THERMOELASTIC STRESS AND DISPLACEMENT IN A THIN ROD DUE TO AN INSTANTANEOUS HEAT SOURCE

by

HWANG HWEI SIANG
B.S., Chung Yuan Christian College, China, 1970

A MASTER'S THESIS

submitted in partial fulfillment of the requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1976

Approved by:

[Signature]
Major Professor
TABLE OF CONTENTS

NOMENCIATURE ........................................................................................................ 1

CHAPTER

1 INTRODUCTION ..................................................................................................... 1

2 TEMPERATURE DISTRIBUTION IN A THIN ROD INDUCED BY AN INSTANTANEOUS
   HEAT SOURCE DISTRIBUTED OVER A FINITE PORTION OF THE ROD .................. 4
   2.1 Derivation of the Heat Conduction Equation ..................................................... 4
   2.2 Methods of solution .......................................................................................... 6
       2.2.1 Solution of Heat Conduction Equation for a point
            impulsive heat source .................................................................................. 6
       2.2.2 Solution of Heat Conduction Equation for a thin,
            rod of finite length due to an instantaneous heat
            source distributed over a finite portion of the rod .................................... 10
   2.3 Evaluation of the analytical solution of temperature variation ................. 11
       2.3.1 Evaluation of an infinite series ................................................................. 11
       2.3.2 Numerical example and result .................................................................. 14

3 THERMAL DEFORMATION AND STRESS DUE TO THE GIVEN TEMPERATURE VARIATION 15
   3.1 Derivation of the Governing Differential Equation of
       Thermoelasticity for a thin rod ......................................................................... 15
   3.2 Solution of the thermal stress and deformation associated
       with the given temperature variation ............................................................... 17
   3.3 Examining the convergence of the analytical solution of
       thermal stress and deformation ..................................................................... 26
4 SOLUTIONS OF DIFFUSION EQUATION, DISPLACEMENT EQUATION AND STRESS WAVE EQUATION BY FINITE DIFFERENCE METHODS 

4.1 On the solution of the Diffusion Equation 

4.1.1 The Explicit form of the Diffusion Equation 

4.1.2 The implicit form of the Diffusion Equation 

4.2 On the solution of Displacement Equation and Stress Wave Equation 

4.3 Numerical example and results 

5 CONCLUSIONS 

BIBLIOGRAPHY 

APPENDICES 

ACKNOWLEDGEMENT
ILLEGIBLE

THE FOLLOWING DOCUMENT (S) IS ILLEGIBLE DUE TO THE PRINTING ON THE ORIGINAL BEING CUT OFF

ILLEGIBLE
NOMENCLATURE

An Coefficient of equation (12)
Bn Coefficient of equation (13)
Ce Coefficient of thermal expansion
E Young's modulus
G_1(n,0), G_2(n,0) Define in section 3.3
G_1'(n,0)
F_1(n,m), F_2(n,m) Define in section 3.3
F_3(n,m), F_4(n,m)
F_1'(n,m)
F_3'(n,m)
H(x), H(x-x_0) Heaviside step function
H(x-x_1)
k thermal conductivity
L span length of rod
L{ } Laplace transformation operator
Q_0 Dimensionless heat quantity, q_0α/κT_0
q_0 Heat quantity per unit time and volume
q(x,t) Heat generation
s dummy variable used in Laplace transformation
t time variable
T(x,t) Temperature
T_0 Reference temperature
\[ u(x,t) \] Displacement
\[ \bar{U}(\xi, \tau) \] Dimensionless displacement
\[ U(\xi, \tau) \] Laplace transformation of \[ U(\xi, \tau) \] with respect to \( \tau \).
\[ x_0, x_1 \] Partical point on bar
\[ x \] Axial coordinate
\[ x' \] Relative axial coordinate
\[ V \] Velocity of elastic wave propagation \( (E/\rho)^{1/2} \)
\[ \alpha \] Thermal diffusivity
\[ \beta \] Defined in equation (41)
\[ \psi_1(s) \] Coefficient of equation (35), defined by equation (36)
\[ \psi_2(s) \] Coefficient of equation (35), defined by equation (37)
\[ \varepsilon(x,t) \] Strain
\[ \lambda_1, \lambda_2, \lambda_3, \lambda_4 \] Defined by equation (39)
\[ \eta \] Dummy variable
\[ \rho \] Density
\[ \Theta(\xi, \tau) \] Dimensionless temperature, \( T/T_0 \)
\[ \sigma(x,t) \] Stress
\[ \tau \] Dimensionless time variable, \( at/L^2 \)
\[ \xi \] Dimensionless length, \( x/L \)
\[ \xi' \] Relative dimensionless length, \( x'/L \)
\[ \xi_0, \xi_1 \] Dimensionless form of \( x_0, x_1 \)
\[ \delta(x-x'), \delta(\tau) \] Dirac's Delta function
\[ \delta(t), \delta(\xi-\xi') \]
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Coordinate system</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>An instantaneous heat source distributed on the thin rod</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>Temperature distribution, $x_0 = 0.2$ and $x_1 = 0.3$</td>
<td>39</td>
</tr>
<tr>
<td>4</td>
<td>Temperature distribution, $x_0 = 1.0$ and $x_1 = 1.1$</td>
<td>40</td>
</tr>
<tr>
<td>5</td>
<td>The explicit form</td>
<td>31</td>
</tr>
<tr>
<td>6</td>
<td>The implicit form</td>
<td>32</td>
</tr>
<tr>
<td>7</td>
<td>Typical relaxation pattern</td>
<td>34</td>
</tr>
<tr>
<td>8</td>
<td>Temperature distribution</td>
<td>41</td>
</tr>
<tr>
<td>9</td>
<td>Displacement distribution</td>
<td>42</td>
</tr>
<tr>
<td>10</td>
<td>Stress distribution</td>
<td>43</td>
</tr>
</tbody>
</table>
LIST OF TABLES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>τ influences the convergence of the series</td>
<td>13</td>
</tr>
<tr>
<td>2</td>
<td>Temperatures solved by evaluating analytical solution, ( x_0=0.2 ) and ( x_1=0.3 )</td>
<td>44</td>
</tr>
<tr>
<td>3</td>
<td>Temperatures solved by evaluating analytical solution, ( x_0=1.0 ) and ( x_1=1.1 )</td>
<td>46</td>
</tr>
<tr>
<td>4</td>
<td>Temperatures solved by evaluating analytical solution, ( x_0=0.7 ) and ( x_1=0.8 )</td>
<td>48</td>
</tr>
<tr>
<td>5</td>
<td>Temperature distribution (F) solved by implicit finite difference method</td>
<td>50</td>
</tr>
<tr>
<td>6</td>
<td>Temperature (F), displacement and stress solved by finite difference method</td>
<td>51</td>
</tr>
</tbody>
</table>
CHAPTER 1

INTRODUCTION

The determination either of the temperature distribution or of the subsequent mechanical responses exposed to the action of heat source in solids has been paid considerable attention both in research and engineering [1,2]. Research on this topic and the development of appropriate methods of analysis are some of the most interesting areas of current technical activity, not only in connection with nuclear reactors, but also in fields such as those of welding, metal cutting, metallurgical processes, high-speed flight, power plant design and so forth.

Apparently, the theory of heat flow due to a heat source originated in connection with arc welding. Rosenthal, P. [4] in 1935 was the first to apply the exact theory of heat flow due to a heat source to arc welding, by using the experimentally established principle of a quasi-stationary state [3,5]. A particular case of the general solution was treated independently by Boulton and Lance Martin [3,6] in 1936. Bruce [7] applied the method of instantaneous sources to another particular case of welding and, in 1941, Mahla [8] extended this method to a three dimensional case. More recently, more complete investigations of thermal stresses in an infinite cylinder due to steady-state or transient temperature variation have since been made by a number of authors [9,10,11,12,13,14,15,16,17]. In reviewing the literature, attention was centered on the case of temperature variation and thermal stress in rods resulting in many efforts which have been made by different investigators [18,19,20,21,22].
In the present research, a simple problem of thermal stress and displacement in a thin finite rod has been considered. The heat source is instantaneously generated over a finite portion of the rod, one end of the rod is fixed with the other free, and both ends are kept at zero temperature. This research is of intrinsic interest itself because the rod problem represents the simplest of all engineering structures.

The problem is approached from the standpoint of classical linear, uncoupled, thermoelastic theory. The analysis is composed of two distinct problems; i.e., heat conduction neglecting the mechanical coupling effect and elasticity regarding the inertia effect. Furthermore, the material is assumed to be homogeneous and isotropic with respect to both its thermal and mechanical responses, and its physical properties are independent of temperature.

At first, this research is concerned with the derivation of the temperature distribution field. Assuming the temperature gradients in the cross section of the rod to be negligible and also that heat losses through the surface to the surroundings medium is not considered, one obtains one uncoupled heat conduction equation. The partial differential equation is solved for the finite long rod by the technique of Laplace transformation method. A "long-time" solution [23] is obtained. Associated with the given temperature variation, an elementary thermoelastic theory was applied to derive the governing differential equations under the thermal load. Laplace transformation has been found convenient for the solution of this problem. In addition, the effect of this temperature induced the thermal
deformation and the propagating stress wave in the rod will be studied by using the finite difference approximation methods.
CHAPTER 2

TEMPERATURE DISTRIBUTION IN A THIN ROD INDUCED

BY A LINE HEAT SOURCE

2.1 Derivation of the Heat Conduction Equation

The coordinate system to be considered is described in Fig. 1.

Fig. 1 Coordinate system

For analysis of the above described thin rod, the linear, uncoupled heat conduction theory will be used subjected to the following assumptions:

1. Any conversion of mechanical energy into heat is neglected.

2. The material is assumed to be homogeneous and isotropic with respect to its thermal response, and all physical properties are regarded to be independent of temperature.

3. The temperature gradient in the cross section of the rod
is neglected and also the loss through the surface to the surrounding medium is not considered in the present investigation.

Under these assumptions the energy balance equation for an element of the rod leads to the Fourier Heat Conduction Equation [24].

\[
\frac{\partial^2 T}{\partial x^2} - \frac{1}{\alpha} \frac{\partial T}{\partial x} = -\frac{q(x,t)}{k} \tag{1}
\]

where \(q(x,t)\) is the heat generation. For a point impulsive heat source at zero time

\[
q(x,t) = q_0 \delta(t) \delta(x-x') \tag{2}
\]

and for an instantaneous line heat source at zero time

\[
q(x,t) = q_0 \delta(t) [H(x-x_0) - H(x-x_1)] \tag{3}
\]

where \(\alpha\) is the thermal diffusivity, \(k\) is the thermal conductivity, \(q_0\) is the quantity of heat generated by the heat source per unit time and volume, \(t\) is the time variable, \(x\) is the position variable, \(H(x)\) is the Heaviside step function and \(\delta(t)\) is the well-known Dirac's Delta function.

Equation (1) and equation (3) constitute the equations of heat conduction to be solved in the present investigation.
2.2 Method of solution

The equation of heat conduction for an isotropic rod of finite length due to a point impulsive heat source at zero time is first solved by using the Laplace transformation method. With the aid of integration with respect to the space coordinate, the solution for a thin rod of finite length due to a constant instantaneous line heat source can be obtained.

2.2.1 Solution of Heat Conduction Equation for a point impulsive heat source

Consider a thin rod of finite length as shown in Fig. 2.

![Diagram of thin rod with instantaneous line heat source](image)

**Fig. 2** An instantaneous heat source distributed on the thin rod
The rod is instantly heated by a point heat source at \( x' \).

Immediately remove the point heat source and let the rod cool naturally.

The differential equation governing the heat conduction in this investigation as derived in section 2.1 is

\[
\frac{\partial^2 T}{\partial x^2} - \frac{1}{\alpha} \frac{\partial T}{\partial t} = -\frac{q_0}{k} \delta(t) \delta(x-x') \tag{4}
\]

where \( x \leq x' \leq x \). Introduction of dimensionless quantities \( \xi = x/L \), \( \tau = at/L^2 \), \( \vartheta = T/T_0 \), reduces equation (4) to

\[
\frac{\partial^2 \vartheta}{\partial \xi^2} - \frac{\partial \vartheta}{\partial \tau} = \frac{q_0 \alpha}{kT_0} \delta(\tau) \delta(\xi-\xi')
\]

\[= -Q_0 \delta(\tau) \delta(\xi-\xi') \tag{5}
\]

where \( Q_0 = q_0 \alpha/kT_0 \), \( \xi' = x'/L \) and \( T_0 \) is a reference temperature.

