ULTIMATE LOAD CAPACITY OF STEEL BEAMS WITH WEB OPENINGS
BY THE FINITE ELEMENT METHOD

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INTRODUCTION

In the past few years, openings in the webs of steel beams for access to or passing of utility components have been a much discussed subject.

In the construction of multistory steel buildings, openings through the webs of wide flange beams are frequently necessary to accommodate the passage of ductwork for heating, ventilation and air-conditioning. This helps in reducing the height of each story and thus significant savings are realized through the reduction of the materials used.

When a portion of the web is removed, the beam may be weakened in the vicinity of the opening and often it becomes essential to reinforce the hole. A considerable amount of analytical and experimental work has been done on this topic yielding several theoretical solutions which have been verified.

The purpose of this study was to use the finite element method of analysis to observe the behavior of steel beams with reinforced eccentric web openings under ultimate load conditions, using a computer program developed at the Illinois Institute of Technology (1). The objective was to obtain the ultimate load data, the yield patterns and the modes of failure and to compare them with the results of an experimental program carried out at Kansas State University (2).

Five W-shape steel beams with eccentric rectangular web openings were tested, all but one being reinforced at the opening. In all cases, the moment to shear ratio was 30 inches and the eccentricity was 2 inches. The variables in the analysis were the reinforcing area, the opening length and the opening height.
LITERATURE REVIEW

In 1932, Muskhelishvilli (3) introduced the application of the conformal mapping technique and complex integration to plane problems of the theory of elasticity and in particular to the problem of the stress distribution in a plane or thin plate which is weakened by any type of hole.

In 1950, Joseph and Brock (4) used this complex variable method to obtain an exact solution for the stress concentrations around small openings of several shapes subjected to pure bending. The problem of stress concentration due to such openings has been much studied and many papers concerning this problem have been published (5,6,7).

Using the same method of complex variables Heller, Brock and Bart (8), in 1958, presented a solution for the stress around a rectangular opening with rounded corners in a uniformly loaded plate. In 1962 (9), they modified the procedure and obtained stress distributions due to bending with shear.

Snell (10), in 1962, used the finite element method to study the effects of various reinforcing configurations for rectangular openings in plates subjected to uniaxial tension. His work was published in 1965.

In 1963, McCutcheon, So and Gersovitz (11) submitted a report describing the test program carried out at McGill University. Tests were performed on beams with unreinforced circular openings and theoretical versus measured strains were indicated.

Segner (12,13) conducted tests on six A36 steel WF beams having rectangular openings of various sizes. The openings were all centered on the neutral axis of the beams and many different reinforcing schemes were investigated. His theoretical approach was based on the theory that a member having such openings centered on the neutral axis acts as a
Vierendeel truss and thus has a point of contraflexure at mid-length of each opening above and below the opening in the tee-section.

An analytical method for calculating stresses around elliptical holes in a wide-flange beam under a uniform load was presented by Bower (14,15) in 1966. The applicability of the analysis depends on the hole size and on the magnitude of the M/V ratio at the hole. Later in the same year, he conducted tests on simply supported wide-flange beams with and without cantilever action having circular or rectangular web openings loaded by concentrated loads.

A comparison of the stresses from the elasticity solution to the experimental tangential stress measurements, for cases of pure bending and bending with shear, was conducted by Redwood (16), in 1967. There seemed to be a good agreement for cases of pure bending but the stresses were underestimated when shear was present.

In the same year the ultimate moment capacity of a beam was investigated by Redwood and McCutcheon (17) using several test specimens. The parameters investigated were shape, number of openings, and the ratio of moment to shear at the opening. The results showed that large reductions in ultimate capacity occur and that the size of those reductions increases with the amount of shear present. Openings that were circular seemed to perform better than rectangular openings and the presence of a second opening nearby seemed to further reduce the strength of the beam.

