THE OBSTACLE PROBLEM FOR SECOND ORDER ELLIPTIC OPERATORS IN NONDIVERGENCE FORM

by

KUBROM HISHO TEKA

B.Sc., University of Asmara, 2001
M.S., Kansas State University, 2011

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

DOCTOR OF PHILOSOPHY

Department of Mathematics
College of Arts and Sciences

KANSAS STATE UNIVERSITY
Manhattan, Kansas
2012

Abstract

We study the obstacle problem with an elliptic operator in nondivergence form with principal coefficients in VMO. We develop all of the basic theory of existence, uniqueness, optimal regularity, and nondegeneracy of the solutions. These results, in turn, allow us to begin the study of the regularity of the free boundary, and we show existence of blowup limits, a basic measure stability result, and a measure-theoretic version of the Caffarelli alternative proven in Caffarelli's 1977 paper "The regularity of free boundaries in higher dimensions."[3] Finally, we show that blowup limits are in general not unique at free boundary points.

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Acknowledgments

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Dedication

To my mom

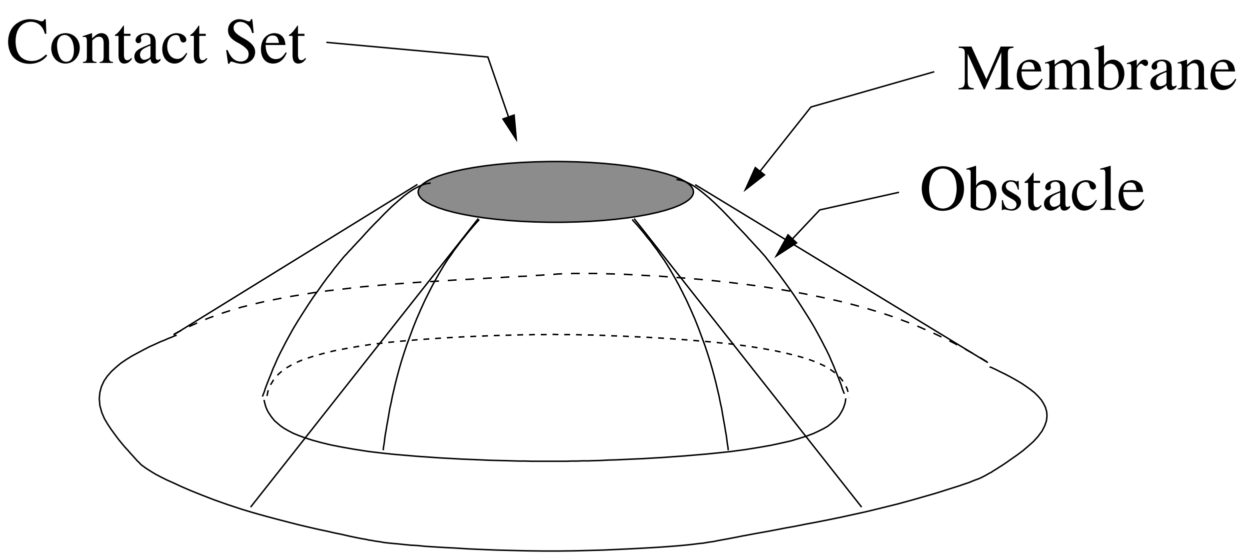
Chapter 1

Introduction

In this chapter we give the physical motivation for the obstacle problem, and then we describe the obstacle problem for second order elliptic operators in nondivergence form.

1.1 Original Physical Motivation

What happens when we pull an elastic membrane down over an obstacle?



To formulate what is happening mathematically:

Assume the membrane is given by the graph

$$u : B_1 \subset \mathbb{R}^n \longrightarrow \mathbb{R},\ u \equiv 0 \quad \text{on} \quad \partial B_1,\ \text{and}\ \varphi : B_1 \subset \mathbb{R}^n \longrightarrow \mathbb{R},\ \varphi < 0 \quad \text{on} \quad \partial B_1.$$

We want to find a function u (the "membrane") which minimizes the Area integral:

$$I_1(u) := \int_{B_1} \sqrt{1+|\nabla u|^2} \qquad \text{among } \ u \ \text{ satisfying :}$$

- $u = 0$ on ∂B_1 (i.e. the membrane is "pinned down") and
- $u \geq \varphi$ in B_1 (i.e. the membrane is above the obstacle).

From the calculus of variations, it easily follows that functions which minimize I_1 in a neighborhood among functions with fixed boundary data satisfy the minimal surface equation. Observe that for a small deflection of the membrane, $|\nabla u|^2$ is the first important term in the Taylor expansion of $\sqrt{1+|\nabla u|^2}$. (i.e. $\sqrt{1+x} \approx 1 + \frac{1}{2}x$, for x small.) Thus, we want to find a function u which minimizes

$$I_2(u) := \int_{B_1} |\nabla u|^2 \qquad \text{i.e. Energy - The Dirichlet Integral}$$

among u satisfying:

- $u = 0$ on ∂B_1 (i.e. the membrane is "pinned down") and
- $u \geq \varphi$ in B_1 (i.e. the membrane is above the obstacle).

It is easy to show that functions which locally minimize I_2 satisfy Laplace's equation. Linearizing the Area integral is very standard in the study of the obstacle problem. Mainly because it adds technical simplification- it changes the operator from nonlinear to linear- without altering the real difficulties of the problem.

Therefore, the obstacle problem involves finding a function u which solves the problem:

$$\text{Minimize} \quad \int_{B_1} |\nabla u|^2 \, dx \ \text{ among all functions } u \in K_\varphi,$$

where we define K_φ to be the closed convex set:

$$K_\varphi := \{u \in W_0^{1,2}(B_1), \ u \geq \varphi\}$$

For the definition of the Banach spaces $W^{k,p}$ and $W_0^{k,p}$ see chapter 1.2.

If we define the "height function" $w := u - \varphi$, and we let $f := -\Delta\varphi$, then w satisfies:

$$\Delta w = \chi_{\{w>0\}} f \ .$$

Since the problem above is variational, there is no difficulty in establishing existence and uniqueness of solutions. Regularity of the solution has been studied by many authors, and in the case where φ is smooth, Frehse showed in 1972 that the solutions would belong to $C^{1,1}$. Finally in Caffarelli's famous Acta paper in 1977, the regularity of the free boundary was addressed in the case where f was Hölder continuous and positive.

Since the obstacle problem was formulated, but especially in the last 15 years, there has been interest in extending some of these results to related problems. Ki-Ahm Lee studied the case where the Laplacian is replaced with a fully nonlinear (but smooth) operator. Blank studied the case where the function f was not assumed to be Hölder continuous. Many people (Blanchet, Caffarelli, Dolbeault, Monneau, Petrosyan, Shahgholian, Weiss, ...) have recently studied the case where the Laplacian is replaced with the Heat Operator.

Here we study the case where the Laplacian is replaced with a general second order elliptic operator in nondivergence form.

1.2 Elliptic Operators in Nondivergence Form

We study strong solutions of the obstacle-type problem:

$$Lw := a^{ij} D_{ij} w = \chi_{\{w>0\}} \ \text{ in } \ B_1 \ , \tag{1.1}$$

where we look for $w \geq 0$. (We use Einstein summation notation throughout this dissertation.) A strong solution to a second order partial differential equation is a twice weakly differentiable function which satisfies the equation almost everywhere. (See chapter 9 of[9].)

We will assume that the matrix $\mathcal{A} = (a^{ij})$ is symmetric and strictly and uniformly elliptic, i.e.

$$\mathcal{A} \equiv \mathcal{A}^T \ \text{ and } \ 0 < \lambda I \leq \mathcal{A} \leq \Lambda I \ , \tag{1.2}$$

or, in coordinates:

$$a^{ij} \equiv a^{ji} \ \text{ and } \ 0 < \lambda |\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \Lambda|\xi|^2 \ \text{ for all } \xi \in \mathbb{R}^n, \ \xi \neq 0 \ .$$

Our motivations for studying this type of problem are primarily theoretical, although as observed in[12] the mathematical modeling of numerous physical and engineering phenomena can lead to elliptic problems with discontinuous coefficients. Although we do not want (or even need) any further assumptions for many of our results about the regularity of solutions to our obstacle problem, it turns out that the question of existence of solutions will require us to assume some regularity of our a^{ij}. In fact, there is an important example due to C. Pucci (found in[14]) which shows that the strict uniform ellipticity of the a^{ij} (i.e. Equation (1.2)) is in general not even enough to guarantee the existence of a solution to the corresponding partial differential equation. On the other hand, the space of vanishing mean oscillation (VMO) turns out to be a suitable setting for existence results and a priori estimates as was shown in papers by Chiarenza, Frasca, and Longo (see[7] and[8]), and it will also turn out to be an appropriate setting for getting some initial results about the regularity of the free boundary. It is worth noting that there are results due to Meyers which require a little bit less smoothness of the coefficients if one is content to work in L^p spaces with p close to 2 (see[13]), but in this case, one cannot use the Sobolev embedding to get continuity of a first derivative except in dimension two. In any case, we will assume that the a^{ij} belong to VMO when proving existence, and again when we turn to study the regularity of the free boundary.

After showing existence of nontrivial solutions when the a^{ij} belong to VMO, we turn to some of the basic questions in the introductory theory of the obstacle problem. Namely, we follow Caffarelli's treatment (see[6] and[1]), and show nondegeneracy and optimal regularity of the solutions. Once we have these tools, it becomes time to turn to a study of the free boundary regularity. We begin with some useful approximation results and an important measure stability theorem, and from there we establish a number of standard theorems about free boundary regularity.

We will use the following basic notation throughout this dissertation:

χ_D	the chacteristic function of the set D
$\overline{D}$	the closure of the set D
∂D	the boundary of the set D
x	$(x_1, x_2, \ldots, x_n)$
x'	$(x_1, x_2, \ldots, x_{n-1}, 0)$
$B_r(x)$	the open ball with radius r centered at the point x
B_r	$B_r(0)$

For Sobolev spaces and Hölder spaces, we will follow the conventions found within Gilbarg and Trudinger's book. In particular for $1 \leq p \leq \infty$, $W^{k,p}(\Omega)$ will denote the Banach space of functions which are k times weakly differentiable, and whose derivatives of order k and below belong to $L^p(\Omega)$, and for $0 < \alpha \leq 1$, $C^{k,\alpha}(\overline{\Omega})$ will denote the Banach space of functions which are k times differentiable on $\overline{\Omega}$ and whose k^{th} derivatives are uniformly α-Hölder continuous. (See[9] for more details.)

When we are studying free boundary regularity, we will frequently assume

$$0 \in \partial\{w > 0\}\ . \tag{1.3}$$

We will make use of the following terminology. We define:

$$\begin{aligned} &\Omega(w) := \{w > 0\}, \\ &\Lambda(w) := \{w = 0\}, \text{ and} \\ &FB(w) := \partial\Omega(w) \cap \partial\Lambda(w)\ . \end{aligned} \tag{1.4}$$

We will omit the dependence on w when it is clear. Note also that "Λ" and "Δ" each have double duty and it is necessary to interpret them based on their context. We use "Λ" for both the zero set and for one of the constants of ellipticity, and we use "Δ" for the both the Laplacian of a function and for the symmetric difference of two sets in $\mathbb{R}^n$. (If $A, B \subset \mathbb{R}^n$, then $A\Delta B := \{A \setminus B\} \cup \{B \setminus A\}$.)