For the rod with a finite length \( L \), the boundary conditions and initial conditions must be provided in order to describe the problem completely. In the present investigation we suppose that the system is at rest initially. Thus the initial conditions are

\[ T = 0 \quad \text{for} \quad 0 \leq x \leq L \quad \text{at} \quad t = 0 \tag{6} \]

or in terms of dimensionless quantities

\[ \vartheta = 0 \quad \text{for} \quad 0 \leq \xi \leq 1 \quad \text{at} \quad \tau = 0 \tag{7} \]
Prescribed temperature at both ends are

\[ T = 0 \quad \text{for} \quad x = 0, L \quad (8) \]

or

\[ \Theta = 0 \quad \text{for} \quad \xi = 0, 1 \quad (9) \]

Taking the Laplace transform of equation (5) and introducing the initial condition, i.e., Eq. (7), we obtain

\[ \frac{d^2 \hat{\Theta}}{d\xi^2} - s \hat{\Theta} = -Q_0 \delta(\xi-\xi') \quad (10) \]

with the boundary condition

\[ \hat{\Theta}(0,s) = \hat{\Theta}(1,s) = 0 \quad (11) \]

where the notation

\[ \hat{\Theta}(\xi,s) = L[\Theta(\xi,\tau)] = \int_0^\infty e^{-st} \Theta(\xi,\tau) d\tau \]

is introduced and \( \hat{\Theta}(\xi,s) \) is referred to as the Laplace transform of \( \Theta(\xi,s) \) with respect to \( \tau \). As a solution of Eq. (10) satisfying Eq. (11), we assume

\[ \hat{\Theta}(\xi,s) = \sum_{n=1}^{\infty} A_n \sin(n\pi\xi) \quad (12) \]

where \( A_n \) is independent of \( \xi \). We also assume

\[ \delta(\xi-\xi') = \sum_{n=1}^{\infty} B_n \sin(n\pi\xi) \quad (13) \]

Since the set of functions \( \sin(n\pi\xi) \) is orthogonal over \((0,1)\), both sides of equation (13) are multiplied by \( \sin(n\pi\xi) \) and integrated
to obtain

\[ B_n = \int_0^1 \delta(\xi - \xi') \sin(n\pi\xi) \]

\[ = 2 \sin(n\pi\xi') \]  \hspace{1cm} (14)

where \( n = 1, 2, 3, 4, 5, \ldots \infty \)

and also

\[ \delta(\xi - \xi') = 2 \sum_{n=1}^\infty \frac{\sin(n\pi\xi')\sin(n\pi\xi)}{n^2\pi^2} \]  \hspace{1cm} (15)

where \( n = 1, 2, 3, 4, 5, \ldots \infty \)

Hence, substituting equation (12) and equation (15) into equation (10),
we have

\[ A_n = \frac{2Q_0 \sin(n\pi\xi')}{s + n^2\pi^2} \]

where \( n = 1, 2, 3, 4, 5, \ldots \infty \)

and

\[ \Theta(\xi, s) = 2 \sum_{n=1}^\infty \frac{Q_0 \sin(n\pi\xi')\sin(n\pi\xi)}{s + n^2\pi^2} \]  \hspace{1cm} (16)

By inversion, the temperature distribution for an impulsive point heat
source is
\[ \Theta(\xi, \tau) = 2 \sum_{n=1}^{\infty} Q_0 \sin(n\pi \xi') \sin(n\pi \xi) e^{-n^2 \pi^2 \tau} \] (17)

2.2.2 Solution of Heat Conduction Equation for a thin rod of finite length due to an instantaneous line heat source

For an instantaneous line heat source distributed over \( \xi_0 \leq \xi' \leq \xi \) we integrate the preceding result of equation (17) with respect to \( \xi' \) between the limits \( \xi' = \xi_0, \xi = \xi_1 \) and get

\[ \Theta(\xi, \tau) = \int_{\xi_0}^{\xi_1} \left[ 2 \sum_{n=1}^{\infty} Q_0 \frac{\sin(n\pi \xi)}{n\pi} e^{-n^2 \pi^2 \tau} \sin(n\pi \xi') \right] d\xi' \]

\[ = 2 \sum_{n=1}^{\infty} Q_0 \frac{\sin(n\pi \xi)}{n\pi} \left[ \cos(n\pi \xi_0) - \cos(n\pi \xi_1) \right] e^{-n^2 \pi^2 \tau} \] (18)

Equation (18) is the solution for temperature variation in terms of nondimensional quantities.
2.3 Evaluation of analytical solution of temperature variation

2.3.1 Evaluation of an infinite series

Since the temperature variation $\Theta(\xi, \tau)$ is expressed in terms of exponential and trigonometric functions, the numerical calculation and convergence of the infinite series must be studied. For convenience, let us repeat equation (18)

$$\Theta(\xi, \tau) = 2Q_0 \sum_{n=1}^{\infty} \frac{\sin(n\pi \xi)}{n\pi} e^{-n^2 \pi^2 \tau} [\cos(n\pi \xi_0) - \cos(n\pi \xi_1)]$$

Let

$$\text{sum} = \sum_{n=1}^{\infty} \frac{\sin(n\pi \xi)}{n\pi} e^{-n^2 \pi^2 \tau} [\cos(n\pi \xi_0) - \cos(n\pi \xi_1)] \quad (19)$$

From the properties of trigonometric functions, the absolute values of $\sin(n\pi \xi)$ and $\cos(n\pi \xi_0) - \cos(n\pi \xi_1)$ should be less than or equal to 1 and 2 respectively, i.e., $|\sin(n\pi \xi)| \leq 1$ and $|\cos(n\pi \xi_0) - \cos(n\pi \xi_1)| \leq 2$.

Therefore, equation (19) can be expressed as

$$\text{sum} \leq 2 \sum_{n=1}^{\infty} \frac{e^{-n^2 \pi^2 \tau}}{n\pi} \quad (20)$$

Upon examining the above equation, the right hand side of the inequality converges very rapidly if $\tau$ is very large, but, obviously, it is inconvenient for calculations pertaining to small times.

Expanding the exponential function $e^{-n^2 \pi^2 \tau}$ in series form, we
find that

\[ e^{-n^2 \pi^2 \tau} = \frac{1}{1 + (n^2 \pi^2 \tau) + (n^2 \pi^2 \tau)^2/2! + (n^2 \pi^2 \tau)^4/4! + \cdots} \]

From the above equation, we obtain

\[ e^{-n^2 \pi^2 \tau} < \frac{1}{n^2 \pi^2 \tau} \]  \hspace{1cm} (21)

Inequalities given by equation (20) and equation (21) can be combined in the following form

\[ \text{sum} < 2 \sum_{n=1}^{\infty} \frac{1}{n^3 \pi^3 \tau} \]  \hspace{1cm} (22)

Therefore, the series sum and the temperature variation \( \Theta(\xi, \tau) \) are convergent, as can be easily proved by combining equation (20) and (22) and applying a comparison test [25].

For the purpose of computation, some particular values, i.e., \( \xi = 0.5, \xi_0 = 1.0, \xi_1 = 1.1 \) were substituted in equation (18).

Table (1) reveals that the larger the value \( \tau \), the less terms one needs to calculate of the series for a required convergent value.
<table>
<thead>
<tr>
<th>n</th>
<th>time = 1.00 sec.</th>
<th>time = 5.00 sec.</th>
<th>time = 10.00 sec.</th>
<th>time = 50.00 sec.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.463740E-01</td>
<td>0.453935E-01</td>
<td>0.441759E-01</td>
<td>0.336131E-01</td>
</tr>
<tr>
<td>2</td>
<td>0.697310E-02</td>
<td>0.595711E-02</td>
<td>0.234661E-02</td>
<td>0.170531E-02</td>
</tr>
<tr>
<td>3</td>
<td>0.697310E-02</td>
<td>0.695636E-02</td>
<td>0.235468E-02</td>
<td>0.170532E-02</td>
</tr>
<tr>
<td>4</td>
<td>0.372074E-01</td>
<td>0.249781E-01</td>
<td>0.136481E-01</td>
<td>0.166757E-01</td>
</tr>
<tr>
<td>5</td>
<td>0.169719E-01</td>
<td>0.314621E-02</td>
<td>0.657653E-03</td>
<td>0.166755E-01</td>
</tr>
<tr>
<td>6</td>
<td>0.190779E-01</td>
<td>0.314445E-02</td>
<td>0.657002E-03</td>
<td>0.166756E-01</td>
</tr>
<tr>
<td>7</td>
<td>0.234975E-01</td>
<td>0.750134E-02</td>
<td>0.182195E-02</td>
<td>0.166755E-01</td>
</tr>
<tr>
<td>8</td>
<td>-0.110176E-01</td>
<td>-0.119356E-02</td>
<td>0.269767E-03</td>
<td>0.166755E-01</td>
</tr>
<tr>
<td>9</td>
<td>-0.110170E-01</td>
<td>-0.119295E-02</td>
<td>0.269881E-03</td>
<td>0.166755E-01</td>
</tr>
<tr>
<td>10</td>
<td>-0.110247E-01</td>
<td>-0.119403E-02</td>
<td>0.269781E-03</td>
<td>0.166755E-01</td>
</tr>
<tr>
<td>11</td>
<td>0.823180E-02</td>
<td>0.231327E-03</td>
<td>0.324594E-03</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>0.823161E-02</td>
<td>0.231073E-03</td>
<td>0.324593E-03</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>0.316855E-02</td>
<td>0.295647E-04</td>
<td>0.323166E-03</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>0.482835E-02</td>
<td>0.216582E-04</td>
<td>0.322566E-03</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>0.482777E-02</td>
<td>0.216272E-04</td>
<td>0.322566E-03</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>0.183557E-03</td>
<td>0.121557E-05</td>
<td>0.322566E-03</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>0.226229E-02</td>
<td>0.236477E-05</td>
<td></td>
<td>0.322583E-03</td>
</tr>
<tr>
<td>18</td>
<td>0.226012E-02</td>
<td>0.288275E-05</td>
<td></td>
<td>0.322583E-03</td>
</tr>
<tr>
<td>19</td>
<td>0.805473E-03</td>
<td>0.159633E-05</td>
<td>0.322588E-03</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>0.804938E-03</td>
<td>0.159598E-05</td>
<td>0.322583E-03</td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>0.804093E-03</td>
<td>0.159466E-05</td>
<td>0.322583E-03</td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>0.493160E-03</td>
<td>0.163477E-05</td>
<td></td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>0.211947E-03</td>
<td>0.163155E-05</td>
<td></td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>0.211731E-03</td>
<td>0.163155E-05</td>
<td></td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>0.163050E-03</td>
<td>0.163151E-05</td>
<td></td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>0.360825E-04</td>
<td>0.163107E-05</td>
<td></td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>0.360621E-04</td>
<td>0.163107E-05</td>
<td></td>
<td></td>
</tr>
<tr>
<td>28</td>
<td>0.210579E-04</td>
<td>0.163107E-05</td>
<td></td>
<td></td>
</tr>
<tr>
<td>29</td>
<td>0.371346E-05</td>
<td>0.163107E-05</td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>0.371366E-05</td>
<td>0.163107E-05</td>
<td></td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>0.122155E-04</td>
<td>0.163107E-05</td>
<td></td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>0.149779E-05</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>33</td>
<td>0.149410E-05</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>34</td>
<td>0.391155E-05</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>0.157317E-05</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>36</td>
<td>0.156955E-05</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>37</td>
<td>0.351350E-05</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>38</td>
<td>0.791607E-06</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>39</td>
<td>0.789100E-06</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>0.792929E-06</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>41</td>
<td>0.372576E-06</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>42</td>
<td>0.371566E-06</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>43</td>
<td>0.182057E-06</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>44</td>
<td>0.246520E-07</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>45</td>
<td>0.246014E-07</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>46</td>
<td>0.501237E-07</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>47</td>
<td>0.744581E-07</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>48</td>
<td>0.744074E-07</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>49</td>
<td>0.542959E-07</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>0.542687E-07</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1 \( \tau \) influences the convergence of the series

* denoting a required convergent value
2.3.2 Numerical example and result

For numerical calculations a finite rod of span length \( L = 1.5 \) ft., made of copper, has been assumed. The temperature has been calculated at points along the longitudinal axis, with \( k = 224.0 \) Btu/hr-ft\(^\circ\)F, \( \rho = 558.0 \) Ib/ft\(^3\), \( C_e = 0.091 \) Btu/Ib\(^\circ\)F, \( a = 4.42 \) ft\(^2\)/hr, and \( E = 1.872 \times 10^3 \) Lbf/ft\(^2\).

The numerical calculations of the dimensionless quantities shown in equation (18) were carried out using an IBM 370/158 computer at the Computing Center of Kansas State University.

The results were found by summing terms for evaluation; for instance \( n = 26 \), at \( \xi = 0.5 \), \( \tau = 0.002728 \), gave very good convergence. Throughout the computations, the dimensionless quantity \( Q_0 \) has been taken as \( 0.019731 \), this choice was assumed so that a unit quantity of heat is introduced.

Part of the numerical values of the dimensionless temperature have been illustrated in Fig. (3), Fig. (4), table (2) and table (3). Thus table (2) and Fig. (3) show the variation of the dimensionless temperature v.s. the dimensionless space variable \( \xi \) along the longitudinal axis of the rod as a line heat source is induced between \( \xi_0 = 0.133 \) and \( \xi_1 = 0.120 \), while table (3) and Fig. (4) illustrate the temperature distribution as a line heat source is induced between \( \xi_0 = 0.667 \) and \( \xi_1 = 0.733 \) at various instants after the thermal shock.
CHAPTER 3

THERMAL STRESS AND DEFORMATION DUE TO THE GIVEN TEMPERATURE VARIATION

3.1 Derivation of Governing Differential Equation of thermoelasticity for a thin rod

We shall confine ourselves to an elastic, isotropic homogeneous thin rod, with respect to both its mechanical and thermal properties, and assume that plane cross sections remain plane, and that only axial stress is present, being uniformly distributed over the cross section. Let \( u = u(x,t) \) be the longitudinal component of the displacement at a point \( x \) and at any time \( t \). As in the linear theory of elasticity the strain-displacement relation can be derived directly from purely geometrical considerations.

