, it allows for more stable classification for those students who are on the boundary between Action and Process or Process and Object. For students on the border, a small change in classification could shift them from an Action rating to a Process rating. The modified scale dampens these shifts.

Each student was also given an overall rating for Confidence and Willingness to Participate. These ratings were given to reflect an average for the "confidence" and "willingness" ratings in the coded transcripts. In order for these ratings to conform to the modified APOS scale, the system that coded students with a 1, 2, or 3 was modified to admit the intermediate levels of 1.5 and 2.5. This allows for a finer rating system as well as makes it more comparable to the ratings of conceptual understanding.

Summary of Findings

APOS Ratings

Of the 20 Engineering seniors, one (5%) was rated at Action, two (10%) were rated at Action/Process, seven (35%) were rated at Process, four (20%) were rated at Process/Object, and six (30%) were rated at Object. Of the nine Mathematics Education majors, two (22.2%) were rated as Action, one (11.1%) was rated as Action/Process, one (11.1%) was rated as Process, three (33.3%) were rated as Process/Object, and two (22.2%) were rated as Object. In the previous longitudinal study, 25% were rated at Action, 12.5% at Action/Process, 25% at Process, 12.5% at Process/Object, and 25% were rated at Object. These results are summarized in the following graph.

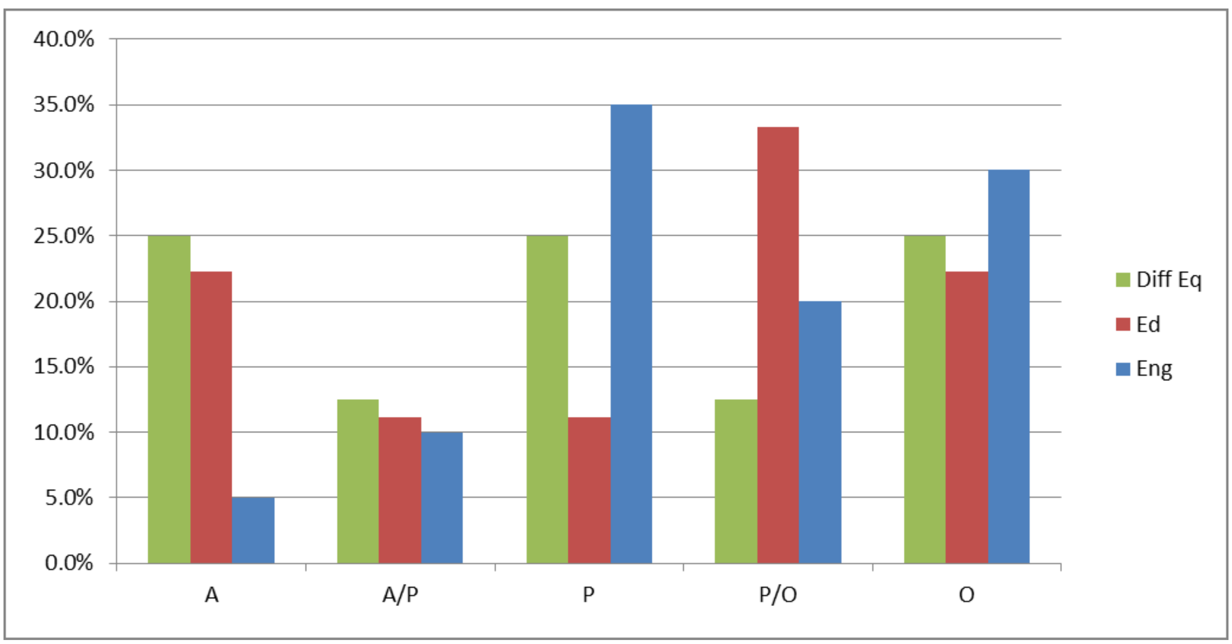

Figure 5.1 Comparison of the two majors with Diff Eq students from longitudinal study

Confidence and Willingness

Overall, the 20 Engineering seniors averaged 2.03 Confidence while the nine Mathematics Education seniors averaged 1.56 Confidence. So the Engineering majors were moderately confident on average whereas the Mathematics Education majors were a little below average confidence as a group. Both groups fared a little better than average in their levels of Willingness. The Engineering majors averaged a 2.3 rating for Willingness and the Mathematics Education majors averaged a 2.17 rating. The following two tables and two graphs summarize the findings for Confidence and Willingness.

Table 5.1 Distribution of the two majors in Confidence ratings

	1	**1.5**	**2**	**2.5**	**3**
Engineers	2 (10%)	6 (30%)	4 (20%)	5 (25%)	3 (15%)
Math Ed	3 (33.3%)	3 (33.3%)	2 (22.2%)	1 (11.1%)	0 (0%)

Figure 5.2 Distribution of students in the Confidence ratings

Table 5.2 Distribution of students in the Willingness ratings

	1	**1.5**	**2**	**2.5**	**3**
Engineers	1 (5%)	1 (5%)	7 (35%)	7 (35%)	4 (20%)
Math Ed	0 (0%)	0 (0%)	6 (66.7%)	3 (33.3%)	0 (0%)

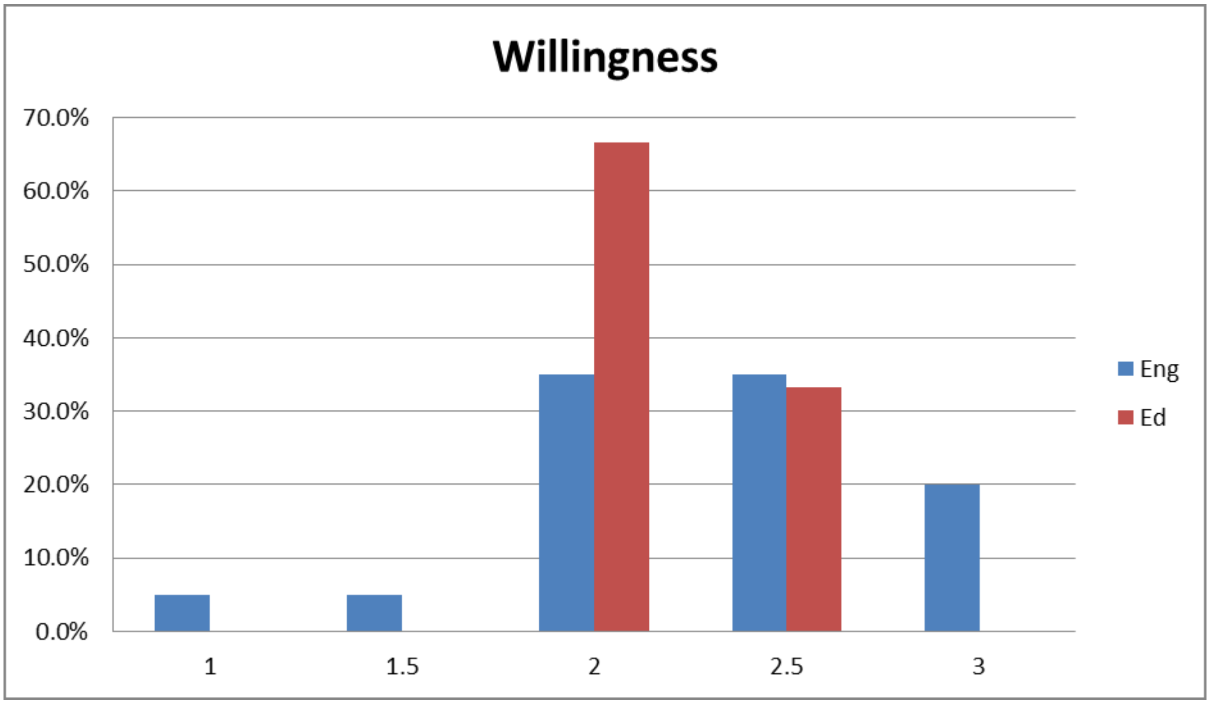

Figure 5.3 Distribution of students in the Willingness ratings

Discussion of Results

Discussion of APOS Ratings

As can easily be seen in the results, the differences among each major are larger than those between the majors. Both majors had students rating in each of the five modified APOS levels from Action to Object. Both majors also mostly ranged over each of the five levels of Confidence, as well, except that no Mathematics Education seniors rated as a 3 in Confidence. Most of the Willingness levels, however, saw only Engineering majors, with the Mathematics Education majors only falling in the 2 and 2.5 levels.

When comparing the APOS ratings of the seniors with those of the students in Differential Equations, it seems that the Engineers as a group have moved to the right, toward the Object side of the scale. While the percent in the Object level has increased from 25% to 30%, the percent in Process or above has increased from 62.5% to 85%. All three levels Process or above saw an increase in the percent of students at that level. This seems to indicate that senior Engineering majors have a deeper conceptual understanding of the Calculus topics of “function” and “accumulation” despite having not taken more mathematics classes after Differential Equations. This would indicate that back transfer is taking place.

It is not as clear whether the group of Mathematics Education majors exhibited a similar movement. There seems to be a general upward movement except at the upper end, but overall it is not as obvious as with the Engineering majors. While the lower levels of understanding saw decreases among the Mathematics Education majors, so did most of the upper levels. The only level that saw an increase was the Process/Object level. Most of the levels did not change much when compared to the levels of the Differential Equation students, which may not represent any significant change when considering the numbers involved. Action, Action/Process and Object decreased by less than 3%, so most of the movement would have been students moving from Process to Process/Object. Overall, the Engineering seniors understand the concepts more deeply and exhibit more movement up the APOS ratings than do the Mathematics Education seniors.

Somewhat surprisingly, there are still students from both majors that have not advanced beyond the lower two levels of understanding. 33.3% of Mathematics Education seniors are still in the Action and Action/Process levels while 15% of Engineering seniors are in those two levels. This shows that there is still a significant portion of seniors who do not truly understand the basic Calculus topics that they are using in their majors and presumably in their careers in the future. The eighth Mathematics Education student, who was rated at Action, illustrates this when asked about maximizing the shaded area in Question C, "I don't remember what it is, I've seen it before and I've had to learn it for a test, but I can't recall it at all."

These students must be finding success in their mathematics and mathematical science classes since, to get to be seniors in either major, the student must have gotten better than passing grades in many such classes in the past. These students are likely "learning" the mathematics by rote memorization without achieving a real understanding of the concepts involved, but they are finding success at this method in as much as they are achieving their degree.

Not only are they passing through college using rote memorization without a strong conceptual understanding, they may not even realize that there are deeper concepts involved, i.e. that rote memorization is not the only way to learn math. This is again illustrated by the eighth Mathematics Education senior when answering Question C:

ED8: In this equation... See this is why I don't think you should... if you're going to be a math teacher you shouldn't go over um... struggling for math because then you forget things that you've done before.

This student is expressing the frustration that her cognitive load has surpassed her ability to memorize that many facts and procedures. It seems that her view on learning mathematics is the common one among students that is memorizing what procedures and formulae to use and when to use them. This is the same students who, when asked to draw the graph of the parametric equations in Question B, drew the unit circle and explained that she knew that this was the graph because "that is what the teacher told me it was".

This student has not progressed beyond the Action level of understanding and this is probably at least partially because either she does not think she can understand these concepts more deeply and therefore must use rote memorization or because she does not realize there is more to understand more deeply about these concepts and therefore rote memorization is the only thing there is.

Discussion of Confidence and Willingness

The Confidence and Willingness ratings differed between the two groups by more than the APOS ratings did. Engineering majors, as a group, rated just above average confidence at 2.03 out of 3. Mathematics Education majors, as a group, rated well below average confidence at 1.56 out of 3. So, overall, Engineering majors are more confident on average when discussing mathematics problems than are the Mathematics Education majors despite both groups having higher conceptual understanding than the average student in Differential Equations.

While the averages of the Willingness rating did not differ very much between the two groups (2.3 versus 2.17 out of 3), the Mathematics Education majors varied far less in their Willingness than did the Engineering majors. The Engineering majors were spread across all 5 ratings while the Mathematics Education majors were only rated at the 2 and 2.5 levels. This may reflect different training or different philosophy and culture between the two majors.

Anecdotal Observations

The researcher made several observations that were not specifically measured for or that originate from subgroups of students that are too small to make proper scientific conclusions.