We will also be using the BMO and the VMO spaces frequently, and we gather the relevent definitions here. (See[12].) For an integrable function f on a set $S \subset \mathbb{R}^n$ we will let

$$f_S := \fint_S f \ .$$

1.2.1 Definition (BMO and BMO norm)**.** If $f \in L^1_{loc}(\mathbb{R}^n)$, and

$$||f||_* := \sup_B \frac{1}{|B|} \int_B |f(x) - f_B| \, dx \tag{1.5}$$

is finite, then f is in the space of bounded mean oscillation, or "$f \in \text{BMO}(\mathbb{R}^n)$." We will take $|| \cdot ||_*$ as our BMO norm.

1.2.2 Definition (VMO and VMO-modulus)**.** Next, for $f \in \text{BMO}$, we define

$$\eta_f(r) := \sup_{\rho \leq r, \ y \in \mathbb{R}^n} \ \frac{1}{|B_\rho|} \int_{B_\rho(y)} |f(x) - f_{_{B_\rho(y)}}| \, dx \ , \tag{1.6}$$

and if $\eta_f(r) \to 0$ as $r \to 0$, then we say that f belongs to the space of vanishing mean oscillation, or "$f \in \text{VMO}$." $\eta_f(r)$ is referred to as the VMO-modulus of the function f.

Since we will need it later, it seems worthwhile to collect some of Caffarelli's results here for the convenience of the reader. These results can be found in[3] and[6]. We start with a definition which will allow us to measure the "flatness" of a set.

1.2.3 Definition (Minimum Diameter)**.** Given a set $S \in \mathbb{R}^n$, we define the minimum diameter of S (or $m.d.(S)$) to be the infimum among the distances between pairs of parallel hyperplanes enclosing S.

1.2.4 Theorem (Caffarelli's Alternative)**.** *Assume γ is a positive number, $w \geq 0$, and*

$$\Delta w = \gamma \chi_{_{\{w>0\}}} \ \textit{in } B_1 \quad \textit{and} \quad 0 \in FB(w) \ .$$

There exists a modulus of continuity $\sigma(\rho)$ depending only on n such that either

a. 0 is a Singular Point of $FB(w)$ in which case
$m.d.(\Lambda \cap B_\rho) \leq \rho \, \sigma(\rho)$ *, for all $\rho \leq 1$, or*

b. 0 *is a Regular Point of* $FB(w)$ *in which case*

there exists a ρ_0 *such that* $m.d.(\Lambda \cap B_{\rho_0}) \geq \rho_0\ \sigma(\rho_0)$, *and for all* $\rho < \rho_0$, $m.d.(\Lambda \cap B_\rho) \geq C\rho\ \sigma(\rho_0)$.

Furthermore, in the case that 0 *is regular, there exists a* ρ_1 *such that for any* $x \in B_{\rho_1} \cap \partial\Omega(w)$, *and any* $\rho < 2\rho_1$, *we have*

$$m.d.(\Lambda \cap B_\rho(x)) \geq C\rho\sigma(2\rho_1). \tag{1.7}$$

So the set of regular points is an open subset of the free boundary, and at any singular point the zero set must become "cusp-like." Examples of solutions with singular points exist and can be found in[11], and in[6] Caffarelli has shown that these singular points must lie in a C^1 manifold.

1.2.5 Theorem (Behavior Near a Regular Point)**.** *Suppose that* w *satisfies the assumptions of the Theorem (1.2.4) but with the domain* B_1 *replaced with the domain* B_M, *and suppose* 0 *is a regular point of* $FB(w)$.

Given $\rho > 0$, *there exists an* $\epsilon = \epsilon(\rho)$ *and an* $M = M(\rho)$, *such that if* $m.d.(\Lambda(w) \cap B_1) > 2n\rho$, *then in an appropriate system of coordinates the following are satisfied for any* x *such that* $|x'| < \rho/16$ *and* $-1 < x_n < 1$, *and for any unit vector* τ *with* $\tau_n > 0$ *and* $||\tau'|| \leq \rho/16$:

a. $D_\tau w \geq 0$.

b. All level surfaces $\{w = c\}$, $c > 0$, *are Lipschitz graphs:*

$$x_n = f(x', c) \quad with \quad ||f||_{_{Lip}} \leq \frac{C(n)}{\rho}\ .$$

c. $D_{e_n} w(x) \geq C(\rho) d(x, \Lambda)$.

d. For $||\tau'|| \leq \rho/32$, $D_\tau w \geq C(\rho) d(x, \Lambda)$.

1.2.6 Theorem ($C^{1,\alpha}$ Regularity of Regular Points)**.** *Suppose that* w *satisfies the assumptions of Theorem (1.2.5) but in* B_1 *again, and suppose* 0 *is a regular point of* $FB(w)$. *There* *exists a universal modulus of continuity $\sigma(\rho)$ such that if for one value of ρ, say ρ_0, we have*

$$m.d.(\Lambda \cap B_{\rho_0}) > \rho_0 \sigma(\rho_0),$$

then in a ρ_0^2 neighborhood of the origin, the free boundary is a $C^{1,\alpha}$ surface $x_n = f(x')$ with

$$||f||_{C^{1,\alpha}} \leq \frac{C(n)}{\rho_0} \; . \tag{1.8}$$

1.2.7 Remark. Note that by the last theorem, the $C^{1,\alpha}$ norm of the free boundary will decay in a universal way at any regular point under the standard quadratic rescaling if we are allowed to rotate the coordinates.

Finally, there are two results due to Chiarenza, Frasca, and Longo which will be of fundamental importance throughout this work, so we will state them here. These results can be found in[7] and[8].

1.2.8 Theorem (Interior Regularity (Taken from Theorem 4.2 of[7])). *Let $D \subset \mathbb{R}^n$ be open, let $p \in (1, \infty)$, assume $a^{ij} \in VMO(D)$ and satisfies Equation (1.2), and let*

$$Lu := a^{ij} D_{ij} u$$

for all $x \in D$. Assume finally that $D'' \subset\subset D' \subset\subset D$. Then there exists a constant C such that

$$||u||_{W^{2,p}(D'')} \leq C(||u||_{L^p(D')} + ||Lu||_{L^p(D')}) \; . \tag{1.9}$$

The constant C depends on $n, \lambda, \Lambda, p, dist(\partial D'', D')$, and quantities which depend only on the a^{ij}. (In particular, C depends on the VMO-modulus of the a^{ij}.)

1.2.9 Theorem (Boundary Regularity (Taken from Theorem 4.2 of[8])). *Let $p \in (1, \infty)$ and assume that $u \in W^{2,p}(B_1) \cap W_0^{1,p}(B_1)$. Then there exists a constant C such that*

$$||u||_{W^{2,p}(B_1)} \leq C(||u||_{L^p(B_1)} + ||Lu||_{L^p(B_1)}) \; . \tag{1.10}$$

The constant C depends on n, λ, Λ, p, and quantities which depend only on the a^{ij}.

1.2.10 Remark ($C^{1,1}$ domains are good enough)**.** We wrote the last result with balls because we will not apply it on any other type of set, but in[8], they prove the result for arbitrary bounded $C^{1,1}$ domains. Of course for a $C^{1,1}$ domain, the constant C will have dependance on the regularity of the boundary.

1.2.11 Corollary (Boundary Regularity II)**.** *Let $p \in (1, \infty)$ and assume that $u, \psi \in W^{2,p}(B_1)$, and $u - \psi \in W_0^{1,p}(B_1)$. Then there exists a constant C such that*

$$||u||_{W^{2,p}(B_1)} \leq C(||u||_{L^p(B_1)} + ||Lu||_{L^p(B_1)} + ||\psi||_{W^{2,p}(B_1)}) \ . \tag{1.11}$$

The constant C depends on n, λ, Λ, p, and quantities which depend only on the a^{ij}.

Chapter 2

Existence Theory when $a^{ij} \in$ VMO

We assume

$$a^{ij} \in \text{VMO}. \tag{2.1}$$

With this assumption coupled with our assumption given in Equation (1.2) we hope to show the existence of a nonnegative solution to Equation (1.1) with nonnegative continuous Dirichlet data, ψ, given on ∂B_1. In order to ease our exposition later, we will assume that we have extended ψ to be a nonnegative continuous function onto all of B_2.

Next, let $\phi(x)$ denote a standard mollifier with support in B_1, and set $\phi_\epsilon(x) := \epsilon^{-n}\phi(x/\epsilon)$. In order to approximate the Heaviside function, we let $\Phi_\epsilon(t)$ be a function which satisfies

$$\begin{array}{ll} 1. & 0 \leq \Phi_\epsilon(t) \leq 1, \ \forall t \in \mathbb{R}. \\ 2. & \Phi_\epsilon(t) \equiv 0 \ \text{ if } \ t \leq 0. \\ 3. & \Phi_\epsilon(t) \equiv 1 \ \text{ if } \ t \geq \epsilon. \\ 4. & \Phi_\epsilon(t) \ \text{ is monotone nondecreasing.} \\ 5. & \Phi_\epsilon \in C^\infty. \end{array} \tag{2.2}$$

We define $a^{ij}_\epsilon := a^{ij} * \phi_\epsilon$, we define $\psi_\epsilon := \psi * \phi_\epsilon$, and finally, we let w_ϵ denote the solution to the problem

$$\begin{aligned} a^{ij}_\epsilon(x) D_{ij} u(x) &= \Phi_\epsilon(u(x)) \quad \text{in} \ \ B_1 \\ u(x) &= \psi_\epsilon(x) \quad \text{on} \ \ \partial B_1 \ . \end{aligned} \tag{2.3}$$

2.0.12 Lemma (Existence of a Solution to the Semilinear PDE)**.** *The boundary value problem (2.3) has a nonnegative solution in $C^\infty(\overline{B_1})$.*

Proof. We will show that the solution, w_ϵ, exists by a fairly standard method of continuity argument below. Using the weak maximum principle it also follows that $w_\epsilon \geq 0$. By Schauder theory it follows that any $C^{2,\alpha}$ solution is automatically C^∞, so it will suffice to get a $C^{2,\alpha}$ solution.

We let S be the set of $t \in [0,1]$ such that the following problem is solvable in $C^{2,\alpha}(\overline{B}_1)$:

$$\begin{aligned} a_\epsilon^{ij}(x)D_{ij}u(x) = t\Phi_\epsilon(u(x)) \quad &\text{in} \;\; B_1 \\ u(x) = \psi_\epsilon(x) \quad &\text{on} \;\; \partial B_1 \; . \end{aligned} \tag{2.4}$$

Equation (2.4) is solvable for $t = 0$ by Schauder Theory. (See chapter 6 of[9].) Thus, S is nonempty.

Claim 1: S as a subset of $[0,1]$ is open.

Proof. We define $L^t(u)$ as a map from the Banach space $C^{2,\alpha}(\overline{B}_1)$ to the Banach space $\mathbf{Y}$ which we define as the direct sum $C^{\alpha}(\overline{B}_1) \oplus C^{2,\alpha}(\overline{\partial B_1})$. (The new norm can be taken as the square root of the sums of the squares of the individual norms.) Our precise definition of $L^t(u)$ is then

$$L^t(u) \;:=\; (\; a_\epsilon^{ij} D_{ij} u - t\Phi_\epsilon(u) \;,\; u \;) \;.$$

Doing calculus in Banach space one can verify that, $[DL^t(u)]v$ is equal to

$$(\; a_\epsilon^{ij} D_{ij} v - t\Phi_\epsilon'(u) v \;,\; v \;) \;,$$

and since $\Phi_\epsilon(t)$ is monotone nondecreasing and smooth we know that the first component of this expression has the form:

$$a_\epsilon^{ij}(x) D_{ij} v(x) - tc(x)v(x) \;\; \text{and} \;\; c(x) \geq 0 \; \forall x \; .$$

By Schauder theory again (see chapter 6 of[9]) the problem

$$\begin{aligned} a_\epsilon^{ij} D_{ij} v - tcv = f \quad &\text{in} \;\; B_1 \\ v = g \quad &\text{on} \;\; \partial B_1 \; . \end{aligned} \tag{2.5}$$

has a unique solution for any pair $(f,\; g) \in \mathbf{Y}$ which satisfies the usual a priori estimates. In other words

$$[DL^t(u)]^{-1} : \mathbf{Y} \to C^{2,\alpha}(\overline{B}_1) \text{ is a bounded 1-1 map.}$$

Therefore, by the infinite dimensional implicit function theorem in Banach spaces, S is open.