In 1968, Redwood and McCutcheon (18) reported on tests to failure of wide-flanged steel beams containing one or two unreinforced openings. Different shear to moment ratios were investigated. The openings were of various shapes but all had the same height which was 57% of the beam depth. The experimental results indicated that under pure bending the moment capacity
of the beams with one or two openings could be calculated based on the plastic modulus of the net section through the opening. The presence of shear reduced the moment capacity of the beam at the opening below that for pure bending. The reduction was a function of the opening shape, dimension, the spacing of openings and the shear to moment ratio. When a single rectangular opening was present, the moment capacity of the beam was reduced to approximately 40% of $M_p$, but when an identical opening was added at a spacing equal to the opening depth, only a small additional reduction resulted. At an $M/V$ ratio of 0.425, the moment capacity reduced to 64% to 72% for both simple and double circular openings.

Also in 1968, Bower (19) suggested criteria for elastic, plastic and buckling design. He concluded that, for elastic design, beams with web holes should be designed using the same basic factors of safety against yielding as in the AISC specifications, except that the maximum allowable bending and shear stresses should be computed using the actual stresses causing yielding at the hole rather than nominal beam stresses. For plastic design, beams with web holes should be designed using the AISC load factor of 1.70, except that the maximum allowable loads should be computed using the actual ultimate strength of the beam at the hole rather than the strength of the gross beam. For large spacings of holes, the effects of each hole should be computed individually. For more than two adjacent holes, the Vierendeel frame analysis should be used while for geometrically dissimilar adjacent holes a frame analysis may be used. With regards to buckling, the same type of buckling analysis should be used as in case of beams without web holes, so long as the edge of the hole is at least four inches from the edge of bearing. When a hole is located in a region of pure bending, the possibility of vertical flange buckling could be checked by assuming that the compression T-section at the hole acts as a column.
In the same year Redwood (20) presented a method of deriving an interaction curve relating moments and shears, as well as a simplified, slightly conservative solution to obtain an approximation to the interaction curve. Bower (21) provided the information on the plastic behavior of beams with web openings along with equations predicting their ultimate strength. As the load on the beam increases, the first yielding in the vicinity of the hole occurs at the corners because of the stress concentration at that location.

In 1969, Cheng (22) experimentally analyzed the stresses around a rectangular web opening in a W shape beam using the photostress method and electrical resistance strain gage technique.

Several papers (23,24,25,26) have been published concerning the plastic behavior and ultimate strength of beams with web openings.

In 1970, Congdon and Redwood (27) conducted an investigation concerning the plastic analysis of reinforced openings through beam webs based on the assumptions of perfectly plastic material behavior. A series of tests were carried out to determine the effects of M/V ratio, reinforced area, hole aspect ratio, ratio of hole depth to beam depth, and location of reinforcement on the beam behavior. The experimental results confirmed the assumption of simple stress distribution at failure. An equation for obtaining the area of reinforcement to take up the maximum shear capacity was also provided. The effects due to one-sided reinforcement for the web opening were not much different from those obtained from reinforcements on both sides of the web at critical points.

An analytical method of determining the moment carrying capacity of beams with eccentric rectangular web holes was given by Richard (28) in 1971.
The effects of varying the opening eccentricity, opening length and opening height were investigated. By increasing the opening eccentricity, the moment carrying capacity for high shear values increased. As the opening length and opening height increased, the moment carrying capacity decreased. When the opening height became larger than the opening length, the moment carrying capacity of the beam increased. It was also found that the shear forces were unequally distributed across unequal web areas.

In 1972, Cooper and Snell (29) performed tests on beams with reinforced web openings and confirmed the validity of the Vierendeel Analysis for the estimation of the normal stresses in the vicinity of the hole. Ultimate load tests were run on three beams and the results were consistent with the predictions obtained from the ultimate strength theory for reinforced web openings presented in Ref. (26).

