# COBORDISM THEORY OF SEMIFREE CIRCLE ACTIONS ON COMPLEX $N$-SPIN MANIFOLDS

by

MUHAMMAD NAEEM AHMAD

M.S., Kansas State University, 2006

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AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
requirements for the degree

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Department of Mathematics
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# Abstract


In this work, we study the complex $N$-Spin bordism groups of semifree circle actions and elliptic genera of level $N$.

The notion of complex $N$-Spin manifolds (or simply $N$-manifolds) was introduced by Höhn in [Höh91]. Let the bordism ring of such manifolds be denoted by $\Omega_*^{U,N}$ and the ideal in $\Omega_*^{U,N} \otimes \mathbb{Q}$ generated by bordism classes of connected complex $N$-Spin manifolds admitting an effective circle action of type $t$ be denoted by $I_*^{N,t}$. Also, let the elliptic genus of level $n$ be denoted by $\varphi_n$. It is conjectured in [Höh91] that

$$I_*^{N,t} = \bigcap_{\substack{n|N \\ n \nmid t}} \ker(\varphi_n).$$

Our work gives a complete bordism analysis of rational bordism groups of semifree circle actions on complex $N$-Spin manifolds via traditional geometric techniques. We use this analysis to give a determination of the ideal $I_*^{N,t}$ for several $N$ and $t$, and thereby verify the above conjectural equation for those values of $N$ and $t$. More precisely, we verify that the conjecture holds true for all values of $t$ with $N \leq 9$, except for case $(N,t) = (6,3)$ which remains undecided. Moreover, the machinery developed in this work furnishes a mechanism with which to explore the ideal $I_*^{N,t}$ for any given values of $N$ and $t$.

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# Introduction

In this work, we study the complex $N$-Spin bordism groups of semifree circle actions and elliptic genera of level $N$.

The motivation for the bordism theory of semifree circle actions on Spin manifolds comes from a question of E. Witten (see [Wit85]) who is interested in the Dirac operator having coefficients in the tangent bundle $\tau(M)$ of an even dimensional Spin manifold $M$. The ordinary index of this operator is given by

$$\langle \hat{\mathcal{A}}(M)\mathrm{ch}(\tau(M)), \sigma_M \rangle,$$

where $\hat{\mathcal{A}}$ denotes the total $\hat{A}$-class of $\tau(M)$, and $\mathrm{ch}(\tau(M))$ the Chern character of the complexification of $\tau(M)$. He shows that the index is constant for a homogeneous space with a nontrivial circle action. Moreover, he realizes that figuring out the same for a Spin manifold with a nontrivial circle action can be explored through the study of bordism groups of circle actions on Spin manifolds. In [Bor87] L. Borsari carries out this study. For simplicity she considers only semifree circle actions on Spin manifolds. Let $I_*$ denote the ideal in $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$ generated by bordism classes of Spin manifolds admitting a semifree circle action of odd type, and $x_{4n}, n \geq 3$ the sequence of bordism classes of complex projective bundles $[\mathbb{CP}(\xi_n)]$, where $\xi_n$ is a suitable complex 4-bundle. In the course of her study she comes up with inclusions

$$\langle x_{12}, x_{16}, \ldots, x_{4n}, \ldots \rangle \subset I_* \subset \ker(L) \cap \ker(\hat{A}),$$

where the $\hat{A}$-genus $\hat{A}$ and signature $L$ are viewed as ring homomorphisms from $\Omega_*^{\mathrm{SO}} \otimes \mathbb{Q}$ to $\mathbb{Q}$. She conjectures that $I_*$ coincides with the ideal $\langle x_{12}, x_{16}, \ldots, x_{4n}, \ldots \rangle$.

In [Höh91], G. Höhn initiates a study of complex $N$-Spin manifolds (or simply $N$-manifolds) with circle actions of type $t$, where $t$ is an element of $\mathbb{Z}/N\mathbb{Z}$. He uses it to explore the elliptic genera $\varphi_N$ of level $N$. In his work he comes up with an inclusion

$$I_*^{N,t} \subset \bigcap_{\substack{n|N \\ n \nmid t}} \ker(\varphi_n),$$

where $I_*^{N,t}$ denotes the ideal in the rational complex $N$-Spin bordism ring $\Omega_*^{U,N} \otimes \mathbb{Q}$ generated by bordism classes of complex $N$-Spin manifolds admitting a semifree circle action of type $t$, and $\varphi_n$ the elliptic genus of level $n$ viewed as a ring homomorphism from $\Omega_*^{U,N} \otimes \mathbb{Q}$ to $\mathbb{Q}$. He verifies that the equality holds when $t = 1$, and based on that he conjectures that the equality is also true for an arbitrary $t \in \mathbb{Z}/N\mathbb{Z}$.

In [Och87], S. Ochanine finds all multiplicative genera which vanish on the ideal $I_*$ and solves the conjecture of Borsari.

In this work, we explore the conjecture of Höhn via traditional geometric techniques in bordism theory similar to those considered by Borsari [Bor87] and Höhn [Höh91].

Denote by $\Omega_n^U(SF, F)$ the bordism group of semifree circle actions on compact $n$-dimensional $U$-manifolds which are free on the boundary. Also, denote by $\Omega_n^{U,N}(SF, F)$ the bordism group of semifree circle actions on compact $n$-dimensional complex $N$-Spin manifolds which are free on the boundary. We have the forgetful homomorphism

$$\Omega_n^{U,N}(SF, F) \to \Omega_n^U(SF, F).$$

Our main result states that this homomorphism is an isomorphism when tensored with the rationals. We establish this result using traditional methods in bordism theory. We use this result to find a computable characterization of the rational bordism group $\Omega_n^{U,N,t}(SF, F)$ of semifree circle actions of type $t$ on compact $n$-dimensional complex $N$-Spin manifolds which are free on the boundary. More precisely, let $\Omega_{2k}^U(BU(p) \times BU(q))$ denote the $2k$-dimensional singular $U$-bordism group of the product of classifying spaces $BU(p)$ and $BU(q)$, we find a rational isomorphism (see Corollary 3.2.13)

$$\Omega_n^{U,N,t}(SF, F) \to \bigoplus_{\substack{k+2p+2q=n \\ p-q \equiv t \pmod{N}}} \Omega_k^U(BU(p) \times BU(q)).$$

For $t|N$, we also characterize the bordism group $\Omega_n^{U,N,t}(F)$ of free circle actions of type $t$ on closed $n$-dimensional complex $N$-Spin manifolds by finding a rational isomorphism (see Theorem 3.3.11)

$$\Omega_n^{U,N,t}(F) \to \Omega_{n-1}^U(\mathbb{CP}_\infty).$$

We use these characterizations to find a condition under which the conjecture of Höhn holds true. This condition involves finding some bordism classes $[U_1]$, $[U_2]$, $\ldots$, $[U_m]$ satisfying a certain inclusion (see Theorem 4.1.4).

We make computations to construct these bordism classes and thereby verify the conjecture of Höhn for several values of $N$ and $t$. The computations grow longer as $N$ increases and we handle them by writing computer programs in *Mathematica*, for bigger values of $N$. As $N$ grows higher the computations become intractable. More precisely, we show that the conjecture holds true for all values of $t$ with $N \leq 9$, except for case $(N,t) = (6,3)$ which remains unanswered. This is the extent of our current knowledge.

The contents of this work are organized as follows:

First chapter is devoted to preliminaries on complex $N$-Spin manifolds with circle actions. Following [Höh91], we formulate the definition of a complex $N$-Spin structure on a $U(n)$-bundle over a $CW$-complex (see Definition 1.2.1). A complex $N$-Spin manifold is a $U$-manifold whose first Chern class is divisible by $N$. Let $\Omega_n^U(X,A)$ denote the $n$-dimensional $U$-bordism group of a finite $CW$-pair $(X,A)$. We define the complex $N$-Spin bordism group

$\Omega_n^{U,N}(X,A)$ of the pair $(X,A)$ and show in Theorem 1.2.18 that the forgetful homomorphism $\Omega_n^{U,N}(X,A) \to \Omega_n^U(X,A)$ is a rational isomorphism. In the last part of the first chapter we talk about the smooth circle action on a connected complex $N$-manifold. It turn out that the type of such an action is an element of the group $\mathbb{Z}/N\mathbb{Z}$ (see [Höh91] or Definition 1.3.2).

The second chapter is dedicated to complex elliptic genera and multiplicative sequences. We begin that chapter by setting up notation needed in the remaining work. Following the notation of [Höh91], we give a definition of the genus $\varphi_V$, with respect to a given basis $\Omega_*^U \otimes \mathbb{C}$, belonging to a variety $V$ of the weighted projective space $\mathbb{CP}^{a_1,\dots,a_n}$ with weights $a_1$, $\dots, a_n \in \mathbb{N}$ in Definition 2.1.6. Next, we define the universal elliptic genus $\varphi_{ell}$ which can be regarded as the genus belonging to the weighted projective variety $\mathbb{CP}^{1,2,3,4}$ with respect to a certain basis of $\Omega_*^U \otimes \mathbb{C}$ (see [Höh91] or Section 2.2). Next, we define the complex elliptic genus $\varphi_N$ of level $N$ in Definition 2.3.1. In [Höh91] Höhn defines a variety $C_N$, in the weighted projective space $\mathbb{CP}^{1,2,3,4}$ — with weights $1, 2, 3$,and $4$ — whose annihilator ideal $I_N$ is a radical ideal (see Theorem 2.3.3. The ideal $I_N$ has the primary decomposition

$$I_N = \bigcap_{\substack{n>1 \\ n|N}} P_n,$$

where ideals $P_n$ (independent of $N$) correspond to certain irreducible components of the curve $C_N$. We use these ideals to define the ideal $I_{N,t}$ for $t|N$ as follows

$$I_{N,t} = \bigcap_{\substack{n|N \\ n\nmid t}} P_n.$$

We compute ideals $I_N, P_n$, and $I_{N,t}$ for several values of $N$ and $t$ by designing a few computer programs in *Mathematica* and MAGMA (see Section 2.3). It turns out that the size of these ideals become intractable as $N$ grows large. Moreover, after proving a few results by doing some analysis of elliptic genera and invoking a theorem of Hirzebruch [Hir98] on elliptic genera, we find a computable condition under which the conjecture of Höhn holds true. This condition involves the existence of bordism classes $[V_1], [V_2], \ldots, [V_m] \in I_*^{N,t}$ satisfying the inclusion of ideals $I_{N,t} \subset \langle \varphi_{\mathrm{ell}}[V_1], \varphi_{\mathrm{ell}}[V_2], \ldots \varphi_{\mathrm{ell}}[V_m] \rangle$ (see Proposition 2.3.9).

In the third chapter, we describe the bordism analysis needed to construct elements of $I_*^{N,t}$ satisfying the above condition. Let $\Omega_n^U(F)$ and $\Omega_n^U(SF)$ denote the bordism groups of free and semifree circle actions on closed $n$-dimensional $U$-manifolds, respectively. Also, denote by $\Omega_n^U(SF, F)$ the bordism group of semifree circle actions on compact $n$-dimensional manifolds which are free on the boundary. One has a long exact sequence (see [HT72]):

$$\cdots \longrightarrow \Omega_n^U(F) \longrightarrow \Omega_n^U(SF) \longrightarrow \Omega_n^U(SF, F) \longrightarrow \Omega_{n-1}^U(F) \longrightarrow \cdots$$

where the maps are either the obvious ones or boundary maps. One may also work in the category of complex $N$-Spin manifolds to obtain a long exact sequence

$$\cdots \longrightarrow \Omega_n^{U,N}(F) \longrightarrow \Omega_n^{U,N}(SF) \longrightarrow \Omega_n^{U,N}(SF, F) \longrightarrow \Omega_{n-1}^{U,N}(F) \longrightarrow \cdots$$

Restricting to circle actions of a fixed type $t$ gives yet another long exact sequence

$$\cdots \longrightarrow \Omega_n^{U,N,t}(F) \longrightarrow \Omega_n^{U,N,t}(SF) \longrightarrow \Omega_n^{U,N,t}(SF, F) \longrightarrow \Omega_{n-1}^{U,N,t}(F) \longrightarrow \cdots$$

This sequence remains exact after tensoring each bordism group in it with the rationals. We set off to compute the bordism groups involved in this sequence, after tensoring with the rationals. Let $\gamma_p$ and $\gamma_q$ denote the universal complex vector bundles over the classifying spaces $BU(p)$ and $BU(q)$, respectively. We introduce the bordism group $\Omega_k^{U,N}(BU(p) \times BU(q), \gamma_p \oplus \gamma_q)$ of triples $(M, f, s)$, where $f : M \to BU(p) \times BU(q)$ is a continuous map on an $(k-2p-2q)$-dimensional $U$-manifold $M$ and $s$ an $N$-structure on the stable bundle representing the $U$-structure of $\tau(M) \oplus f^*(\gamma_p \oplus \gamma_q)$. The bordism group $\Omega_k^U(BU(p) \times BU(q), \gamma_p \oplus \gamma_q)$ can be defined similarly. We define homomorphisms $\Omega_n^{U,N}(SF, F) \to \bigoplus_{k+2p+2q=n} \Omega_k^{U,N}(BU(p) \times BU(q), \gamma_p \oplus \gamma_q)$ and $\Omega_n^U(SF, F) \to \bigoplus_{k+2p+2q=n} \Omega_k^U(BU(p) \times BU(q), \gamma_p \oplus \gamma_q)$, and show that they are isomorphisms (see Proposition 3.2.4 and Proposition 3.2.5). Next, we define homomorphism $\Omega_k^{U,N}(BU(p) \times BU(q), \gamma_p \oplus \gamma_q) \to \Omega_k^{U,N}(D(\gamma_p \oplus \gamma_q), S(\gamma_p \oplus \gamma_q))$ and show that it is also an isomorphism (see Proposition 3.2.6). We put together all the above isomorphisms to obtain our main theorem (see Theorem 3.2.11). After that we begin to compute the rational bordism group $\Omega_n^{U,N,t}(SF,F)\otimes\mathbb{Q}$. For $[M]\in\Omega_n^U(SF,F)$ and a connected component of the fixed-point set of the circle action on $M$, the normal bundle $\upsilon(X_i)$ of $X_i$ in $M$ has the decomposition $\upsilon(X_i)=\upsilon_i^+\oplus\upsilon_i^-$ so that $\lambda\in S^1\subset\mathbb{C}^*$ acts on complex vector bundles $\upsilon_i^+$ and $\upsilon_i^-$ by multiplication with $\lambda$ and $\lambda^{-1}$, respectively (see [Höh91]). We denote an element $[M,f]$ of $\Omega_n^U(BU(p)\times BU(q))$ by $[M,E_1\oplus E_2]$, where complex vector bundles $E_1$ and $E_2$ of ranks $p$ and $q$ are pullbacks of universal bundles over classifying spaces $BU(p)$ and $BU(q)$ under the first and the second components of $f$, respectively. We prove that the homomorphism

$$\Omega_n^{U,N}(SF,F)\to\bigoplus_{k+2p+2q=n}\Omega_k^U(BU(p)\times BU(q))$$

given by $[M]\mapsto\sum[X_i,\upsilon_i^+\oplus\upsilon_i^-]$ is a rational isomorphism (see Theorem 3.2.12). The type of a circle action on a complex $N$-Spin manifold is the sum of rotation numbers of a fixed-point component modulo $N$ (see [Höh91]). It follows that the homorphism

$$\Omega_n^{U,N,t}(SF,F)\to\bigoplus_{\substack{k+2p+2q=n\\ p-q\equiv t\pmod N}}\Omega_k^U(BU(p)\times BU(q))$$

given by $[M]\mapsto\sum[X_i,\upsilon_i^+\oplus\upsilon_i^-]$ is a rational isomorphism. After computing the rational bordism group $\Omega_n^{U,N,t}(SF,F)\otimes\mathbb{Q}$, we sought to compute the rational bordism group $\Omega_n^{U,N,t}(F)\otimes\mathbb{Q}$ of free circle actions of a fixed type $t$ on $n$-dimensional closed complex $N$-Spin manifolds for $t|N$. For this purpose, we introduce the notion of a complex $N$-Spin$^{c,t}$ structure on manifolds and develop its fundamental properties (see Section 3.3). We also define the notion of complex $N$-Spin$^{c,t}$ manifolds and the bordism group $\Omega_n^{U,N^{c,t}}$ of such closed $n$-dimensional manifolds. For $t|N$, the forgetful homomorphism $\Omega_n^{U,N^{c,t}}\to\Omega_n^U(\mathbb{CP}_\infty)$ is a rational isomorphism (see Theorem 3.3.6), and the homomorphism $\Omega_n^{U,N,t}(F)\to\Omega_{n-1}^{U,N^{c,t}}$ given by $[M]\to[M/S^1]$ is an isomorphism(see Proposition 3.3.10). It follows that for $t|N$, the homomorphism $\Omega_n^{U,N,t}(F)\to\Omega_{n-1}^U(\mathbb{CP}_\infty)$ given by $[M]\to[M/S^1]$ is a rational isomorphism.

The fourth chapter is devoted to the determination of the ideal $I_*^{N,t}$ in $\Omega_*^{U,N} \otimes \mathbb{Q}$ generated by bordism classes of connected complex $N$-Spin manifolds admitting a semifree circle action of type $t$ and exploration of the conjecture.

# Chapter 1

# Complex $N$-Spin Bordism and Circle Actions

In this chapter we will review the theory of complex $N$-Spin manifolds and the types of circles actions on such manifolds initiated by G. Höhn in [Höh91]. We will then introduce the notion of singular $N$-bordism theory and see that the singular $N$-bordism and the singular $U$-bordism modules coincide when tensored with the rationals $\mathbb{Q}$.

First we sum up few details about the $U$-bordism theory which will be necessary and motivational for the subsequent study. The $U$-bordism theory was initiated by J. W. Milnor [Mil60] and S. P. Novikov [Nov60]. For a comprehensive understanding of the results pertaining the $U$-bordism theory we refer to R. E. Stong [Sto68], Chapter VII.

## 1.1 $U$-Bordism

A $U$-manifold can be defined in different equivalent ways. We will adopt the method of P. E. Conner and E. E. Floyd [CF66b] to work our way through the definition of a $U$-bordism group.

In what follows $BU(n)$ will stand for a classifying space of the topological group $U(n)$ and $\gamma_n$ for a universal $U(n)$-bundle with fiber $\mathbb{C}^n$. The universal bundle $\gamma_n$ can also be considered as an $O(2n)$-bundle.

**Proposition 1.1.1.** (See for instance, [CF66b], p. 12) *Let $\xi$ be an $O(2n)$-bundle over a $CW$-complex with fiber $\mathbb{R}^{2n}$. Then there is a one to one correspondence between the following two sets:*

*(i) the set of homotopy classes of $O(2n)$-bundle maps $\xi \longrightarrow \gamma_n$, and*

*(ii) the set of homotopy classes of maps $J : E(\xi) \longrightarrow E(\xi)$ for which the identity $J^2 = -1$ holds and which map every fiber of $\xi$ orthogonally onto itself.*

**Definition 1.1.2.** Let $\xi$ be an $O(2n)$-bundle over a $CW$-complex. A *complex structure* $\Phi$ on $\xi$ is a homotopy class of $O(2n)$-bundle maps $\phi : \xi \longrightarrow \gamma_n$.

**Notation.** We will denote by $U(\xi)$ the set of complex structures on $\xi$.

**Remark.** Let $m$ and $n$ be two positive integers. Then there exists the cartesian product bundle $\gamma_m \times \gamma_n \longrightarrow BU(m) \times BU(n)$ which may be considered as a $U(m+n)$-bundle having the fiber $\mathbb{C}^{m+n}$, and there exists a unique homotopy class of $U(m+n)$-bundle maps $\psi : \gamma_m \times \gamma_n \longrightarrow \gamma_{m+n}$ which cover a homotopy class of maps $\bar{\psi} : BU(m) \times BU(n) \longrightarrow BU(m+n)$.

Let $B$ be a $CW$-complex and let $\xi_1$, $\xi_2$ be $O(2m)$, $O(2n)$-bundles over $B$, respectively. The Whitney sum $\xi_1 \oplus \xi_2$ is the restriction of $\xi_1 \times \xi_2$ to the diagonal. The composition of two complex structures is defined as below.

**Definition 1.1.3.** Let $\Phi_1 \in U(\xi_1)$ and $\Phi_2 \in U(\xi_2)$ be two complex structures represented by $\phi_1 : \xi_1 \longrightarrow \gamma_m$ and $\phi_2 : \xi_2 \longrightarrow \gamma_n$, then $\Phi_1 \circ \Phi_2 \in U(\xi_1 \oplus \xi_2)$ is defined as the complex structure represented by the composition $\phi_1 \circ \phi_2$ in

$$\xi_1 \oplus \xi_2 \longrightarrow \xi_1 \times \xi_2 \xrightarrow{\phi_1 \times \phi_2} \gamma_m \times \gamma_n \xrightarrow{\psi} \gamma_{m+n},$$

where $\psi$ is as given in the above remark.

**Remark.** ([CF66b], p. 18) Clearly $U$-structures $\Phi_1$ and $\Phi_2$ could be defined in terms of bundle maps $J_1$ and $J_2$ of Proposition 1.1.1. Define $J_1 \circ J_2$ on $\xi_2 \oplus \xi_2$ by

$$(J_1 \circ J_2)(x, y) = (J_1(x), J_2(y)).$$

A passage to homotopy classes again gives $\Phi_1 \circ \Phi_2$.

**Notation.** Let $k$ be a positive integer and $B$ be a space. We will denote by $\epsilon^k_{\mathbb{R}}$ the trivial $O(k)$-bundle

$$B \times \mathbb{R}^k \longrightarrow B$$

given by the projection onto the first coordinate.

**Notation.** Define the two bundle maps

$$J_0, \ -J_0 : B \times \mathbb{R}^2 \longrightarrow B \times \mathbb{R}^2$$

by

$$J_0(x; s, t) = (x; -t, s) \text{ and } -J_0(x; s, t) = (x, t, -s).$$

Then $J_0, -J_0 \in U(\epsilon^2_{\mathbb{R}})$ will be denoted by $\Phi_0, -\Phi_0$, respectively.

**Proposition 1.1.4.** (See for instance, [CF66b], p. 19) *Let $\xi$ be an $O(2n)$-bundle over a $CW$-complex $B$ with $\dim B \leq 2n - 2$. Then the assignment*

$$\Phi \longrightarrow \Phi_0 \circ \Phi$$

*defines a one to one correspondence between the sets of complex structures $U(\xi)$ and $U(\epsilon^2_{\mathbb{R}} \oplus \xi)$.*

We can identify $U(\xi)$ with $U(\epsilon^2_{\mathbb{R}} \oplus \xi)$ via the correspondence given in Proposition 1.1.4.

**Definition 1.1.5.** Let $\xi$ be an $O(2n)$-bundle over a $CW$-complex $B$ with $\dim B \leq 2n - 2$, and let $\Phi$ be a complex structure on $\xi$. The complex structure $-\Phi$ on $\xi$ is defined to be a unique element of $U(\xi)$ such that

$$\Phi_0 \circ (-\Phi) = (-\Phi_0) \circ \Phi \quad \text{in} \quad U(\epsilon^2_{\mathbb{R}} \oplus \xi).$$

From now on a manifold will mean a smooth (i.e. $C^\infty$) manifold and $\tau(M)$ will denote the tangent bundle of a smooth manifold $M$.

**Definition 1.1.6.** A *$U$-manifold* is defined to be a pair $(M^k, \Phi)$ consisting of a smooth manifold $M$ and a complex structure $\Phi \in U(\epsilon_{\mathbb{R}}^{2n-k} \oplus \tau(M))$, where $2n \geq k+2$.

**Remarks.** The definition of a $U$-manifold is independent of $n$ by Propisition 1.1.4, if $2n \geq k+2$. We will often denote a $U$-manifold $(M, \Phi)$ simply by $M$ and $\Phi$ by $\Phi(M)$. The negative $-(M^k, \Phi)$ of a $U$-manifold $(M^k, \Phi)$ is defined to be $(M^k, -\Phi)$.