Claim 2: S is closed.

Proof. This step is accomplished using a priori estimates. We know that $0 \leq t\Phi_\epsilon(u(x)) \leq 1$. So we have $||a_\epsilon^{ij}(x)D_{ij}u(x)||_{L^\infty(B_1)} \leq 1$, and so for any p we have (see Chapter 9 of[9]),

$$||u||_{W^{2,p}(B_1)} \leq C\left(1 + ||\psi_\epsilon||_{C^0(\partial B_1)}\right) \leq C.$$

By the Sobolev embedding,

$$||u||_{C^{1,\alpha}(\overline{B_1})} \leq C||u||_{W^{2,p}(B_1)} \leq C, \text{ and so } ||t\Phi_\epsilon(u)||_{C^{1,\alpha}(\overline{B_1})} \leq C.$$

Consequently, by Schauder theory again, $u \in C^{3,\alpha}$ and $||u||_{C^{3,\alpha}(\overline{B_1})} \leq C$. Now by Arzela-Ascoli, if $t_k \subset S$ with $t_k \to t_\infty \in [0,1]$, then the corresponding solutions u_{t_k} must converge uniformly together with their 1st and 2nd derivatives to a $C^{3,\alpha}$ function. This function must then solve the t_∞ problem as the left hand sides and right hand sides of the equations in (2.4) are converging uniformly. Thus, S is closed, and hence S must be the entire set, $[0,1]$. ∎

2.0.13 Theorem (Existence of a Solution to the Free Boundary Problem)**.** *Assume Equation (1.2) holds, assume that $a^{ij} \in VMO$, and assume that ψ is nonnegative, continuous, and belongs to $W^{2,p}(B_1)$ for all $p \in (1,\infty)$. Then there exists a nonnegative function $w \in W^{2,p}(B_1)$ which solves Equation (1.1) and satisfies $w - \psi \in W^{2,p}(B_1) \cap W_0^{1,p}(B_1)$ for all $p \in (1,\infty)$. In other words, w satisfies:*

$$\begin{aligned} a^{ij}(x)D_{ij}w(x) &= \chi_{\{w>0\}}(x) \quad \text{in} \;\; B_1 \\ w(x) &= \psi(x) \quad \text{on} \;\; \partial B_1 \, . \end{aligned} \tag{2.6}$$

Proof. We let w_ϵ denote the solution to the problem (2.3), and we view the a_ϵ^{ij} as elements of VMO, and observe that the VMO-moduli $\eta_{a_\epsilon^{ij}}$'s (see Equation (1.6)) are all dominated by the VMO-modulus of the corresponding a^{ij}. (This fact is alluded to in Remark 2.2 of[7].) In fact, we can verify that all of the dependencies on the a^{ij} of the constant within Corollary

(1.2.11) remain under control as we send ϵ to zero. At this point we can invoke this theorem to get a uniform bound on the $W^{2,p}(B_1)$ norm of all of the w_ϵ's. Standard functional analysis allows us to choose a subsequence $\epsilon_n \downarrow 0$, an $\alpha < 1$, and a $w \in W^{2,p}(B_1) \cap C^{1,\alpha}(\overline{B_1})$ such that w_{ϵ_n} converges to w strongly in $C^{1,\alpha}(\overline{B_1})$ and weakly in $W^{2,p}(B_1)$. It remains to show that w satisfies Equation (2.6) .

The fact that $w(x) = \psi(x)$ on ∂B_1 follows immediately from the uniform convergence of the w_{ϵ_n}. Next we need to show that the PDE is satisfied almost everywhere. Everywhere that $w(x) > 0$ it follows easily by the uniform convergence of the w_{ϵ_n} that $\Phi_{\epsilon_n}(w_{\epsilon_n}(x))$ converges to 1. To show that $\Phi_{\epsilon_n}(w_{\epsilon_n}(x))$ converges to 0 almost everywhere on the set $\Lambda := \{w = 0\}$ we assume the opposite in order to derive a contradiction. So, we can assume that there is a new subsequence (still labeled with ϵ_n for convenience), such that

$$0 < \gamma \leq \int_\Lambda \Phi_{\epsilon_n}(w_{\epsilon_n}(x))\, dx$$

for all n. Using this fact we have:

$$\begin{aligned}
0 &< \gamma \\
&\leq \int_\Lambda \Phi_{\epsilon_n}(w_{\epsilon_n})\, dx \\
&= \int_\Lambda a^{ij}_{\epsilon_n} D_{ij} w_{\epsilon_n}\, dx \\
&= \int_\Lambda (a^{ij}_{\epsilon_n} - a^{ij}) D_{ij} w_{\epsilon_n}\, dx + \int_\Lambda a^{ij}(D_{ij} w_{\epsilon_n} - D_{ij} w)\, dx + \int_\Lambda a^{ij} D_{ij} w\, dx \\
&=: I + II + III.
\end{aligned}$$

Integral I converges to zero by using Hölder 's inequality coupled with the strong convergence of a^{ij}_ϵ to a^{ij} in all of the L^p spaces. Integral II converges to zero by using the weak convergence in $W^{2,p}$ of w_{ϵ_n} to w. Finally, integral III is identically zero because the fact that $w \equiv 0$ on Λ guarantees that $D^2 w$ will be zero almost everywhere on Λ. Thus $\Phi_\epsilon(w_\epsilon)$ converges to $\chi_{\{w>0\}}$ pointwise a.e., and as an immediate corollary to this statement, $\Phi_\epsilon(w_\epsilon)$ (and therefore also $a^{ij}_\epsilon D_{ij} w_\epsilon$) converges weakly to $\chi_{\{w>0\}}$ in $L^p(B_1)$ for any $1 < p < \infty$.

Again, by Corollary (1.2.11), we know $D_{ij}w_\epsilon$ is uniformly bounded in $L^p, 1 < p < \infty$. In particular,

$$||D_{ij}w_\epsilon||_{L^3(B_1)} \leq C \ .$$

Now let g be an arbitrary function in $L^3(B_1)$, then:

$$\begin{aligned}
&\int_{B_1} \left[(a_\epsilon^{ij} D_{ij} w_\epsilon)g - (a^{ij} D_{ij} w)g\right] \ dx \\
&= \int_{B_1} \left[(a_\epsilon^{ij} D_{ij} w_\epsilon)g - (a^{ij} D_{ij} w_\epsilon)g\right] \ dx + \int_{B_1} \left[(a^{ij} D_{ij} w_\epsilon)g - (a^{ij} D_{ij} w)g\right] \ dx \\
&= I + II.
\end{aligned}$$

For any fixed i, j, we can apply the Hölder inequality to see that the function $a^{ij}g$ is an element of $L^{3/2}(B_1)$, and then it follows that $II \to 0$ from the fact that $D_{ij}w_\epsilon$ convereges to $D_{ij}w$ weakly in $L^3(B_1)$. On the other hand

$$I \leq ||D_{ij}w_\epsilon||_{L^3(B_1)}||g||_{L^3(B_1)}||a_\epsilon^{ij} - a^{ij}||_{L^3(B_1)} \leq C||a_\epsilon^{ij} - a^{ij}||_{L^3(B_1)} \to 0.$$

Hence, $a_\epsilon^{ij} D_{ij} w_\epsilon$ converges weakly to $a^{ij} D_{ij} w$ in $L^3(B_1)$. By uniqueness of weak limits, it follows that $a^{ij} D_{ij} w = \chi_{\{w>0\}}$ a.e.

■

Chapter 3

Basic Results and Comparison Theorems

In this chapter we will not need to make any assumptions about the regularity of the a^{ij} besides the most basic ellipticity. In spite of our weak hypotheses, we will still be able to show all of the basic regularity and nondegeneracy theorems that we would expect. The fact that we do not need $a^{ij} \in \text{VMO}$ for any result in this chapter will allow us to prove a better measure stability theorem in the next chapter.

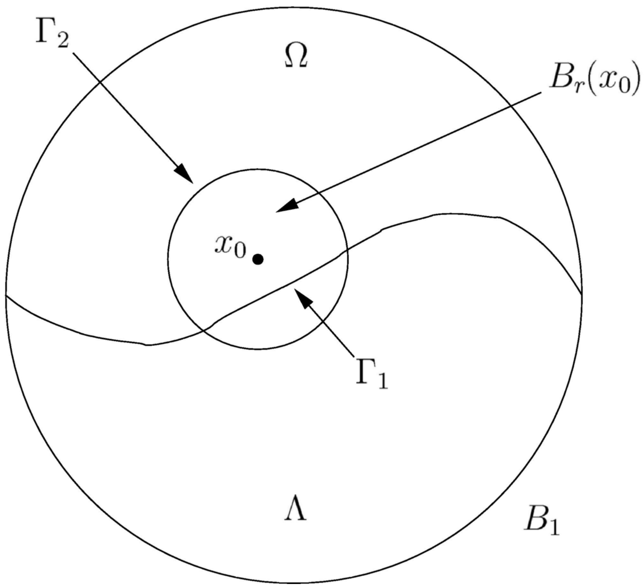
3.0.14 Theorem (Nondegeneracy)**.** *Let w solve (1.1). If $B_r(x_0) \subset B_1$ and $x_0 \in \overline{\Omega}$, then*

$$\sup_{x \in B_r(x_0)} w(x) \geq Cr^2 \,, \tag{3.1}$$

with $C = C(n, \Lambda)$.

Proof. By continuity we can assume that $x_0 \in \Omega$. Define $\Omega_r := B_r(x_0) \cap \Omega$, $\Gamma_1 := FB \cap B_r(x_0)$, and $\Gamma_2 := \partial B_r(x_0) \cap \Omega$. Let $\gamma := \frac{1}{2n}||a^{ij}||_{L^\infty(\Omega_r)}$, and set

$$v(x) := w(x) - w(x_0) - \gamma|x - x_0|^2 \,. \tag{3.2}$$



Now for $x \in \Omega_r$ we compute:

$$
\begin{aligned}
Lv &= a^{ij}D_{ij}w - a^{ij}D_{ij}(\gamma|x-x_0|^2) \\
&= 1 - 2\gamma a^{ij}\delta_{ij} \\
&= 1 - 2\gamma \sum a^{ii} \\
&\geq 1 - 2n\gamma ||a^{ij}||_{L^\infty(\Omega_r)} \\
&\geq 0\ .
\end{aligned}
$$

So now by observing that $v(x_0) = 0$, by using the weak maximum principle of Aleksandrov (see Theorem 9.1 of[9]), and by observing that $v \leq 0$ on Γ_1 we get

$$
\begin{aligned}
0 &\leq \sup_{\Omega_r} v \\
&\leq \sup_{\partial\Omega_r} v^+ \\
&= \sup_{\Gamma_2} v \\
&= \sup_{\Gamma_2} w - w(x_0) - \gamma r^2 \\
&\leq \sup_{\overline{B_r(x_0)}} w - w(x_0) - \gamma r^2\ .
\end{aligned}
$$

Now by rearranging terms and observing $w(x_0) \geq 0$ we are done. ■

3.0.15 Remark (Nontrivial Solutions)**.** As a simple consequence of nondegeneracy, we can take Dirichlet data on ∂B_1 which is positive but small everywhere, to guarantee that we have a solution to our problem which has a nontrivial zero set and a nontrivial free boundary. (The origin must be in the zero set in this case.)