These results must be considered as anecdotal observations; however the observations often have interesting implications for future research or shed light on certain groups of students.

One repeated observation is that specific groups seem to be more likely to have a deeper conceptual understanding of the topics. One of these groups was returning students. There were only one or two of these students interviewed in this study, but these students conformed to the researcher's prior anecdotal experience. There are a number of reasons that these students might have deeper conceptual understanding. These students have gone out into the real world and had many experiences that the average college student has not had yet. These experiences could offer a new context in which to apply their mathematical knowledge, and thus offer a different context from which back transfer can happen. Another possibility is that these experiences occur before they learn the concepts the first time and therefore could represent preparation for future learning (forward transfer). A third possibility could simply be that they are more likely than the average student to be paying for their education and are therefore more serious about gaining a deeper understanding than the other students are.

Another group that appears stronger on average was computer programmers. No students were asked specifically if they were computer programmers; however, there were at least four Engineering seniors who identified themselves as computer programmers. These students seemed particularly strong, averaging just above Process/Object (one Process, one Process/Object, and two Object). At least one third of Engineering seniors who rated at Object identified themselves as computer programmers. These students related functions to methods or subroutines in programming languages. The following is an excerpt illustrating an occasion of this:

Int: Is this, the act of taking a derivative, is that a function?

EN8: I have no idea.

Int: Do you think of that as a function?

EN8: I think of it more as almost like an operator.

Int: Ok, so what's an operator?

EN8: Um... It just tells you to do something with whatever follows it. I don't know. I think of more in programming language where d/dx... if you say... usually I write that out d/dx and then in the parentheses after that I'll put the function I want to do... I guess I think of it more as the function is the argument of the...

Int: So it's like a command that you're doing?

EN8: Yeah, it's like a command, but at the same time I guess they call those functions in programming sometimes too, so...

This type of comparison was common among the computer programmers where they would compare the derivative or similar function to the methods used in programming languages. This could be a clear example of context-assisted back transfer. They have learned about "function" in the Calculus sequence, but by seeing the same concept used in the context of programming languages, the student gains a deeper understanding of the abstract concept. It is somewhat telling that the student related back to a context that they were more comfortable and more interested in rather than relating back to the mathematical context in which they learned the concept originally.

Also illustrated in the above excerpt is another very common pattern among many of the students interviewed including Engineering seniors as well as Mathematics Education seniors. Many in both groups, including most of the Object level students, wanted to make a distinction between "functions" which take numbers to numbers and "operators" or "transformations" which take functions to functions or transform one function into another function. They often make this distinction even when they understand that the two ideas are essentially the same thing. The following is another example of a student making a distinction between these two ideas:

Int: What if we think about the process of taking a derivative? We take f to f'. Is the process of taking a derivative a function?

EN17: Um... I would say no. Because, um... You could say maybe y equals... d/dx that's f, but then that would be... but if you're saying y is a function of, um... you know, x and f is a function of x. You could say that. But, um... just the process of taking a derivate is... I think of it as more of a... um... transformation... than anything else.

Int: So what's the difference?

EN17: Um... Well, when you... when you take a derivative of something, you're um... you're altering it or transforming it. Um... So for example, if you just take... x and you take three times x, that's not necessarily a function, it's just... uh, an operation. Um... you know, on x.

Int: So a function takes a number and assigns to it a number and a transformation takes something and changes it into something else?

EN17: Right.

The following is an example of the same behavior but where the student refers to "operators" instead of "transformations":

Int: So if I assign each function to its derivative… is the act of taking the derivative a function?

EN6: Uh, no it's an operator.

Int: Ok, what's the difference?

EN6: I... an operator is something that you apply to functions.

Int: Ok, so the only difference between functions and operators is that operators apply to functions?

EN6: Sure, like when you apply a function to... you apply functions to values and get out single values and you apply an operator to a function to get out another function.

Int: But didn't you just say I could have a function applying to erasers to give me markers?

EN6: Sure, like, but you apply an operator to the group... like... you know, so you apply the function to... a single eraser to get out another single thing. But an operator takes entire functions.

In the first case and third case (EN6 and EN8), the students were rated as having an Object level understanding of the concepts and in the second example (EN17), the student was rated as having a Process level understanding. So even when the students have a strong conceptual understanding of "function", they still are more comfortable making the superfluous distinction between "functions" and "operators" even when they recognize that these are two words for essentially the same idea. This probably has to do with the fact that much of the treatment of functions of functions, such as differentiation, happens in Differential Equations where it is referred to as the "differentiation operator" or "derivative operator" and when they learn the similar concepts of "transformations" such as the Laplace Transform.

The students have learned that this is how the mathematicians refer to functions that take functions to functions and so they must be different concepts somehow even if they recognize that they are conceptually the same. This is a recurring theme that shows up in all levels of students. Students often believe that mathematicians want an overly formal way of saying things just right even when they run counter to intuition. The following excerpt is from an Object level student illustrating this:

Int: Is the process of taking a derivative a function?

EN15: I wouldn't say yes if it was on a math test, but in a broader sense of the word function, I would say yes. Like I wouldn't say it's like a function the same way that f of x is a function. Like if you asked me that when I was taking Calc I, then I would say no. If you asked me that in an intellectual conversation over the dinner table, I would say that.

Int: So you'd say maybe it's not technically a function, but you view it as a function?

EN15: Um... Kind of, yeah.

Int: Alright, why?

EN15: Well, because I think of the word function as something that does something. Like, you do something to x to get y. Um... So, that kind of... You would define that mathematically as an operation. But um... I kind of think of it as a function in terms of um... linguistics.

Int: So as far as this goes, a function is something that turns something into something else?

EN15: mm... I guess. Sure.

Int: Ok, so because this changes f into f', it's a function?

EN15: Yeah. Yes. I always think of it as like the... like I said, that's not the... the definition that you... I would give in a math class because I know that's not what they want to hear. I like to get... give the right answers on math tests. Yeah, like if you change f, you get... different f primes. So, I mean, I would... I would say that it's an operation on a math test. But... it relates f to f prime.

This same belief that there is a formal, "correct" answer to a question when it is asked by a mathematician and an intuitive answer when actually thinking about the problem was also illustrated by the following student who was rated as Process/Object:

Int: What is a function?

ED5: Uh... As far as a formal answer goes... I couldn't uh... I couldn't give you other... much other than... it's a relationship between two... two um... variables, where you input one... you put... you have an input for one variable and that gives you the output for another.

Since this student was rated as Process/Object, he obviously had a rather deep understanding of function; however, when asked by a mathematician what a function is, instead of giving an answer that drew upon his understanding and experiences, he tried to give "a formal answer" and indicated that he did not really understand very well. This belief that the teacher is looking for a particular, "correct" answer must be a strong one as both of the previous examples of students exhibiting this behavior (EN15 and ED5) were rated as having a 2.5 out of 3 Confidence rating as well as ED5 achieving a Process/Object rating and EN15 an Object rating. These students are confident that they either know the concepts or confident that they can reason through it if they do not know it, and they actually have a relatively deep understanding of the concepts, and yet they still believe that the teacher is looking for a different answer than they as the students are.

Engineering versus Mathematics Education

Other than that an Engineering context effects more back transfer than does Mathematics Education, there are a number of possibilities that could explain why the Engineering seniors seem stronger than the Mathematics Education seniors in terms of conceptual understanding, Confidence, Willingness. One possibility is that self-selection is affecting it. Perhaps those who are interested in Engineering begin at a higher level of understanding or with an stronger ability to increase their understanding. In this case, the differences in amounts of back transfer between the two groups do not necessarily indicate that Mathematics Education admits less back transfer; the Mathematics Education seniors could have the same amounts of back transfer or even greater and still end with less understanding if they started out far behind the Engineers to begin with.

Along the same line, perhaps Engineering is a harder major and weeds out the poor students, thereby increasing the percent of good students remaining. These two possibilities are either not likely or do not have a strong influence on the final outcome as evinced by the significant percent of Engineering seniors who have not progressed past the Action and Action/Process levels. About fifteen percent of those senior-level Engineers interviewed are still at these two lower levels of understanding. These students clearly did not start out with a higher level of understanding as they are still at the lowest levels of understanding. They also were clearly not weeded out of the major since they are nearly all the way through the program. So if Engineering is trying to weed out students who do not understand the mathematical concepts very deeply, they are not doing a very good job of it. As a matter of fact, along these lines, in *Talking About Leaving*, Seymour and Hewitt (1997) studied 335 SME (Science, Mathematics, and Engineering) majors from a variety of four-year institutions by using interviews and focus groups. They found that students switching out of SME majors did not differ significantly from those who did not switch in terms of "performance, attitude, or behavior". Switching majors was more likely caused in the students by whether or not they developed coping strategies, luck, and satisfaction with their instruction and major.

Another possibility is that the historically different cultures of the two majors affect the understanding of the students in them. The two majors have developed historically in very different ways and even today have very different demographics and gender composition. One difference between the two is that Engineering tends to be competitive and individual whereas Mathematics Education tends to be more community oriented. It is possible that this difference in competition causes the differences between the students when looking at these few metrics of conceptual understanding, Confidence and Willingness. This difference between individuality and community could possibly explain why the Mathematics Education students are much more uniform when measured in Willingness to Participate when compared to the Engineers who range across all five levels.

The gender differences between the two majors (Engineering is dominated by men while Mathematics Education has a larger portion of women) may also play a role. Many studies have been conducted on the personality differences between the two genders. Among many other results, they have found that men tend to be more assertive and aggressive, as well as less anxious (Feingold, 1994). Men also tend to perform more effectively in competitive environments (Gneezy, Niederle, & Rustichini, 2003), and have higher confidence even when answering questions they do not necessarily know the answers to (Lundeberg, Fox, & Puncoshar, 1994). These differences in self-efficacy may lead to lower performances as a self-fulfilling prophecy (Feingold, 1994). Given the gender differences between the two majors, this might explain some of the differences in the levels of understanding, Confidence, and Willingness.

One possibility that could further support the existence of back transfer is simply that Engineering majors tend to see many more differential equations than Mathematics Education majors do. Mathematics Education majors take higher level mathematics classes, but these do not usually involve differential equations. Engineering majors, on the other hand, take mathematical science classes which often involve differential equations. One can see in Figure 2.1 that no students are rated in the Object level of understanding until Differential Equations. That class is the first place that heavily stresses functions as objects in terms of using functions as inputs, getting functions and families of functions as answers, and using processes such as differentiation and Laplace transforms in the way one uses functions. It could be that differential equations are a particularly effective tool for understanding functions and so the group that uses them more will exhibit a deeper understanding.

Limitations of the Study

All studies have their limitations and this one is no different. The limitations of this study fall into five categories:

- Self-selection issues.
- Demographic differences between the two groups.
- Different sizes of the two groups.
- Comparison with the previous longitudinal study.
- Limited scope of the study.

Since this study was run on a volunteer basis, there is the always-present issue with who chooses to participate. There is no real way to correct for this without making the study a mandatory part of a class from each major. However, in this study, the number of participants from each group represents roughly half of each class from which each major was recruited. So while the study was strictly voluntary, the numbers represent a large cross section of the class.

There are self-selection issues not just with who volunteered to participate in the study, but also the larger self-selection issues with who chooses to become Engineering majors versus Mathematics Education majors. This was discussed above in the Engineering versus Mathematics Education section. This is an issue that would be impossible to adjust for since the researcher clearly has no control over who decides to become an Engineering major.

Related to self-selection problems are the demographic issues involved both with this study as well as the differences between the two majors. Again, this was discussed above in the Engineering versus Mathematics Education section, but along with those issues, there are the demographic issues specific to this study. Of the nine Mathematics Education seniors interviewed, roughly half were male (5 out of 9) compared to the 20 Engineering seniors of which all 20 were male. This is, of course, related to the historically different gender structure of the two majors where Engineering is still an overwhelmingly male major, but since Engineering is not one hundred percent male, it would have made for a stronger study to have female Engineering volunteers participate in the study. In the end, however, this likely does not affect the results in a significant way.

