SOME LIMIT BEHAVIORS FOR THE LS ESTIMATORS IN ERRORS-IN-VARIABLES REGRESSION MODEL

by

SHU CHEN

B.S., Shandong University at Weihai, China, 2009

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Major Professor
Dr. Weixing Song

Abstract

There has been a continuing interest among statisticians in the problem of regression models wherein the independent variables are measured with error and there is considerable literature on the subject. In the following report, we discuss the errors-in-variables regression model: $y_i = \beta_0 + \beta_1 x_i + \beta_2 z_i + \epsilon_i, X_i = x_i + u_i, Z_i = z_i + v_i$ with i.i.d. errors (ϵ_i, u_i, v_i), for $i = 1, 2, ..., n$ and find the least square estimators for the parameters of interest. Both weak and strong consistency for the least square estimators $\hat{\beta}_0$, $\hat{\beta}_1$, and $\hat{\beta}_2$ of the unknown parameters β_0, β_1, and β_2 are obtained. Moreover, under regularity conditions, the asymptotic normalities of the estimators are reported.

Table of Contents

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Chapter 1

INTRODUCTION

The relationship between a random scalar y and x is often investigated through the classical linear regression model, $y = \beta_0 + \beta_1 x + \epsilon$, where β_0, β_1 are unknown parameters, and ϵ accounts for the uncontrollable errors with $E\epsilon = 0$, $E\epsilon^2 = \sigma_\epsilon^2$. However, in the real application, sometimes x cannot be observed directly. As an example, consider the relationship between the yield of corn and available nitrogen in the soil. Assume that the above classical linear regression model is an adequate approximation to the relationship between yield and nitrogen. The coefficient β_1 is the amount that yield is increased when soil nitrogen increases one unit. To estimate the available soil nitrogen, it is necessary to sample the soil of the experimental plot and to perform a laboratory analysis on the selected sample. As a result of the sampling and of the laboratory analysis, we do not observe x but observe an estimate of x. Therefore, we represent the observed nitrogen by a surrogate X of x which is available and relates to x in an additive way, $X = x + u$, where u is the measurement error introduced by sampling and laboratory analysis, and independent of x. Then we have the linear errors-in-variables model: $y = \beta_0 + \beta_1 x + \epsilon$, $X = x + u$ where ϵ and u are independent with $E\epsilon = Eu = 0$, $E\epsilon^2 = \sigma_\epsilon^2$, and $Eu^2 = \sigma_u^2$.

One may think it would be simpler to directly regress Y on X without considering the measurement error, but theoretical argument show that if measurement error does present in the predictor X, the resulting least square estimator will be biased. The claim is also supported by the following simulation study. Suppose $x \sim N(0, 1), u \sim N(0, 1)$ and $\epsilon \sim$

$N(0,1)$. We generated n random numbers from these distribution, and select $\beta_0 = 1$ and $\beta_1 = 2$ to find observations on $y = \beta_0 + \beta_1 x + \epsilon$, and $X = x + u$. In the simulation, we choose $n = 100, 200, 300, 400$ and 500. For each scenario, we calculate the least square of estimator (LSE) of β_1 by regressing y directly on X, and the bias-corrected estimator of β_1 by taking the measurement error into account. The least square estimator of β_1 has the form of S_{yX}/S_{XX}, and the bias-corrected estimator of β_1 has the form of $S_{yX}/(S_{XX} - 1)$, where S_{yX} is the sample covariance between the observations on y and X, and S_{XX} is the sample variance of the observations on X. We repeat each setup 500 times, and the mean square errors (MSE) of the estimator are reported in the following table:

n	MSE-LSE	MSE-Bias Corrected
100	0.9811421	0.25507402
200	1.0104561	0.08794780
300	1.0031359	0.05439582
400	0.9980504	0.03900376
500	0.9991646	0.02875133

Comparing the two sets of MSE values, it is clear that the MSE values of the least square estimator of β_1 are all larger than the MSE values of the bias-corrected estimator of β_1, and the MSE of the bias-corrected estimator of β_1 decreases when n increases. Thus, the bias-corrected estimator is more efficient than the least square estimator in the mean square of error sense.

Due to its important role in practical application, the estimation of the regression parameters β_0, β_1, and the variances of ϵ and u of the linear errors-in-variables model has been a long lasting research topics in statistical study, and still received attentions from researchers even for today. For a comprehensive introduction to the estimation problems in the errors-in-variables model, see Fuller (1987). Cui (1997) proved asymptotic normality of some M-estimates in the errors-in-variables model; Liu and Chen (2005) discussed the consistency of least square estimators for the linear errors-in-variables regression model and concluded that both weak and strong consistency are equivalent. They also proved that the following condition is sufficient and necessary for $\hat{\beta}_1$ being a strong and weak consistent estimate of β_1:

$$\lim_{n\to\infty} \frac{S_n}{n} = \infty,$$

where $S_n = \sum_{i=1}^n (x_i - \bar{x}_n)^2$. Recently, Miao et al. (2011) proposed some more precise consistency and asymptotic normality results for the least square estimators of $\hat{\beta}_0$ and $\hat{\beta}_1$ when the contaminated predictor x is one-dimensional and nonrandom.

In the previous corn yield example, we also know that, in addition to the nitrogen, there are many other factors can affect the yield of corn. Therefore, simple linear errors-in-variables model might not fit the data very well. To build a better model, one may incorporate other variables into the model. Let z denotes the true amount of precipitation received by the corn, so in addition to the variable x, the nitrogen content, we can add z to construct a multiple linear regression model. The true value of z is a random in nature due to the fact that the precipitation is greatly affected by the temperature, air pressure, and many other uncontrollable factors. Like nitrogen content, the precipitation z is hard to measure, instead, an estimate Z can be made through some special instrument from the weather station. Z and z can be modeled as $Z = z + v$, where v accounts for the measurement error. Therefore, it might be more proper to consider the following linear errors-in-variables model:

$$\begin{cases} y = \beta_0 + \beta_1 x + \beta_2 z + \epsilon, \\ X = x + u, \\ Z = z + v, \end{cases} \tag{1.1}$$

the predictor x is assumed to be fixed, while z is random. Both x and z are one-dimensional predictors, and cannot be observed directly. Surrogates X and Z of x and z are available, and they are related in additive manner described in (1.1). The error terms ϵ, u, v and the random predictor z are assumed to be independent with $E\epsilon = Eu = Ev = 0$ and $Ez = \mu_z$. Moreover, $E\epsilon^2 = \sigma_\epsilon^2$, $Eu^2 = \sigma_u^2$, $Ev^2 = \sigma_v^2$ and $\text{Var}(z) = \sigma_z^2$ are all positive and finite. Assuming that x and z are one-dimensional can greatly simplify the notation in the following argument, the results obtained in this report surely can be easily extended to the multidimensional case.

Suppose a sample of size n, (y_i, X_i, Z_i), $i = 1, 2, \ldots, n$ is obtained from model (1.1). Then

$$\begin{aligned} y_i &= \beta_0 + \beta_1(X_i - u_i) + \beta_2(Z_i - v_i) + \epsilon_i \\ &= \beta_0 + \beta_1 X_i + \beta_2 Z_i + \epsilon_i - \beta_1 u_i - \beta_2 v_i, \qquad i = 1, 2, ...n \end{aligned}$$

this is a multiple linear regression model of y_i on X_i and Z_i if we treat $\epsilon_i - \beta_1 u_i - \beta_2 v_i$ as the error terms. Without further emphasis, we always assume that ϵ_i, $i = 1, 2, \ldots, n$ are independent and identically distributed, same understanding applies for u_i, and v_i. It is easy to show that the least square estimators of β_0, β_1, and β_2 are given by

$$\begin{aligned} &\hat{\beta}_0^* = \bar{y}_n - \hat{\beta}_1 \bar{X}_n - \hat{\beta}_2^* \bar{Z}_n \\ &\hat{\beta}_1 = \frac{\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 \sum_{i=1}^n (X_i - \bar{X}_n)(y_i - \bar{y}_n)}{\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 \sum_{i=1}^n (X_i - \bar{X}_n)^2 - [\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2} \\ &\quad - \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n) \sum_{i=1}^n (Z_i - \bar{Z}_n)(y_i - \bar{y}_n)}{\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 \sum_{i=1}^n (X_i - \bar{X}_n)^2 - [\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2} \\ &\hat{\beta}_2^* = \frac{\sum_{i=1}^n (X_i - \bar{X}_n)^2 \sum_{i=1}^n (Z_i - \bar{Z}_n)(y_i - \bar{y}_n)}{\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 \sum_{i=1}^n (X_i - \bar{X}_n)^2 - [\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2} \\ &\quad - \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n) \sum_{i=1}^n (X_i - \bar{X}_n)(y_i - \bar{y}_n)}{\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 \sum_{i=1}^n (X_i - \bar{X}_n)^2 - [\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2} \end{aligned}$$

where $\bar{y}_n, \bar{X}_n$ and $\bar{Z}_n$ are the sample means of y_i's, X_i's, Z_i's, respectively. In the errors-in-variables model with only random predictors, the least square estimator is not consistent. Similar phenomenon happens in the errors-in-variables model (1.1). To correct the bias, we modify the least square estimator $\hat{\beta}_2^*$ as follows

$$\begin{aligned} \hat{\beta}_2 &= \frac{\sum_{i=1}^n (X_i - \bar{X}_n)^2 \sum_{i=1}^n (Z_i - \bar{Z}_n)(y_i - \bar{y}_n)}{[\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - n\sigma_v^2] \sum_{i=1}^n (X_i - \bar{X}_n)^2 - [\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2} \\ &- \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n) \sum_{i=1}^n (X_i - \bar{X}_n)(y_i - \bar{y}_n)}{[\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - n\sigma_v^2] \sum_{i=1}^n (X_i - \bar{X}_n)^2 - [\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2} \end{aligned}$$

The estimator defined above is an unbiased estimator of the regression coefficient β_2 which is called bias-corrected estimator. In the sequel, we always use $\hat{\beta}_2$ to denote the bias-corrected estimator of β_2. In the definition of $\hat{\beta}_0^*$, the estimator of β_2 should be replaced by the bias-corrected estimator defined above, so we have $\hat{\beta}_0 = \bar{y}_n - \hat{\beta}_1 \bar{X}_n - \hat{\beta}_2 \bar{Z}_n$ which is corrected for attenuation. Hence, the LS estimators in this report means $\hat{\beta}_0$, $\hat{\beta}_1$, and $\hat{\beta}_2$.

The objective of this report is to obtain both weak and strong consistency for the least square estimators $\hat{\beta}_0$, $\hat{\beta}_1$, and $\hat{\beta}_2$ for the parameters of interest. In addition, under regularity conditions, we get the asymptotic normality of the estimators.

Chapter 2

MAIN RESULTS AND PROOFS

Throughout this chapter, we shall use $\xrightarrow{a.s.}$, $\xrightarrow{p}$, and $\xrightarrow{d}$ to represent the convergence almost surely, convergence in probability and convergence in distribution, respectively. Section 2.1 shows that $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\beta}_2$ are all convergent almost surely; the results for convergence in probability are reported in Section 2.2; and finally, the asymptotic normality of these estimators will be discussed in Section 2.3.

2.1 Convergence almost surely

Let X_n be a sequence of random variables and X be a random variable. We say that X_n converges almost surely to X or $X_n \xrightarrow{a.s.} X$, if $P\{\lim_{n\to\infty} X_n = X\} = 1$. The following theorems state the almost sure convergence of $\hat{\beta}_i s$.

Theorem 2.1.1. *Assume that in model (1.1),*

$$E\left|\epsilon_1\right|^p < \infty, \quad E\left|u_1\right|^p < \infty, \quad E\left|v_1\right|^{p\vee 4} < \infty, \quad E\left|z_1\right|^p < \infty \tag{2.1}$$

for $p \geq 2$ and

$$\lim_{n\to\infty} S_n/n^{2-2/p} = \infty. \tag{2.2}$$

Then $\sqrt{S_n}n^{-1/p}(\hat{\beta}_1 - \beta_1) \xrightarrow{a.s.} 0$.

Remark 2.1.1. *In particular, if $p = 2$, under the same assumption of Theorem 2.1.1, we have $n^{-1/2}\sqrt{S_n}(\hat{\beta}_1 - \beta_1) \xrightarrow{a.s.} 0$. This result is also obtained in Liu and Chen (2005).*

Theorem 2.1.2. *In addition to the assumptions in Theorem 2.1.1, suppose that*

$$a_n = \frac{n^{1/2}}{(\log n)^{1/2+\gamma}} \wedge n^{1-2/p} \quad (\gamma > 0), \tag{2.3}$$

then $a_n(\hat{\beta}_2 - \beta_2) \xrightarrow{a.s.} 0.$

Theorem 2.1.3. *Suppose all the assumptions in Theorem 2.1.1 and Theorem 2.1.2 hold. If we further assume that*

$$\frac{a_n n^{1/p}}{\sqrt{S_n}} |\bar{x}_n| = O(1), \tag{2.4}$$

then $a_n(\hat{\beta}_0 - \beta_0) \xrightarrow{a.s.} 0.$

To facilitate the proofs of the theorems above, we will derive some new expressions for $\hat{\beta}_1 - \beta_1$, $\hat{\beta}_2 - \beta_2$, and $\hat{\beta}_0 - \beta_0$. For $\hat{\beta}_1 - \beta_1$, a simple algebra leads to

$$\begin{aligned}\hat{\beta}_1 - \beta_1 = & \frac{\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 \sum_{i=1}^n (X_i - \bar{X}_n)(y_i - \bar{y}_n)}{\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 \sum_{i=1}^n (X_i - \bar{X}_n)^2 - [\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2} \\ & - \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n) \sum_{i=1}^n (Z_i - \bar{Z}_n)(y_i - \bar{y}_n)}{\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 \sum_{i=1}^n (X_i - \bar{X}_n)^2 - [\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2} \\ & - \frac{\beta_1 \sum_{i=1}^n (Z_i - \bar{Z}_n)^2 \sum_{i=1}^n (X_i - \bar{X}_n)^2 - \beta_1 [\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2}{\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 \sum_{i=1}^n (X_i - \bar{X}_n)^2 - [\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2}.\end{aligned}$$

From (1.1), we have

$$\begin{aligned} & y_i - \bar{y}_n - (X_i - \bar{X}_n)\beta_1 = (y_i - X_i\beta_1) - (\bar{y}_n - \bar{X}_n\beta_1) \\ = & \beta_2(Z_i - \bar{Z}_n) + (\epsilon_i - \bar{\epsilon}_n) - \beta_1(u_i - \bar{u}_n) - \beta_2(v_i - \bar{v}_n) \\ = & \beta_2(z_i - \bar{z}_n) + (\epsilon_i - \bar{\epsilon}_n) - \beta_1(u_i - \bar{u}_n),\end{aligned}$$

thus, one can rewrite

$$\hat{\beta}_1 - \beta_1 = \frac{B_{n1} - B_{n2}}{1 - B_{n3}}, \tag{2.5}$$

where

$$\begin{aligned}B_{n1} = & \frac{\beta_2 \sum_{i=1}^n (x_i - \bar{x}_n)(z_i - \bar{z}_n) - \beta_1 \sum_{i=1}^n (x_i - \bar{x}_n)(u_i - \bar{u}_n) + \beta_2 \sum_{i=1}^n (u_i - \bar{u}_n)(z_i - \bar{z}_n)}{\sum_{i=1}^n (X_i - \bar{X}_n)^2} \\ & - \frac{\beta_1 \sum_{i=1}^n (u_i - \bar{u}_n)^2 - \sum_{i=1}^n (x_i - \bar{x}_n)(\epsilon_i - \bar{\epsilon}_n) - \sum_{i=1}^n (u_i - \bar{u}_n)(\epsilon_i - \bar{\epsilon}_n)}{\sum_{i=1}^n (X_i - \bar{X}_n)^2},\end{aligned}$$

$$
\begin{aligned}
B_{n2} = & \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sum_{i=1}^n (X_i - \bar{X}_n)^2 \sum_{i=1}^n (Z_i - \bar{Z}_n)^2} \Big[\beta_2 \sum_{i=1}^n (z_i - \bar{z}_n)^2 - \beta_1 \sum_{i=1}^n (z_i - \bar{z}_n)(u_i - \bar{u}_n) \\
& + \beta_2 \sum_{i=1}^n (v_i - \bar{v}_n)(z_i - \bar{z}_n) - \beta_1 \sum_{i=1}^n (v_i - \bar{v}_n)(u_i - \bar{u}_n) + \sum_{i=1}^n (z_i - \bar{z}_n)(\epsilon_i - \bar{\epsilon}_n) \\
& + \sum_{i=1}^n (v_i - \bar{v}_i)(\epsilon_i - \bar{\epsilon}_n)\Big], \\
B_{n3} = & \frac{[\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2}{\sum_{i=1}^n (X_i - \bar{X}_n)^2 \sum_{i=1}^n (Z_i - \bar{Z}_n)^2}.
\end{aligned}
$$

For $\hat{\beta}_2 - \beta_2$, we can show that it equals

$$
\begin{aligned}
& \frac{\sum_{i=1}^n (X_i - \bar{X}_n)^2 \sum_{i=1}^n (Z_i - \bar{Z}_n)(y_i - \bar{y}_n)}{[\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - n\sigma_v^2] \sum_{i=1}^n (X_i - \bar{X}_n)^2 - [\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2} \\
- & \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n) \sum_{i=1}^n (X_i - \bar{X}_n)(y_i - \bar{y}_n)}{[\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - n\sigma_v^2] \sum_{i=1}^n (X_i - \bar{X}_n)^2 - [\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2} \\
- & \frac{\beta_2 \sum_{i=1}^n (Z_i - \bar{Z}_n)^2 \sum_{i=1}^n (X_i - \bar{X}_n)^2 - \beta_2 [\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2}{[\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - n\sigma_v^2] \sum_{i=1}^n (X_i - \bar{X}_n)^2 - [\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2} \\
+ & \frac{\beta_2 n \sum_{i=1}^n (X_i - \bar{X}_n)^2 \sigma_v^2}{[\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - n\sigma_v^2] \sum_{i=1}^n (X_i - \bar{X}_n)^2 - [\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2}.
\end{aligned}
$$

From (1.1), we have $y_i - \bar{y}_n - (Z_i - \bar{Z}_n)\beta_2 = \beta_1 (X_i - \bar{X}_n) + (\epsilon_i - \bar{\epsilon}_n) - \beta_1 (u_i - \bar{u}_n) - \beta_2 (v_i - \bar{v}_n)$. Therefore, we can rewrite $\hat{\beta}_2 - \beta_2$ as

$$
\begin{aligned}
& \frac{\sum_{i=1}^n (X_i - \bar{X}_n)^2 \sum_{i=1}^n (Z_i - \bar{Z}_n)[\beta_2 (X_i - \bar{X}_n) + (\epsilon_i - \bar{\epsilon}_n) - \beta_1 (u_i - \bar{u}_n) - \beta_2 (v_i - \bar{v}_n)]}{[\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - n\sigma_v^2] \sum_{i=1}^n (X_i - \bar{X}_n)^2 - [\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2} \\
- & \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n) \sum_{i=1}^n (X_i - \bar{X}_n)[\beta_2 (X_i - \bar{X}_n) + (\epsilon_i - \bar{\epsilon}_n) - \beta_1 (u_i - \bar{u}_n)]}{[\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - n\sigma_v^2] \sum_{i=1}^n (X_i - \bar{X}_n)^2 - [\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2} \\
- & \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n) \sum_{i=1}^n (X_i - \bar{X}_n)[\beta_2 (v_i - \bar{v}_n)] - \beta_2 n \sum_{i=1}^n (X_i - \bar{X}_n)^2 \sigma_v^2}{[\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - n\sigma_v^2] \sum_{i=1}^n (X_i - \bar{X}_n)^2 - [\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2}.
\end{aligned}
$$

Some rearrangements finally lead to

$$
\begin{aligned}
\hat{\beta}_2-\beta_2 \;=\; & \frac{\sum_{i=1}^n (Z_i-\bar{Z}_n)(\epsilon_i-\bar{\epsilon}_n)/\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2}{1-[\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)]^2/\sum_{i=1}^n (X_i-\bar{X}_n)^2[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]} \\
& -\frac{\beta_1\sum_{i=1}^n (Z_i-\bar{Z}_n)(u_i-\bar{u}_n)/\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2}{1-[\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)]^2/\sum_{i=1}^n (X_i-\bar{X}_n)^2[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]} \\
& -\frac{\beta_2[\sum_{i=1}^n (Z_i-\bar{Z}_n)(v_i-\bar{v}_n)-n\sigma_v^2]/\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2}{1-[\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)]^2/\sum_{i=1}^n (X_i-\bar{X}_n)^2[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]} \\
& -\frac{\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n (X_i-\bar{X}_n)(\epsilon_i-\bar{\epsilon}_n)}{1-[\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)]^2/\sum_{i=1}^n (X_i-\bar{X}_n)^2[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]} \\
& \times\frac{1}{\sum_{i=1}^n (X_i-\bar{X}_n)^2[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]} \\
& +\frac{\beta_1\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n (X_i-\bar{X}_n)(u_i-\bar{u}_n)}{1-[\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)]^2/\sum_{i=1}^n (X_i-\bar{X}_n)^2[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]} \\
& \times\frac{1}{\sum_{i=1}^n (X_i-\bar{X}_n)^2[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]} \\
& +\frac{\beta_2\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n (X_i-\bar{X}_n)(v_i-\bar{v}_n)}{1-[\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)]^2/\sum_{i=1}^n (X_i-\bar{X}_n)^2[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]} \\
& \times\frac{1}{\sum_{i=1}^n (X_i-\bar{X}_n)^2[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]}.
\end{aligned}
$$

For the sake of brevity, denote C_{n1}, C_{n2}, C_{n3}, C_{n4}, C_{n5}, C_{n6}, and C_{n7} as follows

$$
\begin{aligned}
C_{n1} &= \frac{\sum_{i=1}^n (Z_i-\bar{Z}_n)(\epsilon_i-\bar{\epsilon}_n)}{\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2}, \\
C_{n2} &= \frac{\beta_1\sum_{i=1}^n (Z_i-\bar{Z}_n)(u_i-\bar{u}_n)}{\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2}, \\
C_{n3} &= \frac{\beta_2[\sum_{i=1}^n (Z_i-\bar{Z}_n)(v_i-\bar{v}_n)-n\sigma_v^2]}{\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2}, \\
C_{n4} &= \frac{\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n (X_i-\bar{X}_n)(\epsilon_i-\bar{\epsilon}_n)}{\sum_{i=1}^n (X_i-\bar{X}_n)^2[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]}, \\
C_{n5} &= \frac{\beta_1\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n (X_i-\bar{X}_n)(u_i-\bar{u}_n)}{\sum_{i=1}^n (X_i-\bar{X}_n)^2[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]}, \\
C_{n6} &= \frac{\beta_2\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n (X_i-\bar{X}_n)(v_i-\bar{v}_n)}{\sum_{i=1}^n (X_i-\bar{X}_n)^2[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]}, \\
C_{n7} &= \frac{[\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)]^2}{\sum_{i=1}^n (X_i-\bar{X}_n)^2[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]}.
\end{aligned}
$$

Then we have

$$
\hat{\beta}_2-\beta_2=\frac{C_{n1}-C_{n2}-C_{n3}-C_{n4}+C_{n5}+C_{n6}}{1-C_{n7}}. \tag{2.6}
$$

Finally, for $\hat{\beta}_0 - \beta_0$, we can show that

$$\hat{\beta}_0 - \beta_0 = (\beta_1 - \hat{\beta}_1)\bar{X}_n + (\beta_2 - \hat{\beta}_2)\bar{Z}_n + \bar{\epsilon}_n - \beta_1\bar{u}_n - \beta_2\bar{v}_n. \tag{2.7}$$

To prove the theorems above, we also need two lemmas, which are stated in the following.

