SEARCH TECHNIQUES
AND WAGE INCENTIVE PLANS

by

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ABSTRACT

This thesis has two sections. Part 1 deals with the literature survey and the development of new techniques to handle search problems. Since the effectiveness of the search procedure is characterized by its rate of convergence, much of research work has been and are still being done to reduce the computation time. An attempt was made to solve one-dimensional search problems for convex functions by bisecting the enveloping cone of the function and then rotating it till the bisector becomes vertical. The generalization of this new method for any unimodal function by coupling with Fibonacci search was also discussed. This approach essentially cuts down the total number of experiments required to reach at optimum. A new method for multi-dimensional search problems based on the intersection of quadratics passing through the line-optimums in coordinate directions was developed and exemplified along with the comparison with other standard methods to show its efficiency.

In the second section, a case study was made with a view to show how operations research technique can be applied to formulate and solve certain wage incentive problems. Since the basic problem in an incentive scheme is to define the base level efficiency from which the incentive should start and also the incentive rates, the problem was formulated with the objective as to minimize the variance between the optimum base
level efficiency and the current different efficiencies of various departments. A constraint was that the total incentive to be paid to the workers must not exceed the current overtime expenses. This problem was solved by Generalized Reduced Gradient method and separable programming.
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CHAPTER I
LITERATURE SURVEY ON SEARCH TECHNIQUES

1.1 Introduction

Search for the optimum is the main objective of all decision problems, whether constrained or unconstrained. In fact, constrained problems can be converted into unconstrained ones. The simplest form of search is that for a function having only one variable. In general, two policies, viz, sequential and simultaneous searches are normally used. Some research work has also been done combining these two policies. All these techniques are available in different literatures. Section 1.2 deals with the literature survey on one-dimensional search. In addition, the quadratic and cubic interpolation methods are also briefly discussed. The literature review on multi-dimensional search is provided in section 1.3.

Although there exist different types of searches depending on the nature and the objective of the search, this chapter does not cover all of them.

1.2 One Dimensional Search

For a maximum of an unimodal function, the second order search in the sense that the information is given by pairs of observation, was attempted by Kieffer1. He determined the
interval containing this maximum without postulating any regularity conditions involving continuity, derivatives, etc. This is essentially known as Fibonacci search method which has theoretical connections with problems dating all the way back to Euclid.

A less powerful method known as Dichotomous Search\(^2\) reduces the interval of uncertainty by placing pairs of experiments successively in the remaining interval. The effectiveness of this method grows exponentially with the number of experiments. In a Fibonacci scheme each new experiment serves to reduce the interval of uncertainty but in a dichotomous scheme it takes two new experiments to cut down the interval of uncertainty.

Another technique which is nearly as effective as the Fibonacci does not require any knowledge in advance the number of experiments to be carried out. This is familiar as Golden Section Method\(^2\) which essentially divides a segment into two unequal parts so that the ratio of the whole to the larger is equal to the larger to the smaller. Euclid, himself, did this simply by a ruler and a compass.

When the variable does not assume continuous values within a given interval of uncertainty but instead is confined to a finite number of discrete points, the Lattice Search Technique as Kieffer\(^2\) calls them is used. In this case the number of points are to be finite and arrangable in
some order that will make the criterion of effectiveness unimodal.

Oliver and Wilde\textsuperscript{3} pointed out that Kiefer's original technique (Fibonacci Search) is asymmetric in the sense that the last two experiments are not located symmetrically with respect to each other. The modified procedure developed by Oliver et. al. is symmetric since it permits the last experiment to be placed symmetrically with respect to the most effective previous experiment.

Avriel and Wilde\textsuperscript{4} developed a minimum search plan using 'Block Search Strategy' technique that can be used for any number of experiments and for any number of blocks in the sequence. For one experiment per block, it reduces to the Fibonacci strategy. The 'Block Search Strategy' is optimal in the sense that for a required final interval of uncertainty and for any given number of simultaneous experiments and blocks, it has the largest possible starting interval.

According to Berman\textsuperscript{5} his method which uses Fibonacci numbers is optimal because 1) it does not postulate any regularity conditions, 2) it is simpler, 3) it often requires fewer number of evaluations, 4) it is self-correcting i.e. an error in any particular evaluation will not affect the final result. He also exemplified some possible application of his method.

When an arbitrary probability density function for the
distribution of maximum is given, the problem of estimating the optimal interval containing the location of the maximum of a unimodal function was investigated by Heymann\textsuperscript{6}. The statistical information gained by the search is used for such estimation. He found that the strategies had to be different in accordance with the odd or even number of experiments.

The minimax block search strategy presented by Avriel and Wilde\textsuperscript{4} was further improved by them in a latter publication\textsuperscript{7} and it was shown that this method is an excellent approximation of the previous one. This nearly optimal minimax golden block search method has the advantage that the number of function evaluations need not be specified in advance.

When some bound on the rate of change of function of one variable is available, Shubert's\textsuperscript{8} method can be used to locate the maximum of the function defined over a closed interval.

Wilde and Beamer\textsuperscript{9} presented a minimax search strategy for locating the boundary point of a region on a line joining a feasible point to an infeasible point. These strategies, as it were claimed, could be useful subroutines for many multi-dimensional optimization algorithms.

Gottfried\textsuperscript{10} showed that for a given interval of uncertainty, the minimax separation between two points consider-
ing the distinguishibility of the function values, the
search should be terminated when the interval of uncertainty
is less than \((\epsilon \varphi)/(2-\varphi)\) where \(\varphi\) is the golden ratio.

One of the new additions in the development of search
procedures for one dimensional problems was made by Fox, et.
al.\(^{11}\) Their method finds three points bracketing the
minimum, fits a quadratic through them to yield a fourth
point, then fits successive cubic through four points dis-
carding one at each time, until certain stop criteria are
met. No gradient evaluations are required. This procedure
is claimed to take 1/2 to 3/4th less computer time than
others.

When all experiments must be run at the same time, it
is necessary to use a simultaneous search plan. It is less
effective than the sequential plans but the experimenter,
at times, is forced to use a simultaneous plan. The inter-
val is divided into \((n+1)\)'equal interval and the function
is evaluated at \(n\) points. The best value of function is
picked up, the interval bracketing this best value is again
divided into \((n+1)\) divisions and the process is repeated till
it meets the stopping criterion. For two experiments only,
simultaneous plan is just as good as a sequential one.

Wilde\(^2\) suggested that for even number of experiments,
search by uniform pairs which is, essentially comes under
simultaneous search plan is the best way as far as the deployment of the experiments are concerned.

If the objective function is continuous and convex in the interval of uncertainty, it is often possible to obtain a good estimate of the optimum value (12) of the objective function by using a quadratic approximation of the function to locate the optimum point. But if in addition to the above, the derivative of the function is available then cubic interpolation provides a good estimate of the location of the optimum point.

1.3 Multi Dimensional Search

The problem of locating the optimum on a multi-dimensional response surface is more important than uni-dimensional search since the problems encountered in the real world usually involve multi-dimensions.

With the object of finding the optimum in these types of problems Cauchy\textsuperscript{13} first introduced the method of steepest descent which, as a matter of fact, forms the basis for all the searches currently in use. It was an intuitively attractive idea of climbing the steepest path but because of the inherent difficulties (slow convergence due to interaction among the variables) associated with each new direction being normal to the old direction, the method is not very
efficient.

In the modified steepest descent method\textsuperscript{14} the step size of the steepest descent is multiplied by 0.9 and the process is continued. After, say, four such repetitions of this procedure, one step of full length is taken. In this case, the successive directions will not be mutually orthogonal.

The method of rotating the co-ordinates, as devised by Rosenbrock\textsuperscript{15} is very effective at finding the optimum of a function. Instead of taking a fixed step in each direction, Rosenbrock rotates the co-ordinate system so that one axis points along the direction of ridge as estimated by the previous trial. The other axes are arranged in directions normal to the first.

For straight type of ridges Partan Method\textsuperscript{16} would appear efficient. This technique which does not use gradients can be extended to ellipsoidal functions of any number of independent variables. For non ellipsoidal functions, the Partan will work but if the function is not radially similar on every possible cross section, it will not work.

A variant of this method was discovered independently by Powell\textsuperscript{17}. It is based on the theorem which is that because the function $f(x)$ is quadratic in the independent variables, any line which passes through the optimum point $x^{*}$ intersects the members of the family of contours $f(x) = c$
(constant) at equal angles. The corollary is that of the normal at \( t' \) to the contour \( f(x) = f(t) \) is parallel to the normal at \( t' \) to \( f(x) = f(t') \), then the lines joining \( t \) to \( t' \) pass through \( Y \). This method gives second order convergence.

The method of sectioning or one at a time method\(^8\) will not always reach the maximum, even when the contours are convex. Its practical value is extremely limited. It is good for circular contours only.

The pattern search technique of Hooke and Jeeves\(^9\) has had reasonable practical success, probably due to its ability to follow a curved ridge when necessary. Mugele's "poor man's optimizer"\(^10\) scheme also is able to track the curved ridges. In these methods gradient evaluations are not needed.

In case of defined gradient, Fletcher and Powell's method\(^21\) which essentially is a simplified version of Davidon's (1959) method, provides quadratic convergence and it is superior to Powell's and Partan method both in that it uses the information determined by previous iterations and also in that each iteration is quick and simple to carry out. Further more it yields the curvature of the function at the optimum.

Fletcher and Reeves\(^22\) conjugate gradient method is as effective as that of Fletcher and Powell's method. In the
latter method, storage space for H (Hessian) matrix is to be provided while in Fletcher and Reeve's method, storage is required for only three vectors and time for manipulating H matrix is saved. So in problems, where 'n', the number of variables is large, this method may be preferred to Fletcher and Powell's method.

When derivatives are not available, Powell's method furnishes faster convergence in dealing with many variables. The first iteration is same as that for changing one parameter at a time. This latter method is next modified to generate conjugate directions by making each iteration define a new direction \( \mathbf{y} \) and choosing the linearly independent directions for next iterations.