3.0.16 Theorem (Weak Comparison Principle)**.** *Let w_k, $k = 1, 2$ solve (1.1). If $w_1 \leq w_2$ on ∂B_1, then $w_1 \leq w_2$ in B_1.*

Proof. Set $v := w_1 - w_2$, and suppose for the sake of obtaining a contradiction that

$$\max_{x \in B_1} v(x) = v(x_0) = m > 0 \; . \tag{3.3}$$

Now we let

$$S^m := \{x | v(x) = m\} \; . \tag{3.4}$$

Since v is a continuous function, there exists a number $\sigma > 0$, such that $v \geq m/2$ on the σ-neighborhood of S^m. We will denote this set by S^m_σ. Now if S^m_σ extends to the boundary of the set B_1, then we contradict the fact that $v \leq 0$ on ∂B_1, and thus,

$$S^m_\sigma \subset\subset B_1, \;\; \text{and} \;\; v < m \text{ on } \partial S^m_\sigma \; . \tag{3.5}$$

Now on this set, since $w_2 \geq 0$, we must have that $w_1 \geq m/2 > 0$. Thus, we have

$$Lv = Lw_1 - Lw_2 = 1 - Lw_2 \geq 0 \quad \text{in } S^m_\sigma. \tag{3.6}$$

By applying the ABP estimate (see[9] Theorem 9.1) we can conclude that

$$m = \max_{x \in S^m_\sigma} v(x) \leq \max_{x \in \partial S^m_\sigma} v(x) \; , \tag{3.7}$$

but this equation contradicts the fact that $v < m$ on ∂S^m_σ.

Now we let w_3 denote the solution to (1.1) with boundary data equal to $w_1 + \epsilon$. By the first part of the proof, we can conclude that $w_2 \leq w_3$ in B_1. It remains to show that

$w_3 \leq w_1 + \epsilon$. Suppose not. Then the function $u := w_3 - w_1 - \epsilon$ has a positive maximum, m, at a point x_1. Now after observing that $w_3(x) > 0$ in a neighborhood of where $u = m$ the proof is identical to the proof of the first part. ■

3.0.17 Corollary (Uniqueness)**.** *Any solution to (1.1) with given values on ∂B_1 is unique.*

Proof. Let w_1 and w_2 be any two different solutions with fixed values on ∂B_1. Then by applying the weak comparison principle twice, we have $w_1 = w_2$. ■

3.0.18 Lemma (Bound on $B_{1/2}$)**.** *If $w \geq 0$ satisfies Equations (1.1) and (1.3), then $w(x) \leq C(n, \lambda, \Lambda)$ in $\overline{B_{1/2}}$.*

Proof. Write $w := w_1 + w_2$, where

$$\begin{aligned} Lw_1 = \chi_{\{w>0\}} \quad &\text{in} \ \ B_1 \\ w_1 \equiv 0 \quad &\text{on} \ \ \partial B_1 \ , \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} Lw_2 = 0 \quad &\text{in} \ \ B_1 \\ w_2 = w \quad &\text{on} \ \ \partial B_1 \ . \end{aligned} \tag{3.9}$$

Then $w_1 \leq 0$ in B_1 by the maximum principle. On the other hand, by the ABP estimate (Theorem 9.1[9]) we have, $w_1|_{B_1} \geq -C$. Also, by Corollary 9.25[9], along with the fact that $w_1(0) + w_2(0) = w(0) = 0$ we have:

$$w_2|_{B_{\frac{1}{2}}} \leq \ \sup_{B_{\frac{1}{2}}} w_2 \leq \ C \inf_{B_{\frac{1}{2}}} w_2 \leq \ Cw_2(0) = \ -Cw_1(0) \leq C.$$

Hence $w|_{B_{\frac{1}{2}}} \leq C$. ■

3.0.19 Theorem (Parabolic Bound)**.** *If $w \geq 0$ satisfies (1.1) and (1.3), then*

$$w(x) \leq 4C(n, \lambda, \Lambda)|x|^2 \ \text{in} \ B_{1/2},$$

where the constant $C(n, \lambda, \Lambda)$ is the exact same constant as the constant appearing in the statement of the previous lemma.

Proof. Suppose not. Then, $w(\tilde{x}) > 4C(n, \lambda, \Lambda)|\tilde{x}|^2$ for some $\tilde{x} \in B_{1/2}$, and since $0 \in FB$, we must have $\tilde{x} \neq 0$. Now set $\lambda := 2|\tilde{x}|$ so that if $x := \lambda^{-1}\tilde{x}$, then we have $x \in \partial B_{1/2}$. Define:

$$w_\lambda(x) := \lambda^{-2} w(\lambda x) \ . \tag{3.10}$$

Clearly w_λ satisfies (1.1) and (1.3) in B_1. So by the lemma above:

$$w_\lambda(x) \leq C(n, \lambda, \Lambda) \ \text{ in } \ \overline{B_{1/2}} \ . \tag{3.11}$$

On the other hand,

$$\lambda^2 w_\lambda(x) = w(\lambda x) = w(\tilde{x}) > 4C(n, \lambda, \Lambda)|\tilde{x}|^2 = C(n, \lambda, \Lambda)\lambda^2,$$

and so

$$w_\lambda(x) > C(n, \lambda, \Lambda) \ , \tag{3.12}$$

which contradicts Equation (3.11) . ∎

Chapter 4

Approximation and Measure Stability

So far, except to prove our existence theorem, we have not made any assumptions about our a^{ij} beyond ellipticity. In order to prove regularity theorems about the free boundary in the next chapter, we will need to assume once again that the $a^{ij} \in \text{VMO}$. In this chapter, on the other hand, we will not assume $a^{ij} \in \text{VMO}$, but many of our hypotheses anticipate that assumption later. We start with a basic approximation result.

4.0.20 Lemma (First Approximation)**.** *Let w solve (1.1). Then there exist positive constants $\gamma \leq 1$ and $C(q, n, \lambda, \Lambda)$ such that if*

$$\left(\fint_{B_1} |a^{ij}(x) - \delta^{ij}|^q \, dx \right)^{1/q} \leq \epsilon \tag{4.1}$$

for some $0 < \epsilon < 1$ and $q > n$, then we may find a function $h \in W^{2,2}(B_{3/4})$ with $\Delta h = 0$ in $B_{3/4}$, and such that

$$||w - h||_{L^\infty(B_{1/2})} \leq C(\epsilon^{\gamma/4} + ||\chi_{\{w>0\}}||_{L^n(B_{3/4})}) \leq C(\epsilon^{\gamma/4} + 1) \tag{4.2}$$

Proof. By the Hölder estimate for operators in general nondivergence form (see Corollary 9.24 in[9])

$$||w||_{C^\gamma(\overline{B_{3/4}})} \leq C. \tag{4.3}$$

Let h solve

$$\begin{aligned} \Delta h &= 0 \quad \text{in} \quad B_{3/4} \\ h &\equiv w \quad \text{on} \quad \partial B_{3/4} \, . \end{aligned} \tag{4.4}$$

Since h is harmonic and $w \in C^{\gamma}(\partial B_{3/4})$, by Lemma 1.35 of[10] we have:

$$||h||_{C^{\gamma/2}(\overline{B_{3/4}})} \leq C||w||_{C^{\gamma}(\partial B_{3/4})}, \tag{4.5}$$

where $C = C(n, \gamma)$. Now, let $\tilde{y} \in \partial B_{(\frac{3}{4}-s)}$ and let y be the closest point of $\partial B_{3/4}$ to $\tilde{y}$. Then, since $h|_{\partial B_{3/4}} = w$,

$$|w(\tilde{y}) - h(\tilde{y})| \leq |w(\tilde{y}) - w(y)| + |h(y) - h(\tilde{y})| \leq Cs^{\gamma} + Cs^{\gamma/2} \leq Cs^{\gamma/2}.$$

Thus,

$$||w - h||_{L^{\infty}(\partial B_{(3/4-s)})} \leq Cs^{\gamma/2} \; . \tag{4.6}$$

Fix $\tilde{x} \in B_{(3/4-s)}$ and define $v(x) := \frac{h(\tilde{x}+sx)-h(\tilde{x})}{s^{(\gamma/2)}}$.

It follows from (4.3) and (4.5) that $|v(x)| \leq C, \; \forall |x| \leq 1$. Also, since v is harmonic, by Proposition 1.13 of[10]:

$$|D^2 v(0)| \leq C \; \max|v| \leq C \text{ in } \overline{B_{(3/4-s)}} \; . \tag{4.7}$$

Since $D^2 v(x) = \frac{D^2 h(\tilde{x}+sx)}{s^{\gamma/2-2}}$, we have:

$$|D^2 h(\tilde{x})| = |D^2 v(0)| s^{\gamma/2-2} \leq Cs^{\gamma/2-2}. \tag{4.8}$$

Now since h is harmonic, for any $x \in B_{3/4}$:

$$L(w - h) = \chi_{\{w>0\}} + (\delta_{ij} - a^{ij}) D_{ij} h. \tag{4.9}$$

Thus, by using the ABP estimate (as in Lemma 1 of[5]), and then by using Equations (4.1), (4.6), and (4.8) we get:

$$\begin{aligned}
||w - h||_{L^{\infty}(B_{1/2})} &\leq ||w - h||_{L^{\infty}(B_{3/4-s})} \\
&\leq ||w - h||_{L^{\infty}(\partial B_{3/4-s})} \\
&\quad + C(||\chi_{\{w>0\}}||_{L^n(B_{3/4-s})} + ||(\delta^{ij} - a^{ij}) D_{ij} h||_{L^n(B_{3/4-s})}) \\
&= C(s^{\gamma/2} + 1 + s^{\gamma/2-2}\epsilon) \\
&= C(1 + \epsilon^{\gamma/4})
\end{aligned}$$

for $s := \min\{\epsilon^{1/2}, 1/4\}$. ∎

4.0.21 Theorem (Basic L^p Estimate)**.** *Suppose $w \geq 0$ solves (1.1). Given $n < p < \infty$, there exists $\theta = \theta(p)$ such that if:*

$$\sup_{B_1} |a^{ij} - \delta^{ij}| \leq \theta, \tag{4.10}$$

then

$$||D^2 w||_{L^p(B_{7/8})} \leq C\left(||w||_{L^\infty(\partial B_1)} + 1\right) \tag{4.11}$$

Proof. This theorem is a very slight adaptation of Theorem 1 in Section 4 of[5]. The only real difference is the fact that Caffarelli lists continuity of the right hand side of his PDE in his assumptions, whereas our right hand side is a characteristic function.