For small displacements, the strain \( \varepsilon(x,t) \) at any point \( x \) and at any time \( t \) is connected with the displacement vector by the relation

\[
\varepsilon(x,t) = \frac{\partial u(x,t)}{\partial x}
\]  

(23)

If only a one dimensional problem is considered, then \( \sigma(y,t) \) and \( \sigma(z,t) \) can be neglected, i.e., \( \sigma(y,t) = \sigma(z,t) = 0 \)

For the present investigation, only thermal loading is considered. If we use the assumptions of section 2.1 and further assume that there is no coupling of the temperature and strain fields, i.e., no mechanical energy due to the strain is converted into heat, the governing equations of thermoelasticity for an isotropic homogeneous thin rod are derived.
Since the thermal expansion contributes to part of the direct strain, the temperature will appear explicitly in the stress-strain relation, which now has the form

\[ \sigma(x,t) = E[\varepsilon(x,t) - C_\epsilon T(x,t)] \]  \hspace{1cm} (24)

with \( E \) denoting Young's modulus, \( C_\epsilon \) denoting the coefficient of linear thermal expansion and \( \sigma(x,t) \) denoting the stress at a point and at time \( t \).

When there exist sudden changes of temperature in this rod, the influence of inertia cannot be neglected; we have then to investigate the equations of motion.

The basic differential equations governing the extensional motion of a thin, homogeneous rod are the equation of motion \([23]\).

\[ \frac{\partial \sigma}{\partial x} = \rho \left( \frac{\partial^2 u}{\partial t^2} \right) \] \hspace{1cm} (25)

and the equation of compatibility

\[ \frac{\partial^2 u}{\partial x \partial t} = \frac{\partial^2 u}{\partial t \partial x} \] \hspace{1cm} (26)

Substitution of equations (23) and (24) into equation (25) to eliminate \( \sigma \) leads to the displacement equation, where \( \rho \) is the density of the material of the rod.
\[ \frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} + C e \left( \frac{\partial T}{\partial x} \right) \] (27)

where \( V = (E/\rho)^{1/2} \) is the velocity of elastic wave propagation.

Successive substitutions from equation (23) into equation (24) and from equation (26) into equation (25) to eliminate \( u \) lead to the stress equation.

\[ \frac{\partial^2 \sigma}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \sigma}{\partial t^2} + \rho C e \frac{\partial^2 T}{\partial t^2} \] (28)

3.2 Solution of the thermal stress and deformation

We suppose that the system was initially at rest and in the stress free state thus the initial conditions are

\[ u = \frac{\partial u}{\partial t} = 0 \quad \text{for} \quad 0 \leq x \leq L, \quad t = 0 \] (29)

\[ \sigma = \frac{\partial \sigma}{\partial t} = 0 \quad \text{for} \quad 0 \leq x \leq L, \quad t = 0 \] (30)

Since one end of the rod is kept fixed and the other free and there are no tractions on the lateral surface, the boundary conditions are

\[ u = 0 \quad \text{at} \quad x = 0 \] (31)

\[ \frac{\partial u}{\partial x} = 0 \quad \text{at} \quad x = L \] (32)
Introducing dimensionless quantities \( \xi = \frac{x}{L}, \ U = \frac{u}{L}, \ \Theta = \frac{T}{T_0}, \ \tau = \frac{at}{L^2} \), displacement equation (27) becomes

\[
\frac{\partial^2 U}{\partial \xi^2} = \frac{\alpha^2}{V^2 L^2} \frac{\partial^2 U}{\partial \tau^2} + C e T_0 \frac{\partial \Theta}{\partial \xi} \tag{33}
\]

with

\[
U(\xi, 0) = \frac{\partial U(\xi, 0)}{\partial \tau} = 0 \tag{34}
\]

Taking the Laplace transformation of equation (33) and using equation (34), we obtain

\[
\frac{d^2 \bar{U}}{d \xi^2} - \frac{\alpha^2}{V^2 L^2} s^2 \bar{U} = C e T_0 \left( \frac{\partial \Theta}{\partial \xi} \right) \tag{35}
\]

where \( \bar{U}(\xi, s) = \int_0^\infty e^{-st} U(\xi, \tau) d\tau \) is the Laplace transformation of \( U(\xi, \tau) \) with respect to \( \tau \).

Differentiating equation (22) with respect to \( \xi \) and substituting in the above equation we rewrite the equation as

\[
\frac{d^2 \bar{U}}{d \xi^2} - \frac{\alpha^2}{V^2 L^2} s^2 \bar{U} = 2 Q_0 C e T_0 \sum_{n=1}^{\infty} \frac{\cos(n \pi \xi)}{s + n^2 \pi^2} \left[ \cos(n \pi \xi_0) - \cos(n \pi \xi_1) \right] \tag{36}
\]
with the boundary conditions

\[ \overline{U} = 0 \quad \text{at} \quad \xi = 0 \]  
\[ \frac{d\overline{U}}{d\xi} = 0 \quad \text{at} \quad \xi = 1 \]  

(37)  
(38)

General solution of equation (36) is

\[ \overline{U}(\xi, s) = C_1 e^{\psi_2(s) \xi} + C_2 e^{-\psi_2(s) \xi} - \sum_{n=1}^{\infty} \frac{\psi_1(s)}{n^2 \pi^2 + \psi_2^2} \cos(n\pi \xi) \]  

(39)

where

\[ \psi_1(s) = \frac{2 Q_0 C e^{T_0}}{s + n^2 \pi^2} [\cos(n\pi \xi_0) - \cos(n\pi \xi_1)] \]  

(40)

\[ \psi_2(s) = \frac{a s}{V L} \]  

(41)

Solution of equation (39) satisfying equation (37) and (38) is

\[ \overline{U}(\xi, s) = \left[ \frac{1}{1 + e^{2\psi_2}} \sum_{n=1}^{\infty} \frac{\psi_1}{n^2 \pi^2 + \psi_2^2} \right] e^{\psi_2 \xi} + \left[ \frac{e^{2\psi_2}}{1 + e^{2\psi_2}} \sum_{n=1}^{\infty} \frac{\psi_1}{n^2 \pi^2 + \psi_2^2} \right] e^{-\psi_2 \xi} - \sum_{n=1}^{\infty} \frac{\psi_1}{n^2 \pi^2 + \psi_2^2} \cos(n\pi \xi) \]  

\[ = \left[ \frac{\psi_2 \xi + e^{(2-\xi)\psi_2}}{1 + e^{2\psi_2}} \right] \sum_{n=1}^{\infty} \frac{\psi_1}{n^2 \pi^2 + \psi_2^2} - \left[ \frac{e^{2\psi_2}}{1 + e^{2\psi_2}} \sum_{n=1}^{\infty} \frac{\psi_1}{n^2 \pi^2 + \psi_2^2} \right] \cos(n\pi \xi) \]  

\[ = \left[ \frac{\cosh(1-\xi)\psi_2}{\cosh(\psi_2)} \right] \sum_{n=1}^{\infty} \frac{\psi_1}{n^2 \pi^2 + \psi_2^2} - \left[ \frac{e^{2\psi_2}}{1 + e^{2\psi_2}} \sum_{n=1}^{\infty} \frac{\psi_1}{n^2 \pi^2 + \psi_2^2} \right] \cos(n\pi \xi) \]
Substituting Eq. (40) and Eq. (41) into the above equation, we obtain

\[
\bar{U}(\xi,s) = \frac{\cosh(\alpha(1-\xi)\nu s / \nu L)}{\cosh(\alpha \nu s / \nu L)} \sum_{n=1}^{\infty} \frac{2Q_0 C T_0 \left[ \cos(n^2 \xi_0) - \cos(n \xi_1) \right]}{(s + n^2 \pi^2)(n^2 \pi^2 + a^2 s^2 / \nu^2 \nu^2)}
\]

\[
- \sum_{n=1}^{\infty} \frac{2Q_0 C T_0 \left[ \cos(n^2 \xi_0) - \cos(n \xi_1) \right] \cos(n \xi)}{(n^2 \pi^2 + a^2 s^2 / \nu^2 \nu^2)(s + n^2 \pi^2)}
\]  \quad (42)

Letting

\[
\lambda_1 = \frac{\alpha}{\nu L}
\]

\[
\lambda_2 = \lambda_1 (1 - \xi)
\]

\[
\lambda_3 = \frac{2Q_0 C T_0 \lambda^2}{\lambda^2_1}
\]

\[
\lambda_4 = \frac{n^2 \pi^2}{\lambda^2_1}
\]  \quad (43)

and rearranging Eq. (43) we obtain the simple form as

\[
\bar{U}(\xi,s) = \lambda_3 \sum_{n=1}^{\infty} \frac{\cosh(\lambda_2 s) \left[ \cos(n^2 \xi_0) - \cos(n \xi_1) \right]}{\cosh(\lambda_1 s)} \frac{(s^2 + \lambda_4)}{(s + n^2 \pi^2)}
\]

\[
- \lambda_3 \sum_{n=1}^{\infty} \frac{\cos(n^2 \xi_0) - \cos(n \xi_1) \cos(n \xi)}{(s^2 + \lambda_4)(s + n^2 \pi^2)}
\]  \quad (44)
Inversion

\[ L^{-1} \left[ \frac{1}{(s^2 + \lambda_u)(s+n^2\pi^2)} \right] \]

\[ = -\frac{\cos(\sqrt{\lambda_u} \tau)}{n^4\pi^4 + \lambda_u} + \frac{n^2\pi^2 \sin(\sqrt{\lambda_u} \tau)}{(n^4\pi^4 + \lambda_u)(\sqrt{\lambda_u})} + \frac{e^{-n^2\pi^2\tau}}{n^4\pi^4 + \lambda_u} \]

\[ = \frac{\beta \sin(\sqrt{\lambda_u} \tau - \phi)}{n^4\pi^4 + \lambda_u} + \frac{e^{-n^2\pi^2\tau}}{n^4\pi^4 + \lambda_u} \]  \hspace{1cm} (45)

Where

\[ \beta \cos(\phi) = n^2\pi^2 / \sqrt{\lambda_u}, \quad \beta \sin(\phi) = 1, \quad \phi = \tan^{-1}(\sqrt{\lambda_u}/n^2\pi^2) \]

Also

\[ L^{-1} \left[ -\frac{\tanh(\lambda_2 s)}{s^2 \tanh(\lambda_1 s)} \right] \]

\[ = \tau - \frac{8\lambda_1 \beta}{\pi^2} \sum_{m=1}^{\infty} \frac{\sin((2m-1)\pi \xi/2) \sin((2m-1)\pi \tau/2\lambda_1)}{(2m-1)^2} \]  \hspace{1cm} (46)

And

\[ L^{-1} \left[ \frac{s^2}{(s^2 + \lambda_u)(s+n^2\pi^2)} \right] \]

\[ = -\frac{\lambda_u \beta}{n^4\pi^4 + \lambda_u} \left[ \sin(\sqrt{\lambda_u} \tau - \phi) \right] + \frac{n^4\pi^4}{n^4\pi^4 + \lambda_u} e^{-n^2\pi^2\tau} \]

\[ \hspace{1cm} (47) \]
Hence, by the application of Eq. (46) and Eq. (47) and the convolution theorem of the Laplace transform [26], we obtain

\[
\chi(\xi, \tau) = L^{-1}\left[ \frac{\cosh(\lambda_1 s)}{\cosh(\lambda_1 s)} \frac{1}{(s^2 + \lambda_4)(s + n^2 \pi^2)} \right]
\]

\[
= L^{-1}\left[ \frac{\cosh(\lambda_1 s)}{s^2 \cosh(\lambda_1 s)} \frac{s^2}{(s^2 + \lambda_4)(s + n^2 \pi^2)} \right]
\]

\[
= \int_0^\tau \left[ \frac{-\lambda_4 \sin(\sqrt{\lambda_4} \tau - \phi)}{n^4 \pi^4 + \lambda_4} + \frac{n^4 \pi^4 e^{-n^2 \pi^2 \tau}}{n^4 \pi^4 + \lambda_4} \right] X
\]

\[
\times \left\{ (\tau - \eta) - \frac{8\lambda_4}{\pi^2} \sum_{m=1}^{\infty} \frac{\sin[(2m-1)\pi (\tau - \eta)/2]}{(2m-1)^2} \right\} d\eta
\]

(48)

where \( \eta \) is a dummy variable

After integrating Eq. (48) from zero to \( \tau \) we obtain

\[
\chi(\xi, \tau) = -\frac{n^2 \pi^2}{n^4 \pi^4 + \lambda_4} \left\{ -\frac{1}{\sqrt{\lambda_4}} \left[ \sin(\sqrt{\lambda_4} \tau - \phi) + \sin \phi \right] + \cos(\phi) \right\}
\]

\[
+ \frac{n^2 \pi^2}{n^4 \pi^4 + \lambda_4} \left\{ \tau - \frac{1}{n^2 \pi^2} [1 - e^{-n^2 \pi^2 \tau}] \right\}
\]
\[
- \frac{8n^3n^4\lambda_1}{(n^4n^4+n_4)^{1/2}} \sum_{m=1}^\infty \frac{\sin[(2m-1)\pi \xi/(2m-1)]}{\{n^4n^4+[1+(2m-1)\pi^2/4\lambda_1]^2\}^{1/2}} \cdot X
\]

\[
X \left\{ \sin\left(\frac{2m-1}\pi \xi \lambda_1 \right) - \tan^{-1}\left(\frac{2m-1}{2\lambda_1n^2\pi}\right) + e^{-n^2\pi^2\tau} \sin\left[\tan^{-1}\left(\frac{2m-1}{2\lambda_1n^2\pi}\right)\right] \right\}
\]

\[
+ \frac{4\lambda_4\lambda_4}{(n^4n^4+n_4)^{1/2}} \sum_{m=1}^\infty \frac{\sin[(2m-1)\pi \xi/(2m-1)]}{(2m-1)^2} \cdot X
\]

\[
\times \left\{ \sin\left(\sqrt{\lambda_4} \xi - \phi \right) + \sin\left[\pm + (2m-1)\pi \xi/(2\lambda_1)\right] \right\}
\]

\[
- \frac{\sin\left(\sqrt{\lambda_4} \xi + \phi \right) + \sin\left[-(2m-1)\pi \xi/(2\lambda_1)\right]}{n+(2m-1)/2}
\]