Sequential simplex method is also useful to handle these types of problems. It was introduced by Nelder and Mead. It has the same convergence rate as that of Powell's method.

For minimizing a sum of squares of non-linear functions Powell's generalized least square method does not require evaluation of derivatives. This method has the comparable convergence with the classical procedure and the number of times the individual terms of the sum of squares have to be calculated is approximately proportional to the number of variables.

In a review paper Fletcher discussed the efficiency of the three different methods, viz, Davis, Swam and Campey
method (DSC method), Powell's method and Smith's method using some standard test functions as a basis for comparisons. All these three methods do not require any calculation of derivatives. DSC method is simple and effective for large numbers of variables and when the minimum cannot be represented adequately by a quadratic where as on the basis of function evaluations the most efficient method is that of Powell. However, for large number of variables it is less favorable than DSC method. The Smith's method is generally inferior to other methods and is acceptable only when 'n' is small (2, 3, 4).

Branen\textsuperscript{27} showed that a return function with a given probability distribution can be maximized using an iterative method which is somewhat analogous to Newton's iterative method.

Box\textsuperscript{28} proved that as the number of variables increases, Fletcher and Powell's method is most consistently successful when the gradient is available. Powell's method and Fletcher and Powell's method work substantially better with 5, 10, 20 dimension test functions than other methods though it assumes quadratic optimum characteristic. He has also pointed out that simplex method perform better than Powell's method in case of two-dimensions but lesser and lesser successful as the dimension increases.
Curtis and Powell\textsuperscript{29} discussed in detail on exchange algorithms for calculating minimax approximation with a view to provide a deep insight into the convergence of this method.

Powell's method has been criticized by Zangwill\textsuperscript{30} who in his counter-example showed that Powell's method not only does not converge to the minimum of a quadratic in a finite number of iterations but it will not converge in any number of iterations. He made some modification of Powell's method which can be useful strictly for convex function.

The variation matrix method developed by Davidon\textsuperscript{31} which uses the inverse matrix of second derivative of any function is the generalized form of variable metric method (Davidon, 1959). The algorithm is simpler and in quadratic cases, gradient evaluations are half the number made in variable metric algorithm.

An algorithm for non-linear minimax approximation was described by Osborne and Watson\textsuperscript{33} in 1969. This algorithm was illustrated by the evaluation of several approximation to the solution of Blasius equation.

An improved procedure presented by Palmer\textsuperscript{34} to generate orthogonal search vectors for use in Rosenbrock's (1960) and Swann's (1964) optimization method was shown to make considerable savings in time and in storage requirements. It also
deals more satisfactorily with certain cases in which the original method fails.

Pearson\textsuperscript{35} did an extensive numerical comparison among Newton-Raphson Method, Fletcher and Reeves method and the DFP method. His conclusion was that for well-behaved function Fletcher and Reeves method is simple and fast while for the penalty function methods, the variable metric algorithms are much better and operate more efficiently with reset. The generalized Newton-Raphson algorithm always required fewer iterations and when it can be used, it proves to be the quickest method.

Based on Davidon's method, Mielle et al.\textsuperscript{36} proposed a new accelerated gradient for finding the minimum of a function. He included one extra form $\langle \delta \mathbf{x} \rangle$ in the step length calculation that takes into account the change in position vector from the iteration preceding that under consideration. He showed that, as compared to Fletcher and Reeves method, his method takes 25\% to 40\% less computation time and uses 50\% to 60\% less number of iterations.

The DFP method uses the approximate form of inverse of the Hessian $H$ matrix of objective function $f$ using only the gradient of $f$. Greenstadt\textsuperscript{37} showed that by solving certain variational problems, formulas for successive correction to $H$ matrix can be developed that closely resembles Davidon's and satisfies DFP's condition.
Using the Greenstadt’s variational approach, Goldfarb\textsuperscript{38} developed a new rank - two variable metric method. Like DFP method it preserves the positive definiteness of the H - matrix.

Extension of Davidon’s method for minimization problem in Hilbert space was demonstrated by Tokumaru, et al.\textsuperscript{39} by solving optimal control problems.

Chazan and Miranker\textsuperscript{40} described an algorithm which is suitable for execution on a parallel computer. A non-gradient method similar to Powell's method was used and was shown that the algorithm terminates at minimum for quadratics and converges for strictly convex twice continuously differentiable function.

The variable metric algorithm was further simplified by Fletcher\textsuperscript{41} and it was claimed to be superior to Fletcher and Powell's method since it requires less number of gradient and function evaluations. In this method an approximation of H matrix to G\textsuperscript{-1} matrix is kept and is updated in each iteration.

To account for the efficiency of different techniques, Huang and Levy\textsuperscript{42} tested two different quadratically convergent algorithms (viz, DFP, McCormick, Pearson, generalized Fletcher and Powell etc) through several numerical examples. All algorithms behave identically in case of quadratic function if high-precision arithmatic together with high accuracy in
the one-dimension search is employed. They give same sequence of points, same minimum point and require same number of iterations. For the non-quadratic functions, the results show that some of the algorithms behave identically and so any of them can be considered as a representative of the entire class.

A new method for minimizing a sum of squares of non-linear functions was devised by Peckham⁴³. It was claimed to be more efficient than other methods in that fewer function evaluations are required.

In DFP method the objective function $F(x)$ is assumed strictly convex but Powell pointed out in his survey⁴⁴ of recent development of unconstrained minimization that some better algorithms have now been developed. The most useful work is that which explores algorithms that avoid subproblem of minimizing a function of one variable on every iteration (e.g. large computation time, more number of function evaluations, may not have function improvement and may go beyond the constraints in case of constrained optimization). The algorithms that provide the above features are due to independent work of Davidson⁴¹, Piacco and McCormick⁴⁵, Murtagh and Sargent⁷⁰, Wolfe⁷¹, Bard⁷² and Powell⁷³.

In 1970 Hoshino⁴⁶ found that Davis, Swann and Campey minimization process may generate undesirable zig-zag searches. He proposed a simple modified algorithm and tested it on
some standard test functions. The number of linear searches required were found less.

A general convergence theorem for iterative methods for unconstrained minimization problem was provided by Ortega and Rheinboldt\textsuperscript{47}. The key point is the concept of an essentially gradient related sequence which includes the previously studied gradient-related sequences as well as sequences that arise from univariate relaxation methods.

Cohen\textsuperscript{48} discussed the rate of convergence of several conjugate gradient algorithms to minimize non-linear, non-quadratic real valued function and pointed out that in a neighborhood of the minimum that the error, when starting from a point of reinitialization decreases by order 2 after 'n' steps.

Under the assumption of strict convexity, the projection method of conjugate direction for solving unconstrained minimization was presented by McCormick and Ritter\textsuperscript{49}. It was shown that it converges with (n-1) step superlinear rate.

Without making an initial estimate of the Go (the current estimate of the inverse of H matrix), the matrix used in variable metric algorithms, Mament, et al.\textsuperscript{50} presented a method that uses $x_i Z_i x_i^T$ matrix where $Z_i$ is a diagonal matrix and $x_i$ has maximal rank. The rank of $x_i$ increases by one at each iteration. This pseudo-Newton-Raphson algorithm as
called by the authors, was shown to have finite convergence for quadratic functions and asymptotic convergence for a fairly general class of functions.

Unconstrained optimal control problem can be solved using a gradient algorithm in terms of numerical integration formula, the precision of which is controlled adaptively by a test that ensures convergence. This was shown by Klessig and Polak\(^5\). Their empirical results exhibit that their algorithm is considerably faster than its precision counterpart.

The rate of convergence of Zoutendijk's\(^5\) two procedures were studied and hence two modified methods were developed by Pinonneau and Polak\(^5\). It is shown that under convexity assumption their method converge linearly while Zoutendijk's procedure converge sublinearly.

The method of changing one variable at a time is not an efficient method since the searches are made along the coordinate directions in sequence and the search path tends to a closed loop. On this loop the gradient of the objective function is bounded away from zero. According to Powell\(^5\) this field alone is rather unimportant. What is important is the success of the algorithms depend on the properties that are not shared by the method that changes one variable at a time.
The conditions under which Huang's conjugate gradient method generates descent directions were discussed by Spedicato\textsuperscript{55}. Bounds for the condition number of the inverse Hessian matrix were estimated for the case of a symmetric matrix.

Adachi\textsuperscript{56} also found the same thing, i.e., for quadratic functions, search directions are same for all algorithms and they are independent of parameters. They generate unique sequence of minimizing points for the given initial conditions if the objective function is quadratic.

In minimizing interior penalty function, most of the computational time is spent on one-dimensional search. Lasdon et al.\textsuperscript{57} presented a method that performs this search on barrier function which is significantly faster than current techniques. This method exploits the special structure of barrier functions.

Algorithms for changing the step size efficiently was proposed by Krogh\textsuperscript{58} in the year 1973. He compared the good and bad features of approximately 10 different ways for changing the step size. He also provided an efficient algorithm for the difference formulations of a frequently used halving and doubling process.

Sayama and Takamatsu\textsuperscript{59} found that with the increase in dimensions, the disadvantage of DFP method is the computer
storage problem that increases with number of iterations. In this paper this disadvantage was shown to overcome by formulating the direction of one-dimensional search by means of integral kernels to have a new computation scheme. This may be used for large number of dimensions as well as to obtain high precision for problems having ten number of dimensions.

Bertsekas and Mitter\textsuperscript{60} proposed a new algorithm, the E - subgradient method, a large step, double iterative algorithm which converges rapidly under very general assumption for optimization problems with non-differentiable cost-functions. They discussed the application of this algorithm in some non-linear problems and optimum control and showed that E - subgradient method contains as a special case of a mini-max algorithm.