Let w_k solve:

$$\begin{aligned} Lw_k &= f_k \quad \text{in} \ \ B_1 \\ w_k &\equiv w \quad \text{on} \ \ \partial B_1 \ , \end{aligned} \tag{4.12}$$

where $f_k = \phi_{2^{-k}} * \chi_{\{w>0\}}$, where ϕ_ϵ is a mollifier as in the second section, and where we extend $\chi_{\{w>0\}}$ to be zero outside of B_1. Then $0 \leq f_k \leq 1$, $f_k \in C^\infty$ and for $p < \infty$ we have $f_k \overset{L^p}{\to} \chi_{\{w>0\}}$ in B_1. Set $u_{jk} := w_j - w_k$. With this definition we have:

$$\begin{aligned} Lu_{jk} &= f_j - f_k \quad \text{in} \ \ B_1 \\ u_{jk} &\equiv 0 \quad \text{on} \ \ \partial B_1 \ , \end{aligned} \tag{4.13}$$

Since $f_j - f_k$ is continuous, by Theorem 1 in Section 4 of[5] we get

$$||D^2 u_{jk}||_{L^p(B_{7/8})} \leq C\left(||u_{jk}||_{L^\infty(B_1)} + ||f_j - f_k||_{L^p(B_1)}\right), \tag{4.14}$$

and then by the ABP estimate,

$$||u_{jk}||_{L^\infty(B_1)} \leq \left(\sup_{\partial B_1} u_{jk} + C||f_j - f_k||_{L^n(B_1)}\right) \to 0, \ \text{as} \ j,k \to \infty. \tag{4.15}$$

Hence $D^2 w_j$ is a Cauchy sequence in $L^p(B_{7/8})$ and so it converges in L^p to $D^2 w$. Caffarelli's theorem in the continuous case gives us

$$||D^2 w_j||_{L^p(B_{7/8})} \leq C\left(||w_j||_{L^\infty(\partial B_1)} + ||f_j||_{L^p(B_1)}\right),$$

and so by taking limits we can now say

$$||D^2 w||_{L^p(B_{7/8})} \leq C \left(||w||_{L^\infty(\partial B_1)} + 1 \right). \tag{4.16}$$

■

Now we need a technical compactness lemma which we will need to prove measure stability in this chapter and which we will use again when we prove the existence of blow up limits in the next chapter.

4.0.22 Lemma (Basic Compactness Lemma)**.** *Fix $\gamma > 0$, $1 < p < \infty$ and let $\sigma(r)$ be a modulus of continuity. Assume that we are given the following:*

1. $0 < \lambda I \leq a^{ij,k}(x) \leq \Lambda I$, *for a.e.* x.
2. $w_k \geq 0$ *with* $L^k w_k := a^{ij,k} D_{ij} w_k = \chi_{\{w_k > 0\}}$ *in* B_1.
3. $0 \in FB_k$, *so* $w_k(0) = |\nabla w_k(0)| = 0$.
4. $||w_k||_{W^{2,p}(B_1)} \leq \gamma$.
5. A^{ij} *is a symmetric, constant matrix with* $0 < \lambda I \leq A^{ij} \leq \Lambda I$, *and such that* $||a^{ij,k} - A^{ij}||_{L^1(B_1)} < \sigma(1/k)$.

Then for any $\alpha < 1$ and any $p < \infty$ there exists a function $w_\infty \in W^{2,p}(B_1) \cap C^{1,\alpha}(\overline{B_1})$ and a subsequence of the w_k (which we will still refer to as w_k for ease of notation) such that

A. $w_k \to w_\infty$ *strongly in* $C^{1,\alpha}(\overline{B_1})$,

B. $w_k \rightharpoonup w_\infty$ *weakly in* $W^{2,p}(\overline{B_1})$, *and*

C. $A^{ij} D_{ij} w_\infty = \chi_{\{w_\infty > 0\}}$ *and* $0 \in FB_\infty := \partial\{w_\infty = 0\} \cap B_1$.

Proof. By using the fourth assumption, we immediately have both A and B from elementary functional analysis and the Sobolev Embedding Theorem. We also note that our assumptions of uniform ellipticity actually force a uniform L^∞ bound on all of the $a^{ij,k}$ and the A^{ij}.

That bound, together with the fact that $a^{ij,k} \xrightarrow{L^1} A^{ij}$, allow us to interpolate to any strong convergence in L^q. In other words, by using the fact that

$$||u||_{L^q} \leq ||u||_{L^1}^{(1/q)} \cdot ||u||_{L^\infty}^{(1-(1/q))}$$

(see for example Equation (7.9) in[9]), we can assert that for $q < \infty$ we have $a^{ij,k} \xrightarrow{L^q} A^{ij}$. From this equation it follows that for any $\varphi \in L^\infty$ we have

$$a^{ij,k}\varphi \xrightarrow{L^q} A^{ij}\varphi \; . \tag{4.17}$$

4.0.23 Remark (A Possible Improvement)**.** It seems to be worth observing that if we were to assume that the $a^{ij,k} \in \mathrm{VMO}$ and we removed the assumption of uniform ellipticity, then we could still use the theorem of John and Nirenberg to get strong convergence in L^q. On the other hand, too many of the other proofs rely on the uniform ellipticity of the elliptic operators for us to tackle this issue in this work.

Returning to the proof and letting S be an arbitrary subset of B_1 we have

$$\begin{aligned}
\int_S a^{ij,k} D_{ij} w_k &= \int_S (a^{ij,k} D_{ij} w_k - A^{ij} D_{ij} w_k + A^{ij} D_{ij} w_k) \\
&= \int_S (a^{ij,k} - A^{ij}) D_{ij} w_k + \int_S (A^{ij} D_{ij} w_k - A^{ij} D_{ij} w_\infty + A^{ij} D_{ij} w_\infty) \\
&= \int_S (a^{ij,k} - A^{ij}) D_{ij} w_k + \int_S A^{ij} (D_{ij} w_k - D_{ij} w_\infty) + \int_S A^{ij} D_{ij} w_\infty \\
&= I + II + \int_S A^{ij} D_{ij} w_\infty .
\end{aligned}$$

The integral I now goes to zero by combining Equation (4.17) with the fourth assumption and then using Hölder 's inequality. The integral II goes to zero by using B. Thus we can conclude

$$\int_S a^{ij,k} D_{ij} w_k \to \int_S A^{ij} D_{ij} w_\infty \tag{4.18}$$

for arbitrary $S \subset B_1$, and in particular, the convergence is also pointwise a.e.

Now we claim: $\chi_{\{w_k > 0\}} \to \chi_{\{w_\infty > 0\}}$ a.e in B_1. Since we already know that $a^{ij,k} D_{ij} w_k \to A^{ij} D_{ij} w_\infty$ a.e. and since $a^{ij,k} D_{ij} w_k = \chi_{\{w_k > 0\}}$ a.e., if we show our claim, then it will immediately imply that

$$A^{ij} D_{ij} w_\infty = \chi_{\{w_\infty > 0\}} \quad \text{a.e.} \tag{4.19}$$

Since we obviously have $||\chi_{\{w_k>0\}}||_{L^p(B_1)} \leq C$ for all $p \in (1, \infty]$, elementary functional analysis implies the existence of a function $g \in L^\infty(B_1)$ with $0 \leq g \leq 1$ such that

$$\chi_{\{w_k>0\}} \rightharpoonup g \text{ in } L^p, 1 < p < \infty \, . \tag{4.20}$$

Now, wherever we had $w_\infty > 0$, it is immediate that $\chi_{\{w_k>0\}}$ converges pointwise (and therefore weakly) to 1 by the uniform convergence of w_k to w_∞. In particular, $g \equiv 1$ on $\{w_\infty > 0\}$.

Next we show that $g \equiv 0$ in $\{w_\infty = 0\}^\circ$. So, we suppose that $\overline{B_r(x_0)} \subset \{w_\infty = 0\}$, and we claim that $w_k \equiv 0$ in $\overline{B_{r/2}(x_0)}$ for k sufficiently large. Suppose not. Then applying Theorem (3.0.14) (the nondegeneracy result) to the offending w_k's, we have a sequence $\{x_k\} \subset \overline{B_r(x_0)}$ such that $w_k(x_k) \geq C(r/2)^2$. On the other hand, $w_\infty(x_k) \equiv 0$ (since $\overline{B_r(x_0)} \subset \{w_\infty = 0\}$) and this fact contradicts the uniform convergence of w_k to w_∞.

At this point we have $g(x) \equiv 1$ for $x \in \{w_\infty > 0\}$, and $g(x) \equiv 0$ for $x \in \{w_\infty = 0\}^\circ$ and so g agrees with $\chi_{\{w_\infty>0\}}$ on this set. By the arguments above, the convergence to g is actually pointwise on this set. Now we finish this proof by showing that the set $\mathcal{P} := \{x : |\chi_{\{w_\infty>0\}} - g| \neq 0\}$ has measure zero, and it follows from the preceding arguments that $\partial\{w_\infty = 0\} \subset \mathcal{P}$.

We will show that $\mathcal{P}$ has measure zero by showing that it has no Lebesgue points. To this end, let $x_0 \in \mathcal{P}$ and let r be positive, but small enough so that $B_r(x_0) \subset B_1$. Define $W_\infty(x) := r^{-2} w_\infty(x_0 + rx)$ and define $W_j(x) := r^{-2} w_j(x_0 + rx)$, and observe that all of the convergence we had for w_j to w_∞ carries over to convergence for W_j to W_∞, except that now everything is happening on B_1.

From our change of coordinates, it follows that $0 \in \partial\{W_\infty = 0\}$ and since $W_\infty \geq 0$, there exists a sequence $\{x_k\} \to 0$ such that $W_\infty(x_k) > 0$ for all k. Now fix k so that $x_k \in B_{1/8}$, and then take J sufficiently large to ensure that if $i, j \geq J$ then the following hold:

$$||W_j - W_\infty||_{L^\infty(B_1)} \leq \frac{W_\infty(x_k)}{2} \ , \quad \text{and} \quad ||W_i - W_j||_{L^\infty(B_1)} \leq \frac{\tilde{C}}{10} \tag{4.21}$$

where $\tilde{C}$ is a constant which will be determined from the nondegeneracy theorem, and which will be named momentarily. The existence of such a J follows from the fact that W_j converges to W_∞ in $C^{1,\alpha}(\overline{B_1})$.

We use the first estimate in Equation (4.21) to guarantee that $W_J(x_k) > 0$. We apply Theorem (3.0.14) to W_J at x_k to guarantee the existence of a point $\tilde{x} \in B_{1/2}$ such that

$$W_J(\tilde{x}) \geq C(3/8)^2 \ . \tag{4.22}$$

Putting this equation together with the second convergence statement in Equation (4.21) and letting $\tilde{C}$ be defined by the constant on the right hand side of Equation (4.22) we see that for $i \geq J$ we have:

$$W_i(\tilde{x}) \geq \frac{9\tilde{C}}{10} \ . \tag{4.23}$$

Since all of the W_i's satisfy a uniform $C^{1,\alpha}$ estimate, there exists an $\tilde{r} > 0$ such that $W_i(y) \geq \tilde{C}/2$ for all $y \in B_{\tilde{r}}(\tilde{x})$ once $i \geq J$. From this fact we conclude that $B_{\tilde{r}}(\tilde{x}) \subset \{W_\infty > 0\}$.

Scaling back to the original functions, we conclude that within $B_r(x_0)$ is a ball, B, with radius equal to $r\tilde{r}$ such that $B \subset \{w_\infty > 0\} \subset \mathcal{P}^c$. Since this type of statement will be true for any r sufficiently small, we are guaranteed that x_0 is *not* a Lebesgue point of $\mathcal{P}$. Since x_0 was arbitrary, we can conclude that $\mathcal{P}$ has measure zero.