The third limitation of this study was that only nine Mathematics Education seniors were evaluated in the study compared to the 20 Engineering seniors that participated. The university where this study took place has a strong Engineering program while the numbers of Mathematics Education majors are much smaller. This is why there is a discrepancy in the numbers of students interviewed in the two majors; it has only to do with the total number of students available to draw from in each major. As the main goal of this study was to measure back transfer, the small numbers of Mathematics Education seniors does not hinder this task since there was still a small amount of back transfer measured with the Mathematics Education majors and a larger amount of back transfer measured with the Engineering majors. The second goal of this study was to measure differences between the two majors. This goal is likely also not hindered by the smaller numbers as the findings were sufficiently different between the two groups that they would not have been greatly swayed by the addition of a few more students. This is especially likely when considering the students who would volunteer for this type of interview would likely be those with higher levels of understanding, confidence, and willingness to participate, each of which the Mathematics Education majors measured lower in than the Engineering majors.

The fourth limitation of this study is actually a limitation of the previous longitudinal study upon which this study draws. The previous longitudinal study did not differentiate between majors; it lumped all students together in each level of understanding. Therefore, the baseline that this study compares the two majors against is an imperfect baseline as it consists of both majors and is being used to compare against each major individually. Given the distribution of majors at the university in which this study took place, most of the students in Differential Equations would be Engineering majors. In light of the differences in ratings between the two majors found in this study, this limitation would appear to unfairly handicap the Mathematics Education majors given that their baseline would consist mostly of Engineering majors. Therefore, it is possible that the amount of back transfer actually present among the Mathematics Education majors was greater than was measured; however the goal was to measure whether back transfer was occurring and that appears to be the case regardless and, regardless of the baseline, there is still a large measured difference between the two groups of seniors.

The fifth limitation of this study is the limited scope that the researcher was able to take on with this study. The goal was to measure whether back transfer was happening and, if so, whether context affected the amount of back transfer. Therefore this study aimed to measure a different level of understanding possessed by seniors compared to students in Differential Equations as well as compare Engineering seniors with Mathematics Education seniors. The goal was not to explain how this back transfer occurred or to explain why different contexts correspond to different levels of back transfer. Answers to those questions would require much more extensive and directed research now that the existence of context-dependent back transfer has been established and would constitute a possible area of future research.

Recommendations for Future Research

This research provides a launching point for future research in many different directions. The most obvious question that should be researched in the future is "what causes the difference between the two majors in terms of Confidence and Willingness and what about the different contexts causes different levels of back transfer?" Since we want our students to have a deeper understanding of the concepts and back transfer can offer a pathway in achieving this, it is important to know more thoroughly how and why back transfer occurs.

More to the point, a question for future research would be "can we increase the likelihood of back transfer?" Even if we do not discover exactly how and why back transfer occurs, we can still achieve the benefits if we discover how to increase back transfer. For example, since Engineering majors seem to benefit from more back transfer than do Mathematics Education majors, perhaps it would be beneficial for Mathematics Education majors to have more experience applying the mathematical concepts in the context of real world, scientific situations. Perhaps having them take more mathematical science classes would increase the depth of understanding exhibited by seniors. Similarly, in regards to the anecdotal observation that computer programmers seem to have a deeper understanding of function, perhaps all majors would benefit from taking more computer science classes so they can become more comfortable seeing functions in that context. Or since differential equations seem to be an important contributor to a deeper mathematical understanding of "function", then future research could be aimed at having students see more differential equations contexts and measuring the effect on understanding.

Since Engineering seniors rated higher in all three categories, conceptual understanding, Confidence, and Willingness, how are these ideas correlated? Do confidence and willingness facilitate ability? Does ability increase confidence and willingness? Were Engineering majors more confident and willing to participate because they had higher levels of understanding, did they have higher levels of understanding because they believed more that they could understand, or is does it just have something to do with the difference between Engineering majors and Mathematics Education majors? It would be instructive to further explore this relationship since we would be provided another method of facilitating learning if, by raising confidence and willingness, we could push conceptual understanding.

Future studies could also be done that attempt to negate the limitations of this study. For example, simply repeating the study would enlarge the sample and lessen the potential effects of the demographic differences as well as the population size issues. The previous longitudinal study was limited by not differentiating between majors. This could be rectified by a study that interviews a group of students in Differential Equations, and then interviews them again in a couple of years when they are seniors. This would have the advantage of measuring back transfer in individual students and not just in the group overall.

Summary

The goal of this research was to measure if back transfer was occurring and, if so, to measure the effect of applying the mathematics in different contexts has on this back transfer. Engineering seniors and Mathematics Education seniors were interviewed in one-on-one, hour-long, conversation-style interviews. Each student was then rated on their conceptual understanding of "function" and "accumulation" on a modified APOS scale, their confidence in their ability to answer mathematical questions, and their willingness to participate in the discussion of mathematical topics they may or may not understand. The overall distribution of each major on the modified APOS scale was compared against students in Differential Equations from a previous longitudinal study used as a baseline to see if seniors had higher levels of understanding of Calculus concepts despite not having taken any more Calculus. The levels of understanding of the two majors were compared to see if the two different contexts resulted in different amounts of back transfer.

The first result of this study was that it does appear that back transfer is occurring. Seniors in both majors exhibited deeper levels of understanding on average of "function" and "accumulation" than students who were just finishing the Calculus sequence. Since the conceptual understanding of Calculus concepts increased in both groups even without taking more Calculus seems to imply that applying this knowledge in context of later major-related classes deepens the understanding of these concepts.

It also appears that the context in which the mathematical knowledge is applies affects the amount of back transfer occurring. Engineering majors seemed to experience more back transfer than did Mathematics Education majors. This is possibly indicates that real-world contexts of mathematical science classes allow for deeper understanding than do abstract, higher-level mathematics classes, or it may indicate that differential equations, specifically, facilitate back transfer effectively. As further evidence that different contexts lead to different amounts of back transfer, the Anecdotal Observations section above indicates that computer programmers may have a deeper understanding of “function” because they have seen it and relate it in terms of programming languages, though this would take more research to verify.

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Appendix A - Modified APOS Rubric

Table A.1 Rubric for rating conceptual understanding from previous longitudinal study

Level	Description	Function Example	Accumulation Example
Action	Students can “do the math” to arrive at an answer, but their only strategy is to follow algorithms they have been taught without any vision of what lies ahead, what the steps mean, or why we do those steps. If the student does not know how to do a problem they will run through their catalog of procedures until either they find one that works or they give up. Functions, integrals, etc. are static formulae that are evaluated by means of an algorithm.	A function is an equation or a graph that passes the Vertical Line Test. Functions are static; they are analyzed by putting numbers into the equation or looking at points on the graph. The inverse of a function is a different function. Different representations are different things.	An integral is understood by taking an antiderivative and then plugging in a number. An integral is “area under the curve” but this is only understood as something the teacher said they were supposed to memorize and probably cannot be applied to evaluating integrals. The student cannot dynamically visualize how an integral changes as x changes; if they do not have an explicit way to obtain the equation of the integral, they can only plug in different x’s to analyze it. Given the graph of a hemisphere, students will try to obtain the equation and then integrate using antiderivatives.
Action/Process	Students still follow prescribed steps but they show some understanding and adaptability. They can recognize mistakes and contradictions, but they may not be able to explain them or figure out what went wrong. Alternatively they may skip back and forth between Action and Process.	A function passes the Vertical Line Test, but this does not have to be a graph. The VLT can also be applied to tables and equations. The inverse is related to the original function by switching x’s and y’s. The student can begin to see how varying the x’s will affect the y’s. They can begin to relate representations.	Integrals can vaguely be estimated from a graph of the original function, but the student will likely resort to pointwise calculations to figure out a rough sketch. The students will recognize that the integral can be gotten from moving the x’s and visualizing how the area under the curve changes, but they will have a lot of trouble actually doing so.
Process	The student can reason through a problem, recognize the reason for steps, connect different representations, etc. The object in question, though, is still an	A function is a separate notion from an equation or graph. Functions take an input to at most one output; the inverse undoes this process. Inputs and outputs do	The student can visualize an integral from the graph of a function by varying the x’s and seeing how the area accumulates under the graph. Given a hemisphere the student will recognize that it will be easier to

	intangible process rather than an object with properties that exhibit themselves as that process.	not have to be numbers. Graphs and equations depict outcomes of a process.	evaluate the integral as a fraction of a circle, unless this strategy will be difficult - such as when the integral does not come out as a nice fraction of the circle – in which case they will switch to a formulaic approach. An integral is not a function until it is evaluated.
Process/ Object	Students begin to be able to talk about the concept in question as a thing and not just as an abstract process.	They can talk about functions as objects but they may still have trouble manipulating them as such.	Integrals are functions even when still written in integral notation (i.e. before the antiderivative is found).
Object	The concept is now something that they can manipulate. The processes and representations are properties that the object has.	Functions are objects that can be manipulated just like numbers or variables. A function can take other functions as inputs and outputs. Graphs and equations are representations of a single entity.	An integral is an object; it does not have to be evaluated to be a function. Actions and processes can be performed on integrals (addition, etc.).

Appendix B - Interview Protocol

Senior Interview Protocol S11

1. *Prepare for the interview at least 5 minutes before the scheduled time. Unlock the conference room and leave the door open. Set out the IC Recorder, two copies of the Informed Consent form and a pad of paper for students to write or draw on as needed when they answer the questions.*

2. *When the student arrives, introduce yourself and welcome the student by name. Close the door to the conference room. Ask for permission to record the interview. If permission is granted, start the recorder.*

3. *Explain the purpose of the interview:*
   We are interviewing students to better understand what students are learning about math. This is prompted by a desire to better prepare students for later courses. I will ask you some questions about your knowledge of functions and integrals so we can better understand what and how students have learned from math courses. Some of these questions have right and wrong answers while other questions may not have a particular right answer, but are intended just to give an idea of how you think about the material. This interview should take approximately an hour. Your participation is completely voluntary and your grade will not be affected by your decision to participate or not. You will receive $10 for your time for participating in this interview and you may also benefit by improvements in instruction in mathematics and by having a chance to go over some mathematical ideas with an instructor. In the event we include any of your comments in any discussion or publication, your privacy will be maintained by the use of a pseudonym. We have two copies of an Informed Consent Form for you to sign; one for our records and one for you to keep.

4. *Have them read and sign the form. If they decline to sign the form, thank them for their time and terminate the interview. Otherwise sign and date the form as witness and then proceed to the questions below.*

5. *Mathematical questions. The purpose is to get a sense of what level of understanding of functions and accumulations (integrals) that different students are at. If a student gets stuck answering a question, you may offer a hint if appropriate to get them unstuck. Don't let the interview stretch over an hour. If students make mistakes, please correct them either at the end of the problem or at the end of the interview as appropriate.*

Protocol

(a) *Questions about function.*

- What is a function?
- Is differentiation a function?
- What is its domain? Range?
- Is $\int_0^{1.3}(x^4 + k)dx$ a function? of what? Domain? Range? What does it look like?
- Is the Laplace transform a function? of what? Domain? Range?

(b)

- Graph x(t) = cos t, y(t) = sin t, $0 \le t \le 2\pi$.
- Graph x(t) = cos 2t, y(t) = sin 2t, $0 \le t \le \pi$.
- Are these the same functions?

(c) *Give the student the graph with the quarter circle*

- At what height does the line have to be placed in order to maximize the shaded area?

(d)

- What is math?
- What is the role of theorems in math?
- What is the role of logic/deductive reasoning in math?
- What is the role of math in physics?

(e)

- Given that a_g = 32ft/s/s, how fast is a stone falling 1 second after being dropped?
- After 2, 3, 4 sec?
- Which models the situation: v = 32t or t = $\frac{1}{32}$v
- Are these the same function?

(f) *A car starts out moving 10mph, but it continuously slows down by 1mph every hour. So after 1 hour, it is only going 9mph, and so on.*

- How long does it take for the car to go 5 miles?
- What is the farthest the car travels?

(g) *A car starts out moving 10mph, but it continuously slows down by 1mph for every mile it has traveled. So after 1 mile, it is only going 9mph, and so on.*

- What is the farthest the car travels?
- How long does it take for the car to go 5 miles?
- What is the farthest the car travels?