**Definition 1.1.7.** Let $f : \xi \longrightarrow \eta$ be an $O(2n)$-bundle map. The function

$$f^* : U(\eta) \longrightarrow U(\xi)$$

is defined as: if $\Phi \in U(\eta)$ is represented by $\phi : \eta \longrightarrow \gamma_n$, then $f^*\Phi$ is represented by

$$\phi \circ f : \xi \longrightarrow \gamma_n.$$

**Remark.** The boundary of a $U$-manifold is a $U$-manifold: Let $M$ be a smooth manifold. Equip $M$ with a Riemannian metric. Consider a vector field over $\partial M^k$ which assigns to each $x \in \partial M^k$ an outward unit tangent vector $v(x)$ to $M^k$ at $x$. This vector field gives rise to a bundle map $f : \epsilon_{\mathbb{R}}^1 \oplus \tau(\partial M^k) \longrightarrow \tau(M^k)$. So we get $\mathrm{id}\oplus\mathrm{f} : \epsilon_{\mathbb{R}}^{\mathrm{2n-k+1}} \oplus \tau(\partial \mathrm{M}^\mathrm{k}) \longrightarrow \epsilon_{\mathbb{R}}^{\mathrm{2n-k}} \oplus \tau(\mathrm{M}^\mathrm{k})$. We denote $(\mathrm{id} \oplus \mathrm{f})^*\Phi$ by $\partial\Phi$. The boundary $\partial(M^k, \Phi)$ of the $U$-manifold $(M^k, \Phi)$ is defined to be the $U$-manifold $(\partial M^k, \partial\Phi)$.

**Remark.** The product of two $U$-manifold is a $U$-manifold: Let $M^m$ and $M'^n$ be two $U$-manifolds. Since $\tau(M^m \times M'^n) \cong \tau(M^m) \times \tau(M'^n)$, therefore, $\epsilon_{\mathbb{R}}^{2r+2s-m-n} \oplus \tau(M^m \times M'^n)$ and $(\epsilon_{\mathbb{R}}^{2r-m} \oplus \tau(M^m)) \times (\epsilon_{\mathbb{R}}^{2s-n} \oplus \tau(M'^n))$ are isomorphic under a canonical isomorphism $h$, say. Let $\Phi(M^m) \in U(\epsilon^{2r-m} \oplus \tau(M^m))$ and $\Phi(M'^n) \in U(\epsilon_{\mathbb{R}}^{2s-n} \oplus \tau(M'^n))$ be two complex structures, then $h^*(\Phi(M^m) \circ \Phi(M'^n)) \in U(\epsilon_{\mathbb{R}}^{2r+2s-m-n} \oplus \tau(M^m \times M'^n))$. The complex structure $\Phi(M^m \times M'^n)$ is defined to be $(-1)^{mn} h^*(\Phi(M^m) \circ \Phi(M'^n))$.

**Definition 1.1.8.** Let $f : M^k \longrightarrow M'^k$ be a diffeomorphism of a $U$-manifold $M^k$ onto a $U$-manifold $M'^k$. Denote

$$\mathrm{id} + \mathrm{df} : \epsilon_{\mathbb{R}}^{\mathrm{2n-k}} \oplus \tau(\mathrm{M^k}) \longrightarrow \epsilon_{\mathbb{R}}^{\mathrm{2n-k}} \oplus \tau(\mathrm{M'^k})$$

after an abuse of notation by df. The diffeomorphism $f$ is said to be a *$U$-diffeomorphism* if

$$(\mathrm{df})^* : \mathrm{U}(\epsilon_{\mathbb{R}}^{\mathrm{2n-k}} \oplus \tau(\mathrm{M'^k})) \longrightarrow \mathrm{U}(\epsilon_{\mathbb{R}}^{\mathrm{2n-k}} \oplus \tau(\mathrm{M^k}))$$

satisfies the identity $(\mathrm{df})^*\Phi(\mathrm{M'^k}) = \Phi(\mathrm{M^k})$.

In the following we will see how the $U$-bordism is defined.

**Definition 1.1.9.** Let $k$ be a positive integer. A closed $U$-manifold $M^k$ is said to *bord* if there exists a compact $U$-manifold $W^{k+1}$ such that its boundary $\partial W^{k+1}$ is $U$-diffeomorphic to the $U$-manifold $M^k$.

**Definition 1.1.10.** Two closed $U$-manifolds $M_1^k$ and $M_2^k$ are said to be *bordant* if the disjoint union $M_1^k \dot{\cup} -M_2^k$ bords.

The relation of bordism is an equivalence relation on the class of closed $k$-dimensional $U$-manifolds. The resulting set $\Omega_k^U$ is an abelian group under addition induced by disjoint union. The weak direct sum $\Omega_*^U = \sum \Omega_k^U$ is a graded ring under the multiplication induced by direct product.

**Definition 1.1.11.** Let $k$ be a non-negative integer and $(X, A)$ be a pair of spaces. Consider pairs $(M^k, f)$ such that $M^k$ is a compact $U$-manifold and $f : (M^k, \partial M^k) \longrightarrow (X, A)$. A pair $(M^k, f)$ *bords* if there exists a compact $(k+1)$-dimensional $U$-manifold $W^{k+1}$ and $F : W^{k+1} \longrightarrow X$ such that

(i) $M^k$ is a submanifold of $\partial W^{k+1}$ and $\Phi(M^k)$ is the restriction of $\partial\Phi(W^{k+1})$, and

(ii) $F \mid M^k = f$ and $F(\partial W^{k+1} - M^k) \subset A$.

**Definition 1.1.12.** Two pairs $(M_1^k, f_1)$ and $(M_2^k, f_2)$ are said to be *bordant* if the disjoint union $(M_1^k, f_1) \dot{\cup} (-M_2^k, f_2)$ bords.

The bordism is an equivalence relation and the set of equivalence classes forms an abelian group under addition induced by disjoint union. The bordism class containing $(M^k, f)$ will be denoted by $[M^k, f]$ and the abelian group of bordism classes will be denoted by $\Omega_k^U(X, A)$. The weak direct sum $\Omega_*^U(X, A) = \sum \Omega_k^U(X, A)$ can be given the structure of a graded $\Omega_*^U$-module.

**Definition 1.1.13.** Let $(M^k, \Phi)$ be a $U$-manifold and let $\Phi$ be represented by

$$\phi : \epsilon_{\mathbb{R}}^{2n-k} \oplus \tau(M^k) \longrightarrow \gamma_n$$

which covers

$$\bar{\phi} : M^k \longrightarrow BU(n).$$

Let $c_i \in H^{2i}(BU(n))$, $i = 1, \ldots, r$ denote the universal Chern classes. The unique classes $\bar{\phi}^*(c_i) \in H^{2i}(M^k)$ — denoted by $c_i(M^k)$ — are called the *Chern classes* of the $U$-manifold $M^k$.

**Remark.** A $U$-manifold is naturally oriented: Let $(M^k, \Phi)$ be as in the above definition. The standard orientation of the universal bundle $\gamma_n$ induces an orientation on $\epsilon_{\mathbb{R}}^{2n-k} \oplus \tau(M^k)$ and so an orientation on $\tau(M^k)$.

**Notation.** Let $M^k$ be a compact $U$-manifold. The orientation class of $H_k(M^k, \partial M^k)$ will be denoted by $[M^k]$.

**Definition 1.1.14.** Let $M^k$ be $U$-manifold and let $f : (M^k, \partial M^k) \longrightarrow (X, A)$. Let a partition $\omega = (i_1, \ldots, i_r)$ be given and let $x \in H^s(X, A)$. One has $c_\omega = c_{i_1} \ldots c_{i_r} \in H^{2(i_1 + \ldots + i_r)}(M^k)$, $f^*(x) \in H^s(M^k, \partial M^k)$, and hence $c_\omega \cdot f^*(x) \in H^*(M^k, \partial M^k)$. A *characteristic number* of the pair $(M^k, f)$ is defined to be $c_\omega \cdot f^*(x)[M^k]$.

**Remark.** The characteristic number $c_\omega \cdot f^*(x)[M^k]$ is tacitly assumed to be zero if the number $2i_1 + \ldots + 2i_r + s$ is not equal to $k$, and it is a bordism invariant.

**Definition 1.1.15.** A characteristic number of pair $(M^k, f)$ is also called a *characteristic number* of the bordism class $[M^k, f] \in \Omega_k(X, A)$.

**Proposition 1.1.16.** (See for instance [CF66b], p. 22) *Let $(X, A)$ be a CW-pair and $H_*(X, A)$ be free abelian. Two elements of $\Omega_k(X, A)$ are equal if and only if they have the same characteristic numbers for every $\omega$ and every $x \in H^*(X, A)$.*

**Theorem 1.1.17.** (See for instance, [Sto68], p. 110) *We have*

$$\Omega_*^U \otimes \mathbb{Q} \cong \mathbb{Q}[[\mathbb{CP}_1], [\mathbb{CP}_2], [\mathbb{CP}_3], \ldots].$$

**Theorem 1.1.18.** (See for instance, [Sto68], p. 143) *Let $(X, A)$ be a CW-pair. Then the $\Omega_*^U \otimes \mathbb{Q}$-module $\Omega_*^U(X, A) \otimes \mathbb{Q}$ is free, and*

$$\Omega_*^U(X, A) \otimes \mathbb{Q} \cong H_*(X, A; \mathbb{Q}) \otimes (\Omega_*^U \otimes \mathbb{Q}).$$

In the next section we will study the bordism theory of Complex $N$-Spin manifolds.

## 1.2 Complex $N$-Spin Bordism

The notion of a complex $N$-Spin manifold was introduced by G. Höhn in [Höh91]. He used the bordism theory of complex $N$-Spin manifolds with certain circle actions to study elliptic genera. The notion of a complex $N$-Spin manifold is a generalization of the notion of a Spin manifold. More precisely, a complex $N$-Spin manifold is a complex Spin manifold for $N = 2$. For an expository introduction to Spin theory we refer to H. B. Lawson and M. L. Michelsohn [LM89]. Here we will put together few details concerning the definition and immediate properties of complex $N$-Spin manifolds from [Höh91]. Furthermore, we will introduce the notion of the complex $N$-bordism groups of a pair of spaces and develop its basic properties which will be useful to prove our main results in Chapter 3.

In the sequel we will use the identifications $\widehat{U}(1) := \mathbb{R}/N\mathbb{Z} \cong S^1$ and $U(1) \cong \mathbb{R}/\mathbb{Z} \cong S^1$ without further mentioning.

**Definition 1.2.1.** [Höh91] Let $\xi$ be a principal $U(n)$-bundle over a finite connected $CW$-complex $B$, and let $P$ be the associated principal $S^1$-bundle belonging to the representation

$$\Lambda^n : U(n) \longrightarrow U(1) \cong S^1.$$

A *complex $N$-Spin structure (or simply $N$-structure)* on $\xi$ is a pair $(Q, \pi)$ :

(i) $Q$ is a principal $\widehat{U}(1)$-bundle over $B$, and

(ii) $\pi : Q \longrightarrow P$ is a map such that the diagram

$$\begin{tikzcd} Q \times \widehat{U}(1) \arrow[r] \arrow[d, "\pi \times \lambda_N"] & Q \arrow[d, "\pi"] \arrow[dr] & \\ & & B \\ P \times U(1) \arrow[r] & P \arrow[ur] & \end{tikzcd}$$

is commutative. Where $\lambda_N : \widehat{U}(1) \longrightarrow U(1)$ is the canonical $N$-fold covering map given by $g \longmapsto g^N$, and the horizonal maps are the natural actions of $\widehat{U}(1)$ and $U(1)$ on $Q$ and $P$, respectively.

**Remark.** The map $\pi : Q \longrightarrow P$ is an $N$-fold covering.

**Definition 1.2.2.** Let $\xi$ be a principal $U(n)$-bundle over a finite connected $CW$-complex $B$, and let $(Q, \pi)$ and $(Q', \pi')$ be two $N$-structures on $\xi$. The $N$-structure $(Q', \pi')$ is said to be *equivalent* to the $N$-structure $(Q, \pi)$ if there exists a principal $\widehat{U}(1)$-bundle isomorphism

$$f : Q' \longrightarrow Q$$

such that $\pi \circ f = \pi'$.

**Remark.** It is possible that the $\widehat{U}(1)$-bundles $Q$ and $Q'$ have the same first Chern classes and hence are isomorphic but the two $N$-structures $(Q, \pi)$ and $(Q', \pi')$ on $\xi$ are not equivalent.

**Definition 1.2.3.** An *$N$-structure on a complex vector bundle* is an $N$-structure of the associated principal $U(n)$-bundle.

**Definition 1.2.4.** [Höh91] A *complex $N$-Spin manifold (or simply $N$-manifold)* is a $U$-manifold whose first Chern Class is divisible by $N$.

**Definition 1.2.5.** An *$N$-structure on a connected complex $N$-Spin manifold* is an equivalence class of an $N$-structure belonging to the stable tangent bundle which represents its $U$-structure. If a complex $N$-Spin manifold is not connected then $N$-structure is chosen on each component.

**Remark.** ([Höh91], p. 11) The above definition is independent of the principal bundle representing the $U$-structure.

**Definition 1.2.6.** Let $X$ and $Y$ be two complex $N$-Spin manifolds with $N$-structures. The complex $N$-Spin manifolds $X$ and $Y$ are said to be *differentiably equivalent* if there exists a $U$-diffeomorphism

$$f : X \longrightarrow Y$$

such that the $N$-structure on $X$ pulled back by $f$ from the $N$-structure on $Y$ is the given $N$-structure on $X$. We will conveniently call $f$ an *$N$-diffeomorphism.*

**Proposition 1.2.7.** (See [Höh91], p. 11) *Let $X$ and $Y$ be two complex $N$-Spin manifolds with $N$-structures. The cartesian product $X \times Y$ is a complex $N$-Spin manifold with a canonical $N$-structure.*

**Proposition 1.2.8.** (See [Höh91], p. 11) *Let $X$ be a complex $N$-Spin manifold with an $N$-structure. The boundary $\partial X$ of $X$ is a complex $N$-Spin manifold with a canonical $N$-structure.*

**Proposition 1.2.9.** (See [Höh91], p. 11) *Let $X$ be a complex $N$-Spin manifold. If $X$ is simply connected, then $X$ has a unique $N$-structure.*

**Definition 1.2.10.** Let $I = [0, 1]$ be given a $U$-structure so that for any $U$-manifold $X$, the $U$-structure induced on the component $X \times \{0\}$ from $\partial(X \times I)$ is the $U$-structure of $X$. We will call such a $U$-structure on $I$ to be a *natural $U$-structure* on $I$.

**Remark.** By Proposition 1.2.9 the $U$-manifold $I$ with a natural $U$-structure has a unique $N$-structure.

**Definition 1.2.11.** Let the $U$-manifold $I$ be equipped with a natural $U$-structure. The unique $N$-structure on $I$ will be called a *natural $N$-structure* on the $U$-manifold $I$.

Unless mentioned otherwise, the complex $N$-Spin manifold $I$ will always be assumed to have a natural $N$-structure on it.

**Definition 1.2.12.** The *negative of a closed complex $N$-Spin manifold* $X$ with an $N$-structure — denoted by $-X$ — is defined to be the manifold $X$ with an $N$-structure induced (to the induced U-structure) from the component $X \times \{1\}$ of $\partial(X \times I)$.

Now we give the definition of $N$-bordism (see [Höh91]). It will be defined analogous to $U$-bordism.

**Definition 1.2.13.** Let $k$ be a positive integer. A closed complex $N$-Spin manifold $M^k$ is said to *bord* if there exists a compact complex $N$-Spin manifold $W^{k+1}$ such that its boundary $\partial W^{k+1}$ is $N$-diffeomorphic to the complex $N$-Spin manifold $M^k$.

**Definition 1.2.14.** Two closed complex $N$-Spin manifolds $M_1^k$ and $M_2^k$ are said to be *bordant* if the disjoint union $M_1^k \,\dot{\cup}\, -M_2^k$ bords.

The relation of bordism is an equivalence relation on the class of closed $k$-dimensional complex $N$-Spin manifolds. The resulting set $\Omega_k^{U,N}$ is an abelian group under addition induced by disjoint union. The weak direct sum $\Omega_*^{U,N} = \sum \Omega_k^{U,N}$ is a graded ring under the multiplication induced by direct product.

**Remark.** ([Höh91], p. 12) Let $M_1^k$ and $M_2^k$ be two complex $N$-Spin manifolds. If $M_1^k$ and $M_2^k$ are bordant as complex $N$-Spin manifolds then they are also bordant as $U$-manifolds. Thus the Chern numbers of a complex $N$-Spin manifold depend only on the $N$-bordism class. The Chern numbers are also independent of the chosen $N$-structure. Let $\rho_*^N : \Omega_*^{U,N} \longrightarrow \Omega_*^U$ be a map which is defined by forgetting the $N$-structure. The map $\rho_*^N$ is a ring homomorphism which is neither surjective nor injective.

**Theorem 1.2.15.** ([Höh91], p. 14) *Let $\rho_*^N$ be the map as defined in the above remark. Then the map*

$$\rho_*^N \otimes \mathbb{Q} : \Omega_*^{U,N} \otimes \mathbb{Q} \longrightarrow \Omega_*^U \otimes \mathbb{Q}$$

*is a ring isomorphism.*

So far in this section we have reviewed the theory of complex $N$-Spin manifolds from [Höh91]. Now we introduce the notion of the $N$-bordism of a pair of spaces and discuss the some of its immediate properties which will be useful in the subsequent chapters.

**Definition 1.2.16.** Let $k$ be a non-negative integer and $(X, A)$ be a pair of spaces. Consider pairs $(M^k, f)$ such that $M^k$ is a compact complex $N$-Spin manifold and $f : (M^k, \partial M^k) \longrightarrow (X, A)$. A pair $(M^k, f)$ *bords* if there exists a compact $(k+1)$-dimensional complex $N$-Spin manifold $W^{k+1}$ and $F : W^{k+1} \longrightarrow X$ such that

(i) $M^k$ is contained in $\partial W^{k+1}$ as a submanifold whose $N$-structure is induced by the $N$-structure of $W^{k+1}$, and

(ii) $F \mid M^k = f$ and $F(\partial W^{k+1} - M^k) \subset A$.

**Definition 1.2.17.** Two pairs $(M_1^k, f_1)$ and $(M_2^k, f_2)$ are said to be *bordant* if the disjoint union $(M_1^k, f_1) \dot\cup (-M_2^k, f_2)$ bords.

The bordism is an equivalence relation and the set of equivalence classes forms an abelian group under addition induce by disjoint union. The bordism class containing $(M^k, f)$ will be denoted by $[M^k, f]_N$ or simply by $[M^k, f]$ when there is no ambiguity and the abelian group of bordism classes will be denoted by $\Omega_k^{U,N}(X, A)$. The weak direct sum $\Omega_*^{U,N}(X, A) = \sum \Omega_k^{U,N}(X, A)$ can be given the structure of a graded $\Omega_*^{U,N}$-module.

**Remark.** It is a routine matter to check that $\Omega_K^{U,N}(\,\cdot\,)$ becomes a generalized homology theory (see for instance [CF64], p. 9). Hence it becomes trivial when tensored with the rationals (see for instance [Dol62]). So we get, by Theorem 1.2.15, the following result about the bordism groups.

**Theorem 1.2.18.** *Let $(X, A)$ be a finite CW-pair. The forgetful homomorphism*

$$\Omega_n^{U,N}(X, A) \otimes \mathbb{Q} \longrightarrow \Omega_n^U(X, A) \otimes \mathbb{Q}$$

*is an isomorphism.*

The next section is devoted to the types of circle action on a complex $N$-Spin manifold.

## 1.3 Circle Actions on Complex $N$-Spin Manifolds

The types of a circle action on a complex $N$-Spin manifold play a key role in defining the ideals needed to characterize the kernels of elliptic genera. The idea of the types of a circle action on a complex $N$-Spin manifold was introduced by G. Höhn [Höh91] which is analogous to the idea of the types of a circle action on a Spin manifold (see for instance, [AH70]). A smooth circle action on a Spin manifold is either of even or odd type while the type $t$ of a circle action on a complex $N$-Spin manifold corresponds to an element $t \in \mathbb{Z}/N\mathbb{Z}$. Höhn [Höh91] extensively used the idea of a type of circle action on a complex $N$-Spin manifold to study the elliptic genera. We put together here few definitions and results related to the types of circle actions from [Höh91]

**Definition 1.3.1.** Let $X^k$ be a connected complex $N$-Spin manifold with an $N$-structure, and let

$$\alpha : S^1 \times X \longrightarrow X$$

be a smooth action of circle $S^1$. We say that the *circle action respects the $U$-structure* if for all $\lambda \in S^1$, the induced map

$$\mathrm{T}\alpha : \mathrm{S}^1 \times \tau(\mathrm{X}) \oplus \epsilon_{\mathbb{R}}^{\mathrm{2n-k}} \longrightarrow \tau(\mathrm{X}) \oplus \epsilon_{\mathbb{R}}^{\mathrm{2n-k}},$$

given by

$$\mathrm{T}\alpha(\lambda) = \mathrm{D}\alpha(\lambda) \oplus \mathrm{id}$$

commutes with the map

$$J : \tau(X) \oplus \epsilon_{\mathbb{R}}^{2n-k} \longrightarrow \tau(X) \oplus \epsilon_{\mathbb{R}}^{2n-k}$$

which defines the $U$-structure.

**Remark.** ([Höh91], p. 16) Let $X^k$ be a connected complex $N$-Spin manifold with a principal $S^1$-bundle $Q$ representing its $N$-structure. Let $P$ be the principal $S^1$-bundle associated to the complex determinant bundle of a stable tangent bundle of $X^k$ representing its $U$-structure. Let $\pi : Q \longrightarrow P$ be an $N$-fold covering map given as in Definition 1.2.1. If $q \in Q$, then any path in $P$ with the initial point $\pi(q)$ has a unique lifting with the initial point $q$.

The type of a smooth circle action $\alpha$ on a connected complex $N$-Spin manifold $X^k$ with principal $S^1$-bundles $P$ and $Q$ given as in the above remark is defined as:

**Definition 1.3.2.** Let $p \in P$, and let

$$\gamma : S^1 \longrightarrow P$$

given by

$$\lambda \longmapsto \det(T\alpha(\lambda))p$$

be a closed path. Let $q$ be an element of $Q$ with $\pi(q) = p$. The *type of circle action* $\alpha$ is defined to be the element $t \in \mathbb{Z}/N\mathbb{Z} \subset S^1$ such that $t\gamma'(0) = \gamma'(1)$, where

$$\gamma' : [0,1] \longrightarrow Q$$

is the unique lifting of $\gamma$ with $\pi \circ \gamma' = \gamma$ and $\gamma'(0) = q$.

**Remark.** The type $t$ of a smooth circle action on a connected complex $N$-Spin manifold is independent of the chosen $q$, but it generally depends on the chosen $N$-structure.

**Remark.** (See [Höh91], p. 17) Let $X^k$ be a compact $U$-manifold with a smooth circle action. Let $x \in X$. The circle acts on the stable tangent space $\tau_x(X^k) \oplus (\epsilon_{\mathbb{R}}^{2n-k})_x$ as in Definition 1.3.1, and there exist integers $m_1, \ldots, m_n$ such that $\lambda \in S^1$ acts by the diagonal matrix $(\lambda^{m_1}, \ldots, \lambda^{m_n})$.

**Definition 1.3.3.** The numbers $m_1, \ldots, m_n$ defined in the above remark are called the *rotation numbers.*

**Remark.** The rotation numbers $m_1, \ldots, m_n$ depend (up to an order) only on the fixed point component.

The following theorem states conditions under which the type of a circle action is independent of a chosen $N$-structure, and the residue class modulo $N$ of the sum of rotation numbers is independent of a chosen fixed point component.

**Theorem 1.3.4.** ([Höh91], p. 17) *Let $X^k$ be a connected complex $N$-Spin manifold with a circle action. If $[X^k] \neq 0$ in $\Omega_*^{U,N} \otimes \mathbb{Q}$, i.e., not all Chern numbers are zero, then the type of the circle action is independent of a chosen $N$-structure and it is equal to the residue class modulo $N$ of the sum of rotation numbers of any fixed point component.*

**Proposition 1.3.5.** ([Höh91], p. 18) *Let $X$ be a complex $N$-Spin manifold. If*

$$\alpha : S^1 \times X \longrightarrow X$$

*such that*

$$(\lambda, x) \longmapsto \alpha(\lambda)x$$

*is a circle action of type $t$, then for $k \in \mathbb{Z}$*

$$\alpha^k : S^1 \times X \longrightarrow X$$

*given by*

$$(\lambda, x) \longmapsto \alpha(\lambda^k)x$$

*is a circle action of type $kt$.*

**Remark.** (See [Höh91], p. 18) Let $\Omega_*^{U,N}(\tilde{S}^1)$ denote the bordism group of complex $N$-Spin manifolds with circle action. One has the decomposition

$$\Omega_*^{U,N}(\tilde{S}^1) = \bigoplus_{t \in \mathbb{Z}/N\mathbb{Z}} \Omega_*^{U,N,t}(\tilde{S}^1)$$

regarding the type of circle action.

**Notation.** We will denote $\Omega_*^{U,N,t}(\tilde{S}^1)$ simply by $\Omega_*^{U,N,t}$.

**Remark.** One has that $\Omega_*^{U,N,*}$ is an $\mathbb{N} \times \mathbb{Z}/N\mathbb{Z}$ bigraded algebra over $\Omega_*^{U,N}$.