Lemma 2.1.1. *(Miao et al. (2011)) Suppose that $\{K_i, i = 1, 2, ...\}$ is a sequence of i.i.d. random variables with $EK_1 = 0$ and $E|K_1|^p < \infty$, $(p \geq 2)$ and $\{a_{i,n}, i = 1, ..., n, n = 1, ...\}$ is a sequence of non-random weighted coefficients with $\sum_{i=1}^n a_{i,n}^2 = 1$ for all $n \geq 1$. Then $n^{-1/p}\sum_{i=1}^n a_{i,n}K_i \xrightarrow{a.s.} 0$.*

Lemma 2.1.2. *(Durrett (2005)) Let $\{T_i, i = 1, 2, ...\}$ be i.i.d. random variables with $ET_1 = 0$ and $ET_i^2 = \sigma^2 < \infty$ and Let $M_n = T_1 + T_2 + ... + T_n$. Then $M_n/(n^{1/2}(\log n)^{1/2+\gamma}) \xrightarrow{a.s.} 0$ for any $\gamma \geq 0$.*

Now, we can prove Theorem 2.1.1.

Proof of Theorem 2.1.1: From condition (2.1) and the strong law of large numbers, we have

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^n z_i \xrightarrow{a.s.} \mu_z, \quad \frac{1}{n}\sum_{i=1}^n Z_i \xrightarrow{a.s.} \mu_z,\\
&\frac{1}{n}\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - \sigma_v^2 \xrightarrow{a.s.} \sigma_z^2, \quad \frac{1}{n}\sum_{i=1}^n v_i \xrightarrow{a.s.} 0,\\
&\frac{1}{n}\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 \xrightarrow{a.s.} \sigma_Z^2, \quad \frac{1}{n}\sum_{i=1}^n (z_i - \bar{z}_n)^2 \xrightarrow{a.s.} \sigma_z^2,\\
&\frac{1}{n}\sum_{i=1}^n (u_i - \bar{u}_n)(Z_i - \bar{Z}_n) \xrightarrow{a.s.} \mathrm{Cov}(u, Z) = 0,\\
&\frac{1}{n}\sum_{i=1}^n (u_i - \bar{u}_n)(z_i - \bar{z}_n) \xrightarrow{a.s.} \mathrm{Cov}(u, z) = 0, \qquad (2.8)\\
&\frac{1}{n}\sum_{i=1}^n (v_i - \bar{v}_n)(z_i - \bar{z}_n) \xrightarrow{a.s.} \mathrm{Cov}(v, z) = 0,\\
&\frac{1}{n}\sum_{i=1}^n (v_i - \bar{v}_n)(u_i - \bar{u}_n) \xrightarrow{a.s.} \mathrm{Cov}(v, u) = 0,
\end{aligned}$$

$$\frac{1}{n}\sum_{i=1}^{n}(z_i-\bar{z}_n)(\epsilon_i-\bar{\epsilon}_n)\xrightarrow{a.s.}\mathrm{Cov}(z,\epsilon)=0,$$
$$\frac{1}{n}\sum_{i=1}^{n}(v_i-\bar{v}_n)(\epsilon_i-\bar{\epsilon}_n)\xrightarrow{a.s.}\mathrm{Cov}(v,\epsilon)=0,$$

by the independence of ϵ, u, v and z. Some of the above results will be used in the proof of Theorem 2.1.2 and 2.1.3. In addition, by (2.1), (2.2), the strong law of large numbers, and the finiteness of σ_ϵ^2, σ_u^2, σ_v^2 and σ_z^2, we get

$$\begin{aligned}
\frac{1}{\sqrt{S_n}n^{1/p}}\sum_{i=1}^{n}(u_i-\bar{u}_n)^2 &\le \frac{1}{\sqrt{S_n}n^{1/p}}\sum_{i=1}^{n}u_i^2=\frac{n}{\sqrt{S_n}n^{1/p}}\times\frac{1}{n}\sum_{i=1}^{n}u_i^2\xrightarrow{a.s.}0,\\
\frac{1}{\sqrt{S_n}n^{1/p}}\sum_{i=1}^{n}(z_i-\bar{z}_n)^2 &\le \frac{1}{\sqrt{S_n}n^{1/p}}\sum_{i=1}^{n}z_i^2=\frac{n}{\sqrt{S_n}n^{1/p}}\times\frac{1}{n}\sum_{i=1}^{n}z_i^2\xrightarrow{a.s.}0,\\
\frac{1}{\sqrt{S_n}n^{1/p}}\sum_{i=1}^{n}(v_i-\bar{v}_n)^2 &\le \frac{1}{\sqrt{S_n}n^{1/p}}\sum_{i=1}^{n}v_i^2=\frac{n}{\sqrt{S_n}n^{1/p}}\times\frac{1}{n}\sum_{i=1}^{n}v_i^2\xrightarrow{a.s.}0,\\
\frac{1}{\sqrt{S_n}n^{1/p}}\sum_{i=1}^{n}(\epsilon_i-\bar{\epsilon}_n)^2 &\le \frac{1}{\sqrt{S_n}n^{1/p}}\sum_{i=1}^{n}\epsilon_i^2=\frac{n}{\sqrt{S_n}n^{1/p}}\times\frac{1}{n}\sum_{i=1}^{n}\epsilon_i^2\xrightarrow{a.s.}0,
\end{aligned}\tag{2.9}$$

which yield

$$\begin{aligned}
\left|\frac{1}{\sqrt{S_n}n^{1/p}}\sum_{i=1}^{n}(u_i-\bar{u}_n)(\epsilon_i-\bar{\epsilon}_n)\right| &\le \frac{1}{2\sqrt{S_n}n^{1/p}}\sum_{i=1}^{n}\left[(u_i-\bar{u}_n)^2+(\epsilon_i-\bar{\epsilon}_n)^2\right]\xrightarrow{a.s.}0,\\
\left|\frac{1}{\sqrt{S_n}n^{1/p}}\sum_{i=1}^{n}(u_i-\bar{u}_n)(z_i-\bar{z}_n)\right| &\le \frac{1}{2\sqrt{S_n}n^{1/p}}\sum_{i=1}^{n}\left[(u_i-\bar{u}_n)^2+(z_i-\bar{z}_n)^2\right]\xrightarrow{a.s.}0,\\
\left|\frac{1}{\sqrt{S_n}n^{1/p}}\sum_{i=1}^{n}(u_i-\bar{u}_n)(v_i-\bar{v}_n)\right| &\le \frac{1}{2\sqrt{S_n}n^{1/p}}\sum_{i=1}^{n}\left[(u_i-\bar{u}_n)^2+(v_i-\bar{v}_n)^2\right]\xrightarrow{a.s.}0.
\end{aligned}\tag{2.10}$$

Furthermore, $\sum_{i=1}^{n}(X_i-\bar{X}_n)^2$ has the following decomposition, $\sum_{i=1}^{n}(X_i-\bar{X}_n)^2 = \sum_{i=1}^{n}(x_i-\bar{x}_n)^2+2\sum_{i=1}^{n}(x_i-\bar{x}_n)(u_i-\bar{u}_n)+\sum_{i=1}^{n}(u_i-\bar{u}_n)^2$, from Cauchy-Schwarz inequality, we can show that

$$\left|\frac{1}{S_n}\sum_{i=1}^n (X_i-\bar{X}_n)^2-1\right| \le \frac{1}{S_n}\left[2\sum_{i=1}^n |(x_i-\bar{x}_n)(u_i-\bar{u}_n)| + \sum_{i=1}^n (u_i-\bar{u}_n)^2\right]$$
$$\le \frac{1}{S_n}\left[2\sqrt{\sum_{i=1}^n (x_i-\bar{x}_n)^2}\sqrt{\sum_{i=1}^n (u_i-\bar{u}_n)^2}+\sum_{i=1}^n (u_i-\bar{u}_n)^2\right]$$
$$= 2\sqrt{\frac{1}{S_n}\sum_{i=1}^n (u_i-\bar{u}_n)^2}+\frac{1}{S_n}\sum_{i=1}^n (u_i-\bar{u}_n)^2 \xrightarrow{a.s.} 0.$$

In the above argument, we used the condition (2.2) and the first equation of (2.9). Hence, we have

$$\frac{1}{S_n}\sum_{i=1}^n (X_i-\bar{X}_n)^2 \xrightarrow{a.s.} 1. \tag{2.11}$$

This fact will be used frequently in the subsequent proofs.

Finally, let us denote $a_{i,n}=(x_i-\bar{x}_n)/\sqrt{\sum_{i=1}^n (x_i-\bar{x}_n)^2}$, $K_i=(Z_i-EZ)$. Then $(a_{i,n}, 1\le i\le n, n\ge 1)$ and $(K_i, i\ge 1)$ satisfy the assumptions of Lemma 2.1.1 with $p=2$. Therefore, we get

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n \left[\frac{(x_i-\bar{x}_n)}{\sqrt{\sum_{i=1}^n (x_i-\bar{x}_n)^2}}\right](Z_i-EZ) \xrightarrow{a.s.} 0. \tag{2.12}$$

Similarly, we have

$$\begin{aligned}
\frac{1}{\sqrt{S_n}n^{1/p}}\sum_{i=1}^n (x_i-\bar{x}_n)(z_i-Ez) &= \frac{1}{n^{1/p}}\sum_{i=1}^n \left[\frac{(x_i-\bar{x}_n)}{\sqrt{\sum_{i=1}^n (x_i-\bar{x}_n)^2}}\right](z_i-Ez) \xrightarrow{a.s.} 0,\\
\frac{1}{\sqrt{S_n}n^{1/p}}\sum_{i=1}^n (x_i-\bar{x}_n)(u_i-Eu) &= \frac{1}{n^{1/p}}\sum_{i=1}^n \left[\frac{(x_i-\bar{x}_n)}{\sqrt{\sum_{i=1}^n (x_i-\bar{x}_n)^2}}\right](u_i-Eu) \xrightarrow{a.s.} 0,\\
\frac{1}{\sqrt{S_n}n^{1/p}}\sum_{i=1}^n (x_i-\bar{x}_n)(v_i-Ev) &= \frac{1}{n^{1/p}}\sum_{i=1}^n \left[\frac{(x_i-\bar{x}_n)}{\sqrt{\sum_{i=1}^n (x_i-\bar{x}_n)^2}}\right](v_i-Ev) \xrightarrow{a.s.} 0,\\
\frac{1}{\sqrt{S_n}n^{1/p}}\sum_{i=1}^n (x_i-\bar{x}_n)(\epsilon_i-E\epsilon) &= \frac{1}{n^{1/p}}\sum_{i=1}^n \left[\frac{(x_i-\bar{x}_n)}{\sqrt{\sum_{i=1}^n (x_i-\bar{x}_n)^2}}\right](\epsilon_i-E\epsilon) \xrightarrow{a.s.} 0.
\end{aligned} \tag{2.13}$$

Now, we can start to prove $\sqrt{S_n}B_{n1}/n^{1/p} \xrightarrow{a.s.} 0$, first note that

$$
\begin{aligned}
&\frac{\sqrt{S_n}}{n^{1/p}} B_{n1} = \frac{B_{n1}/\sqrt{S_n} n^{1/p}}{1/S_n} \\
= \;& \frac{\frac{1}{\sqrt{S_n} n^{1/p}}[\beta_2 \sum_{i=1}^n (x_i - \bar{x}_n)(z_i - \bar{z}_n) - \beta_1 \sum_{i=1}^n (x_i - \bar{x}_n)(u_i - \bar{u}_n)]}{S_n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2} \\
&+ \frac{\frac{1}{\sqrt{S_n} n^{1/p}}[\beta_2 \sum_{i=1}^n (u_i - \bar{u}_n)(z_i - \bar{z}_n) - \beta_1 \sum_{i=1}^n (u_i - \bar{u}_n)^2]}{S_n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2} \\
&+ \frac{\frac{1}{\sqrt{S_n} n^{1/p}}[\sum_{i=1}^n (x_i - \bar{x}_n)(\epsilon_i - \bar{\epsilon}_n) + \sum_{i=1}^n (u_i - \bar{u}_n)(\epsilon_i - \bar{\epsilon}_n)]}{S_n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2} \\
= \;& \frac{\frac{1}{\sqrt{S_n} n^{1/p}}[\beta_2 \sum_{i=1}^n (x_i - \bar{x}_n)(z_i - Ez + Ez - \bar{z}_n) - \beta_1 \sum_{i=1}^n (x_i - \bar{x}_n)(u_i - Eu + Eu - \bar{u}_n)]}{S_n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2} \\
&+ \frac{\frac{1}{\sqrt{S_n} n^{1/p}}[\beta_2 \sum_{i=1}^n (u_i - \bar{u}_n)(z_i - \bar{z}_n) - \beta_1 \sum_{i=1}^n (u_i - \bar{u}_n)^2]}{S_n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2} \\
&+ \frac{\frac{1}{\sqrt{S_n} n^{1/p}}[\sum_{i=1}^n (x_i - \bar{x}_n)(\epsilon_i - E\epsilon + E\epsilon - \bar{\epsilon}_n) + \sum_{i=1}^n (u_i - \bar{u}_n)(\epsilon_i - \bar{\epsilon}_n)]}{S_n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2} \\
= \;& \frac{\frac{1}{\sqrt{S_n} n^{1/p}}[\beta_2 \sum_{i=1}^n (x_i - \bar{x}_n)(z_i - Ez) - \beta_1 \sum_{i=1}^n (x_i - \bar{x}_n)(u_i - Eu)]}{S_n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2} \\
&+ \frac{\frac{1}{\sqrt{S_n} n^{1/p}}[\beta_2 \sum_{i=1}^n (u_i - \bar{u}_n)(z_i - \bar{z}_n) - \beta_1 \sum_{i=1}^n (u_i - \bar{u}_n)^2]}{S_n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2} \\
&+ \frac{\frac{1}{\sqrt{S_n} n^{1/p}}[\sum_{i=1}^n (x_i - \bar{x}_n)(\epsilon_i - E\epsilon) + \sum_{i=1}^n (u_i - \bar{u}_n)(\epsilon_i - \bar{\epsilon}_n)]}{S_n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2}.
\end{aligned}
$$

Then from the first equation in (2.9), the first, second equations in (2.10), (2.11), the first, second, and fourth equations in (2.13), we have

$$\frac{\sqrt{S_n}}{n^{1/p}} B_{n1} \xrightarrow{a.s.} 0. \tag{2.14}$$

In order to prove $\sqrt{S_n} B_{n2}/n^{1/p} \xrightarrow{a.s.} 0$, let us prove $\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)/\sqrt{S_n} n^{1/p} \xrightarrow{a.s.} 0$ first, since

$$\frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sqrt{S_n} n^{1/p}} = \frac{\sum_{i=1}^n (x_i - \bar{x}_n + u_i - \bar{u}_n)(z_i - \bar{z}_n + v_i - \bar{v}_n)}{\sqrt{S_n} n^{1/p}}$$

$$= \frac{1}{\sqrt{S_n} n^{1/p}} \sum_{i=1}^{n} (x_i - \bar{x}_n)(z_i - \bar{z}_n) + \frac{1}{\sqrt{S_n} n^{1/p}} \sum_{i=1}^{n} (x_i - \bar{x}_n)(v_i - \bar{v}_n)$$
$$+ \frac{1}{\sqrt{S_n} n^{1/p}} \sum_{i=1}^{n} (u_i - \bar{u}_n)(z_i - \bar{z}_n) + \frac{1}{\sqrt{S_n} n^{1/p}} \sum_{i=1}^{n} (u_i - \bar{u}_n)(v_i - \bar{v}_n)$$

Therefore, from the second, third equations in (2.10), the first, third equations in (2.13), one can obtain

$$\frac{\sum_{i=1}^{n} (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sqrt{S_n} n^{1/p}} \xrightarrow{a.s.} 0. \tag{2.15}$$

Similarly we have

$$\frac{\sum_{i=1}^{n} (X_i - \bar{X}_n)(\epsilon_i - \bar{\epsilon}_n)}{\sqrt{S_n} n^{1/p}} \xrightarrow{a.s.} 0, \quad \frac{\sum_{i=1}^{n} (X_i - \bar{X}_n)(v_i - \bar{v}_n)}{\sqrt{S_n} n^{1/p}} \xrightarrow{a.s.} 0,$$
$$\frac{\sum_{i=1}^{n} (X_i - \bar{X}_n)(u_i - \bar{u}_n)}{\sqrt{S_n} n^{1/p}} \xrightarrow{a.s.} 0. \tag{2.16}$$

Which will be used in the proof of Theorem 2.1.2.