Numerical experiments on Dual Matrix algorithms are done by Huang and Chambliss\textsuperscript{61} for function minimization. The four algorithms were characterised by the simultaneous use of two matrices and by the property that the one-dimensional search for the optimal step size is not needed for convergence. For quadratic function with \( n \) variables it needs at most \( (n+1) \) number of iterations. These algorithms were tested on four non-quadratic test-functions and exhibited satisfactory convergence properties and compare favorably
with the corresponding quadratically convergent algorithms using one-dimensional search procedure to obtain optimal step size. The reverse one out of 4 algorithms was found best. It requires least number of iterations and least sensitive to step size.

Larichev and Gorvits\textsuperscript{62} carried out similar kind comparison test among different search methods viz, steepest descent, accelerated Partan method, conjugate gradient and Davidon's method using several test functions. Davidon's was the best found in terms of minimum function value and number of iterations.

The modified one-at-a-time optimization procedure introduced by Findlay\textsuperscript{63} is based on assuming that a partial optimal value of one variable is a linear function of the other independent variables. The essence of this method is to observe the effects of each variable combined with some interactions of that variable. The number of trials required was found more than Rosenbrock method but less than gradient method in his study.

Baranger and Temam\textsuperscript{64} in 1975 discussed at length about non-convex optimization problems. The main result is that for almost all values of the parameter, the optimization problem possesses at least one solution.

The algorithm for unconstrained optimization that do not use line searches was developed by Davidon\textsuperscript{65}. This
method uses the $JJ^T$ instead of using $H$ matrix and only store and update the Jacobian matrix $J$.

Exact solution of one-dimensional search for solving problems using DFP method is not always necessary to cover the practical situation where only approximate solutions to the line searches can be found. Lenard\textsuperscript{66} discovered a class of methods which have $n$-step quadratic convergence rate when restarted even if the line search is not exact.

An algorithm for unconstrained minimization of a function of $n$ variables that does not require the evaluation of partial derivatives was presented by Mifflin\textsuperscript{67}. It is a second order extension of the method of local variations which makes the algorithm an approximate Newton method. Its convergence is superlinear for a twice continuously differentiable strongly convex function.

Best\textsuperscript{68} recently developed a method that was claimed to have cubic rate of convergence. The procedure involves 'n' step optimization using any appropriate optimization procedure which is followed by a special step and then another 'n' iterations of the underlying algorithm followed by a second special step. This pattern is then repeated. The special step is interpreted as an approximation to Newton step. After a certain number of iteration this step size procedure will always use a step size of one.

With the object of comparing the different techniques
of unconstrained optimization effectively Shanno and Phua took into account the overhead as well as function evaluations. This new method eliminates much of the machine dependency of earlier criteria.
REFERENCES


CHAPTER II
A NEW SEARCH TECHNIQUE

2.1 Introduction

The problems involving optimization of only one dimension are rarely encountered in real world. On the other hand, almost all search oriented multi-dimensional optimization problems, whether constrained or unconstrained need one dimensional search for its solution. In fact a large part of the computation time of solving multi-dimensional problems is taken by the one-dimensional search. So cutting down the computation time of one-dimensional search has the direct bearing on the reduction of computation time of multi-dimensional problems since these types of problems use one-dimensional search more than once.

Of the many techniques currently used for one dimensional search, Fibonacci search is the most powerful technique followed by Golden section because they do not assume any regularity conditions i.e. convexity, continuity, existence of derivative of function etc. Fibonacci method converges faster than any other method. It is apparent from Fig. 2.1 which shows the relationship between the interval of uncertainty and the number of experiments, that for the first few experiments, the rate of convergence is very fast but after that (say, about 9 experiments), as the interval of uncertainty
THIS BOOK CONTAINS NUMEROUS PAGES WITH DIAGRAMS THAT ARE CROOKED COMPARED TO THE REST OF THE INFORMATION ON THE PAGE.

THIS IS AS RECEIVED FROM CUSTOMER.
Figure 2.1 Interval of uncertainty versus number of experiments in Fibonacci Search.
Figure 2.2 One Dimensional search for finding the maximum by this Method.
becomes smaller and smaller, the rate of convergence becomes asymptotic.

A new approach to solve one-dimensional problem is discussed in section 2.2 and how this asymptotic convergence rate can be overcome using the combination of the new method and Fibonacci method, has been discussed in section 2.3. The new method for solving multi-dimensional problems is provided in section 2.4 with two examples and the comparison of this method with the standard methods has been made and presented in section 2.5.

2.2 Method of Bisecting the Envelope of One-Dimensional Function

In case of convex function, peak value can be obtained by bisecting the envelope i.e. the tangent cone of the function and rotating the cone along the curve till the bisector becomes vertical. The point of intersection of the bisector and the abscissa gives the optimum point since the tangent at the point of intersection of the bisector with the curve becomes horizontal. Even when the function is unknown, this method can be used to determine the optimum.

**Method:**

Let \( y = f(x) \) be the convex function as shown in Fig. 2.2 by the curve BGC in an interval bc.

AB and AC are the two tangents at B & C respectively to form the enveloping cone. Thus \( \tan \phi_1 \) and \( \tan \phi_2 \), are known
when the functions are known or they can be calculated numerically by running two experiments one at b and the other at b+Δx and other two experiments one at c and the other at c+Δx.

Now, \( \tan \theta = \frac{f(b)-f(c)}{c-b} \) \( \therefore \theta = \tan^{-1} \left[ \frac{f(b)-f(c)}{c-b} \right] \)

Using \( f(b) \), \( f(c) \) and slopes of AB and AC, the eqns. of AB and AC can be determined and solving them co-ordinate of A can be calculated.

Now \( \alpha_2 = \Phi_2 + \theta \) and \( \alpha_1 = \Phi_1 - \theta \) and since AP bisects \( \angle A \), \( \beta_1 = \beta_2 = 90^\circ - \left( \frac{\alpha_1 + \alpha_2}{2} \right) \)

\( \therefore \) The inclination of AP = \( \gamma = \left[ 180^\circ - (\Phi_1 + \beta_1) \right] \)

The angle to be rotated = \( \Delta \gamma = [90^\circ - \gamma] \)

When slope of AB is less than slope of AC i.e. when \( \alpha_2 \) is greater than \( \alpha_1 \), the optimum lies in the obtuse angle side of AP when \( \angle \gamma \) is acute. On the other hand if \( \alpha_2 \) is less than \( \alpha_1 \) i.e. when slope of AB is greater than slope of AC with the \( \angle \gamma \) being acute, the optimum lies on the acute angle side of the AP.

Case I \( \angle \alpha_2 > \angle \alpha_1 \)

This case is shown in Fig. 2.2. The cone ABC is rotated along the curve maintaining AC always tangent to the curve through an angle \( \Delta \gamma \) to make the bisector AD vertical. In that case, A will be shifted to A'.
Figure 2.3A  One Dimensional search for finding the minimum
by this Method.
The amount of shift from point A is given by

\[ S = \text{Shift} = AC \cos \Phi_1 - AC \cos (\Phi_1 - \Delta \gamma) = AC \left[ \cos \Phi_1 - \cos (\Phi_1 - \Delta \gamma) \right] = \frac{Y_A - f(c)}{\sin \Phi_1} \left[ \cos \Phi_1 - \cos (\Phi_1 - \Delta \gamma) \right] \]

Since \( X_A \) and \( Y_A \), the co-ordinates of A are known

\[ \therefore X_{\text{optimum}} = X_A - S \]

\[ Y_{\text{optimum}} = f(X_{\text{optimum}}) \]

**Case II**

When \( \alpha_2 \) is less than \( \alpha_1 \), and \( \Delta \gamma \) is acute, the optimum lies within the inner triangle. Using the same procedure i.e. knowing the points B and C, the parameters \( \Phi_1, \Phi_2, \alpha_1, \alpha_2, \beta_1, \beta_2 \), co-ordinate of A and the angle \( \Delta \gamma \) are determined.

It is important to note here that, as in the previous case, if the cone is rotated through \( \Delta \gamma \), the optimum will be obtained within the triangle ACD which is not true.

In this case the optimum lies within the \( \Delta \) ADF and the angle of rotation required is \( \Delta \gamma / 2 \) (i.e. the rotation required by the bisector \( AD' \) of the angle DAF of \( \Delta \) ADF till this new bisector becomes vertical).

\[ \text{Shift} = AB \cos (\alpha_1 - \Theta) - AB \left[ \cos (\alpha_1 - \Theta - \frac{\Delta \gamma}{2}) \right] \]

\[ X_{\text{optimum}} = X_A - \text{Shift} \]

**Examples:**

Two problems were solved to illustrate the application
of this methodology for both the cases. Example 1 is a
maximization problem and example 2 is a minimization problem.
The detail calculations etc. are provided in Appendix 1.

The results are summarized below.

<table>
<thead>
<tr>
<th>Function</th>
<th>Optimum Point by the New Method</th>
<th>Optimum Point by Other Method</th>
<th>Difference in Function Evaluation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max: $5x^2 + 4xy + 8y^2 - 16x + 8y - 16 = 0$</td>
<td>$x = 1.5939$</td>
<td>$x = 1.6$</td>
<td>$0.0005$</td>
</tr>
<tr>
<td>Min: $y = e^x - 5x$</td>
<td>$x = 1.5645$</td>
<td>$x = 1.6$</td>
<td>$0.0047$</td>
</tr>
<tr>
<td></td>
<td>$y = -3.0422$</td>
<td>$y = -3.0469$</td>
<td></td>
</tr>
</tbody>
</table>

It may be noted here that this method gives optimum in
one step while the other standard methods requires several
iterations to reach the optimum.

2.3 Generalization of this Method

It has been shown that for convex functions this method
works well but for non-convex unimodal functions having in-
flection points, this method can be effectively used in com-
bination with Fibonacci method.

It is true that near the optimum, the function is convex.
Outside this convex region bracketing the optimum, noise
Figure 2.3B  Combination of Fibonacci Search and this 
Method to reach optimum.

Note:
1) AB is the original interval of uncertainty. Initial 
reduction from AB to CD can be done by Fibonacci Search.