End of the Interview

(a) *Other comments. Are there any other comments or questions you would like to make about any aspect of the course? (Ask follow-up questions or provide answers (if you know the answers) as appropriate. You may tell the student you will refer questions to Prof. Bennett and he will get back to them if you don't feel you can adequately address a question (say about why we are doing labs in the course))*

Final Questions

i. How many more semesters until graduation?
ii. What math classes have you taken?
iii. What math-related science classes have you taken?

(b) *Thank the student for participating. Gently correct any errors they made that haven't been addressed already. Let them know they are always welcome to email any additional comments or suggestions about the course.*

(c) *Stop the recorder. Have the student fill out a receipt form with their*

- Name
- Home Address
- WID Number
- Date
- Signature

Once you have the completed receipt, give the student $10 and thank them again. Place the white copy of the completed receipt in the money envelope and leave the yellow copy in the receipt book.

(d) *As soon as you have time (since sometimes there are two interviews back to back), please write up notes on the interview, transfer the recording to the computer system and erase the session from the IC Recorder.*

(e) *Write the name of the student/interviewee on each piece of scratch paper used and file the paper with his/her consent form*.

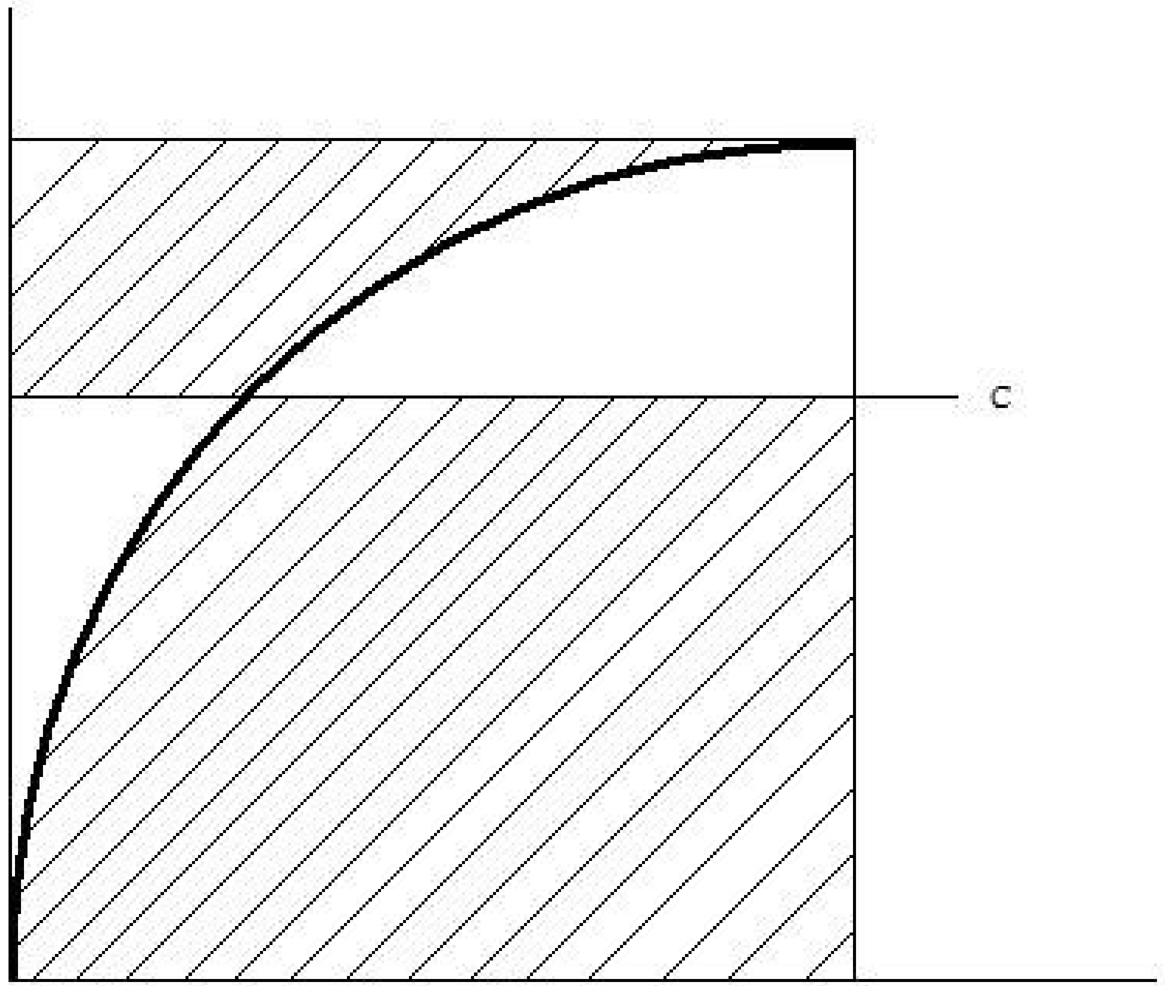

Figure B.1 Graph given to students for question (c)

Appendix C - Sample Interview

The following is the full text from the interview with the first Engineering major. This student was rated at Process with a Confidence of 1.5 and a Willingness of 2.

Question A:

Researcher: So, first question is... um... what is a function?

Student: A function uh.. do you want me to tell you what the definition is? or write the definition

Researcher: um... you can just tell me, yeah, what, what the definition of a function is, or how you think about a function.

Student: Yeah, it's been a while since I've had any technical or.. I guess.. uh.. what the school's definition is.. uh..

Researcher: So, what is your definition of a function?

Student: It should be... a... a function... uh...

Researcher: make you think way back

Student: yeah Describing colors to a blind person. It's a formula using ...uh I think of a function as you've got like your what it's called and then like equal and then you've got your formula of the rest of the...

Researcher: So a function is an equation then?

Student: Yeah, an equation. um...

Researcher: So If we think about... um... So is any equation a function?

Student: Off the top of my head, I want to say yeah, but it seems like a trick question. I mean, I think there's got to be some

Researcher: Mathematicians like to ask trick questions.

Student: Yeah.

Researcher: So, y equals x squared: that's a function, right? It's a parabola.

Student: Doesn't it have to be like differentiable at points. I don't know, something. Maybe not. Or continuous. It has to be continuous doesn't it? Maybe not. Because you can have to sum of... yeah. So this would look like...

Researcher: So is that a function?

Student: Uh... yeah. I think it is.

Researcher: Ok. So it doesn't have to be continuous?

Student: I think that's right.

Researcher: Ok. What about...

Student: Well, but that is defined for all points, so... I think it's about right. I don't...

Researcher: Alright.

Student: It has been so long; I never thought about this.

Researcher: And you probably never really thought about it like this?

Student: No.

Researcher: But I just want to figure out... because you use functions all the time. I want to figure out what you really think about all of them. I don't really care about what the mathematicians say a function is. Is this a function?

Student: I remember there is something and I can't remember like (inaudible) but it had to only have one value of y for every, uh...

Researcher: Ok, so every x there can only one value of y. So this one has two so it can't be a function.

Student: Right.

Researcher: So it is the same thing as saying it has to pass the Vertical Line Test. I don't know if you remember that, but... But, yeah, every x has to have at most one y. Alright, so, So a function, then, is an equation where every x goes to at most one y. Um... or no x goes to two or more y's or whatever you want to say. So, um, If I take the derivative of a function you know, I get f prime.

Student: Yeah.

Researcher: My f... goes to f prime when I take my derivative. So it only goes one derivative. There's only one derivative. So is differentiation - just the act of doing a derivative - the d/dx - is that a function?

Student: Uh... I don't know. I'd almost say it's an operation.

Researcher: Ok.

Student: I've come to think of it more as like multiplying or adding two numbers together. It's like something you perform over... on a function.

Researcher: Ok, so what's the difference?

Student: Uh... I guess you perform the differentiation on a function and not like a... um... a number.

Researcher: Ok, so the function you're thinking of is an equation that takes x's to y's and it can't take it to more than one y. And these are numbers I'm plugging in?

Student: So what's the question?

Researcher: Is the act of taking the derivative a function?

Student: I'm not sure.

Researcher: That's fine, I mean.. I'm not looking for...

Student: Yeah.

Researcher: necessarily a mathematical answer to this. I just want to know do you think about that as a function?

Student: Yeah, I think of it more as an operation or something.

Researcher: Ok, and that's different than a function, right?

Student: No... you have operators in functions.

Researcher: Ok, so there's like addition in functions but that's

Student: Yeah.

Researcher: Ok, so that could be part of the function? Maybe. Alright, so what about... um...

Student: I guess if it's part of a function, it could be a function. So,

Researcher: Say we have a differential equation. Is this a function? So, it's an equation.

Student: Yeah... But you have a y prime on... I would say no because you have a y prime on both sides, you'd have to...

Researcher: So it matters what... um... how the equation is written?

Student: I think so. You're like defining itself or something.

Researcher: So, Is this a function?

Student: Yeah.

Researcher: Why? What's the difference?

Student: Well, I can easily say that's y equals x squared.

Researcher: Ok, so you can rewrite this one as a function, so

Student: Yeah.

Researcher: Ok. So...

Student: If I'm not so sure on it, I could do some math.

Researcher: Right. We don't have to do math. Ok, so you a function then is something like this, so I plug in an x and I get at most one y. Do these have, do I have to plug in a number to this?

Student: Uh... Usually you plug in a value for x and find your y.

Researcher: Ok, but do those x and y's have to be numbers?

Student: Well, no you could have uh.. um.. I guess you, yeah you could call them variables, pretty much, they'd just be something that would be

Researcher: But can I put anything in here? Could I even... Could I put a function in there?

Student: Yeah, I think so.

Researcher: So when I was putting functions into this, you wanted to call it an operator. Is this an operator now... or an operation, or whatever?

Student: I mean I guess, I guess you could call it a function.

Researcher: Is this still a function?

Student: Yeah.

Researcher: Ok. So what's the difference between that and that?

Student: Uh... There really nothing, I guess.

Researcher: So this could be a function?

Student: Yeah.

Researcher: Ok, so the only difference then is... um...

Student: It's just notation or something.

Researcher: Yeah, ok. So, you're alright with this being a function?

Student: Yeah.

Researcher: Well, if it's a function what are its domain and its range? It must have one if it's a function.

Student: Hmm. The domain and range are... I forgot about those.

Researcher: So domain is what you plug into the function, range is what you get out.

Student: Right, ok. It's... hm... Uh...

Researcher: So, if it is a function, then it has to have a domain and range, right? So either it's not a function or it has a domain and range.

Student: I can't think of any domain and range for it.

Researcher: Well, so what's the domain of this function?

Student: It would be x.

Researcher: So the domain is everything we plug in for x, right, so we can plug in all real numbers in this case. We can think about that as the parabola. So it's what we can plug in here and get an answer out. What can we plug into here and get an answer out?

Student: I mean... You can... I guess you can plug in any real numbers...

Researcher: Ok, so you can take the derivative of any real numbers, then?

Student: Yeah.

Researcher: So I can plug functions in right? So any differentiable function I can plug in and take the derivative of right?

Student: Mm Hmm.

Researcher: So would that make the functions the domain?

Student: Yeah. I think so.

Researcher: So I plug in function and I get functions back out? So I can plug function... so I can plug functions into functions and functions back out?

Student: Yeah.

Researcher: So is the Laplace transform a function? Remember you did that... You take the function and transform into some other function. So is L, the Laplace transform, is that a function?

Student: I think so. Uh... Well I think it works the same way that taking the derivative would.

Researcher: Ok.

Student: I mean, we're... I'm not sure.

Question B:

Researcher: Let's... uh... do something different, then. Is, um, so I'm going to make you remember some old stuff again. So Here we have a set of um a parametric set of equations. Um, so x of t is running... is cosine of t and y is sine of t and then t is running between zero and two pi. Can you graph that? Ok and how do you know that?

Student: Um.. I don't know, it's just... Your x is the... uh, x and y. Y starts at zero, x starts at one and it's just the unit circle.

Researcher: Ok. So what if I...

Student: ..circle though...

Researcher: Yeah, you're right. So what if I do Um... This. Now it's cosine of two t and sine of two t. But t is running between zero and pi now.

Student: It's uh... It's still going to be the uh.. full circle

Researcher: ok Why what's the difference?

Student: Um.. A... what's the word, it's like compression or expansion, I think but you're still going effectively from zero to two pi because you've still got your two in there.