Therefore, the inverse transform of Eq. (44) is the solution of the displacement equation, i.e.,

\[
U(\xi,\tau)
\]

\[
= \lambda_3 \sum_{n=1}^\infty \chi(\xi,\tau) \left[ \cos(n\pi \xi_0) - \cos(n\pi \xi_1) \right]
\]

\[
- \lambda_3 \sum_{n=1}^\infty \frac{1}{n^4n^4 + \lambda_4} \cdot X \left( \beta \sin(\sqrt{\lambda_4} \xi - \phi) + e^{-n^2\pi^2\tau} \right)
\]

\[
\times \left[ \cos(n\pi \xi_0) - \cos(n\pi \xi_1) \right] \cos(n\pi \xi)
\]

(50)

Where \( \chi(\xi,\tau) \) is shown in Eq. (49) rearranging Eq. (50) to combine the summation of single and double forms we obtain
\[ U(\xi, \tau) \]
\[ = \lambda_3 \sum_{n=1}^{\infty} \frac{8n^4 \pi^4 \lambda_1}{\pi^2 (n^4 \pi^4 + \lambda_4) (2m-1)^2 \{n^4 \pi^4 + [(2m-1)\pi/2\lambda_1]^2\}^{1/2}} \times \]
\[ \times \{ \sin[\frac{(2m-1)\pi \tau}{2\lambda_1}] - \tan^{-1}(\frac{2m-1}{2\lambda_1 n^2 \pi}) + e^{-n^2 \pi^2 \tau} \sin[\tan^{-1}(\frac{2m-1}{2\lambda_1 n^2 \pi})] \} \]
\[ - \frac{4\lambda_4 \lambda_2^2}{\pi^3 (n^4 \pi^4 + \lambda_4) (2m-1)^2} \times \]
\[ \times \{ -\frac{\sin(\sqrt{\lambda_4 \pi \tau} - \phi) + \sin(\phi + (2m-1)\pi \tau/2\lambda_1)}{n + (2m-1)/2} - \frac{\sin(\sqrt{\lambda_4 \pi \tau} - \phi) + \sin(i - (2m-1)\pi \tau/2\lambda_1)}{n - (2m-1)/2} \} \times \]
\[ \times \sin[(2m-1)\pi \xi/2)] [\cos(n \pi \xi_0) - \cos(n \pi \xi_1)] \]  

(51)

From equation (51) it is evident that \( \frac{3U}{3\xi} = 0 \) for \( \xi = 1 \). Also, it can be shown that \( U(0, \tau) = 0 \) and \( U(\xi, 0) = \frac{3U(\xi, 0)}{3\xi} = 0 \), for \( 0 \leq \xi \leq 1 \). Thus the solution satisfies the boundary and initial conditions. The normal stress can be easily verified from equation (18), equation (24) and equation (51) to be

\[ \frac{\sigma(\xi, \tau)}{E} = \frac{3U(\xi, \tau)}{3\xi} = C e T_0 \Theta(\xi, \tau) \]
\[ \lambda_3 \sum_{n=1}^{\infty} \left( \frac{n\pi\left[\sin(\sqrt{\lambda_4}\tau) + e^{-n^2\pi^2\tau}\right]}{n^4\pi^4 + \lambda_4} - \frac{2Q_0C_Tn\pi\pi}{n\pi\lambda_3} \right) \times \]

\[ X \{ \sin(n\pi\xi)[\cos(n\pi\xi_0)-\cos(n\pi\xi_1)] \} - \]

\[ - \lambda_3 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{8n^4\pi^4\lambda_1}{\pi^2(n^8\pi^8+\lambda_4)(2m-1)^2(n^8\pi^8+[2m-1]^2/2\lambda_1)^{1/2}} \]

\[ \times \left\{ \sin(\sqrt{\lambda_4}\tau-\phi) + \sin[\phi+(2m-1)\pi\tau/2\lambda_1] - \frac{\sin(\sqrt{\lambda_4}\phi)+\sin[\phi-(2m-1)\pi\tau/2\lambda_1]}{n + (2m+1)/2} - \frac{\sin(\sqrt{\lambda_4}\phi)+\sin[\phi-(2m-1)\pi\tau/2\lambda_1]}{n - (2m-1)/2} \right\} \]

\[ \pi^3(n^8\pi^8+\lambda_4)(2m-1)^2 \]

\[ X \left\{ \frac{(2m-1)^\pi}{2} \cos\left[\frac{(2m-1)\pi\xi}{2}\right][\cos(n\pi\xi_0)-\cos(n\pi\xi_1)] \right\} \]

(52)

By equation (52), \( \sigma = 0 \) at \( \xi = 1 \). Hence the condition of end \( \xi = 1 \) being stress-free is satisfied. From the expression of U and \( \sigma \), it is evident that both U and \( \sigma \) remain bounded as \( \tau \to \infty \) which is expected.
3.3 Examining the convergence of the analytical solutions
of thermal stress and deformation.

An examination of solution of the displacement and stress obtained
in section 3.2 shows that all solutions are expressed in the forms
of combinations of single and double infinite series. Therefore, a study of
the convergence of the single and double infinite series becomes necessary.

For instance, the expression of the displacement solution is

\[
U(\xi, \tau) = \sum_{n=1}^{\infty} \frac{G_1(n,0)}{G_2(n,0)} \sum_{m=1}^{\infty} \frac{F_1(n,m)}{F_2(n,m)} \left[ \frac{F_3(n,m)}{F_4(n,m)} \right]
\]

where \( G_1(n,0), G_2(n,0), F_1(n,m), F_2(n,m), F_3(n,m) \) and \( F_4(n,m) \) are
polynomials in the variable \( m \) and \( n \) and

\[
G_1(n,0) = \lambda_3 [\beta \sin(\sqrt{\lambda_4} \tau - \phi) + e^{-n^2\pi^2\tau} ] [1 - \cos(n\pi \xi)] [\cos(n\pi \xi_0) - \cos(n\pi \xi_1)]
\]

\[
G_2(n,0) = n^{h_4} n^{h_4} + \lambda_4
\]

\[
F_1(n,m) = 8\lambda_3 \int_0^{\pi/2} \int_0^{\pi/2} \sin(\frac{(2m-1)\pi}{\lambda_1} - \frac{\pi}{\lambda_1} - \frac{2m-1}{2}) \sin(\tan(\frac{n\pi x^2}{\lambda_1} - \frac{\pi}{\lambda_1} - \frac{2m-1}{2}) ) X \sin([2-1)/(2\pi)] \cos(n\pi \xi_0) - \cos(n\pi \xi_1)]
\]

\[
F_2(n,m) = \pi^2 (n^{h_4} + \lambda_4)(2m-1)^2 [n^{h_4} + [(2m-1)\pi/\lambda_1]^{1/2}
\]

\[
F_3(n,m) = 4\lambda_4 \lambda_4 \int_0^{\pi/2} \int_0^{\pi/2} \sin(\sqrt{\lambda_4} (\tau - \phi) + \sin(\psi - (2m-1)\pi/\lambda_1) - \sin(\sqrt{\lambda_4} (\tau - \phi) + \sin(\psi - (2m-1)\pi/\lambda_1) ) \sin([2-1)/(2\pi)] \cos(n\pi \xi_0) - \cos(n\pi \xi_1)]
\]

\[
F_4(n,m) = \pi^2 (n^{h_4} + \lambda_4)(2m-1)^2
\]
Since the degrees of $G_1(n,0)$, $F_1(n,m)$ and $F_3(n,m)$ are lower than that of $G_2(n,0)$, $F_2(n,m)$ and $F_4(n,m)$ respectively, series $\sum_{n=1}^{\infty} G_2(n,0)$, $\sum_{n=1}^{\infty} F_2(n,m)$ and $\sum_{n=1}^{\infty} F_4(n,m)$ are convergent, as can be easily proved by applying a comparison test [26].

Examining the solution of stress, we can express the equation (52) as the following form

$$\frac{\sigma(\xi, \tau)}{E} = 8 \sum_{n=1}^{\infty} \frac{G_1'(n,0)}{G_2(n,0)} \sum_{n=1}^{\infty} \frac{F_1'(n,m)}{F_2(n,m)} \frac{F_3'(n,m)}{F_4(n,m)} - C e^{2\theta_0}(\xi, \tau)$$

where $G_1'(n,0)$, $F_3'(n,m)$ and $F_3'(n,m)$ are the derivatives of $G_1(n,0)$, $F_1(n,m)$ and $F_3(n,m)$ with respect to the independent variable, respectively.

$$G_1'(n,0) = \lambda_3[2 \sin(\sqrt{\lambda_4} \tau - \phi) + e^{-n^2 \pi^2 x}]\left[\cos(n \pi \xi_0) - \cos(n \pi \xi_1)\right][n \pi \sin(n \pi \xi)]$$

$$F_1'(n,m) = 8 \lambda_3 n^2 \sin[\frac{(2m-1) \pi}{2 \lambda_1} - \tan^{-1}(2m-1)] e^{-n^2 \pi^2 x} \sin[\tan^{-1}(2m-1)] X$$

$$\cos(n \pi \xi_0) - \cos(n \pi \xi_1) \left[\cos((2m-1) \pi \xi/2) \cos((2m-1) \pi \xi/2)\right]$$

$$F_3'(n,m) = 4 \lambda_4 \lambda_1^2 B \left(\frac{\sin(\sqrt{\lambda_4} \tau - \phi) + \sin[\tau + (2m-1) \pi \xi/2 \lambda_1]}{n + (2m-1)/2} - \frac{\sin(\sqrt{\lambda_4} \tau - \phi) + \sin[\tau - (2m-1) \pi \xi/2 \lambda_1]}{n + (2m-1)/2} \right) [\cos(n \pi \xi_0) - \cos(n \pi \xi_1)] X$$

$$X [(2m-1) \pi \xi/2] \cos[(2m-1) \pi \xi/2]$$

Comparing the degrees of $G_1'(n,0)$, $F_1'(n,m)$ and $F_3'(n,m)$ to that of $G_1(n,0)$, $F_1(n,m)$ and $F_3(n,m)$ respectively, we find that the degrees of $G_1'(n,0)$, $F_1'(n,m)$ and $F_3'(n,m)$ are greater than that of $G_1(n,0)$
\( F_1(n,m) \) and \( F_3(n,m) \). Therefore, the convergence of the series \( \sum_{n=1}^{\infty} \frac{G_1'(n,0)}{I_2(n,m)} \), 
\( \sum_{n=1}^{\infty} \frac{F_1'(n,m)}{I_2(n,m)} \) and \( \sum_{n=1}^{\infty} \frac{F_2'(n,m)}{I_2(n,m)} \) should be slower than that of the series 
\( \sum_{n=1}^{\infty} \frac{G_1(n,0)}{I_2(n,m)} \), \( \sum_{n=1}^{\infty} \frac{F_1(n,m)}{I_2(n,m)} \) and \( \sum_{n=1}^{\infty} \frac{F_2(n,m)}{I_2(n,m)} \) respectively. By repeated trials of numerical evaluation, even when \( n \) and \( m \) up to 300, the required convergent result of \( \sum_{n=1}^{\infty} \frac{G_1'(n,0)}{G_2(n,0)} \), \( \sum_{n=1}^{\infty} \frac{F_1'(n,m)}{F_2(n,m)} \) and 
\( \sum_{n=1}^{\infty} \frac{F_3'(n,m)}{F_4(n,m)} \) is not reached, which indicated that the convergence is very slow.

For our purpose of obtaining the results of thermal stress and deformation, a finite difference approximate method based on the result of the given temperature variation is introduced instead of directly evaluating the analytical solutions of stress and deformation, i.e., equation (46) and equation (47) respectively.
CHAPTER 4

SOLUTIONS OF THE DIFFUSION, DISPLACEMENT AND STRESS

WAVE EQUATION BY FINITE-DIFFERENCE METHODS

In the present investigation, finite difference approximations to the solution of the diffusion equation are so chosen as to give an explicit computational program for the unknown function. The solution then proceeds step by step, i.e., it matches in a direction normal to the boundary along which the initial condition is specified, guided by boundary conditions along the transverse boundaries of the open region. Then based on the given temperature field, implicit computational programs are used for solving the displacement equation and stress wave equation. An implicit finite difference approximation procedure which is unconditionally stable for the solution of the diffusion equation is also presented. Results obtained by this method are compared with that evaluated analytical solution and with that obtained by an explicit finite difference method.