2) To reach at the optimum point, from CD, this Method 
can be used.
in terms of inflection points exists. So the problem can be divided into two parts. In the first part noise can be eliminated using Fibonacci search which provides fast rate of convergence before it becomes asymptotic. In the second part, the new method can be used to calculate the optimum.

In the above discussion the problem is how to ascertain the domain of convex region that brackets the optimum. The researcher from his experience and the knowledge of the experiment can assume a certain percentage (say 10% to 15%) of the interval of uncertainty for this purpose leaving the rest of it 85%-90%) for Fibonacci Search (Ref. Fig. 2.3B).

Thus, by the process of coupling this method with Fibonacci search, we can overcome the asymptotic disadvantage of Fibonacci Search and cut down the total number of experiments. This is essentially an economic advantage.
APPENDIX 1

Example 1:

Maximization problem.

Max. \( 5x^2 + 4xy + 8y^2 - 16x + 8y - 16 = 0 \)

Differentiating \( 10x + 4y + 16y' = 16 + 8y' = 0 \)

or \( \frac{dy}{dx} = \frac{8 - 5x}{8x - 4y} \)

\( \frac{dy}{dx} \bigg|_{4,0} = \frac{8 - 20}{8} = -2 \quad \Rightarrow 45^\circ \)

\( \frac{dy}{dx} \bigg|_{0,1} = \frac{8}{8} = \frac{1}{3} = 33.69^\circ \)

\( \tan \theta = \frac{0 - 1}{4 - 0} = 0.25 = 14.036^\circ \)

\( \therefore d_2 = 33.69^\circ + 14.036^\circ = 47.726^\circ \)

\( \therefore d_1 = 45^\circ - 14.036^\circ = 30.964^\circ \)

\( \beta = \beta_2 = 90^\circ - \frac{47.726}{2} = 50.655^\circ \)

\( \therefore \gamma = 180^\circ - (45^\circ + 50.655^\circ) = 84.345^\circ \)

\( \therefore \) Angle to be rotated = 5.655°

Co-ordinate of A: \[ \frac{x}{x - 4} = -1 \quad \ldots \ldots \text{eqn. of AC} \]

\( \text{[Fig. 2.2]} \)

\( \frac{y - 1}{x} = \frac{2}{3} \quad \ldots \ldots \text{eqn. of AB} \)

\( \begin{align*}
\text{Solving: } & x_A = 1.6 \\
& y_A = 2.2
\end{align*} \)

\( \therefore \) Due to rotation along the curve maintaining the tangency the lateral (L.H. side) shift = \( AC \cos 45^\circ - AC \cos (45^\circ - 5.655^\circ) \)

\[ = AC \left[ \cos 45^\circ - \cos 39.345^\circ \right] \]

\[ = AC \times -0.0662357 \]

\[ = \frac{y_A}{\sin 45^\circ} \times -0.0662357 = -0.20607 \]
\[ X_0 = X_{\text{optimum}} = X_A - 2.0607 = 1.59393 \]

exact \[ X_A = 1.6 \]

**Example 2**

Minimization problem.

Min. \( y = e^x - 5x \)

at \( x = -1 \)
\[ y = e^{-1} - 5 = 5.36788 \]

at \( x = 3 \)
\[ y = e^3 - 15 = 5.08554 \]

\[ \frac{dy}{dx} = e^x - 5 \]

\[ \frac{dy}{dx} \bigg|_{x=1.5} = e^{-1} - 5 = -4.63212 = \tan 77.813^\circ \]

\[ \frac{dy}{dx} \bigg|_{x=3.5} = e^3 - 5 = 15.08554 = \tan 86.207^\circ \]

Ref. Fig. 2.3 A, Slope of \( BC = \frac{5.08554 - 5.36788}{3+1} = 0.070585 \)

\[ \text{Slope of } BC = \tan 4.0375^\circ \]

\[ d_1 = 86.207 + 4.038 = 90.245^\circ \]

\[ d_2 = 77.813 - 1.038 = 73.78^\circ \]

The optimum will be on the concave side of the axis of the cone.

\[ \beta = \frac{180^\circ - (d_1 + d_2)}{2} = 79.024^\circ \]

\[ \phi = 180^\circ - (d_1 + \beta) = 90^\circ - 90.245 - 79.024 = 81.767^\circ \]

\[ \text{Slope of } AD = \phi + 1.038 = 85.805 \]

\[ \tan 85.805^\circ = 13.6337 \]

Equation of \( AB \):

\[ y = 5.08554 \times 15.08554 - x - 3 \]

Equation of \( AC \):

\[ y = 5.36788 - 4.63212 \times 1 \]

Solving \( X_A = 2.0746 \) and \( y_A = -8.3746 \)
When AB is rotated and translated through \( \frac{\pi}{2} \), 'A' moves away from AF and 'G' moves toward AF.

\[ \text{Horizontal shift} = AB \cos 86.207^\circ - AB \cos (86.207^\circ - 2.0975^\circ) \]

Now \( AB = \sqrt{(3-2.0746)^2 + (5.0855 - 5.8746)^2} = 13.985 \)

Hence Horizontal shift required = 13.985 [0.066152 - 0.10262] = 0.51

\[ x_{opt} = 2.0746 - 0.51 = 1.5645 \]

and \( y_{opt} = -3.0422 \)
REFERENCES


CHAPTER III

LITERATURE SURVEY ON WAGE INCENTIVE PLANS

3.1 Introduction

Although there is no dearth of literature available on wage incentive plans, I have not found any literature that deals with the application of operations research on the formulation and solution of decision problems with regard to wage incentive. The probable reason may be that the decisions like the base level efficiency or the incentive rate etc. are, in most of the cases, settled between the management and the union accross the table. In section 3.2 the standard techniques of wage incentives are discussed briefly. Section 3.3 provides the general literature survey on different types of incentive plans.

Since each plan has to be tailored to suit a particular condition of each organization and it should be such as to satisfy other objectives of the organization like the employment condition, wage structure and quality of the product. There is a scope for application of standard optimization techniques for the optimum choice of the incentive plan.

3.2 Standard Techniques of Wage Incentive - Payment by Results.

Usually, payment by results, are classified in four
main groups in accordance with whether worker's earnings vary 1) in the same proportion as output
2) proportionally less than output
3) proportionally more than output
4) in proportions which differ at different levels of output.

The most common system of payment is the straight piece-work system that comes under the category 1. It may be applied to individuals or to group of workers, the worker is paid at a specified rate per unit of output. Direct labor cost per unit of output remains constant when output increases above standard but the total unit costs decrease because fixed and semi-variable overhead unit costs decrease. Variations in workers earnings and direct labor costs are shown in Fig. 3.1.

When it is difficult to set the job standards accurately, the worker usually shares with his employer the gains or losses that result due to change of output. All schemes under category 2 have this characteristic, i.e., they all possess less motivating reward than straight piece work system. Under the Halsey System, the worker is guaranteed a minimum wage even when his output falls below standard. But if the job is completed in less than standard time, the worker is paid at his time rate for the actual time taken and, in addition, receives a bonus payment at his time rate
Figure 3.1  Straight piece-work system.
Figure 3.2  Halsey System.
A  Earnings (A₁ for low task and A₂ for standard task)
B  Direct labor costs per unit of output (B₁ for low
task and B₂ for standard task)
C  Earnings on straight piece-work system with
guaranteed time rate.

Figure 3.3  The Rowan System.
Figure 3.4 The Birth Variable Sharing System.
Output as % of standard.

A  Earnings as a % of standard.
B  Direct labor costs per unit of output.
C  Earnings on a straight piece-work system
    with a guaranteed time rate.

Figure 3.5  The Bedaux System.
Output as % of standard.

A  Earnings as % of time rate.
B  Direct labor costs per unit of output.
C  Earnings on a straight piece-work system with a guaranteed time rate.

Figure 5.6  High piece rate System.
for a specified percentage of time saved (usually varies from 30% to 70%). The variations in worker's earning and direct labor costs are shown in Fig. 3.2. In the Rowan System bonus is similarly paid for any time saved. The bonus takes the form of a percentage of the worker's time rate. This percentage is equal to the proportion which the time saved forms of standard time. The characteristics of the earnings and direct labor cost curves for the low task and standard task under this system is shown in Fig. 3.3. The Birth variable sharing system is similar to the Halsey and Rowan Systems but does not provide for a guaranteed time rate. The worker's pay is ascertained by multiplying the standard hour by the number of hours actually taken to do the job, taking the square root of the product and multiplying by the worker's hourly rate. The characteristics of the earnings and direct labor cost curves for low task and for standard task are shown in Fig. 3.4. Under the Bedaux system, each minute of allowed time is called a point, thus making in all 480 points in an 8-hour day. A standard number of points is specified for the completion of each job. The worker receives, in addition to his hourly or daily rate, a bonus which is, under the original Bedaux system, equal to 75% of the number of points earned in excess of 60 per hour multiplied by one sixtieth of the worker's hourly rate. Fig. 3.5 shows the variations in earnings and direct labor
costs under this system.

In category 3, the high piece-rate system provides the worker's earnings in proportion to output as under straight piece-work but the increment in earnings for each increase in output is greater. The characteristics of this system are shown in Figure 3.6.

A great many varieties of systems under category 4 have been developed. The most important ones are a) the Taylor Differential Piece-Rate System, b) the Merrick Differential Piece-Rate System, c) the Gantt Task System, and d) the Emerson Empiric or Efficiency System.

In all these systems earnings vary from minimum to maximum at different levels of output. Earnings for part of the range may vary proportionally less than output and for another part proportionally more, or more usually in the same proportion as output.

3.3 Review on Different Types of Plans

Increased labor productivity is the fundamental requirement for an increased material standard of living. Holt showed a simple mathematical model that there exists a definite relationship between overall efficiency and labor productivity. Other input factors held constant, efficiency rises with the increase in labor productivity. By this
model, it is also possible to calculate the amount of investment to be made for the replacement of equipment when the rise in labor productivity is known.