Researcher: Ok So it will still make it all the way around once.

Student: Yeah.

Researcher: Ok. So, I got the exact same graph for those two. Um... Are they the same function?

Student: No.

Researcher: Why not?

Student: Well, because one is cosine of t, the other one is cosine of two t. I don't know if they have a different, uh.. range.

Researcher: Well, they have a different domain.

Student: Or domain, yeah.

Researcher: Ok. So, because one goes between zero and two pi and one goes between zero and pi, so different... even thought they give me the exact same graph, because they have a different domain, they're different functions?

Student: Yeah.

Researcher: Ok. So, so, a function is different than a graph then? Right? Or are they the same thing?

Student: The graph's just a visible.. visual representation of a function.

Researcher: But two different functions can have the same graph?

Student: Right.

Researcher: Ok, can two different functions have the same equation?

Student: Can two, ok, yeah. uhhh... hmmm... Yeah, no, I tend to use function and equation interchangeably, but it's.. uh... not the same thing. These are definitely different equations. But, the... if the graph is a function, then they both yield that same function, I guess. What was the question again? Is a...

Researcher: Well... uh.. well.. Is, so... Can two different functions have the same graph?

Student: I think so.

Researcher: So these two things have the same graph; are they two different functions?

Student: I don't know, they're different equations.

Researcher: Are function and equation the same thing? Let's say I have um... f(x) equals three But what if I think of this same function, but instead I change it to r of theta instead of f of x. All I've done is change the name, but let's consider this in polar coordinates, then.

Student: mm hmm.

Researcher: What does that look like? So this one's in Cartesian and this one's in polar.

Student: Well, isn't it in... f of x equals...

Researcher: So r of theta... r of theta equals three.

Student: Yeah.

Researcher: So, in polar coordinates, that's just a circle of radius three, right? Because my theta can be anything and my radius is always three.

Student: mm hmm.

Researcher: So these are the same equations, though, I just changed the names. I could have just as easily written this as f of x. I considered it in polar coordinates instead Cartesian coordinates. So they're effectively the same equation. In fact, I could have done the same thing with any equation. So I can write f of x equals x and r of theta equals r... or that might be theta... theta. So all I've done is change the x to a theta and the f to an r, so it's the same equation, but I think about this one in Cartesian coordinates, it's a line, and if I think about this one in... um... polar coordinates, there's this, what, spiral.

Student: Yeah.

Researcher: Those are the same equations, I've just changed the names.

Student: You've also changed the... the...

Researcher: coordinate system.

Student: coordinate system.

Researcher: Ok. So, so is the coordinate system part of the equation?

Student: I think it's going to determine

Researcher: It'll determine what it will look like.

Student: Yeah.

Researcher: But it's effectively the same equation that's giving me two different functions, right? It's just because I'm considering it in two different um... backgrounds, basically.

Student: Yeah.

Researcher: Ok, here I have two different functions - or at least that's what we decided earlier; that these are two different functions - um... give me the same graph. So function is not the same as a graph and a function is not the same thing as an equation, because two different equations gave me the same... or, two different equations... the same equation gave me two different functions the same graph can give me two different functions. um... and I can come up with multiple equations for any function. So function and graph and function and equation must be different concepts, right? So what's the difference? What is a function if it's not one of these things I've used?

Student: It... Something you have an input for and you get an output. I don't...

Researcher: Ok. So it's just something that all we can say about it is we put an input in and we get an output. We're still restricting, right, to the... uh... one input can't go to two different outputs.

Student: Right.

Researcher: So, this is something we can put an input into and get an output.

Student: mm hmm.

Researcher: And it only gives us one output. So that's a function?

Student: Yeah.

Researcher: And... that's a function?

Student: I think so. I'm fuzzy on Laplace transforms.

Researcher: um... So, so is a function... So a function is just this abstract concept then... of being able to plug something into something and getting something out? It's a way of relating an input to an output?

Student: Yeah.

Researcher: So if it's just this abstract concept, how are we putting it into differentiation? Can we do differentiation on an abstract concept?

Student: Uh... I don't know.

Researcher: Ok. So, you're right an equation is an abstract concept where we relate an input to an output where one input goes to at most one output.

Student: mm hmm.

Researcher: But because we've made this relationship, this relationship in itself can have umm... properties. Like, derivatives. So... I know it's torture, but... You've never had to think about that stuff really before?

Student: Not for a long time.

Researcher: Usually when we taught you that stuff we didn't ask you questions like that.

Student: Yeah.

Question C:

Researcher: Um... So here I have a graph.

Student: mm hmm.

Researcher: Um... I don't know anything about this, um, I just know that this is my graph here. and I'm bounded by this box and if I put this line at height c, I'm look at the shaded region below it and the shaded region above it... so above this graph and below the line. So, it's bounded by the box and bounded this line down here. Um... So... If that's all I know about this, Can I decide where to put this line... what height, c, to put this line to, um, minimize the shaded area? Or, maximize the shaded area. Is there a way to do that?

Student: Um... We could probably write a computer program, or...

Researcher: Alright, what would that computer program do?

Student: It'd basically be, I mean, it would be calculating the function... um...

Researcher: Oh ok, so you'd figure out the points and stuff and you'd...

Student: Yeah.

Researcher: Reverse engineer the function and then what would you do with function? Once you had the equation for the function.

Student: You would do, uh, yeah you could, let's see... To minimize the area, we're talking about an integral. And for under the graph is easy enough; you'd just take the integral of the function. Now for the area above that line, if we're just talking about the boxed-in region you could manipulate that function so... I don't... The integral is area under the curve... so... I guess you could, uh... Subtract that area from that height.

Researcher: Ok, so you'd basically just be taking the integral of this minus the function.

Student: Yeah.

Researcher: Ok, and so if we just took the integral of this function, though, we would grab all this extra piece.

Student: mm hmm.

Researcher: So we'd have to take the integral of the function up to this point and take the integral of that function, right?.

Student: Yeah.

Researcher: Ok. So, you would just set up this integral where you'd have to do a couple of things to get it right, but then you could just take an integral and evaluate it. So, that would require knowing what the function is, but also knowing what all of these bound are.

Student: Yeah.

Researcher: Can we do it without any of that knowledge? Or is that just asking too much?

Student: Um... Well, you could. I mean... It would not be easy or necessarily very accurate.

Researcher: Ok.

Student: So this...

Researcher: How are you thinking about it? Like estimates?

Student: Yeah, you could like measure it or estimate it.

Researcher: How would you estimate it?

Student: Um... hmm...

Researcher: So how would you estimate the integral?

Student: Well... I mean... or... I don't know.

Researcher: Well, so one way we could estimate it is we could just start taking chunks off.

Student: Yeah.

Researcher: So if we started doing rectangles... we could figure out the size of the rectangles easily enough, right? If we knew...

Student: mm hmm.

Researcher: If we could just measure them. So we have these little rectangles; we'll be a little bit off because it's just an estimate using all these little chunks, but... So if we have these rectangles, and we go all the way up to here... We could do the same thing here. Alright, so we could get our estimate that way. Would we just have to put the line at different places and hopefully hit the one that happens to be the smallest?

Student: You could do it that way, but... you could... You could like start zeroing in on it that way, maybe, if you got close and you could move the other direction if you started to get, you know, started to get bigger or something.

Researcher: So you're thinking about moving the line and seeing what's happening to the area?

Student: Yeah.

Researcher: Ok. So how would you move the line? Would you just start moving one of the directions and see if it went up or down?

Student: Well, I don't know, if you're talking about having the smallest, uh, shaded area on there um... You're going to going to want the, um, line c to be down here because under that...

Researcher: So, if I put it all the way at the bottom, it will be the smallest? Yeah, because I'll only have just this bit... this bit.

Student: Yeah, I think I'm going to go yeah. It looks like it. I think at the bottom will be the smallest.

Researcher: What happens if I move this, um, line up just a little bit? What happens to the, uh, shaded area?

Student: The shaded area will get... it looks like it's bigger because this section here is, is uh, wider than this side here.

Researcher: Ok.

Student: So this, this area here is going to be increasing. The area under the curve is going to be increasing faster than this will be decreasing. Basically.

Researcher: So you thought about this little bit of area that I just increased by.

Student: mm hmm.

Researcher: And you've said I've gained that but I've lost this.

Student: Yeah.

Researcher: So, I gained more than I lost, so the integral went up?

Student: Yeah.

Researcher: Ok. Um... Let me just change the question slightly. What if I wanted to maximize... this area? So we just saw that it went up as I moved it up.

Student: mm hmm.

Researcher: Will it keep moving up if I move it all the way up. So you say it's minimized down here. Will it be maximized up here?

Student: Um... Not entirely, there a point... I think there's going to be a point right about... right around there where you're going to be losing more... uh...

Researcher: Ok, so you're thinking about moving this up now.

Student: Where this is basically half way...

Researcher: Ok, so you want this to be halfway across so, whatever this line is, so that it's halfway across. How did you decide it's halfway?

Student: Well... I guess that it's not exactly a straight line, though, so that's not right.

Researcher: You said that the function is not a straight line?

Student: Right, yeah, well I'm saying it's not a, uh... it curves, it's not a...

Researcher: It's not linear.

Student: Yeah, it's not linear.

Researcher: Right, so um, where would this go then? How does that affect that? Because why did you say it was... why did you originally say it was halfway?

Student: Well, I was... I originally said it was halfway because it's just width-wise you'd be... So the amount I would have just added... If your chunks going up are small enough, it's, it's going to be, uh... It's going to be just strictly based on... how, uh... wide each chunk is.

Researcher: Alright, so I've added this much, but I've lost this much. And so, when... you wanted to say when I'm adding the same amount as I'm losing I'm at my maximum.

Student: Yeah.

Researcher: Ok, and you're thinking about just the lengths and because the widths are the same... I just moved my line up... that it only matters about the lengths. And so you said that when the lengths are equal...

Student: Right, if your chunks are small enough...

Researcher: Ok. So, you're right, that's where it would be. Um... and no matter what the graph is that's just where it would be because of that exact reasoning. Um... What does that have to do with an integral? So you wanted to find it with an integral originally. If we had the equation, we would have done it with an integral.

Student: Yeah.

Researcher: In fact you even wanted to make a computer program to find the function and reverse engineer it.

Student: Yeah.

Researcher: What does this have to do with the integral?

Student: Uh... Well I think of the integral as the area under the curve a lot of the times.

Researcher: Well, we're talking about area under the curve. So, even when we were just moving this up a little bit, we were talking about the area under the curve. We were talking about, it moved up a little bit.

Student: Yeah.

Researcher: So, how does that correspond to the integral? If we were to write it out in mathematical symbols, what did we just do? What were we just looking at?

Student: The... The integral is you take the... the limit as the size of your chunks approaches zero. I think.

Researcher: Is your integral?

Student: Is your integral, yeah.

Researcher: So you're thinking about... we were talking about these estimates and so as your estimates get finer and finer, um... you're going toward your integral?

Student: Yeah.

Researcher: So when we move this up, we would be moving it up by a smaller and smaller amount and so it would be closer and closer to moving our integral.

Student: mm hmm.

Researcher: Um... So what does that correspond to: moving that up by an infinitely small amount? I mean we've changed our integral; no we're looking at a different integral.

Student: mm hmm.

Researcher: But it's like an infinitely small amount that we just moved it up, so how would that correspond? What are we looking at?

Student: Um... I'm not sure what you're asking...

Researcher: Well, so we're moving c up and we're watching the integral change.

Student: Yeah.

Researcher: So how did you decide... how would we decide mathematically... if we had this... If we knew what the function was and could take the integral, how could we decide where to put c?

Student: When... I guess we're talking about the rate of change of the integral, or something.

Researcher: Ok, why?

Student: Because we want to know when the rate of change is basically zero.

Researcher: So we were watching the integral increase.

Student: mm hmm.

Researcher: And we want to know when it's zero because then it will start going back down and we know we're at a maximum?

Student: Yeah.

Researcher: So how would we find the rate of change of an integral?

Student: Take the derivative of it...

Researcher: Yeah, so, what we were doing... So what we were doing, then, we were looking at... We, you moved this up a little bit and you said this area increases by this much and we lose this much.

Student: mm hmm.