Frost (30), 1973, conducted an experimental investigation on eight beams to determine their ultimate strength. The web openings were tested with an eccentricity of 1 in. and 2 in. and with the moment to shear ratio at the hole at values of 0 in. and 40 in. The Vierendeel Method was used for the theoretical analysis and several different formulas for determining the shear force distribution were presented and compared.
THE FINITE ELEMENT METHOD

The concept of the finite element method was originally introduced by Turner, et al. in 1956 (31). The method has proved to be quite convenient from an automation point of view, for the solution of problems in continuum mechanics. The first applications were in plane stress problems (32). O. C. Zienkiewicz and Y. K. Cheng (33) also presented the theory necessary for the analysis of a plane elastic continua. The finite element method has since then been extended to axi-symmetric stress analysis, flat plate bending, three-dimensional stress analysis and shell analysis.

The basic concept of the finite element method is that every structure may be considered to be an assemblage of individual structural components or elements. A plane continuum is divided into elements interconnected at a finite number of nodes. Certain approximations have been introduced into the formulation of this discretization of the original continuum and the evaluation of element properties. Judgement is required in making the proper subdivisions, such as element shape and degree of freedom, so that the substitute structure can simulate the actual structure. It is also important to choose a suitable displacement function which can satisfy the requirements of displacement continuity between adjacent elements. All of these factors will determine whether the substitute structure is stiffer or more flexible than the real structure and to what degree the approximation simulates the behavior of the actual structure.

In brief, the finite element analysis of an elastic continuum has the following characteristics:

(1) Structural discretization

(2) The necessity for choosing an appropriate displacement function.
(3) The evaluation of element properties

(4) The assemblage of finite elements and the following of standard displacement method procedures.

In applying this method, the following requirements must be satisfied simultaneously.

(1) Force equilibrium in each element

(2) Displacement compatibility at nodal points between adjacent elements

(3) The internal forces and deformations are related through the geometric and material property characteristics.

Various shapes of finite elements were employed in these analyses. In general, applicable rectangular elements give a little better approximation of stresses and deflections for a given nodal pattern than triangular elements, because they employ a closer deformation approximation. However, the use of quadrilateral elements could entail arithmetical difficulty and consequently a disproportionate increase of computing time in deriving the element characteristics. Because of the greater adaptability of the triangular shape in fitting arbitrary boundary geometrics, triangular elements have been used more widely in the development of general purpose analysis programs.
THE FINITE ELEMENT PROGRAM

Introduction

The computer program used (1) in this project is for the stress analysis of plane structures in the elastic-plastic range by the finite element method. It was developed jointly by the Illinois Institute of Technology Research Institute and the Air Force Flight Dynamics Laboratory. The program can handle bar and triangular plate elements so that it is applicable to trusses and to the analysis of in-plane stresses in reinforced plates. The material behavior is assumed to be isotropic and the user has a choice of three types of stress-strain laws namely Ramberg-Osgood, Goldberg-Richard and the Bilinear laws. In this project the Bilinear law has been used. The program is developed to handle up to ten different materials.

A numerical step by step procedure for obtaining solutions which satisfy the requirements of the incremental theory of plasticity for materials which obey the Mises yield condition and the associated flow rule is used in the program. At each step in the solution, an iterative procedure is used to find the correct values of the strain increments. Changes in plastic strain are accounted for by the addition of fictitious plastic forces to the actual loading on the structure in such a way that the deflections of the structure under the modified loading with assumed elastic behavior are equal to the actual deflections.

Element Properties

The two types of elements used in the analysis (the bar and the triangular plate) are shown in Fig. (1). The coordinates of the end points of the bar and the vertices of the triangle are referred to a fixed coordinate system in a plane. The nodes of each element are numbered in the anti-clockwise
direction as shown in the figure. The geometry of the structure is determined by specifying the \( x \) and \( y \) coordinates of each node, with respect to a fixed set of coordinate axes and by specifying the thickness of the triangular element or the cross-sectional area in the case of a bar element. The cartesian components of the nodal displacements for each of these elements comprise the element displacement vector \( \bar{X} \). The total element strain designated by the vector \( \varepsilon \) can be expressed in terms of the nodal displacement by an equation of the form

\[
\varepsilon = B\bar{X}
\]

The stresses are related to the elastic strains by Hooke's Law

\[
\sigma = C\varepsilon^e
\]

The nodal forces, \( F \), corresponding to given displacement, \( \bar{X} \), are found by the principle of virtual work.