The basis for the incentive scheme for the restricted work (i.e. restricted by the process or the machine performance) should be quite different from that for the unrestricted work. Schieb\(^3\) pointed out that variation in the performance time is precluded by the nature of the operation. He suggested five approaches that should be followed by the Industrial Engineer for the design of incentive schemes in such situations.

Seidel\(^4\) demonstrated a simple technique how much the increase in labor wage incentive can be paid in the next year if the sales, labor force requirement and other cost data are known for the current year and next year. Using this method decisions relating to the incentive rate or increase in labor wages a replacement of equipment can be taken very easily and effectively.

Like Seheib, the disadvantages associated with the straight standard hour system as a basis of incentive plan were also shown by Halty\(^5\) who developed a new system that gives us a mathematical equation to calculate the earning index, taking into account a variable machine incentive allowance.

O'Connor\(^6\) stresses on the unique position of standard time as the most important part of the incentive plan. He
explained the merits and demerits of straight piece work and geared linear plans for incentive plan. When there exists some doubt about the accuracy of the time standard, his recommendation was to adopt his curvilinear type of incentive plan.

Usually labor productivity varies with respect to time in a particular organization. Nassi\textsuperscript{7} showed how these indices with respect to time which are known as 'index of Laspeyres' and 'index of Paasche' can be calculated. He has also shown how to measure the performance index of Method study and standard department in terms of work saved per unit of time.

Incentives also can be applied for quality improvement. This was shown by Mehra, et al.,\textsuperscript{8} by linking the scheme with the acceptance sampling incentive plan. The wage, inclusive of incentive would be computed using game theory approach.

With this same objective, Nandi and Nair\textsuperscript{9} presented a quality incentive plan for an operator which was designed based on cost equations of the sampling plan and management policy without increasing the total cost per lot.

Success of the incentive scheme depends on the consistency of time data among some other factors. Groff\textsuperscript{10} pointed out that the standard output rate obtained by time study is not always optimal since best output rate for standard is simply influenced by the incentive plan for which the data is intended. He presented an incentive plan considering the
effect of selected output response patterns and cost structures on optimal standard level.

Expensive downtime, at times poses a problem to the management, particularly, in line paced operation. James\textsuperscript{11} showed how to alleviate this problem by introducing incentive in the system.
REFERENCES


CHAPTER IV

APPLICATION OF OPERATIONS RESEARCH TECHNIQUES TO FORMULATE AND SOLVE AN INCENTIVE PROBLEM -- A CASE STUDY

4.1 Introduction

This chapter is primarily concerned with the application of some optimization techniques to solve some decision problems regarding wage incentive scheme. This is essentially a case study. In this section the management's problem and policy have been discussed. In section 4.2 and 4.3, the formulation and solution of the problems are provided. In this case study, a situation in a light engineering concern has been considered wherein the management is currently scheduling overtime hours to meet its production schedule. It wants to put a stop to giving overtime and get the same or more production without overtime through the installation of an incentive plan that will eventually improve operator's efficiency, increase machine utilization accompanied by less power consumption.

Management does not want the worker's weekly paycheck to be affected. By having the same output during normal working hours, it hopes to reduce the overhead expenses associated with having the firm work longer hours.

In this particular case, it is proposed that for twelve departments and for two groups of workers in each department, namely skilled and unskilled worker, group incentive plan is suitable.
4.2 Problem Formulation

The following nomenclatures were used for the formulation of the problem:

\[ N_k = \text{Total number of workers in Group } k \text{ in the department } i. \]

\[ J_i = \text{Total number of operations done in department } i. \]

\[ x_l = \text{Proposed base level efficiency in } \% \text{ from which incentive should start.} \]

\[ C_{ki} = \text{Current efficiency in } \% \text{ of Group } k \text{ in the department } i. \]

\[ U_{kij} = \text{Total number of units produced by Group } k \text{ for } j\text{th operation in the department after the implementation of incentive scheme.} \]

\[ H_{ki} = \text{Input labor hours by } k\text{th group of workers in the department } i. \]

\[ t_{kij} = \text{Standard time in hours per unit for } j\text{th operation done in the department } i \text{ by } k\text{th group of worker.} \]

\[ x_{k+1} = \text{Incentive rate per point rise in efficiency per hour (i.e. } \$/\%\text{/hr.) for } k\text{th group of workers.} \]

\[ d_k = \text{Average overtime in } \$ \text{ paid per hour to Group } k. \]

It is also desired that \( x_l \), the proposed base level
efficiency should be same for all groups of workers.

So the desired objective function is to:

Minimize \[ \sum_{k} \sum_{i} N_{ki} (x_{i} - C_{ki})^2 \]

Constraints:

1. Total incentive to be paid must not exceed the total overtime payment, i.e.

\[ \sum_{j} \left[ \frac{U_{kij} t_{kij}}{H_{ki}} - x_1 \right] x_{k+1} \leq d_{k} \quad \text{for } k = 1, 2 \]

Essentially \( U_{kij} t_{kij}/H_{ki} \) gives the new efficiencies of two groups (\( k = 1 \) and 2). If they are defined as \( C_{1i} \) and \( C_{2i} \) then \((C_{1i} - x_1)\) and \((C_{2i} - x_1)\) are the total rise in efficiencies by two groups of workers for \( i \)th department after the implementation of the scheme.

2. Again to have good motivation, the incentive rate should not be less than the 'per hour wage' evaluated on per point basis at the optimum base level efficiency, i.e.

\[ x_{k+1} \geq \frac{w_{k}}{x_1} \]

where \( w_{k} \) is the average wage rate for \( k \)th group of workers.

This is quite clear from the relationship (line BC) shown in Figure 4.1.

The number of workmen for the two groups and for twelve departments are shown in Appendix 3. After time study, the current performance index (P.I.) i.e. \( C_{ik_1} \) and \( C_{ik_2} \) of the
Fig. 4.1 Relationship between the earnings and performance index under the proposed incentive scheme.

Note: Guaranteed minimum wage is $W$; even when the output falls below the base level efficiency $A$. The slope of $BC$ (i.e., the incentive rate) is greater than the slope of $OB$. This provides greater motivation. Constraint 2 is essentially derived from this condition.
two groups for each department are evaluated and provided in the same Appendix 3. It also has been found that the management gets the same output if the two groups of workers work at 105% and 100% P.I. during normal working hours. The wage rate for skilled and unskilled groups of workers are assumed as $4 and $3 per hour respectively. From the past records in the account section, the overtime earning per hour per worker for the two groups are found to be $2 and $1.50 respectively.

So using those data the problem is rewritten as:

\[ z = 10(x_1-78)^2 + 20(x_1-68)^2 + 18(x_1-82)^2 + 28(x_1-85)^2 + 5(x_1-83)^2 + 5(x_1-69)^2 + 40(x_1-95)^2 + 5(x_1-75)^2 + 80(x_1-80)^2 + 10(x_1-96)^2 + 20(x_1-91)^2 + 10(x_1-65)^2 + 30(x_1-71)^2 + 10(x_1-66)^2 + 40(x_1-76)^2 + 32(x_1-75)^2 + 10(x_1-78)^2 + 10(x_1-65)^2 + 60(x_1-89)^2 + 10(x_1-69)^2 + 60(x_1-72)^2 + 5(x_1-90)^2 + 10(x_1-85)^2 + 5(x_1-52)^2 \ldots \text{ eqn. (4.1)} \]

S.T. \((105-x_1)x_2 \leq 2 \ldots \text{ eqn. (4.2)}\)

\((100-x_1)x_3 \leq 1.5 \ldots \text{ eqn. (4.3)}\)

\[ x_2 \geq \frac{4}{x_1} \ldots \text{ eqn. (4.4)} \]

\[ x_3 \geq \frac{3}{x_1} \ldots \text{ eqn. (4.5)} \]

\[ x_1 = c_p = \text{Optimum base level efficiency in %} \]

\[ x_2 = \text{Incentive rate per point rise per hour for skilled group (3%/hr.)} \]

\[ x_3 = \text{Incentive rate per point rise per hour for unskilled group. (3%/hr.)} \]
4.3 Generalized Reduced Gradient Formulation

To solve the above problem by the GRG method, the objective functions and the constraints may be represented by:
Maximize \[ f_o(\bar{x}) \]
Subject to the constraints
\[ f(\bar{x}) = 0 \]
\[ a \leq \bar{x} \leq b \]

Any inequality constraints can be converted into equality constraints using the standard procedure of adding slack variables and changing the sign, if necessary.

The basic underlying principle of this technique is to change the constrained optimization problem into an unconstrained one. This is done by dividing the solution vector components into two groups, independent (\( \bar{x} \)) and dependent (\( \bar{y} \)). The dependent variables denoted by the vector \( \bar{y} \) are solved in terms of independent vector \( \bar{x} \), through the constrain functions.