At present, let us show that $\sqrt{S_n} B_{n2}/n^{1/p} \xrightarrow{a.s.} 0$. A tedious and routing calculation leads to

$$\begin{aligned}
\frac{\sqrt{S_n}}{n^{1/p}} B_{n2} &= \frac{\sqrt{S_n}}{n^{1/p}} \frac{\sum_{i=1}^{n} (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \sum_{i=1}^{n} (Z_i - \bar{Z}_n)^2} \Big[\beta_2 \sum_{i=1}^{n} (z_i - \bar{z}_n)^2 \\
&\quad - \beta_1 \sum_{i=1}^{n} (z_i - \bar{z}_n)(u_i - \bar{u}_n) + \beta_2 \sum_{i=1}^{n} (v_i - \bar{v}_n)(z_i - \bar{z}_n) - \beta_1 \sum_{i=1}^{n} (v_i - \bar{v}_n)(u_i - \bar{u}_n) \\
&\quad + \sum_{i=1}^{n} (z_i - \bar{z}_n)(\epsilon_i - \bar{\epsilon}_n) + \sum_{i=1}^{n} (v_i - \bar{v}_i)(\epsilon_i - \bar{\epsilon}_n) \Big] \\
&= \beta_2 \frac{\sqrt{S_n}}{n^{1/p}} \frac{\sum_{i=1}^{n} (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \sum_{i=1}^{n} (Z_i - \bar{Z}_n)^2} \sum_{i=1}^{n} (z_i - \bar{z}_n)^2 \\
&\quad - \beta_1 \frac{\sqrt{S_n}}{n^{1/p}} \frac{\sum_{i=1}^{n} (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \sum_{i=1}^{n} (Z_i - \bar{Z}_n)^2} \sum_{i=1}^{n} (z_i - \bar{z}_n)(u_i - \bar{u}_n) \\
&\quad + \beta_2 \frac{\sqrt{S_n}}{n^{1/p}} \frac{\sum_{i=1}^{n} (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \sum_{i=1}^{n} (Z_i - \bar{Z}_n)^2} \sum_{i=1}^{n} (v_i - \bar{v}_n)(z_i - \bar{z}_n) \\
&\quad - \beta_1 \frac{\sqrt{S_n}}{n^{1/p}} \frac{\sum_{i=1}^{n} (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sum_{i=1}^{n} (X_i - \bar{X}_n)^2 \sum_{i=1}^{n} (Z_i - \bar{Z}_n)^2} \sum_{i=1}^{n} (v_i - \bar{v}_n)(u_i - \bar{u}_n)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\sqrt{S_n}}{n^{1/p}}\frac{\sum_{i=1}^n(X_i-\bar{X}_n)(Z_i-\bar{Z}_n)}{\sum_{i=1}^n(X_i-\bar{X}_n)^2\sum_{i=1}^n(Z_i-\bar{Z}_n)^2}\sum_{i=1}^n(z_i-\bar{z}_n)(\epsilon_i-\bar{\epsilon}_n)\\
&+\frac{\sqrt{S_n}}{n^{1/p}}\frac{\sum_{i=1}^n(X_i-\bar{X}_n)(Z_i-\bar{Z}_n)}{\sum_{i=1}^n(X_i-\bar{X}_n)^2\sum_{i=1}^n(Z_i-\bar{Z}_n)^2}\sum_{i=1}^n(v_i-\bar{v}_i)(\epsilon_i-\bar{\epsilon}_n)\\
=\ &\beta_2\frac{\sum_{i=1}^n(X_i-\bar{X}_n)(Z_i-\bar{Z}_n)}{\sqrt{S_n}n^{1/p}}\frac{1}{S_n^{-1}\sum_{i=1}^n(X_i-\bar{X}_n)^2}\frac{\frac{1}{n}\sum_{i=1}^n(z_i-\bar{z}_n)^2}{\frac{1}{n}\sum_{i=1}^n(Z_i-\bar{Z}_n)^2}\\
&-\beta_1\frac{\sum_{i=1}^n(X_i-\bar{X}_n)(Z_i-\bar{Z}_n)}{\sqrt{S_n}n^{1/p}}\frac{1}{S_n^{-1}\sum_{i=1}^n(X_i-\bar{X}_n)^2}\frac{\frac{1}{n}\sum_{i=1}^n(z_i-\bar{z}_n)(u_i-\bar{u}_n)}{\frac{1}{n}\sum_{i=1}^n(Z_i-\bar{Z}_n)^2}\\
&+\beta_2\frac{\sum_{i=1}^n(X_i-\bar{X}_n)(Z_i-\bar{Z}_n)}{\sqrt{S_n}n^{1/p}}\frac{1}{S_n^{-1}\sum_{i=1}^n(X_i-\bar{X}_n)^2}\frac{\frac{1}{n}\sum_{i=1}^n(v_i-\bar{v}_n)(z_i-\bar{z}_n)}{\frac{1}{n}\sum_{i=1}^n(Z_i-\bar{Z}_n)^2}\\
&-\beta_1\frac{\sum_{i=1}^n(X_i-\bar{X}_n)(Z_i-\bar{Z}_n)}{\sqrt{S_n}n^{1/p}}\frac{1}{S_n^{-1}\sum_{i=1}^n(X_i-\bar{X}_n)^2}\frac{\frac{1}{n}\sum_{i=1}^n(v_i-\bar{v}_n)(u_i-\bar{u}_n)}{\frac{1}{n}\sum_{i=1}^n(Z_i-\bar{Z}_n)^2}\\
&+\frac{\sum_{i=1}^n(X_i-\bar{X}_n)(Z_i-\bar{Z}_n)}{\sqrt{S_n}n^{1/p}}\frac{1}{S_n^{-1}\sum_{i=1}^n(X_i-\bar{X}_n)^2}\frac{\frac{1}{n}\sum_{i=1}^n(z_i-\bar{z}_n)(\epsilon_i-\bar{\epsilon}_n)}{\frac{1}{n}\sum_{i=1}^n(Z_i-\bar{Z}_n)^2}\\
&+\frac{\sum_{i=1}^n(X_i-\bar{X}_n)(Z_i-\bar{Z}_n)}{\sqrt{S_n}n^{1/p}}\frac{1}{S_n^{-1}\sum_{i=1}^n(X_i-\bar{X}_n)^2}\frac{\frac{1}{n}\sum_{i=1}^n(v_i-\bar{v}_i)(\epsilon_i-\bar{\epsilon}_n)}{\frac{1}{n}\sum_{i=1}^n(Z_i-\bar{Z}_n)^2}
\end{aligned}
$$

So that, from the fifth, sixth, eighth, ninth, tenth, eleventh, twelfth, equations of (2.8), (2.11) and (2.15), we can get the desired result

$$
\frac{\sqrt{S_n}}{n^{1/p}}B_{n2}\xrightarrow{a.s.}0. \tag{2.17}
$$

Finally, because

$$
\begin{aligned}
&\frac{\sum_{i=1}^n(X_i-\bar{X}_n)(Z_i-\bar{Z}_n)}{\sqrt{\sum_{i=1}^n(X_i-\bar{X}_n)^2}\sqrt{\sum_{i=1}^n(Z_i-\bar{Z}_n)^2}}\\
=\ &\frac{\sum_{i=1}^n(X_i-\bar{X}_n)(Z_i-\bar{Z}_n)}{\sqrt{n}\sqrt{\sum_{i=1}^n(X_i-\bar{X}_n)^2}}\frac{1}{\sqrt{\frac{1}{n}\sum_{i=1}^n(Z_i-\bar{Z}_n)^2}}\\
=\ &\left[\frac{\sum_{i=1}^n(x_i-\bar{x}_n)(Z_i-\bar{Z}_n)}{\sqrt{n}\sqrt{\sum_{i=1}^n(x_i-\bar{x}_n)^2}}+\frac{\sum_{i=1}^n(u_i-\bar{u}_n)(Z_i-\bar{Z}_n)}{\sqrt{n}\sqrt{\sum_{i=1}^n(x_i-\bar{x}_n)^2}}\right]\frac{\sqrt{\sum_{i=1}^n(x_i-\bar{x}_n)^2}}{\sqrt{\sum_{i=1}^n(X_i-\bar{X}_n)^2}}\\
&\times\frac{1}{\sqrt{\frac{1}{n}\sum_{i=1}^n(Z_i-\bar{Z}_n)^2}}
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\frac{(x_i-\bar{x}_n)}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x}_n)^2}}\Big)(Z_i-EZ+EZ-\bar{Z}_n)+\frac{1}{n}\frac{\sum_{i=1}^{n}(u_i-\bar{u}_n)(Z_i-\bar{Z}_n)}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x}_n)^2/n}}\right] \\
&\quad \times\frac{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x}_n)^2}}{\sqrt{\sum_{i=1}^{n}(X_i-\bar{X}_n)^2}}\frac{1}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(Z_i-\bar{Z}_n)^2}} \\
&= \left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\Big(\frac{(x_i-\bar{x}_n)}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x}_n)^2}}\Big)(Z_i-EZ)+\frac{\frac{1}{n}\sum_{i=1}^{n}(u_i-\bar{u}_n)(Z_i-\bar{Z}_n)}{\sqrt{S_n/n}}\right] \\
&\quad \times\frac{1}{\sqrt{S_n^{-1}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2}}\frac{1}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(Z_i-\bar{Z}_n)^2}},
\end{aligned}
$$

then from the condition (2.2), the fifth, seventh equations in (2.8), (2.11) and (2.12), we can show that

$$B_{n3}\xrightarrow{a.s.}0. \tag{2.18}$$

Combining (2.5), (2.14), (2.17) and (2.18), we can obtain that

$$\frac{\sqrt{S_n}}{n^{1/p}}(\hat{\beta}_1-\beta_1)=\frac{\frac{\sqrt{S_n}}{n^{1/p}}B_{n1}-\frac{\sqrt{S_n}}{n^{1/p}}B_{n2}}{1-B_{n3}}\xrightarrow{a.s.}0.$$

This completes the proof of Theorem 2.1.1. □

Remark: Theorem 2.1.1 shows that the almost sure convergence rate is the same as the one in the fixed design errors-in-variables model discussed in Miao et al. (2011). So adding a random component z to the fixed design errors-in-variables model does not affect the almost sure convergence rate of $\hat{\beta}_1$. ■

Proof of Theorem 2.1.2: Let $T_i = Z_i\epsilon_i$, $M_n=\sum_{i=1}^{n}T_i$, then $T_i, i=1,2,\dots,n$ and M_n satisfy the assumptions of Lemma 2.1.2. Therefore we have

$$\frac{\sum_{i=1}^{n}Z_i\epsilon_i}{n^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\gamma}}=\frac{M_n}{n^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\gamma}}\xrightarrow{a.s.}0.$$

From the condition (2.3), we get

$$\left|\frac{a_n}{n}\sum_{i=1}^{n}Z_i\epsilon_i\right|\le\left|\frac{\sum_{i=1}^{n}Z_i\epsilon_i}{n^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\gamma}}\right|=\left|\frac{M_n}{n^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\gamma}}\right|\xrightarrow{a.s.}0. \tag{2.19}$$

Similarly, we can show that

$$
\begin{aligned}
&\left|\frac{a_n}{n}\sum_{i=1}^{n}\epsilon_i\right| \le \left|\frac{\sum_{i=1}^{n}\epsilon_i}{n^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\gamma}}\right| \xrightarrow{a.s.} 0, \quad \left|\frac{a_n}{n}\sum_{i=1}^{n}v_i\right| \le \left|\frac{\sum_{i=1}^{n}v_i}{n^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\gamma}}\right| \xrightarrow{a.s.} 0,\\
&\left|\frac{a_n}{n}\sum_{i=1}^{n}Z_iu_i\right| \le \left|\frac{\sum_{i=1}^{n}Z_iu_i}{n^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\gamma}}\right| \xrightarrow{a.s.} 0, \quad \left|\frac{a_n}{n}\sum_{i=1}^{n}z_iv_i\right| \le \left|\frac{\sum_{i=1}^{n}z_iv_i}{n^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\gamma}}\right| \xrightarrow{a.s.} 0,\\
&\left|\frac{a_n}{n}\sum_{i=1}^{n}u_i\right| \le \left|\frac{\sum_{i=1}^{n}u_i}{n^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\gamma}}\right| \xrightarrow{a.s.} 0, \qquad (2.20)\\
&\left|\frac{a_n}{n}\sum_{i=1}^{n}[(v_i-\bar{v}_n)^2-\sigma_v^2]\right| \le \left|\frac{\sum_{i=1}^{n}[(v_i-\bar{v}_n)^2-\sigma_v^2]}{n^{\frac{1}{2}}(\log n)^{\frac{1}{2}+\gamma}}\right| \xrightarrow{a.s.} 0.
\end{aligned}
$$

Here and now, we can start to prove $a_nC_{n1} \xrightarrow{a.s.} 0$, note that

$$a_nC_{n1} = a_n\frac{\sum_{i=1}^{n}(Z_i-\bar{Z}_n)(\epsilon_i-\bar{\epsilon}_n)}{\sum_{i=1}^{n}(Z_i-\bar{Z}_n)^2-n\sigma_v^2} = \frac{\frac{a_n}{n}\sum_{i=1}^{n}Z_i\epsilon_i-\bar{Z}_na_n\bar{\epsilon}_n}{\frac{1}{n}\sum_{i=1}^{n}(Z_i-\bar{Z}_n)^2-\sigma_v^2}.$$

From the second, third equations in (2.8), (2.19) and the first equation in (2.20), we have

$$a_nC_{n1} \xrightarrow{a.s.} 0. \qquad (2.21)$$

For C_{n2}, since

$$a_nC_{n2} = \beta_1a_n\frac{\sum_{i=1}^{n}(Z_i-\bar{Z}_n)(u_i-\bar{u}_n)}{\sum_{i=1}^{n}(Z_i-\bar{Z}_n)^2-n\sigma_v^2} = \beta_1\frac{\frac{a_n}{n}\sum_{i=1}^{n}Z_iu_i-\bar{Z}_na_n\bar{u}_n}{\frac{1}{n}\sum_{i=1}^{n}(Z_i-\bar{Z}_n)^2-\sigma_v^2}.$$

Then from the second, third equations in (2.8), the third and fifth equations in (2.20), we get

$$a_nC_{n2} \xrightarrow{a.s.} 0. \qquad (2.22)$$

For C_{n3}, simple calculation shows that a_nC_{n3} equals

$$\beta_2a_n\frac{[\sum_{i=1}^{n}(Z_i-\bar{Z}_n)(v_i-\bar{v}_n)-n\sigma_v^2]}{\sum_{i=1}^{n}(Z_i-\bar{Z}_n)^2-n\sigma_v^2} = \beta_2\frac{\frac{a_n}{n}\sum_{i=1}^{n}z_iv_i-\bar{z}_na_n\bar{v}_n+\frac{a_n}{n}\sum_{i=1}^{n}[(v_i-\bar{v}_n)^2-\sigma_v^2]}{\frac{1}{n}\sum_{i=1}^{n}(Z_i-\bar{Z}_n)^2-\sigma_v^2}.$$

Therefore, from the first, third equations in (2.8), the second, fourth, sixth equations in (2.20), we obtain that

$$a_nC_{n3} \xrightarrow{a.s.} 0. \qquad (2.23)$$

Note that $|a_n C_{n4}|$ can be written as

$$
\begin{aligned}
&\left| a_n \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n) \sum_{i=1}^n (X_i - \bar{X}_n)(\epsilon_i - \bar{\epsilon}_n)}{\sum_{i=1}^n (X_i - \bar{X}_n)^2 [\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - n\sigma_v^2]} \right| \\
&= \left| \frac{\frac{a_n}{n} \sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n) \sum_{i=1}^n (X_i - \bar{X}_n)(\epsilon_i - \bar{\epsilon}_n)}{\sum_{i=1}^n (X_i - \bar{X}_n)^2} \frac{1}{\frac{1}{n} \sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - \sigma_v^2} \right| \\
&\leq \left| \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sqrt{S_n} n^{\frac{1}{p}}} \right| \left| \frac{\sqrt{S_n}}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2}} \right| \left| \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(\epsilon_i - \bar{\epsilon}_n)}{\sqrt{S_n} n^{\frac{1}{p}}} \right| \\
&\times \left| \frac{\sqrt{S_n}}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2}} \right| \left| \frac{1}{\frac{1}{n} \sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - \sigma_v^2} \right|.
\end{aligned}
$$

By the third equation in (2.8), (2.11), (2.15) and the first equation in (2.16), we get

$$a_n C_{n4} \xrightarrow{a.s.} 0. \tag{2.24}$$

For $a_n C_{n5}$, as

$$
\begin{aligned}
&\left| \beta_1 a_n \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n) \sum_{i=1}^n (X_i - \bar{X}_n)(u_i - \bar{u}_n)}{\sum_{i=1}^n (X_i - \bar{X}_n)^2 [\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - n\sigma_v^2]} \right| \\
&= \left| \beta_1 \frac{\frac{a_n}{n} \sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n) \sum_{i=1}^n (X_i - \bar{X}_n)(u_i - \bar{u}_n)}{\sum_{i=1}^n (X_i - \bar{X}_n)^2} \frac{1}{\frac{1}{n} \sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - \sigma_v^2} \right| \\
&\leq \left| \beta_1 \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sqrt{S_n} n^{\frac{1}{p}}} \right| \left| \frac{\sqrt{S_n}}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2}} \right| \left| \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(u_i - \bar{u}_n)}{\sqrt{S_n} n^{\frac{1}{p}}} \right| \\
&\times \left| \frac{\sqrt{S_n}}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2}} \right| \left| \frac{1}{\frac{1}{n} \sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - \sigma_v^2} \right|.
\end{aligned}
$$

Thus, noting the third equation in (2.8), (2.11), (2.15) and the third equation in (2.16), we have

$$a_n C_{n5} \xrightarrow{a.s.} 0. \tag{2.25}$$

To show that $a_n C_{n6} \xrightarrow{a.s.} 0$, first we have

$$
\begin{aligned}
&\left| \beta_2 a_n \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n) \sum_{i=1}^n (X_i - \bar{X}_n)(v_i - \bar{v}_n)}{\sum_{i=1}^n (X_i - \bar{X}_n)^2 [\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - n\sigma_v^2]} \right| \\
&= \left| \beta_2 \frac{\frac{a_n}{n} \sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n) \sum_{i=1}^n (X_i - \bar{X}_n)(v_i - \bar{v}_n)}{\sum_{i=1}^n (X_i - \bar{X}_n)^2} \frac{1}{\frac{1}{n} \sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - \sigma_v^2} \right|
\end{aligned}
$$

$$\leq \left|\beta_2 \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sqrt{S_n} n^{\frac{1}{p}}}\right| \left|\frac{\sqrt{S_n}}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2}}\right| \left|\frac{\sum_{i=1}^n (X_i - \bar{X}_n)(v_i - \bar{v}_n)}{\sqrt{S_n} n^{\frac{1}{p}}}\right|$$
$$\times \left|\frac{\sqrt{S_n}}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2}}\right| \left|\frac{1}{\frac{1}{n}\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - \sigma_v^2}\right|,$$

then by the third equation in (2.8), (2.11), (2.15) and second equation in (2.16), we have

$$a_n C_{n6} \xrightarrow{a.s.} 0. \tag{2.26}$$

Finally, let's prove $C_{n7} \xrightarrow{a.s.} 0$. Because

$$\begin{aligned}
& \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2}\sqrt{[\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - n\sigma_v^2]}} \\
= & \frac{\sum_{i=1}^n (x_i - \bar{x}_n + u_i - \bar{u}_n)(Z_i - \bar{Z}_n)}{\sqrt{n}\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2}} \frac{1}{\sqrt{\frac{1}{n}\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - \sigma_v^2}} \\
= & \left[\frac{\sum_{i=1}^n (x_i - \bar{x}_n)(Z_i - \bar{Z}_n)}{\sqrt{n}\sqrt{\sum_{i=1}^n (x_i - \bar{x}_n)^2}} + \frac{\sum_{i=1}^n (u_i - \bar{u}_n)(Z_i - \bar{Z}_n)}{\sqrt{n}\sqrt{\sum_{i=1}^n (x_i - \bar{x}_n)^2}}\right] \\
& \times \frac{\sqrt{\sum_{i=1}^n (x_i - \bar{x}_n)^2}}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2}} \frac{1}{\sqrt{\frac{1}{n}\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - \sigma_v^2}} \\
= & \left[\frac{1}{\sqrt{n}}\sum_{i=1}^n \Big(\frac{(x_i - \bar{x}_n)}{\sqrt{\sum_{i=1}^n (x_i - \bar{x}_n)^2}}\Big)(Z_i - EZ + EZ - \bar{Z}_n)\right. \\
& \left. + \frac{1}{n}\frac{\sum_{i=1}^n (u_i - \bar{u}_n)(Z_i - \bar{Z}_n)}{\sqrt{\sum_{i=1}^n (x_i - \bar{x}_n)^2 / n}}\right] \frac{\sqrt{\sum_{i=1}^n (x_i - \bar{x}_n)^2}}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2}} \frac{1}{\sqrt{\frac{1}{n}\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - \sigma_v^2}}
\end{aligned}$$

$$\begin{aligned}
= & \left[\frac{1}{\sqrt{n}}\sum_{i=1}^n \Big(\frac{(x_i - \bar{x}_n)}{\sqrt{\sum_{i=1}^n (x_i - \bar{x}_n)^2}}\Big)(Z_i - EZ) + \frac{\frac{1}{n}\sum_{i=1}^n (u_i - \bar{u}_n)(Z_i - \bar{Z}_n)}{\sqrt{S_n/n}}\right] \\
& \times \frac{1}{\sqrt{\frac{1}{S_n}\sum_{i=1}^n (X_i - \bar{X}_n)^2}} \frac{1}{\sqrt{\frac{1}{n}\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - \sigma_v^2}}.
\end{aligned}$$

So that, from (2.2), the third, seventh equations in (2.8), (2.11), and (2.12), we obtain that

$$C_{n7} \xrightarrow{a.s.} 0. \tag{2.27}$$

Consequently, combining (2.6), (2.21) to (2.27), we have the desired result $a_n(\hat{\beta}_2 - \beta_2) \xrightarrow{a.s.} 0$. This completes the proof of Theorem 2.1.2. □

Remark: If we focus on the convergence rate a_n in the Theorem 2.1.2, we can tell that if $p = 2, 3$, then $a_n = n^{1-2/p}$, otherwise $a_n = n^{1/2}/(\log n)^{1/2+\gamma}$. ■

Proof of Theorem 2.1.3: By the condition (2.4) and the Theorem 2.1.1, we have

$$\begin{aligned} \left|\frac{a_n \bar{x}_n n^{1/p}}{\sqrt{S_n}}\Big(\frac{\sqrt{S_n}}{n^{1/p}}(\beta_1 - \hat{\beta}_1)\Big)\right| &= \left|\frac{a_n \bar{x}_n n^{1/p}}{\sqrt{S_n}}\right| \left|\Big(\frac{\sqrt{S_n}}{n^{1/p}}(\beta_1 - \hat{\beta}_1)\Big)\right| \\ = \quad O(1)\left|\Big(\frac{\sqrt{S_n}}{n^{1/p}}(\beta_1 - \hat{\beta}_1)\Big)\right| &\xrightarrow{a.s.} 0. \end{aligned} \tag{2.28}$$

In addition, we have

$$\begin{aligned} & a_n(\hat{\beta}_0 - \beta_0) = a_n(\beta_1 - \hat{\beta}_1)\bar{X}_n + a_n(\beta_2 - \hat{\beta}_2)\bar{Z}_n + a_n\bar{\epsilon}_n - a_n\beta_1\bar{u}_n - a_n\beta_2\bar{v}_n \\ = \quad & a_n(\beta_1 - \hat{\beta}_1)\bar{x}_n + a_n(\beta_1 - \hat{\beta}_1)\bar{u}_n + a_n(\beta_2 - \hat{\beta}_2)\bar{z}_n + a_n(\beta_2 - \hat{\beta}_2)\bar{v}_n \\ & + a_n\bar{\epsilon}_n - \beta_1 a_n\bar{u}_n - \beta_2 a_n\bar{v}_n \\ = \quad & \frac{a_n \bar{x}_n n^{1/p}}{\sqrt{S_n}}\Big(\frac{\sqrt{S_n}}{n^{1/p}}(\beta_1 - \hat{\beta}_1)\Big) + \frac{n^{1/p}}{\sqrt{n}}\frac{\sqrt{n}}{\sqrt{S_n}}\Big(\frac{\sqrt{S_n}}{n^{1/p}}(\beta_1 - \hat{\beta}_1)\Big)\frac{a_n}{n}\sum_{i=1}^n u_i \\ & + [a_n(\beta_2 - \hat{\beta}_2)]\bar{z}_n + [a_n(\beta_2 - \hat{\beta}_2)]\bar{v}_n + \frac{a_n}{n}\sum_{i=1}^n \epsilon_i - \beta_1\frac{a_n}{n}\sum_{i=1}^n u_i - \beta_2\frac{a_n}{n}\sum_{i=1}^n v_i. \end{aligned}$$

Accordingly, from the condition (2.2), the first, fourth equations in (2.8), the first, second, fifth equations of (2.20), (2.28), the Theorem 2.1.1 and 2.1.2, we can show that

$$a_n(\hat{\beta}_0 - \beta_0) \xrightarrow{a.s.} 0.$$

This completes the proof of Theorem 2.1.3. □

Remark: Theorem 2.1.3 reveals that the almost sure convergence rate is different with the one in the fixed design errors-in-variables model discussed in Miao et al. (2011). However, if we pay attention to the convergence rate a_n in the Theorem 2.1.3, the assumption (2.4) that a_n needs to be satisfied is the same as the one in the fixed design errors-in-variables model in Miao et al. (2011). ■

2.2 Convergence in probability

Let X_n be a sequence of random variables and let X be a random variable. We say that X_n converges in probability to X, $X_n \xrightarrow{P} X$, if for every $\epsilon > 0$, $P\{|X_n - X| > \epsilon\} \longrightarrow 0$ as $n \longrightarrow \infty$. The following theorems state the convergence in probability of $\hat{\beta}_i s$.