**Elastic-Plastic Analysis:**

In the elastic range of material behavior the equilibrium equations for a structure composed of plate and bar elements of the type considered can be written as

\[
F = K\bar{X}
\]

where the force and displacement vectors now have as their components, the cartesian components of force and displacement at all the nodes and \( K \) is the assembled stiffness matrix for the whole structure. The solution of Eq. (1) for the unknown displacement is given symbolically by \( \bar{X} = K^{-1} F \).

The displacements known, the element strain and stresses can be obtained. However, when the stresses reach the intensity required to cause plastic flow, it becomes necessary to determine the increments of plastic strain caused by the load increment. The material is assumed to obey the Mises yield condition and the associated flow rule. For plane stress the following equations apply:
\[
\bar{\sigma} = (\sigma_x^2 - \sigma_y + \frac{\sigma_z^2}{3} + \frac{3\tau_{xy}^2}{2})^{1/2} = H(\bar{\varepsilon}^p) \tag{2}
\]

\[
\Delta \varepsilon^p = \frac{2}{\sqrt{3}} \left( \Delta \varepsilon_x^p + \Delta \varepsilon_y^p + \Delta \varepsilon_z^p + \frac{1}{4} \gamma_{xy}^p \right) \tag{3}
\]

\[
\begin{aligned}
\Delta \varepsilon_x^p &= \frac{\Delta \varepsilon^p}{2\bar{\sigma}} (2\sigma_x - \sigma_y) \\
\Delta \varepsilon_y^p &= \frac{\Delta \varepsilon^p}{2\bar{\sigma}} (2\sigma_y - \sigma_x) \\
\Delta \gamma_{xy}^p &= 3 \frac{\Delta \varepsilon^p}{\bar{\sigma}} \tau_{xy}
\end{aligned}
\tag{4}
\]

where \( \bar{\sigma} \) and \( \bar{\varepsilon}^p \) are the effective stress and the effective plastic strain, respectively, and where \( H(\bar{\varepsilon}^p) \) is the stress-plastic strain relation for uniaxial stress.

If it is assumed that the response of the structure to the removal of a load increment will be completely elastic then Equation (1) can be modified to account for plastic flow as follows

\[
KX = F + F^P \tag{5}
\]

where \( X \) and \( F \) are the displacement and load after the application of the increment and \( F^P \) is the vector of plastic forces corresponding to the plastic strains. The plastic strain increments caused by the increment of load must satisfy Equations (4) and for an element undergoing plastic flow the stresses must satisfy the yield condition (Equation (2)).

In general, the following steps give the iterative method used to obtain solution. For the details refer to the program in Appendix I:

1. An increment is given to the applied loads.
2. New values of displacement are found from Equation (5) using the current values of the plastic forces (these will be zero for the first step).
3. The displacements are used to compute total strains, elastic strains, stresses, and the effective stress.

4. If the new value of the effective stress is greater than the largest previous value, the element is plastic and the effective stress is used to determine a new value of the effective strain.

5. Plastic strain increments computed from Equations (4) are added to the current values of the plastic strain and new values of the plastic forces are calculated.

6. If the increment in effective plastic strain is sufficiently small the iteration is complete and a return to step 1 is made, if not a return is made to step 2 and a new cycle begun.

This procedure is applied to each of the elements and the decision to start a new step (apply a load increment) is based on the largest plastic strain increment found among all the elements.

An important feature of the method is the way in which the effective plastic strain is computed from the new value of the effective stress at each iteration. If the inverse of Equation (2) is used to give \( \varepsilon^{p} \) as a function of \( \bar{\sigma} \) the solution may become unstable. This becomes obvious when one considers the case of the elastic, perfectly plastic material for which the inverse of the function \( \Pi(\varepsilon^{p}) \) does not exist. To avoid this difficulty, the total strain \( \varepsilon_{t} \) is taken equal to the sum of the value of \( \varepsilon^{p} \) computed in the previous iteration and \( \bar{\sigma}/E \).