**Notation.** The ideal in $\Omega_*^{U,N} \otimes \mathbb{Q}$ generated by the connected complex $N$-Spin manifolds with effective circle action of type $t$ will be denoted by $I_*^{N,t}$.

**Remark.** Let

$$\rho_t : \Omega_*^{U,N,t} \longrightarrow \Omega_*^{U,N}$$

given by

$$[X^k, \alpha] \longmapsto [X^k]$$

be a map which forgets the circle action. One has that

$$I_*^{N,t} = \mathrm{Im}(\rho_{\mathrm{t}} \otimes \mathbb{Q} \mid_{{}^{\mathrm{eff}}\Omega_*^{\mathrm{U,N,t}} \otimes \mathbb{Q}}).$$

**Remark.** Let $[X^k] \neq 0$ in $\Omega_*^{U,N} \otimes \mathbb{Q}$. By Theorem 1.3.4, the type of a circle action on $X^k$ is independent of a chosen $N$-structure on it , therefore, one does not need to specify a certain $N$-structure on $X^k$ while investigating the ideals $I_*^{N,t}$. Moreover, $[X^k] \in I_*^{N,t}$ does not necessarily mean that $X^k$ has an effective circle action of type $t$ but one can find a natural number $m$ and a complex $N$-Spin manifold $Y^k$ with an effective circle action of type $t$ so that $[mX^k] = [Y^k]$.

**Proposition 1.3.6.** ([Höh91], p. 19) *Let $N$ be a positive integer and $t \in \mathbb{Z}/N\mathbb{Z}$. We have*

$$I_*^{N,t} = I_*^{N,\gcd(N,t)}.$$

In the view of the above proposition we will normally consider only those ideals $I_*^{N,t}$ for which an integer representing the class of $t$ is a divisor of $N$.

**Remark.** Let $t$ and $s$ be two positive integers. By an abuse of notation we will also denote by $t$ and $s$ the equivalence classes of the integers $t$ and $s$ in $\mathbb{Z}/N\mathbb{Z}$, respectively. If $t \mid s$, then we have

$$I_*^{N,1} \subset I_*^{N,t} \subset I_*^{N,s} \subset I_*^{N,0}.$$

# Chapter 2

# Complex Elliptic Genera and Multiplicative Sequences

In [Höh91], G. Höhn introduced the notion of the universal complex elliptic genus and showed that the complex elliptic genus of level $N$ factors over the universal complex elliptic genus. He then exploited this relationship to characterize the ideal $I_*^{N,1}$ in terms of the kernels of elliptic genera. We will use similar considerations in this thesis to take the investigation further to study the ideals $I_*^{N,t}$ for different values of $N$ and $t$. For a comprehensive introduction to the theory of genera we refer to F. Hirzebruch [Hir66]. We will here summarize a few definitions and preliminaries pertaining to the elliptic genera which will be important to prove our main results later on.

## 2.1 Complex Genera

Recall from Chapter 1 that the rational bordism ring $\Omega_*^U \otimes \mathbb{Q}$ of $U$-manifolds is isomorphic to the graded polynomial ring $\mathbb{Q}[[\mathbb{CP}_1], [\mathbb{CP}_2], [\mathbb{CP}_3], \ldots]$, where $[\mathbb{CP}_k]$ has weight $2k$.

**Definition 2.1.1.** Let $\Lambda$ be a graded commutative algebra with identity over the rationals $\mathbb{Q}$. A *complex genus* is a graded algebra homomorphism

$$\Omega_*^U \otimes \mathbb{Q} \longrightarrow \Lambda$$

so that $\varphi(1) = 1$ holds.

**Definition 2.1.2.** Let $\varphi$ be a complex genus. The power series

$$g(y) = \sum_{n=0}^{\infty} \frac{\varphi([\mathbb{CP}_n])}{n+1} y^{n+1} \in \Lambda[[y]]$$

is called the *logarithm of the genus* $\varphi$.

**Definition 2.1.3.** Let $\varphi$ be a complex genus, and let $f(x)$ be the inverse function of the logarithm $g(y)$ of $\varphi$, i.e. $f(g(y)) = y$, $g(f(x)) = x$. The power series

$$Q(x) = \frac{x}{f(x)} = 1 + a_1 x + a_2 x^2 + ... \in \Lambda[[x]]$$

is called the *characteristic power series of the genus* $\varphi$.

**Definition 2.1.4.** Let $\varphi$ be a complex genus, and let $Q(x)$ be the characteristic power series of $\varphi$. Let $c_i$ be the $i$-th symmetric function in $x_j$. The sequence of polynomials $K_0 = 1, K_1(c_1)$, $\ldots$, $K_n(c_1, ..., c_n)$ with the polynomial $K_n \in \Lambda[c_1, c_2, \ldots, c_n]$, homogenous of weight $n$ in $c_1, \ldots, c_n$ ($c_i$ is assigned the weight $i$), determined by the equation

$$Q(x_1) \ldots Q(x_n) = \sum_{i=0}^{n} K_i(c_1, \ldots, c_i) + \sum_{i=n+1}^{\infty} K_i(c_1, \ldots, c_n, 0, \ldots, 0)$$

is called the *multiplicative sequence of polynomials of the genus* $\varphi$.

**Remark.** The equation

$$\sum_{i=0}^{\infty} K_i(c_1, ..., c_i) z^i = \sum_{i=0}^{\infty} K_i(c_1', \ldots, c_i') z^i \cdot \sum_{i=0}^{\infty} K_i(c_1'', \ldots, c_i'') z^i$$

holds if the equation

$$(1 + c_1 z + c_2 z^2 + \ldots) = (1 + c_1' z + c_2' z^2 + \ldots)(1 + c_1'' z + c_2'' z^2 + \ldots)$$

holds. The power series $Q(x)$ can be recovered from the polynomials $K_i$ by the equation $Q(x) = \sum_{i=0}^{\infty} K_i(x, 0, ..., 0)$. Moreover, each of the logarithm, the characteristic power series, and the multiplicative sequence of polynomials of $\varphi$ determines the genus $\varphi$ if the coefficients of $\Lambda$ are suitably graded. The genus of a $U$-manifold $X^{2n}$ is

$$\varphi(X^{2n}) = K_n(c_1, c_2, \ldots, c_n)[X^{2n}] \in \Lambda,$$

where $c(X^{2n}) = 1 + c_1 + c_2 + \ldots + c_n \in H^*(X^{2n})$ is the total Chern class of $X^{2n}$. Notice that the genus of $X^{2n}$ is a $\Lambda$-linear combination of the Chern numbers of $X^{2n}$.

Analogous to the genera of $\Omega_*^U \otimes \mathbb{Q}$ with values in an algebra over $\mathbb{Q}$, one can also consider the genera of $\Omega_*^U \otimes \mathbb{C}$ with values in an algebra over $\mathbb{C}$.

**Definition 2.1.5.** Let $a_1, \ldots, a_n \in \mathbb{N}$. The *weighted projective space with weights* $a_1, \ldots, a_n \in \mathbb{N}$ — denoted by $\mathbb{CP}^{a_1,\ldots,a_n}$ — is defined by $(\mathbb{C}^n \setminus \{0\})/\sim$, where $(x_1, \ldots, x_n) \sim (y_1, \ldots, y_n)$ if and only if there exists a $\lambda \in \mathbb{C}^*$ with $y_i = \lambda^{a_i} x_i$ for all $i = 1, \ldots, n$.

Let $V$ (not necessarily irreducible) be a projective variety in $\mathbb{CP}^{a_1,\ldots,a_n}$ with weighted homogeneous annihilator ideal $I(V) \subset \mathbb{C}[x_1, ..., x_n]$, where $x_i$ has weight $a_i$. Let the graded coordinate algebra $\mathbb{C}[x_1, ..., x_n]/I(V)$ of $V$ be denoted by $K(V)$, and let $\pi : \mathbb{C}[x_1, \ldots, x_n] \longrightarrow K(V)$ denote the projection map. Now with these notations one can associate a genus to the variety $V$ as:

**Definition 2.1.6.** Let $[X_1], [X_2], \ldots$ be a basis of $\Omega_*^U \otimes \mathbb{C}$, and let $a_1 < a_2 < \ldots < a_n$. The *genus belonging to the variety* $V$ is a graded algebra homomorphism

$$\varphi_V : \Omega_*^U \otimes \mathbb{C} \to K(V)$$

defined by

$$\varphi_V = \pi \circ \lambda_{a_1,\ldots,a_n},$$

where the map

$$\lambda_{a_1,\ldots,a_n} : \Omega_*^U \otimes \mathbb{C} \to \mathbb{C}[x_1, ..., x_n]$$

is given by

$$[X_i] \longmapsto \begin{cases} x_j & \text{if } i = a_j, \\ 0 & \text{otherwise.} \end{cases}$$

**Remark.** The genus $\varphi_V$ depends upon the choice of a basis sequence and the choice of a coordinate system for $\mathbb{CP}^{a_1,\ldots,a_n}$.

Observe that kernel of $\varphi_V$ is the ideal $\lambda^{-1}_{a_1,\dots,a_n}(I(V))$.

**Proposition 2.1.7.** *Let $V$ and $W$ be two projective varieties in $\mathbb{CP}^{a,\dots,a_n}$ with a fixed coordinate system. Then with a fixed basis sequence we have*

$$\ker \varphi_{V\cup W} = \ker \varphi_V \cap \ker \varphi_W.$$

**Remark.** The Chern numbers of an $N$-manifold specify its rational bordism class in $\Omega^{U,N}_* \otimes \mathbb{Q}$ and in $\Omega^U_* \otimes \mathbb{Q}$, and by Theorem 1.2.15 we have $\Omega^{U,N}_* \otimes \mathbb{Q} \cong \Omega^U_* \otimes \mathbb{Q}$. So one can identify the genera of $\Omega^{U,N}_* \otimes \mathbb{Q}$ and $\Omega^U_* \otimes \mathbb{Q}$ with one another.

## 2.2 The Universal Complex Elliptic Genus and its Multiplicative Sequence

In [Höh91], G. Höhn introduced the universal complex elliptic genus and computed the first few terms of its multiplicative sequence. This section is devoted to present the definitions and preliminaries related to the universal complex elliptic genus which are required to make certain calculations necessary to accomplish the main results.

The universal complex elliptic genus is defined as the genus belonging to the power series which is the solution of a certain second order differential equation. First we state the following result about the differential equations.

**Lemma 2.2.1.** (See Hoehn [Höh91], p. 37) *Let $S(y) = y^4 + q_1 y^3 + q_2 y^2 + q_3 y + q_4$ be a standard polynomial in $y$ with coefficients $q_1$ to $q_4$, and let $Q(x) = 1 + a_1 x + a_2 x^2 + \dots$ be a power series in $x$. If we put $h(x) := \frac{f'(x)}{f(x)}$, the logarithmic derivative of $f(x) = \frac{x}{Q(x)}$, then the second order differential equation*

$$(h'(x))^2 = S(h(x)) \tag{2.1}$$

*has a unique solution $h(x) \in \mathbb{Q}[q_1, q_2, q_3, q_4][[x]][x^{-1}]$. This in turn determines the power series $Q(x) \in \mathbb{Q}[q_1, q_2, q_3, q_4][[x]]$. If $q_1$ to $q_4$ are assigned weights 1 to 4, then the coefficients $a_n$ of $Q(x)$ are homogeneous polynomials of weight $n$ in $q_1$ to $q_4$.*

**Definition 2.2.2.** Let $\Omega_*^U \otimes \mathbb{Q}$ be the rational bordism ring of $U$-manifolds. The *universal complex elliptic genus*

$$\varphi_{ell} : \Omega_*^U \otimes \mathbb{Q} \to \mathbb{Q}[A, B, C, D]$$

is the complex elliptic genus having the characteristic power series $Q(x)$ belonging to the solution of the differential equation (2.1).

**Remark.** The indeterminates $q_1$ to $q_4$ become the homogeneous polynomials in $A, B, C$, and $D$ as follows (the indeterminates $A$ to $D$ are provided with the weights 1 to 4):

$$q_1 = 2A, \quad q_2 = \frac{3}{2}A^2 - \frac{1}{4}B, \quad q_3 = \frac{1}{2}A^3 - \frac{1}{4}AB + 4C, \tag{2.2}$$
$$q_4 = \frac{1}{16}A^4 - \frac{1}{16}A^2B + 2AC + \frac{1}{64}B^2 - 2D.$$

Conversely, if $q_1$ to $q_4$ are given, then one can determine $A$ to $D$ as follows:

$$A = \frac{1}{2}q_1, \quad B = \frac{3}{2}q_1^2 - 4q_2, \quad C = \frac{1}{32}q_1^3 - \frac{1}{8}q_1q_2 + \frac{1}{4}q_3, \tag{2.3}$$
$$D = \frac{3}{128}q_1^4 - \frac{1}{8}q_1^2q_2 + \frac{1}{8}q_1q_3 + \frac{1}{8}q_2^2 - \frac{1}{2}q_4.$$

**Remark.** By Lemma 2.2.1 the coefficients of $Q(x)$ are homogeneous in $q_1$ to $q_4$, and so by the above remark they are also homogeneous in $A, B, C$, and $D$. For first few coefficients of the power series we have

$$\begin{aligned}
a_1 &= \frac{1}{2}A,\\
a_2 &= \frac{1}{2^4 \cdot 3}(6A^2 - B),\\
a_3 &= \frac{1}{2^5 \cdot 3}(2A^3 - AB + 16C),\\
a_4 &= \frac{1}{2^9 \cdot 3^2 \cdot 5}(60A^4 - 60A^2B + 1920AC + 7B^2 - 1152D),\\
a_5 &= \frac{1}{2^{10} \cdot 3^2 \cdot 5}(12A^5 - 20A^3B + 960A^2C + 7AB^2 - 1152AD + 32CB).
\end{aligned}$$

**Remark.** The value of $\varphi_{ell}$ on a $U$-manifold $X^{2n}$ is a homogeneous polynomial of weight $n$ in $A, B, C$, and $D$. For first few polynomials of the associated multiplicative sequence we have

$$K_1 = \frac{1}{2}Ac_1,$$

$$K_2 = \frac{1}{2^4 \cdot 3}(2Bc_2 + (6A^2 - B)c_1^2),$$

$$K_3 = \frac{1}{2^5 \cdot 3}(48Cc_3 + (2AB - 48C)c_2c_1 + (2A^3 - AB + 16C)c_1^3),$$

$$\begin{aligned} K_4 = {} & \frac{1}{2^9 \cdot 3^2 \cdot 5}((-8B^2 + 4608D)c_4 + (5760AC + 8B^2 - 4608D)c_3c_1 + (24B^2 - 2304D)c_2^2 + \\ & (120A^2B - 5760AC - 28B^2 + 4608D)c_1^2c_2 + (60A^4 - 60A^2B + 1920AC + 7B^2 - 1152D)c_1^4), \end{aligned}$$

$$\begin{aligned} K_5 = {} & \frac{1}{2^{10} \cdot 3^2 \cdot 5}(960BCc_5 + (-8AB^2 + 4608AD - 960BC)c_4c_1 + \\ & (8AB^2 + 2880A^2C - 4608AD + 480BC)c_3c_1^2 + (24AB^2 - 2304AD)c_2^2c_1 + \\ & (40A^3B - 2880A^2C - 28AB^2 + 4608AD - 160BC)c_2c_1^3 + \\ & (12A^5 - 20A^3B + 960A^2C + 7AB^2 - 1152AD + 32BC)c_1^5). \end{aligned}$$

The special choice of the polynomials $q_1$ to $q_4$ in $A$ to $D$ determines itself by the values of $\varphi_{ell}$ on the manifolds $W_1$ to $W_4$.

G. Höhn in [Höh91] constructed a basis sequence $[W_1], [W_2], [W_3], [W_4], [W_5], \ldots$ of $\Omega_*^U \otimes \mathbb{Q}$. The values of the universal complex elliptic genus on this basis sequence are given in the following result.

**Theorem 2.2.3.** (See [Höh91], p. 40) *The universal elliptic genus on the elements of the basis sequence $[W_1], [W_2], [W_3], [W_4], [W_5], \ldots$ of $\Omega_*^U \otimes \mathbb{Q}$ takes the values: $\varphi_{ell}([W_1]) = A$, $\varphi_{ell}([W_2]) = B$, $\varphi_{ell}([W_3]) = C$, $\varphi_{ell}([W_4]) = D$, and $\varphi_{ell}([W_n]) = 0$ for all $n \geq 5$.*

Observe that we have

$$\ker \varphi_{ell} = \langle [W_5], [W_6], \ldots \rangle.$$

**Remark.** Let $\mathbb{C}[A,B,C,D]$ be the graded coordinate algebra of the twisted projective space $\mathbb{CP}^{1,2,3,4}$. The universal complex elliptic genus can be regarded as the genus $\varphi_{ell}$ : $\Omega^U_* \otimes \mathbb{C} \to \mathbb{C}[A,B,C,D]$ belonging to the weighted projective variety $\mathbb{CP}^{1,2,3,4}$ and the basis sequence $[W_1],[W_2],[W_3],[W_4],[W_5],[V_6], \ldots,$ where $[V_n]$ are the bordism classes of manifolds from $\ker \varphi_{ell} \mid \Omega^U_n \otimes \mathbb{C}$ such that $s_n([V_n]) \neq 0$, $n \geq 6$. Such bordism classes of manifolds exist because if $[V_n']$ is in $\Omega^U_n \otimes \mathbb{C}$ with $s_n([V_n']) \neq 0$, then $\varphi([V_n'])$ is a polynomial $P([W_1],[W_2],[W_3],[W_4])$ of weight $n$ in $A$ to $D$. Put $[V_n] := [V_n'] - P([W_1],[W_2],[W_3],[W_4])$, then $\varphi_{ell}([V_n]) = 0$ and $s_n([V_n]) = s_n([V_n']) \neq 0$, because for $n \geq 5$, $P([W_1],[W_2],[W_3],[W_4])$ is the sum of the different powers of the bordism classes of manifolds.

## 2.3 The Elliptic Genera of Level $N$

This section is devoted to set up the notations and preliminaries to work with the elliptic genera $\varphi_N$ of level $N$. Before getting to the definition of these genera we need to discuss the following remarks whose proofs can be found in [Jun89].

**Remark.** Let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im(z)} > 0\}$ be the complex upper half plane, and let $\tau \in \mathbb{H}$. Consider the lattice $L = 2\pi i(\mathbb{Z}\tau + \mathbb{Z})$ in $\mathbb{C}$, and let $\alpha \in \mathbb{C}/L$ be a non-zero $N$-division point of the associated elliptic curve. There is a unique elliptic function $h(x)$ of $L$ with the divisor $N \cdot (0) - N \cdot (\alpha)$ and the normalization $h(x) = x^N + O(x^{N+1})$ of the Taylor expansion around the origin. The function $f(x) = \sqrt[N]{h(x)}$ is well-defined if $f(x) = x + O(x^2)$.

In the notations of the above remark a complex elliptic genus of level $N$ is defined as follows.

**Definition 2.3.1.** The *complex elliptic genus of level $N$* — denoted by $\varphi_N$ — is the complex elliptic genus having the characteristic power series $Q(x) = \frac{x}{f(x)} = 1 + b_1 x + b_2 x^2 + \cdots$, where

$f(x)$ is as in the above remark the function belonging to the lattice $L = 2\pi i(\mathbb{Z}\tau + \mathbb{Z})$ and the primitive $N$-division point $\frac{2\pi i}{N}$.

The coefficient $b_k$ of the power series $Q(x)$ is a modular form of weight $k$ and the value of $\varphi_N$ on a $U$-manifold $X^{2n}$ is then modular form of weight $n$ to the modular group $\Gamma_1(N)$, where the modular subgroup

$$\Gamma_1(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \pmod{N}\}$$

of $SL_2(\mathbb{Z})$ acts on $\mathbb{H}$ under the action

$$\sigma : SL_2(\mathbb{Z}) \times \mathbb{H} \to \mathbb{H}$$

given by

$$(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau) \mapsto \frac{a\tau + b}{c\tau + d}.$$

The the orbit space of this action is denoted by $\mathbb{H}/\Gamma_1(N)$.

**Remark.** (See [Höh91], p. 43) The function $f(x)$ in the definition of $\varphi_N$ is determined by either of the following two differential equations:

(i) The first differential equation is of order 2 and has the form

$$\left(\frac{f'}{f}\right)'^{\,2} = S\left(\frac{f'}{f}\right), \tag{2.4}$$

where $S(y) = y^4 + q_1 y^3 + q_2 y^2 + q_3 y + q_4$ is a polynomial of degree 4, and the coefficients $q_i$ are modular forms of weight $i$ to $\Gamma_1(N)$. Since the power series $f(x)$ to $\varphi_N$ satisfies the differential equation (2.1) with special $q_i$, $\varphi_N$ factorizes over $\varphi_{ell}$.

(ii) The second differential equation is of order 2 and has the form

$$\frac{1}{f^N} + d_{2N} f^N = T_N\left(\frac{f'}{f}\right), \tag{2.5}$$

where $T_N(y) = y^N + d_1 y^{N-1} + ... + d_{N-1} y + d_N$ is a polynomial of degree $N$ and $d_i$ are the modular forms of weights $i$ to $\Gamma_1(N)$.

The polynomial $T_N(y)$ in the above remark is a so-called Zolotarev-polynomial. It is characterized by the equations $d_{N-1} = 0$, and $T_N^2(y) = 4d_{2N}$ for $y \neq 0$ with $T_N'(y) = 0$, and another equation which gives a relationship between $S(y)$ and $T_N(y)$ given by

$$S(y)T_N'(y)^2 = N^2 y^2 (T_N(y)^2 - 4d_{2N}). \tag{2.6}$$

This differential equation gives rise to two (depending on $N$) weighted homogeneous relations $R_{N-1}$ and $R_{N+1}$ to the coefficients $q_1$ to $q_4$ of $S(y)$ or in $A$ to $D$.

**Notation.** We will denote the variety $V(\langle R_{N-1}, R_{N+1}\rangle) \subset \mathbb{CP}^{1,2,3,4}$ by $C_N$.

**Theorem 2.3.2** ([Jun89], Chapter 3)**.** *The mapping*

$$\Phi : \dot{\bigcup_{\substack{n \mid N \\ n>1}}} \overline{\mathbb{H}/\Gamma_1(n)} \to \mathbb{CP}^{1,2,3,4},$$

*which for each $n|N$, $n > 1$ assigns to a pair $(\tau, n)$ the point $(q_1{:}q_2{:}q_3{:}q_4)$, where $q_1$ to $q_4$ are the coefficients of the polynomial $S(y)$ of the differential equation (2.4), is surjective to the curve $C_N$. The images $\Phi(\overline{\mathbb{H}/\Gamma_1(n)})$ from a decomposition of $C_N$ into irreducible components. The image $\Phi(\overline{\mathbb{H}/\Gamma_1(n)})$ is independent of $N$.*

**Notation.** We will denote the projection $\mathbb{C}[A, B, C, D] \longrightarrow \mathbb{C}[A, B, C, D]/I(C_N)$ by $\pi_N$.

The kernel of $\pi_N$ is the annihilator ideal $I(C_N)$ of the curve $C_N$ and therefore by Hilbert Nullstellensatz the radical of $\langle R_{N-1}, R_{N+1}\rangle$.