Researcher: Is that just looking at the derivative of an integral?

Student: Um... I think so.

Researcher: Ok.

Student: Yeah.

Researcher: So... Ok. You're right.

Student: Because you'd pretty much looking at where that function is... the rate of change is like a 45 degree angle where x basically, I think... or, I mean...

Researcher: Yeah, I mean, right.

Student: Because below this point it's going be...

Researcher: It will be increasing.

Student: And then after that point, it's going to be tapering off.

Researcher: So, yeah, so what we were doing was move it up by a little bit and that gave us an estimate for how much did our integral change which was just our estimate for the rate of change of our integral and so if we made that smaller and smaller... Well, actually we were looking at the rate of change... Are we looking at how much did the estimate of our area change. And so as we made those smaller, the estimate of our area went to the integral and that amount of change went to the derivative. So we were looking for when the derivative... was zero. Right. Ok. Good. So, Is that normally the way you think about it?

Student: Not exactly.

Researcher: Yeah, derivatives of integrals.

Question D:

So, um... Next question is maybe actually a little harder to answer. What is math?

Student: Um. It has to do with the... the first thing that comes to mind is the manipulation of numbers.

Researcher: Ok.

Student: Um... but that's not... it may be a little over simplifying, I suppose.

Researcher: Ok. So what else might it be if it's not just manipulation of numbers?

Student: It deals with functions, I mean, you can do math on things that aren't numbers, really. You can assign uh, values, you know, for letters and stuff... um... I mean, you do get constant, sort of, you can treat it as a constant. It's still, I guess it will still be sort of a pseudo number.

Researcher: Right. Was this math that we did? Just the idea of trying to figure out... without equations or anything, where this thing would be maximized?

Student: Yeah.

Researcher: Alright, so even though that has nothing to do with numbers or equations, or... all we have is this shape and some shaded area.

Student: Well, but we were trying to figure out the amount which is quantitative.

Researcher: Ok. Um...

Student: Whether or not you can actually assign a number to it, I suppose...

Researcher: Yeah, and we can't assign a number to this because we don't know... I mean, can make this any size I want and any number I wanted. So what I'm doing is I'm just looking at this shaded area is... just takes up more space than this shaded area.

Student: mm hmm.

Researcher: And I'm looking at when does it take up the most space and I'm not using equations. We're not using numbers... Is that still math? Or is that something else?

Student: I'd say it's math.

Researcher: Ok. So math doesn't have to have numbers or equations at all? So what is not math?

Student: Well I'd say there you're still comparing... um... you're comparing the value, I guess, of something of one thing to another thing.

Researcher: Ok so we could assign value to it and so...

Student: Yeah.

Researcher: Ok. Um... What is... what is the role of theorems in math?

Student: Theorems? Theorems are...

Researcher: Yeah, Fundamental Theorem of Calculus and we have Pythagorean Theorem.

Student: I forget... See I think theorem and I think of theory and theories are things that aren't necessarily proven, but is that the same with theorems? I can't... They're sort of pre-established um... pretty much I want to say proof, I guess, but it's something that somebody's come up with before. That someone was sort of proven.

Researcher: So it's some idea or fact or something that somebody's come up with and that we use to do something with? Um... So... What do you mean they're sort of proven.

Student: I... I don't know, I want it to be absolutely...

Researcher: You mean like in physics where we, um... so how do we prove something in physics? Or Engineering, for that matter.

Student: Um... You usually use... uh... we usually use math to go in and find out the equations for something. And um.. make sure everything adds up or, you know, calculates out.

Researcher: So, um... but how can we find the equations in physics? How are the equations found originally in physics? Like, you know, force equals mass times acceleration, and all this...

Student: Well there's testing.

Researcher: Ok.

Student: Taking tests and recording the data.

Researcher: Alright. So in physics, we basically try something a bunch of times and see if it works and if it works then we call it a theory and say that's what's going to always happen. Is that what we do in math? Do we just try triangles a bunch of times and come up with the Pythagorean Theorem?

Student: I'm going to say yeah, but there's also... there's other ways, I suppose, of...

Researcher: Would you call this a theorem? That putting this where these two lines are equal, or so where it hits it in the middle is always going to maximize the area? ...or maximize the area of this graph?

Student: Yeah.

Researcher: Is that... would that be a theorem?

Student: Yeah.

Researcher: So it's something that we've come up with that we could potentially do somewhere? Um... Did we just try a bunch of... how did we come up with... how did we decide this was true? I mean, we didn't try a bunch of examples, we only had the one example. Do we know this is true? Is this going to work for every shape or is there some shape out there that this won't work for?

Student: It would be at another... I mean if it's Yeah, if it's not, like, that shape right here.

Researcher: So, if it's a different shape, then it will be... well it will be a different, but will it be... But, if we say it's going to be halfway between these two lines... um... then it's going to be true or is that going to depend on the shape?

Student: As long as it... um... as long as it doesn't do something weird like loop back around or something and as long as it's a function.

Researcher: Ok. So as long as this is a function, it will always be the case where this hits the function right in the middle of the box? For every single function? I mean, we've only looked at one; how do we know it works for everything?

Student: Because it would be, um... You know it's the same thing as this. That when it hits halfway, then the other half is going to be... um...

Researcher: Yeah, so we'll be gaining as much as we're losing.

Student: mm hmm.

Researcher: Ok. So, we only tried this for one function. In fact, we don't even know what this function is.

Student: It looks like a unit circle.

Researcher: It is, but we don't know that. um... So, um... I mean, it could be an upside down parabola or something.

Student: Yeah.

Researcher: But uh... And we didn't even calculate anything for this. But we still know if it's true?

Student: I want to say yes, but I... from experience in... I mean I have good experience with other functions in math and working on other things. Because, I mean, to prove it you would want to try it on as many functions as possible.

Researcher: Will you ever find one that this doesn't work for?

Student: I guess if it's still a function... I'm going to go with that why it's called a theorem, because there might be some function that it doesn't work for.

Researcher: So, let's say there's some different shape. It will... if it doesn't work on it, then it will have to be the case that we move it up a little bit and the amount is the amount we change, but that's not the maximum. That we've either increased it or... or, no, we've increased it by doing that. By making the amount we gain as much... the same amount as much as we lose when we total this up.

Student: mm hmm.

Researcher: Can that happen? We lose the same amount we gain, but the whole thing goes up? The total goes up?

Student: Well, if you're losing the same amount you're gaining then the total shouldn't be changing.

Researcher: And that's all we did: found a place that we were losing the same amount as we were gaining.

Student: Right.

Researcher: Yeah. So, there won't ever be a function that that's not true for. Because the logic says that if that happens, then you must be at a maximum. Or at least a local maximum.

Student: mm hmm.

Researcher: So... The key here was we didn't try a bunch of... functions to find out if it was true; we only tried one function. But we used logic on it.

Student: mm hmm.

Researcher: To decide that had to be the case. And in fact, that's going to be the case for every single function we try.

Student: mm hmm.

Researcher: Because it will be the same logic. We never actually used the shape of this function when we were deciding this.

Student: Yeah.

Researcher: We only used the logic of how functions behave. So we do know this is true for every single function. Without even having to try it on any other functions.

Student: Because we determined that independent of the...

Researcher: Independent of the function, exactly. So... Um... So in that case at least, theorems were something that we, um... know are true then? Or at least we know this one is.

Student: I think so, yeah.

Researcher: But in Physics, it's not necessarily the case? We just try it a bunch of times and that seems to be the case?

Student: Well... Yeah, I think, because we don't necessarily... there's some things in Physics that we're still learning about.

Researcher: Right, and stuff gets overturned. Einstein came in and said Newton was wrong and all that.

Student: Yeah. And there's some stuff I've seen lately of people saying Einstein was wrong and stuff.

Researcher: Yeah. Um... Alright, so... Is... so we really just used logic here; we didn't calculate anything. Is logic math? What's the role of logic in math?

Student: Yeah. I would say that logic is math. Um... Or math is logic, I suppose. Yeah, because um... logic is... argumentative, I guess. Right? I mean you're saying that...

Researcher: Yeah. Math people like to argue.

Student: Yeah. But I mean like this... um... I haven't ever taken an actual logic course, but... but, um... I mean that's what... you talk about computer logic, you talk about... I'm a computer engineer, so we talk about math.

Researcher: Right.

Student: I mean you're talking about those AND's and OR's and NAND's and those are all... that's all math basically... or logic that...

Researcher: Right.

Student: IF and NOT and...

Researcher: Yeah. So... alright... So... um... What's the role of math in physics?

Student: I don't know. It's kind of... um... I guess you're actually... I mean measuring what happens... assigning... assigning values...

Researcher: Ok, so there's definitely some calculations going on.

Student: Yeah.

Researcher: The number... so you said what's happening to the numbers in the functions.

Student: Yeah.

Researcher: Is the logic they use in Physics math?

Student: I want to say yeah, I think so.

Researcher: So Physics is just applied math?

Student: Yeah.

Researcher: Ok.

Student: That's kind of how I've always thought of it.

Researcher: Makes us we like to claim all...

Student: Yeah.

Researcher: All the different...

Student: Music’s math, there's a lot of... everything.

Researcher: That's true. Um...

Question E:

Researcher: Ok, so let's um... move onto a word problem. Um... So... If... if we have a car starts out at some point and it is moving at, um... it starts out moving at ten miles an hour

Student: mm hmm.

Researcher: and, um... it continuously slows down continuously, um... by one mile an hour every hour. So after one hour, it's going nine miles an hour. After two hours, eight miles.

Student: mm hmm.

Researcher: So... and it's doing it continuously. How long does it take for the car to go five miles?

Student: Um... approximately thirty minutes.

Researcher: Ok, how did you come up with that?

Student: Well, it's going ten miles an hour for... I mean, you're talking continuously, right? So it's not like a step, like piecewise, right? So it's not like when it hits... when it hits one hour it, it goes down to nine miles an hour. You're talking a continuous decrease? Um... It's... it's not exactly...

Researcher: So if it was a step function, though, it would be exactly.

Student: It would be exactly thirty minutes.

Researcher: Ok, so, but if it's continuous, how do we do it?

Student: If it's continuous, you could, uh... write the equation for, uh... What is it, uh...

Researcher: So we need a distance to...

Student: Yeah, your distance is... I guess you set d equals, uh... what is it? Speed times...

Researcher: Rate times time.

Student: Rate times time. And your...

Researcher: or velocity times time or whatever want...

Student: Your velocity is changing.

Researcher: Ok.

Student: Your...

Researcher: So what's our velocity equal to?

Student: And your velocity equals... c. And time... Ok, so time... At ten hours your velocity will be zero. So it's going to be... ten t minus ten. I think. So, ten t minus ten.

Researcher: Ok, so how did you come up with that?

Student: No, wait, I'm sorry. I'm doing this wrong. Uh...

Researcher: So if time is zero, what should that be?

Student: Uh, time... yeah, if time is zero... if time is zero, we should have... ten.

Researcher: mm hmm.

Student: And it's decreasing... minus... uh... Is it minus t?

Researcher: Because t is measured in hours?

Student: Yeah, if you're doing time in hours.

Researcher: Ok, so if after one, it would be ten minus one. After half an hour, it would be ten minus a half.

Student: Yeah.

Researcher: Ok, so velocity is equal to this.

Student: Yeah.

Researcher: So what's my distance?

Student: Um... Just plug that in there... for... so you have ten t minus t squared. So... Um... That's not true. That's not true. Dang it.

Researcher: So this equation is when we have a constant velocity.

Student: Yeah.

Researcher: So... Um... So how do we normally go from velocity to distance?

Student: Take the integral.

Researcher: So how do we take the integral?

Student: You get that. Well... Right? No, over two, right?

Researcher: Yeah, exactly. Right, so yeah. Ten t minus t squared over two. So... What is the farthest the car travels?

Student: Um... It would be... Well, the car stops at ten hours.

Researcher: Ok.

Student: So... One hundred minus a hundred over two... fifty... So, fifty... miles.

Researcher: Ok. Good. So... Um...

Question F:

Researcher: Let me just change the question slightly. Um... So suppose we start again going ten miles an hour... and we're slowing down continuously again... by one mile an hour... every... but instead of hour, we do it every mile. So it's pretty much the same thing except for... instead of slowing down by one mile an hour every hour, we do it by every mile.