Theorem 2.2.1. *Under the model (1.1), assume that*

$$E\,|\epsilon_1|^p < \infty, \quad E\,|u_1|^p < \infty, \quad E\,|v_1|^p < \infty, \quad E\,|z_1|^p < \infty \quad (p \geq 2),$$

and

$$\lim_{n\to\infty} \frac{S_n}{n} = \infty, \quad \lim_{n\to\infty} \frac{b_n}{\sqrt{S_n}} = 0, \quad \lim_{n\to\infty} \frac{nb_n}{S_n} = 0. \tag{2.29}$$

Then $b_n(\hat{\beta}_1 - \beta_1) \xrightarrow{p} 0$.

Theorem 2.2.2. *Assume that in (1.1),*

$$E\,|\epsilon_1|^p < \infty, \quad E\,|u_1|^p < \infty, \quad E\,|v_1|^{p\vee 4} < \infty, \quad E\,|z_1|^p < \infty \quad (p \geq 2),$$

and

$$\lim_{n\to\infty} \frac{S_n}{n} = \infty, \quad \lim_{n\to\infty} \frac{b_n^2}{n} = 0. \tag{2.30}$$

Then $b_n(\hat{\beta}_2 - \beta_2) \xrightarrow{p} 0$.

Theorem 2.2.3. *Under the assumptions of Theorem 2.2.1 and Theorem 2.2.2, suppose that*

$$\frac{b_n \bar{x}_n}{\sqrt{S_n}} \to 0, \quad \text{and} \quad \frac{nb_n \bar{x}_n}{S_n} \to 0, \tag{2.31}$$

then $b_n(\hat{\beta}_0 - \beta_0) \xrightarrow{p} 0$.

Proof of Theorem 2.2.1: From the assumption in Theorem 2.2.1, and the weak law of large numbers, we have the following equations

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n} z_i \xrightarrow{p} \mu_z, \quad \frac{1}{n}\sum_{i=1}^{n} Z_i \xrightarrow{p} \mu_z, \quad \frac{1}{n}\sum_{i=1}^{n} u_i \xrightarrow{p} 0,\\
&\frac{1}{n}\sum_{i=1}^{n}(Z_i-\bar{Z}_n)^2-\sigma_v^2 \xrightarrow{p} \sigma_z^2, \quad \frac{1}{n}\sum_{i=1}^{n}(u_i-\bar{u}_n)^2 \xrightarrow{p} \sigma_u^2,\\
&\frac{1}{n}\sum_{i=1}^{n}(Z_i-\bar{Z}_n)^2 \xrightarrow{p} \sigma_Z^2, \quad \frac{1}{n}\sum_{i=1}^{n}(z_i-\bar{z}_n)^2 \xrightarrow{p} \sigma_z^2,\\
&\frac{1}{n}\sum_{i=1}^{n}(u_i-\bar{u}_n)(Z_i-\bar{Z}_n) \xrightarrow{p} \mathrm{Cov}(u,Z)=0,\\
&\frac{1}{n}\sum_{i=1}^{n}(u_i-\bar{u}_n)(z_i-\bar{z}_n) \xrightarrow{p} \mathrm{Cov}(u,z)=0, \qquad (2.32)\\
&\frac{1}{n}\sum_{i=1}^{n}(v_i-\bar{v}_n)(z_i-\bar{z}_n) \xrightarrow{p} \mathrm{Cov}(v,z)=0,\\
&\frac{1}{n}\sum_{i=1}^{n}(v_i-\bar{v}_n)(u_i-\bar{u}_n) \xrightarrow{p} \mathrm{Cov}(v,u)=0,\\
&\frac{1}{n}\sum_{i=1}^{n}(z_i-\bar{z}_n)(\epsilon_i-\bar{\epsilon}_n) \xrightarrow{p} \mathrm{Cov}(z,\epsilon)=0,\\
&\frac{1}{n}\sum_{i=1}^{n}(v_i-\bar{v}_n)(\epsilon_i-\bar{\epsilon}_n) \xrightarrow{p} \mathrm{Cov}(v,\epsilon)=0,\\
&\frac{1}{n}\sum_{i=1}^{n}(u_i-\bar{u}_n)(\epsilon_i-\bar{\epsilon}_n) \xrightarrow{p} \mathrm{Cov}(u,\epsilon)=0
\end{aligned}
$$

by the independence of ϵ, u, v and z. Some of the above results will be used in the proof of Theorem 2.2.2, 2.2.3 and asymptotic normality. Besides, by the third equation in (2.29), the weak law of large numbers, and the finiteness of σ_ϵ^2, σ_u^2, σ_v^2 and σ_z^2, we can show that

$$
\begin{aligned}
&\frac{b_n}{S_n}\sum_{i=1}^{n}(u_i-\bar{u}_n)^2 \le \frac{b_n}{S_n}\sum_{i=1}^{n} u_i^2 = \frac{nb_n}{S_n}\times\frac{1}{n}\sum_{i=1}^{n} u_i^2 \xrightarrow{p} 0,\\
&\frac{b_n}{S_n}\sum_{i=1}^{n}(v_i-\bar{v}_n)^2 \le \frac{b_n}{S_n}\sum_{i=1}^{n} v_i^2 = \frac{nb_n}{S_n}\times\frac{1}{n}\sum_{i=1}^{n} v_i^2 \xrightarrow{p} 0,
\end{aligned}
\qquad (2.33)
$$

$$\frac{b_n}{S_n}\sum_{i=1}^{n}(z_i-\bar{z}_n)^2 \le \frac{b_n}{S_n}\sum_{i=1}^{n} z_i^2 = \frac{nb_n}{S_n}\times\frac{1}{n}\sum_{i=1}^{n} z_i^2 \xrightarrow{p} 0,$$
$$\frac{b_n}{S_n}\sum_{i=1}^{n}(\epsilon_i-\bar{\epsilon}_n)^2 \le \frac{b_n}{S_n}\sum_{i=1}^{n} \epsilon_i^2 = \frac{nb_n}{S_n}\times\frac{1}{n}\sum_{i=1}^{n} \epsilon_i^2 \xrightarrow{p} 0,$$

which yield also

$$\begin{aligned}
\left|\frac{b_n}{S_n}\sum_{i=1}^{n}(u_i-\bar{u}_n)(\epsilon_i-\bar{\epsilon}_n)\right| &\le \frac{b_n}{2S_n}\sum_{i=1}^{n}\left[(u_i-\bar{u}_n)^2+(\epsilon_i-\bar{\epsilon}_n)^2\right] \xrightarrow{p} 0,\\
\left|\frac{b_n}{S_n}\sum_{i=1}^{n}(u_i-\bar{u}_n)(z_i-\bar{z}_n)\right| &\le \frac{b_n}{2S_n}\sum_{i=1}^{n}\left[(u_i-\bar{u}_n)^2+(z_i-\bar{z}_n)^2\right] \xrightarrow{p} 0,\\
\left|\frac{b_n}{S_n}\sum_{i=1}^{n}(u_i-\bar{u}_n)(v_i-\bar{v}_n)\right| &\le \frac{b_n}{2S_n}\sum_{i=1}^{n}\left[(u_i-\bar{u}_n)^2+(v_i-\bar{v}_n)^2\right] \xrightarrow{p} 0.
\end{aligned} \tag{2.34}$$

Furthermore, noting the proof of the (2.11) in Theorem 2.1.1. Here, we used the first equation in (2.29), and the fifth equation in (2.32). Hence, we have

$$\frac{1}{S_n}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2 \xrightarrow{p} 1. \tag{2.35}$$

This fact is very important in the proof of the convergence in probability and asymptotic normality.

In addition, by using the Markov's inequality, from the second equation in (2.29), so that $\mathrm{Var}\Big((b_n/S_n)\sum_{i=1}^{n}(x_i-\bar{x}_n)z_i\Big) = (b_n^2/S_n^2)\mathrm{Var}\Big(\sum_{i=1}^{n}(x_i-\bar{x}_n)z_i\Big) = (b_n^2/S_n)\mathrm{Var}(z_1) \to 0$, which implies

$$\frac{b_n}{S_n}\sum_{i=1}^{n}(x_i-\bar{x}_n)z_i \xrightarrow{p} 0. \tag{2.36}$$

Similarly we can show that

$$\begin{aligned}
&\frac{b_n}{S_n}\sum_{i=1}^{n}(x_i-\bar{x}_n)u_i \xrightarrow{p} 0, \quad \frac{b_n}{S_n}\sum_{i=1}^{n}(x_i-\bar{x}_n)v_i \xrightarrow{p} 0,\\
&\frac{b_n}{S_n}\sum_{i=1}^{n}(x_i-\bar{x}_n)\epsilon_i \xrightarrow{p} 0, \quad \frac{1}{\sqrt{n}\sqrt{S_n}}\sum_{i=1}^{n}(x_i-\bar{x}_n)Z_i \xrightarrow{p} 0.
\end{aligned} \tag{2.37}$$

To prove $b_n B_{n1} \xrightarrow{p} 0$, first note that

$$
\begin{aligned}
b_n B_{n1} &= \frac{b_n B_{n1}/S_n}{1/S_n} \\
= & \frac{\frac{b_n}{S_n}[\beta_2 \sum_{i=1}^n (x_i - \bar{x}_n)(z_i - \bar{z}_n) - \beta_1 \sum_{i=1}^n (x_i - \bar{x}_n)(u_i - \bar{u}_n)]}{S_n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2} \\
& + \frac{\frac{b_n}{S_n}[\beta_2 \sum_{i=1}^n (u_i - \bar{u}_n)(z_i - \bar{z}_n) - \beta_1 \sum_{i=1}^n (u_i - \bar{u}_n)^2]}{S_n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2} \\
& + \frac{\frac{b_n}{S_n}[\sum_{i=1}^n (x_i - \bar{x}_n)(\epsilon_i - \bar{\epsilon}_n) + \sum_{i=1}^n (u_i - \bar{u}_n)(\epsilon_i - \bar{\epsilon}_n)]}{S_n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2} \\
= & \frac{\frac{b_n}{S_n}[\beta_2 \sum_{i=1}^n (x_i - \bar{x}_n) z_i - \beta_1 \sum_{i=1}^n (x_i - \bar{x}_n) u_i + \beta_2 \sum_{i=1}^n (u_i - \bar{u}_n)(z_i - \bar{z}_n)]}{S_n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2} \\
& - \frac{\frac{b_n}{S_n}[\beta_1 \sum_{i=1}^n (u_i - \bar{u}_n)^2 - \sum_{i=1}^n (x_i - \bar{x}_n)\epsilon_i - \sum_{i=1}^n (u_i - \bar{u}_n)(\epsilon_i - \bar{\epsilon}_n)]}{S_n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2}.
\end{aligned}
$$

Then from the first equation in (2.33), the first, second equations in (2.34), (2.35), (2.58), the first, third equations in (2.37), we have

$$
b_n B_{n1} \xrightarrow{p} 0. \tag{2.38}
$$

In order to prove $b_n B_{n2} \xrightarrow{p} 0$, let us prove $b_n \sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)/S_n \xrightarrow{p} 0$ first. Since we have

$$
\begin{aligned}
& \frac{b_n \sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{S_n} \\
= & \frac{b_n \sum_{i=1}^n (x_i - \bar{x}_n + u_i - \bar{u}_n)(z_i - \bar{z}_n + v_i - \bar{v}_n)}{S_n} \\
= & \frac{b_n}{S_n} \sum_{i=1}^n (x_i - \bar{x}_n)(z_i - \bar{z}_n) + \frac{b_n}{S_n} \sum_{i=1}^n (x_i - \bar{x}_n)(v_i - \bar{v}_n) \\
& + \frac{b_n}{S_n} \sum_{i=1}^n (u_i - \bar{u}_n)(z_i - \bar{z}_n) + \frac{b_n}{S_n} \sum_{i=1}^n (u_i - \bar{u}_n)(v_i - \bar{v}_n).
\end{aligned}
$$

So that from the the second, third equations in (2.34), (2.58), and the second equation in (2.37), one can obtain

$$
\frac{b_n \sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{S_n} \xrightarrow{p} 0. \tag{2.39}
$$

Now, let us show the main proof. A boring calculation results in

$$\begin{aligned}
b_n B_{n2} &= b_n \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sum_{i=1}^n (X_i - \bar{X}_n)^2 \sum_{i=1}^n (Z_i - \bar{Z}_n)^2} \Big[\beta_2 \sum_{i=1}^n (z_i - \bar{z}_n)^2 - \beta_1 \sum_{i=1}^n (z_i - \bar{z}_n)(u_i - \bar{u}_n) \\
&\quad + \beta_2 \sum_{i=1}^n (v_i - \bar{v}_n)(z_i - \bar{z}_n) - \beta_1 \sum_{i=1}^n (v_i - \bar{v}_n)(u_i - \bar{u}_n) \\
&\quad + \sum_{i=1}^n (z_i - \bar{z}_n)(\epsilon_i - \bar{\epsilon}_n) + \sum_{i=1}^n (v_i - \bar{v}_i)(\epsilon_i - \bar{\epsilon}_n) \Big] \\
&= \beta_2 b_n \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sum_{i=1}^n (X_i - \bar{X}_n)^2 \sum_{i=1}^n (Z_i - \bar{Z}_n)^2} \sum_{i=1}^n (z_i - \bar{z}_n)^2 \\
&\quad - \beta_1 b_n \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sum_{i=1}^n (X_i - \bar{X}_n)^2 \sum_{i=1}^n (Z_i - \bar{Z}_n)^2} \sum_{i=1}^n (z_i - \bar{z}_n)(u_i - \bar{u}_n) \\
&\quad + \beta_2 b_n \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sum_{i=1}^n (X_i - \bar{X}_n)^2 \sum_{i=1}^n (Z_i - \bar{Z}_n)^2} \sum_{i=1}^n (v_i - \bar{v}_n)(z_i - \bar{z}_n) \\
&\quad - \beta_1 b_n \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sum_{i=1}^n (X_i - \bar{X}_n)^2 \sum_{i=1}^n (Z_i - \bar{Z}_n)^2} \sum_{i=1}^n (v_i - \bar{v}_n)(u_i - \bar{u}_n) \\
&\quad + b_n \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sum_{i=1}^n (X_i - \bar{X}_n)^2 \sum_{i=1}^n (Z_i - \bar{Z}_n)^2} \sum_{i=1}^n (z_i - \bar{z}_n)(\epsilon_i - \bar{\epsilon}_n) \\
&\quad + b_n \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sum_{i=1}^n (X_i - \bar{X}_n)^2 \sum_{i=1}^n (Z_i - \bar{Z}_n)^2} \sum_{i=1}^n (v_i - \bar{v}_i)(\epsilon_i - \bar{\epsilon}_n) \\
&= \beta_2 \frac{b_n \sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{S_n} \frac{1}{S_n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2} \frac{\frac{1}{n} \sum_{i=1}^n (z_i - \bar{z}_n)^2}{\frac{1}{n} \sum_{i=1}^n (Z_i - \bar{Z}_n)^2} \\
&\quad - \beta_1 \frac{b_n \sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{S_n} \frac{1}{S_n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2} \frac{\frac{1}{n} \sum_{i=1}^n (z_i - \bar{z}_n)(u_i - \bar{u}_n)}{\frac{1}{n} \sum_{i=1}^n (Z_i - \bar{Z}_n)^2} \\
&\quad + \beta_2 \frac{b_n \sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{S_n} \frac{1}{S_n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2} \frac{\frac{1}{n} \sum_{i=1}^n (v_i - \bar{v}_n)(z_i - \bar{z}_n)}{\frac{1}{n} \sum_{i=1}^n (Z_i - \bar{Z}_n)^2} \\
&\quad - \beta_1 \frac{b_n \sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{S_n} \frac{1}{S_n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2} \frac{\frac{1}{n} \sum_{i=1}^n (v_i - \bar{v}_n)(u_i - \bar{u}_n)}{\frac{1}{n} \sum_{i=1}^n (Z_i - \bar{Z}_n)^2} \\
&\quad + \frac{b_n \sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{S_n} \frac{1}{S_n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2} \frac{\frac{1}{n} \sum_{i=1}^n (z_i - \bar{z}_n)(\epsilon_i - \bar{\epsilon}_n)}{\frac{1}{n} \sum_{i=1}^n (Z_i - \bar{Z}_n)^2} \\
&\quad + \frac{b_n \sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{S_n} \frac{1}{S_n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2} \frac{\frac{1}{n} \sum_{i=1}^n (v_i - \bar{v}_i)(\epsilon_i - \bar{\epsilon}_n)}{\frac{1}{n} \sum_{i=1}^n (Z_i - \bar{Z}_n)^2}.
\end{aligned}$$

Therefore, from the sixth, seventh, ninth, tenth, eleventh, twelfth, thirteenth equations in

(2.32), (2.35), and (2.39), we can obtain that

$$b_n B_{n2} \xrightarrow{p} 0. \tag{2.40}$$

Finally, as we have

$$\begin{aligned}
&\frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2}\sqrt{\sum_{i=1}^n (Z_i - \bar{Z}_n)^2}} \\
= \;& \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sqrt{n}\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2}} \frac{1}{\sqrt{\frac{1}{n}\sum_{i=1}^n (Z_i - \bar{Z}_n)^2}} \\
= \;& \left[\frac{\sum_{i=1}^n (x_i - \bar{x}_n)(Z_i - \bar{Z}_n)}{\sqrt{n}\sqrt{\sum_{i=1}^n (x_i - \bar{x}_n)^2}} + \frac{\sum_{i=1}^n (u_i - \bar{u}_n)(Z_i - \bar{Z}_n)}{\sqrt{n}\sqrt{\sum_{i=1}^n (x_i - \bar{x}_n)^2}}\right] \frac{\sqrt{\sum_{i=1}^n (x_i - \bar{x}_n)^2}}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2}} \\
&\times \frac{1}{\sqrt{\frac{1}{n}\sum_{i=1}^n (Z_i - \bar{Z}_n)^2}} \\
= \;& \left[\frac{1}{\sqrt{n}}\sum_{i=1}^n \Big(\frac{(x_i - \bar{x}_n)}{\sqrt{\sum_{i=1}^n (x_i - \bar{x}_n)^2}}\Big)(Z_i) + \frac{1}{n}\frac{\sum_{i=1}^n (u_i - \bar{u}_n)(Z_i - \bar{Z}_n)}{\sqrt{\sum_{i=1}^n (x_i - \bar{x}_n)^2/n}}\right] \\
&\times \frac{\sqrt{\sum_{i=1}^n (x_i - \bar{x}_n)^2}}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2}} \frac{1}{\sqrt{\frac{1}{n}\sum_{i=1}^n (Z_i - \bar{Z}_n)^2}} \\
= \;& \left[\frac{1}{\sqrt{n}\sqrt{S_n}}\sum_{i=1}^n (x_i - \bar{x}_n)(Z_i) + \frac{\frac{1}{n}\sum_{i=1}^n (u_i - \bar{u}_n)(Z_i - \bar{Z}_n)}{\sqrt{S_n/n}}\right] \\
&\times \frac{1}{\sqrt{S_n^{-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2}} \frac{1}{\sqrt{\frac{1}{n}\sum_{i=1}^n (Z_i - \bar{Z}_n)^2}},
\end{aligned}$$

thus from the first equation in (2.29), the sixth, eighth equations in (2.32), (2.35), and the fourth equation in (2.37), one can get

$$B_{n3} \xrightarrow{p} 0. \tag{2.41}$$

Combining (2.5), (2.38), (2.40), and (2.41), we can get the fancy result

$$b_n(\hat{\beta}_1 - \beta_1) = \frac{b_n B_{n1} - b_n B_{n2}}{1 - B_{n3}} \xrightarrow{p} 0.$$

This completes the proof of Theorem 2.2.1. $\square$

Remark: Theorem 2.2.1 shows that the rate of convergence in probability is the same as the one in the fixed design errors-in-variables model discussed in Miao et al. (2011). So adding a random component z to the fixed design errors-in-variables model does not affect the rate of convergence in probability of $\hat{\beta}_1$. ■