**Theorem 2.3.3** (Höhn [Höh91])**.** *For the annihilator ideal of the curve $C_N$, we have*

$$I(C_N) = \mathrm{Rad}(\langle R_{N-1}, R_{N+1}\rangle) = \langle R_{N-1}, R_{N+1}\rangle.$$

From now on we will denote the ideal $I(C_N)$ simply by $I_N$. For first few ideals $I_N$ we have[1]

$$
\begin{aligned}
I_2 &= \langle A,\, C\rangle,\\
I_3 &= \langle 125\,B^2 + 1152\,AC - 384\,D,\, 18\,A^2 - B\rangle,\\
I_4 &= \langle 125\,B^2C + 768\,AC^2 - 384\,CD,\, 125\,AB^2 + 768\,A^2C - 384\,AD,\, 8\,A^3 - AB + 4\,C\rangle,\\
I_5 &= \langle 15625\,B^4 + 480000\,AB^2C + 1843200\,A^2C^2 + 36864\,BC^2 - 96000\,B^2D - 1474560\,ACD +\\
&\quad 147456\,D^2,\, 6250\,A^2B^2 - 125\,B^3 + 32000\,A^3C - 640\,ABC + 4096\,C^2 - 19200\,A^2D +\\
&\quad 384\,BD,\, 1250\,A^4 - 250\,A^2B + 67\,B^2 + 1600\,AC - 192\,D\rangle,\\
I_6 &= \langle 15625\,B^4C + 288000\,AB^2C^2 + 884736\,A^2C^3 + 24576\,BC^3 - 96000\,B^2CD -\\
&\quad 884736\,AC^2D + 147456\,CD^2,\, 15625\,AB^4 + 288000\,A^2B^2C + 884736\,A^3C^2 +\\
&\quad 24576\,ABC^2 - 96000\,AB^2D - 884736\,A^2CD + 147456\,AD^2,\, 2250\,A^3B^2 - 125\,AB^3 +\\
&\quad 10368\,A^4C - 576\,A^2BC + 500\,B^2C + 4608\,AC^2 - 6912\,A^3D + 384\,ABD -\\
&\quad 1536\,CD,\, 2592\,A^5 - 720\,A^3B + 407\,AB^2 + 5760\,A^2C - 128\,BC - 1152\,AD\rangle.
\end{aligned}
$$

The ideal $I_N$ has a primary decomposition

$$I_N = \bigcap_{\substack{n>1\\ n\mid N}} P_n$$

where the ideal $P_n$ corresponds to the irreducible component $\Phi(\overline{\mathbb{H}/\Gamma_1(n)})$ of the curve $C_N$ as in the Theorem 2.3.2. The ideal $P_n$ is independent of the multiples $N$ of $n$. For first few

[1]The computations were accomplished with the help of computer programs in *Mathematica* and MAGMA.

ideals $P_n$ we have [2]

$$P_2 = I_2,$$

$$P_3 = I_3,$$

$$P_4 = \langle 2\,AC - D, 8\,A^3 - AB + 4\,C\rangle,$$

$$P_5 = I_5,$$

$$\begin{aligned} P_6 = {} & \langle 108000\,A^4BC + 15625\,B^3C + 129024\,A^3C^2 + 235904\,ABC^2 + 40960\,C^3 - 12000\,AB^2D - \\ & 129024\,A^2CD - 48000\,BCD + 36864\,AD^2,\ 15625\,B^4 + 288000\,AB^2C + 884736\,A^2C^2 + \\ & 24576\,BC^2 - 96000\,B^2D - 884736\,ACD + 147456\,D^2,\ 125\,AB^3 - 3456\,A^4C + \\ & 1344\,A^2BC - 500\,B^2C - 7680\,AC^2 - 384\,ABD + 1536\,CD,\ A^2B^2 + 384/125A^3C + \\ & 128\,ABC - 512\,C^2 - 1152\,A^2D,\ 2592\,A^5 - 720\,A^3B + 407\,AB^2 + 5760\,A^2C - \\ & 128\,BC - 1152\,AD\rangle. \end{aligned}$$

**Definition 2.3.4.** For $t \mid N$, we define the ideal $I_{N,t}$ as the following intersection of ideals

$$I_{N,t} = \bigcap_{\substack{n \mid N \\ n \nmid t}} P_n.$$

[2]The computations were accomplished with the help of a computer program in MAGMA.

For first few ideals $I_{N,t}$ we have[3]

$$
\begin{aligned}
I_{2,1} &= I_2 = P_2,\\
I_{3,1} &= I_3 = P_3,\\
I_{4,1} &= I_4 = P_2 \cap P_4,\\
I_{4,2} &= P_4,\\
I_{5,1} &= I_5 = P_5,\\
I_{6,1} &= I_6,\\
I_{6,2} &= P_3 \cap P_6 = \langle 15625\, B^4 + 288000\, AB^2C + 884736\, A^2C^2 + 24576\, BC^2 - 96000\, B^2D - \\
&\quad 884736\, ACD + 147456\, D^2,\ 2250\, A^3B^2 - 125\, AB^3 + 10368\, A^4C - 576\, A^2BC + 500\, B^2C \\
&\quad +4608\, AC^2 - 6912\, A^3D + 64/375\, ABD - 1536\, CD,\ 2592\, A^5 - 720\, A^3B + 407\, AB^2 + \\
&\quad 5760\, A^2C - 128\, BC - 1152\, AD\rangle,\\
I_{6,3} &= P_2 \cap P_6 = \langle 15625\, B^4C + 288000\, AB^2C^2 + 884736\, A^2C^3 + 24576\, BC^3 - 96000\, B^2CD - \\
&\quad 884736\, AC^2D + 147456\, CD^2,\ 108000\, A^4BC + 15625\, B^3C + 129024\, A^3C^2 + 235904\, ABC^2 \\
&\quad +40960\, C^3 - 12000\, AB^2D - 129024\, A^2CD - 48000\, BCD + 36864\, AD^2,\ 375\, AB^3 - 10368\, A^4C + \\
&\quad 4032\, A^2BC - 1500\, B^2C - 23040\, AC^2 - 1152\, ABD + 4608\, CD,\ 375\, A^2B^2 + 1152\, A^3C + \\
&\quad 128\, ABC - 512\, C^2 - 1152\, A^2D,\ 2592\, A^5 - 720\, A^3B + 407\, AB^2 + 5760\, A^2C - 128\, BC \\
&\quad -1152\, AD\rangle.
\end{aligned}
$$

[3]The computations were accomplished with the help of a computer program in MAGMA.

**Definition 2.3.5.** Let $[W_1], [W_2], [W_3], [W_4], [V_5], [V_6], \ldots$ be a basis sequence of $\Omega_*^U \otimes \mathbb{C}$ with $\langle [V_5], [V_6], \ldots \rangle = \ker \varphi_{ell}$ and $[W1]$ to $[W_4]$ have the same meaning as in Theorem 2.2.3. We define by $\psi_{N,t}$ the genus to the variety $C_{N,t} = V(I_{N,t}) \subset \mathbb{CP}^{1,2,3,4}$.

The definition is independent of the chosen basis manifolds from $\ker_{ell}$.

The following situation exists in $A, B, C, D$-coordinate system for $\mathbb{CP}^{1,2,3,4}$

$$\begin{tikzcd}
\Omega_*^U \otimes \mathbb{C} \arrow[r, "\varphi_{ell}"] \arrow[dr, "\psi_{N,t}"'] & \mathbb{C}[A,B,C,D] \arrow[d, "\pi_{N,t}"] \\
& K(C_{N,t}) \cong \mathbb{C}[A,B,C,D]/I(C_{N,t})
\end{tikzcd}$$

that is,

$$\psi_{N,t} = \pi_{N,t} \circ \varphi_{ell}. \tag{2.7}$$

**Remarks.** Let $N \geq 2$ be an integer.

1. The ideal $I_{N,1}$ coincides with the ideal $I_N$, and the genus $\psi_{N,1}$ coincides with the genus — say $\psi_N$ — to the variety $C_N = V(I_N)$.

2. The complex elliptic genus $\varphi_N$ of level $N$ coincides with the genus to the variety $D_N = V(P_N)$.

3. If $N$ is a prime number, then the ideals $I_{N,1}, I_N$ and $P_N$ coincide with each other, and so do the genera $\psi_{N,1}, \psi_N$ and $\varphi_N$.

The following lemma gives a precise relationship between $\psi_{N,t}$ and $\varphi_N$, in general.

**Lemma 2.3.6.** *We have*

$$\psi_{N,t} = \bigcap_{\substack{n \mid N \\ n \nmid t}} \varphi_n.$$

*Proof.* By Proposition 2.1.7 and Theorem 2.3.2, we have $\ker \psi_{N,t} = \ker \varphi_{C_{N,t}} = \bigcap_{\substack{n \mid N \\ n \nmid t}} \varphi_{\Phi(\overline{\mathbb{H}/\Gamma_1(n)})}$.

Now $[X] \in \ker \varphi_{\Phi(\overline{\mathbb{H}/\Gamma_1(n)})} \Leftrightarrow \varphi_{ell}([X]) \in I(\Phi(\overline{\mathbb{H}/\Gamma_1(n)}))$. The point $P \in \Phi(\overline{\mathbb{H}/\Gamma_1(n)})$ corresponding to $\tau \in \overline{\mathbb{H}/\Gamma_1(n)}$ has the parametrization: $P(\tau) = (A(\tau){:}B(\tau){:}C(\tau){:}D(\tau))$. So $\varphi_{ell}([X])(P(\tau)) = \varphi_n([X])(\tau)$, if $\varphi_{ell}([X]) \in I(\Phi(\overline{\mathbb{H}/\Gamma_1(n)})) \Leftrightarrow \varphi_n([X]) = 0$. □

**Theorem 2.3.7.** (Hirzebruch [Hir98], p. 58) *Let $X$ be an $N$-manifold with $S^1$-action. If the type $t$ of circle action is $\not\equiv 0 \pmod N$, then we have $\varphi_N(X) = 0$.*

In [Hir98], the theorem is formulated only for complex manifolds, however, it is pointed out in introduction that it remains true for the $U$-manifolds with equivariant $S^1$-actions.

Now we have the following important result which is an immediate consequence of the above theorem.

**Corollary 2.3.8.** *The ideals $I_*^{N,t}$ with $t \not\equiv 0 \pmod N$ satisfy the inclusion*

$$I_*^{N,t} \subset \bigcap_{\substack{n \mid N \\ n \nmid t}} \ker \varphi_n.$$

*Proof.* For $n \mid N$, an $N$-manifold $X$ with a circle action of type $t \not\equiv 0 \pmod N$ is also an $n$-manifold with a circle action of the type $t \not\equiv 0 \pmod n$ if $n \nmid t$. So we have $[X] \in \bigcap_{\substack{n \mid N \\ n \nmid t}} \ker \varphi_n$ by Theorem 2.3.7. □

**Conjecture** (Höhn [Höh91], p. 55). *For all pairs $(N,t)$ with $N \geq 2$ and $t \not\equiv 0 \pmod N$, we have*

$$I_*^{N,t} = \bigcap_{\substack{n \mid N \\ n \nmid t}} \ker \varphi_n.$$

We find here a condition under which the conjecture is true:

**Proposition 2.3.9.** *If there exist bordism classes* $[V_1]$, $[V_2]$, $\ldots$, $[V_m] \in I_*^{N,t}$ *such that*

$$I_{N,t} \subset \langle \varphi_{ell}[V_1], \varphi_{ell}[V_2], \ldots, \varphi_{ell}[V_m] \rangle.$$

*Then we have*

$$I_*^{N,t} = \bigcap_{\substack{n \mid N \\ n \nmid t}} \ker \varphi_n.$$

*Proof.* If the sequence $[W_5], [W_6], \ldots$ have the same meaning as in Theorem 2.2.3, then $\ker \varphi_{ell} = \langle [W_5], [W_6], \ldots \rangle$. It is also shown in [Höh91] that $\langle [W_5], [W_6], \ldots \rangle \subset I_*^{N,1} \subset I_*^{N,t}$ for all $t \neq 0$. So we get $\langle [V_1], [V_2], \ldots [V_m], [W_5], [W_6], \ldots \rangle \subset I_*^{N,t}$. By Corollary 2.3.8 and Lemma 2.3.6 we have

$$I_*^{N,t} \subset \bigcap_{\substack{n \mid N \\ n \nmid t}} \ker \varphi_n = \ker(\psi_{N,t}).$$

Hence we get

$$\langle [V_1], [V_2], \ldots [V_m], [W_5], [W_6], \ldots \rangle \subset I_*^{N,t} \subset \bigcap_{\substack{n \mid N \\ n \nmid t}} \ker \varphi_n = \ker(\psi_{N,t}).$$

Now it remains to show that $\ker(\psi_{N,t}) \subset \langle [V_1], [V_2], \ldots [V_m], [W_5], [W_6], \ldots \rangle$. Let $[V]$ be a bordism class in $\ker(\psi_{N,t})$. Since $\psi_{N,t} = \pi_{N,t} \circ \varphi_{ell}$, we have $\varphi_{ell}([V]) \in \ker(\pi_{N,t})$, and hence $\varphi_{ell}([V]) \in I_{N,t}$, and so $\varphi_{ell}([V]) \in \langle \varphi_{ell}[V_1], \varphi_{ell}[V_2], \ldots \varphi_{ell}[V_m] \rangle$, by hypothesis. There exist scalars $a_i$ and non-negative integers $n$ and $n_{ij}$ such that $\varphi_{ell}([V]) = \sum_{i=1}^{n} a_i \prod_{j=1}^{m} (\varphi_{ell}[V_j])^{n_{ij}} = \varphi_{ell}(\sum_{i=1}^{n} a_i \prod_{j=1}^{m} [V_j]^{n_{ij}})$ which implies $[V] - \sum_{i=1}^{n} a_i \prod_{j=1}^{m} [V_j]^{n_{ij}} \in \ker(\varphi_{ell}) = \langle [W_5], [W_6], \ldots \rangle$. Hence $[V] \in \sum_{i=1}^{n} a_i \prod_{j=1}^{m} [V_j]^{n_{ij}} + \langle [W_5], [W_6], \ldots \rangle \subset \langle [V_1], [V_2], \ldots, [V_m] + [W_5], [W_6], \ldots \rangle$. This completes the proof. $\square$

In [Höh91], Höhn showed that the conjecture is true in the case $t \equiv 1 \pmod{N}$.

**Theorem 2.3.10** (Höhn [Höh91]). *We have*

$$I_*^{N,1} = \bigcap_{\substack{n \mid N \\ n > 1}} \ker \varphi_n.$$

In Chapter 4 we will verify the above conjecture in several additional cases of pairs $(N, t)$ — by constructing bordism classes satisfying the condition of Proposition 2.3.9 — after we introduce certain notions and prove our main results related to bordism theory in Chapter 3 necessary to work with our proofs of those cases of the conjecture.

# Chapter 3

# Bordism of Semifree Circle Actions

This chapter is devoted to the study of smooth circle actions on smooth manifolds from the bordism theory viewpoint. We will use the machinery developed in this chapter in the next chapter to produce the bordism classes of certain $N$-manifolds with semifree circle actions of certain types.

## 3.1 Families of Subgroups of the Circle

In what follows, $\mathcal{F}$ and $\mathcal{G}$ will denote two families of the subgroups of the circle $S^1$ such that $\mathcal{G} \subset \mathcal{F}$. The $U$-bordism group $\Omega_n^U(S^1, \mathcal{F}, \mathcal{G})$ of all $U$-structure preserving $(\mathcal{F}, \mathcal{G})$-free smooth $S^1$ actions on $n$-dimensional compact $U$-manifolds is defined by A. Hattori and H. Taniguchi in [HT72]. For the sake of completeness we will include here the definition of this $U$-bordism group. First we start with the following definition.

**Definition 3.1.1.** Let $M$ be a compact $U$-manifold. A $U$-structure preserving smooth circle action on $M$ is called $(\mathcal{F}, \mathcal{G})$-*free* if

(i) the action is effective on the each component of $M$,

(ii) the isotropy group at each point $x \in M$ belongs to $\mathcal{F}$, and

(iii) the isotropy group at each point $x \in \partial M$ belongs to $\mathcal{G}$.

**Remark.** If $\mathcal{G} = \emptyset$, then necessarily $\partial M = \emptyset$, and $M$ is a closed manifold.

**Definition 3.1.2.** An $(\mathcal{F}, \emptyset)$-free circle action on a closed $U$-manifold is called *$\mathcal{F}$-free.*

**Definition 3.1.3.** A manifold $M^n$ with an $(\mathcal{F}, \mathcal{G})$-free circle action is said to *bord* if there exists manifold $W^{n+1}$ with an $(\mathcal{F}, \mathcal{G})$-free circle action such that

(i) $M^n$ is equivariantly $U$-diffeomorphic to an equivariant $U$-submanifold $M_1^n$ of $\partial W^{n+1}$, and

(ii) the isotropy group at each $x \in \partial W^{n+1} \setminus M_1^n$ belongs to $\mathcal{G}$.

**Definition 3.1.4.** Two manifolds $M_1^n$ and $M_2^n$ with $(\mathcal{F}, \mathcal{G})$-free circle actions are said to be *bordant* if the disjoint union $M_1^n \,\dot{\cup}\, -M_2^n$ bords.

This is an equivalence relation and the set of equivalence classes forms an abelian group under addition induced by disjoint union. The bordism class containing $M^n$ with a circle action $\alpha$ will be denoted by $[M^n, \alpha]$ or simply by $[M^n]$, and the abelian group of bordism classes will be denoted by $\Omega_n^U(\mathcal{F}, \mathcal{G})$.

**Notation.** When $\mathcal{G} = \emptyset$ then $\Omega_n^U(\mathcal{F}, \mathcal{G})$ will be simply denoted by $\Omega_n^U(\mathcal{F})$.

**Remarks.** Let $\mathcal{F}$ and $\mathcal{G}$ be two families of the subgroups of circle such that $\mathcal{G} \subset \mathcal{F}$.

1. A $\mathcal{G}$-free circle action $M$ is also $\mathcal{F}$-free and there is a homomorphism

$$i_U : \Omega_n^U(\mathcal{G}) \longrightarrow \Omega_n^U(\mathcal{F}), \quad [M] \longmapsto [M].$$

2. An $\mathcal{F}$-free circle action $M$ is also $(\mathcal{F}, \mathcal{G})$-free and so there is a homomorphism

$$j_U : \Omega_n^U(\mathcal{F}) \longrightarrow \Omega_n^U(\mathcal{F}, \mathcal{G}), \quad [M] \longmapsto [M].$$

3. The boundary $\partial M$ of an $(\mathcal{F}, \mathcal{G})$-free circle action $M$ is $\mathcal{G}$-free and there is a homomorphism

$$\partial_U : \Omega_n^U(\mathcal{F}, \mathcal{G}) \longrightarrow \Omega_n^U(\mathcal{G}), \quad [M] \longmapsto [\partial M].$$

The following result is noted by A. Hattori and H. Taniguchi in [HT72].

**Theorem 3.1.5.** *Let $\mathcal{F}$ and $\mathcal{G}$ be two families of the subgroups of the circle such that $\mathcal{G} \subset \mathcal{F}$. The sequence*

$$\cdots \longrightarrow \Omega_n^U(\mathcal{G}) \xrightarrow{i_U} \Omega_n^U(\mathcal{F}) \xrightarrow{j_U} \Omega_n^U(\mathcal{F}, \mathcal{G}) \xrightarrow{\partial_U} \Omega_{n-1}^U(\mathcal{G}) \longrightarrow \cdots$$

*is exact.*

Let $M$ be a compact complex $N$-Spin manifold. We will say that a smooth circle action on $M$ preserving its underlying $U$-structure is $(\mathcal{F}, \mathcal{G})$-free if it is so on the underlying $U$-manifold. In the following definition whenever we consider a complex $N$-Spin manifold as a $U$-manifold, it will be with respect to the map forgetting the underlying complex $N$-Spin structure.

**Definition 3.1.6.** An $(\mathcal{F}, \mathcal{G})$-free circle action on $M^n$ is said to *bord* if there exists a complex $N$-Spin manifold $W^{n+1}$ with a $(\mathcal{F}, \mathcal{G})$-free circle action and a complex $N$-Spin submanifold $M_1^n$ of $\partial W^{n+1}$ such that

(i) $M_1^n$ is an equivariant $U$-submanifold $M_1^n$ of $\partial W^{n+1}$ and $M^n$ is equivariantly $U$-diffeomorphic to $M_1^n$,

(ii) $M^n$ is $N$-diffeomorphic $M_1^n$,

(iii) the isotropy group at each $x \in \partial W^{n+1} \setminus M_1^n$ belongs to $\mathcal{G}$.

Now the complex $N$-Spin bordism group (or simply $N$-bordism group) $\Omega_n^{U,N}(\mathcal{F}, \mathcal{G})$ of all $(\mathcal{F}, \mathcal{G})$-free circle actions on compact $n$-dimensional complex $N$-Spin manifolds can be defined along the lines of definition of the group $\Omega_n^U(\mathcal{F}, \mathcal{G})$. We have the following result analogous to the above theorem.

**Theorem 3.1.7.** *Let $\mathcal{F}$ and $\mathcal{G}$ be two families of the subgroups of the circle such that $\mathcal{G} \subset \mathcal{F}$. The sequence*

$$\cdots \longrightarrow \Omega_n^{U,N}(\mathcal{G}) \xrightarrow{i_N} \Omega_n^{U,N}(\mathcal{F}) \xrightarrow{j_N} \Omega_n^{U,N}(\mathcal{F}, \mathcal{G}) \xrightarrow{\partial_N} \Omega_{n-1}^{U,N}(\mathcal{G}) \longrightarrow \cdots$$

*is exact, where the maps $i_N$, $j_N$, and $\partial_N$ have the same meanings as in Theorem 3.1.5.*

Recall from Chapter 1 that the type of a circle action on a connected $N$-manifold is an element $t$ of the group $\mathbb{Z}/N\mathbb{Z}$. The group $\Omega_n^{U,N}(\mathcal{F},\mathcal{G})$ decomposes as

$$\Omega_n^{U,N}(\mathcal{F},\mathcal{G}) = \bigoplus_{t\in\mathbb{Z}/N\mathbb{Z}} \Omega_n^{U,N,t}(\mathcal{F},\mathcal{G}),$$

where the group $\Omega_n^{U,N,t}(\mathcal{F},\mathcal{G})$ is defined by requiring the circle action to be of type $t$. Furthermore, each map in Theorem 3.1.7 preserves the type $t$ of the circle action and we have the following result.

**Corollary 3.1.8.** *Let $\mathcal{F}$ and $\mathcal{G}$ be two families of the subgroups of the circle such that $\mathcal{G}\subset\mathcal{F}$. The sequence*

$$\cdots \longrightarrow \Omega_n^{U,N,t}(\mathcal{G}) \xrightarrow{i_{N,t}} \Omega_n^{U,N,t}(\mathcal{F}) \xrightarrow{j_{N,t}} \Omega_n^{U,N,t}(\mathcal{F},\mathcal{G}) \xrightarrow{\partial_{N,t}} \Omega_{n-1}^{U,N,t}(\mathcal{G}) \longrightarrow \cdots$$

*is exact, where the maps $i_{N,t}$, $j_{N,t}$, and $\partial_{N,t}$ are the restrictions to the groups $\Omega_n^{U,N,t}(\mathcal{G})$, $\Omega_n^{U,N,t}(\mathcal{F})$, and $\Omega_n^{U,N,t}(\mathcal{F},\mathcal{G})$ of the maps $i_N$, $j_N$, and $\partial_N$, respectively.*

In this thesis, we will mainly focus our attention on the semifree circle actions, i.e. $\mathcal{F}$-free actions for which $\mathcal{F}=\{S^1,1\}$, where 1 denotes the identity subgroup of the group $S^1$. We will use the following notations subsequently.

**Notations.** Let $\mathcal{F}=\{S^1,1\}$ and $\mathcal{G}=\{1\}$.

1. $\Omega_n^U(F) := \Omega_n^U(\mathcal{G})$ denotes the bordism group of all closed $n$-dimensional $U$-manifolds with free circle actions.

2. $\Omega_n^U(SF) := \Omega_n^U(\mathcal{F})$ denotes the bordism group of all closed $n$-dimensional $U$-manifolds with of semifree circle actions.

3. $\Omega_n^U(SF,F) := \Omega_n^U(\mathcal{F},\mathcal{G})$ denotes the bordism group of all compact $n$-dimensional $U$-manifolds with semifree circle actions which are free on the boundary.

The group $\Omega_n^{U,N}(SF,F)$ will also be called the *relative group*.

Similarly, we will use notations $\Omega_n^{U,N}(F)$, $\Omega_n^{U,N}(SF)$, and $\Omega_n^{U,N}(SF,F)$ for bordism groups of circle actions on complex $N$-Spin manifolds. Also, we have the notations $\Omega_n^{U,N,t}(F)$, $\Omega_n^{U,N,t}(SF)$, and $\Omega_n^{U,N,t}(SF,F)$ for bordism groups of circle actions on complex $N$-Spin manifolds having a fixed type $t$.

Now the following three results follow directly, for the special families $\mathcal{F}$ and $\mathcal{G}$ of the subgroups of $S^1$, from Theorem 3.1.5, Theorem 3.1.7, and Corollary 3.1.8, respectively.