Therefore on this basis the constraints may be rewritten as:
\[ \bar{f}(\bar{x}) = \bar{F}(\bar{x}, \bar{y}) = 0 \]
Solving \( \bar{y} = \Phi(\bar{x}) \)

The objective function also is rewritten in terms of \( \bar{x} \) and \( \bar{y} \) and substituting the value of \( \bar{y} \) in that one gets
\[ f_o(\bar{x}) = f_o(\bar{x}, \bar{y}) = f_o(\bar{x}, \Phi(\bar{x})) = F(\bar{x}) \]
Hence the problem is to maximize

\[ F(\bar{x}) \]

Subject to \( \bar{a} \leq \bar{x} \leq \bar{b} \)

Since \( F(\bar{x}) = f_0(x, y) \)

\[ \therefore \text{The reduced gradient can be evaluated as:} \]

\[ \frac{\partial F}{\partial \bar{x}} = \frac{\partial f_0}{\partial \bar{x}} + \frac{\partial f_0}{\partial \bar{y}} \cdot \frac{\partial \bar{y}}{\partial \bar{x}} \]

\[ \frac{\partial \bar{y}}{\partial \bar{x}} \]

is determined indirectly from the constraints.

\[ \therefore f(\bar{x}) = f(\bar{x}, y) = 0 \]

\[ \therefore \frac{\partial f}{\partial \bar{x}} + \frac{\partial f}{\partial \bar{y}} \cdot \frac{\partial \bar{y}}{\partial \bar{x}} = 0 \]

or \( \frac{\partial \bar{y}}{\partial \bar{x}} = -\left[ \frac{\partial f}{\partial \bar{y}} \right]^{-1} \left[ \frac{\partial f}{\partial \bar{x}} \right] \)

\[ \therefore \bar{g} = \frac{\partial F}{\partial \bar{x}} = \frac{\partial f_0}{\partial \bar{x}} - \frac{\partial f_0}{\partial \bar{y}} \left[ \frac{\partial f}{\partial \bar{y}} \right]^{-1} \left[ \frac{\partial f}{\partial \bar{x}} \right] \]

The conditions that determine an optimum solution, \( \bar{x}^* \)

are as given below (for all \( j \))

\[ \frac{\partial F}{\partial x_j^*} = 0 \text{ if } a_j < x_j^* < b_j \]

\[ \frac{\partial F}{\partial x_j^*} \leq 0 \text{ if } x_j^* = a_j \]

\[ \frac{\partial F}{\partial x_j^*} \geq 0 \text{ if } x_j^* = b_j \]
Slope: -ve satisfies condition $\frac{\partial F}{\partial x_j^*} = 0$ if $x_j^* = a_j$

Slope = 0 satisfies condition $\frac{\partial F}{\partial x_j^*} = 0$ if $a_j < x_j^* < b_j$

Slope: +ve satisfies condition $\frac{\partial F}{\partial x_j^*} = 0$ if $x_j^* = b_j$

Fig. 4.2 Graphical representation of the optimum conditions used in GRG technique.
These conditions are graphically represented in Fig. 4.2.

The underlying assumptions for this algorithm are that for a given iteration

1) There exists a set of dependent variables contained within the boundary conditions

2) The Jacobian $\frac{\partial f}{\partial y}$ is non-singular.

Using the above information, the basic GRG algorithms can be stated in five steps which are provided in the flow chart (Appendix 4).

Theoretically, the stopping condition is when the projected reduced gradient $P_i^o = 0$, $i = j$, ..., $N-M$, where $N$ is the number of variables in the original objective function and $M$ is the number of constraints, $N-M$ being the reduced dimension.

In practice, the following three stopping criteria are employed.

1) $\| \hat{p}^o \| = \sqrt{\sum_{i=1}^{N-M} (P_i^o)^2} < \epsilon_1$

2) $P_i^o < \epsilon_2$, $i = 1, 2, \ldots, (N-M)$

3) $|f_0(\bar{x}^i) - f_0(\bar{x})^o| < \epsilon_3$
TABLE 4.1
RESULTS OBTAINED BY CGR METHOD USING THREE DIFFERENT STARTING POINTS

<table>
<thead>
<tr>
<th>Run No.</th>
<th>Starting Values</th>
<th>Solution</th>
<th>Function Value</th>
<th>Norm of Reduced Gradient</th>
<th>$\Delta F_1$</th>
<th>ETA</th>
<th>No. of Iterns.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$x_1=85%$</td>
<td></td>
<td>Joy Abandoned</td>
<td></td>
<td></td>
<td></td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>$x_2=.03$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x_3=.03$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$x_1=60%$</td>
<td>$x_1=79.27$</td>
<td></td>
<td></td>
<td>$0.4276 \times 10^5$</td>
<td>1.5</td>
<td>10 44 23</td>
</tr>
<tr>
<td></td>
<td>$x_2=.05$</td>
<td>$x_2=.078$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x_3=.05$</td>
<td>$x_3=.05$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$x_1=75%$</td>
<td>$x_1=79.27$</td>
<td></td>
<td></td>
<td>$0.4276 \times 10^5$</td>
<td>0.0</td>
<td>0.0 0.0 7</td>
</tr>
<tr>
<td></td>
<td>$x_2=.04$</td>
<td>$x_2=.051$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$x_3=.04$</td>
<td>$x_3=.04$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: 1) In Run # 2, the termination occurred since the same function values are obtained in the last two iterations before it meets the other stopping criteria.

2) $\Delta F_1 = \sum |P_i(a_i - x_i)|$ when $P_i < 0$  
   $= \sum |P_i(b_i - x_i)|$ when $P_i > 0$

Where $P_i$ is the gradient of the function with

ETA = Max. $|P_i(a_i - x_i)|$ for $P_i < 0$
   or Max. $|P_i(b_i - x_i)|$ for $P_i > 0$

respect to the variable.
4.4 Solution of the Incentive Problem by G.R.G. Method

The incentive problem as formulated in section 4.2 was solved by GRG method using the GREG program which was developed by Abadie and his associates of Electricite de France. The program was run thrice using three different starting values (Table 4.1). The number of iterations required, the optimum value of the variables and the value of the objective function etc. are given in Table 4.1. Appendices 5-7 are the computer printout for the three runs which provide the other informations like the stopping criteris etc.

The variable \( x_3 \), i.e. the incentive rate for unskilled group, assumes the same optimal value as the starting value in both the feasible runs although the function values are same. Hence it may be concluded that the objective function is very flat near the optimum.

4.5 Separable Programming

Separable programming is a special case of non-linear programming. When the objective function and the constraints are constructed or can be constructed of separable functions, this method can be used effectively. The basic principle is to approximate the non-linear function to piecewise linear functions and thereby changing the problem into a restricted linear programming problem.
Thus the incentive problem given by eqns 4.1 to 4.5 can be changed to separable programming problem which may be defined as:

\[ C(\bar{x}) = \sum_{i=1}^{m} f_i(x_i) \]

Subject to constraints:

\[ \sum_{i=1}^{m} g_{ki}(x_i) \leq b_k \quad k = 1, 2, \ldots, p \]
\[ x_i \geq 0 \quad i = 1, 2, \ldots, m \]

Partitioning each variable \( x_i \) into \( n_i \) divisions and approximating the functions \( f_i(x_i) \) and \( g_{ki}(x_i) \) one can write as:

\[ x_i = x_i^o + \sum_{j=1}^{n_i} \Delta x_i^j D_i^j \]
\[ f_i(x_i) = f_i(x_i^o) + \sum_{j=1}^{n_i} f_i^j D_i^j \]
\[ g_{ki}(x_i) = g_{ki}(x_i^o) + \sum_{j=1}^{n_i} g_{ki}^j D_i^j \]

\[ k = 1, 2, \ldots, p, \text{ and } i = 1, 2, \ldots, m \]
\( x_i^o \) = lower boundary of variable \( x_i, i = 1, 2, \ldots, m \)
\( x_i^o \) may or may not be equal to zero

\( f_i(x_i) \) and \( g_{ki}(x_i) \) are the corresponding values of the objective function and
constraints at \( x_i^0 \). These values may or may not be equal to zero.

\( D_i^j \) represents a variable created for the \( j \)th partition of variable \( x_i \).

Thus the original problem can be written as:

\[
\text{Optimize (max. or min.)} \\
C = \sum_{i=1}^{m} \sum_{j=1}^{n_i} f_i^j D_i^j + \sum_{i=1}^{m} f_i(x_i^0) \\
\text{Subject to:} \sum_{i=1}^{m} \sum_{j=1}^{n_i} g_{ki}^j D_i^j b_k - \sum_{L=1}^{m} g_{ki}^j(x_i^0), \quad k = 1, 2, \ldots, p \\
\text{Grid equation:} \quad x_i - \sum_{j=1}^{n_i} \Delta x_i^j D_i^j = x_i^0, \quad L = 1, 2, \ldots, m \\
0 \leq D_i^j \leq 1, \quad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n_i \\
x_i \geq 0, \quad i = 1, 2, \ldots, m
\]

4.6 Solution of the Incentive Problem by Separable Method

The incentive problem as formulated in section 4.2 and defined by the eqns.4.1 to 4.5 can be separated as follows:

The objective function on expansion yields

\[ \text{Min} = 543 \, x_1^2 - 86082 \, x_1 + 3454417 \ldots \text{ eqn 4.6} \]
The constraints are separated according to the principle of separable programming by taking logarithms on both sides.

Thus the constraints are:
\[
\begin{align*}
\log (105 - x_1) + \log x_2 & \leq \log 2 \quad \text{eqn. 4.7} \\
\log (100 - x_1) + \log x_3 & \leq \log 1.5 \quad \text{eqn. 4.8} \\
\log x_2 + \log x_1 & \geq \log 4 \quad \text{eqn. 4.9} \\
\log x_3 + \log x_1 & \geq \log 3 \quad \text{eqn. 4.10}
\end{align*}
\]

The starting value of the variables \(x_1, x_2\) and \(x_3\) are 75%, $.06, and $.04 respectively. The upper bounds are 90%, $.12 and $.09 and the number of partitions required for linearisation are 20, 10 and 10 respectively.

The linearized form of the non-linear components of the objective function and of constraints are furnished in Appendix 8.

Since \(f(x_i)\) and \(g_{ki}(x_i)\) where \(x_i\) is the starting value of the \(i\)th variable and \(f\) and \(g\) stand for objective function and constraint respectively, are not zero so the right hand side of constraints and also the \(d_{ij}\) function are to be adjusted.