Student: Yeah.

Researcher: What is the farthest the car travels?

Student: Ten miles...

Researcher: Ok. So it slows down by every... every mile it slows down one mile an hour, so at ten miles it's stopped.

Student: Yeah.

Researcher: Ok. How long does it take to do that?

Student: Infinity?

Researcher: So why is that?

Student: I don't know, just off the top of my head. Does it ever, uh...

Researcher: So it goes ten miles but it takes infinitely long to do it?

Student: Well in theory. There is a point where the car is actually not going to be able to move that slow.

Researcher: Sure. But... why did you come with that it takes infinitely long to do? Why did you jump to that so quick?

Student: Um... I don't know. Because it's going to, uh... Because as the car approaches that, it's going to keep getting slower and slower and slower. So... it would never...

Researcher: So when it's going to be one...

Student: Yeah.

Researcher: So it will never actually get there. Ok. How long does it take for the car to go five miles? What happens?

Student: Um... Uh... I don't know, I could write something out. You want me to...

Researcher: Sure.

Student: Um... Shoot. So distance is... Velocity times time. That's still the same.

Researcher: Well... What did we decide for velocity? Because that was when it was constant, right?

Student: Yeah. That was when it was constant.

Researcher: So we had to take the integral of velocity.

Student: Yeah.

Researcher: So distance equals the integral of velocity.

Student: Ok.

Researcher: We'll just make the time so the integral will be v dt. So what's our velocity?

Student: Velocity is... Well velocity changes based on our distance.

Researcher: mm hmm. So if we were out at this, what would we come up with?

Student: Uh... it equals... ten, I guess, minus... your... distance.

Researcher: So ten minus d.

Student: Yeah.

Researcher: Ok. How do we solve that? Because our velocity depends on distance and distance depends on velocity.

Student: Yeah, I know. That's the problem. I'm not sure.

Researcher: Ok. So... What if I said that if we look at this, this tells us... that...

Student: mm hmm. Oh, yeah, right, so... Um... D prime equals v, we also have v equals ten minus d. So I mean it just turns into a differential equation, right?

Researcher: mm hmm.

Student: Where you'd... set d prime equal to... ten minus d.

Researcher: mm hmm.

Student: Or you could... put the terms together or something. So we get d prime plus t... I forget, do you move the ten on the left or the right or... Is it minus ten.. equal zero. So that would give you... that would give you homogenous or whatever.

Researcher: Or I think the ten would go on this side. Then you could do it, um... Well, I guess you could do it homogenous...

Student: Yeah.

Researcher: Um, you could do linear, you could do homogenous, um, you could do separable. Or not separable, but exact, so um...

Student: then you could do diff eq.

Researcher: Yeah. And then... so if you end up coming... so if you did that, you'd end up coming up, if I remember correctly, d is equal to, uh... ten e to the... negative ten t. I think. So at zero, you'd be going ten. At uh... infinity, you'd be going... That's not right.

Student: At infinity...

Researcher: Oh, yeah, at infinity, this would go to zero and then you'd be going zero. No, wait, that's still not right.

Student: One over e to the ten t. At infinity it's going to be one over a really, really, really large number, right?

Researcher: That should be, um... Yeah.

Student: So it would be ten thousand... very, very small fraction, which would...

Researcher: This should be actually one minus this. I think. Is that right? Ok.

Student: Because at zero, you're at... Or ten minus that. Ten minus ten... Right because it's your distance.

Researcher: Yeah, because it's not my velocity, it's my distance. So, yeah, ten minus... This is what it ends up coming out to be, not this. It doesn't matter that much.

Student: Yeah.

Researcher: This is what your answer would be. And so... And that's why at, um... and then we can just solve it for five equals that and solve minus that. And that's why it takes infinitely long to, um... That's another way of looking at it. It takes infinitely long to make this thing drop off of there and get to ten.

Student: Yeah.

Researcher: So notice how I changed one word and it became a much harder equation.

Student: Yeah.

Researcher: Ok. So. I think we're out of time. So... so... Um... I think I'm going to stop torturing you. So... um...

Student: It's helped me a bunch...

Appendix D - Interview Excerpt 1

The following is an excerpt from the interview of the fourth Engineering major consisting of Question A and Question B from the protocol. This student was rated as having an Object level understanding, and a 3 on both Confidence and Willingness.

Question A:

Researcher: Ok. So... let's start. So the first question is one you might not have thought about too much... um... what is a function?

Student: Um... let's see... A function... I think a function is usually an equation or something expressed in a... in terms of two variables, uh, in order to depict a line... graph or maybe a matrix. The way a function has to be defined in the programming sense is an algorithm to take care of... take care of a certain process. Something like that so I guess you can think of two different ways...

Researcher: Ok. Are those two different... so is that... two different definitions of functions are...

Student: I mean it's, it's essentially... they're... they're essentially related, I mean... you have an equation... and then it's also an algorithm in a sense.

Researcher: Ok.

Student: That, that, you know, gives you a certain output based on... certain inputs. So... I guess I got more of a programming sense of that.

Researcher: Ok. Um... So... Does a function have to have an equation?

Student: It doesn't have to. Um, it can make use of... um... other... other processes or... other... algorithms, not necessarily equation based. Um...

Researcher: Ok. So... Um... So a function... is, is something that you do, then?

Student: Right, it's... I, I view function as something that... that um... that's used to obtain, obtain a desired result.

Researcher: Ok.

Student: It, it takes an input and gives you a certain output.

Researcher: Alright. Okay. Um... Do those inputs and outputs have to be numbers?

Student: No. They don't. Um, they could be complex, I guess, complex values. Um... they could be... they can be... I guess... you can... uh, it's programming because it could be arrays. I guess those are numbers. But... Uh, there are also objects, um... but I guess there is a lot of programming applications for inputs not being specific numbers because you could have file formats into the functions, so... I guess... I'm guaranteeing you know something about programming, so...

Researcher: Right.

Student: So...

Researcher: Ok. So... in... in... So you're thinking of a function as you can take things... um... like objects and then do something with them?

Student: Right.

Researcher: Ok. So... If I think about it that way then I can take f as a thing and take the derivative and it gives me back the derivative of the function. So I take some function and send it to its derivative.

Student: mm hmm.

Researcher: So... Is this, the process of taking a derivative, is that a function?

Student: mm... let's see... I feel like it can be. Because you could program something to calculate a derivative and therefore defining it as a function that... gives you a derivative. I mean it's not associated with, you know... function f. But...

Researcher: Right.

Student: Um, I believe... and in a programming sense it can... it can be defined as a function. But, in a mathematical sense, I'm not sure if that's the correct way to define it.

Researcher: But you want to define it as a function because takes some input and does something to it and gives an output?

Student: Right.

Researcher: Ok. And so, so functions, then, can... can... have functions as inputs?

Student: Um... Yes, I guess that's... that's what I set myself up for there. Yeah.

Researcher: Ok. So... um... Let me ask a different thing. Um... Is that a function? The integral from zero...

Student: Yes.

Researcher: to one point three...

Student: Yes.

Researcher: Ok what is that a function of?

Student: Uh... it's a function of the equation x to the fourth and... evaluated...

Researcher: Well, so what are my inputs and where are my outputs?

Student: Uh, you're inputting this equation, uh... of this function... of the polynomial function. And the output is... its integral evaluated at the points.

Researcher: Ok. So I'm integrating that as a function? Ok.

Student: So I feel like I'm getting further and further away from the proper definition of function, but...

Researcher: Well, I mean, it... it all depends on... I'm just asking what... what do you think of a...

Student: I, I, it... I may have a misconception because you can define function in certain programming languages and maybe that's... maybe that's not the right term for it in a mathematical sense because functions in math are... more of... you know, associated with numbers, polynomials, or certain... certain... equations, I guess. So...

Researcher: Well then at least in a computer programming sense, you want to say integral is a function?

Student: You can... yes, you can create...

Researcher: It takes some input to something... you get some output.

Student: uh huh.

Question B:

Researcher: Ok. Um… Let's say I have this... um... set of parametric equations. So x is... x of t is equal to cosine of t, y of t is equal to sine of t...

Student: mm hmm.

Researcher: and t is running between zero and two pi.

Student: mm hmm.

Researcher: Can you graph that?

Student: Um... Let's see... I guess it would be... x t versus y t... t prime... mm... cosine... So I... I can graph each individual function, obviously, but... um... I mean, the two together... I feel like I'm forgetting something... as far as x of t, y of t. Um... I guess my initial thought would be... since it's x of t versus y of t, I would be graphing cosine of t, uh, on the x axis and sine t on the y. But... I feel like that's not the... quite right.

Researcher: Ok, so what if we just take some points? So what happens when t is zero?

Student: When t is zero we've got cosine is equal to one, sine's zero.

Researcher: mm hmm. So we get the point one, zero, right?

Student: Yeah. So you get one zero there and then... One zero... Yeah.

Researcher: would be... over here.

Student: Right, right, right. Right, right. One zero. And um... And then we're going to do... then that's... it's evaluated... um... and pi over two... Ok. pi over two, so we got... I guess... I want to make... I want to make this pi over two... So x... That's probably not right... Ok... Oh, ok. No, that's right. This is t. And... Cosine pi over two is radical two over two is cosine of pi... or sine pi over two... So I guess we can... So that's radical... and then pi is...

Researcher: Ok, so, so what do you get? You...

Student: Uh, I'm just evaluating for different values of t.

Researcher: So you have... t across the bottom. What's this axis here?

Student: Um...

Researcher: Are you doing x and y on the same axis? Is that what you're doing?

Student: Yeah, I guess that's what I'm doing.

Researcher: Ok. Um... So... Um... When we do parametric equations, we do it... so we plug in... so if we just plug in zero for these, we get x of zero is one, like you said, and y of zero is zero. So we get the point one, zero...

Student: Ok.

Researcher: when t is zero. When t is pi over two, we get x of pi over two is zero, and um... y of pi over two is one.

Student: Oh, got you. Right.

Researcher: So we get x is zero and y is one, so we get the point zero, one.

Student: mm hmm.

Researcher: And so... when t is, um, pi, then we do x of pi is is, you know, over there and then...

Student: Ok.

Researcher: And so what do we get for our graph?

Student: It's a circle.

Researcher: Yeah, ok. So we get a circle. How many times does it go around?

Student: It just goes around once.

Researcher: Ok.

Student: Zero to two pi.

Researcher: Alright.

Student: I knew that.

Researcher: Um... alright... so... what if I change that slightly and go x of t is equal to cosine of two t and y of t is equal to sine of two t. But now t is running between zero and pi. What's the graph of that look like?

Student: Same thing... So, it's the same thing. Yeah.

Researcher: It's a circle going around once?

Student: Yeah. So it's... Yeah. You go twice as fast around it. That's the problem.

Researcher: Ok. So it's the same circle...

Student: mm hmm.

Researcher: um... so...

Student: If it's from zero to two pi, it would be twice, so...

Researcher: Ok, so since we're going from zero to pi, it's... it's once around the circle. So we get the exact same graph, right?

Student: Right.

Researcher: Are those the same functions?

Student: No.

Researcher: So two different graphs can have... or two different functions can have the same graph?

Student: mm hmm. This is a different interval, so yes.

Researcher: Um... So... Um, so the outputs are the same, though... Right?

Student: Right.

Researcher: So, two different functions can give the exact same outputs?

Student: Sure.

Researcher: Ok. Um... so if two different functions can have the same... graph, can two different functions have the same equation?

Student: Uh... that's not necessarily true. I mean, there's more than one way to find... um... there can be more than one way to find certain... certain lines... certain graphs.

Researcher: So we can give more than one equation to a function? Like y equals x squared or y minus x squared equals zero.

Student: Right.

Researcher: So what if I look at... um... so if I write f of x is equal to x. That's just a line, right?

Student: mm hmm.

Researcher: But if I just change the names of those to r of theta is equal to theta... then I get like a spiral.

Student: mm hmm.

Researcher: Because I'm thinking of the one in Cartesian and thinking of the other in polar coordinates.

Student: Yes.

Researcher: But they're basically the same equation, I've just changed the names.

Student: Right.

Researcher: So are those the same functions?