**Theorem 3.1.9.** *The sequence*

$$\cdots \longrightarrow \Omega_n^U(F) \xrightarrow{i_U} \Omega_n^U(SF) \xrightarrow{j_U} \Omega_n^U(SF,F) \xrightarrow{\partial_U} \Omega_{n-1}^U(F) \longrightarrow \cdots$$

*is exact.*

**Theorem 3.1.10.** *The sequence*

$$\cdots \longrightarrow \Omega_n^{U,N}(F) \xrightarrow{i_N} \Omega_n^{U,N}(SF) \xrightarrow{j_N} \Omega_n^{U,N}(SF,F) \xrightarrow{\partial_N} \Omega_{n-1}^{U,N}(F) \longrightarrow \cdots$$

*is exact.*

**Corollary 3.1.11.** *The sequence*

$$\cdots \longrightarrow \Omega_n^{U,N,t}(F) \xrightarrow{i_{N,t}} \Omega_n^{U,N,t}(SF) \xrightarrow{j_{N,t}} \Omega_n^{U,N,t}(SF,F) \xrightarrow{\partial_{N,t}} \Omega_{n-1}^{U,N,t}(F) \longrightarrow \cdots$$

*is exact.*

## 3.2 The Group $\Omega_n^{U,N}(SF,F)\otimes\mathbb{Q}$

In this section we will see how to compute the group $\Omega_n^{U,N,t}(SF,F)\otimes\mathbb{Q}$. We will begin by studying the groups $\Omega_n^U(SF,F)$ and $\Omega_n^{U,N}(SF,F)$ and then rationalize to compute $\Omega_n^{U,N}(SF,F)\otimes\mathbb{Q}$.

### 3.2.1 The Group $\Omega^U_n(SF, F)$

In this subsection we will study the group $\Omega^U_n(SF, F)$ of bordism classes of compact $n$-dimensional $U$-manifolds with semifree circle actions which are free on the boundary.

**Proposition 3.2.1.** *Let $[M] \in \Omega^U_n(SF, F)$ and $X$ be an $n$-dimensional compact equivariant submanifold with boundary which lies in the interior of $M$. If the circle action is free on $M \setminus X$ then $[M] = [X]$ in $\Omega^U_n(SF, F)$.*

*Proof.* The result follows directly from ([CF66a], 5.2). □

**Proposition 3.2.2.** *Let $[M] \in \Omega^U_n(SF, F)$ and $\{F_i\}$ be the totality of connected components of the fixed point set of the circle action on $M$. If $D_i$ is an equivariant closed tubular neighborhood of $F_i$ with respect to an equivariant Hermitian metric on $M$, then we have*

$$[M] = \sum_i [D_i]$$

*in $\Omega^U_n(SF, F)$.*

*Proof.* The result is a direct consequence of Proposition 3.2.1 because the circle action restricted to $M \setminus \cup_i D_i$ is free. □

In the following we identify $S^1$ with the complex numbers $\lambda = e^{2\pi\iota\omega}$, $\omega \in \mathbb{Z}$, of absolute value 1. The fixed point set $X^{S^1}$ of an $S^1$-action $\alpha$ on a compact differentiable manifold $X^n$ is the disjoint union of compact differentiable sub-manifolds of various dimensions.

Let $Y \subset X^{S^1}$ be a fixed point component with normal bundle $\nu$ and let $p \in Y$. With respect to the stably almost complex structure, the $l = (n + k)/2$ complex dimensional vector space $\tau_p(X) \oplus (\epsilon^k_{\mathbb{R}})_p = \tau_p(Y) \oplus \nu_p \oplus (\epsilon^k_{\mathbb{R}})_p$ becomes a complex $S^1$-module, which decomposes into summands $E^i_p$, $i \in \mathbb{Z}$, where an element $\lambda \in S^1$ acts on $E^i_p$ by multiplication with $\lambda^i$. With respect to a basis of eigenvectors of the action of $\lambda \in S^1$ on the stable tangent bundle one can get a diagonal matrix $\mathrm{diag}(\lambda^{m_1}, \lambda^{m_2}, \ldots, \lambda^{m_l})$. The numbers $m_1, m_2,$

$\ldots, m_l$ are called the rotation numbers and the rotation number $m_\nu$ with $m_\nu = i$ is complex dimension of $E_p^i$.

The decomposition of $\tau_p(X) \oplus (\epsilon_{\mathbb{R}}^k)_p$ leads to a decomposition of $\tau(X) \oplus \epsilon_{\mathbb{R}}^k|_Y$ into eigenspace bundles $E^i$. Thus the rotation numbers $m_1, m_2, \ldots, m_l$ depend (up to the order) only on the fixed point component. The eigenspace bundle $E^0$ is just the stable tangent bundle $\tau(Y) \oplus \epsilon_{\mathbb{R}}^k$ of $Y$, in particular $Y$ is also stably almost complex.

Let $M$ be a $U$-manifold with a semifree circle action respecting the $U$-structure which is free on the boundary. Let $X_i$ be a connected component of the fixed point set of the circle action. Then $X_i$ has a natural $U$-structure and the normal bundle $\upsilon(X_i)$ of $X_i$ in $M$ becomes a complex vector bundle on which $S^1$ acts by $U$-structure preserving isomorphisms of complex vector bundle. Moreover, there is a decomposition of the normal bundle $\upsilon(X_i) = \upsilon_i^+ \oplus \upsilon_i^-$ such that $\lambda \in S^1 \subset \mathbb{C}^*$ acts on the complex bundles $\upsilon_i^+$ and $\upsilon_i^-$ by multiplication with $\lambda$ and $\lambda^{-1}$, respectively (see, for instance [Höh91]).

In the sequel we will denote a generic element $[M, f]$ of $\Omega_n^U(BU(p) \times BU(q))$ by $[M, E \oplus G]$, where the complex bundles $E$ and $G$ of complex ranks $p$ and $q$ are the pullbacks of the universal bundles over the classifying spaces $BU(p)$ and $BU(q)$ under the first and the second components of $f$, respectively. Consider the homomorphism

$$\mu : \Omega_n^U(SF, F) \longrightarrow \bigoplus_{k+2p+2q=n} \Omega_k^U(BU(p) \times BU(q))$$

defined by

$$\mu[M] = \sum_i [X_i, \upsilon_i^+ \oplus \upsilon_i^-],$$

where the summation is taken over the connected components of the fixed point set. It is a well-defined homomorphism.

**Proposition 3.2.3.** *The homomorphism*

$$\mu : \Omega_n^U(SF, F) \longrightarrow \bigoplus_{k+2p+2q=n} \Omega_k^U(BU(p) \times BU(q))$$

*is an isomorphism.*

*Proof.* We will prove the result by exhibiting an inverse for $\mu$. Define the homomorphism

$$\delta : \bigoplus_{k+2p+2q=n} \Omega_k^U(BU(p) \times BU(q)) \longrightarrow \Omega_n^U(SF, F)$$

by sending an element $[X, E \oplus G]$ of $\Omega_{2k}^U(BU(p) \times BU(q))$ to $[D(E \oplus G)]$, where $E$ and $G$ are vector bundles over X of complex ranks $p$ and $q$ respectively, and $D(E \oplus G)$ is the disk bundle with respect to an equivariant metric on the vector bundle $E \oplus G$. It is a well-defined homomorphism. Clearly $\mu \circ \delta =$ identity and by Proposition 3.2.2 it follows that $\delta \circ \mu =$ identity. Thus $\mu$ is an isomorphism. □

### 3.2.2 The Group $\Omega_n^{U,N}(SF, F)$

In this subsection we will begin by studying the group $\Omega_n^{U,N}(SF, F)$ of bordism classes of semifree circle actions on compact $n$-dimensional $N$-manifolds which are free on the boundary. More precisely, we will prove several results concerning $\Omega_n^{U,N}(SF, F)$ necessary to compute this group tensored with $\mathbb{Q}$ at the end of this section.

Let $\gamma_p$ and $\gamma_q$ be the universal complex vector bundles (of real ranks $2p$ and $2q$) over $BU(p)$ and $BU(q)$, respectively. We denote by $\Omega_k^{U,N}(BU(p) \times BU(q), \gamma_p \oplus \gamma_q)$ the bordism group consisting of 4-tuples $(M, f, g, s)$ such that $f : M \longrightarrow BU(p)$ and $g : M \longrightarrow BU(q)$ are continuous map on a $(k - 2p - 2q)$-dimensional $U$-manifold $M$ and $s$ an $N$-structure on $\tau(M) \oplus f^*(\gamma_p) \oplus g^*(\gamma_q)$ (here by an $N$-structure we mean an $N$-structure on the stable bundle representing the $U$-structure). The bordism between the elements of $\Omega_k^{U,N}(BU(p) \times BU(q), \gamma_p \oplus \gamma_q)$ is defined as: Two 4-tuples $(M_1, f_1, g_1, s_1)$ and $(M_2, f_2, g_2, s_2)$

are said to be bordant if there is a third 4-tuple $(W, f, g, s)$ with $W$ an $(k-2p-2q+1)$-dimensional $U$-manifold such that $\partial W = M_1 \cup -M_2$, $f|M_1 = f_1$, $f|M_2 = f_2$, $g|M_1 = g_1$, $g|M_2 = g_2$, and the $N$-structures $s_1$ and $s_2$ are induced by $s$. The direct sum

$$\Omega_*^{U,N}(BU(p) \times BU(q), \gamma_p \oplus \gamma_q) = \bigoplus_k \Omega_k^{U,N}(BU(p) \times BU(q), \gamma_p \oplus \gamma_q)$$

has a natural $\Omega_*^{U,N}$-module structure, as usual.

Now we intend to show that groups $\Omega_n^{U,N}(SF, F)$ and $\bigoplus_{k+2p+2q=n} \Omega_k^{U,N}(BU(p) \times BU(q), \gamma_p \oplus \gamma_q)$ are isomorphic. First we need to define a homomorphism

$$\nu : \Omega_n^{U,N}(SF, F) \longrightarrow \bigoplus_{k+2p+2q=n} \Omega_k^{U,N}(BU(p) \times BU(q), \gamma_p \oplus \gamma_q).$$

Let $[M, \alpha]$ be an element of $\Omega_n^{U,N}(SF, F)$ with the given $N$-structure on $M$ denoted by $s$ and $X^{k-2p-2q}$ the union of the $(k-2p-2q)$-dimensional components of the fixed point set of circle action $\alpha$. Also denote by $\upsilon_{p,q}$ the normal bundle of $X^{k-2p-2q}$ in $M$. The $N$-structure $s$ on $M$ induces an $N$-structure $s_{p,q}$ on $\tau(X^{k-2p-2q}) \oplus \upsilon_{p,q}$ because $\tau(M)|X^{k-2p-2q} \cong \tau(X^{k-2p-2q}) \oplus \upsilon_{p,q}$. Now define

$$\nu : \Omega_n^{U,N}(SF, F) \longrightarrow \bigoplus_{k+2p+2q=n} \Omega_k^{U,N}(BU(p) \times BU(q), \gamma_p \oplus \gamma_q)$$

by

$$\nu([M, \alpha]) = \bigoplus_{k+2p+2q=n} [X^{k-2n-2p}, f_p, g_p, s_{p,q}],$$

where $f_p$ and $g_q$ are the classifying maps of $\upsilon_{p,q}^+$ and $\upsilon_{p,q}^-$, respectively. It is a routine matter to see that $\nu$ is a well-defined homomorphism.

**Proposition 3.2.4.** *The homomorphism*

$$\nu : \Omega_n^{U,N}(SF, F) \longrightarrow \bigoplus_{k+2p+2q=n} \Omega_k^{U,N}(BU(p) \times BU(q), \gamma_p \oplus \gamma_q)$$

*is an isomorphism.*

*Proof.* We will prove the result by showing that the homomorphism $\nu$ above has an inverse. Let $[M,f,g,s] \in \Omega_k^{U,N}(BU(p)\times BU(q), \gamma_p \oplus \gamma_q)$ and set $E := f^*(\gamma_p)$, $F := g^*(\gamma_q)$, then $f$ and $g$ give rise to the maps $\hat{f} : D(E) \longrightarrow D(\gamma_p)$ and $\hat{g} : D(F) \longrightarrow D(\gamma_q)$ between the disk bundles which may be assumed transverse regular to the zero sections $BU(p)$ and $BU(q)$ of $\gamma_p$ and $\gamma_q$, respectively. So one gets $\upsilon(M,E) = f^*(\upsilon(BU(p),\gamma_p)) = f^*(\gamma_p) = E$ and $\upsilon(M,F) = g^*(\upsilon(BU(q),\gamma_q)) = g^*(\gamma_q) = F$, where $\upsilon$ denotes normal bundle. If $p : E\oplus F \longrightarrow M$ denotes the projection, then by using the identity $\tau(E\oplus F) = p^*(\tau(M)\oplus E\oplus F)$ we have that the $N$-structure $s$ induces an $N$-structure $p^*(s)$ on $\tau(E\oplus F)$ which restricts to an $N$-structure on the disk bundle $D(E\oplus F)$. If we denote by $\beta$ the complex multiplication on the bundle $E$ and complex multiplication after inverse on $F$, then the assignment $[M,f,g,s] \longrightarrow [D(E\oplus F),\beta]$ gives an inverse of $\nu$. $\square$

From now on we will denote a generic element $[M,f,g,s]$ of $\Omega_k^{U,N}(BU(p)\times BU(q),\gamma_p\oplus\gamma_q)$ by $[M,h,s]$, where $h$ is the map $(f,g) : M \longrightarrow BU(p)\times BU(q)$. The bordism group $\Omega_k^{U}(BU(p)\times BU(q),\gamma_p\oplus\gamma_q)$ can be defined similar to $\Omega_k^{U,N}(BU(p)\times BU(q),\gamma_p\oplus\gamma_q)$. For more details on this group $\Omega_k^{U}(BU(p)\times BU(q),\gamma_p\oplus\gamma_q)$ we refer to [LW75].

The following result can be proved on the lines of Proposition 3.2.4.

**Proposition 3.2.5.** *The homomorphism*

$$\nu' : \Omega_n^U(SF,F) \longrightarrow \bigoplus_{k+2p+2q=n} \Omega_k^U(BU(p)\times BU(q),\gamma_p\oplus\gamma_q)$$

*is an isomorphism, where $\nu'$ has similar meaning as $\nu$.*

Next, we want to show that the groups $\Omega_k^{U,N}(BU(p)\times BU(q),\gamma_p\oplus\gamma_q)$ and $\Omega_k^{U,N}(D(\gamma_p\oplus\gamma_q), S(\gamma_p\oplus\gamma_q))$ are isomorphic, where $D(\gamma_p\oplus\gamma_q)$ and $S(\gamma_p\oplus\gamma_q))$ denote the disk and sphere bundles of $\gamma_p\oplus\gamma_q$, respectively. Define

$$\rho : \Omega_k^{U,N}(BU(p)\times BU(q),\gamma_p\oplus\gamma_q) \to \Omega_k^{U,N}(D(\gamma_p\oplus\gamma_q), S(\gamma_p\oplus\gamma_q))$$

by

$$\rho[M,f,s] = [D(f^*(\gamma_p \oplus \gamma_q)), \hat{f}|D(f^*(\gamma_p \oplus \gamma_q)), \pi^*(s)|D(f^*(\gamma_p \oplus \gamma_q))],$$

where $\hat{f}$ is the map from $f^*(\gamma_p \oplus \gamma_q)$ to $\gamma_p \oplus \gamma_q$ induced by $f$. Since $\tau(f^*(\gamma_p \oplus \gamma_q)) = \pi^*(\tau(M) \oplus f^*(\gamma_p \oplus \gamma_q))$, where $\pi : f^*(\gamma_p \oplus \gamma_q) \longrightarrow M$ is the projection, the $N$-structure $s$ induces an $N$-structure $\pi^*(s)$ on $f^*(\gamma_p \oplus \gamma_q)$. Clearly, $\rho$ is a well-defined homomorphism.

**Proposition 3.2.6.** *The homomorphism*

$$\rho : \Omega_k^{U,N}(BU(p) \times BU(q), \gamma_p \oplus \gamma_q) \longrightarrow \Omega_k^{U,N}(D(\gamma_p \oplus \gamma_q), S(\gamma_p \oplus \gamma_q))$$

*is an isomorphism.*

*Proof.* We will prove the result by exhibiting an inverse for $\rho$. Recall the definition of group $\Omega_k^{U,N}(D(\gamma_p \oplus \gamma_q), S(\gamma_p \oplus \gamma_q))$ from Chapter 1. A generic element of $\Omega_k^{U,N}(D(\gamma_p \oplus \gamma_q), S(\gamma_p \oplus \gamma_q))$ is represented here by a bordism class $[W,F,s]$, where $W$ is a $k$-dimensional compact $N$-manifold with the given $N$-structure $s$ and $F$ is a continuous map $(W, \partial W) \longrightarrow (D(\gamma_p \oplus \gamma_q), S(\gamma_p \oplus \gamma_q))$. We may choose $F$ to be transverse regular to the zero section $BU(p) \times BU(q)$ of $\gamma_p \oplus \gamma_q$. Put $M := F^{-1}(BU(p) \times BU(q))$ and $f := F|M$. Since $\upsilon(M,W) = f^*(\upsilon(BU(p) \times BU(q), \gamma_p \oplus \gamma_q)) = f^*(\gamma_p \oplus \gamma_q)$ and $\tau(W)|M = \tau(M) \oplus \upsilon(M,W)$, therefore $\tau(W)|M = \tau(M) \oplus f^*(\gamma_p \oplus \gamma_q)$. It follows that the restriction of $s$ induces an $N$-structure $\hat{s}$ on $\tau(M) \oplus f^*(\gamma_p \oplus \gamma_q)$. Now define

$$\phi : \Omega_k^{U,N}(D(\gamma_p \oplus \gamma_q), S(\gamma_p \oplus \gamma_q)) \rightarrow \Omega_k^{U,N}(BU(p) \times BU(q), \gamma_p \oplus \gamma_q)$$

by

$$\phi[W,F,s] = [M,f,\hat{s}].$$

Clearly $\phi$ is a homomorphism and $\phi \circ \rho$ =identity. We need to show that $\rho \circ \phi$=identity. Observe first that

$$\rho \circ \phi[W,F,s] = \rho[M,f,\hat{s}] = [D(f^*(\gamma_p \oplus \gamma_q)), \hat{f}|D(f^*(\gamma_p \oplus \gamma_q)), \pi^*(\hat{s})|D(f^*(\gamma_p \oplus \gamma_q))].$$

Now we must construct a bordism between $(D(f^*(\gamma_p \oplus \gamma_q)), \hat{f}|D(f^*(\gamma_p \oplus \gamma_q)), \pi^*(\hat{s})|D(f^*(\gamma_p \oplus \gamma_q)))$ and $(W, F, s)$ to show that the both represent the same bordism class in group $\Omega_k^{U,N}(D(\gamma_p \oplus \gamma_q), S(\gamma_p \oplus \gamma_q))$. Recall from Chapter 1 that the unit interval $I$ has the natural $N$-structure corresponding to a stable $U$-structure. If one identifies $D(f^*(\gamma_p \oplus \gamma_q))$ with a tubular neighborhood of $M$ in $W$. Then, after smoothing a corner at $W \times 1$ and introducing a corner at $S(f^*(\gamma_p \oplus \gamma_q)) \times 1$, one can interpret $(W \times I, f, s)$ as a bordism between $(W, F, s)$ and $(D(f^*(\gamma_p \oplus \gamma_q)), F|D(f^*(\gamma_p \oplus \gamma_q)), s|D(f^*(\gamma_p \oplus \gamma_q)))$. The maps $\hat{f}$ and $F$ are homotopic, so $\rho \circ \phi[W, F, s] = [W, F, s]$ and this finishes the proof. □

Let us remark that the techniques used in proving the above result are similar to those of Lazarov and Wasserman [LW75] to show the following result which is also useful to state here.

**Proposition 3.2.7.** *The homomorphism*

$$\rho' : \Omega_k^U(BU(p) \times BU(q), \gamma_p \oplus \gamma_q) \longrightarrow \Omega_k^U(D(\gamma_p \oplus \gamma_q), S(\gamma_p \oplus \gamma_q))$$

*is an isomorphism, where $\rho'$ has similar meaning as $\rho$.*

We conclude this subsection by combining Proposition 3.2.4 with Proposition 3.2.6 and Proposition 3.2.5 with Proposition 3.2.7 into following two results.

**Proposition 3.2.8.** *We have an isomorphism*

$$\rho \circ \nu : \Omega_n^{U,N}(SF, F) \longrightarrow \bigoplus_{k+2p+2q=n} \Omega_k^{U,N}(D(\gamma_p \oplus \gamma_q), S(\gamma_p \oplus \gamma_q)).$$

**Proposition 3.2.9.** *We have an isomorphism*

$$\rho' \circ \nu' : \Omega_n^U(SF, F) \longrightarrow \bigoplus_{k+2p+2q=n} \Omega_k^U(D(\gamma_p \oplus \gamma_q), S(\gamma_p \oplus \gamma_q)).$$

Since an $N$-bordism in $\Omega_n^{U,N}(SF, F)$, $\Omega_k^{U,N}(BU(p) \times BU(q),\ \gamma_p \oplus \gamma_q)$, and $\Omega_k^{U,N}(D(\gamma_p \oplus \gamma_q), S(\gamma_p \oplus \gamma_q)$ is also a $U$-bordism in $\Omega_n^U(SF, F)$, $\Omega_k^{U,N}(BU(p) \times BU(q), \gamma_p \oplus \gamma_q)$, and $\Omega_k^U(D(\gamma_p \oplus \gamma_q), S(\gamma_p \oplus \gamma_q)$, respectively. Therefore, the forgetful homomorphisms

$$\Omega_n^{U,N}(SF, F) \longrightarrow \Omega_n^U(SF, F),$$

$$\Omega_k^{U,N}(BU(p)\times BU(q),\gamma_p\oplus\gamma_q)\longrightarrow\Omega_k^{U}(BU(p)\times BU(q),\gamma_p\oplus\gamma_q),$$

and

$$\Omega_k^{U,N}(D(\gamma_p\oplus\gamma_q),S(\gamma_p\oplus\gamma_q)\longrightarrow\Omega_k^{U}(D(\gamma_p\oplus\gamma_q),S(\gamma_p\oplus\gamma_q))$$

are well-defined.

**Proposition 3.2.10.** *The diagrams*

$$\begin{tikzcd}
\Omega_n^{U,N}(SF,F) \arrow[r, "\nu"] \arrow[d] & \bigoplus\limits_{k+2p+2q=n}\Omega_k^{U,N}(BU(p)\times BU(q),\gamma_p\oplus\gamma_q) \arrow[d] \\
\Omega_n^{U}(SF,F) \arrow[r, "\nu\prime"] & \bigoplus\limits_{k+2p+2q=n}\Omega_k^{U}(BU(p)\times BU(q),\gamma_p\oplus\gamma_q)
\end{tikzcd}$$

*and*

$$\begin{tikzcd}
\bigoplus\limits_{k+2p+2q=n}\Omega_k^{U,N}(BU(p)\times BU(q),\gamma_p\oplus\gamma_q) \arrow[r, "\rho"] \arrow[d] & \bigoplus\limits_{k+2p+2q=n}\Omega_k^{U,N}(D(\gamma_p\oplus\gamma_q),S(\gamma_p\oplus\gamma_q)) \arrow[d] \\
\bigoplus\limits_{k+2p+2q=n}\Omega_k^{U,N}(BU(p)\times BU(q),\gamma_p\oplus\gamma_q) \arrow[r, "\rho'"] & \bigoplus\limits_{k+2p+2q=n}\Omega_k^{U,N}(D(\gamma_p\oplus\gamma_q),S(\gamma_p\oplus\gamma_q))
\end{tikzcd}$$

*are commutative, where vertical arrows are the forgetful homomorphisms.*

*Proof.* Since $\nu'$ and $\rho'$ have the definitions similar to $\nu$ and $\rho$, respectively, and the vertical arrows are forgetful homomorphisms. Therefore, the result follows by using the definitions of maps involved in the diagram. □

### 3.2.3 Rational Bordism Groups

In this subsection we will tensor the bordism groups with $\mathbb{Q}$. It will give us more traction on these groups and would help by using the results of previous section to compute $\Omega_n^{U,N}(SF,F)\otimes\mathbb{Q}$.

Recall from Chapter 1 that $U$-bordism and $N$-bordism groups coincide when tenosred with the rationals. In particular, the forgetful homomorphism $\Omega_k^{U,N}(D(\gamma_p\oplus\gamma_q),S(\gamma_p\oplus\gamma_q))\otimes\mathbb{Q}\longrightarrow$

$\Omega^U_k(D(\gamma_p \oplus \gamma_q), S(\gamma_p \oplus \gamma_q)) \otimes \mathbb{Q}$ is an isomorphism, by Theorem 1.2.18. The following result follows using this fact about rationalized bordism groups together with Proposition 3.2.10.