Thus the original incentive problem is represented as:

Maximize:
\[
F = (-Z) = \sum_{j=1}^{20} \Delta f(D_j^3 - 543(x_i) + 86082 x_1 - 3454417
\]
S.T.
\[\sum_{j=1}^{10} \Delta g_1 D_1^j + \sum_{j=1}^{10} \Delta g_2 D_2^j \leq \log 2 - g_1(x_1^0, x_2^0)\]
\[\sum_{j=1}^{20} \Delta g_2 D_1^j + \sum_{j=1}^{10} \Delta g_2 D_3^j \leq \log 1.5 - g_2(x_1^0, x_2^0)\]
\[\sum_{j=1}^{10} \Delta g_3 D_2^j + \sum_{j=1}^{20} \Delta g_3 D_4^j \geq \log 4 - g_3(x_1^0, x_2^0)\]
\[\sum_{j=1}^{10} \Delta g_4 D_3^j + \sum_{j=1}^{20} \Delta g_4 D_1^j \geq \log 3 - g_4(x_1^0, x_3^0)\]

Grid equations:
\[x_1 - \sum_{j=1}^{20} \Delta x_1 D_1^j = 75\]
\[x_2 - \sum_{j=1}^{10} \Delta x_2 D_2^j = 0.06\]
\[x_3 - \sum_{j=1}^{10} \Delta x_3 D_3^j = 0.04\]
\[x_1, x_2, x_3 \geq 0, \quad 0 < D_i^j \leq 1 \quad i = 1, 2, 3 \quad \& \quad j = 1, 2, 3 \ldots ... 20\]

The linear equations, the grid equations are given in details in Appendix 8. The problem then eventually was solved by linear programming using MPS/360 program, the results are given at the end of Appendix 8.

The results are summarized in Table 4.2 and the value of the objective function also was given in the same table after manipulating the constant terms using eqn. 4.6.
TABLE 4.2
Results Obtained Using Separable Programming

<table>
<thead>
<tr>
<th>Variable</th>
<th>Lower bound</th>
<th>Upper bound</th>
<th>Starting Value</th>
<th>Number of partitions</th>
<th>Number of iterations for linear programming solution</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>75%</td>
<td>90%</td>
<td>75%</td>
<td>20</td>
<td></td>
<td>79.5</td>
</tr>
<tr>
<td>$x_2$</td>
<td>.06</td>
<td>.12</td>
<td>.06</td>
<td>10</td>
<td>9</td>
<td>.06</td>
</tr>
<tr>
<td>$x_3$</td>
<td>.04</td>
<td>.09</td>
<td>.04</td>
<td>10</td>
<td></td>
<td>.04</td>
</tr>
</tbody>
</table>

value of obj. function = $- [6466000 - 3454417 - 543(75)^2]$

$= - [-42792] = .42792 \times 10^5$
4.7 Conclusion

The GRG method appears to be a very powerful tool for handling optimization of non-linear objective function subjected to non-linear constraints. As can be seen from the results (Table 4.1) the convergence rate is quite fast and only a very small amount of computer time and computer memory are needed to solve the problem.

Separable programming is also a powerful non-linear programming technique since it will yield, as with any other non-linear technique, at least a local optimum solution, if it exists. Its only pitfall is precision, but this is of little consequence since most engineering problems need only a good approximation.

As contrast to the GRG technique, separable programming requires more manipulation since a large number of new variables have to be introduced and it can solve only certain non-linear programming problems.

As far as our given incentive problem is concerned, separable programming and GRG yield nearly the (Table 4.3) same objective function value and the base efficiency but the incentive rates obtained by separable programming are higher than those by GRG method. Since higher motivation will be generated by higher values of incentive rates, so results of separable programming may be recommended with a insignificant change in the value of objective function.
Table 4.3  Comparison of the results obtained by G.R.G. method and Separable Programming

<table>
<thead>
<tr>
<th>Method</th>
<th>Starting values</th>
<th>No of iterations</th>
<th>Optimum solns.</th>
<th>Optimum func. values</th>
</tr>
</thead>
<tbody>
<tr>
<td>G. R.G.</td>
<td>$x_1 = 75%$</td>
<td>7</td>
<td>$x_1 = 79.27%$</td>
<td>$x_2 = 0.05($)$</td>
</tr>
<tr>
<td></td>
<td>$x_2 = 0.04($)$</td>
<td></td>
<td></td>
<td>$4276 \times 10^5$</td>
</tr>
<tr>
<td></td>
<td>$x_3 = 0.04($)$</td>
<td></td>
<td></td>
<td>$x_3 = 0.04($)$</td>
</tr>
<tr>
<td>Separable programming</td>
<td>$x_1 = 75%$</td>
<td>9</td>
<td>$x_1 = 79.5%$</td>
<td>$x_2 = 0.078($)$</td>
</tr>
<tr>
<td></td>
<td>$x_2 = 0.06($)$</td>
<td></td>
<td></td>
<td>$4279 \times 10^5$</td>
</tr>
<tr>
<td></td>
<td>$x_3 = 0.04($)$</td>
<td></td>
<td></td>
<td>$x_3 = 0.074($)$</td>
</tr>
</tbody>
</table>
APPENDIX 2

CURRENT MANPOWER AND PERFORMANCE INDEX OF TWO GROUPS OF WORKERS FOR VARIOUS DEPARTMENTS

<table>
<thead>
<tr>
<th>Dept.</th>
<th>No. of men</th>
<th>Efficiency (%)</th>
<th>No. of men</th>
<th>Efficiency (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Power Press</td>
<td>10</td>
<td>78</td>
<td>30</td>
<td>71</td>
</tr>
<tr>
<td>Auto Screw Cutting</td>
<td>20</td>
<td>68</td>
<td>20</td>
<td>66</td>
</tr>
<tr>
<td>Drilling</td>
<td>18</td>
<td>82</td>
<td>40</td>
<td>76</td>
</tr>
<tr>
<td>Milling</td>
<td>28</td>
<td>85</td>
<td>32</td>
<td>75</td>
</tr>
<tr>
<td>Plating</td>
<td>5</td>
<td>83</td>
<td>10</td>
<td>78</td>
</tr>
<tr>
<td>Painting</td>
<td>5</td>
<td>69</td>
<td>10</td>
<td>65</td>
</tr>
<tr>
<td>Sub. Assem.</td>
<td>40</td>
<td>95</td>
<td>60</td>
<td>89</td>
</tr>
<tr>
<td>Spring Mfg.</td>
<td>5</td>
<td>75</td>
<td>10</td>
<td>69</td>
</tr>
<tr>
<td>Assembly</td>
<td>80</td>
<td>80</td>
<td>60</td>
<td>72</td>
</tr>
<tr>
<td>Salvage</td>
<td>10</td>
<td>96</td>
<td>5</td>
<td>90</td>
</tr>
<tr>
<td>Grinding</td>
<td>20</td>
<td>91</td>
<td>10</td>
<td>85</td>
</tr>
<tr>
<td>Heat Treatment</td>
<td>10</td>
<td>65</td>
<td>5</td>
<td>62</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>251</strong></td>
<td></td>
<td><strong>292</strong></td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX 3

COMPUTER FLOW CHART FOR G.R.G METHOD
(STEP 1) SELECT INITIAL STARTING POINT, $x^0$

IS $x^0$ FEASIBLE?

YES

(STEP 1.1) COMPUTE THE REDUCED GRADIENT

$$g^0 - \frac{\partial f_0}{\partial x} - \frac{\partial f_0}{\partial y} \left( \frac{\partial F}{\partial y} \right)^{-1} \frac{\partial F}{\partial x}$$

NO

SELECT A FEASIBLE $x^0$

4

(STEP 1.2) DETERMINE THE PROJECTED REDUCED GRADIENT.

$$\forall_i, p^0_i = \begin{cases} 0 & \text{IF } x^0_i = \text{LOWER BOUND AND } g^0_i \leq 0 \\ 0 & \text{IF } x^0_i = \text{UPPER BOUND AND } g^0_i \geq 0 \\ g^0_i & \text{OTHERWISE} \end{cases}$$

STOP

CHECK THE STOPPING CONDITION.

$$\forall_i, p^0_i = 0$$

Computer flow diagram for GRG algorithm
(STEP 1.3) COMPUTE THE DIRECTION OF MOVEMENT, \( \overrightarrow{h} \), FOR

\[ \overrightarrow{x} \]

A SIMPLE EXAMPLE IS \( \overrightarrow{h} = \overrightarrow{p} \)

(STEP 2) COMPUTE THE DIRECTION OF MOVEMENT, \( \overrightarrow{k} \), FOR

\[ \overrightarrow{y} \]

(STEP 2.1)

\[ \overrightarrow{k} = - \left[ \frac{\partial \overrightarrow{F}}{\partial \overrightarrow{y}} \right]^{-1} \left[ \frac{\partial \overrightarrow{F}}{\partial \overrightarrow{x}} \right] \overrightarrow{h} \]

(STEP 2.2) USE A ONE-DIMENSIONAL SEARCH TO \( \max_{\theta} f_{\theta}(\overrightarrow{x} + \theta \overrightarrow{h}, \overrightarrow{y} + \theta \overrightarrow{k}) \)

(STEP 3) CALCULATE \( \overrightarrow{x}' = \overrightarrow{x} + \theta \overrightarrow{h}, \overrightarrow{y}' = \overrightarrow{y} + \theta \overrightarrow{k} \). PROJECT \( \overrightarrow{x}' \) INTO \( P \).

\[ \forall j, x'_j = \begin{cases} \text{UPPER BOUND IF } x'_j + \theta h'_j \geq \text{UPPER BOUND} \\ \text{LOWER BOUND IF } x'_j + \theta h'_j \leq \text{LOWER BOUND} \\ x'_j + h'_j \text{ OTHERWISE} \end{cases} \]

(continued)
(STEP 4.1) SET $\theta = \frac{1}{2} \theta$

(SOLVE $\bar{F}(\bar{x}^1, \bar{y}^1) = 0$)

DOES $\bar{y}^1$ EXIST?