4.1 On the solution of the Diffusion Equation

In section 2.1 the liberation of source as just described means that a quantity of energy is instantaneously liberated at \( t = 0 \); this may be taken to imply that at \( t = 0 \) an instantaneous rise of temperature of an amount \( aq_0/k \) will take place \([23]\) and therefore an equivalent problem is considered instead of solving the equation (22), equation (51) and equation (52) directly [see Appendix I], i.e.,
\[
\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}
\]

\( T = 0 \quad \text{for } x = 0, L \text{ at } t \geq 0 \) (52)

\( T = 0 \quad \text{for } x < x_0, x > x_1 \text{ at } t = 0 \)

\( = \frac{a q_0}{k} \quad \text{for } x_0 \leq x \leq x_1 \text{ at } t = 0 \) (53)

\( \frac{\partial T}{\partial t} = 0 \quad \text{for } 0 \leq x \leq L \text{ at } t = 0 \)

4.1.1 The explicit form of the Diffusion Equation

In order to approximate the solution of equation (52) and equation (53), a network of grid points is first established throughout the region \( 0 \leq x \leq L, 0 \leq t \), with grid spacings \( \Delta x, \Delta t \). In this problem, it is easy to ensure that grid points lie on the boundaries of \( x \) and \( t \). For any grid point \((i,j)\) that does not have \( i=0 \), or \( j=0 \). The derivatives of equation (52) are now replaced by the finite-difference forms suggested by using the second central difference in the \( x \) direction and first forward difference in the \( t \) direction.

\[
\frac{T_{i,j+1} - T_{i,j}}{\Delta t} = \alpha \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2}
\]

(53)

Letting \( r = \frac{\alpha \Delta t}{(\Delta x)^2} \), this may be rewritten as

\[
T_{i,j+1} = rT_{i-1,j} + (1-2r)T_{i,j} + rT_{i+1,j}
\]

(54)
In Fig. (5) the crosses and circles indicate those grid points involved in the time and space differences respectively.

![Diagram showing grid points and time steps](image)

**Fig. 5** the explicit form

It has been established that the calculations will be stable [27] and the solution of equation (54) will closely approximate that of equation (52) provided that \( r \leq 1/2 \). Furthermore, it was proven [27] that the solution of equation (54) will converge to that of equation (52) as both the time and space increments \( \Delta t \) and \( \Delta x \) approach zero assuming that inequality \( r \leq 1/2 \) is satisfied. This method has the advantages of being simple and easy to program.

4.1.2 The implicit form of the Diffusion equation

The stability restrictions inherent in explicit methods require very small steps in the \( t \) direction. Therefore, an implicit method, in which stability for all \( r > 0 \) is ensured, is applied to obtain a solution to compare with the "long time" analytical solution.

Using the second central difference in the \( x \) direction and the
first backward difference (suggested by Crank and Nicolso [1947] and Laasonen [1949]) in the $t$ direction at the point $(i,j+1)$, the difference equation is

$$\frac{T_{i,j+1} - T_{i,j}}{\Delta t} = \frac{T_{i+1,j+1} - 2T_{i,j+1} + T_{i-1,j+1}}{(\Delta x)^2} \cdot a$$

or

$$(1+2r)T_{i,j+1} = T_{i,j} + r(T_{i+1,j+1} + T_{i-1,j+1}) \quad (55)$$

where $r = a\Delta t / (\Delta x)^2$

That is, the above relation exists between the values of $T$ at the four points shown in the space-time grid of Fig.(6)

![Fig. 6 The implicit form](image)

This method is unconditionally stable. However, the use of this method requires the solution of a large number of simultaneous, linear, algebraic equations at each time step. Iterative methods are usually utilized to accomplish this solution.
4.2 On the solution of the Displacement Equation and Stress Wave Equation

Equation (27) is a hyperbolic partial differential equation. Following our usual approach, we select a network of points \((i,j)\) with spacing \(\Delta x\) and \(\Delta t\) and approximate the governing equation as follows:

\[
\frac{U_{i-1,j} - 2U_{i,j} + U_{i+1,j}}{(\Delta x)^2} = \frac{1}{v^2} \frac{U_{i,j-1} - 2U_{i,j} + U_{i,j+1}}{(\Delta t)^2} + \frac{Ce}{2(\Delta x)} \frac{T_{i+1,j} - T_{i-1,j}}{2(\Delta x)}
\]

or

\[
U_{i,j+1} = R_1^2 U_{i-1,j} + 2(1-R_1^2) U_{i,j} + R_1^2 U_{i+1,j} - U_{i,j-1} + R_2 (T_{i+1,j} - T_{i-1,j})
\]

where \(R_1 = V\Delta t/\Delta x,\ R_2 = CeR_1^2\Delta x/2\). This is an explicit recurrence formula.

Basing on this explicit recurrence formula, an improved implicit method [28] is introduced. The method is carried on an approximate difference for \(\partial^2 u/\partial x^2\) at the time steps \(j-1\) and \(j+1\) respectively. Then, an average recurrence formula between time steps \(j-1\) and \(j+1\) are shown as

\[
\frac{\partial^2 u}{\partial x^2} \frac{\partial^2 u}{\partial t^2} = \frac{1}{2(\Delta x)^2} (U_{i-1,j-1} - 2U_{i,j-1} + U_{i+1,j-1} + U_{i-1,j+1} - 2U_{i,j+1} + U_{i+1,j+1}) -
\]
\[
- \frac{1}{(\Delta t)^2} \left( U_{i,j-1} - 2U_{i,j} + U_{i,j+1} \right)
\]

Introducing,
\[
Ce \frac{\partial T}{\partial x} = Ce \frac{T_{i+1,j} - T_{i-1,j}}{2(\Delta x)}
\]
equation (27) can be approximated as
\[
\frac{R_i^2}{2} U_{i-1,j-1} - (1+R_i^2)U_{i,j-1} + \frac{R_i^2}{2} U_{i+1,j-1} +
\]
\[
+ \frac{R_i^2}{2} U_{i-1,j+1} - (1+R_i^2)U_{i,j+1} + \frac{R_i^2}{2} U_{i+1,j+1} + 2U_{i,j}
\]

\[
= R_2 (T_{i+1,j} - T_{i-1,j})
\]

This is an implicit recurrence formula. Fig. (7) is the typical relaxation pattern for equation (57). The use of this method requires the solution of a large number of simultaneous, linear, algebraic equations at each time step. Iterative methods are utilized to accomplish this solution.

Fig. 7 Typical relaxation pattern
Having established the values of $U$ at any time step, we can apply the following approximate formula to study the solution of the stress $\sigma$.

$$\frac{\sigma}{E} = \frac{3u}{3x} - C'eT$$

$$= \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x} - T_{i,j}$$
4.3 Numerical example and results

For numerical calculations, the same example presented in section 2.3.2 is carried out. All the computer runs were performed on the IBM 360 machine.

Part of the numerical results of the temperature, stress and displacement in the thin rod have been illustrated in Fig. 8 through 10 and table 4 through table 6. Fig. 8 and table 4 show the variation of temperature, which are carried out by explicit and implicit finite difference methods respectively. The result of the analytical solution is also compared with the results carried out by finite difference methods in Fig. 8. Fig. 9 and Fig. 10 show the results for an elastic wave traveling through a bar in the direction of continuously decreasing temperature. A typical time and space increments that lead to excellent result was \( \Delta x = 0.05, \Delta t = 0.00005 \). Fig. 10 shows two compression stress waves traveling in two opposite direction after a thermal shock. The wave velocity measured from Fig. 10 is about 1800 \( \text{ft/sec} \). Which is very close to the value \( (E/\rho)^{1/2} \).
CONCLUSIONS

The mathematical model with its solutions and the popular finite difference methods for an isotropic finite rod due to an instantaneous heat source have been studied. Obviously, Laplace transformation method is convenient for the solution of this problem. Although this solution is formally exact in that it is able to satisfy all the boundary and initial conditions as well as the governing differential equations, it will give rise to considerable difficulties in the numerical evaluation, for the convergence of the analytical solution with single and double series forms is very slow.

For our purpose, the finite difference method with carefully selected time and space increments has the advantage of investigating the stress wave pattern in the thin rod after a thermal shock is introduced.

Based on the study of this research, the following research is recommended and suggested.

1. In the present analysis, the edges of the rod are assumed to be fixed and freely supported. Analyses involving boundary conditions, such as clamped and simple edges have not been considered and have been left open for future investigations.

2. Although the idea of an instantaneous heat source is an idealization, experimental approximations are still of value. An
application of the laser beam technique has been done successfully by many investigators [29]. Therefore it is significant to pursue such techniques so that an experiment which closely approximate the present problem can be carried out in the near future.

The present analysis is concerned with the isotropic, homogeneous thin rod. Therefore, the study of the propagation of a stress wave in a nonhomogeneous thin rod and the effect of temperature gradients on the propagation of a stress wave due to an instantaneous heat source provides a new task for future investigation.
THIS BOOK CONTAINS NUMEROUS PAGES WITH DIAGRAMS THAT ARE CROOKED COMPARED TO THE REST OF THE INFORMATION ON THE PAGE.

THIS IS AS RECEIVED FROM CUSTOMER.
Fig. 4  Temperature Distribution, $x_0=1.0$ and $x_1=1.1$ (in dimensionless quantity)
Fig. 8 Temperature Distribution

- by implicit finite difference method.
- by explicit finite difference method.
- by evaluating the analytical solution.

\( x (\text{ft.}) \)

\( t = 1.0 \text{ sec.} \)
| AT TIME = | 1.000000 SECOND |
|===========|------------------|
| 0.000000 | 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 |
| 0.000000 | 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 |
| AT TIME = | 3.000000 SECOND |
| 0.000000 | 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 |
| 0.000000 | 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 |
| AT TIME = | 5.000000 SECOND |
| 0.000000 | 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 |
| 0.000000 | 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 |
| AT TIME = | 7.000000 SECOND |
| 0.000000 | 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 |
| 0.000000 | 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 |
| AT TIME = | 9.000000 SECOND |
| 0.000000 | 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 |
| 0.000000 | 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 |
| AT TIME = | 11.000000 SECOND |
| 0.000000 | 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 |
| 0.000000 | 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 |
| AT TIME = | 13.000000 SECOND |
| 0.000000 | 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 |
| 0.000000 | 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 |
| AT TIME = | 15.000000 SECOND |
| 0.000000 | 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 |
| 0.000000 | 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 |

Table 2 / Temperatures solved by evaluating analytical solution, \( x_0 = 0.2 \) and \( x_1 = 0.3 \)

**TEMPERATURE DISTRIBUTION (F)**

- THE SECTION LENGTH IS 0.1 FT.
- \( x_0 = 0.26 \) AND \( x_1 = 0.30 \)
- \( x = 0.1333 \) AND \( x = 0.2000 \)

\[ T(0,0) = 1.000000 \text{ SECOND} \]
### Table 3: Temperatures solved by evaluating analytical solution, \( x_0 = 1.0 \) and \( x_1 = 1.1 \)

**Temperature Distribution (F)**

<table>
<thead>
<tr>
<th>Time (seconds)</th>
<th>Temperature (°F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0000</td>
<td>0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000</td>
</tr>
<tr>
<td>3.0000</td>
<td>0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000</td>
</tr>
<tr>
<td>5.0000</td>
<td>0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000</td>
</tr>
<tr>
<td>7.0000</td>
<td>0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000</td>
</tr>
<tr>
<td>9.0000</td>
<td>0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000</td>
</tr>
<tr>
<td>11.0000</td>
<td>0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000</td>
</tr>
<tr>
<td>13.0000</td>
<td>0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000</td>
</tr>
<tr>
<td>15.0000</td>
<td>0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000</td>
</tr>
</tbody>
</table>
Table 4 Temperatures solved by evaluating analytical solution, \( x_0 = 0.7 \) and \( x_1 = 0.8 \)

Temperature Distribution (F)

- THE SECTION LENGTH IS 0.1 FT.
- \( x_0 = 0.7 \) AND \( x_1 = 0.8 \)
- \( x_{20} = 0.4567 \) AND \( x_{11} = 0.5433 \)