**Theorem 3.2.11.** *The forgetful homomorphism $\Omega^{U,N}_n(SF, F) \otimes \mathbb{Q} \longrightarrow \Omega^U_n(SF, F) \otimes \mathbb{Q}$ is an isomorphism.*

*Proof.* The diagrams in Proposition 3.2.10 remain commutative after tensoring with the rationals. Putting them together after tensoring with $\mathbb{Q}$ we get the commutative diagram

$$\begin{tikzcd}
\Omega^{U,N}_n(SF,F)\otimes\mathbb{Q} \arrow[r, "\nu\otimes\mathbb{Q}"] \arrow[d] & \bigoplus\limits_{k+2p+2q=n} \Omega^{U,N}_k(BU(p)\times BU(q), \gamma_p\oplus\gamma_q)\otimes\mathbb{Q} \arrow[r, "\rho\otimes\mathbb{Q}"] \arrow[d] & \bigoplus\limits_{k+2p+2q=n} \Omega^{U,N}_k(D(\gamma_p\oplus\gamma_q), S(\gamma_p\oplus\gamma_q))\otimes\mathbb{Q} \arrow[d] \\
\Omega^{U}_n(SF,F)\otimes\mathbb{Q} \arrow[r, "\nu'\otimes\mathbb{Q}"] & \bigoplus\limits_{k+2p+2q=n} \Omega^{U}_k(BU(p)\times BU(q), \gamma_p\oplus\gamma_q)\otimes\mathbb{Q} \arrow[r, "\rho'\otimes\mathbb{Q}"] & \bigoplus\limits_{k+2p+2q=n} \Omega^{U}_k(D(\gamma_p\oplus\gamma_q), S(\gamma_p\oplus\gamma_q))\otimes\mathbb{Q}
\end{tikzcd}$$

where vertical arrows are the forgetful homomorphism and horizontal arrows are isomorphisms defined in the previous subsection. Since the right hand vertical arrow is an isomorphism by the remarks above, it follows that the left hand vertical arrow is also an isomorphism. $\square$

Now the following result gives a way to compute the group $\Omega^{U,N}_n(SF, F) \otimes \mathbb{Q}$.

**Theorem 3.2.12.** *We have an isomorphism*

$$\vartheta \otimes \mathbb{Q} : \Omega^{U,N}_n(SF, F) \otimes \mathbb{Q} \longrightarrow \bigoplus_{2k+2p+2q=n} \Omega^U_{2k}(BU(p) \times BU(q)) \otimes \mathbb{Q}.$$

*with $\vartheta$ given by*

$$\vartheta[M] = \sum_i [X_i, \upsilon_i^+ \oplus \upsilon_i^-],$$

*where the summation is taken over the connected components of the fixed point set, and $\upsilon_i^-, \upsilon_i^+$ have the same meaning as in Proposition 3.2.3.*

*Proof.* The result follows from Theorem 3.2.11 and Proposition 3.2.3. $\square$

Note that we have the decompositions

$$\Omega^{U,N}_n(SF, F) \otimes \mathbb{Q} = \bigoplus_{t \in \mathbb{Z}/N\mathbb{Z}} \Omega^{U,N,t}_n(SF, F) \otimes \mathbb{Q}$$

and

$$\bigoplus_{2k+2p+2q=n} \Omega^U_{2k}(BU(p)\times BU(q))\otimes\mathbb{Q} = \bigoplus_{\substack{2k+2p+2q=n \\ p-q\equiv t \pmod N}} \Omega^U_{2k}(BU(p)\times BU(q))\otimes\mathbb{Q}$$

Recall from Chapter 1 that the type of an $S^1$-action on an $N$-manifold is the sum of rotation numbers of a fixed point component modulo $N$. So summands involving $t$ in above two decompositions correspond to each other under the isomorphism of groups in Theorem 3.2.12. Thus we have

**Corollary 3.2.13.** *The homorphism*

$$\Omega^{U,N,t}_n(SF,F)\otimes\mathbb{Q} \longrightarrow \bigoplus_{\substack{2k+2p+2q=n \\ p-q\equiv t \pmod N}} \Omega^U_{2k}(BU(p)\times BU(q))\otimes\mathbb{Q}$$

*given by*

$$[M] \longmapsto \sum_i [X_i, \upsilon_i^+ \oplus \upsilon_i^-]$$

*is an isomorphism.*

## 3.3 The Group $\Omega^{U,N}_n(F)\otimes\mathbb{Q}$

In this section we will investigate the rational bordism group $\Omega^{U,N}_n(F)\otimes\mathbb{Q}$ of $n$-dimensional closed $N$-manifolds with free circle actions.

### 3.3.1 Complex $N$-Spin$^{c,t}$ Structures

Let $N$ be a positive integer. In this subsection we introduce the notion of a complex $N$-Spin$^{c,t}$ manifold which is required to characterize the rational $N$-bordism groups having a free circle action of type $t$. Throughout this section we will use the identifications $U(1)\cong S^1\cong \mathbb{R}/\mathbb{Z}$ and $\hat{U}(1)\cong \mathbb{R}/N\mathbb{Z}\cong S^1$ without further mentioning. At times we will also use $S^1$ and $U(1)$ separately when it is needed to distinguish between the actions on manifolds and bundles, respectively.

Let $(\mathbf{Z}_N, \cdot)$ denote the group of $N^{th}$ roots of unity generated by $e^{\frac{2\pi\iota}{N}}$. It is isomorphic to the group $(\mathbb{Z}_N, +)$ of residue classes of integers $\mod N$. So we note for a later use that one can identify the cohomology groups $H^n(X; \mathbf{Z}_N)$ and $H^n(X; \mathbb{Z}_N)$ for any space $X$ and any nonnegative integer $n$. Let $\lambda_N : \hat{U}(1) \longrightarrow U(1)$ given by $\lambda \longmapsto \lambda^N$ be the $N$-fold covering of $U(1)$. Note that $\mathbf{Z}_N = \ker \lambda_N$. We denote by $i : \mathbf{Z}_N \longrightarrow \hat{U}(1)$ the inclusion.

**Definition 3.3.1.** Let $N$ and $t$ be positive integers such that $t|N$, and let $\mu_{N/t} : \hat{S}^1 \longrightarrow S^1$ given by $\zeta \longmapsto \zeta^{N/t}$ be the $N/t$-fold covering of $S^1$. Let $\mathbf{Z}_N$ act on $\hat{U}(1) \times \hat{S}^1$ by sending $(\lambda, g, \zeta)$ to $(\lambda g, \lambda^{-t}\zeta)$, where $\lambda \in \mathbf{Z}_N, g \in \hat{U}(1)$, and $\zeta \in \hat{S}^1$. We define $U^{N,c,t}$ as the group $\hat{U}(1) \underset{\mathbf{Z}_N}{\times} \hat{S}^1$.

We will represent a generic element of $U^{N,c,t}$ as $[g, \zeta]$, where $g \in \hat{U}(1)$, and $\zeta \in \hat{S}^1$. Notice that $\mathbf{Z}_{N/t}$ can be considered as a subgroup of $U^{N,c,t}$ generated by $[e^{\frac{2\pi\iota}{N}}, 1] = [e^{\frac{2\pi\iota}{N}}(e^{\frac{2\pi\iota}{N}})^{-1}, ((e^{\frac{2\pi\iota}{N}})^{-1})^{-t}] = [1, e^{\frac{2\pi t\iota}{N}}]$.

Let $\mu : U^{N,c,t} \longrightarrow U(1) \times S^1$ be induced by the map $\lambda_N \times \mu_{N/t} : \hat{U}(1) \times \hat{S}^1 \longrightarrow U(1) \times S^1$. Then we have a short exact sequence:

$$0 \longrightarrow \mathbf{Z}_{N/t} \longrightarrow U^{N,c,t} \xrightarrow{\mu} U(1) \times S^1 \longrightarrow 1 \tag{3.1}$$

**Definition 3.3.2.** Let $\Delta$ be a principal $U(1)$-bundle over a finite connected CW-complex $B$. A *complex $N$-Spin$^{c,t}$ structure (or simply $N^{c,t}$-structure)* on $\Delta$ consists of

1. a principal $U^{N,c,t}$-bundle $\eta$ over $B$,
2. a principal $U(1)$-bundle $\Lambda$ over $B$, and
3. a bundle map $f : E(\eta) \longrightarrow E(\Delta) \times E(\Lambda)$ such that $f(xg) = f(x)\mu(g)$, where $x \in E(\eta)$, and $g \in U^{N,c,t}$.

Let $\xi$ be a principal $U(n)$-bundle over a finite connected CW-complex $B$ and $\Delta$ be the associated principal bundle belonging to the representation $U(n) \longrightarrow U(1)$. A *complex $N$-Spin$^{c,t}$ or simply $N^{c,t}$-structure* on $\xi$ is an $N^{c,t}$-structure on $\Delta$. We will see that every principal $U(n)$-bundle over a finite $CW$-complex admits an $N^{c,1}$-structure.

The coefficient sequence (3.1) determines the exact sequence:

$$H^1(B; U^{N,c,t}) \xrightarrow{\mu_*} H^1(B; U(1)) \oplus H^1(B; S^1) \xrightarrow{\hat{c}_{N/t}} H^2(B, \mathbf{Z}_{N/t}) \tag{3.2}$$

The map $\hat{c}_{N/t}$ sends a pair $(\Delta, \Lambda)$ to the cohomology class $\rho_{N/t}(c_1(\Delta)) - t\rho_{N/t}(c_1(\Lambda))$, where $\rho_{N/t} : H^2(B; \mathbb{Z}) \longrightarrow H^2(B; \mathbf{Z}_{N/t})$ is the $\mod N/t$ reduction.

It follows by the exactness of sequence (3.2) that the bundle $\xi$ (equivalently $\Delta$) can be given an $N^{c,t}$-structure if and only if there exists a cohomology class $u$ in $H^2(B; \mathbb{Z})$ such that $\rho_{N/t}(c_1(\Delta)) = t\rho_{N/t}(u)$. Since the first Chern class $c_1(\xi)$ of $\xi$ is the same as the first Chern class $c_1(\Delta)$ of determinant line bundle $\Delta$, one can take $u = c_1(\xi)$ to see that the principal $U(n)$-bundle $\xi$ admits an $N^{c,1}$-structure.

**Proposition 3.3.3.** *Let $E_1$ and $E_2$ be two complex vector bundles over a finite CW-complex $B$. Then a choice of $N^{c,t}$-structures on two of the three bundles $E_1, E_2, E_1 \oplus E_2$ uniquely determines an $N^{c,t}$-structure on the third one.*

*Proof.* The bundles $E_1$ and $E_2$ can be provided with Hermitian metrics to look at the associated principal $U(n)$-bundles. Let $u_1, u_2 \in H^2(B; \mathbb{Z})$ be two cohomology classes such that $\rho_{N/t}(c_1(E_1)) = t\rho_{N/t}(u_1)$ and $\rho_{N/t}(c_1(E_2)) = t\rho_{N/t}(u_2)$. Since $c_1(E_1 \oplus E_2) = c_1(E_1) + c_2(E_2)$, therefore, $\rho_{N/t}(c_1(E_1 \oplus E_2)) = \rho_{N/t}(c_1(E_1) + c_2(E_2)) = \rho_{N/t}(c_1(E_1)) + \rho_{N/t}(c_2(E_2))$, and so $\rho_{N/t}(c_1(E_1 \oplus E_2)) = t\rho_{N/t}(u_1) + t\rho_{N/t}(u_2) = t\rho_{N/t}(u_1 + u_2)$ with $u_1 + u_2 \in H^2(B; \mathbb{Z})$. Hence a choice of $N^{c,t}$-structures on $E_1$ and $E_2$ uniquely determines an $N^{c,t}$-structure on $E_1 \oplus E_2$. Similarly the equations $\rho_{N/t}(c_1(E_1)) = \rho_{N/t}(c_1(E_1 \oplus E_2)) - \rho_{N/t}(c_1(E_2))$ and $\rho_{N/t}(c_1(E_2)) = \rho_{N/t}(c_1(E_1 \oplus E_2)) - \rho_{N/t}(c_1(E_1))$ can be used to prove the other two cases. □

### 3.3.2 Complex $N$-Spin$^{c,t}$ Bordism Groups

**Definition 3.3.4.** A *complex $N$-Spin$^{c,t}$ manifold or simply $N^{c,t}$-manifold* is a $U$-manifold together with a given $N^{c,t}$-structure on the principal $U(n)$-bundle belonging to its stable tangent bundle representing its $U$-structure.

We define the bordism group $\Omega_n^{U,N^{c,t}}$ of connected $N^{c,t}$-manifolds analogous to $\Omega_n^{U,N}$. Let $\psi : B \longrightarrow BU(n)$ be a classifying map of a $U(n)$-bundle $\xi$ and consider the map $\bar{c}_1 - t\bar{c}_1 : BS^1 \times K(\mathbb{Z},2) \longrightarrow K(\mathbb{Z}_{N/t},2)$, where $\bar{c}_1$ is the first Chern class reduction $\mod N/t$. Let $\phi_{N/t}$ be the pullback, with the map $\bar{c}_1 - t\bar{c}_1$, of the fibration $p_{N/t} : PK(\mathbb{Z}_{N/t},2) \longrightarrow K(\mathbb{Z}_{N/t},2)$ having the fiber homotopy type $\Omega K(\mathbb{Z}_{N/t},2) \cong K(\mathbb{Z}_{N/t},1)$. Further pulling back the fibration $\phi_{N/t}$ with det $\times$id to $BU(n) \times K(\mathbb{Z},2)$ gives the fibration $(\hat{\phi}_{N/t})_n : \widehat{BU}(n) \longrightarrow BU(n) \times K(\mathbb{Z},2)$:

$$\begin{tikzcd}
 & & \widehat{BU}(n) \arrow[r] \arrow[d, "(\hat{\phi}_{N/t})_n"] \arrow[ddl, "f"'] & \widehat{BS}^1 \arrow[r] \arrow[d, "\phi_{N/t}"] & PK(\mathbb{Z}_{N/t},2) \arrow[d, "p_{N/t}"] \\
 & & BU(n)\times K(\mathbb{Z},2) \arrow[r, "\det\times\mathrm{id}"] \arrow[dl, "\pi_1"] & BS^1\times K(\mathbb{Z},2) \arrow[r, "\bar{c}_1 - t\bar{c}_1"] & K(\mathbb{Z}_{N/t},2) \\
B \arrow[r, "\psi"] \arrow[uurr, dashed, "\hat{\psi}"] & BU(n) & & &
\end{tikzcd}$$

An equivalence class of an $N^{c,t}$-structure on $\xi$ corresponds to a fiber homotopy class of a lifting $\hat{\psi}$ of $\psi$ regarded as fibration, where $f = \pi_1 \circ (\hat{\phi}_{N/t})_n$.

The fibrations $(\hat{\phi}_{N/t})_n : \widehat{BU}(n) \longrightarrow BU(n) \times K(\mathbb{Z},2)$ induce the fibration $\hat{\phi}_{N/t} : \widehat{BU} \longrightarrow BU$ over the stable classifying space:

$$\begin{tikzcd}
\widehat{BU}(n) \arrow[r, "k_n"] \arrow[d, "(\hat{\phi}_{N/t})_n"] & \widehat{BU}(n+1) \arrow[r] \arrow[d, "(\hat{\phi}_{N/t})_{n+1}"] & \widehat{BU} \arrow[r] \arrow[d, "\hat{\phi}_{N/t}"] & \widehat{BS}^1 \arrow[d, "\phi_{N/t}"] \\
BU(n) \times K(\mathbb{Z},2) \arrow[r, "i_n \times \mathrm{id}"] & BU(n+1) \times K(\mathbb{Z},2) \arrow[r] & BU \times K(\mathbb{Z},2) \arrow[r, "\det\times\mathrm{id}"] & BS^1 \times K(\mathbb{Z},2)
\end{tikzcd}$$

If ones sets $B_{2n} := B_{2n+1} := \widehat{BU}(n)$, then one gets the following commutative diagram:

$$\begin{tikzcd}
... \arrow[r] & B_{2n} \arrow[r, "g_{2n} := id"] \arrow[d, "f_{2n}"] & B_{2n+1} \arrow[r, "g_{2n+1} := k_n"] \arrow[d, "f_{2n+1}"] & B_{2n+2} \arrow[r] \arrow[d, "f_{2n+2}"] & ... \\
... \arrow[r] & BO(2n) \arrow[r, "j_{2n}"] & BO(2n+1) \arrow[r, "j_{2n+1}"] & BO(2n+2) \arrow[r] & ...
\end{tikzcd} \tag{3.3}$$

Here $j_k : BO(k) \longrightarrow BO(k+1)$ is the natural map which belongs to the addition of trivial real line bundle, and $f_k$ is the fibration defined by $f_{2n} := h_n \circ (\hat{\phi}_{N/t})_n$, $f_{2n+1} := j_{2n} \circ f_{2n}$, where $h_n = q_n \circ (\pi_n)_1$ such that $(\pi_n)_1 : BU(n) \times K(\mathbb{Z}, 2) \longrightarrow BU(n)$ is the projection on the first component and $q_n : BU(n) \longrightarrow BO(2n)$ is the natural map belonging to the "forgetting" of complex structure.

**Theorem 3.3.5.** *The cobordism theory $\Omega_n(B, f)$, belonging to the above family of fibrations $f_k : B_k \longrightarrow BO(k)$, corresponds to the cobordism group $\Omega_n^{U,N^{c,t}}$.*

*Proof.* An $N^{c,t}$-manifold $X$ with an $N^{c,t}$-structure corresponds to a unique stable equivalence class of the homotopy classes of the liftings $f_l : B_l \longrightarrow BO(l)$ to the classifying maps of the normal bundle of $X$:

An $N^{c,t}$-structure on the stable tangent bundle induces an $N^{c,t}$-structure on the normal bundle: Embed $X$ differentiably in $\mathbb{R}^{2s}$, $s$ large, then

$$(\tau(X) \oplus \epsilon_{\mathbb{R}}^{k}) \oplus \nu \cong \epsilon_{\mathbb{R}}^{2s}.$$

Provide the tangent bundle $\epsilon_{\mathbb{R}}^{2s}$ of $\mathbb{R}^{2s}$ with the canonical $U$- and $N^{c,t}$-structures and restrict them to $X$. The normal bundle $\nu$ can be provided with a $U$-structure (see [CF66b], p. 21) and by Proposition 3.3.3 one gets the $N^{c,t}$-structure. The relationship between $U$-structures and liftings is clear, we have discussed the relationship between the $N^{c,t}$-structures and liftings previously.

The set of equivalence classes $\Omega_n(B, f)$ corresponds to set of equivalence classes $\Omega_n^{U,N^{c,t}}$. $\square$

Let $M$ be an $N^{c,t}$-manifold. Let $f : M \longrightarrow \mathbb{CP}_\infty$ denote the classifying map of the principal $U(1)$-bundle over $M$ which exists due to the second axiom of definition of $N^{c,t}$-structure. The map $\Omega_n^{U,N^{c,t}} \longrightarrow \Omega_n^U(\mathbb{CP}_\infty)$ which sends $[M]$ to $[M, f]$ is called the forgetful homomorphism.

**Theorem 3.3.6.** *For $t|N$, the forgetful homomorphism*

$$\Omega_n^{U,N^{c,t}} \longrightarrow \Omega_n^U(\mathbb{CP}_\infty)$$

*is an isomorphism after tensoring with the rationals $\mathbb{Q}$.*

*Proof.* Let $T\widehat{BU}_k$ denote the Thom space of the pullback of universal vector bundle $\gamma_k$ over $BO(k)$ to $\widehat{BU}([\frac{k}{2}])$ with $f_k = (j_{k-1}) \circ h_{[\frac{k}{2}]} \circ (\hat{\phi}_{N/t})_{[\frac{k}{2}]} : \widehat{BU}([\frac{k}{2}]) \longrightarrow BO(k)$. Then by Theorem 3.3.5 and the generalized Pontrjagin-Thom Theorem (see [Sto68], page 18) one gets:

$$\Omega_n^{U,N^{c,t}} \cong \lim_{k\to\infty} \pi_{n+k}(T\widehat{BU}_k, \infty).$$

Since the Thom space $T\widehat{BU}_k$ is $(k-1)$-connected, therefore, by Serre theorem (cf. [MS74]), after tensoring with $\mathbb{Q}$, the Hurewicz homomorphism $\pi_{n+k}(T\widehat{BU}_k, \infty)\otimes\mathbb{Q} \longrightarrow H_{n+k}(T\widehat{BU}_k, \infty, \mathbb{Z})\otimes \mathbb{Q}$ is an isomorphism for $n + k < 2k - 1$, so for $k > n + 1$. Since the map $f_k$ is induced by a complex vector bundle, the bundle $f_k^*(\gamma_k)$ is oriented , and so Thom isomorphism gives $H_{n+k}(T\widehat{BU}_k, \infty, \mathbb{Z}) \cong H_n(\widehat{BU}([\frac{k}{2}]), \mathbb{Z})$. Now by applying the Leray spectral sequence to the fibration $(\hat{\phi}_{N/t})_{[\frac{k}{2}]} : \widehat{BU}([\frac{k}{2}]) \longrightarrow BU([\frac{k}{2}]) \times K(\mathbb{Z}, 2)$ with the fiber $K(\mathbb{Z}_{N/t}, 1)$ and $H_*(K(\mathbb{Z}_{N/t}, 1), \mathbb{Q}) \cong \mathbb{Q}$ one has $H_n(\widehat{BU}([\frac{k}{2}]), \mathbb{Q}) \cong H_n(BU([\frac{k}{2}] \times K(\mathbb{Z}, 2)), \mathbb{Q})$.

Consequently, we have for $k > n + 1$
$\pi_{n+k}(T\widehat{BU}_k, \infty)\otimes\mathbb{Q} \cong H_{n+k}(T\widehat{BU}_k, \infty, \mathbb{Z})\otimes\mathbb{Q} \cong H_n(\widehat{BU}([\frac{k}{2}]), \mathbb{Q}) \cong H_n(BU[\frac{k}{2}]\times K(\mathbb{Z}, 2), \mathbb{Q}) \cong \bigoplus_{i+j=n} (H_i(BU[\dfrac{k}{2}], \mathbb{Q}) \otimes H_j(\mathbb{CP}_\infty, \mathbb{Q}))$. The last isomorphism is by the Knuth theorem.

Since $\Omega_n^U(\mathbb{CP}_\infty)\otimes\mathbb{Q} \cong \bigoplus_{i+j=n} (\Omega_i^U \otimes\mathbb{Q})\otimes H_j(\mathbb{CP}_\infty, \mathbb{Q})$, and $\Omega_i^U \cong \lim_{k\to\infty} \pi_{i+k}(TBU_k, \infty)$, and by using the previous arguments and applying the Serre theorem and the Thom isomorphism we get $\pi_{i+k}(TBU_k, \infty) \otimes \mathbb{Q} \cong H_{i+k}(TBU_k, \infty, \mathbb{Z}) \otimes \mathbb{Q} \cong H_i(BU[\frac{k}{2}], \mathbb{Q})$, for $k > i + 1$. More precisely, the isomorphisms involved in the context of this proof are forgetful isomorphisms. Thus the result follows. □

### 3.3.3 Relation Between Bordism Groups $\Omega_n^{U,N,t}(F)$ and $\Omega_n^{U,N^{c,t}}$

Let $G$ and $H$ be two compact Lie groups. The following two Lemmas are due to R. H. Szczarba [Szc64].

**Lemma 3.3.7.** *Let $\xi = (E(\xi), \pi, B(\xi))$ be a principal $G$-bundle with $E(\xi)$ compact and assume that $H$ acts (on the left) on $E(\xi)$ such that*

1. *$(hx)g = h(xg)$ for all $h \in H, x \in E(\xi)$, and $g \in G$,*
2. *the induced action of $H$ on $B(\xi)$ has no fixed points, and*
3. *the spaces $E(\xi)/H$ and $B(\xi)/H$ are Hausdorff.*

*Then the triple $(E(\xi)/H, \pi', B(\xi)/H)$ is a principal $G$-bundle, where $\pi'$ is induced by $\pi$.*

*Furthermore, if $G$ acts on a space $F$ we may form the $F$-bundle associated with $\xi$ and $\xi/H = (E(\xi)/H, \pi', B(\xi)/H)$. Then $H$ acts on $E(\xi) \underset{G}{\times} F$ by acting on the first factor and one has $(E(\xi) \underset{G}{\times} F)/H = (E(\xi)/H) \underset{G}{\times} F$.*

**Lemma 3.3.8.** *Let $M$ be a connected oriented manifold having a free circle action and let $\pi : M \longrightarrow M/S^1$ be the orbit projection. Then*

$$\tau(M)/S^1 \cong \tau(M/S^1) \oplus \epsilon,$$

*where $\epsilon$ denotes the trivial line bundle over $M/S^1$.*

**Lemma 3.3.9.** *Let $M$ be an $N$-manifold and let $\alpha$ be a free circle action on $M$, and let $t \neq 0$ be such that $t|N$. Then $\alpha$ has type $t$ if and only if the orbit space $M/S^1$ can be given an $N^{c,t}$-structure with principal $U(1)$-bundle $\pi : M \longrightarrow M/S^1$ and principal $N^{c,t}$-bundle given by the composition $E \longrightarrow M \longrightarrow M/S^1$, where $E \longrightarrow M$ is the $\hat{U}(1)$-bundle over $M$ determined by the given $N$-structure on $M$.*

*Proof.* Let the $N$-structure on $M$ be given by the following commutative diagram:

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & K \arrow[dl] \\ & M & \end{tikzcd}$$

where $K$ is the $U(1)$-bundle associated to the stable tangent bundle representing the $U$-structure on $M$, $E$ is a $\hat{U}(1)$-bundle, and $f$ is the $N$-covering map.