NO \rightarrow (STEP 4.1)

YES \rightarrow (STEP 4.2)

CHECK $f_0(x, y) = f_0(x, y)$

NO \rightarrow (STEP 4.1)

YES \rightarrow (STEP 5)

SET $\bar{x}^0 = \bar{x}^1$

4

(continued)
APPENDIX X: 4

COMPUTER PRINTOUT OF THE RESULTS OBTAINED

BY G.R.G. METHOD
NOMBRE DE VARIABLES NATURELLES 3
NOMBRE TOTAL DE VARIABLES 9
NOMBRE DE CONTRAINTES 4

EPSILON DE NEWTON: 0.10000E-02
EPSILON TEST GRADIENT: 0.10000E-02
EPSILON ECHELLE: 0.606220E-05

<table>
<thead>
<tr>
<th></th>
<th>BORNE INFERIEURE</th>
<th>VARIABLE NATURELLE</th>
<th>BORNE SUPERIEURE</th>
</tr>
</thead>
<tbody>
<tr>
<td>XII</td>
<td>0.5699999743300E+02</td>
<td>0.8500000000000E+02</td>
<td>0.9900000000000E+02</td>
</tr>
<tr>
<td>XII</td>
<td>0.9999999743300E-02</td>
<td>0.2599999746830E-01</td>
<td>0.2000000000000E+01</td>
</tr>
<tr>
<td>XII</td>
<td>0.5699999743300E-02</td>
<td>0.2599999746830E-01</td>
<td>0.2000000000000E+01</td>
</tr>
</tbody>
</table>

VALEUR DES CONTRAINTES

<table>
<thead>
<tr>
<th></th>
<th>V1(1)</th>
<th>V1(2)</th>
<th>V1(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>-0.135999961E5333E+01</td>
<td>-0.1349992370635E+01</td>
<td>-0.1349992370635E+01</td>
</tr>
<tr>
<td>C1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX: 5

Computer printout of the results obtained
by G.R.G. Method
### Gradient réduit généralisé

<table>
<thead>
<tr>
<th>Nombre de variables naturelles</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nombre total de variables</td>
<td>10</td>
</tr>
<tr>
<td>Nombre de contraintes</td>
<td>4</td>
</tr>
</tbody>
</table>

| Epsilon de Newton             | 0.1000E-04 |
| Epsilon test gradient         | 0.1000E-02 |

| Fonction effective            | -0.24429700000000E 06 |

<table>
<thead>
<tr>
<th>Régne inférieur</th>
<th>Variable naturelle</th>
<th>Régne supérieur</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_1(1)$</td>
<td>0.50000000000000E 02</td>
<td>$X_1(1)$</td>
</tr>
<tr>
<td>$X_2(2)$</td>
<td>0.6599997913874E-02</td>
<td>$X_2(2)$</td>
</tr>
<tr>
<td>$X_3(3)$</td>
<td>0.9599997913874E-02</td>
<td>$X_3(3)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Valeur des contraintes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1(1)$</td>
</tr>
<tr>
<td>$C_1(2)$</td>
</tr>
<tr>
<td>$C_1(3)$</td>
</tr>
<tr>
<td>$C_1(4)$</td>
</tr>
</tbody>
</table>
IT 1   PHI-C.202153636333336E C7  ITERATION  SPECIALE  RASE  DEGENEREE
IT 2   PHI-0.1083077333333333E C7  DIR. GRAD.  NO 3  YN 0.18E 07  DELTAI 0.76E 07  ETA 0.12E 17  NCD 3  NCN 20  NITA241
IT 3   PHI-C.10832073333333336E C7  DIR. GRAD.  NO 3  YN 0.15E 07  DELTAI 0.76E 07  ETA 0.10E 17  NCD 0  NCN 3  NITA 12
IT 4   PHI-C.10833073333333336E C7  DIR. GRAD.  NO 3  YN 0.15E 07  DELTAI 0.76E 07  ETA 0.10E 17  NCD 0  NCN 3  NITA 12
IT 5   PHI-C.10834073333333336E C7  DIR. GRAD.  NO 3  YN 0.15E 07  DELTAI 0.76E 07  ETA 0.10E 17  NCD 0  NCN 3  NITA 12
IT 6   PHI-C.10835073333333336E C7  DIR. GRAD.  NO 3  YN 0.15E 07  DELTAI 0.76E 07  ETA 0.10E 17  NCD 0  NCN 3  NITA 12
IT 7   PHI-C.10836073333333336E C7  DIR. GRAD.  NO 3  YN 0.15E 07  DELTAI 0.76E 07  ETA 0.10E 17  NCD 0  NCN 3  NITA 12
IT 8   PHI-C.10837073333333336E C7  DIR. GRAD.  NO 3  YN 0.15E 07  DELTAI 0.76E 07  ETA 0.10E 17  NCD 0  NCN 3  NITA 12
IT 9   PHI-C.10838073333333336E C7  DIR. GRAD.  NO 3  YN 0.15E 07  DELTAI 0.76E 07  ETA 0.10E 17  NCD 0  NCN 3  NITA 12
IT 10  PHI-C.10839073333333336E C7  DIR. GRAD.  NO 3  YN 0.15E 07  DELTAI 0.76E 07  ETA 0.10E 17  NCD 0  NCN 3  NITA 12

SOLUTION OBTENUE...A L ITERATION 10

IT 11  PHI-C.202153636333336E C7  DIR. GRAD.  NO 4  YN 0.16E 07  DELTAI 0.76E 07  ETA 0.11E 17  NCD 3  NCN 14  NITA 52
IT 12  PHI-C.202153636333336E C7  DIR. GRAD.  NO 3  YN 0.17E 07  DELTAI 0.76E 07  ETA 0.11E 17  NCD 3  NCN 14  NITA 52
IT 13  PHI-C.202153636333336E C7  DIR. GRAD.  NO 3  YN 0.17E 07  DELTAI 0.76E 07  ETA 0.11E 17  NCD 3  NCN 14  NITA 52
IT 14  PHI-C.202153636333336E C7  DIR. GRAD.  NO 3  YN 0.17E 07  DELTAI 0.76E 07  ETA 0.11E 17  NCD 3  NCN 14  NITA 52
IT 15  PHI-C.202153636333336E C7  DIR. GRAD.  NO 3  YN 0.17E 07  DELTAI 0.76E 07  ETA 0.11E 17  NCD 3  NCN 14  NITA 52
IT 16  PHI-C.202153636333336E C7  DIR. GRAD.  NO 3  YN 0.17E 07  DELTAI 0.76E 07  ETA 0.11E 17  NCD 3  NCN 14  NITA 52
IT 17  PHI-C.202153636333336E C7  DIR. GRAD.  NO 3  YN 0.17E 07  DELTAI 0.76E 07  ETA 0.11E 17  NCD 3  NCN 14  NITA 52
IT 18  PHI-C.202153636333336E C7  DIR. GRAD.  NO 3  YN 0.17E 07  DELTAI 0.76E 07  ETA 0.11E 17  NCD 3  NCN 14  NITA 52
IT 19  PHI-C.202153636333336E C7  DIR. GRAD.  NO 3  YN 0.17E 07  DELTAI 0.76E 07  ETA 0.11E 17  NCD 3  NCN 14  NITA 52
IT 20  PHI-C.202153636333336E C7  DIR. GRAD.  NO 3  YN 0.17E 07  DELTAI 0.76E 07  ETA 0.11E 17  NCD 3  NCN 14  NITA 52

SOLUTION OBTENUE...A L ITERATION 20

IT 21  PHI-C.4276176796531250E C7  DIR. GRAD.  NO 4  YN 0.13E 08  DELTAI 0.53E 07  ETA 0.98E 17  NCD 1  NCN 6  NITA 42
IT 22  PHI-C.4276176796531250E C7  DIR. GRAD.  NO 5  YN 0.34E 04  DELTAI 0.23E 04  ETA 0.23E 04  NCD 0  NCN 1  NITA 1
IT 23  PHI-C.4276176796531250E C7  DIR. GRAD.  NO 1  YN 0.15E 01  DELTAI 0.10E 02  ETA 0.44E 02  NCD 0  NCN 1  NITA 1
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APPENDIX X: 6

COMPUTER PRINTOUT OF THE RESULTS OBTAINED

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<tr>
<td>NOMBRE DE CONTRAINTEES</td>
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| EPSILON DE NEWTON            | 0.1000E-04 |
| EPSILON TEST GRADIENT        | 0.1000E-02 |

**FONCTION ECONOMIQUE**

\[-0.5264200000000000E+05\]

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**BORNE SUPERIEURE**

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<td>IT 7</td>
<td>PHI-0.42763769531250E 05</td>
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<td>LE 1</td>
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LES VARIABLES ARTIFICIELLES SONT TOUTES ANNULÉES

EUREE DU CALCUL 12 CENTISECONDES
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<table>
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APPENDIX 7

COMPUTER PRINTOUT OF THE RESULTS OBTAINED

BY SEPARABLE PROGRAMMING
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### SECTION 2 - COLUMNS

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SEARCH TECHNIQUES
AND WAGE INCENTIVE PLANS

by

ROBINDRA N. PAL

B.M.E. JADAVPUR UNIVERSITY, CALCUTTA, INDIA. 1965

AN ABSTRACT OF THE MASTER'S THESIS

submitted in partial fulfillment of the
requirements for the degree

MASTER OF SCIENCE

Department of Industrial Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas
1976
ABSTRACT

This thesis has two sections. Part I deals with the literature survey and the development of new techniques to handle search problems. Since the effectiveness of the search procedure is characterized by its rate of convergence, much of research work has been and are still being done to reduce the computation time. An attempt was made to solve one-dimensional search problems for convex functions by bisecting the enveloping cone of the function and then rotating it till the bisector becomes vertical. The generalization of this new method for any unimodal function by coupling with Fibonacci search was also discussed. This approach essentially cuts down the total number of experiments required to reach at optimum. A new method for multi-dimensional search problems based on the intersection of quadratics passing through the line-optimums in co-ordinate directions was developed and exemplified along with the comparison with other standard methods to show its efficiency.

In the second section, a case study was made with a view to show how operations research technique can be applied to formulate and solve certain wage incentive problems. Since the basic problem in an incentive scheme is to define the base level efficiency from which the incentive should start and also the incentive rates, the problem was formulated with the objective as to minimize the variance between the optimum base
level efficiency and the current different efficiencies of various departments. A constraint was that the total incentive to be paid to the workers must not exceed the current overtime expenses. This problem was solved by Generalized Reduced Gradient method and separable programming.