Finally we observe that the nondegeneracy theorem implies immediately that 0 remains in the free boundary in the limit. ■

4.0.24 Theorem (Basic Measure Stability Result)**.** *Suppose $w \in W^{2,p}(B_1)$ satisfies (1.1) and (1.3), assume $\epsilon > 0$, $p, q > n$, and $||a^{ij} - \delta^{ij}||_{L^q(B_1)} < \epsilon$, and let u denote the solution to*

$$\begin{aligned} \Delta u &= \chi_{\{u>0\}} \quad \text{in} \ \ B_1 \\ u &\equiv w \quad \text{on} \ \ \partial B_1 \ . \end{aligned} \tag{4.24}$$

Then there is a modulus of continuity σ whose definition depends only on $\lambda, \Lambda, p, q, n,$ and $||w||_{W^{2,p}(B_1)}$ such that

$$|\{\Lambda(u) \;\Delta\; \Lambda(w)\} \cap B_1| \leq \sigma(\epsilon). \tag{4.25}$$

(Here we use "Δ" first to denote the Laplacian and next to denote the symmetric difference between two sets: $A \Delta B = \{A \setminus B\} \cup \{B \setminus A\}$.)

Proof. Let $\gamma := ||w||_{W^{2,p}(B_1)}$, and suppose the theorem is false. Then there exist w_k, u_k and $a^{ij,k}$ such that:

1. $L^k w_k = a^{ij,k} D_{ij} w_k = \chi_{\{w_k > 0\}}$ in B_1.
2. $0 \in \mathrm{FB}, w_k(0) = |\nabla w_k(0)| = 0$.
3. $0 < \lambda I \leq a^{ij,k} \leq \Lambda I$.
4. $||a^{ij,k} - \delta^{ij}||_{L^q(B_1)} < \frac{1}{2^k}$.
5. $\Delta u_k = \chi_{\{u_k > 0\}}$ in B_1 and $u_k \equiv w_k$ on ∂B_1.
6. $||w_k||_{W^{2,p}(B_1)} \leq \gamma$.

But,

$$|\Lambda(u_k) \Delta \Lambda(w_k) \cap B_1| \geq \gamma > 0 \text{ for some fixed } \gamma > 0. \tag{4.26}$$

We invoke the last lemma to guarantee the existence of a function w_∞ which satisfies:

$$\Delta w_\infty = \chi_{\{w_\infty > 0\}} \quad \text{a.e.} \tag{4.27}$$

and has $0 \in FB_\infty$.

Now we will use Equation (4.26) to get to a contradiction. We have

$$\begin{aligned}
0 &< \gamma \\
&\leq |\Lambda(u_k)\Delta\Lambda(w_k) \cap B_1| \\
&= ||\chi_{\{u_k>0\}} - \chi_{\{w_k>0\}}||_{L^1(B_1)} \\
&\leq ||\chi_{\{u_k>0\}} - \chi_{\{w_\infty>0\}}||_{L^1(B_1)} + ||\chi_{\{w_\infty>0\}} - \chi_{\{w_k>0\}}||_{L^1(B_1)} \\
&=: I + II\ .
\end{aligned}$$

Now the argument above combined with Lebesgue's Dominated Convergence Theorem shows immediately that $II \to 0$.

In order to show $I \to 0$, we first note that w_∞ and u_k satisfy the same obstacle problem within B_1, and on ∂B_1 we know that u_k equals w_k which in turn converges in $C^{1,\alpha}$ to w_∞. Now by a well-known comparison principle for the obstacle problem (see for example, Theorem 2.7(a) of[1]) we know that

$$||u_k - w_\infty||_{L^\infty(B_1)} \leq ||u_k - w_\infty||_{L^\infty(\partial B_1)}\ . \tag{4.28}$$

At this point we can quote Corollary 4 of[4] to finally conclude that $I \to 0$ and thereby obtain our contradiction. ∎

4.0.25 Corollary (Uniform Stability)**.** *Suppose $w \in W^{2,p}(B_1)$ satisfies (1.1) and (1.3), assume $\epsilon > 0$, $p, q > n$, and $||a^{ij} - \delta^{ij}||_{L^q(B_1)} < \epsilon$, and let u denote the solution to*

$$\begin{aligned}
\Delta u = \chi_{\{u>0\}} &\quad in \;\; B_1 \\
u \equiv w &\quad on \;\; \partial B_1\ .
\end{aligned} \tag{4.29}$$

Then there is a modulus of continuity σ whose definition depends only on $\lambda, \Lambda, p, q, n$, and $||w||_{W^{2,p}(B_1)}$ such that

$$||u - w||_{L^\infty(B_1)} \leq \sigma(\epsilon). \tag{4.30}$$

Proof. By Calderon-Zygmund theory, if the Laplacian of $u - w$ is small in L^r, then $u - w$ will be small in $W^{2,r}$. (See Corollary 9.10 in[9].) If $r > n/2$, then smallness in $W^{2,r}$ guarantees smallness in L^∞ by applying the Sobolev Embedding Theorem.

$$\begin{aligned}\Delta(u-w) &= \chi_{\{u>0\}} - (\delta^{ij} - a^{ij} + a^{ij})D_{ij}w \\ &= (\chi_{\{u>0\}} - \chi_{\{w>0\}}) + (a^{ij} - \delta^{ij})D_{ij}w \\ &=: I + II.\end{aligned}$$

The fact that I is small in any L^r follows from the fact that it is bounded between -1 and 1 (to get control of its L^∞ norm), and is as small as we like in L^1 by Theorem (4.0.24). In order to guarantee that II is small in L^r for some $r > n/2$, we first observe that $D_{ij}w$ is bounded in L^p for some $p > n$, and $||a^{ij} - \delta^{ij}||_{L^q(B_1)}$ is as small as we like by our hypotheses. Now we simply apply Hölder's inequality. ■

Chapter 5

Regularity of the Free Boundary

We turn now to a study of the free boundary in the case where the $a^{ij} \in$ VMO. We will show the existence of blowup limits and it will follow from this result together with the measure stability result from the previous chapter, that a form of the Caffarelli Alternative will hold in a suitable measure theoretic sense.

5.0.26 Theorem (Existence of Blowup Limits)**.** *Assume w satisfies (1.1) and (1.3), assume a^{ij} satisfies (1.2) and belongs to VMO, and define the rescaling*

$$w_\epsilon(x) := \epsilon^{-2} w(\epsilon x).$$

Then for any sequence $\{\epsilon_k\} \downarrow 0$, there exists a subsequence (which we will still call $\{\epsilon_k\}$ to simplify notation) and a symmetric matrix $A = (A^{ij})$ with

$$0 < \lambda I \leq A \leq \Lambda I$$

such that for all $1 \leq i, j \leq n$ we have

$$⨍_{B_{\epsilon_k}} a^{ij}(x)\, dx \to A^{ij}\ , \tag{5.1}$$

and on any compact set, $w_{\epsilon_k}(x)$ converges strongly in $C^{1,\alpha}$ and weakly in $W^{2,p}$ to a function $w_\infty \in W^{2,p}_{loc}(\mathbb{R}^n)$, which satisfies:

$$A^{ij} D_{ij} w_\infty = \chi_{\{w_\infty > 0\}} \quad on\ \mathbb{R}^n, \tag{5.2}$$

and has 0 in its free boundary.

5.0.27 Remark (Nonuniqueness of Blowup Limits)**.** Notice that the theorem does not claim that the blowup limit is unique. In fact, it is relatively easy to produce nonuniqueness, and we will give such an example in the next chapter.

Proof. Because the matrix $a^{ij}(x)$ satisfies $0 < \lambda I \leq a^{ij}(x) \leq \Lambda I$ for all x, it is clear that if we define the matrix

$$A_r^{ij} := \fint_{B_r} a^{ij}(x)\, dx, \tag{5.3}$$

then this matrix must also satisfy the same inequality. Of course, since all of the entries are bounded, we can take a subsequence of the radii ϵ_k such that each scalar $A^{ij}_{\epsilon_k}$ converges to a real number A^{ij}. With this subsequence, we already know that we satisfy Equation (5.1), but because $a^{ij}(x) \in \text{VMO}$, we also know:

$$\fint_{B_{\epsilon_k}} \left|a^{ij}(x) - A^{ij}_{\epsilon_k}\right|\, dx \leq \eta(\epsilon_k) \to 0\ ,$$

where η is just taken to be the maximum of all of the VMO-moduli for each of the a^{ij}'s, and by the triangle inequality this leads to

$$\fint_{B_{\epsilon_k}} \left|a^{ij}(x) - A^{ij}\right|\, dx \to 0\ . \tag{5.4}$$

Now we observe that if $a^{ij,k}(x) := a^{ij}(\epsilon_k x)$ then the rescaled function $w_k := w_{\epsilon_k}$ satisfies the equation:

$$a^{ij,k}(x) D_{ij} w_k(x) = \chi_{\{w_k > 0\}}(x)\ , \tag{5.5}$$

and

$$\fint_{B_1} \left|a^{ij,k}(x) - A^{ij}\right|\, dx \leq \eta(\epsilon_k) \to 0\ . \tag{5.6}$$

By combining Theorem (3.0.19) with Corollary (1.2.11) we get the existence of a constant $\gamma < \infty$ so that $||w_k||_{W^{2,p}(B_1)} \leq \gamma$ for all k. At this point we satisfy all of the hypotheses of Lemma (4.0.22), and applying that lemma gives us exactly what we need. ∎

5.0.28 Theorem (Caffarelli's Alternative in Measure (Weak Form)). *Under the assumptions of the previous theorem, the limit*

$$\lim_{r \downarrow 0} \frac{|\Lambda(w) \cap B_r|}{|B_r|} \tag{5.7}$$

exists and must be equal to either 0 *or* $1/2$.

Proof. We will suppose that

$$\limsup_{r \downarrow 0} \frac{|\Lambda(w) \cap B_r|}{|B_r|} > 0 \tag{5.8}$$

and show that in this case the limit exists and is equal to $1/2$. It follows immediately from this assumption that there exists a sequence $\{\epsilon_k\} \downarrow 0$ such that (for some $\delta > 0$) we have

$$\frac{|\Lambda(w_{\epsilon_k}) \cap B_1|}{|B_1|} > \delta \tag{5.9}$$

for all k. (Here again we use the quadratic rescaling: $w_s(x) := s^{-2}w(sx)$, and we will even shorten "w_{ϵ_k}" to "w_k" henceforth.) We can now apply the last theorem to extract a subsequence (still called "ϵ_k"), and to guarantee the existence of a symmetric positive definite matrix A^{ij} with all of its eigenvalues in $[\lambda, \Lambda]$, and a $w_\infty \in W^{2,p}_{loc}(\mathbb{R}^n)$, such that if $a^{ij,k}(x) := a^{ij}(\epsilon_k x)$, then

$$⨍_{B_1} \left| a^{ij,k}(x) - A^{ij} \right| \, dx \to 0 \, . \tag{5.10}$$

and

$$A^{ij} D_{ij} w_\infty = \chi_{\{w_\infty > 0\}} \quad \text{on } \mathbb{R}^n, \tag{5.11}$$

and 0 is in $FB(w_\infty)$.

Now we make an orthogonal change of coordinates on $\mathbb{R}^n$ to diagonalize the matrix A^{ij}, and then we dilate the individual coordinates by strictly positive amounts depending only on λ and Λ so that in the new coordinate system we have $A^{ij} = \delta^{ij}$. Now of course, there are new functions, and the constants may change by positive factors that we can control, but all of the equations above remain qualitatively unchanged, and we will abuse notation (in a manner similar to the fact that we have not bothered to rename the subsequences), by continuing to refer to our new functions in the new coordinate system as w_k and w_∞, and by continuing to refer to the "new" $a^{ij,k}$ as $a^{ij,k}$, etc.