Suppose that the circle action $\alpha$ has type $t$. Then the connected $N/t$-covering $\hat{S}^1$ of $S^1$ acts freely on $E$ and commutes with the right $\hat{U}(1)$-action on $E$, and is compatible with $f : E \longrightarrow K$. Define an action $\hat{\alpha}$ of $\hat{U}(1) \times \hat{S}^1$ on $E$ by sending $(x, (g, \lambda))$ to $\lambda^{-1}xg$, where $x \in E$, $g \in \hat{U}(1)$, and $\lambda \in \hat{S}^1$.

This is a free action of $\hat{U}(1) \times \hat{S}^1$ on $E$ and it induces a free action of $U^{N,c,t}$ on $E$.

We prove that $\pi \circ p : E \longrightarrow M \longrightarrow M/S^1$ is a $U^{N,c,t}$-bundle over $M/S^1$.

Clearly $\pi \circ p(\lambda^{-1}xg) = \pi \circ p(x)$, so $\pi \circ p$ induces a surjective map $E/U^{N,c,t} \longrightarrow M/S^1$. We see that this map is also injective. If $\pi \circ p(x) = \pi \circ p(y)$, then $p(x) = p(\hat{\lambda}y)$ for some $\lambda \in S^1$, where $\hat{\lambda}$ is sent to $\lambda$ under the $N/t$-covering map $\mu_{N/t} : \hat{S}^1 \longrightarrow S^1$. Also, $E$ is a $U^{N,c,t}$-bundle, so $x = \hat{\lambda}yg$ for some $g \in U^{N,c,t}$. This proves the injectivity. Hence $\pi \circ p : E \longrightarrow M/S^1$ is a $U^{N,c,t}$-bundle.

Now we define a map

$$\mu : E \longrightarrow K/S^1 \times M$$

as

$$\mu(x) = (\bar{\pi}(f(x)), p(x)),$$

where $\bar{\pi} : K \longrightarrow K/S^1$ is the orbit projection.

Let us show that the map $\mu$ is equivariant. For $x \in E, g \in \hat{U}(1)$, and $\lambda \in \hat{S}^1$, we have

$$\begin{aligned}
\mu(\lambda^{-1}xg) &= (\bar{\pi}(f(\lambda^{-1}xg)), p(\lambda^{-1}xg)) \\
&= (\bar{\pi}(f(x).\lambda_N(g)), \alpha(\mu_{N/t}(\lambda^{-1}), x)) \\
&= \mu(x).(\lambda_N(g), \mu_{N/t}(\lambda)),
\end{aligned}$$

where $\lambda_N : \hat{U}(1) \longrightarrow U(1)$ is the $N$-covering of $U(1)$ given by $g \longmapsto g^N$.

Furthermore, we have the following commutative diagram:

$$\begin{tikzcd}
E \arrow[rr, "\mu"] \arrow[dr] & & K/S^1 \times M \arrow[dl] \\
& M/S^1. &
\end{tikzcd}$$

So we have an $N^{c,t}$-structure on $K/S^1$ and hence on $M/S^1$.

Conversely assume that the $N$-structure on $M$ induces an $N^{c,t}$-structure on $M/S^1$ as in the following diagram:

$$\begin{tikzcd}
E \arrow[rr, "\mu"] \arrow[dr, "\pi\circ p"'] & & K/S^1 \times M \arrow[dl] \\
& M/S^1, &
\end{tikzcd}$$

where $\mu$ sends an element $x$ of $E$ to the element $(\bar{\pi}(f(x)), p(x))$ of $K/S^1 \times M$.

Let us consider the following commutative pullback diagram:

$$\begin{tikzcd}
E \arrow[drr, "\bar{\pi}\circ f"] \arrow[ddr, "p"'] & & \\
& \pi^*(K/S^1) \arrow[r] \arrow[d] & K/S^1 \arrow[d] \\
& M \arrow[r, "\pi"'] & M/S^1.
\end{tikzcd}$$

Define a map $\nu : E \longrightarrow \pi^*(K/S^1)$ by $\nu(x) = (\bar{\pi}(f(x)), p(x))$, for $x \in E$.

Because the map $\mu : E \longrightarrow K/S^1 \times M$ is an $N$-covering, therefore, the diagram

$$\begin{tikzcd}
E \arrow[rr, "\nu"] \arrow[dr, "p"'] & & \pi^*(K/S^1) \arrow[dl] \\
& M &
\end{tikzcd}$$

defines an $N$-structure on $M$ which, under the identification $K \cong \pi^*(K/S^1)$, coincides with the given $N$-structure on $M$.

We need to show that $\alpha$ is of type $t$. Let $x \in E$, and map induced by $\alpha$

$$\alpha_{\nu(x)} : \mathbb{R} \longrightarrow \pi^*(K/S^1)$$

given by

$$\alpha_{\nu(x)}(s) = (\bar{\pi}(f(x)), \alpha(e^{2\pi\iota s}, p(x))).$$

Also, define

$$\hat{\alpha}_{\nu(x)} : \mathbb{R} \longrightarrow E$$

by

$$\hat{\alpha}_{\nu(x)}(s) = x.[1, e^{\frac{2\pi t\iota s}{N}}]$$

where $[1, e^{\frac{2\pi\iota t s}{N}}] \in U^{N,c,t}$. We see that $\hat{\alpha}_{\nu(x)}(0) = x$, and $\nu(\hat{\alpha}_{\nu(x)}(s)) = \nu(x.[1, e^{\frac{2\pi\iota t s}{N}}]) = \mu(x.[1, e^{\frac{2\pi\iota t s}{N}}]) = (\bar{\pi}(f(x)), \alpha(e^{2\pi\iota s}, p(x))) = \alpha_{\nu(x)}(s)$. Therefore, $\hat{\alpha}_{\nu(x)}$ is a lifting of $\alpha_{\nu(x)}$, and $\hat{\alpha}_{\nu(x)}(1) = x.[1, e^{\frac{2\pi\iota t}{N}}]$. Thus the circle action $\alpha$ is of type $t$. $\square$

The above Lemma can be translated in terms of bordism language as follows.

**Proposition 3.3.10.** *For $t \neq 0$ and $t|N$, we have an isomorphism*

$$\Omega_n^{U,N,t}(F) \longrightarrow \Omega_{n-1}^{N^{c,t}}$$

*given by*

$$[M] \longrightarrow [M/S^1].$$

A direct consequence of Theorem 3.3.6 and Proposition 3.3.10 is

**Theorem 3.3.11.** *If $t \neq 0$ and $t|N$, then the homomorphism*

$$\Omega_n^{U,N,t}(F) \longrightarrow \Omega_{n-1}^{U}(\mathbb{CP}_\infty)$$

*given by*

$$[M] \longmapsto [M/S^1, f]$$

*is an isomorphism after tensoring with the rationals $\mathbb{Q}$, where $f$ is the classifying map of the $S^1$-bundle $M \longrightarrow M/S^1$.*

# Chapter 4

# Applications

In this chapter we will discuss few applications of theory developed in the previous chapter to explore the ideal $I_*^{N,t}$ in the rational bordism ring $\Omega_*^{U,N} \otimes \mathbb{Q}$ generated by bordism classes of complex $N$-Spin manifolds admitting an effective circle action of type $t$. More precisely, we will find a sufficient condition to determine the ideal $I_*^{N,t}$ and show that the condition holds true for several values of $N$ and $t$, and thereby verify the conjecture of Höhn for those values of $N$ and $t$. More precisely, we will see that the conjecture holds true for all values of $t$ with $N \leq 9$, except for case $(N, t) = (6, 3)$ which remains unanswered.

First we set up a general framework for computations to explore the ideal $I_*^{N,t}$ by exploiting characterizations, obtained in the previous chapter, of the bordism groups in the exact sequence of Corollary 3.1.11.

## 4.1 Exact Sequence

In what follows we will denote with an abuse of notation a map $f \otimes \mathbb{Q}$ simply by $f$ without mentioning it further. The sequence in Corollary 3.1.11 remains exact after tensoring with the rationals $\mathbb{Q}$:

The sequence

$$\cdots \longrightarrow \Omega_n^{U,N,t}(F)\otimes\mathbb{Q} \xrightarrow{i_{N,t}} \Omega_n^{U,N,t}(SF)\otimes\mathbb{Q} \xrightarrow{j_{N,t}} \Omega_n^{U,N,t}(SF,F)\otimes\mathbb{Q} \xrightarrow{\partial_{N,t}} \Omega_{n-1}^{U,N,t}(F)\otimes\mathbb{Q} \longrightarrow \cdots \tag{4.1}$$

is exact.

Composing the isomorphisms of Corollary 3.2.13 and Theorem 3.3.11 suitably with the maps $j_{N,t}$ and $\partial_{N,t}$ of sequence (4.1) we get maps $\bar{j}_{N,t}$ and $\bar{\partial}_{N,t}$, respectively, which we will denote with an abuse of notation simply by $j_{N,t}$ and $\partial_{N,t}$, respectively. Thus we have

**Proposition 4.1.1.** *The sequence*

$$\cdots \longrightarrow \Omega_n^{U,N,t}(SF)\otimes\mathbb{Q} \xrightarrow{j_{N,t}} \bigoplus_{\substack{k+2p+2q=n \\ p-q\equiv t \pmod N}} \Omega_k^U(BU(p)\times BU(q))\otimes\mathbb{Q}$$
$$\xrightarrow{\partial_{N,t}} \Omega_{n-2}^U(\mathbb{CP}_\infty)\otimes\mathbb{Q} \longrightarrow \cdots$$

*is exact.*

Now we recall from [Höh91], the details about the complex twisted projective bundles. Let $E$ and $F$ be complex vector bundles of rank $p$ and $q$ over a $U$-manifold $B$. The $U$-structure on $B$ and the complex structure on $E\oplus F$ provides a $U$-structure $\sigma^*(E\oplus F)\oplus\sigma^*TB$ on $E\oplus F \xrightarrow{\sigma} B$ and - after a choice of a Hermitian metric - on the disk bundle $D(E\oplus F)$, and the sphere bundle $S(E\oplus F = \partial D(E\oplus F)$. Consider on $E\oplus F$ the $S^1$-action $\alpha$ : $S^1\times E\oplus F \to E\oplus F$ given by $\alpha(\lambda,(p,e_p,f_p)) = (p,\lambda e_p,\lambda^{-1}f_p)$. This $S^1$-action restricted to $S(E\oplus F)$ is free and respects the $U$-structure. The quotient $S(E\oplus F)/\alpha$ as a differentiable manifold is the complex projective bundle $\mathbb{CP}(E\oplus\overline{F})$, where $\overline{F}$ is the complex bundle conjugate to $F$. We have the following sequence of complex vector bundles over the complex projective bundle $\mathbb{CP}(E \oplus \overline{F})$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S & \longrightarrow & \pi^*(E \oplus \overline{F}) & \longrightarrow & Q & \longrightarrow & 0 \\ & & & & \downarrow & & & & \\ & & & & \mathbb{CP}(E \oplus \overline{F}) & & & & \\ & & & & \downarrow{\scriptstyle \pi} & & & & \\ & & E \oplus F & \xrightarrow{\sigma} & B & & & & \end{array} \tag{4.2}$$

where $S$ is the tautological line bundle and $Q$ the quotient bundle.

**Definition 4.1.2.** The twisted projective bundle $\widetilde{\mathbb{CP}}(E \oplus F)$ of two complex vector bundles $E$ and $F$ over a $U$-manifold $B$ is the bundle $\mathbb{CP}(E \oplus \overline{F})$ with the stably almost complex structure

$$T\widetilde{\mathbb{CP}}(E \oplus F) := S^* \otimes \pi^* E \oplus \overline{S^*} \otimes \pi^* F \oplus \pi^* TB \tag{4.3}$$

The orientation of $\widetilde{\mathbb{CP}}(E \oplus F)$ induced by this "twisted" stably almost complex structure is $(-1)^q$ times the usual orientation of $\mathbb{CP}(E \oplus \overline{F})$.
The cohomology of $\widetilde{\mathbb{CP}}(E \oplus F)$ is the cohomology of $\mathbb{CP}(E \oplus \overline{F})$:

$$H^*(\widetilde{\mathbb{CP}}(E \oplus F)) \simeq H^*(B)[t]/ < t^{p+q} + c_1(E \oplus \overline{F})t^{p+q-1} + ... + c_{p+q}(E \oplus \overline{F}) >, \tag{4.4}$$

where $t = c_1(S^*)$ is the first Chern class of the dual bundle of the tautological line bundle $S$.
The $S^1$-bundle $S(E \oplus F)$ over $\widetilde{\mathbb{CP}}(E \oplus F)$ under the given action $\alpha$ has the first Chern class $-t$. If $f$ is the $S^1$-principal bundle classifying map, then $[\widetilde{\mathbb{CP}}(E \oplus F), f]$ defines an element in $\Omega_*^U(\mathbb{CP}_\infty)$. The bordism group $\Omega_*^U(\mathbb{CP}_\infty)$ as a $\Omega_*^U$-module is isomorphic to $\Omega_{*+1}^U(F)$. The stably almost complex structure on $\widetilde{\mathbb{CP}}(E \oplus F)$ was chosen such that $[\widetilde{\mathbb{CP}}(E \oplus F), f]$ under the above isomorphism of $S^1$-manifolds is equivalent to $[S(E \oplus F), \alpha]$.

Let $\upsilon_i^-$ and $\upsilon_i^+$ have the same meanings as in the remarks preceding Proposition 3.2.3 and $\widetilde{\mathbb{CP}}((\upsilon_i^+ \oplus \epsilon_{\mathbb{C}}^1) \oplus \upsilon_i^-)$ denote their twisted projective bundle. We have

**Proposition 4.1.3.** (See [HT72]) *Define*

$$k_U : \Omega_n^U(SF, F) \longrightarrow \Omega_n^U(SF)$$

*by*

$$[M] \longmapsto \sum_i [\widetilde{\mathbb{CP}}((\upsilon_i^+ \oplus \epsilon_{\mathbb{C}}^1) \oplus \upsilon_i^-)],$$

*where* $\epsilon_{\mathbb{C}}^1$ *is the trivial complex line bundle. Then* $k_U \circ j_U = \mathrm{id}$*, where* $j_U$ *is the inclusion map* $\Omega_n^U(SF) \longrightarrow \Omega_n^U(SF, F)$*.*

Since $[W] = \mathrm{id}[W] = k_U \circ j_U[W] = k_U(j_U[W]) = k_U[W]$, therefore, the equation

$$[W] = \sum_i [\widetilde{\mathbb{CP}}((\upsilon_i^+ \oplus \epsilon_{\mathbb{C}}^1) \oplus \upsilon_i^-)] \tag{4.5}$$

holds in $\Omega_n^U(SF)$.

Let

$$\varepsilon_{N,t} : \bigoplus_{\substack{2k+2p+2q=n \\ p-q \equiv t \pmod N}} \Omega_{2k}^U(BU(p) \times BU(q)) \otimes \mathbb{Q} \longrightarrow \Omega_n^U \otimes \mathbb{Q}$$

be the linear map defined by sending $[X, E \oplus F]$ to $[\widetilde{\mathbb{CP}}((E \oplus \epsilon_{\mathbb{C}}^1) \oplus F)]$.

**Theorem 4.1.4.** *Suppose there exist bordism classes* $[U_1]$, $[U_2]$, $\ldots$, $[U_m] \in \mathrm{Im}(\varepsilon_{N,t}|_{\ker(\partial_{N,t})})$ *such that*

$$I_{N,t} \subset \langle \varphi_{\mathrm{ell}}[U_1], \varphi_{\mathrm{ell}}[U_2], \ldots, \varphi_{\mathrm{ell}}[U_m] \rangle.$$

*Then we have*

$$I_*^{N,t} = \bigcap_{\substack{n \mid N \\ n \nmid t}} \ker \varphi_n$$

*Proof.* Let $[U_i'] \in \ker(\partial_{N,t})$ be such that $\varepsilon_{N,t}([U_i']) = [U_i]$. By the exactness of sequence in Proposition 4.1.1, there exists a $[V_i] \in \Omega_n^{U,N,t}(SF) \otimes \mathbb{Q}$ such that $j_{N,t}([V_i]) = [U_i']$. By equation (4.5) the Chern numbers of $[V_i]$ and $[U_i]$ are the same, and hence $\varphi_{\mathrm{ell}}[V_i] = \varphi_{\mathrm{ell}}[U_i]$. So there exist bordism clasees $[V_1], [V_2], \ldots, [V_m] \in I_*^{N,t}$ such that $I_{N,t} \subset \langle \varphi_{\mathrm{ell}}[V_1], \varphi_{\mathrm{ell}}[V_2], \ldots, \varphi_{\mathrm{ell}}[V_m] \rangle$. Thus the result follows by Proposition 2.3.9. □

## 4.2 The Ideals $I_*^{N,t}$

In this section we study the ideal $I_*^{N,t}$ in the ring $\Omega^{N,t} \otimes \mathbb{Q}$ generated by bordism classes of $N$-manifolds with a semifree circle action of type $t$. More precisely, we verify the conjecture of Höhn (see Chapter 3) for several values of $N$ and $t$ by constructing the bordism classes of manifolds satisfying the sufficient condition of Lemma 4.1.4. We will begin with the case $N = 4$ and $t = 2$.

**Theorem 4.2.1.** *Let $\varphi_4$ denote the elliptic genus of level 4. Then we have*

$$I_*^{4,2} = \ker(\varphi_4).$$

*Proof.* Consider the linear map

$$\partial_{4,2} : \bigoplus_{\substack{k+p+q=4 \\ p-q \equiv 2 \pmod 4}} \Omega_{2k}^U(BU(p) \times BU(q)) \otimes \mathbb{Q} \longrightarrow \Omega_6^U(\mathbb{CP}_\infty) \otimes \mathbb{Q}.$$

We intend to compute its kernel. Let $E_i$ and $F_i$ be complex vector bundles over $U$-manifold $B_i$, $1 \leq i \leq 4$, where $B_1$ and $B_2$ are zero dimensional $U$-manifolds with the number of elements $r$ and $s$ respectively, and $B_3$ and $B_4$ are 2- dimensional manifolds. If the bundles $E_1, E_2, E_3, E_4$ are of $3, 1, 2, 0$ complex ranks, and $F_1, F_2, F_3, F_4$ are of $1, 3, 0, 2$ complex ranks, then

$$[E_1 \oplus F_1, B_1] + [E_2 \oplus F_2, B_2] + [E_3 \oplus F_3, B_3] + [E_4 \oplus F_4, B_4]$$

is a well-defined element of $\bigoplus_{\substack{k+p+q=4 \\ p-q \equiv 2 \pmod 4}} \Omega_{2k}^U(BU(p) \times BU(q))$ and its rational image under $\partial_{4,2}$ is given by

$$[\widetilde{\mathbb{CP}}(E_1 \oplus F_1), h_1] + [\widetilde{\mathbb{CP}}(E_2 \oplus F_2), h_2] + [\widetilde{\mathbb{CP}}(E_3 \oplus F_3), h_3] + [\widetilde{\mathbb{CP}}(E_4 \oplus F_4), h_4]. \tag{4.6}$$

Consider the total Chern classes $c(B_3) = 1 + a_1(B_3) + a_2(B_3)$, $c(B_4) = 1 + b_1(B_4) + b_2(B_4)$, $c(E_3) = 1 + e_1(E_3) + e_2(E_3) + e_3(E_3)$, $c(F_4) = 1 + f_1(F_4) + f_2(F_4)$, and $d$ denote the first

Chern class of the dual bundle of the tautological line bundle. By using the equation (4.4), the generalized Chern numbers of the expression in (4.6) are given by

$$
\begin{aligned}
c_3 &= 2r - 2s + 2a_2 - 2b_2,\\
c_2c_1 &= 2a_1^2 + 2a_2 - 2b_1^2 - 2b_2,\\
c_1^3 &= -8r + 8s + 6a_1^2 + 2e_1^2 - 8e_2 - 6b_1^2 - 2f_1^2 + 8f_2,\\
c_2d &= -a_2 + a_1e_1 - b_2 + b_1f_1,\\
c_1^2d &= 4r + 4s - a_1^2 - e_1^2 + 4e_2 + 2a_1e_1 - b_1^2 - f_1^2 + 4f_2 + 2b_1f_1,\\
c_1d^2 &= -2r + 2s + e_1^2 - 2e_2 - 1a_1e_1 - f_1^2 + 2f_2 + b_1f_1,\\
d^3 &= 1r + 1s + 0a_1^2 - e_1^2 + e_2 + -f_1^2 + f_2.
\end{aligned}
$$

An element of $\Omega_6^U(\mathbb{CP}_\infty)$ is zero if the all of its generalized Chern numbers are zero. So each solution of the system, obtained from the above system by putting left sides of equations equal to zero, determines an element of the kernel of $\partial_{4,2}$. The solution set of the system is spanned by $(-5,-3,-14,2,-10,-2,0,-12,0,0,0,2)$, $(-1,-1,0,0,0,1,0,0,0,0,1,0)$, $(1,1,3,0,1,0,0,3,0,1,0,0)$, $(7,3,20,-2,12,2,0,16,2,0,0,0)$, $(-1,0,-1,1,-1,0,1,0,0,0,0,0)$ with respect to the coordinates $(r,s,a_1^2,a_2,e_1^2,e_2,a_1e_1,b_1^2,b_2,f_1^2,f_2,b_1f_1)$.

Consider the linear map

$$\varepsilon_{4,2} : \bigoplus_{\substack{k+p+q=4 \\ p-q\equiv 2 \pmod 4}} \Omega_{2k}^U(BU(p)\times BU(q))\otimes\mathbb{Q} \longrightarrow \Omega_8^U\otimes\mathbb{Q}.$$

We intend to compute $\mathrm{Im}(\varepsilon_{4,2}\mid_{\ker(\partial_{4,2})})$. Let $E_i$, $F_i$ , $1\le i\le 4$ be as above, then the Chern numbers of the bordism class

$$[\widetilde{\mathbb{CP}}((E_1\oplus\epsilon_{\mathbb{C}}^1)\oplus F_1)] + [\widetilde{\mathbb{CP}}((E_2\oplus\epsilon_{\mathbb{C}}^1)\oplus F_2)] + [\widetilde{\mathbb{CP}}((E_3\oplus\epsilon_{\mathbb{C}}^1)\oplus F_3)] + [\widetilde{\mathbb{CP}}((E_4\oplus\epsilon_{\mathbb{C}}^1)\oplus F_4)]$$

are given by

$$\begin{aligned}
c_4 &= 3r - s + 3a_2 - b_2, \\
c_3c_1 &= 6r + 2s + 3a_1^2 + 9a_2 - b_1^2 + b_2, \\
c_2^2 &= -4r - 4s + 9a_1^2 + 6a_2 + 1e_1^2 - 3e_2 + b_1^2 - 2b_2 + f_1^2 - 3f_2, \\
c_2c_1^2 &= -18r + 2s + 21a_1^2 + 9a_2 + 6e_1^2 - 18e_2 + b_1^2 + b_2 + 2b_1f_1, \\
c_1^4 &= -81r - s + 54a_1^2 + 27e_1^2 - 81e_2 + 6b_1^2 + 3f_1^2 - f_2 + 8b_1f_1.
\end{aligned}$$

One obtains that the image $\mathrm{Im}(\varepsilon_{4,2} \mid_{\ker(\partial_{4,2})})$ is spanned by $[U_1] = (1, 4, -1, 0, 0)$ and $[U_2] = (0, 3, 13, 28, 64)$, where the coordinates are the Chern numbers $(c_4, c_3c_1, c_2^2, c_2c_1^2, c_1^4)$.

Using the remark preceding Theorem 2.2.3, we have $\varphi_{\mathrm{ell}}([U_1]) = AC - D/2$, and $\varphi_{\mathrm{ell}}([\mathbb{CP}_3]) = (2C)/3 - (AB)/6 + (4A^3)/3$. Since $I_{4,2} = \langle 2AC - D, 8A^3 - AB + 4C\rangle$ by remarks preceding Definition 2.3.5 we see that $I_{4,2} \subset \langle \varphi_{\mathrm{ell}}([U_1]), \varphi_{\mathrm{ell}}([\mathbb{CP}_3])\rangle$. Theorem 4.1.4 finishes the proof. □

We will see that the conjecture also hold true for a few higher values of $N$. We can give proofs for those cases of higher values of $N$ along lines of the above proof. However, to simplify their presentation we will use the idea of Hilbert-Poincaré series of a graded ideal of $\mathbb{C}[A, B, C, D]$.