<table>
<thead>
<tr>
<th>Time (sec)</th>
<th>Temperature (F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0000</td>
<td>0.000000 0.000430 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000</td>
</tr>
<tr>
<td>3.0000</td>
<td>0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000</td>
</tr>
<tr>
<td>5.0000</td>
<td>0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000</td>
</tr>
<tr>
<td>7.0000</td>
<td>0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000</td>
</tr>
<tr>
<td>9.0000</td>
<td>0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000</td>
</tr>
<tr>
<td>11.0000</td>
<td>0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000</td>
</tr>
<tr>
<td>13.0000</td>
<td>0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000</td>
</tr>
<tr>
<td>15.0000</td>
<td>0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000</td>
</tr>
<tr>
<td>Time (Seconds)</td>
<td>Values</td>
</tr>
<tr>
<td>---------------</td>
<td>-------------</td>
</tr>
<tr>
<td>17</td>
<td>0.000000</td>
</tr>
<tr>
<td></td>
<td>0.000026</td>
</tr>
<tr>
<td></td>
<td>0.000109</td>
</tr>
<tr>
<td></td>
<td>0.000354</td>
</tr>
<tr>
<td></td>
<td>0.000905</td>
</tr>
<tr>
<td></td>
<td>0.001831</td>
</tr>
<tr>
<td></td>
<td>0.002929</td>
</tr>
<tr>
<td>19</td>
<td>0.000000</td>
</tr>
<tr>
<td></td>
<td>0.000040</td>
</tr>
<tr>
<td></td>
<td>0.000149</td>
</tr>
<tr>
<td></td>
<td>0.000428</td>
</tr>
<tr>
<td></td>
<td>0.000895</td>
</tr>
<tr>
<td></td>
<td>0.001471</td>
</tr>
<tr>
<td></td>
<td>0.002850</td>
</tr>
<tr>
<td></td>
<td>0.003518</td>
</tr>
<tr>
<td>Time</td>
<td>Temperature Distribution (°F)</td>
</tr>
<tr>
<td>-------</td>
<td>-------------------------------</td>
</tr>
<tr>
<td>Time = 1.0000</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000</td>
</tr>
<tr>
<td>0.000000 0.000003 0.000024 0.000185 0.001137 0.005127 0.014396 0.017454</td>
<td></td>
</tr>
<tr>
<td>0.014396 0.005127 0.001137 0.000185 0.000024 0.000003 0.000000 0.000000 0.000000</td>
<td></td>
</tr>
<tr>
<td>0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.0</td>
<td></td>
</tr>
<tr>
<td>Time = 3.0000</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000</td>
</tr>
<tr>
<td>0.000027 0.000113 0.0000419 0.0001315 0.0003404 0.0006999 0.010967 0.012696</td>
<td></td>
</tr>
<tr>
<td>0.010967 0.0006999 0.0003404 0.0001315 0.0000419 0.00000113 0.0000027 0.000006</td>
<td></td>
</tr>
<tr>
<td>0.000001 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.0</td>
<td></td>
</tr>
<tr>
<td>Time = 4.9999</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000 0.000000</td>
</tr>
<tr>
<td>0.000144 0.0000412 0.0001041 0.0002293 0.0004339 0.0006943 0.0009262 0.010204</td>
<td></td>
</tr>
<tr>
<td>0.00262 0.00643 0.00393 0.000293 0.0001041 0.000412 0.000144 0.000045</td>
<td></td>
</tr>
<tr>
<td>0.000013 0.000003 0.000001 0.000000 0.000000 0.000000 0.000000 0.0</td>
<td></td>
</tr>
<tr>
<td>Time = 5.0000</td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td>0.000011 0.000028 0.000062 0.00125 0.030240 0.000436 0.000748</td>
</tr>
<tr>
<td>0.001210 0.001844 0.002643 0.003556 0.004386 0.005300 0.006580 0.006606</td>
<td></td>
</tr>
<tr>
<td>0.005860 0.005300 0.004586 0.003556 0.002643 0.001844 0.001210 0.000748</td>
<td></td>
</tr>
<tr>
<td>0.000436 0.000240 0.000125 0.000062 0.000028 0.000011 0.0</td>
<td></td>
</tr>
</tbody>
</table>
Table 6  Temperature (F), displacement and stress solved
by finite difference method
* segment length is 0.05 feet *

<table>
<thead>
<tr>
<th>TIME</th>
<th>0.0</th>
<th>SECOND</th>
</tr>
</thead>
<tbody>
<tr>
<td>TEMPERATURE</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>DISPLACEMENT</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>STRESS</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>TIME</th>
<th>0.000050</th>
<th>SECOND</th>
</tr>
</thead>
<tbody>
<tr>
<td>TEMPERATURE</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>DISPLACEMENT</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-3.3396E-12</td>
<td>-0.86701E-12</td>
</tr>
<tr>
<td></td>
<td>-0.39569E-10</td>
<td>-0.84116E-10</td>
</tr>
<tr>
<td></td>
<td>-0.31735E-08</td>
<td>-0.33770E-08</td>
</tr>
<tr>
<td></td>
<td>-0.17891E-09</td>
<td>-0.38411E-10</td>
</tr>
<tr>
<td></td>
<td>0.95030E-12</td>
<td>0.52378E-12</td>
</tr>
<tr>
<td>STRESS</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.67492E-11</td>
<td>-0.85601E-11</td>
</tr>
<tr>
<td></td>
<td>-0.65503E-09</td>
<td>-0.13924E-08</td>
</tr>
<tr>
<td></td>
<td>-0.14503E-07</td>
<td>-0.12004E-06</td>
</tr>
<tr>
<td></td>
<td>-0.29598E-08</td>
<td>-0.13924E-08</td>
</tr>
<tr>
<td></td>
<td>-0.14503E-10</td>
<td>-0.95280E-11</td>
</tr>
</tbody>
</table>
### AT TIME = 0.000100 SECOND

**TEMPERATURE**

<table>
<thead>
<tr>
<th></th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.01973</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**DISPLACEMENT**

<p>| | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>-0.37882E-11</td>
<td>0.94365E-11</td>
<td>-0.1967E-10</td>
<td>0.3936E-10</td>
<td>-0.7761E-10</td>
<td>0.1517E-09</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.29492E-09</td>
<td>-0.5646E-09</td>
<td>0.1071E-08</td>
<td>-0.2003E-08</td>
<td>0.3676E-08</td>
<td>-0.6577E-08</td>
<td>0.1135E-07</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.2017E-08</td>
<td>0.9045E-15</td>
<td>0.3963E-08</td>
<td>0.1135E-07</td>
<td>0.6577E-08</td>
<td>0.3676E-08</td>
<td>0.2003E-08</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1071E-08</td>
<td>0.5646E-09</td>
<td>-0.2949E-09</td>
<td>0.1517E-09</td>
<td>0.7761E-10</td>
<td>0.3936E-10</td>
<td>-0.2003E-08</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1071E-08</td>
<td>0.5646E-09</td>
<td>0.2949E-09</td>
<td>0.1517E-09</td>
<td>0.7761E-10</td>
<td>0.3936E-10</td>
<td>-0.2003E-08</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**STRESS**

<p>| | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7576E-10</td>
<td>0.9436E-10</td>
<td>0.1588E-09</td>
<td>0.2992E-09</td>
<td>0.5793E-09</td>
<td>0.1123E-08</td>
<td>0.2164E-08</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.4127E-08</td>
<td>0.7773E-08</td>
<td>0.1438E-07</td>
<td>0.2005E-07</td>
<td>0.4574E-07</td>
<td>0.7675E-07</td>
<td>0.2787E-07</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.6993E-07</td>
<td>0.3767E-08</td>
<td>0.6645E-07</td>
<td>0.2787E-07</td>
<td>0.7675E-07</td>
<td>0.4574E-07</td>
<td>0.2005E-07</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1438E-07</td>
<td>0.7773E-08</td>
<td>0.2164E-08</td>
<td>0.1123E-08</td>
<td>0.5793E-09</td>
<td>0.2992E-09</td>
<td>0.4127E-08</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1399E-09</td>
<td>0.5646E-09</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### AT TIME = 0.000150 SECOND

**TEMPERATURE**

<table>
<thead>
<tr>
<th></th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
<th>0.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.01973</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**DISPLACEMENT**

<p>| | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>-0.2203E-10</td>
<td>0.5475E-10</td>
<td>-0.1087E-09</td>
<td>0.2023E-09</td>
<td>0.3793E-09</td>
<td>0.6666E-09</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.1179E-08</td>
<td>0.2044E-08</td>
<td>0.3655E-08</td>
<td>0.5650E-08</td>
<td>0.8249E-08</td>
<td>0.1287E-07</td>
<td>0.1976E-07</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-0.1252E-07</td>
<td>0.6677E-14</td>
<td>0.1252E-07</td>
<td>0.1676E-07</td>
<td>0.1287E-07</td>
<td>0.8249E-08</td>
<td>0.5650E-08</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.3456E-08</td>
<td>0.2044E-08</td>
<td>0.1179E-08</td>
<td>0.4672E-09</td>
<td>0.3716E-09</td>
<td>0.2044E-09</td>
<td>0.1179E-08</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.6392E-10</td>
<td>0.4039E-10</td>
<td>0.3342E-10</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**STRESS**

<p>| | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4566E-09</td>
<td>0.5475E-09</td>
<td>0.8524E-09</td>
<td>0.1476E-08</td>
<td>0.2625E-08</td>
<td>0.4642E-08</td>
<td>0.8092E-08</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1378E-07</td>
<td>0.2276E-07</td>
<td>0.3605E-07</td>
<td>0.5369E-07</td>
<td>0.7227E-07</td>
<td>0.7939E-07</td>
<td>0.3512E-08</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.1348E-07</td>
<td>0.6694E-07</td>
<td>0.1544E-07</td>
<td>0.3512E-08</td>
<td>0.7939E-07</td>
<td>0.7227E-07</td>
<td>0.5369E-07</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.3605E-07</td>
<td>0.2276E-07</td>
<td>0.1378E-07</td>
<td>0.8339E-08</td>
<td>0.4624E-08</td>
<td>0.2587E-08</td>
<td>0.3512E-08</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.7244E-09</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### AT TIME = 0.000300 SECOND

<table>
<thead>
<tr>
<th>TEMPERATURE</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>DISPLACEMENT</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>STRESS</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

### AT TIME = 0.000350 SECOND

<table>
<thead>
<tr>
<th>TEMPERATURE</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>DISPLACEMENT</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>STRESS</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td></td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>AT TIME = 0.000400 SECOND</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>TEMPERATURE</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.000000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| DISPLACEMENT |
| 0.0          |
| -0.282066E-09 |
| -0.57234E-08  |
| -0.87310E-08  |
| -0.11749E-07  |
| -0.14523E-07  |
| -0.16634E-07  |
| -0.17579E-07  |
| -0.16492E-07  |
| -0.15140E-07  |
| -0.17238E-07  |
| -0.17933E-07  |
| -0.17297E-07  |
| -0.15570E-07  |
| -0.13317E-07  |
| -0.11226E-07  |
| 0.95800E-08   |
| 0.83504E-08   |
| 0.79646E-08   |

| STRESS         |
| -0.37542E-07   |
| -0.57234E-07   |
| -0.59134E-07   |
| -0.60256E-07   |
| -0.57925E-07   |
| -0.48014E-07   |
| -0.33537E-07   |
| -0.34786E-08   |
| -0.26278E-09   |
| -0.38290E-07   |
| -0.69933E-09   |
| -0.32062E-07   |
| -0.17353E-07   |
| -0.39363E-07   |
| -0.29149E-07   |
| -0.39275E-07   |
| -0.17346E-07   |
| -0.32578E-07   |
| -0.12032E-08   |
| -0.39203E-07   |
| -0.29744E-07   |

| AT TIME = 0.000450 SECOND |
| TEMPERATURE          |
| 0.3                  |
| 0.000000            |
| 0.000000            |
| 0.000000            |
| 0.000000            |
| 0.000000            |
| 0.000000            |
| 0.000000            |
| 0.000000            |
| 0.000000            |
| 0.000000            |
| 0.000000            |
| 0.000000            |
| 0.000000            |
| 0.000000            |
| 0.000000            |
| 0.000000            |
| 0.000000            |
| 0.000000            |
| 0.000000            |
| 0.000000            |
| 0.000000            |
| 0.000000            |
| 0.000000            |

| DISPLACEMENT |
| 0.3          |
| -0.40445E-09 |
| -0.79499E-09 |
| -0.11514E-07 |
| -0.14434E-07 |
| -0.16328E-07 |
| -0.16843E-07 |
| -0.15846E-07 |
| -0.17933E-07 |
| -0.16731E-07 |
| -0.10941E-07 |
| -0.12397E-07 |
| -0.14759E-07 |
| -0.14767E-07 |
| -0.15777E-08 |
| -0.22759E-10 |
| 0.39628E-08   |
| 0.16814E-07   |
| -0.14685E-07 |
| -0.12569E-07 |
| -0.11214E-07 |
| -0.19789E-07 |
| -0.13806E-07 |
| -0.15828E-07 |
| -0.10518E-07 |
| -0.14562E-07 |
| -0.14215E-07 |

| STRESS         |
| -0.80349E-07  |
| -0.79499E-07  |
| -0.76697E-07  |
| -0.64865E-07  |
| -0.48441E-07  |
| -0.24084E-07  |
| -0.46198E-08  |
| -0.30501E-07  |
| -0.41928E-07  |
| -0.28920E-07  |
| -0.72436E-08  |
| -0.38093E-07  |
| -0.23621E-07  |
| -0.51689E-07  |
| -0.35575E-07  |
| -0.45473E-07  |
| -0.55329E-07  |
| -0.52256E-07  |
| -0.22678E-07  |
| -0.36440E-07  |
| -0.44074E-08  |
| -0.12623E-07  |
| -0.48525E-07  |
| -0.34564E-07  |
| -0.18597E-07  |
| -0.30732E-07  |
| -0.19610E-07  |
| -0.30567E-07  |
| -0.22652E-07  |
| -0.13033E-07  |
| 0.0           |
BIBLIOGRAPHY

[1] Boley, B. A.


[7] Bruce, W. A.
    Heat Flow in Arc Welding. The Welding J., vol. 20, no. 10,
    October, 1941, res. suppl., pp. 459.

[9] Barrekete, E. S.
    Thermoelastic Stresses in Beams. J. of App. Mech., 27, pp.465-573,

[10] Barton, M. V.
    The Circular Cylinder with a Band of Uniform Pressure on a Finite
    Length of the Surface. Tran. ASME, 63, pp.a, 97-A, 110, Sept.,
    1941.

    Thermal Stress in Tube with Axial Temperature Gradient, AFC Report

[12] Gatewood, B. E.
    Thermal Stresses in Long Cylindrical Bodies. Phil. Mag., Ser. 7,
    32, pp.282-301, 1941.