Proof of Theorem 2.2.2: Using the Markov's inequality again, from the second equation in (2.30), we have $\text{Var}\big((b_n/n)\sum_{i=1}^n Z_i\epsilon_i\big) = (b_n^2/n^2)\text{Var}(\sum_{i=1}^n Z_i\epsilon_i)$
$= (b_n^2/n)\text{Var}(Z_1\epsilon_1) \to 0$, which implies

$$\frac{b_n}{n}\sum_{i=1}^n Z_i\epsilon_i \xrightarrow{p} 0. \tag{2.42}$$

Identically, we have

$$\begin{aligned} &\frac{b_n}{n}\sum_{i=1}^n \epsilon_i \xrightarrow{p} 0, \quad \frac{b_n}{n}\sum_{i=1}^n Z_i u_i \xrightarrow{p} 0, \\ &\frac{b_n}{n}\sum_{i=1}^n u_i \xrightarrow{p} 0, \quad \frac{b_n}{n}\sum_{i=1}^n v_i \xrightarrow{p} 0. \end{aligned} \tag{2.43}$$

Next, because $E\,|v_1|^{p\vee 4} < \infty$, $E\big((b_n/n)\sum_{i=1}^n [Z_iv_i - \sigma_v^2]\big) = 0$, and $\text{Var}\big((b_n/n)\sum_{i=1}^n [Z_iv_i - \sigma_v^2]\big) = (b_n^2/n^2)\text{Var}\big(\sum_{i=1}^n [Z_iv_i - \sigma_v^2]\big) = (b_n^2/n)\text{Var}(Z_1v_1) \to 0$, which imply

$$\frac{b_n}{n}\sum_{i=1}^n [Z_iv_i - \sigma_v^2] \xrightarrow{p} 0. \tag{2.44}$$

Moreover, as we have

$$\text{Var}\Big(\frac{\sqrt{b_n}\sum_{i=1}^n (x_i - \bar{x}_n)Z_i}{\sqrt{S_n}\sqrt{n}}\Big) = \frac{b_n}{nS_n}\text{Var}(\sum_{i=1}^n (x_i - \bar{x}_n)Z_i) = \frac{b_n}{\sqrt{n}} \times \frac{1}{\sqrt{n}}\text{Var}(Z_1) \to 0,$$

which implies

$$\frac{\sqrt{b_n}\sum_{i=1}^n (x_i - \bar{x}_n)Z_i}{\sqrt{S_n}\sqrt{n}} \xrightarrow{p} 0. \tag{2.45}$$

Similarly we get

$$\frac{\sqrt{b_n}\sum_{i=1}^n (x_i - \bar{x}_n)\epsilon_i}{\sqrt{S_n}\sqrt{n}} \xrightarrow{p} 0, \frac{\sqrt{b_n}\sum_{i=1}^n (x_i - \bar{x}_n)u_i}{\sqrt{S_n}\sqrt{n}} \xrightarrow{p} 0, \frac{\sqrt{b_n}\sum_{i=1}^n (x_i - \bar{x}_n)v_i}{\sqrt{S_n}\sqrt{n}} \xrightarrow{p} 0. \tag{2.46}$$

Finally, note that

$$\text{Var}(\frac{\sqrt{b_n}\sum_{i=1}^n u_iZ_i}{\sqrt{S_n}\sqrt{n}}) = \frac{b_n}{nS_n}\text{Var}(\sum_{i=1}^n u_iZ_i) = \frac{b_n}{\sqrt{n}} \times \frac{1}{\sqrt{n}}\frac{n}{S_n}\text{Var}(u_1Z_1) \to 0,$$

which implies

$$\frac{\sqrt{b_n}\sum_{i=1}^n u_i Z_i}{\sqrt{S_n}\sqrt{n}} \xrightarrow{p} 0. \tag{2.47}$$

Immediately, we can show that

$$\begin{aligned}
&\frac{\sqrt{b_n}\sum_{i=1}^n u_i \epsilon_i}{\sqrt{S_n}\sqrt{n}} \xrightarrow{p} 0, \quad \frac{\sqrt{b_n}\sum_{i=1}^n u_i^2}{\sqrt{S_n}\sqrt{n}} \xrightarrow{p} 0, \quad \frac{\sqrt{b_n}\sum_{i=1}^n u_i v_i}{\sqrt{S_n}\sqrt{n}} \xrightarrow{p} 0, \\
&\frac{\sqrt{b_n}\sum_{i=1}^n u_i}{\sqrt{S_n}\sqrt{n}} \xrightarrow{p} 0, \quad \frac{\sqrt{b_n}\sum_{i=1}^n v_i}{\sqrt{S_n}\sqrt{n}} \xrightarrow{p} 0, \quad \frac{\sqrt{b_n}\sum_{i=1}^n \epsilon_i}{\sqrt{S_n}\sqrt{n}} \xrightarrow{p} 0.
\end{aligned} \tag{2.48}$$

To prove $b_n C_{n1} \xrightarrow{p} 0$, as

$$b_n C_{n1} = b_n \frac{\sum_{i=1}^n (Z_i - \bar{Z}_n)(\epsilon_i - \bar{\epsilon}_n)}{\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - n\sigma_v^2} = \frac{\frac{b_n}{n}\sum_{i=1}^n Z_i \epsilon_i - \bar{Z}_n \frac{b_n}{n}\sum_{i=1}^n \epsilon_i}{\frac{1}{n}\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - \sigma_v^2}.$$

Thus from the second, fourth equations in (2.32), (2.42), and the first equation in (2.43), we have

$$b_n C_{n1} \xrightarrow{p} 0. \tag{2.49}$$

For C_{n2}, because we have

$$b_n C_{n2} = \beta_1 b_n \frac{\sum_{i=1}^n (Z_i - \bar{Z}_n)(u_i - \bar{u}_n)}{\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - n\sigma_v^2} = \beta_1 \frac{\frac{b_n}{n}\sum_{i=1}^n Z_i u_i - \bar{Z}_n \frac{b_n}{n}\sum_{i=1}^n u_i}{\frac{1}{n}\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - \sigma_v^2}.$$

So that from the second, fourth equations in (2.32), the second, third equations in (2.43), one can obtain

$$b_n C_{n2} \xrightarrow{p} 0. \tag{2.50}$$

For C_{n3}, simple calculation shows that $b_n C_{n3}$ equals

$$\beta_2 b_n \frac{[\sum_{i=1}^n (Z_i - \bar{Z}_n)(v_i - \bar{v}_n) - n\sigma_v^2]}{\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - n\sigma_v^2} = \beta_2 \frac{\frac{b_n}{n}\sum_{i=1}^n [Z_i v_i - \sigma_v^2] - \bar{Z}_n \frac{b_n}{n}\sum_{i=1}^n v_i}{\frac{1}{n}\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - \sigma_v^2}.$$

Therefore, from the second, fourth equations in (2.32), the fourth equation in (2.43), and (2.44), we can show that

$$b_n C_{n3} \xrightarrow{p} 0. \tag{2.51}$$

In order to prove $b_n C_{n4} \xrightarrow{p} 0$, note that

$$
\begin{aligned}
& b_n \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n) \sum_{i=1}^n (X_i - \bar{X}_n)(\epsilon_i - \bar{\epsilon}_n)}{\sum_{i=1}^n (X_i - \bar{X}_n)^2 [\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - n\sigma_v^2]} \\
= & \frac{\frac{b_n}{n} \sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n) \sum_{i=1}^n (X_i - \bar{X}_n)(\epsilon_i - \bar{\epsilon}_n)}{\sum_{i=1}^n (X_i - \bar{X}_n)^2} \frac{1}{\frac{1}{n}\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - \sigma_v^2} \\
= & \frac{\sqrt{b_n} \sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sqrt{S_n}\sqrt{n}} \frac{\sqrt{S_n}}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2}} \frac{\sqrt{b_n} \sum_{i=1}^n (X_i - \bar{X}_n)(\epsilon_i - \bar{\epsilon}_n)}{\sqrt{S_n}\sqrt{n}} \\
& \times \frac{\sqrt{S_n}}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2}} \frac{1}{\frac{1}{n}\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - \sigma_v^2} \\
= & \left[\frac{\sqrt{b_n} \sum_{i=1}^n (x_i - \bar{x}_n)(Z_i - \bar{Z}_n)}{\sqrt{S_n}\sqrt{n}} + \frac{\sqrt{b_n} \sum_{i=1}^n (u_i - \bar{u}_n)(Z_i - \bar{Z}_n)}{\sqrt{S_n}\sqrt{n}} \right] \frac{\sqrt{S_n}}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2}} \\
& \times \left[\frac{\sqrt{b_n} \sum_{i=1}^n (x_i - \bar{x}_n)(\epsilon_i - \bar{\epsilon}_n)}{\sqrt{S_n}\sqrt{n}} + \frac{\sqrt{b_n} \sum_{i=1}^n (u_i - \bar{u}_n)(\epsilon_i - \bar{\epsilon}_n)}{\sqrt{S_n}\sqrt{n}} \right] \\
& \times \frac{\sqrt{S_n}}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2}} \frac{1}{\frac{1}{n}\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - \sigma_v^2} \\
= & \left[\frac{\sqrt{b_n} \sum_{i=1}^n (x_i - \bar{x}_n)(Z_i)}{\sqrt{S_n}\sqrt{n}} + \frac{\sqrt{b_n} \sum_{i=1}^n u_i Z_i - \bar{Z}_n \sqrt{b_n} \sum_{i=1}^n u_i}{\sqrt{S_n}\sqrt{n}} \right] \frac{\sqrt{S_n}}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2}} \\
& \times \left[\frac{\sqrt{b_n} \sum_{i=1}^n (x_i - \bar{x}_n)(\epsilon_i)}{\sqrt{S_n}\sqrt{n}} + \frac{\sqrt{b_n} \sum_{i=1}^n u_i \epsilon_i - \bar{u}_n \sqrt{b_n} \sum_{i=1}^n \epsilon_i}{\sqrt{S_n}\sqrt{n}} \right] \\
& \times \frac{\sqrt{S_n}}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2}} \frac{1}{\frac{1}{n}\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - \sigma_v^2}.
\end{aligned}
$$

Accordingly, by the second, third, fourth equations in (2.32), (2.35), and (2.45), the first equation in (2.46), (2.47), the first, fourth, sixth equations in (2.48), we have

$$
b_n C_{n4} \xrightarrow{p} 0. \tag{2.52}
$$

For $b_n C_{n5}$, since we have

$$
\begin{aligned}
& \beta_1 b_n \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n) \sum_{i=1}^n (X_i - \bar{X}_n)(u_i - \bar{u}_n)}{\sum_{i=1}^n (X_i - \bar{X}_n)^2 [\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - n\sigma_v^2]} \\
= & \beta_1 \frac{\frac{b_n}{n} \sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n) \sum_{i=1}^n (X_i - \bar{X}_n)(u_i - \bar{u}_n)}{\sum_{i=1}^n (X_i - \bar{X}_n)^2} \frac{1}{\frac{1}{n}\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 - \sigma_v^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \beta_1 \frac{\sqrt{b_n}\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)}{\sqrt{S_n}\sqrt{n}} \frac{\sqrt{S_n}}{\sqrt{\sum_{i=1}^n (X_i-\bar{X}_n)^2}} \frac{\sqrt{b_n}\sum_{i=1}^n (X_i-\bar{X}_n)(u_i-\bar{u}_n)}{\sqrt{S_n}\sqrt{n}} \\
&\quad \times \frac{\sqrt{S_n}}{\sqrt{\sum_{i=1}^n (X_i-\bar{X}_n)^2}} \frac{1}{\frac{1}{n}\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-\sigma_v^2} \\
&= \beta_1 \left[\frac{\sqrt{b_n}\sum_{i=1}^n (x_i-\bar{x}_n)(Z_i-\bar{Z}_n)}{\sqrt{S_n}\sqrt{n}} + \frac{\sqrt{b_n}\sum_{i=1}^n (u_i-\bar{u}_n)(Z_i-\bar{Z}_n)}{\sqrt{S_n}\sqrt{n}} \right] \frac{\sqrt{S_n}}{\sqrt{\sum_{i=1}^n (X_i-\bar{X}_n)^2}} \\
&\quad \times \left[\frac{\sqrt{b_n}\sum_{i=1}^n (x_i-\bar{x}_n)(u_i-\bar{u}_n)}{\sqrt{S_n}\sqrt{n}} + \frac{\sqrt{b_n}\sum_{i=1}^n (u_i-\bar{u}_n)(u_i-\bar{u}_n)}{\sqrt{S_n}\sqrt{n}} \right] \\
&\quad \times \frac{\sqrt{S_n}}{\sqrt{\sum_{i=1}^n (X_i-\bar{X}_n)^2}} \frac{1}{\frac{1}{n}\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-\sigma_v^2} \\
&= \beta_1 \left[\frac{\sqrt{b_n}\sum_{i=1}^n (x_i-\bar{x}_n)(Z_i)}{\sqrt{S_n}\sqrt{n}} + \frac{\sqrt{b_n}\sum_{i=1}^n u_i Z_i - \bar{Z}_n\sqrt{b_n}\sum_{i=1}^n u_i}{\sqrt{S_n}\sqrt{n}} \right] \frac{\sqrt{S_n}}{\sqrt{\sum_{i=1}^n (X_i-\bar{X}_n)^2}} \\
&\quad \times \left[\frac{\sqrt{b_n}\sum_{i=1}^n (x_i-\bar{x}_n)(u_i)}{\sqrt{S_n}\sqrt{n}} + \frac{\sqrt{b_n}\sum_{i=1}^n u_i^2 - \bar{u}_n\sqrt{b_n}\sum_{i=1}^n u_i}{\sqrt{S_n}\sqrt{n}} \right] \\
&\quad \times \frac{\sqrt{S_n}}{\sqrt{\sum_{i=1}^n (X_i-\bar{X}_n)^2}} \frac{1}{\frac{1}{n}\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-\sigma_v^2}.
\end{aligned}
$$

Hence, by the second, third, fourth equations in (2.32), (2.35), and (2.45), the second equation in (2.46), (2.47), the second, fourth equations in (2.48), we get

$$
b_n C_{n5} \xrightarrow{p} 0. \tag{2.53}
$$

To show that $b_n C_{n6} \xrightarrow{p} 0$, first we have

$$
\begin{aligned}
&\beta_2 b_n \frac{\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n (X_i-\bar{X}_n)(v_i-\bar{v}_n)}{\sum_{i=1}^n (X_i-\bar{X}_n)^2[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]} \\
&= \beta_2 \frac{\frac{b_n}{n}\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n (X_i-\bar{X}_n)(v_i-\bar{v}_n)}{\sum_{i=1}^n (X_i-\bar{X}_n)^2} \frac{1}{\frac{1}{n}\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-\sigma_v^2} \\
&= \beta_2 \frac{\sqrt{b_n}\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)}{\sqrt{S_n}\sqrt{n}} \frac{\sqrt{S_n}}{\sqrt{\sum_{i=1}^n (X_i-\bar{X}_n)^2}} \frac{\sqrt{b_n}\sum_{i=1}^n (X_i-\bar{X}_n)(v_i-\bar{v}_n)}{\sqrt{S_n}\sqrt{n}} \\
&\quad \times \frac{\sqrt{S_n}}{\sqrt{\sum_{i=1}^n (X_i-\bar{X}_n)^2}} \frac{1}{\frac{1}{n}\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-\sigma_v^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \beta_2\left[\frac{\sqrt{b_n}\sum_{i=1}^n(x_i-\bar{x}_n)(Z_i-\bar{Z}_n)}{\sqrt{S_n}\sqrt{n}}+\frac{\sqrt{b_n}\sum_{i=1}^n(u_i-\bar{u}_n)(Z_i-\bar{Z}_n)}{\sqrt{S_n}\sqrt{n}}\right]\frac{\sqrt{S_n}}{\sqrt{\sum_{i=1}^n(X_i-\bar{X}_n)^2}} \\
&\quad\times\left[\frac{\sqrt{b_n}\sum_{i=1}^n(x_i-\bar{x}_n)(v_i-\bar{v}_n)}{\sqrt{S_n}\sqrt{n}}+\frac{\sqrt{b_n}\sum_{i=1}^n(u_i-\bar{u}_n)(v_i-\bar{v}_n)}{\sqrt{S_n}\sqrt{n}}\right] \\
&\quad\times\frac{\sqrt{S_n}}{\sqrt{\sum_{i=1}^n(X_i-\bar{X}_n)^2}}\frac{1}{\frac{1}{n}\sum_{i=1}^n(Z_i-\bar{Z}_n)^2-\sigma_v^2} \\
&= \beta_2\left[\frac{\sqrt{b_n}\sum_{i=1}^n(x_i-\bar{x}_n)(Z_i)}{\sqrt{S_n}\sqrt{n}}+\frac{\sqrt{b_n}\sum_{i=1}^n u_iZ_i-\bar{Z}_n\sqrt{b_n}\sum_{i=1}^n u_i}{\sqrt{S_n}\sqrt{n}}\right]\frac{\sqrt{S_n}}{\sqrt{\sum_{i=1}^n(X_i-\bar{X}_n)^2}} \\
&\quad\times\left[\frac{\sqrt{b_n}\sum_{i=1}^n(x_i-\bar{x}_n)(v_i)}{\sqrt{S_n}\sqrt{n}}+\frac{\sqrt{b_n}\sum_{i=1}^n u_iv_i-\bar{u}_n\sqrt{b_n}\sum_{i=1}^n v_i}{\sqrt{S_n}\sqrt{n}}\right] \\
&\quad\times\frac{\sqrt{S_n}}{\sqrt{\sum_{i=1}^n(X_i-\bar{X}_n)^2}}\frac{1}{\frac{1}{n}\sum_{i=1}^n(Z_i-\bar{Z}_n)^2-\sigma_v^2},
\end{aligned}
$$

then, by the second, third, fourth equations in (2.32), (2.35), and (2.45), the third equation in (2.46), (2.47), the third, fourth, fifth equations in (2.48), we obtain

$$
b_nC_{n6}\xrightarrow{p}0. \tag{2.54}
$$

Finally, let us prove $C_{n7}\xrightarrow{p}0$. Note that it is very similarly as to prove $B_{n3}\xrightarrow{p}0$, since

$$
\begin{aligned}
&\frac{\sum_{i=1}^n(X_i-\bar{X}_n)(Z_i-\bar{Z}_n)}{\sqrt{\sum_{i=1}^n(X_i-\bar{X}_n)^2}\sqrt{[\sum_{i=1}^n(Z_i-\bar{Z}_n)^2-n\sigma_v^2]}} \\
&= \frac{\sum_{i=1}^n(x_i-\bar{x}_n+u_i-\bar{u}_n)(Z_i-\bar{Z}_n)}{\sqrt{n}\sqrt{\sum_{i=1}^n(X_i-\bar{X}_n)^2}}\frac{1}{\sqrt{\frac{1}{n}\sum_{i=1}^n(Z_i-\bar{Z}_n)^2-\sigma_v^2}} \\
&= \left[\frac{\sum_{i=1}^n(x_i-\bar{x}_n)(Z_i-\bar{Z}_n)}{\sqrt{n}\sqrt{\sum_{i=1}^n(x_i-\bar{x}_n)^2}}+\frac{\sum_{i=1}^n(u_i-\bar{u}_n)(Z_i-\bar{Z}_n)}{\sqrt{n}\sqrt{\sum_{i=1}^n(x_i-\bar{x}_n)^2}}\right] \\
&\quad\times\frac{\sqrt{\sum_{i=1}^n(x_i-\bar{x}_n)^2}}{\sqrt{\sum_{i=1}^n(X_i-\bar{X}_n)^2}}\frac{1}{\sqrt{\frac{1}{n}\sum_{i=1}^n(Z_i-\bar{Z}_n)^2-\sigma_v^2}} \\
&= \left[\frac{1}{\sqrt{n}}\sum_{i=1}^n\left(\frac{(x_i-\bar{x}_n)}{\sqrt{\sum_{i=1}^n(x_i-\bar{x}_n)^2}}\right)Z_i+\frac{1}{n}\frac{\sum_{i=1}^n(u_i-\bar{u}_n)(Z_i-\bar{Z}_n)}{\sqrt{\sum_{i=1}^n(x_i-\bar{x}_n)^2/n}}\right] \\
&\quad\times\frac{\sqrt{\sum_{i=1}^n(x_i-\bar{x}_n)^2}}{\sqrt{\sum_{i=1}^n(X_i-\bar{X}_n)^2}}\frac{1}{\sqrt{\frac{1}{n}\sum_{i=1}^n(Z_i-\bar{Z}_n)^2-\sigma_v^2}}
\end{aligned}
$$

$$= \left[\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Big(\frac{(x_i - \bar{x}_n)}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x}_n)^2}} \Big) Z_i + \frac{\frac{1}{n}\sum_{i=1}^{n}(u_i - \bar{u}_n)(Z_i - \bar{Z}_n)}{\sqrt{S_n/n}} \right]$$
$$\times \frac{1}{\sqrt{S_n^{-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2}} \frac{1}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(Z_i - \bar{Z}_n)^2 - \sigma_v^2}}.$$

Consequently, from the first equation in (2.30), the fourth, eighth equations in (2.32), (2.35), and the fourth equation in (2.37), we have

$$C_{n7} \xrightarrow{p} 0. \tag{2.55}$$

Combining (2.6), (2.49) to (2.55), we have the wistful outcome

$$b_n(\hat{\beta}_2 - \beta_2) = \frac{b_n C_{n1} - b_n C_{n2} - b_n C_{n3} - b_n C_{n4} + b_n C_{n5} + b_n C_{n6}}{1 - C_{n7}} \xrightarrow{p} 0.$$

This completes the proof of Theorem 2.2.2. □

Proof of Theorem 2.2.3: First, because we have $\mathrm{Var}\big((b_n/n)\sum_{i=1}^{n}\epsilon_i\big)$ $= (b_n^2/n^2)\mathrm{Var}(\sum_{i=1}^{n}\epsilon_i) = (b_n^2/n)\mathrm{Var}(\epsilon_1) \to 0$, which implies

$$b_n\bar{\epsilon}_n = \frac{b_n}{n}\sum_{i=1}^{n}\epsilon_i \xrightarrow{p} 0. \tag{2.56}$$

Similarly we have

$$b_n\bar{v}_n = \frac{b_n}{n}\sum_{i=1}^{n} v_i \xrightarrow{p} 0, \quad b_n\bar{u}_n = \frac{b_n}{n}\sum_{i=1}^{n} u_i \xrightarrow{p} 0. \tag{2.57}$$

In order to prove $b_n(\hat{\beta}_1 - \beta_1)\bar{x}_n \xrightarrow{p} 0$, if we note the formula (2.5), it is the same as to show that $b_n B_{n1}\bar{x}_n - b_n B_{n2}\bar{x}_n/1 - B_{n3} \xrightarrow{p} 0$. Because we already know that $B_{n3} \xrightarrow{p} 0$ by the (2.41), so we need to prove $b_n B_{n1}\bar{x}_n \xrightarrow{p} 0$ and $b_n B_{n2}\bar{x}_n \xrightarrow{p} 0$ respectively.