Now we let u_k denote the solution to

$$\begin{aligned} \Delta u_k = \chi_{\{u_k>0\}} \quad & \text{in} \quad B_1 \\ u_k \equiv w_k \quad & \text{on} \quad \partial B_1 \ . \end{aligned} \tag{5.12}$$

Using Equations (5.9) and (5.10) and applying our measure stability result to u_k and w_k we can make $|\Lambda(u_k)\Delta\Lambda(w_k)|$ as small as we like for k sufficiently large. In particular, we now have:

$$\frac{|\Lambda(u_k) \cap B_1|}{|B_1|} > \frac{\delta}{2} \ . \tag{5.13}$$

Since w_k converges uniformly to w_∞ on every compact set, it follows that u_k converges uniformly to w_∞ on ∂B_1, and now we start arguing exactly as in the last paragraph of the proof of our measure stability theorem. In particular, Equation (4.28) holds, and Corollary 4 of[4] then gives us

$$\frac{|\Lambda(w_\infty) \cap B_1|}{|B_1|} > \frac{\delta}{2} \ . \tag{5.14}$$

Of course now we can invoke the $C^{1,\alpha}$ regularity at regular points (see Theorem (1.2.6)) to guarantee that w_∞ is $C^{1,\alpha}$ at the origin, and this in turn implies that

$$\lim_{r \downarrow 0} \frac{|\Lambda(w_\infty) \cap B_r|}{|B_r|} = \frac{1}{2} \ . \tag{5.15}$$

Now it remains to do two things. First we need to pass this result from w_∞ back to our subsequence of radii for w, but second we will then need to show that we get the same limit along any sequence of radii converging to zero. The first step is a consequence of combining our measure stability theorem with Corollary 4 of[4] again. Indeed, for any $r > 0$,

$$\lim_{k \to \infty} \left(\frac{|\ \Lambda(w_k) \cap B_r\ |}{|\ B_r\ |} - \frac{|\ \Lambda(w_\infty) \cap B_r\ |}{|\ B_r\ |} \right) = 0 \ . \tag{5.16}$$

On the other hand, by our rescaling, this equation becomes

$$\lim_{k \to \infty} \left(\frac{|\ \Lambda(w) \cap B_{(r\epsilon_k)}\ |}{|\ B_{(r\epsilon_k)}\ |} - \frac{|\ \Lambda(w_\infty) \cap B_r\ |}{|\ B_r\ |} \right) = 0 \ , \tag{5.17}$$

which we can combine with Equation (5.15) to ensure that

$$\lim_{k\to\infty} \frac{\mid \Lambda(w) \cap B_{(r\epsilon_k)} \mid}{\mid B_{(r\epsilon_k)} \mid} = \frac{1}{2}\,. \tag{5.18}$$

Finally, we wish to be able to replace "$r\epsilon_k$" with "r" in Equation (5.18). Suppose that we have a different sequence of radii converging to zero (which we can call s_ℓ) such that

$$\lim_{\ell\to\infty} \frac{\mid \Lambda(w) \cap B_{s_\ell} \mid}{\mid B_{s_\ell} \mid} \neq \frac{1}{2}\,. \tag{5.19}$$

At this point we are led to a contradiction in one of two ways. If the limit above does not equal zero (including the case where it simply does not exist), then we can simply use Theorem (5.0.26) combined with Theorem (4.0.24) to get convergence to a global solution with properties which contradict the Caffarelli Alternative (Theorem (1.2.4)). On the other hand, if the limit does equal zero, then we use the continuity of the function:

$$g(r) := \frac{\mid \Lambda(w) \cap B_r \mid}{\mid B_r \mid}$$

to get an interlacing sequence of radii which we can call $\tilde{s}_\ell$ and which converge to zero such that $g(s_\ell) \equiv 1/4$, and then we proceed as in the first case. ∎

5.0.29 Definition (Regular and Singular Free Boundary Points)**.** A free boundary point where Λ has density equal to 0 is referred to as *singular in measure*, and a free boundary point where the density of Λ is $1/2$ is referred to as *regular in measure*. For the rest of this work, we will refer to these free boundary points as simply "singular" or "regular." Note that this definition should be compared with Caffarelli's definition which is given in Theorem (1.2.4).

The theorem above gives us the alternative, but we do not have any kind of uniformity to our convergence. Caffarelli stated his original theorem in a much more quantitative (and therefore useful) way, and so now we will state and prove a similar stronger version. We will need the stronger version in order to show openness and stability under perturbation of the regular points of the free boundary.

5.0.30 Theorem (Caffarelli's Alternative in Measure (Strong Form))**.** *Under the assumptions of the previous theorem, for any* $\epsilon \in (0, 1/8)$, *there exists an* $r_0 \in (0, 1)$, *and a* $\tau \in (0, 1)$ *such that*

if there exists a $t \leq r_0$ *such that*

$$\frac{|\Lambda(w) \cap B_t|}{|B_t|} \geq \epsilon \ , \tag{5.20}$$

then for all $r \leq \tau t$ *we have*

$$\frac{|\Lambda(w) \cap B_r|}{|B_r|} \geq \frac{1}{2} - \epsilon \ , \tag{5.21}$$

and in particular, 0 *is a regular point according to our definition. The* r_0 *and the* τ *depend on* ϵ *and on the* a^{ij}, *but they do not depend on the function* w.

Proof. We start by assuming that we have a t such that Equation (5.20) holds, and by rescaling if necessary, we can assume that $t = r_0$. Next, by arguing exactly as in the last theorem, by assuming that r_0 is sufficiently small, and by defining $s_0 := \sqrt{r_0}$, we can assume without loss of generality that

$$⨍_{B_{s_0}} \left| a^{ij}(x) - \delta^{ij} \right| \ dx \tag{5.22}$$

is as small as we like. Now we will follow the argument given for Theorem 4.5 in[1] very closely.

Applying our measure stability theorem on the ball B_{s_0} we have the existence of a function u which satisfies:

$$\begin{aligned} \Delta u = \chi_{\{u>0\}} \quad & \text{in} \ \ B_{(3s_0)/4} \\ u \equiv w \quad & \text{on} \ \ \partial B_{(3s_0)/4} \ , \end{aligned} \tag{5.23}$$

and so that

$$|\{\Lambda(u) \Delta \Lambda(w)\} \cap B_{r_0}| \tag{5.24}$$

is small enough to guarantee that

$$\frac{|\Lambda(u) \cap B_{r_0}|}{|B_{r_0}|} \geq \frac{\epsilon}{2} \ , \tag{5.25}$$

and therefore

$$m.d.(\Lambda(u) \cap B_{r_0}) \geq C(n) r_0 \epsilon \ . \tag{5.26}$$

Now if r_0 is sufficiently small, then by the $C^{1,\alpha}$ regularity theorem (Theorem (1.2.6)) we conclude that $\partial\Lambda(u)$ is $C^{1,\alpha}$ in an r_0^2 neighborhood of the origin. Furthermore, if we rotate coordinates so that $FB(u) = \{(x', x_n) \mid x_n = f(x')\}$, then we have the following bound (in $B_{r_0^2}$):

$$||f||_{C^{1,\alpha}} \leq \frac{C(n)}{r_0} \ . \tag{5.27}$$

On the other hand, because of this bound, there exists a $\gamma < 1$ such that if $\rho_0 := \gamma r_0 < r_0$, then

$$\frac{|\Lambda(u) \cap B_{\rho_0}|}{|B_{\rho_0}|} \ > \ \frac{1-\epsilon}{2} \ . \tag{5.28}$$

Now by once again requiring r_0 to be sufficiently small, we get

$$\frac{|\Lambda(w) \cap B_{\rho_0}|}{|B_{\rho_0}|} \ > \ \frac{1}{2} - \epsilon \ . \tag{5.29}$$

(So you may note that here our requirement on the size of r_0 will be much smaller than it was before; we need it small both because of the hypotheses within Caffarelli's regularity theorems and because of the need to shrink the L^p norm of $|a^{ij} - \delta^{ij}|$ in order to use our measure stability theorem.)

Now since $\frac{1}{2} - \epsilon$ is strictly greater than ϵ, we can rescale B_{ρ_0} to a ball with a radius *close to* r_0, and then repeat. Since we have a little margin for error in our rescaling, after we repeat this process enough times we will have a small enough radius (which we call τr_0), to ensure that for all $r \leq \tau r_0$ we have

$$\frac{|\Lambda(w) \cap B_r|}{|B_r|} \ > \ \frac{1}{2} - \epsilon \ .$$

$\blacksquare$

5.0.31 Corollary (The Set of Regular Points Is Open)**.** *If we take w as above, then the set of points of $FB(w)$ which are regular in measure is an open subset of $FB(w)$.*

The proof of this corollary is identical to the proof of Corollary 4.8 in[1] except that in place of using Theorem 4.5 of[1] we use Theorem (5.0.30).

5.0.32 Corollary (Persistent Regularity)**.** *Let A^{ij} be a constant symmetric matrix with eigenvalues in $[\lambda, \Lambda]$. Let w satisfy $w \geq 0$,*

$$A^{ij} D_{ij} w = \chi_{\{w>0\}} ,$$

and assume that $FB(w) \cap B_{3/4}$ is $C^{1,\alpha}$. If $a^{ij}(x) \in VMO \cap L^{\infty}(B_1)$, and

$$||a^{ij} - A^{ij}||_{L^q(B_1)}$$

is sufficiently small, then the solution, w_a, to the obstacle problem:

$$w_a \geq 0 \ , \qquad a^{ij}(x) D_{ij} w_a(x) = \chi_{\{w_a>0\}}(x), \qquad w_a = w \text{ on } \partial B_1$$

has a regular free boundary in $B_{1/2}$. (In other words the density of $\Lambda(w_a)$ is equal to $1/2$ at every $x \in FB(w_a) \cap B_{1/2}$.)

Proof. We start by observing that by Theorem (1.2.5) there will be a neighborhood of $FB(w) \cap B_{5/8}$ where $w(x)$ will satisfy:

$$\gamma^{-1} \cdot \text{dist}(x, \Lambda(w))^2 \leq w(x) \leq \gamma \cdot \text{dist}(x, \Lambda(w))^2 \ , \tag{5.30}$$

for a constant $\gamma > 0$. By the same theorem, the size of this neighborhood will be bounded from below by a constant, β, which depends only on the $C^{1,\alpha}$ norm of $FB(w) \cap B_{3/4}$. In other words, Equation (5.30) will hold for all $x \in \Lambda(w)_{\beta} \cap B_{5/8}$. On the other hand, in $\Lambda(w)^c_{\beta} \cap \overline{B_{5/8}}$ the function w will attain a positive minimum. By applying Corollary (4.0.25) to guarantee that

$$||w - w_a||_{L^{\infty}(B_1)}$$

is as small as we like, we can ensure that $w_a > 0$ in $\Lambda(w)^c_{\beta} \cap \overline{B_{5/8}}$, and so $FB(w_a) \subset \Lambda(w)_{\beta}$. By using Theorem (3.0.14) applied to w_a, we can even guarantee that

$$FB(w_a) \cap B_{5/8} \subset FB(w)_{\beta} \ . \tag{5.31}$$

Now fix $0 < \tilde{\epsilon} \ll \epsilon \leq 1/100$. We choose $\tilde{\beta} < \beta$ based on the $C^{1,\alpha}$ norm of $FB(w)$ to ensure that for any $x_0 \in FB(w) \cap B_{5/8}$ and any $r \in (0, \tilde{\beta}]$ we have the inequality:

$$\left| \frac{|B_r(x_0) \cap \Lambda(w)|}{|B_r(x_0)|} - \frac{1}{2} \right| < \epsilon \ . \tag{5.32}$$

Arguing exactly as above and shrinking $||a^{ij} - A^{ij}||_{L^q(B_1)}$ if necessary, we can now guarantee that

$$FB(w_a) \cap B_{5/8} \subset FB(w)_{(\tilde{\epsilon}\tilde{\beta})} \ . \tag{5.33}$$

Now pick an $y_0 \in FB(w_a) \cap B_{1/2}$. Using Equations (5.33) and (5.32) we estimate:

$$\begin{aligned} \frac{|\Lambda(w_a) \cap B_{\tilde{\beta}}(y_0)|}{|B_{\tilde{\beta}}(y_0)|} &\geq \frac{|\Lambda(w) \cap B_{\tilde{\beta}}(y_0)|}{|B_{\tilde{\beta}}(y_0)|} - C(n)\tilde{\epsilon}\tilde{\beta}^n \\ &\geq \frac{1}{2} - \epsilon - C(n)\tilde{\epsilon}\tilde{\beta}^n \\ &\geq 1/4 \ , \end{aligned}$$

as long as we choose our constants sufficiently small. Now by shrinking the value of $\tilde{\beta}$ (if necessary) to be less than the r_0 given in Theorem (5.0.30) we can be sure that y_0 is a regular point of $FB(w_a)$. ∎

Chapter 6

An Important Counter-Example

Now we will give an example of a solution to an obstacle problem of the type we have been studying above which has more than one blowup limit at the origin. The first step will be to construct a convenient discontinuous function in $\mathrm{VMO} \cap L^\infty(B_1)$.