[13] Ignaczak, J.
    Thermal Stresses in a long Cylinder Heated in a Discontinuous

[14] Sokolowski, M.
    The Axially Symmetric Thermoelasticity Problem of the Infinite

    The Stress Distribution in a long Circular Cylinder when a Discon-
    tinuous Pressure is Applied to the Curved Surfaces. Phil. Mag.,
[16] Takeuti, Y.

[17] Hsu, T.R.

[18] Choudhuri, S.K.R.


[20] Das, B.R.

[22] Chu, W.H. and Dodge F.T.

Theory of Thermal Stresses. John Wiley and Sons, Inc.


[26] Spiegel, M.R.

[27] Kersten, R.D.
Engineering Differential Systems

[28] Hsu, Y.P.
Numerical Analysis. Hsien Yeh Publish Co. Taiwan.

[29] Axelrad, D.R. and Hsu, T.R.
APPENDIX I

AN EQUIVALENT SOLUTION OF TEMPERATURE VARIATION

For the purpose of our numerical work we consider an equivalent problem to the problem which was presented in section 2.2.2 instead of solving this problem directly.

The linear partial differential equation is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{a} \frac{\partial T}{\partial t}$$

subject to the boundary and initial conditions

$$T = 0 \quad \text{at } x = 0, \text{ for } t \geq 0$$

$$T = 0 \quad \text{at } x = L, \text{ for } t \geq 0$$

$$T = \frac{q_0 a}{k} \{H(x-x_0) - H(x-x_1)\} \quad \text{at } t = 0, \text{ for } 0 \leq x \leq L$$

where $q_0$, $a$, and $k$ are known constants.

Let us note that only the $x$-direction yields a characteristic value problem; then, with the proper choice of separation constant, the product solution $T(x,t) = X(x)\cdot\tau(t)$ gives
\[ \frac{d^2X}{dx^2} + \lambda^2X = 0; \quad X(0) = X(L) = 0 \quad (A1) \]

\[ \frac{dt}{dt} + \alpha \lambda^2 t = 0 \quad (A2) \]

The solution of equation (A1) is

\[ X_n(x) = A_n \psi_n(x), \quad \psi_n(x) = \sin(\lambda_n x), \text{ characteristic functions,} \]

\[ \lambda_n L = n\pi \quad n = 1, 2, 3, \ldots \ldots \text{ characteristic values,} \]

and the solution of equation (A2) is

\[ \tau_n(t) = C_n e^{-\alpha \lambda_n^2 t} \quad (A3) \]

Hence the product solution becomes

\[ T(x,t) = \sum_{n=1}^{\infty} a_n e^{-\alpha \lambda_n^2 t} \sin(\lambda_n x) \quad (A4) \]

where \( a_n = \frac{A}{n \pi} \)

Finally, introducing the initial condition, \( T(x,0) = \alpha q_0 / k \)

\( (x-x_0) - H(x-x_1) \), into equation (A4) gives
\[ T(x,0) = \frac{aq_0}{k[H(x-x_0)-H(x-x_1)]} \]

\[ = \sum_{n=1}^{\infty} a_n \sin(\lambda_n x) \]  \hspace{1cm} (A5)

Equation (A5) is the Fourier sine series expansion of \( T(x,0) \) over the interval \((0,L)\). Multiplying both sides of equation (A5) by \( \sin(\lambda_m x) \) and integrating the result over the interval \((0,L)\), where \( \sin(\lambda_m x) \) is the \( m \)th term in the set, we have

\[ \int_0^L T(x,0) \sin(\lambda_m x) dx = \sum_{n=1}^{\infty} a_n \int_0^L \sin(\lambda_n x) \sin(\lambda_m x) dx \]  \hspace{1cm} (A6)

where \( \sin(\lambda_m x) \) is the \( m \)th term in the set. Using the orthogonality of the set, we find that all terms in the sum on the right of equation (A6) are zero except the term corresponding to \( n = m \). Hence we obtain

\[ a_n = \frac{2}{L} \int_0^L T(x,0) \sin(\lambda_n x) dx \]

\[ = \frac{2aq_0}{kn\pi} [\cos(\lambda_n x_0) - \cos(\lambda_n x_1)] \]  \hspace{1cm} (A7)

Substituting equation (A7) into equation (A4), we have

\[ T(x,t) = -\frac{2aq_0}{k} \sum_{n=1}^{\infty} \frac{\sin(\lambda_n x)}{n\pi} e^{-\frac{\lambda_n^2 t}{2}} [\cos(\lambda_n x_0) - \cos(\lambda_n x_1)] \]

where \( \lambda_n = n\pi/L \). Hence some solution can be derived in this manner by this method.
APPENDIX II

...THIS PROGRAMM USED TO EVALUATE THE ANALYSIS
SOLUTION OF TEMPERATURE VARIATION

SPL...THE SPAN LENGTH OF ROD     (FT)
CON...THERMAL CONDUCTIVITY     (BTU/HR.FT.F)
CAP...SPECIAL HEAT COEFFICIENT     (BTU/LB.F)
DENS...DENSITY OF THE ROD     (IB/CUBIC FT)
ALPH...THERMAL DIFFUSIVITY     (SQUARE FT/HR.)
TIME...TIME FOR CONTROLLING THE
COMPUTATION OF THIS PROGRAM     (SECOND)
U0...HEAT QUANTITY DUE TO A CONSTANT
INSTANTANEOUS HEAT SOURCE     (BTU/CUBIC FT,SEC.)
X0,X1...PARTICULAR POINTS, BETWEEN THAT
A CONSTANT HEAT QUANTITY U0 IS INDUCED
E...YOUNGS ELASTIC MODULUS     (IBF/SQUARE FT)
COEF...COEFFICIENT OF LINEAR THERMAL
EXPANSION

DIMENSION TEMP(16)
10 FORMAT(F3.1,F5.1,F5.3,F5.1,F4.2,F4.1,F7.1,E9.3,E7.1)
11 FORMAT(5X,'**AT EXACT VALUE**')
12 FORMAT(1X,'**TEMPERATURE DISTRIBUTION (F)*/,//25X,
C'** THE SECTION LENGTH IS 0.1 FT.**')
15 FORMAT(1H1)
14 FORMAT(1X)
17 FORMAT(2F5.2)
18 FORMAT(2X,'X0 =',F5.2,') AND X1 =',F5.2,')
22 FORMAT(2X,'XX0 =',F7.4,') AND XX1 =',F7.4,')
40 FORMAT(1H0,10X,'AT TIME=',F11.4,' SECOND ',//,
C'10X,8F12.0')
MM=15
READ (5,10) SPL,CON,CAP,DENS,ALPH,TIMEF,U0,E,COEF
WRITE(6,15)
WRITE(6,14)
WRITE(6,12)

...A UNIT HEAT QUANTITY IS INTRODUCED
U0=U0/100

...CHANGE TIME UNIT (HOUR) TO TIME UNIT (SECOND)
CON=CON/3600.0
ALPH=ALPH/3600.0
PI=3.141529
Q0=U0*ALPH/CON

...KDUUMYA USED FOR CONTROLLING THE READ CARD
KDUMYA=1

4500 CONTINUE
READ(5,17) X0,X1
WRITE(6,18) X0,X1
XX0=X0/SPL
XX1=X1/SPL
EPI=1.0E-12
WRITE(6,22) XX0,XX1
TIME=1.0

2000 CONTINUE
TD=ALPH*TIME/(SPL**2)
X=0.0
MK=1

3000 CONTINUE
XX=X/SPL
C
C       ...SUMMING THE TERMS
C
SUM=0.0
N=1
S1=0.0

200 CONTINUE
PIN=N*PI
PIN2=PIN**2
COSN=CO(S(PIN*XX0)-COS(PIN*XX1)
SUM=SI(N(PIN*XX)/PIN**EXPL-PIN2*TD)*COSN+SUM
C
C       ...COMPARE THE RESULTS OF SUMMING TERMS (FIVE TERMS)
C
IF (N.GT.1) GO TO 110
S1=SUM
N=N+1
GO TO 200

110 IF (N.GT.2) GO TO 210
S2=SUM
N=N+1
GO TO 200

210 IF (N.GT.3) GO TO 300
S3=SUM
N=N+1
GO TO 200

300 IF (N.GT.4) GO TO 400
S4=SUM
N=N+1
GO TO 200

400 IF (N.GT.5) GO TO 500
S5=SUM
N=N+1
GO TO 200

500 CONTINUE
D1=ABS(S2-S1)
D2=ABS(S3-S1)
D3=ABS(S4-S1)
D4=ABS(S5-S1)
S1=S2
S2=S3
S3=S4
S4=S5
S5=SUM
C(D4.LE.EPI)) GO TO 1000
IF(N.EQ.1)) GO TO 800
N=N+1
GO TO 200
800 WRITE(6,11) N
GO TO 810
1000 CONTINUE
IF(MK.EQ.MM) GO TO 780
GO TO 790
780 S1=0.0
790 CONTINUE
TEMP(MK)=2.0*JO*ABS(S1)
X=X+D*1
MK=MK+1
IF(MK-MM) 3000, 3000, 4000
4000 CONTINUE
WRITE(6,40) TIME,(TEMP(KK),KK=1,MM)
TIME=TIME+2.0
IF(TIME-TIMEF) 2000, 2000, 810
810 CONTINUE
WRITE(6,15)
KDUMYA=KDUMYA+1
C
C \* \* USING L TO CONTROL READ CARD \* \*
C
L=3
IF(KDUMYA-L) 4500, 4500, 4600
4600 CONTINUE
WRITE(6,15)
STOP
END
SAMPLE DATA IS AS FOLLOWS

SPL...1.5
CON...224.0
CAP.....0.981
DENS....553.0
ALPH...4.42
TIMEF...11.0
U0......3500.0
E......1.87E-09
COEF.....9.3E-06
X0......0.2, X1.....0.3
X0......1.0, X1.....1.1
X0......0.7, X1.....0.3
APPENDIX III

C .... THIS PROGRAM USED THE IMPLICIT FINITE DIFFERENCE
C TO SOLVE THE TEMPERATURE VARIATION
C DIMENSION T(31,2), A(29,30)
C COMMON/47/A
C 10 FORMAT(5,3,F5.1)
C 12 FORMAT(2X,'TIME=',F6.4,'/3X,4(5F9.6/,3X))
C 15 FORMAT(/,','====================================================='
C
C C
C ALPH... THERMAL DIFFUSIVITY (SQUARE FT/HR.)
C CON... THERMAL CONDUCTIVITY (BTU/HR,FT,F)
C
C READ(5,10) ALPH, CON
C NV=31
C NN1=NN-1
C NN2=NN-2
C ALPH=ALPH/3600
C CON=CON/3600
C DTAU=0.05
C TIMEF=3.0
C DX=0.050
C R=ALPH*DTAU/(DX*DX)
C
C C
C .... AN UNIT HEAT QUANTITY BE INTRODUCED......
C
C QO=1.0
C TO=ALPH/CON=QO
C TAU=0.0
C
C C
C .... SET AND PRINT INITIAL TEMPERATURES......
C
C DO 100 I=1,31
C
C 100 T(I,1)=0.0
C T(15,1)=TO
C T(16,1)=TO
C T(17,1)=TO
C WRITE(6,15)
C WRITE(6,12) TAU,(T(I,1), I=1, NN)
C
C C
C .... SET BOUNDARY VALUES......
C
C T(1,1)=0.0
C T(NN,1)=0.0
C T(1,2)=0.0
C T(NN,2)=0.0
C
C C
C .... REFORM CALCULATIONS OVER SUCCESSIVE TIME STEPS....
500 CONTINUE
   TAU = TAU + DTAU

   ..... COMPUTE NEW TEMPERATURE ..... 

   DO 200 J = 1, N1
   DO 200 I = 1, N2
   A(I, J) = 1.0

200 CONTINUE
   DO 300 J = 1, N1
   A(I, J) = (-1.0) + 2.0 * R
   IF (I .EQ. 1) GO TO 350
   IF (I .EQ. N2) GO TO 360
   A(I, I+1) = R
   A(I, I-1) = R
   A(I, N1) = T(I+1, 1)
   GO TO 300

350 A(I, I+1) = 0
   A(I, N1) = T(I+1, 1)
   GO TO 300

360 A(I, I-1) = R
   A(I, N1) = T(I+1, 1)
   GO TO 300

300 CONTINUE
   CALL GAUS2 (NN2)
   DO 320 I = 1, N2
      T(I+1, 2) = A(I, N1)

320 CONTINUE

   ..... PRINT TEMPERATURES WHEN APPROPRIATE ..... 

   WRITE (6, 16)
   WRITE (6, 12) TAU, (T(I, 2), I = 1, N1)

   ..... CHANGE NEW TEMPERATURES TO OLD TEMPERATURES AND STORE ..... 