Student: No because they have different parameters. And one's in polar and one's in Cartesian.

Researcher: Ok, I mean I could have called this one f of... If I really wanted to, I could call that f of x is equal to x. And if I consider it in, in, um... polar, then it's a spiral.

Student: Right.

Researcher: So I mean, in some sense that's two different functions having the same equation.

Student: Yes. It's just how you define the parameters.

Researcher: Ok.

Student: So if you're in...

Researcher: So if... if a function is not an equation and it's not a graph, what is it?

Student: Um... It depends on how... A function is, I guess, how you can define it.

Researcher: Ok.

Student: How you define the certain... certain... I guess... variables, or... Function can mean, I guess... however you define the certain... it's not an equation, but... um... I'm going to use equation for lack of better word. Um... based on certain parameter, so, I mean, you could define your very own function... Um... and... create an output on the parameters you want. So it's very flexible in that sense, I suppose.

Researcher: So you said earlier, that a function was something that took inputs to outputs... or it did something to inputs to get outputs.

Student: mm hmm.

Researcher: Um... So...

Student: In a programming sense.

Researcher: Right. Um... And so... here we had... similar inputs, but they were different.

Student: mm hmm.

Researcher: We'll get the exact same outputs, but we got them in a different way. So that was a different function. So the inputs... doing something to the inputs to make outputs... even if that makes the same outputs, you decided that they were doing two different things, so they're different functions. Um... Here we're doing the same thing, we're sending the thing to itself. So here we were sending... whatever we put in there we get out there.

Student: mm hmm.

Researcher: But it's kind of a different background. And so we got different outputs. Um... So... Um... What was my question? So... Well, so a function's not the same thing as a graph. It's not the same thing as an equation. It is this abstract... so you're saying it's this abstract...

Student: I guess.

Researcher: um, action that we're doing to inputs?

Student: Yeah, generalized idea, or... process that we want...

Researcher: So if it's this generalized process... how can we treat it like a thing when we're taking like the derivative of it. Can we take the derivative of an abstract process?

Student: No, because we know the function to be defined as whatever their equation is. A certain polynomial... we know the parameters of an equation.

Researcher: Do we need an equation to take the derivative of it?

Student: Not necessarily... I guess the only thing I've seen the derivative of is an equation. So...

Researcher: So what is a derivative? What does it mean?

Student: It... it means the uh... um... it's the uh... it, like, gives you the points of inflection... um... as far as, like, let's say you have a... um... like if you... it will give the slope of every point along the line.

Researcher: Ok.

Student: I guess... that's right.

Researcher: Well that works if we have a, um, a graph. Because this tells the slope along that graph, right? But, um... if we can take derivatives of functions and functions are things that aren't the same as graph, they're just these abstract... ways of changing inputs to outputs, how do we take the derivative of it? What's the derivative of it mean, then, in that sense?

Student: Well, I mean, I guess we'd only take derivatives of functions that are defined as equations.

Researcher: Ok.

Student: I don't see it as a... abstract object.

Researcher: Why would we want to know the slope of a graph?

Student: You can use it, uh, in a... in a Physics application. You can use it to, uh... if you got an equation that finds certain position, you can find velocity and acceleration by various derivatives.

Researcher: Ok.

Student: And graph them, then.

Researcher: So, so... So basically, then, a velo... a... derivative of position will give us velocity; derivative of velocity will give us acceleration. So derivative is just measuring rate of change...

Student: Right.

Researcher: of something. So we can... I mean, if our function is this abstract way of relating things... then we can measure how that changes, right?

Student: Right.

Researcher: Even without an equation. Because if it's relating two things, we can see how that relationship changes as we change the things that we're relating. And we can measure... or we can talk about how fast it changes... what the rate of change is.

Student: Right and used it... obvious in... in... in... applications beyond Physics because a lot of thing have patterns and equations of functions are associated can... can define a lot of things. And with all of them, you can measure the rate of change by taking derivative of such things.

Researcher: So do we need an equation to take a derivative?

Student: Um...

Researcher: If it's just the rate of something that's changing?

Student: I guess... I guess maybe not, but I wouldn't know how.

Appendix E - Interview Excerpt 2

The following is an excerpt from the interview of the twentieth Engineering major consisting of Question A and Question B from the protocol. This student was rated as having a Process level understanding, and a 1.5 on both Confidence and Willingness.

Question A:

Researcher: Alright, so my first question is what is a function?

Student: A function?

Researcher: mm hmm.

Student: It's uh... a mathematic... mathematic representation of... uh... of behaviors in the plane.

Researcher: Ok, so it's modeling something?

Student: Yeah.

Researcher: Ok. Um... So... Let me ask you this, then. So if I have... so if I think about the process of taking a derivative, I take f and it tells me f prime, so it's, it's... telling me... it's modeling... a behavior here, it's modeling how the derivative is, um, compares to f. So... is the process of taking a derivative a function?

Student: Um, what was your question about this one? Is it...

Researcher: Is the process of taking a derivative a function? So it's modeling how the derivative of the function relates to the original function. Whatever function I put in there.

Student: Yeah.

Researcher: Why?

Student: Why?

Researcher: mm hmm. So the process of taking a derivative is a function?

Student: If you take the derivative function... then, you'd have... you have found out like... the... you are looking at the model on the paper of the function itself. Like... what... what does it cover and what's the area of it and... uh, the... when it's continuous... So I think it's like... the process for that function.

Researcher: So it's a process for that one?

Student: Yeah.

Researcher: So then taking the derivative is, is, um... is a process, you're saying?

Student: Yeah.

Researcher: Ok. Is it a process and not a function? Or is it a process and a function? I mean, is that process a function, or is it just a process?

Student: It's a process for the... for the function.

Researcher: Ok, so it's a process that we do to a function?

Student: Yeah.

Researcher: Ok. Um... Ok. So... Let me ask you something else. Suppose I have the integral from zero to one point three of x to the fourth plus k dx. Is this a function?

Student: This is a function?

Researcher: So the x to the fourth plus k is a function?

Student: Yes.

Researcher: Ok, so is the whole thing a function?

Student: No, it's a process.

Researcher: Ok. Alright, so... And if I evaluate this, though, I get y equals uh one point three to the fifth... over five plus one point three k minus zero.

Student: Then it will become a function.

Researcher: Now it's a function?

Student: Yes.

Researcher: So, here it's not a function, it's a process. But here it's a function?

Student: Yeah.

Researcher: Ok, so so we have to do the process...

Student: To have the function.

Researcher: to have the function. So, um... What about, um, the Laplace transformation. So we have the Laplace transformation, remember it takes some function in t to a complex function of s. So is the Laplace transform... is the Laplace transformation a function?

Student: I think it's still a process.

Researcher: Ok.

Student: Because you have like... looking at the function, we would... units and uh... uh different domains... as we look at it from... Yeah.

Researcher: We change f to g...

Student: Yeah.

Question B:

Researcher: Ok. Um... So, suppose I have a series of parametric equations. x of t is equal to cosine of t. y of t is equal to sine of t. And t goes between zero and two pi. What's the graph of that look like?

Student: Cosine of t and sine of t for... from zero to two pi..

Researcher: mm hmm. Ok.

Student: So, this one is...

Researcher: Cosine.

Student: cosine. The other one is sine.

Researcher: Alright, so you graphed them... alright, so you graphed your cosine... your y versus t... and your x versus t here and the graph of y versus t here.

Student: Yeah.

Researcher: Ok. Um... What if I want them graphed on an x, y axis, though? So... X and y axes... What if I want them graphed like that?

Student: x and y axis?

Researcher: mm hmm. Instead of x and t and... instead of the axes being t and... I have my axes are x and y.

Student: x and y... It will... still the same isn't it?

Researcher: Well, so, for example, if we have t equals zero... x of zero is cosine of zero, which is one... and y of zero is sine of zero, which is zero. So if t equals zero, we get the point one, zero.

Student: Yeah.

Researcher: So, if t is zero we would graph the point one, zero... and then if t is... um... pi over two, we would get the point zero, one. So I'd get...

Student: Oh, yeah.

Researcher: I want the graph of those points and not...

Student: Oh, ok. Like... So both... and... cosine would be my x and the sine would be my y then? cosine would be one... so it would be... minus one, then one... That's what... would be... hmm... so y... so it would be zero one... y of t... so it would be... be... be... so...

Researcher: So then you go... once you hit the point zero, one... you then go to negative one, zero.

Student: Oh, yeah. So you have minus one... so you get minus one... cosine of zero... sine is negative one... so...

Researcher: So your third point should have been negative one, zero.

Student: Yeah. mm hmm. and then this will be zero for two pi... So it give me the... the unit circle. That...

Researcher: Ok, how do you know that?

Student: Because it keeps going.

Researcher: Ok.

Student: Like, from zero to one or... from zero to like minus one...

Researcher: Alright. So what if I instead change this slightly. And instead I have x of t is equal to cosine of two t and y of t is equal to sine of two t where t runs between zero and pi?

Student: It's going to be the same graph, but, like shifted toward the... the y t graph...

Researcher: Ok.

Student: So it would be shifted, since we are multiplying t.

Researcher: Alright. So what about my... x, y graph?

Student: mm... should it goes to... to... zero to one... It... it's making... expand the circle like shrinking...

Researcher: Ok, so we're still going to get the circle?

Student: Yes.

Researcher: Ok. So what if we put in points. What do we get when cosine... when... t is zero?

Student: Should get one.

Researcher: Ok, so we still get one, zero.

Student: Oh... We will still get the same circle.

Researcher: Ok. Yeah, it does... um... shrink, but it does it along the circle instead of outwards.

Student: Yes.

Researcher: So we just go around the circle twice as fast. Ok, so we got the exact same graph for both right?

Student: Yes.

Researcher: Ok. Are those the same functions?

Student: mm... No. You mean this one and this would be different? cosine t, sine t... No, it's different... different functions.

Researcher: Ok, so two different functions can give us the exact same graph?

Student: Yeah.

Researcher: Um... So... What's the difference between graph and function then? If they're not the same thing...

Student: Uh, the graph is... a... the output of the function.

Researcher: Ok.

Student: So it finds... it's... it's... it's like... find the outputs for to find two different functions equal.

Researcher: Ok, so we get the same output for the two different functions...

Student: Yeah.

Researcher: So why are they different functions?

Student: Because you cannot say cosine t equals cosine two t.

Researcher: Ok so...

Student: They're different.

Researcher: So we got the same outputs, but we got them in a different way?

Student: Yeah.

Researcher: And so those are different functions? Um... Can... is... so we decided, then that graph and function aren't the same thing, that graph is just the... it's just drawing the function's outputs. Are graph and equations... are function and equation the same thing?

Student: Function and equation? mm... No.

Researcher: Ok what's the difference?

Student: Um... Well, they... like... The way I'm thinking of it is like the function is... you can... you can change it to whatever you want: multiply it, divide. And the equation should be constant and like... as a... I don't understand what you mean by function... equation. Is it like formula stuff?

Researcher: Well, like, um... I mean, we have, for example we can think of y equals x squared and we know that gives us a parabola. So we know the graph of this function and we know the equation for this function. So, is the equation the same thing as the function?

Student: Yes. Because, yeah.

Researcher: Ok.

Student: I always think they're the same thing because...

Researcher: Can we... is um... Can we have a function that doesn't have an equation?

Student: A function doesn't have equation? No.

Researcher: Um... So suppose, I looked at um... just f of x is equal to x. So you get a line through zero, zero. Um... but if I wanted, I could look at that in polar coordinates... so normally we'd call that r of theta equals... theta, but it doesn't matter what I call it, so I'll just call it f of x equals x. If I look at in polar coordinates, I get... a spiral. Are those the same function?

Student: mm... Yes.

Researcher: Ok. So... We already said that two different functions can give us the same graph, now you're saying that one function can give me two different graphs?

Student: Yeah.

Researcher: Ok. So why is that the same function?

Student: Because... it's... it's up to the... the way you are looking for... at the function.

Researcher: Ok. So because it's in different coordinate...

Student: Yeah.

Researcher: we get different graphs? So why did you decide that was the same function, though?

Student: Because if... if you look at the equation you can set them equal... equal.

Researcher: Ok, so the equations are the same so they're the same function? Ok.