First, similarly as to prove (2.33), by the second equation in (2.31), we get

$$\frac{b_n\bar{x}_n}{S_n}\sum_{i=1}^{n}(u_i - \bar{u}_n)^2 \le \frac{b_n\bar{x}_n}{S_n}\sum_{i=1}^{n} u_i^2 = \frac{nb_n\bar{x}_n}{S_n} \times \frac{1}{n}\sum_{i=1}^{n} u_i^2 \xrightarrow{p} 0,$$

which yield also

$$\left|\frac{b_n\bar{x}_n}{S_n}\sum_{i=1}^{n}(u_i-\bar{u}_n)(\epsilon_i-\bar{\epsilon}_n)\right| \le \frac{b_n\bar{x}_n}{2S_n}\sum_{i=1}^{n}\left[(u_i-\bar{u}_n)^2+(\epsilon_i-\bar{\epsilon}_n)^2\right] \xrightarrow{p} 0,$$
$$\left|\frac{b_n\bar{x}_n}{S_n}\sum_{i=1}^{n}(u_i-\bar{u}_n)(z_i-\bar{z}_n)\right| \le \frac{b_n\bar{x}_n}{2S_n}\sum_{i=1}^{n}\left[(u_i-\bar{u}_n)^2+(z_i-\bar{z}_n)^2\right] \xrightarrow{p} 0,$$
$$\left|\frac{b_n\bar{x}_n}{S_n}\sum_{i=1}^{n}(u_i-\bar{u}_n)(v_i-\bar{v}_n)\right| \le \frac{b_n\bar{x}_n}{2S_n}\sum_{i=1}^{n}\left[(u_i-\bar{u}_n)^2+(v_i-\bar{v}_n)^2\right] \xrightarrow{p} 0.$$

Moreover, by using the Markov's inequality, from the first equation in (2.31), so that $\mathrm{Var}\Big((b_n\bar{x}_n/S_n)\sum_{i=1}^{n}(x_i-\bar{x}_n)z_i\Big) = (b_n^2\bar{x}_n^2/S_n^2)\mathrm{Var}\Big(\sum_{i=1}^{n}(x_i-\bar{x}_n)z_i\Big) = (b_n^2\bar{x}_n^2/S_n)\mathrm{Var}(z_1) \to 0$, which implies

$$\frac{b_n\bar{x}_n}{S_n}\sum_{i=1}^{n}(x_i-\bar{x}_n)z_i \xrightarrow{p} 0.$$

Similarly we can show that

$$\frac{b_n\bar{x}_n}{S_n}\sum_{i=1}^{n}(x_i-\bar{x}_n)u_i \xrightarrow{p} 0, \quad \frac{b_n\bar{x}_n}{S_n}\sum_{i=1}^{n}(x_i-\bar{x}_n)v_i \xrightarrow{p} 0, \quad \frac{b_n\bar{x}_n}{S_n}\sum_{i=1}^{n}(x_i-\bar{x}_n)\epsilon_i \xrightarrow{p} 0.$$

Now, if we come back to the proof of (2.38) and (2.40), we can easily show that $b_nB_{n1}\bar{x}_n \xrightarrow{p} 0$ and $b_nB_{n2}\bar{x}_n \xrightarrow{p} 0$.

In a word, we obtain

$$b_n(\hat{\beta}_1-\beta_1)\bar{x}_n \xrightarrow{p} 0. \tag{2.58}$$

Then we have

$$\begin{aligned}
& b_n(\hat{\beta}_0-\beta_0) \\
= \;& b_n(\beta_1-\hat{\beta}_1)\bar{X}_n + b_n(\beta_2-\hat{\beta}_2)\bar{Z}_n + b_n\bar{\epsilon}_n - b_n\beta_1\bar{u}_n - b_n\beta_2\bar{v}_n \\
= \;& b_n(\beta_1-\hat{\beta}_1)(\bar{x}_n+\bar{u}_n) + b_n(\beta_2-\hat{\beta}_2)\bar{Z}_n + b_n\bar{\epsilon}_n - b_n\beta_1\bar{u}_n - b_n\beta_2\bar{v}_n \\
= \;& b_n(\beta_1-\hat{\beta}_1)\bar{x}_n + b_n(\beta_1-\hat{\beta}_1)\bar{u}_n + b_n(\beta_2-\hat{\beta}_2)\bar{Z}_n + b_n\bar{\epsilon}_n - \beta_1b_n\bar{u}_n - \beta_2b_n\bar{v}_n.
\end{aligned}$$

Therefore, noting the second, third equations in (2.32), (2.56), all equations in (2.57), (2.58) and the Theorem 2.2.1 and 2.2.2, one can show that

$$b_n(\hat{\beta}_0-\beta_0) \xrightarrow{p} 0.$$

This completes the proof of Theorem 2.2.3. □

Remark: Theorem 2.2.3 shows that the result of convergence in probability is different with the one in the fixed design errors-in-variables model discussed in Miao et al. (2011). However, if we focus to the convergence rate b_n in the Theorem 2.2.3, the first equation in the assumption (2.31) that b_n needs to be satisfied is the same as the one talked in the fixed design errors-in-variables model in Miao et al. (2011) while the second one in (2.31) is very similar. ■

2.3 Asymptotic normality

Let X_n be a sequence of random variables and let X be a random variable. Let F_{X_n} and F_X be, respectively, the cdfs of X_n and X. Let $C(F_X)$ denote the set of all points where F_X is continuous. We say that X_n converges in distribution to X, $X_n \xrightarrow{D} X$ if $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$, for all $x \in C(F_X)$. The following theorems state the asymptotic normality of $\hat{\beta}_i s$.

Theorem 2.3.1. *Under the model (1.1), assume that*

$$E\epsilon_1^2 < \infty, \quad Eu_1^2 < \infty, \quad Ev_1^4 < \infty, \quad Ez_1^4 < \infty,$$

and

$$\lim_{n\to\infty} \frac{n}{\sqrt{S_n}} = 0, \quad \lim_{n\to\infty} \max_{1\le i\le n} \frac{|x_i - \bar{x}_n|}{\sqrt{S_n}} = 0. \tag{2.59}$$

Then

$$\frac{\sqrt{S_n}}{\sqrt{Var[\epsilon_1 - \beta_1 u_1 - (\beta_2 - \frac{\beta_2\sigma_v^2}{\sigma_Z^2})v_1 + \frac{\beta_2\sigma_v^2}{\sigma_Z^2}(z_1 - \mu_z)]}}(\hat{\beta}_1 - \beta_1) \xrightarrow{d} N(0,1),$$

where $Var\Big[\epsilon_1 - \beta_1 u_1 - (\beta_2 - \frac{\beta_2\sigma_v^2}{\sigma_Z^2})v_1 + \frac{\beta_2\sigma_v^2}{\sigma_Z^2}(z_1 - \mu_z)\Big] = \sigma_\epsilon^2 + \beta_1^2\sigma_u^2 + \frac{\beta_2^2\sigma_z^2\sigma_v^2}{\sigma_z^2+\sigma_v^2}$.

Theorem 2.3.2. *Assume that in model (1.1), suppose*

$$E\epsilon_1^2 < \infty, \quad Eu_1^2 < \infty, \quad Ez_1^4 < \infty, \quad Ev_1^4 < \infty,$$

and

$$\lim_{n\to\infty} \frac{n}{\sqrt{S_n}} = 0. \tag{2.60}$$

Then

$$\sqrt{n}(\hat{\beta}_2 - \beta_2) \xrightarrow{d} N\Big(0, \frac{Var[(Z_1 - \mu_z)(\epsilon_1 - \beta_1 u_1 - \beta_2 v_1)]}{\sigma_z^4}\Big).$$

Theorem 2.3.3. *Under the assumption of Theorem 2.3.1, suppose that*

$$\frac{n\bar{x}_n^2}{S_n} \to 0, \tag{2.61}$$

then

$$\sqrt{n}(\hat{\beta}_0 - \beta_0) \xrightarrow{d} N\Big(0, \frac{Var(\rho_1)}{\sigma_z^4}\Big),$$

where $\rho_1 = -(Z_1 - \mu_z)(\epsilon_1 - \beta_1 u_1 - \beta_2 v_1)\mu_z - \beta_2\sigma_v^2\mu_z + (\epsilon_1 - \beta_1 u_1 - \beta_2 v_1)\sigma_z^2$.

First and foremost, we need to recall a necessary and sufficient condition for central limit theorem of partial sums of i.i.d. random variables by Gnedenko and Kolmogorov (1954).

Lemma 2.3.1. *Gnedenko and Kolmogorov (1954) [Theorem 2, pp.128 in Gnedenko and Kolmogorov (1954)] Let* $\{\xi_{k,n}, 1 \le k \le n, n \ge 1\}$ *be an infinite array of row wise independent random variables. In order that for some suitably chosen constants* A_n *the distributions of the sums*

$$\eta_n = \xi_{1,n} + ... + \xi_{n,n} - A_n,$$

converge as $n \to \infty$ *to the normal law* $N(0,1)$*, and the summands* $\xi_{k,n}(1 \le k \le n)$ *be infinitesimal, i.e., the requirement that for any* $\lambda > 0$ *as* $n \to \infty$

$$sup_{1\le k\le n} P(|\xi_{k,n}| \ge \lambda) \to 0,$$

it is necessary and sufficient that the conditions

$$\sum_{k=1}^{n} P(|\xi_{k,n}| \ge \lambda) \to 0, \tag{2.62}$$

and

$$\sum_{k=1}^{n} \left(E\Big(\xi_{k,n}^2 \boldsymbol{1}\{|\xi_{k,n}| < \lambda\}\Big) - \Big(E\xi_{k,n}\boldsymbol{1}\{|\xi_{k,n}| < \lambda\}\Big)^2 \right) \to 1, \tag{2.63}$$

be satisfied for every $\lambda > 0$ *, as* $n \to \infty$.

To simplify the proofs of the theorems above, we will again derive some new expressions for $\hat{\beta}_1 - \beta_1$, $\hat{\beta}_2 - \beta_2$, and $\hat{\beta}_0 - \beta_0$. For $\hat{\beta}_1 - \beta_1$, a simple operation results in

$$
\begin{aligned}
&\hat{\beta}_1 - \beta_1 \\
= \;& \frac{\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 \{\sum_{i=1}^n (X_i - \bar{X}_n)[\beta_2(Z_i - \bar{Z}_n) + (\epsilon_i - \bar{\epsilon}_n)]\}}{\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 \sum_{i=1}^n (X_i - \bar{X}_n)^2 - [\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2} \\
& - \frac{\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 \{\sum_{i=1}^n (X_i - \bar{X}_n)[\beta_1(u_i - \bar{u}_n) + \beta_2(v_i - \bar{v}_n)]\}}{\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 \sum_{i=1}^n (X_i - \bar{X}_n)^2 - [\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2} \\
& - \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n) \{\sum_{i=1}^n (Z_i - \bar{Z}_n)[\beta_2(Z_i - \bar{Z}_n) + (\epsilon_i - \bar{\epsilon}_n)]\}}{\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 \sum_{i=1}^n (X_i - \bar{X}_n)^2 - [\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2} \\
& + \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n) \{\sum_{i=1}^n (Z_i - \bar{Z}_n)[\beta_1(u_i - \bar{u}_n) + \beta_2(v_i - \bar{v}_n)]\}}{\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 \sum_{i=1}^n (X_i - \bar{X}_n)^2 - [\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2} \\
= \;& \frac{\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 \{\sum_{i=1}^n (X_i - \bar{X}_n)[(\epsilon_i - \bar{\epsilon}_n) - \beta_1(u_i - \bar{u}_n) - \beta_2(v_i - \bar{v}_n)]\}}{\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 \sum_{i=1}^n (X_i - \bar{X}_n)^2 - [\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2} \\
& - \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n) \{\sum_{i=1}^n (Z_i - \bar{Z}_n)[(\epsilon_i - \bar{\epsilon}_n) - \beta_1(u_i - \bar{u}_n) - \beta_2(v_i - \bar{v}_n)]\}}{\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 \sum_{i=1}^n (X_i - \bar{X}_n)^2 - [\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2},
\end{aligned}
$$

thus, one can rewrite

$$
\hat{\beta}_1 - \beta_1 = (D_{n1} - D_{n2}) \frac{1}{1 - D_{n3}}, \tag{2.64}
$$

where

$$
D_{n1} = \frac{\sum_{i=1}^n (X_i - \bar{X}_n)[(\epsilon_i - \bar{\epsilon}_n) - \beta_1(u_i - \bar{u}_n) - \beta_2(v_i - \bar{v}_n) + (Z_i - \bar{Z}_n)\frac{\beta_2 \sigma_v^2}{\sigma_Z^2}]}{\sum_{i=1}^n (X_i - \bar{X}_n)^2},
$$

$$
\begin{aligned}
D_{n2} &= \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sqrt{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2} \\
&\quad \times \sqrt{n} \Big\{ \frac{\frac{1}{n} \sum_{i=1}^n (Z_i - \bar{Z}_n)[(\epsilon_i - \bar{\epsilon}_n) - \beta_1(u_i - \bar{u}_n) - \beta_2(v_i - \bar{v}_n)]}{\frac{1}{n} \sum_{i=1}^n (Z_i - \bar{Z}_n)^2} + \frac{\beta_2 \sigma_v^2}{\sigma_Z^2} \Big\}, \\
D_{n3} &= \frac{[\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)]^2}{\sum_{i=1}^n (X_i - \bar{X}_n)^2 \sum_{i=1}^n (Z_i - \bar{Z}_n)^2}.
\end{aligned}
$$

Proof of Theorem 2.3.1: By the assumption in Theorem 2.3.1, and the weak law of large numbers, we have

$$\bar{\epsilon}_n = \frac{1}{n}\sum_{i=1}^n \epsilon_i \xrightarrow{p} 0, \quad \bar{u}_n = \frac{1}{n}\sum_{i=1}^n u_i \xrightarrow{p} 0, \quad \bar{v}_n = \frac{1}{n}\sum_{i=1}^n v_i \xrightarrow{p} 0,$$
$$\frac{1}{n}\sum_{i=1}^n (Z_i - \bar{Z}_n)[(\epsilon_i - \bar{\epsilon}_n) - \beta_1(u_i - \bar{u}_n) - \beta_2(v_i - \bar{v}_n)] \xrightarrow{p} -\beta_2\sigma_v^2, \tag{2.65}$$
$$\frac{1}{n}\sum_{i=1}^n (Z_i - \bar{Z}_n)^2 \xrightarrow{p} \sigma_Z^2.$$

Moreover, by the first equation in (2.59), and the weak law of large numbers we have

$$\frac{1}{\sqrt{S_n}}\sum_{i=1}^n (u_i - \bar{u}_n)^2 \le \frac{1}{\sqrt{S_n}}\sum_{i=1}^n u_i^2 \xrightarrow{p} 0, \frac{1}{\sqrt{S_n}}\sum_{i=1}^n (z_i - \bar{z}_n)^2 \le \frac{1}{\sqrt{S_n}}\sum_{i=1}^n z_i^2 \xrightarrow{p} 0,$$
$$\frac{1}{\sqrt{S_n}}\sum_{i=1}^n (v_i - \bar{v}_n)^2 \le \frac{1}{\sqrt{S_n}}\sum_{i=1}^n v_i^2 \xrightarrow{p} 0, \frac{1}{\sqrt{S_n}}\sum_{i=1}^n (\epsilon_i - \bar{\epsilon}_n)^2 \le \frac{1}{\sqrt{S_n}}\sum_{i=1}^n \epsilon_i^2 \xrightarrow{p} 0, \tag{2.66}$$

which yield also

$$\left|\frac{1}{\sqrt{S_n}}\sum_{i=1}^n (u_i - \bar{u}_n)(\epsilon_i - \bar{\epsilon}_n)\right| \le \frac{1}{2\sqrt{S_n}}\sum_{i=1}^n \left[(u_i - \bar{u}_n)^2 + (\epsilon_i - \bar{\epsilon}_n)^2\right] \xrightarrow{p} 0,$$
$$\left|\frac{1}{\sqrt{S_n}}\sum_{i=1}^n (u_i - \bar{u}_n)(v_i - \bar{v}_n)\right| \le \frac{1}{2\sqrt{S_n}}\sum_{i=1}^n \left[(u_i - \bar{u}_n)^2 + (v_i - \bar{v}_n)^2\right] \xrightarrow{p} 0, \tag{2.67}$$
$$\left|\frac{1}{\sqrt{S_n}}\sum_{i=1}^n (u_i - \bar{u}_n)(z_i - \bar{z}_n)\right| \le \frac{1}{2\sqrt{S_n}}\sum_{i=1}^n \left[(u_i - \bar{u}_n)^2 + (z_i - \bar{z}_n)^2\right] \xrightarrow{p} 0.$$

In addition, let us define $\zeta_i = \epsilon_i - \beta_1 u_i - (\beta_2 - \frac{\beta_2\sigma_v^2}{\sigma_Z^2})v_i + \frac{\beta_2\sigma_v^2}{\sigma_Z^2}(z_i - \mu_z)$, so that, $\zeta_1 = \epsilon_1 - \beta_1 u_1 - (\beta_2 - \frac{\beta_2\sigma_v^2}{\sigma_Z^2})v_1 + \frac{\beta_2\sigma_v^2}{\sigma_Z^2}(z_1 - \mu_z)$. Besides, let

$$\begin{aligned}
\xi_{i,n} &= \frac{x_i - \bar{x}_n}{\sqrt{S_n \mathrm{Var}[\epsilon_1 - \beta_1 u_1 - (\beta_2 - \frac{\beta_2\sigma_v^2}{\sigma_Z^2})v_1 + \frac{\beta_2\sigma_v^2}{\sigma_Z^2}(z_1 - \mu_z)]}} \\
&\quad \times \left[\epsilon_i - \beta_1 u_i - (\beta_2 - \frac{\beta_2\sigma_v^2}{\sigma_Z^2})v_i + \frac{\beta_2\sigma_v^2}{\sigma_Z^2}(z_i - \mu_z)\right] \\
&= \frac{x_i - \bar{x}_n}{\sqrt{S_n \mathrm{Var}(\zeta_1)}}\zeta_i,
\end{aligned}$$

then it is enough to check the conditions (2.62) and (2.63) in Lemma 2.3.1 by using the assumption

$$\lim_{n\to\infty} \max_{1\le i\le n} \frac{|x_i - \bar{x}_n|}{\sqrt{S_n}} = 0.$$

We have

$$\sum_{k=1}^{n} E\Big(\xi_{k,n}^2 \mathbf{1}\{|\xi_{k,n}| \geq \lambda\}\Big) \leq \sum_{k=1}^{n} \frac{(x_i - \bar{x}_n)^2}{S_n \mathrm{Var}(\zeta_1)} E\zeta_1^2 \mathbf{1}\left\{ |\zeta_1| \geq \frac{\lambda\sqrt{S_n \mathrm{Var}(\zeta_1)}}{\max_{1\leq i \leq n} |x_i - \bar{x}_n|} \right\} \to 0,$$

which implies, by the fact that $\sum_{i=1}^n E\xi_{i,n}^2 = 1$,

$$\begin{aligned}
&\sum_{k=1}^{n} \left(E\Big(\xi_{k,n}^2 \mathbf{1}\{|\xi_{k,n}| < \lambda\}\Big) - \Big(E\xi_{k,n} \mathbf{1}\{|\xi_{k,n}| < \lambda\}\Big)^2 \right) \\
&= \sum_{i=1}^{n} E\xi_{i,n}^2 - \sum_{k=1}^{n} E\Big(\xi_{k,n}^2 \mathbf{1}\{|\xi_{k,n}| \geq \lambda\}\Big) - \sum_{k=1}^{n} E\Big(\xi_{k,n} \mathbf{1}\{|\xi_{k,n}| \geq \lambda\}\Big)^2 \to 1.
\end{aligned}$$

Furthermore, for any $\lambda > 0$, we get

$$\sum_{k=1}^{n} P(|\xi_{k,n}| \geq \lambda) \leq \sum_{k=1}^{n} \frac{E\xi_{k,n}^2 \mathbf{1}\{|\xi_{k,n}| \geq \lambda\}}{\lambda^2} \to 0.$$