   DO 400 J = 2, 30
      T(I, J) = T(I, 2)

400 CONTINUE
   IF (TAU-TIM <= 1 500, 500, 60)
   600 STOP
   END

C
C
C
SUBROUTINE GAUS2 (N)
DIMENSION A(29, 30)
COMMON /A2/A
N1 = N+1
DO 200 J = 1, N
   DIV = A(J, J)
   S = 1.0 / DIV
   DO 201 K = J, N1
      A(J, K) = A(J, K) / S

201 A(J, K) = A(J, K) / S
DO 202 I=1,N  
   IF(I-J) 203,212,203  
203  A(I,J)=A(I,J)  
   DO 204 K=J+1  
204  A(I,K)=A(I,K)+A(I,J)*A(J,K)  
202  CONTINUE  
203  CONTINUE  
RETURN  
END

SAMPLE DATA IS AS FOLLOWS

ALPH....4.43  
CONV.....224.3
APPENDIX IV

.....THIS PROGRAM USED THE FINITE DIFFERENCE METHODS
    TO SOLVE THERMOELASTICITY PROBLEM

.....A UNIT HEAT QUANTITY OF HEAT IS INTRODUCED

DIMENSION T(31,3),DISPL(31,3),STRESS(31,3),RIGHT(29),
    CLEFT(29),ERROR(29)
COMMON/Z1/T
COMMON/Z4/STRESS
COMMON/Z5/DISPL,TIME,XO,X1
COMMON/Z6/DEnst,ALPH
COMMON/Z7/CDEF
COMMON/Z8/LEFT
COMMON/Z9/RIGHT
COMMON/Z10/ERROR
10 FORMAT(F3.1,F5.1,F5.3,F5.1,F4.2,F4.1,F7.1,E9.3,E7.1)
17 FORMAT(2F5.2)
30 FORMAT(1HL,25X,'TEMPERATURE (F) STRAIN AND STRESS
    (PSI) ARE GIVEN AT SPACING 1F',F13.6,' FT APART'/)
40 FORMAT(1HO,' AT TIME= ',F11.9,' SECOND ',/,,6X,
    'TEMPERATURE',/,,3(4X,3F15.9,/,),4X,7F16.9)
42 FORMAT(6X,'DISPLACEMENT',/,,3(4X,3E15.9,/,),4X,7F16.9)
29 FORMAT(//,'.........................................')
50 FORMAT(6X,STRESS',/,,3(4X,3E15.9,/,),4X,7F16.9)

CON...THERMAL CONDUCTIVITY (BTU/Hr,FT,F)
SPL...THE SPAN LENGTH OF ROD (FT)
CAP...SPECIAL HEAT (BTU/IB,F)
DENS...DENSITY OF THE ROD (IB/CUBIC FT)
ALPH...THERMAL DIFFUSIVITY (SQUARE FT/HR.)
TIME...TIME FOR CONTROLLING THE
      COMPUTATION OF THIS PROGRAM (SECOND)
UO...HEAT QUANTITY DUE TO A CONSTANT
      INSTANTANEOUS (BTU/CUBIC FT,SEC.)
X0,X1...PARTICULAR POINTS, BETWEEN THAT
      A CONSTANT HEAT QUANTITY UO IS INDUCED
E...YOUNG'S ELASTIC MODULUS (IBF/SQUARE FT)
CDEF...COEFFICIENT OF LINEAR THERMAL
      EXPANSION (FT/FT,F)

READ (5,10) SPL,CON,CAP,DENS,ALPH,TIME,UO,E,CDEF
NUM3=31
NUM1=NUM3-1
NUM2=NUM3-2
UO=U0/10}
C
CON=CON/3600.0
ALPH=ALPH/3600.0
DTIME=0.00005
DSPLX=0.050
XX=0.050
READ(5,17) XJ,XI
TIME=0.0
K=1
KK=1
X=0.0
F=UN=ALPH/CON

***** SET INITIAL TEMPERATURES (AN EQUIVALENT PROBLEM WAS REPLACED) *****

DO 62 I=1, NUMB
   T(I,1)=0.0
62 CONTINUE
T(15,1)=F
T(16,1)=F
T(17,1)=F
DO 100 I=1, NUMB
   DISPL(I,K)=3.0
   STRESS(I,K)=3.0
100 CONTINUE

SET BOUNDARY VALUES

T(1,1)=0.0
T(1,2)=3.0
T(1,3)=0.0
T(NUMB,1)=0.0
T(NUMB,2)=3.0
T(NUMB,3)=0.0
DISPL(1,1)=3.0
DISPL(1,2)=3.0
DISPL(1,3)=3.0
WRITE(6,33) XX
WRITE(6,23)
WRITE(6,40) TIME, (T(I,K), I=1, NUMB)
WRITE(6,42) (DISPL(I,K), I=1, NUMB)
WRITE(6,50) (STRESS(I,K), I=1, NUMB)
999 CONTINUE
K=K+1
KK=KK+1
TIME=TIME+DTIME
CALL TEMP (NUMB, K, DTIME)
CALL DISPL (DTIME, DSPLX, K, NUMB)
CALL ELAST (K, NUMB, GT, F, DSPLX)
IF (KK .GT. 2) GO TO 250
WRITE(6,29)
WRITE (6,40) TIME, (T(I,2), I=1, NUMB)
WRITE (6, 42) (DISPL(I, J), I=1, NUMH)
WRITE (6, 50) (STRESS(I, 2), I=1, NUMH)
WRITE (6, 29)
GO TO 260
250 CONTINUE
WRITE (6, 40) TIME, (T(I, 3), I=1, NUMH)
WRITE (6, 42) (DISPL(I, 3), I=1, NUMH)
WRITE (6, 50) (STRESS(I, 3), I=1, NUMH)
WRITE (6, 29)
260 CONTINUE
IF (K-2) 270, 270, 280
283 CONTINUE
DO 301 I = 1, NUMB
  T(I, 1) = T(I, 2)
  T(I, 2) = T(I, 2)
  DISPL(I, 1) = DISPL(I, 2)
  DISPL(I, 2) = DISPL(I, 3)
  STRESS(I, 1) = STRESS(I, 2)
  STRESS(I, 2) = DISPL(I, 3)
301 CONTINUE
K = 2
270 CONTINUE
IF (KX-50) 290, 290, 300
290 CONTINUE
GO TO 999
300 CONTINUE
STOP
END

C * USING EXPLICIT METHOD TO SOLVE TEMPERATURE
C
SUBROUTINE TEMP (NM, K, DTIME)
DIMENSION T(31, 3), A(30, 30)
COMMON/Z1/T
COMMON/Z5/DISPLX, TIME, XJ, XI
COMMON/Z6/DENS, E, ALPH
NM = NM - 1
T(1, K) = .0
T(NM, K) = .0
B = ALPH * DTIME / (DISPLX * DISPLX)
DO 100 I = 2, NUM
  T(I, K) = B * T(I-1, K-1) + (1.0 - 2.0 * B) * T(I, K-1) + B * T(I+1, K-1)
100 CONTINUE
RETURN
END

C * USING IMPROVED IMPLICIT METHOD TO SOLVE DISPLACEMENT
C
SUBROUTINE DISP (DTIME, DISPLX, K, M)
DIMENSION A(30, 31), DISPL(31, 3), T(31, 3)
COMMON/Z1/T
COMMON/Z2/DISPL
COMMON/Z6/ DENS, E, ALPH
COMMON/Z7/CJEF
COMMON/W2/A
M1=M-1
M2=M-2
VELO=SQRT(E/DENS)
RATO=VELO-DTIME/DSPLX
RATO=RATO/2
RA=(1+RATO)
COR=CJEF*DSPLX*RA
DO 100 K=1,M1
DO 900 J=1,K1
AI(I,J)=0.0
100 CONTINUE
IF(K-2) 300,300,200
300 CONTINUE
DO 900 I=1,K1
AI(I,I)=2.0*RA
IF(I.EQ.1) GO TO 500
IF(I.EQ.K1) GO TO 550
AI(I,I+1)=RATO
AI(I,I+1)=RATO
AI(I,M)=COR*(T(I+2,K-1)-T(I,K-1))/2.0-2.0*DISPL(1+1,K-1)
GO TO 700
500 CONTINUE
AI(I,I+1)=RATO
AI(I,M)=G0/2.0-T(I+2,K-1)-2.0*DISPL(I+1,K-1)
GO TO 700
550 CONTINUE
AI(I,I-1)=RATO+2.0
AI(I,M)=COR*T(I,K-1)-2.0*DISPL(I+1,K-1)
700 CONTINUE
800 CONTINUE
GO TO 910
200 CONTINUE
DO 840 I=1,K1
AI(I,I)=-1.0*RA
IF(I.EQ.1) GO TO 505
IF(I.EQ.K1) GO TO 550
AI(I,I+1)=RATO/2.0
AI(I,I-1)=RATO/2.0
AI(I,M)=COR*T(I+2,K-1)-T(I,K-1)/2.0-2.0*DISPL(1+1,K-1)
C=(RATO/2.0)*DISPL(1+1,K-2)-RA*DISPL(I+1,K-2)
C=RATO/2.0*DISPL(I+2,K-2)
GO TO 710
505 CONTINUE
AI(I,I+1)=RATO/2.0
AI(I,M)=COR*(T(I+2,K-1)-T(I,K-1))/2.0-
C-RA*DISPL(I+1,K-2)+RATO/2.0*DISPL(I+2,K-2)
C-2.0*DISPL(I+1,K-1)
GO TO 710
515 AI(I,1)=RATO
A(i,m) = COR * T(i,k-1) - (T(i,k-1) * A(i,k-1)) - (DISPL(i,k-1) * A(i,k-1))
710 CONTINUE
840 CONTINUE
880 CONTINUE
900 DISPL(i+1,k) = A(i,m)
RETURN
END
SUBROUTINE ELAST(K,NUMA,COEF,E,DISPLX)
DIMENSION T(31,3),DISPL(31,3),STRESS(31,3)
COMMON/ZI/ T
COMMON/Z2/DISPL
COMMON/Z4/STRESS
IF(K.EQ.1) 100,100,300
100 CONTINUE
DO 150 I=1,NUMA
IF(I.EQ.1) GO TO 190
IF(I.EQ.NUMA) GO TO 200
STRESS(I,2) = (DISPL(I+1,2) - DISPL(I-1,2)) / (2.0 * DISPLX)
C = COEF * T(I,2)
GO TO 150
190 STRESS(I,2) = DISPL(I+1,2) / DISPLX - COEF * T(I,2)
GO TO 150
200 STRESS(I,2) = -COEF * T(I,2)
150 CONTINUE
GO TO 500
300 CONTINUE
DO 390 I=1,NUMA
IF(I.EQ.1) GO TO 310
IF(I.EQ.NUMA) GO TO 320
STRESS(I,3) = (DISPL(I+1,3) - DISPL(I-1,3)) / (2.0 * DISPLX)
C = COEF * T(I,3)
GO TO 390
310 STRESS(I,3) = DISPL(I+1,3) / DISPLX - COEF * T(I,3)
GO TO 390
320 STRESS(I,3) = -COEF * T(I,3)
390 CONTINUE
500 CONTINUE
RETURN
END
SUBROUTINE GAUS2(N)
DIMENSION A(30,31)
COMMON/Z2/A
NI = N+1
DO 200 J=1,NI
DIV = A(J,J)
S = 1.0 / DIV
DO 201 K=J,NI
201 A(J,K) = A(J,K) * S
DO 202 K=1,NI
202
DO 204 K = 1, 11
204 A(I,K) = A(I,K) + AIJ * A(I,J,K)
202 CONTINUE
200 CONTINUE
RETURN
END

SAMPLE DATA IS AS FOLLOWS

SPL.....1.5
CON.....2.25
CAP.....0.091
DENS.....558.3
ALPH.....4.42
TIMEF.....11.0
U0.....35000.0
E.....1.372E 09
COEF.....9.3E-06
ACKNOWLEDGEMENT

The author wishes to express his sincere thanks to Dr. Hugh S. Walker, for his supervision and guidance in the preparation of this thesis and for his teachings and advice from which the author has acquired most of his ability to analyze and see more clearly various problems. The author also feels deeply indebted to Dr. Chi Lung Huang for the many valuable discussions the author has had with him over the subject of mechanics in general and this thesis in particular.

Finally, the author would like to pay tribute to his wife, Chunmei, who painstakingly checked the formulas of mathematics and computer programs and who, by her encouragement, made this thesis possible.
THERMOELASTIC STRESS AND DEFORMATION IN A THIN ROD DUE TO AN INSTANTANEOUS HEAT SOURCE

by

HWANG HWEI SIANG
B.S., Chung Yuan Christian College, China, 1970

AN ABSTRACT OF A MASTER'S THESIS

submitted in partial fulfillment of the
requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1976
This research presents an analytical and numerical solution for the transient temperature and the associated thermal stress and deformation which arise in an isotropic thin rod of finite length due to an instantaneous heat source distributed over a finite portion of the rod. The problem is approached from the standpoint of classical linear, uncoupled, thermoelastic theory. The material of the rod is assumed to be homogeneous and isotropic with respect to both its thermal and mechanical response, and its physical properties to be independent of temperature.

Assuming the temperature gradients in the cross section of the rod to be negligible, and also that heat loses through the surface to the surroundings medium is not considered. The diffusion equation is solved by the technique of Laplace transformation. A "long-time" solution for the simple boundary conditions (zero temperature boundary) is obtained. Associated with the given temperature variation, an elementary thermoelastic theory was applied to derive the governing differential equations under the thermal load. For the fixed end and free end boundary conditions, the Laplace transformation method leads directly to the solutions in terms of a double infinite series. For the poor convergence of the double infinite series, it gives rise to considerable difficulties in the numerical evaluation. For this research, the finite difference approximate method with carefully selected time and space increments has the advantage of investigate the stress wave and deformation patterns
in the thin rod. A graphical presentation is made of predicted temperature distributions, stress and deformation.