Let $I = \bigoplus_{n\in\mathbb{N}} I^{(n)}$ be a graded ideal in $\mathbb{C}[A, B, C, D]$, where each $I^{(n)}$ containing elements of degree $n$ has the finite dimension, say, $d_n$. The Hilbert-Poincaré series of $I$ in indeterminate $s$ is given by $\sum_{n\in\mathbb{N}} d_n s^n$. We make an observation here which will be useful in formulating a simple proof of the conjecture for a few higher values of $N$: Let $I' \subset I$ be an inclusion of graded ideals in $\mathbb{C}[A, B, C, D]$, and let $I$ be generated by a finite number of elements with the highest degree of generators equal to $n_0$. If $d'_n = d_n$ for all $n \leq n_0$, then we observe that $I' = I$. In fact, from the inclusion of graded ideals we see that ${I'}^{(n)} \subset I^{(n)}$ for all $n$, and so for dimension reasons we must have ${I'}^{(n)} = I^{(n)}$ for $n \leq n_0$. Since the highest degree of generators of $I$ is $n_0$, so it is less than equal to $n_0$ for $I'$ due to the inclusion $I' \subset I$. Thus a generic elements of these ideals is a combination of elements of dimensions less than equal to $n_0$. It follows that ${I'}^{(n)} = I^{(n)}$ for all $n$, and hence the conclusion $I' = I$.

From now on we will be more specific about notations of maps $\varepsilon_{N,t}$ and $\partial_{N,t}$, and will denote them by $\varepsilon_{N,t}^{(n)}$ and $\partial_{N,t}^{(n)}$, respectively, when their domains of definition have degree $n$. We will denote the image $\varphi_{ell}(\mathrm{Im}(\varepsilon_{N,t}^{(n)}|_{\ker(\partial_{N,t}^{(n)})}))$ of the universal elliptic genus by ${I'}_{N,t}^{(n)}$. Let $I'_{N,t} = \bigoplus_{n \in \mathbb{N}} {I'}_{N,t}^{(n)}$ and $I_{N,t} = \bigoplus_{n \in \mathbb{N}} I_{N,t}^{(n)}$ be graded ideals of $\mathbb{C}[A, B, C, D]$ with dimensions of ${I'}_{N,t}^{(n)}$ and $I_{N,t}^{(n)}$ denoted by $d'_n$ and $d_n$, respectively. We remark that $I'_{N,t} \subset I_{N,t}$. In fact, a generic element of $I'_{N,t}$ has the form $\varphi_{ell}[U]$, where $[U] \in \mathrm{Im}(\varepsilon_{N,t}|_{\ker(\partial_{N,t})})$. As in the context of the proof of Theorem 4.1.4, there exists $[V] \in I_*^{N,t}$ such that $\varphi_{ell}[U] = \varphi_{ell}[V]$. By Corollary 2.3.8, we have $[V] \in \bigcap_{\substack{n \mid N \\ n \nmid t}} \ker \varphi_n$, and so $[V] \in \ker \psi_{N,t}$ by Lemma 2.3.6. By equation (2.7), we get $[V] \in \ker(\pi_{N,t} \circ \varphi_{ell})$. Hence $\pi_{N,t}(\varphi_{ell}[V])$ is the zero element and so $\varphi_{ell}[V] \in \ker \pi_{N,t} = I_{N,t}$. Hence $\varphi_{ell}[U] \in I_{N,t}$, and as it was a generic element of $I'_{N,t}$, we have the inclusion $I'_{N,t} \subset I_{N,t}$.

This remark and the above observation will be used in proving the following result which will give a general framework to present a simplified proof of the conjecture for several values of $N$ and $t$.

**Proposition 4.2.2.** *Let $n_0$ be the highest degree of generators of $I_{N,t}$. If $d_n = d'_n$ for $n \leq n_0$. Then*

$$I_*^{N,t} = \bigcap_{\substack{n \mid N \\ n \nmid t}} \ker \varphi_n.$$

*Proof.* Since $I'_{N,t} \subset I_{N,t}$, therefore, hypothesis of the proposition together with the above observation imply that $I'_{N,t} = I_{N,t}$. It follows from this equation and the definition of $I'_{N,t}$ that the sufficient condition of Theorem 4.1.4 pertaining to the existence of bordism classes and inclusion is satisfied with equality. This completes the proof. $\square$

**Theorem 4.2.3.** *The dimensions $d_n$ and $d'_n$ of $I^{(n)}_{N,t}$ and $I'^{(n)}_{N,t}$, respectively, up to the degree $n = 12$, for several values of $N$ and $t$ are given in Table 4.1.*[1]

**Table 4.1**: *Dimensions $d_n$ and $d'_n$ of $I^{(n)}_{N,t}$ and $I'^{(n)}_{N,t}$, respectively.*

| $(N,t)/n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $(4,2)$ | | | 1 | 2 | 3 | 5 | 7 | 10 | 13 | 17 | 21 | 27 |
| $(6,2)$ | | | | | 1 | 1 | 3 | 5 | 7 | 10 | 14 | 18 |
| $(6,3)$ | | | | | 1 | 2 | 4 | 6 **(5)** | 9 | 12 **(11)** | 16 | 21 **(19)** |
| $(8,2)$ | | | | | | | 1 | 1 | 3 | 4 | 7 | 10 |
| $(8,4)$ | | | | | | | 1 | 2 | 4 | 6 | 9 | 13 |
| $(9,3)$ | | | | | | | | 1 | 1 | 3 | 4 | 8 |
| $(10,2)$ | | | | | | | | | 1 | 1 | 3 | 4 |
| $(10,5)$ | | | | | | | | | 1 | 2 | 4 | 6 |

The dimensions $d'_n$ are made bold and given in Table 4.1 only when they differ from the dimensions $d_n$. The entries in the table containing zeros are kept empty. Now we will use the highest degree of generators of $I_{N,t}$ to accomplish the following result. In most cases we can verify that $I_{N,t}$ is generated by three elements whose degrees can be determined. But we will not need to use this fact in the proof, however, we will discuss it later for extra information.

**Theorem 4.2.4.** *For $(N,t) = (6,2)$, $(8,2)$, $(8,4)$, and $(9,3)$, we have*

$$I_*^{N,t} = \bigcap_{\substack{n \mid N \\ n \nmid t}} \ker \varphi_n.$$

*Proof.* The highest degrees of generators of $I_{6,2}$, $I_{8,2}$, $I_{8,4}$, and $I_{9,3}$ are 8, 12, 9, and 12, respectively[2]. The result follows from Proposition 4.2.2 and Theorem 4.2.3. $\square$

[1] See appendix for the *Mathematica* program.

[2] The computations were accomplished with the help of a computer program in MAGMA.

Our technique does not work for higher values of $N$ because computer runs out of memory while running *Mathematica* program to compute dimensions $d'_n$ of $I'^{(n)}_{N,t}$ for higher degrees $n$. For instance, the highest degree of generators of $I_{10,2}$ is 16 and this degree is too big for our *Mathematica* program to compute $d'_{16}$ for $(N,t) = (10,2)$. The technique developed in our work does not generate enough bordism classes in case of $(N,t) = (6,3)$ to verify the sufficient condition of Theorem 4.1.4. It is our conjecture that the technique gives enough bordism classes except when $(N,t) = (4k+2, k+1)$ for $k \geq 1$, however, *Mathematica* program does not work in computations of dimensions $d'_n$ of $I'^{(n)}_{N,t}$ for some degrees $n$ needed for higher values of $N$. Our technique works if $t = 1$ or $t$ does not divide $N$, and *Mathematica* computations can be made to verify the conjecture for values of $N$ up to 9, as well. However, we combine Theorem 2.3.10, Theorem 4.2.1, and Theorem 4.2.4 to state the following result.

**Theorem 4.2.5.** *Let $t$ be a divisor of $N$ such that $t \neq N$ and $2 \leq N \leq 9$ besides $(N,t) = (6,3)$. We have*

$$I_*^{N,t} = \bigcap_{\substack{n \mid N \\ n \nmid t}} \ker \varphi_n.$$

Let $P_{N,t}(s)$ denote the product of the polynomial $(1-s)(1-s^2)(1-s^3)(1-s^4)$ and the Hilbert-Poincarè series of the quotient algebra $\mathbb{C}[A,B,C,D]/I_{N,t}$ in the indeterminate $s$. We conjecture that in most cases the ideal $I_{N,t}$ is generated at most by three elements. More precisely, we have:

**Conjecture 1.** *Let $t$ be a divisor of $N \geq 2$ and $(N,t) \neq (4k+2, k+1)$, $k \geq 1$. Then*

*(i) $P_{N,t}(s)$ is a polynomial of degree $2N - t + 1$ given by the equation*

$$P_{N,t}(s) = 1 - s^{N-1} - s^{N+1} - s^{2N-2t} + s^{2N-t-1} + s^{2N-t+1}, \tag{4.7}$$

*(ii) a minimal generating set of $I_{N,t}$ has a generator in degree $n$ precisely when the polynomial $P_{N,t}(s)$ contains a term $s^n$ with the coefficient $-1$.*

Observe that a cancelation occurs in equation (4.7) when $t = 1$ or $(N,t) = (4,2)$, and in these cases $I_{N,t}$ can be generated by two elements. We have verified the above conjecture up to $N = 18$ with the help of computations.[3]

Notice that our technique seems to break down in verifying the conjectural equation of Höhn [Höh91] when $(N,t) = (4k+2, k+1)$, $k \geq 1$ (cf. $(N,t) = (6,3)$ in Table 4.1). These values of $N$ and $t$ are consistent with two facts, namely, the values of $N$ and $t$ for which $P_{N,t}(s)$ does not satisfy the equation (4.7) and $d'_n$ is smaller than $d_n$. For $(N,t) = (4k+2, k+1)$, $1 \leq k \leq 4$, we have

$$P_{6,3}(s) = 1 - s^5 - s^6 - s^7 + s^9 + s^{10} + s^{11} - s^{12},$$

$$P_{10,5}(s) = 1 - s^9 - s^{10} - s^{11} + s^{14} + s^{17} + s^{19} - s^{20},$$

$$P_{14,7}(s) = 1 - s^{13} - s^{14} - s^{15} + s^{20} + s^{22} - s^{24} + s^{25} + s^{27} - s^{28},$$

$$P_{18,9}(s) = 1 - s^{17} - s^{18} - s^{19} - s^{21} - s^{23} + 2s^{25} + 2s^{26} + 2s^{27} - s^{28} - s^{30}.$$

Let $c_n$ and $c'_n$ denote the coefficients of Hilbert-Poincaré series of the ideals $\bigcap\limits_{\substack{n \mid N \\ n \nmid t}} \ker \varphi_n$ and $\mathrm{Im}(\varepsilon_{N,t}|_{\ker(\partial_{N,t})}) = \bigoplus\limits_{n\in\mathbb{N}} \mathrm{Im}(\varepsilon^{(n)}_{N,t}|_{\ker(\partial^{(n)}_{N,t})})$ in $\Omega^{U,N}_* \otimes \mathbb{Q}$, respectively. Thus $c_n$ is the coefficient of $s^n$ in the Hilbert-Poincarè series of the quotient algebra $\Omega^{U,N}_* \otimes \mathbb{Q}/I^{N,t}_*$ in the indeterminate $s$ times the series $\prod\limits_{i=1}^{\infty}(1 - s^i)$.

**Theorem 4.2.6.** *The coefficients $c_n$ and $c'_n$ up to degree 12 for several values of $N$ and $t$ are given in Table 4.2*[4].

[3] The computations were accomplished with the help of computer programs in *Mathematica* and MAGMA.

[4] See appendix for the *Mathematica* program.

**Table 4.2**: *Values of $c_n$ and $c'_n$*

| $(N,t)/n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $(2,1)$ | 1 | 1 | 3 | 3 | 7 | 9 | 15 | 19 | 30 | 39 | 56 | 73 |
| $(3,1)$ | | 1 | 1 | 3 | 5 | 8 | 12 | 19 | 26 | 38 | 52 | 72 |
| $(4,1)$ | | | 1 | 1 | 4 | 6 | 11 | 15 | 25 | 34 | 50 | 67 |
| $(4,2)$ | | | 1 | 2 | 4 | 7 | 11 | 17 | 25 | 36 | 50 | 70 |
| $(5,1)$ | | | | 1 | 2 | 5 | 8 | 14 | 21 | 32 | 45 | 65 |
| $(6,1)$ | | | | | 2 | 3 | 7 | 11 | 19 | 28 | 43 | 59 |
| $(6,2)$ | | | | | 2 | 3 | 7 | 12 | 19 | 29 | 43 | 61 |
| $(6,3)$ | | | | | 2 | 4 | 8 | 13 **(12)** | 21 | 31 **(30)** | 45 | 64 **(62)** |
| $(7,1)$ | | | | | 1 | 3 | 5 | 10 | 16 | 26 | 38 | 57 |
| $(8,1)$ | | | | | 1 | 2 | 5 | 8 | 15 | 23 | 36 | 52 |
| $(8,2)$ | | | | | 1 | 2 | 5 | 8 | 15 | 23 | 36 | 53 |
| $(8,4)$ | | | | | 1 **(0)** | 2 **(1)** | 5 **(4)** | 9 **(8)** | 16 **(15)** | 25 **(24)** | 38 **(37)** | 56 **(55)** |
| $(9,1)$ | | | | | 1 | 2 | 4 | 8 | 13 | 22 | 33 | 50 |
| $(9,3)$ | | | | | 1 | 2 | 8 | 4 | 13 | 22 | 33 | 51 |
| $(10,1)$ | | | | | 1 | 2 | 4 | 7 | 13 | 20 | 32 | 47 |
| $(10,2)$ | | | | | 1 | 2 | 4 | 7 | 13 | 20 | 32 | 47 |
| $(10,5)$ | | | | | 1 **(0)** | 2 **(0)** | 4 **(0)** | 7 **(2)** | 13 **(8)** | 21 **(16)** | 33 **(28)** | 49 **(44)** |

The coefficients $c'_n$ are given and made bold in the table only when they differ from the coefficients $c_n$. The entries in the table containing zeros are kept empty. This table shows that for $(N,t) = (6,3)$, $(8,4)$, and $(10,5)$ the ideal $I_*^{N,t}$ is not generated by bordism classes of complex $N$-Spin manifolds admitting a semifree circle action of type $t$. In general, it is clear that for $t \geq 6$ the ideal $I_*^{N,t}$ can not be generated by bordism classes of complex $N$-Spin manifolds admitting a semifree circle action of type $t$ because for any non-negative integers $k$, $p$, and $q$ with $k+p+q=5$ we have $p+q \leq 5$ and so $|p-q| \leq 5$ which prevents the existence of $p$ and $q$ for $t|N$ such that $p-q \equiv t \pmod{N}$ thus no 5-dimensional manifold with the required semifree circle actions can exist but $\dim(I_5^{N,t}) \geq 1$. However, Table 4.2 suggests the following conjecture:

**Conjecture 2.** *Let $2 \leq N \leq 9$, $t|N$, and $(N,t) \neq (6,3)$, $(8,4)$, $(10,5)$. Then the ideal $I_*^{N,t}$ is generated by bordism classes of complex $N$-Spin manifolds admitting a semifree circle action of type $t$.*

S. Ochanine [Och87] proved such a result for Spin manifolds admitting a semifree circle action of odd type whose complex case version corresponds to $(N, t) = (2, 1)$ in the above conjecture.

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# Appendix

We provide on pages 74–77 the *Mathematica* program which was used to create Table 4.1 and Table 4.2. The last two rows of the program give values of entries in Table 4.1 and Table 4.2, respectively, with value of $(N, t, n)$ equal to $(\mathrm{dm}, \mathrm{nm}, \mathrm{tm}) = (4, 4, 2)$. We can change values of dm, nm, and tm in the first three rows of the program to obtain the values of other entries in the tables.

```
dm = 4;
nm = 4;
tm = 2;
dimMAX = dm + 1;
part /: part[0] = {{}};
Do[part /: part[n] =
Union[Flatten[
Table[Sort[Join[{i}, #]] & /@ part[n - i], {i, 1, n}], 1]], {n,
1, dimMAX}];
L = (Times @@ # &) /@ Map[c, part[dm], {2}];
fixcomp[d_, n_, t_] :=
Select[Table[{b, p, d - b - p}, {b, 0, d}, {p, 0, d - b}] //
Flatten[#, 1] &, Mod[#[[2]] - #[[3]], n] == t &];
mono[x_, y_, z_] :=
Times @@ # & /@
Flatten[Outer[Join, Map[b, part[x], {2}], Map[e, part[y], {2}],
Map[f, part[z], {2}], 1], 2];
P[k_, p_, q_] := Module[{t}, dim = k + p + q - 1;
su = p + q;
b[0] = 1; cb = Sum[b[i], {i, 0, k}];
e[0] = 1; ce = Sum[e[i], {i, 0, p}];
f[0] = 1; cf = Sum[f[i], {i, 0, q}];
subst =
Join[Table[b[i] -> b[i]*x^i, {i, 1, k}],
Table[e[i] -> e[i]*x^i, {i, 1, p}],
Table[f[i] -> f[i]*x^i, {i, 1, q}], {t -> t*x}];
degli[pol_] := CoefficientList[pol /. subst, x];
cfbar = Sum[(-1)^i*f[i], {i, 0, q}];
cv = ce*cfbar // ExpandAll;
cvli = degli[cv];
ClearAll[t];
t /: t^su = -Sum[cvli[[i + 1]]*t^(su - i), {i, 1, Min[k, su]}];
t /: t = t^1;
Do[t /: t^(su + i + 1) = (t^(su + i)*t) // ExpandAll, {i, 0,
k - 1}];
ces = Expand[Sum[e[i]*(1 + t)^(p - i), {i, 0, p}]];
cfs = Expand[Sum[f[i]*(1 - t)^(q - i), {i, 0, q}]];
ct = Expand[cb*ces*cfs];
ctli = degli[ct];
Table[c[i] = ctli[[i + 1]], {i, 1, dim}];
tmax[pol_] := Coefficient[pol, t, su - 1];
part[dim_] :=
If[dim == 0, {{}},
Union[Flatten[
Table[Sort[Join[{i}, #]] & /@ part[dim - i], {i, 1, dim}], 1]]];
makec[v_] := Times @@ (c[#] & /@ v);
d = -t;
base[dim_] :=
Table[(makec /@ part[dim - i])*d^i, {i, 0, dim}] // Flatten;
```

```
S = If[su == 1, Expand@base[dim], Map[tmax, Expand@base[dim]]];
U = (-1)^q*S];
make[f_, p_] := Times @@ (f[#] & /@ p);
basi[m_] :=
Table[Outer[Times, make[b, #] & /@ part[m - i - j],
make[e, #] & /@ part[i], make[f, #] & /@ part[j]], {i, 0,
m}, {j, 0, m - i}] // Flatten // Sort;
var[k_, p_, q_] :=
Complement[
Union[basi[k] /.
Join[Table[e[i] -> 0, {i, p + 1, k}],
Table[f[j] -> 0, {j, q + 1, k}]]]], {0}] // Sort;
coeff[x_, y_] :=
If[NumberQ[y], x /. ((# -> 0) & /@ Variables[x]), Coefficient[x, y]];
monal[k_, p_, q_] := Outer[coeff, P[k, p, q], var[k, p, q]];
mono[x_, y_, z_] :=
Times @@ # & /@
Flatten[Outer[Join, Map[b, part[x], {2}], Map[e, part[y], {2}],
Map[f, part[z], {2}], 1], 2];
PE[k_, p_, q_] := Module[{t}, dim = k + p + q - 1;
su = p + q;
b[0] = 1; cb = Sum[b[i], {i, 0, k}];
e[0] = 1; ce = Sum[e[i], {i, 0, p}];
f[0] = 1; cf = Sum[f[i], {i, 0, q}];
subst =
Join[Table[b[i] -> b[i]*x^i, {i, 1, k}],
Table[e[i] -> e[i]*x^i, {i, 1, p}],
Table[f[i] -> f[i]*x^i, {i, 1, q}], {t -> t*x}];
degli[pol_] := CoefficientList[pol /. subst, x];
cfbar = Sum[(-1)^i*f[i], {i, 0, q}];
cv = ExpandAll[ce*cfbar];
cvli = degli[cv];
ClearAll[t];
t /: t^su = -Sum[cvli[[i + 1]]*t^(su - i), {i, 1, Min[k, su]}] //
ExpandAll;
t /: t = t^1;
Do[t /: t^(su + i + 1) = ExpandAll[t^(su + i)*t], {i, 0, k - 1}];
ces = ExpandAll[Sum[e[i]*(1 + t)^(p - i), {i, 0, p}]];
cfs = ExpandAll[Sum[f[i]*(1 - t)^(q - i), {i, 0, q}]];
ct = ExpandAll[cb*ces*cfs];
ctli = degli[ct];
Table[c[i] = ctli[[i + 1]], {i, 1, dim}];
tmax[pol_] := Coefficient[pol, t, su - 1];
chrn = Times @@ # & /@ Map[c, part[dim], {2}];
chern = If[su == 1, Expand@chrn, Map[tmax, Expand@chrn]];
S = chern /. {e[p] -> 0};
U = (-1)^q*S];
monall[k_, p_, q_] := Outer[coeff, PE[k, p + 1, q], var[k, p, q]];
```

```
pol = Module[{Q, f, h, q1, q2, q3, q4, S, erg, erg2, erg3, F, fs},
Q = 1 + Sum[a[i]*x^i, {i, 1, dimMAX}] + O[x]^(dimMAX + 1);
f = ExpandAll[x/Q];
h = ExpandAll[D[f, x]/f];
q1 = 2*A;
q2 = 3/2*A^2 - B/4;
q3 = A^3/2 - A*B/4 + 4*C;
q4 = A^4/16 - A^2*B/16 + 2*A*C + B^2/64 - 2*D;
S = y^4 + q1*y^3 + q2*y^2 + q3*y + q4;
erg = ExpandAll[D[h, x]^2] - ExpandAll[S /. {y -> h}];
erg2 = CoefficientList[erg*x^3, x];
erg3 = Flatten[Solve[erg2 == 0, Rest[Variables[Q]]]];
F = Rest[CoefficientList[f, x]];
fs = Table[(-1)^(i + 1)*F[[i]], {i, 1, Length[F]}];
qlist = fs /. erg3;
clist = Table[c[i], {i, 0, dimMAX}];
c[0] = 1;
Ds[a_, b_] :=
Det[Table[
Table[If[b[[i]] - i + j < 0, 0, a[[b[[i]] - i + j + 1]]], {j, 1,
Length[b]}], {i, 1, Length[b]}]];
part /: part[0] = {{}};
Do[part /: part[n] =
Union[Flatten[
Table[Sort[Join[{i}, #]] & /@ part[n - i], {i, 1, n}], 1]], {n,
1, dimMAX}];
parti[n_] := Table[Reverse[part[n][[i]]], {i, 1, Length[part[n]]}];
pol = {};
Do[pa = parti[n];
AppendTo[pol,
Expand[Sum[
Ds[qlist, pa[[i]]]*Ds[clist, pa[[i]]], {i, 1,
Length[pa]}]]], {n, 1, dimMAX}];
pol];
K[n_] := pol[[n]];
Mm = fixcomp[dm, nm, tm];
m = Join @@ # & /@ Transpose[(monal @@ # &) /@ Mm];
Print[Dimensions[m]];
n = NullSpace[m];
Print[Dimensions[n]];
v = Join @@ # & /@ Transpose[(monall @@ # &) /@ Mm];
Print[Dimensions[v]];
H = Transpose[v.Transpose[n]];
ClearAll[c, lh];
lh[j_] := #[[1]] -> #[[2]] & /@ Transpose[{L, v.n[[j]]}];
ST = Table[K[dm] /. lh[i], {i, 1, Length[n]}];
InputForm[%]; Print[ST /. {C -> 0, A -> 0}];
SS = Complement[Union[ST], {0}];
gf[a_, b_, c_, d_] := A^a*B^b*C^c*D^d;
```

```
bas[q_] :=
Rest[Union[
Flatten[Table[
If[k + 2 p + 3 r + 4 l == q, {k, p, r, l}, {}], {k, 0, q}, {p,
0, q/2}, {r, 0, q/2}, {l, 0, q/4}], 3]]] // Sort;
vr[q_] := gf @@ # & /@ bas[q];
coeff[x_, y_] :=
If[NumberQ[y], x /. ((# -> 0) & /@ Variables[x]), Coefficient[x, y]];
SD = Outer[coeff, SS, vr[dim]];
MatrixRank[SD]
MatrixRank[H]
```