Hence we have

$$\sum_{i=1}^{n} \xi_{i,n} \xrightarrow{d} N(0,1). \tag{2.68}$$

Also, since

$$\mathrm{Var}\Big(\frac{\sum_{i=1}^n (x_i - \bar{x}_n) Z_i}{\sqrt{S_n}\sqrt{n}}\Big) = \frac{1}{nS_n} \mathrm{Var}\Big(\sum_{i=1}^{n} (x_i - \bar{x}_n) Z_i\Big) = \frac{1}{n} \mathrm{Var}(Z_1) \to 0,$$

which implies

$$\frac{\sum_{i=1}^n (x_i - \bar{x}_n)(Z_i - \bar{Z}_n)}{\sqrt{S_n}\sqrt{n}} \xrightarrow{p} 0.$$

Therefore, one can show that

$$\begin{aligned}
&\sqrt{S_n} \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sqrt{n} \sum_{i=1}^n (X_i - \bar{X}_n)^2} \\
&= \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sqrt{n}\sqrt{S_n}} \times \frac{S_n}{\sum_{i=1}^n (X_i - \bar{X}_n)^2} \\
&= \left[\frac{\sum_{i=1}^n (x_i - \bar{x}_n)(Z_i - \bar{Z}_n)}{\sqrt{n}\sqrt{S_n}} + \frac{\sum_{i=1}^n (u_i - \bar{u}_n)(Z_i - \bar{Z}_n)}{\sqrt{n}\sqrt{S_n}} \right] \times \frac{1}{S_n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2} \\
&= \left[\frac{\sum_{i=1}^n (x_i - \bar{x}_n)(Z_i - \bar{Z}_n)}{\sqrt{n}\sqrt{S_n}} + \frac{\frac{1}{n} \sum_{i=1}^n (u_i - \bar{u}_n)(Z_i - \bar{Z}_n)}{\frac{\sqrt{S_n}}{\sqrt{n}}} \right] \times \frac{1}{S_n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2}.
\end{aligned}$$

Thus, from the first equation in (2.59), the eighth equation in (2.32), (2.35), and the above result, we obtain

$$\sqrt{S_n}\frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Z_i - \bar{Z}_n)}{\sqrt{n}\sum_{i=1}^n (X_i - \bar{X}_n)^2} \xrightarrow{p} 0. \tag{2.69}$$

Finally, it is very important that by the Central Limit Theorem, we have the following outcomes

$$\begin{aligned}
&\sqrt{n}\Big(\frac{1}{n}\sum_{i=1}^n (Z_i - \mu_z)^2 - \sigma_Z^2\Big) \xrightarrow{d} N\Big(0, \mathrm{Var}(Z_1 - \mu_z)^2\Big),\\
&\sqrt{n}\Big(\frac{1}{n}\sum_{i=1}^n Z_i - \mu_z\Big) \xrightarrow{d} N\Big(0, \mathrm{Var}(Z_1)\Big),\\
&\sqrt{n}\Big(\frac{1}{n}\sum_{i=1}^n (\epsilon_i - \beta_1 u_i - \beta_2 v_i)\Big) \xrightarrow{d} N\Big(0, \mathrm{Var}(\epsilon_1 - \beta_1 u_1 - \beta_2 v_1)\Big),\\
&\sqrt{n}\Big(\frac{1}{n}\sum_{i=1}^n (Z_i - \mu_z)(\epsilon_i - \beta_1 u_i - \beta_2 v_i) + \beta_2\sigma_v^2\Big) \xrightarrow{d} N\Big(0, \mathrm{Var}[(Z_1 - \mu_z)(\epsilon_1 - \beta_1 u_1 - \beta_2 v_1)]\Big).
\end{aligned} \tag{2.70}$$

To prove $\sqrt{S_n}D_{n1}/\sqrt{\mathrm{Var}(\zeta_1)} \xrightarrow{d} N(0,1)$, first note that

$$\begin{aligned}
&\frac{\sqrt{S_n}}{\sqrt{\mathrm{Var}(\zeta_1)}}D_{n1}\\
= &\frac{\sqrt{S_n}}{\sqrt{\mathrm{Var}(\zeta_1)}}\frac{\sum_{i=1}^n (X_i - \bar{X}_n)\Big[(\epsilon_i - \bar{\epsilon}_n) - \beta_1(u_i - \bar{u}_n) - \beta_2(v_i - \bar{v}_n) + (Z_i - \bar{Z}_n)\frac{\beta_2\sigma_v^2}{\sigma_Z^2}\Big]}{\sum_{i=1}^n (X_i - \bar{X}_n)^2}\\
= &\frac{1}{\sqrt{\mathrm{Var}(\zeta_1)}}\frac{\frac{1}{\sqrt{S_n}}\sum_{i=1}^n (X_i - \bar{X}_n)\Big[(\epsilon_i - \bar{\epsilon}_n) - \beta_1(u_i - \bar{u}_n) - \beta_2(v_i - \bar{v}_n) + (Z_i - \bar{Z}_n)\frac{\beta_2\sigma_v^2}{\sigma_Z^2}\Big]}{S_n^{-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2}\\
= &\frac{1}{\sqrt{\mathrm{Var}(\zeta_1)}}\Big[\frac{1}{\sqrt{S_n}}\sum_{i=1}^n (u_i - \bar{u}_n)(\epsilon_i - \bar{\epsilon}_n) - \beta_1\frac{1}{\sqrt{S_n}}\sum_{i=1}^n (u_i - \bar{u}_n)^2\\
&-(\beta_2 - \frac{\beta_2\sigma_v^2}{\sigma_Z^2})\frac{1}{\sqrt{S_n}}\sum_{i=1}^n (u_i - \bar{u}_n)(v_i - \bar{v}_n) + \frac{\beta_2\sigma_v^2}{\sigma_Z^2}\frac{1}{\sqrt{S_n}}\sum_{i=1}^n (u_i - \bar{u}_n)(z_i - \bar{z}_n)\\
&+\frac{1}{\sqrt{S_n}}\sum_{i=1}^n (x_i - \bar{x}_n)\zeta_i\Big]\frac{1}{S_n^{-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2}.
\end{aligned}$$

So that, from the equation (2.35), the first equation in (2.66), all equations in (2.67), and (2.68), we have

$$\frac{\sqrt{S_n}}{\sqrt{\mathrm{Var}(\zeta_1)}}D_{n1} \xrightarrow{d} N(0,1). \tag{2.71}$$

In order to prove $\sqrt{S_n}D_{n2}/\sqrt{\mathrm{Var}(\zeta_1)} \xrightarrow{p} 0$, let us consider

$$\sqrt{n}\Big\{\frac{\frac{1}{n}\sum_{i=1}^n(Z_i-\bar{Z}_n)[(\epsilon_i-\bar{\epsilon}_n)-\beta_1(u_i-\bar{u}_n)-\beta_2(v_i-\bar{v}_n)]}{\frac{1}{n}\sum_{i=1}^n(Z_i-\bar{Z}_n)^2}+\frac{\beta_2\sigma_v^2}{\sigma_Z^2}\Big\}$$

first.

Since we have

$$\begin{aligned}
& \sqrt{n}\Big\{\frac{\frac{1}{n}\sum_{i=1}^n(Z_i-\bar{Z}_n)[(\epsilon_i-\bar{\epsilon}_n)-\beta_1(u_i-\bar{u}_n)-\beta_2(v_i-\bar{v}_n)]}{\frac{1}{n}\sum_{i=1}^n(Z_i-\bar{Z}_n)^2}+\frac{\beta_2\sigma_v^2}{\sigma_Z^2}\Big\} \\
= \; & \sqrt{n}\Big\{\frac{\frac{1}{n}\sum_{i=1}^n(Z_i-\bar{Z}_n)[(\epsilon_i-\bar{\epsilon}_n)-\beta_1(u_i-\bar{u}_n)-\beta_2(v_i-\bar{v}_n)]}{\sigma_Z^2} \\
& \times\Big[\frac{\sigma_Z^2}{\frac{1}{n}\sum_{i=1}^n(Z_i-\bar{Z}_n)^2}-1+1\Big]+\frac{\beta_2\sigma_v^2}{\sigma_Z^2}\Big\} \\
= \; & -\frac{\frac{1}{n}\sum_{i=1}^n(Z_i-\bar{Z}_n)[(\epsilon_i-\bar{\epsilon}_n)-\beta_1(u_i-\bar{u}_n)-\beta_2(v_i-\bar{v}_n)]}{\sigma_Z^2}\frac{1}{\frac{1}{n}\sum_{i=1}^n(Z_i-\bar{Z}_n)^2} \\
& \times\Big(\sqrt{n}\big(\frac{1}{n}\sum_{i=1}^n(Z_i-\mu_z)^2-\sigma_Z^2\big)-\sqrt{n}\big(\frac{1}{n}\sum_{i=1}^n Z_i-\mu_z\big)(\bar{Z}_n-\mu_z)\Big) \\
& +\sqrt{n}\Big(\frac{\frac{1}{n}\sum_{i=1}^n(Z_i-\mu_z)(\epsilon_i-\beta_1u_i-\beta_2v_i)+\beta_2\sigma_v^2}{\sigma_Z^2}\Big) \\
& -\sqrt{n}\Big(\frac{(\bar{\epsilon}_n-\beta_1\bar{u}_n-\beta_2\bar{v}_n)(\frac{1}{n}\sum_{i=1}^n Z_i-\mu_z)}{\sigma_Z^2}\Big) \\
& -\sqrt{n}\Big(\frac{(\bar{Z}_n-\mu_z)[\frac{1}{n}\sum_{i=1}^n(\epsilon_i-\beta_1u_i-\beta_2v_i)]}{\sigma_Z^2}\Big)+\sqrt{n}\Big(\frac{(\bar{Z}_n-\mu_z)(\bar{\epsilon}_n-\beta_1\bar{u}_n-\beta_2\bar{v}_n)}{\sigma_Z^2}\Big).
\end{aligned}$$

Consequently, by the second equation in (2.32), all equations in (2.65), all equations in (2.70) and the Slutsky's Theorem, we get

$$\sqrt{n}\Big\{\frac{\frac{1}{n}\sum_{i=1}^n(Z_i-\bar{Z}_n)[(\epsilon_i-\bar{\epsilon}_n)-\beta_1(u_i-\bar{u}_n)-\beta_2(v_i-\bar{v}_n)]}{\frac{1}{n}\sum_{i=1}^n(Z_i-\bar{Z}_n)^2}+\frac{\beta_2\sigma_v^2}{\sigma_Z^2}\Big\},$$

which converges to a Normal Distribution, and so by the (2.69), we have the result

$$\frac{\sqrt{S_n}}{\sqrt{\mathrm{Var}(\zeta_1)}}D_{n2} \xrightarrow{p} 0. \tag{2.72}$$

As we have proven before, by the equation (2.41), we have

$$D_{n3}=B_{n3} \xrightarrow{p} 0. \tag{2.73}$$

Combining (2.64), (2.71) to (2.73), we obtain

$$\frac{\sqrt{S_n}}{\sqrt{\mathrm{Var}(\zeta_1)}}(\hat{\beta}_1-\beta_1)=\left(\frac{\sqrt{S_n}}{\sqrt{\mathrm{Var}(\zeta_1)}}D_{n1}-\frac{\sqrt{S_n}}{\sqrt{\mathrm{Var}(\zeta_1)}}D_{n2}\right)\frac{1}{1-D_{n3}}\xrightarrow{d}N(0,1).$$

This completes the proof of Theorem 2.3.1. □

For $\hat{\beta}_2-\beta_2$, easy calculation show that it equals

$$\begin{aligned}
&\frac{\sum_{i=1}^n(X_i-\bar{X}_n)^2\sum_{i=1}^n(Z_i-\bar{Z}_n)[\beta_2(X_i-\bar{X}_n)+(\epsilon_i-\bar{\epsilon}_n)-\beta_1(u_i-\bar{u}_n)-\beta_2(v_i-\bar{v}_n)]}{[\sum_{i=1}^n(Z_i-\bar{Z}_n)^2-n\sigma_v^2]\sum_{i=1}^n(X_i-\bar{X}_n)^2-[\sum_{i=1}^n(X_i-\bar{X}_n)(Z_i-\bar{Z}_n)]^2}\\
&-\frac{\sum_{i=1}^n(X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n(X_i-\bar{X}_n)[\beta_2(X_i-\bar{X}_n)+(\epsilon_i-\bar{\epsilon}_n)]}{[\sum_{i=1}^n(Z_i-\bar{Z}_n)^2-n\sigma_v^2]\sum_{i=1}^n(X_i-\bar{X}_n)^2-[\sum_{i=1}^n(X_i-\bar{X}_n)(Z_i-\bar{Z}_n)]^2}\\
&+\frac{\sum_{i=1}^n(X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n(X_i-\bar{X}_n)[\beta_1(u_i-\bar{u}_n)+\beta_2(v_i-\bar{v}_n)]}{[\sum_{i=1}^n(Z_i-\bar{Z}_n)^2-n\sigma_v^2]\sum_{i=1}^n(X_i-\bar{X}_n)^2-[\sum_{i=1}^n(X_i-\bar{X}_n)(Z_i-\bar{Z}_n)]^2}\\
&+\frac{\beta_2 n\sum_{i=1}^n(X_i-\bar{X}_n)^2\sigma_v^2}{[\sum_{i=1}^n(Z_i-\bar{Z}_n)^2-n\sigma_v^2]\sum_{i=1}^n(X_i-\bar{X}_n)^2-[\sum_{i=1}^n(X_i-\bar{X}_n)(Z_i-\bar{Z}_n)]^2}\\
=\;&\frac{\sum_{i=1}^n(X_i-\bar{X}_n)^2\sum_{i=1}^n(Z_i-\bar{Z}_n)[(\epsilon_i-\bar{\epsilon}_n)-\beta_1(u_i-\bar{u}_n)]}{[\sum_{i=1}^n(Z_i-\bar{Z}_n)^2-n\sigma_v^2]\sum_{i=1}^n(X_i-\bar{X}_n)^2-[\sum_{i=1}^n(X_i-\bar{X}_n)(Z_i-\bar{Z}_n)]^2}\\
&-\frac{\sum_{i=1}^n(X_i-\bar{X}_n)^2\sum_{i=1}^n(Z_i-\bar{Z}_n)[\beta_2(v_i-\bar{v}_n)]-\beta_2 n\sum_{i=1}^n(X_i-\bar{X}_n)^2\sigma_v^2}{[\sum_{i=1}^n(Z_i-\bar{Z}_n)^2-n\sigma_v^2]\sum_{i=1}^n(X_i-\bar{X}_n)^2-[\sum_{i=1}^n(X_i-\bar{X}_n)(Z_i-\bar{Z}_n)]^2}\\
&-\frac{\sum_{i=1}^n(X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n(X_i-\bar{X}_n)[(\epsilon_i-\bar{\epsilon}_n)-\beta_1(u_i-\bar{u}_n)-\beta_2(v_i-\bar{v}_n)]}{[\sum_{i=1}^n(Z_i-\bar{Z}_n)^2-n\sigma_v^2]\sum_{i=1}^n(X_i-\bar{X}_n)^2-[\sum_{i=1}^n(X_i-\bar{X}_n)(Z_i-\bar{Z}_n)]^2}.
\end{aligned}$$

Accordingly, we can make some realignment and denote E_{n1}, E_{n2} and E_{n3} in the following

$$\begin{aligned}
E_{n1}&=\frac{\sum_{i=1}^n(Z_i-\bar{Z}_n)[(\epsilon_i-\bar{\epsilon}_n)-\beta_1(u_i-\bar{u}_n)-\beta_2(v_i-\bar{v}_n)]+\beta_2 n\sigma_v^2}{[\sum_{i=1}^n(Z_i-\bar{Z}_n)^2-n\sigma_v^2]},\\
E_{n2}&=\frac{\sum_{i=1}^n(X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n(X_i-\bar{X}_n)[(\epsilon_i-\bar{\epsilon}_n)-\beta_1(u_i-\bar{u}_n)-\beta_2(v_i-\bar{v}_n)]}{[\sum_{i=1}^n(Z_i-\bar{Z}_n)^2-n\sigma_v^2]\sum_{i=1}^n(X_i-\bar{X}_n)^2},\\
E_{n3}&=\frac{[\sum_{i=1}^n(X_i-\bar{X}_n)(Z_i-\bar{Z}_n)]^2}{\sum_{i=1}^n(X_i-\bar{X}_n)^2[\sum_{i=1}^n(Z_i-\bar{Z}_n)^2-n\sigma_v^2]}.
\end{aligned}$$

Then we have

$$\hat{\beta}_2-\beta_2=(E_{n1}-E_{n2})\frac{1}{1-E_{n3}}. \qquad (2.74)$$

Proof of Theorem 2.3.2: From the Markov's inequality, we have $E\Big((\sum_{i=1}^n(x_i-\bar{x}_n)(Z_i-\bar{Z}_n)/n^{1/4}\sqrt{S_n}\Big)=0$ and $\mathrm{Var}\Big((\sum_{i=1}^n(x_i-\bar{x}_n)(Z_i-\bar{Z}_n)/n^{1/4}\sqrt{S_n}\Big)=(1/\sqrt{n})\mathrm{Var}(Z_1)\to 0$, which imply

$$\frac{\sum_{i=1}^n(x_i-\bar{x}_n)(Z_i-\bar{Z}_n)}{n^{1/4}\sqrt{S_n}}\xrightarrow{p}0. \qquad (2.75)$$

Similarly we have

$$\frac{\sum_{i=1}^n (x_i-\bar{x}_n)(\epsilon_i-\bar{\epsilon}_n)}{n^{1/4}\sqrt{S_n}} \xrightarrow{p} 0, \quad \frac{\sum_{i=1}^n (x_i-\bar{x}_n)(v_i-\bar{v}_n)}{n^{1/4}\sqrt{S_n}} \xrightarrow{p} 0,$$
$$\frac{\sum_{i=1}^n (x_i-\bar{x}_n)(u_i-\bar{u}_n)}{n^{1/4}\sqrt{S_n}} \xrightarrow{p} 0. \tag{2.76}$$

To prove $\sqrt{n}E_{n1} \xrightarrow{d} N\Big(0, (\mathrm{Var}[(Z_1-\mu_z)(\epsilon_1-\beta_1u_1-\beta_2v_1)]/\sigma_z^4)\Big)$, as we have

$$\begin{aligned}
&\sqrt{n}E_{n1}\\
=\ & \frac{\sqrt{n}\sum_{i=1}^n (Z_i-\mu_z+\mu_z-\bar{Z}_n)[(\epsilon_i-\bar{\epsilon}_n)-\beta_1(u_i-\bar{u}_n)-\beta_2(v_i-\bar{v}_n)]+\beta_2 n\sigma_v^2}{[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]}\\
=\ & \sqrt{n}\Big(\frac{\frac{1}{n}\sum_{i=1}^n (Z_i-\mu_z)(\epsilon_i-\beta_1u_i-\beta_2v_i)+\beta_2\sigma_v^2}{\frac{1}{n}\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-\sigma_v^2}\Big)\\
& -\sqrt{n}\Big(\frac{(\bar{\epsilon}_n-\beta_1\bar{u}_n-\beta_2\bar{v}_n)(\frac{1}{n}\sum_{i=1}^n Z_i-\mu_z)}{\frac{1}{n}\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-\sigma_v^2}\Big)\\
& -\sqrt{n}\Big(\frac{(\bar{Z}_n-\mu_z)[\frac{1}{n}\sum_{i=1}^n (\epsilon_i-\beta_1u_i-\beta_2v_i)]}{\frac{1}{n}\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-\sigma_v^2}\Big)\\
& +\sqrt{n}\Big(\frac{(\bar{Z}_n-\mu_z)(\bar{\epsilon}_n-\beta_1\bar{u}_n-\beta_2\bar{v}_n)}{\frac{1}{n}\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-\sigma_v^2}\Big).
\end{aligned}$$

Then from the second, fourth equations in (2.32), the first, second, third equations in (2.65), second, third, fourth equations in (2.70), we obtain

$$\sqrt{n}E_{n1} \xrightarrow{d} N\left(0, \frac{\mathrm{Var}[(Z_1-\mu_z)(\epsilon_1-\beta_1u_1-\beta_2v_1)]}{\sigma_z^4}\right). \tag{2.77}$$

In order to prove $\sqrt{n}E_{n2} \xrightarrow{p} 0$, first let us consider

$$\frac{\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n (X_i-\bar{X}_n)(\epsilon_i-\bar{\epsilon}_n)}{[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]\sum_{i=1}^n (X_i-\bar{X}_n)^2}.$$

Note that

$$\begin{aligned}
&\frac{\sqrt{n}\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n (X_i-\bar{X}_n)(\epsilon_i-\bar{\epsilon}_n)}{[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]\sum_{i=1}^n (X_i-\bar{X}_n)^2}\\
=\ & \frac{\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n (X_i-\bar{X}_n)(\epsilon_i-\bar{\epsilon}_n)}{\sqrt{n}S_n}\frac{1}{S_n^{-1}\sum_{i=1}^n (X_i-\bar{X}_n)^2}\\
&\times\frac{1}{\frac{1}{n}\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-\sigma_v^2}
\end{aligned}$$

$$
\begin{aligned}
&= \Big[\frac{\sum_{i=1}^n (x_i-\bar{x}_n)(Z_i-\bar{Z}_n)}{n^{1/4}\sqrt{S_n}} + \frac{\sum_{i=1}^n (u_i-\bar{u}_n)(Z_i-\bar{Z}_n)}{n^{1/4}\sqrt{S_n}}\Big]\Big[\frac{\sum_{i=1}^n (x_i-\bar{x}_n)(\epsilon_i-\bar{\epsilon}_n)}{n^{1/4}\sqrt{S_n}} \\
&\quad + \frac{\sum_{i=1}^n (u_i-\bar{u}_n)(\epsilon_i-\bar{\epsilon}_n)}{n^{1/4}\sqrt{S_n}}\Big] \frac{1}{S_n^{-1}\sum_{i=1}^n (X_i-\bar{X}_n)^2}\frac{1}{\frac{1}{n}\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-\sigma_v^2} \\
&= \Big[\frac{\sum_{i=1}^n (x_i-\bar{x}_n)(Z_i-\bar{Z}_n)}{n^{1/4}\sqrt{S_n}} + \frac{\frac{1}{n}\sum_{i=1}^n (u_i-\bar{u}_n)(Z_i-\bar{Z}_n)}{n^{1/4}\sqrt{S_n}/n}\Big]\Big[\frac{\sum_{i=1}^n (x_i-\bar{x}_n)(\epsilon_i-\bar{\epsilon}_n)}{n^{1/4}\sqrt{S_n}} \\
&\quad + \frac{\frac{1}{n}\sum_{i=1}^n (u_i-\bar{u}_n)(\epsilon_i-\bar{\epsilon}_n)}{n^{1/4}\sqrt{S_n}/n}\Big] \frac{1}{S_n^{-1}\sum_{i=1}^n (X_i-\bar{X}_n)^2}\frac{1}{\frac{1}{n}\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-\sigma_v^2}.
\end{aligned}
$$