We define the function $f_k(x)$ by letting $f_k(x) := \gamma_k(|x|)$ where $\gamma_k(r)$ is defined by

$$\gamma_k(r) := \begin{cases} 2 & \text{for } r \geq e^{-e^{2k+1}} \\ \dfrac{5 + \cos(\pi \log|\log r|)}{2} & \text{for } r < e^{-e^{2k+1}} \, . \end{cases} \tag{6.1}$$

Now we observe the following properties:

1. $$2 \leq f_k \leq 3 \text{ in } B_1,$$

2. $$\text{for any } q < \infty, \quad \lim_{k \to \infty} ||f_k - 2||_{L^q(B_1)} = 0 \ , \quad \text{and}$$

3. $$\lim_{r \downarrow 0} r\gamma_k'(r) = 0 \ .$$

It now follows from a Theorem of Bramanti (using the first and third propery above) that $f_k(x) \in \mathrm{VMO}(B_1)$. Since we were not able to find this result published elsewhere we will include the proof in an appendix. (This proof is due to Bramanti and is found in his

PhD dissertation: Commutators of singular integrals and parabolic equations with VMO coefficients. Ph.D. Thesis, University of Milano, Italy, 1993.[2])

Now we define $a^{ij,k}(x) := f_k(x)\delta^{ij}$, and $p_\beta(x) := \frac{1}{4}((x_n - \beta)_+)^2$. Observe that p_β solves the obstacle problem:

$$2\Delta w = \chi_{\{w>0\}} \ ,$$

and $FB(p_\beta) = \{x_n = \beta\}$. Now for $-1/10 \leq \beta \leq 1/10$ and $k \in \mathbb{N}$, we let $w_{\beta,k}$ denote the solution to the obstacle problem:

$$w \geq 0, \quad a^{ij,k}(x)D_{ij}w = \chi_{\{w>0\}} \text{ in } B_1, \quad w(x) = p_\beta(x) \text{ on } \partial B_1 \ .$$

Now we observe that

$$2\Delta(p_\beta - w_{\beta,k}) = \chi_{\{p_\beta>0\}} - (2\delta^{ij} - a^{ij,k})D_{ij}w_{\beta,k} - \chi_{\{w_{\beta,k}>0\}} \ ,$$

and so

$$\begin{aligned} ||2\Delta(p_\beta - w_{\beta,k})||_{L^p(B_1)} &\leq ||\chi_{\{p_\beta>0\}} - \chi_{\{w_{\beta,k}>0\}}||_{L^p(B_1)} \\ &\quad + ||(2\delta^{ij} - a^{ij,k})D_{ij}w_{\beta,k}||_{L^p(B_1)} \\ &= ||\chi_{\{p_\beta>0\}} - \chi_{\{w_{\beta,k}>0\}}||_{L^p(B_1)} \\ &\quad + ||(2\delta^{ij} - a^{ij,k})D_{ij}w_{\beta,k}||_{L^p(B_{e^{-e^{2k+1}}})} \end{aligned}$$

The first L^p norm can be made as small as we like by letting k be very large and then by using measure stability, and the second L^p norm can be made as small as we like by letting k be very large and by observing that $||w_{\beta,k}||_{L^q(B_{1/2})} \leq C$. Since $(p_\beta - w_{\beta,k}) \in W^{2,p}(B_1) \cap W^{1,p}_0(B_1)$ we can use Lemma 9.17 of[9] to guarantee that $||p_\beta - w_{\beta,k}||_{W^{2,p}(B_1)}$ is as small as we like for any $p < \infty$ and therefore by the Sobolev embedding

$$||p_\beta - w_{\beta,k}||_{L^\infty(B_1)} \ \text{ is as small as we like.} \tag{6.2}$$

(We have not hesitated to increase k.)

Now by using Theorem (5.0.32) along with the nondegeneracy enjoyed by the $w_{\beta,k}$ functions and with (6.2), we can assert that for all $\beta \in [-1/10, 1/10]$, as long as k is sufficiently large, every $x \in FB(w_{\beta,k}) \cap B_{1/2}$ is a regular free boundary point (in the sense of definition (5.0.29)) and

$$x \in \{\beta - 1/100 < x_n < \beta + 1/100\}.$$

Now we claim that there exists a β_0 such that $0 \in FB(w_{\beta_0,k})$, and since our function $f_k(x)$ oscillates between 2 and 3 infinitely many times as we zoom in toward the origin, we can apply Theorem (5.0.26) to guarantee the existence of different blowup limits. To establish the claim, we observe that if it is false, then there is a ball $B_{2^{-m}}$ which never intersects the free boundary for any β, and this situation would contradict measure stability.

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Appendix A

Radial VMO

A.0.33 Theorem (Radial VMO)**.** *Let* $f : (0, R] \to \mathbb{R}$*,* $f \in C^1(0, R]$*, and assume the following:*

1. $f \in L^2(0, R)$
2. $xf(x)^2 \to 0$ *for* $x \to 0^+$
3. $xf'(x) \to 0$ *for* $x \to 0^+$
4. $\frac{1}{r}\int_0^r x\left[f(r) - f(x)\right] f'(x)\, dx \to 0$ *for* $r \to 0^+$.

(Note that if f *is bounded, then it's enough to assume 3).*

Let $u : B_R(0) \subset \mathbb{R}^n \to \mathbb{R}$

$$u(x) = f(|x|).$$

Then $u \in VMO(B_R(0))$.

Before we prove the theorem, let us see the following lemma. We will consider the case $n = 1$. The general case can be handled similarly by radial change of variables. Hence u is an even function on $[-r, r]$.

A.0.34 Lemma. *If* f, u *are as in the above theorem, then*

$$\psi(r) \equiv \frac{1}{2r}\int_{-r}^{r} \left|u(x) - u_{(-r,r)}\right|^2 dx \to 0 \text{ as } r \to 0.$$

Proof. By integration by parts

$$\begin{aligned} f_{(0,r)} :=& \frac{1}{r}\int_0^r f(x)\,dx \\ &= \frac{1}{r}\left[xf(x)\right]_0^r - \frac{1}{r}\int_0^r xf'(x)\,dx \\ &\to f(r) \end{aligned}$$

Then

$$\begin{aligned} \frac{1}{2r}\int_{-r}^r |u(x)-u_{(-r,r)}|^2dx =& \frac{1}{r}\int_0^r |f(x)-f_{(0,r)}|^2dx \\ &\leq \frac{2}{r}\int_0^r |f(x)-f(r)|^2dx + o(1) \text{ for } r\to 0 \\ &\leq \frac{2}{r}[x[f(x)-f(r)]^2]_0^r - \frac{4}{r}\int_0^r x[f(x)-f(r)]^2 f'(x)dx + o(1) \to 0. \end{aligned}$$

■

To prove the theorem, it suffices to show that

$$\eta^*_{2,u}(r) = \sup_{x\in B_R(0), 0<\sigma<r} \frac{1}{|B_\sigma(x)\cap B_R(0)|}\int_{B_\sigma(x)\cap B_R(0)} \left|u(x) - u_{B_\sigma(x)\cap B_R(0)}\right|^2 dx \to 0 \text{ for } r\to 0.$$

Proof. (for $n=1, R=1$) We will write $(a,b)^* := (a,b)\cap(-1,1)$. To bound $\eta^*_{2,u}(r)$ (for $n=1, R=1$), let

$$\psi(x_0,\varepsilon) = \frac{1}{|(x_0-\varepsilon, x_0+\varepsilon)^*|}\int_{(x_0-\varepsilon,x_0+\varepsilon)^*} \left|u(x) - u_{(x_0-\varepsilon,x_0+\varepsilon)^*}\right|^2 dx.$$

Assume $x_0 \geq 0$ (for symmetry) and recall that

$$\int_a^b \left|u(x) - u_{(a,b)}\right|^2 dx = \min_{\lambda\in\mathbb{R}} \int_a^b |u(x)-\lambda|^2\,dx.$$

Let us distinguish the cases:

1. $0 \leq x_0 < 2\varepsilon$. We can take $\varepsilon < \frac{1}{3}$. Then $(x_0-\varepsilon, x_0+\varepsilon) \subset (-3\varepsilon, 3\varepsilon) \subset (-1,1)$ and

   $$\psi(x_0,\varepsilon) \leq \frac{1}{2\varepsilon}\int_{x_0-\varepsilon}^{x_0+\varepsilon} \left|u(x) - u_{(-3\varepsilon,3\varepsilon)}\right|^2 dx \leq 3\frac{1}{6\varepsilon}\int_{-3\varepsilon}^{3\varepsilon} \left|u(x) - u_{(-3\varepsilon,3\varepsilon)}\right|^2 dx \leq 3\psi(3\varepsilon) \to 0$$

   as $\varepsilon \to 0$, by the above Lemma.

2. $2\varepsilon \leq x_0 < 1$. Then $(x_0 - \varepsilon, x_0 + \varepsilon)^* \subset [\varepsilon, 1]$ and

$$\psi(x_0, \varepsilon) \leq \omega_\varepsilon^2(2\varepsilon)$$

where

$$\omega_\varepsilon(h) = \sup_{|x-y|<h;\ x,y\in[\varepsilon,1]} |f(x) - f(y)|.$$

Since $f \in C^1[\varepsilon, 1]$,

$$\omega_\varepsilon(h) \leq h \cdot \max_{x\in[\varepsilon,1]} |f'(x)|.$$

Now, if f' is bounded on $(0, 1]$ we have $\omega_\varepsilon(2\varepsilon) \leq c\varepsilon$, otherewise:

$$\begin{aligned} \omega_\varepsilon(2\varepsilon) &\leq 2\varepsilon |f'(\xi_\varepsilon)| \text{ for some } \xi_\varepsilon \in [\varepsilon, 1], \text{ and} \\ \omega_\varepsilon(2\varepsilon) &\leq 2\xi_\varepsilon |f'(\xi_\varepsilon)| \end{aligned}$$

with $\xi_\varepsilon \to 0$ as $\varepsilon \to 0$, since f' is unbounded near the origin; then (3) implies $2\xi_\varepsilon |f'(\xi_\varepsilon)| \to 0$. In any case, $\omega_\varepsilon(2\varepsilon) \to 0$ for $\varepsilon \to 0$.

We conclude that

$$\sup_{x_0\in[0,1]} \psi(x_0, \varepsilon) \to 0 \text{ for } \varepsilon \to 0,$$

and $u \in VMO(B_1(0))$.

■