By the fourth, eighth, fourteenth equations in (2.32), (2.35), (2.60), (2.75) and the first equation in (2.76), one can get

$$\frac{\sqrt{n}\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n (X_i-\bar{X}_n)(\epsilon_i-\bar{\epsilon}_n)}{[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]\sum_{i=1}^n (X_i-\bar{X}_n)^2} \xrightarrow{p} 0.$$

Second let us consider

$$\frac{\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n (X_i-\bar{X}_n)(v_i-\bar{v}_n)}{[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]\sum_{i=1}^n (X_i-\bar{X}_n)^2}.$$

Similarly we have

$$
\begin{aligned}
&\frac{\sqrt{n}\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n (X_i-\bar{X}_n)(v_i-\bar{v}_n)}{[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]\sum_{i=1}^n (X_i-\bar{X}_n)^2} \\
&= \frac{\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n (X_i-\bar{X}_n)(v_i-\bar{v}_n)}{\sqrt{n}S_n}\frac{1}{S_n^{-1}\sum_{i=1}^n (X_i-\bar{X}_n)^2} \\
&\quad \times \frac{1}{\frac{1}{n}\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-\sigma_v^2} \\
&= \Big[\frac{\sum_{i=1}^n (x_i-\bar{x}_n)(Z_i-\bar{Z}_n)}{n^{1/4}\sqrt{S_n}} + \frac{\sum_{i=1}^n (u_i-\bar{u}_n)(Z_i-\bar{Z}_n)}{n^{1/4}\sqrt{S_n}}\Big]\Big[\frac{\sum_{i=1}^n (x_i-\bar{x}_n)(v_i-\bar{v}_n)}{n^{1/4}\sqrt{S_n}} \\
&\quad + \frac{\sum_{i=1}^n (u_i-\bar{u}_n)(v_i-\bar{v}_n)}{n^{1/4}\sqrt{S_n}}\Big] \frac{1}{S_n^{-1}\sum_{i=1}^n (X_i-\bar{X}_n)^2}\frac{1}{\frac{1}{n}\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-\sigma_v^2} \\
&= \Big[\frac{\sum_{i=1}^n (x_i-\bar{x}_n)(Z_i-\bar{Z}_n)}{n^{1/4}\sqrt{S_n}} + \frac{\frac{1}{n}\sum_{i=1}^n (u_i-\bar{u}_n)(Z_i-\bar{Z}_n)}{n^{1/4}\sqrt{S_n}/n}\Big]\Big[\frac{\sum_{i=1}^n (x_i-\bar{x}_n)(v_i-\bar{v}_n)}{n^{1/4}\sqrt{S_n}} \\
&\quad + \frac{\frac{1}{n}\sum_{i=1}^n (u_i-\bar{u}_n)(v_i-\bar{v}_n)}{n^{1/4}\sqrt{S_n}/n}\Big] \frac{1}{S_n^{-1}\sum_{i=1}^n (X_i-\bar{X}_n)^2}\frac{1}{\frac{1}{n}\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-\sigma_v^2}.
\end{aligned}
$$

So that noting the fourth, eighth, fourteenth equations in (2.32), (2.35), (2.60), (2.75) and the second equation in (2.76), we can show that

$$\frac{\sqrt{n}\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n (X_i-\bar{X}_n)(v_i-\bar{v}_n)}{[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]\sum_{i=1}^n (X_i-\bar{X}_n)^2} \xrightarrow{p} 0.$$

Finally let us consider

$$\frac{\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n (X_i-\bar{X}_n)(u_i-\bar{u}_n)}{[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]\sum_{i=1}^n (X_i-\bar{X}_n)^2}.$$

We have

$$\begin{aligned}
&\frac{\sqrt{n}\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n (X_i-\bar{X}_n)(u_i-\bar{u}_n)}{[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]\sum_{i=1}^n (X_i-\bar{X}_n)^2} \\
= &\frac{\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n (X_i-\bar{X}_n)(u_i-\bar{u}_n)}{\sqrt{n}S_n}\frac{1}{S_n^{-1}\sum_{i=1}^n (X_i-\bar{X}_n)^2} \\
&\times\frac{1}{\frac{1}{n}\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-\sigma_v^2} \\
= &\Big[\frac{\sum_{i=1}^n (x_i-\bar{x}_n)(Z_i-\bar{Z}_n)}{n^{1/4}\sqrt{S_n}}+\frac{\sum_{i=1}^n (u_i-\bar{u}_n)(Z_i-\bar{Z}_n)}{n^{1/4}\sqrt{S_n}}\Big]\Big[\frac{\sum_{i=1}^n (x_i-\bar{x}_n)(u_i-\bar{u}_n)}{n^{1/4}\sqrt{S_n}} \\
&+\frac{\sum_{i=1}^n (u_i-\bar{u}_n)^2}{n^{1/4}\sqrt{S_n}}\Big]\frac{1}{S_n^{-1}\sum_{i=1}^n (X_i-\bar{X}_n)^2}\frac{1}{\frac{1}{n}\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-\sigma_v^2} \\
= &\Big[\frac{\sum_{i=1}^n (x_i-\bar{x}_n)(Z_i-\bar{Z}_n)}{n^{1/4}\sqrt{S_n}}+\frac{\frac{1}{n}\sum_{i=1}^n (u_i-\bar{u}_n)(Z_i-\bar{Z}_n)}{n^{1/4}\sqrt{S_n}/n}\Big]\Big[\frac{\sum_{i=1}^n (x_i-\bar{x}_n)(u_i-\bar{u}_n)}{n^{1/4}\sqrt{S_n}} \\
&+\frac{\frac{1}{n}\sum_{i=1}^n (u_i-\bar{u}_n)^2}{n^{1/4}\sqrt{S_n}/n}\Big]\frac{1}{S_n^{-1}\sum_{i=1}^n (X_i-\bar{X}_n)^2}\frac{1}{\frac{1}{n}\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-\sigma_v^2}.
\end{aligned}$$

Therefore, from the fourth, eighth, fourteenth equations in (2.32), (2.35), (2.60), (2.75) and the third equation in (2.76), we obtain

$$\frac{\sqrt{n}\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n (X_i-\bar{X}_n)(u_i-\bar{u}_n)}{[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]\sum_{i=1}^n (X_i-\bar{X}_n)^2}\xrightarrow{p} 0.$$

In all, we have

$$\begin{aligned}
\frac{\sqrt{n}\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n (X_i-\bar{X}_n)(v_i-\bar{v}_n)}{[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]\sum_{i=1}^n (X_i-\bar{X}_n)^2}&\xrightarrow{p} 0, \\
\frac{\sqrt{n}\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n (X_i-\bar{X}_n)(\epsilon_i-\bar{\epsilon}_n)}{[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]\sum_{i=1}^n (X_i-\bar{X}_n)^2}&\xrightarrow{p} 0, \\
\frac{\sqrt{n}\sum_{i=1}^n (X_i-\bar{X}_n)(Z_i-\bar{Z}_n)\sum_{i=1}^n (X_i-\bar{X}_n)(u_i-\bar{u}_n)}{[\sum_{i=1}^n (Z_i-\bar{Z}_n)^2-n\sigma_v^2]\sum_{i=1}^n (X_i-\bar{X}_n)^2}&\xrightarrow{p} 0.
\end{aligned} \tag{2.78}$$

Hence, from all equations in (2.78), we get

$$\sqrt{n}E_{n2}\xrightarrow{p} 0. \tag{2.79}$$

Similarly as we have proven before, noting the equation (2.55), we know that

$$E_{n3} = C_{n7} \xrightarrow{p} 0. \tag{2.80}$$

Combining the equations (2.74), (2.77), (2.79), (2.80), we can have the desired result that

$$\sqrt{n}\big(\hat{\beta}_2 - \beta_2\big) = (\sqrt{n}E_{n1} - \sqrt{n}E_{n2})\frac{1}{1 - E_{n3}} \xrightarrow{d} N\Big(0, \frac{\mathrm{Var}[(Z_1 - \mu_z)(\epsilon_1 - \beta_1 u_1 - \beta_2 v_1)]}{\sigma_z^4}\Big).$$

This completes the proof of Theorem 2.3.2. $\square$

Last, for $\hat{\beta}_0 - \beta_0$, a simple calculation leads to

$$\hat{\beta}_0 - \beta_0 = (\beta_1 - \hat{\beta}_1)\bar{X}_n + (\beta_2 - \hat{\beta}_2)\bar{Z}_n + \bar{\epsilon}_n - \beta_1\bar{u}_n - \beta_2\bar{v}_n.$$

Thus we have

$$\begin{aligned}
& \sqrt{n}(\hat{\beta}_0 - \beta_0) \\
= \; & \frac{\sqrt{n}\bar{X}_n}{\sqrt{S_n}}\sqrt{\mathrm{Var}(\zeta_1)}\frac{\sqrt{S_n}}{\sqrt{\mathrm{Var}(\zeta_1)}}(\beta_1 - \hat{\beta}_1) + \sqrt{n}(\beta_2 - \hat{\beta}_2)(\bar{Z}_n - \mu_z) \\
& + \sqrt{n}(\beta_2 - \hat{\beta}_2)\mu_z + \sqrt{n}(\bar{\epsilon}_n - \beta_1\bar{u}_n - \beta_2\bar{v}_n).
\end{aligned}$$

Further more, we can define

$$\begin{aligned}
F_{n1} &= \frac{\sqrt{n}\bar{X}_n}{\sqrt{S_n}}\sqrt{\mathrm{Var}(\zeta_1)}\frac{\sqrt{S_n}}{\sqrt{\mathrm{Var}(\zeta_1)}}(\beta_1 - \hat{\beta}_1). \\
F_{n2} &= \sqrt{n}(\beta_2 - \hat{\beta}_2)(\bar{Z}_n - \mu_z). \\
F_{n3} &= \sqrt{n}(\beta_2 - \hat{\beta}_2)\mu_z + \sqrt{n}(\bar{\epsilon}_n - \beta_1\bar{u}_n - \beta_2\bar{v}_n) \\
&= \sqrt{n}\mu_z(\frac{E_{n2} - E_{n1}}{1 - E_{n3}}) + \sqrt{n}(\bar{\epsilon}_n - \beta_1\bar{u}_n - \beta_2\bar{v}_n) \\
&= \sqrt{n}\Big[\frac{\mu_z E_{n2} - \mu_z E_{n1} + (\bar{\epsilon}_n - \beta_1\bar{u}_n - \beta_2\bar{v}_n) - E_{n3}(\bar{\epsilon}_n - \beta_1\bar{u}_n - \beta_2\bar{v}_n)}{1 - E_{n3}}\Big] \\
&= \sqrt{n}\Big[\frac{-\mu_z E_{n1} + (\bar{\epsilon}_n - \beta_1\bar{u}_n - \beta_2\bar{v}_n)}{1 - E_{n3}}\Big] + \sqrt{n}\Big[\frac{\mu_z E_{n2} - E_{n3}(\bar{\epsilon}_n - \beta_1\bar{u}_n - \beta_2\bar{v}_n)}{1 - E_{n3}}\Big].
\end{aligned}$$

Then we have

$$\sqrt{n}(\hat{\beta}_0 - \beta_0) = F_{n1} + F_{n2} + F_{n3}. \tag{2.81}$$

Proof of Theorem 2.3.3: Let us define $\rho_i = -(Z_i - \mu_z)(\epsilon_i - \beta_1 u_i - \beta_2 v_i)\mu_z - \beta_2\sigma_v^2\mu_z + (\epsilon_i - \beta_1 u_i - \beta_2 v_i)\sigma_z^2$, then by the Central Limit Theory, we have

$$\sqrt{n}\Big(\frac{1}{n}\sum_{i=1}^{n}(\epsilon_i - \beta_1 u_i - \beta_2 v_i)\Big) \xrightarrow{p} N\big(0, \mathrm{Var}(\epsilon_1 - \beta_1 u_1 - \beta_2 v_1)\big). \tag{2.82}$$

and

$$\sqrt{n}(\frac{1}{n}\sum_{i=1}^{n}\rho_i) \xrightarrow{d} N\big(0, \mathrm{Var}(\rho_1)\big). \tag{2.83}$$

Let us consider $-\sqrt{n}\mu_z E_{n1} + \sqrt{n}(\bar{\epsilon}_n - \beta_1\bar{u}_n - \beta_2\bar{v}_n)$ first. Because we have

$$\begin{aligned}
& -\sqrt{n}\Big(\frac{\frac{1}{n}\sum_{i=1}^{n}(Z_i - \mu_z)(\epsilon_i - \beta_1 u_i - \beta_2 v_i)\mu_z + \beta_2\sigma_v^2\mu_z}{\frac{1}{n}\sum_{i=1}^{n}(Z_i - \bar{Z}_n)^2 - \sigma_v^2}\Big) + \sqrt{n}(\bar{\epsilon}_n - \beta_1\bar{u}_n - \beta_2\bar{v}_n) \\
= & -\sqrt{n}\Big(\frac{\frac{1}{n}\sum_{i=1}^{n}[(Z_i - \mu_z)(\epsilon_i - \beta_1 u_i - \beta_2 v_i)\mu_z + \beta_2\sigma_v^2\mu_z]}{\frac{1}{n}\sum_{i=1}^{n}(Z_i - \bar{Z}_n)^2 - \sigma_v^2}\Big) \\
& +\sqrt{n}\Big(\frac{\frac{1}{n}\sum_{i=1}^{n}(\epsilon_i - \beta_1 u_i - \beta_2 v_i)(\frac{1}{n}\sum_{i=1}^{n}(Z_i - \bar{Z}_n)^2 - \sigma_v^2 - \sigma_z^2 + \sigma_z^2)}{\frac{1}{n}\sum_{i=1}^{n}(Z_i - \bar{Z}_n)^2 - \sigma_v^2}\Big) \\
= & \sqrt{n}\Big(\frac{\frac{1}{n}\sum_{i=1}^{n}[-(Z_i - \mu_z)(\epsilon_i - \beta_1 u_i - \beta_2 v_i)\mu_z - \beta_2\sigma_v^2\mu_z + (\epsilon_i - \beta_1 u_i - \beta_2 v_i)\sigma_z^2]}{\frac{1}{n}\sum_{i=1}^{n}(Z_i - \bar{Z}_n)^2 - \sigma_v^2}\Big) \\
& +\sqrt{n}\Big(\frac{\frac{1}{n}\sum_{i=1}^{n}(\epsilon_i - \beta_1 u_i - \beta_2 v_i)(\frac{1}{n}\sum_{i=1}^{n}(Z_i - \bar{Z}_n)^2 - \sigma_v^2 - \sigma_z^2)}{\frac{1}{n}\sum_{i=1}^{n}(Z_i - \bar{Z}_n)^2 - \sigma_v^2}\Big).
\end{aligned}$$

So that from the fourth equation in (2.32), (2.82) and (2.83), we get

$$-\sqrt{n}\Big(\frac{\frac{1}{n}\sum_{i=1}^{n}(Z_i - \mu_z)(\epsilon_i - \beta_1 u_i - \beta_2 v_i)\mu_z + \beta_2\sigma_v^2\mu_z}{\frac{1}{n}\sum_{i=1}^{n}(Z_i - \bar{Z}_n)^2 - \sigma_v^2}\Big) + \sqrt{n}(\bar{\epsilon}_n - \beta_1\bar{u}_n - \beta_2\bar{v}_n) \xrightarrow{d} N\Big(0, \frac{\mathrm{Var}(\rho_1)}{\sigma_z^4}\Big). \tag{2.84}$$

Then we have

$$\begin{aligned}
& -\sqrt{n}\mu_z E_{n1} + \sqrt{n}(\bar{\epsilon}_n - \beta_1\bar{u}_n - \beta_2\bar{v}_n) \\
= & -\sqrt{n}\Big(\frac{\frac{1}{n}\sum_{i=1}^{n}(Z_i - \mu_z)(\epsilon_i - \beta_1 u_i - \beta_2 v_i) + \beta_2\sigma_v^2}{\frac{1}{n}\sum_{i=1}^{n}(Z_i - \bar{Z}_n)^2 - \sigma_v^2}\Big)\mu_z \\
& +\sqrt{n}\Big(\frac{(\bar{\epsilon}_n - \beta_1\bar{u}_n - \beta_2\bar{v}_n)(\frac{1}{n}\sum_{i=1}^{n}Z_i - \mu_z)}{\frac{1}{n}\sum_{i=1}^{n}(Z_i - \bar{Z}_n)^2 - \sigma_v^2}\Big)\mu_z \\
& +\sqrt{n}\Big(\frac{(\bar{Z}_n - \mu_z)[\frac{1}{n}\sum_{i=1}^{n}(\epsilon_i - \beta_1 u_i - \beta_2 v_i)]}{\frac{1}{n}\sum_{i=1}^{n}(Z_i - \bar{Z}_n)^2 - \sigma_v^2}\Big)\mu_z \\
& -\sqrt{n}\Big(\frac{(\bar{Z}_n - \mu_z)(\bar{\epsilon}_n - \beta_1\bar{u}_n - \beta_2\bar{v}_n)}{\frac{1}{n}\sum_{i=1}^{n}(Z_i - \bar{Z}_n)^2 - \sigma_v^2}\Big)\mu_z + \sqrt{n}(\bar{\epsilon}_n - \beta_1\bar{u}_n - \beta_2\bar{v}_n).
\end{aligned}$$

From the second, fourth equations in (2.32), the first, second, third equations in (2.65), the second, third equations in (2.70), (2.84), we have

$$-\sqrt{n}\mu_z E_{n1} + \sqrt{n}(\bar{\epsilon}_n - \beta_1 \bar{u}_n - \beta_2 \bar{v}_n) \xrightarrow{d} N\Big(0, \frac{\text{Var}(\rho_1)}{\sigma_z^4}\Big). \tag{2.85}$$

In addition, by the first, second, third equations in (2.65), (2.79), (2.80), and (2.85), we get

$$F_{n3} \quad \xrightarrow{D} \quad N\Big(0, \frac{\text{Var}(\rho_1)}{\sigma_z^4}\Big). \tag{2.86}$$

Besides, noting the condition (2.61), and Theorem 2.3.1, one can obtain

$$F_{n1} \xrightarrow{p} 0. \tag{2.87}$$

Furthermore, from the second equation in (2.32), and Theorem 2.3.2, we get

$$F_{n2} \xrightarrow{p} 0. \tag{2.88}$$

So finally, by (2.81), (2.86) to (2.88), we can show that

$$\sqrt{n}(\hat{\beta}_0 - \beta_0) \xrightarrow{d} N\Big(0, \frac{\text{Var}(\rho_1)}{\sigma_z^4}\Big).$$

This completes the proof of Theorem 2.3.3. □

Remark: Theorem 2.3.3 says that the asymptotic normality is different with the one in the fixed design errors-in-variables model discussed in Miao et al. (2011). But we can see that the assumption (2.61) is the same as the one mentioned in Miao et al. (2011). ■

Chapter 3

FURTHER WORK

For a class of errors-in-variables model in which both random and fixed predictor present, we investigated the large sample properties of the biased corrected estimators for the regression coefficients. Under some regularity conditions, we proved the weak and strong consistency and obtained the asymptotic normality results for the proposed estimators.

In addition to the consistency and the asymptotic normality, the iterated logarithm law, the moderate deviation, and the large deviation principle for the estimator are also important and interesting research topics in probability and statistics theory. For classical linear regression model, these topics have already been thoroughly studied, see Ibragimov and Has'miniskii (1979), Ibragimov and Radavicius (1981), Gao (2001) and the references therein. Relatively few research are done for errors-in-variables linear regression model, the research is even more scarce when both fixed and random predictors present.

Recently, Miao and Yang (2011) and Miao (2010) studied the iterated logarithm law, and large deviation principle in the linear errors-in-variable models when only fixed predictor presents. There is no discussion on the moderate deviation principle. Also, when discussing large deviation principle, the authors assume that the error term ϵ and the measurement error u follow normal distributions. As the continuation of the current report and our future research, we will focus on the following topics in the linear errors-in-variables models when both fixed and random predictor present:

- Develop the iterated logarithm law for the estimators $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\beta}_2$ defined in Chapter

1, and the compare the result with the ones obtained in Miao and Yang (2011).

- Develop the moderate, large deviation principles for $\hat{\beta}_0$, $\hat{\beta}_1$ and $\hat{\beta}_2$ defined in Chapter 1, and make some comparisons with the results obtained in Miao (2010).

- The normality assumptions imposed in Miao (2010) rarely hold in real applications, we will investigate the possibility of removing the normality conditions from the large deviation principles.

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