The stress-strain law can be written in the form

\[
\varepsilon_{t} = \frac{\bar{\sigma}}{E} + \varepsilon^{p}
\]

or

\[
\varepsilon_{t} = \frac{\Pi(\varepsilon^{p})}{E} + \varepsilon^{p}
\]  \hspace{1cm} (6)
The new value of $\varepsilon^P$ can be found from Equation (6) without difficulty.

The criterion used in Step 6 of the iterative procedure given above, to decide whether the plastic strains have been determined with sufficient accuracy, is the size of the ratio of the increment in effective plastic strain to $\bar{\sigma}/E$. This ratio is a measure of the difference between the ordinates to the theoretical stress-strain curve and the curve that is actually being used at that step in the calculations.
LABORATORY EXPERIMENTS

Ultimate load tests were conducted on five A36 steel beams, two of which were W16 x 45 shapes and three were W16 x 40 shapes. Though the length of the beam was not the same in all the cases, the moment-to-shear ratio, M/V, at the centerline of the opening was kept constant at 30 inches. The test set up, as depicted in Fig. 2a, consisted of simple supports at the ends and a concentrated load applied at midspan. A variable, X, describes the variation of the span length as illustrated in the figure and tabulated in Table 1 for the various test specimens. Beams 3 and 4 are not listed in Table 1 since they were subjected to elastic tests only.

The size of an opening, one of the experimental variables, is given by half its length, a, half its depth, h, and the corner radius, r. Table 1 gives these values for each of the test specimens. However the eccentricity of the opening, that is the distance between the mid-depth of the beam and the centerline of the opening was 2 inches for all the beams. The eccentricity was towards the compression flange in Beams 1, 2 and 5. In Beams 6 and 7, it was towards the tension flange.

Except for Beam 1, all Beams were reinforced at the opening with two bars, one above and one below the opening and at a distance 1/4 inch from the edge of the opening. Beam 5 was reinforced on both sides of the web and the others on one side only. Figure 2b shows the general layout of the reinforcement used. The reinforcement was comprised of bars 2 x 1/4 inches which were considered as the practical minimum size in accordance with the AISC Specifications (34). In all cases the reinforcement was extended 3 inches beyond the edges of the opening. This 3 inches was calculated to be sufficient to develop the strength of the bar using a 3/16 inch fillet weld.
Bars of 3 x 1/2 inches, welded to the web and the flanges with a 3/16 inch fillet weld served as bearing stiffeners in all the beams. All except Beam 1 were provided with bearing stiffeners at the supports and at the load point. Beam 1 was provided with bearing stiffeners only at the load point.

In Beams 2 and 5, cover plates were attached with 1/4" fillet welds to both the flanges at the centerline of the beam. The function of these cover plates was to strengthen the beam at the center and force the failure to occur at the opening. Table 1 gives the dimensions of the cover plates. In Beams 1, 6 and 7 no cover plates were used because the opening sizes and the spans used were such that failure occurred at the opening before a plastic hinge could form at mid-span.

The actual static yield stresses were determined from tensile tests on coupons which were cut from the web and flanges. Similar tests were also conducted to obtain the static yield stresses of the reinforcing bars. The average static yield stresses and the maximum deviation from the average are listed in Table 2.

Load was applied in increments with a Tinius-Olsen screw type machine until specified deflections were reached, and the load then allowed to drop off to the static level. A detailed description of the experimental work can be found in Ref. (2). The experimental ultimate loads were measured and the load-deflection curves plotted. The ultimate loads were corrected for strain hardening and the corrected values are given in Table 6.
Idealization

In order to apply a plane stress finite element method to a three-dimensional structure, certain modifications must be made. The procedure followed was to substitute equivalent bar members pin-connected to the appropriate nodes of the plate elements for the flanges and reinforcing bars. These bar members do not have an associated thickness but have one dimensional material properties which can transfer only axial forces.

To find the equivalent flange element, therefore, there is a conflict between maintaining the area of the flange and maintaining the moment of inertia of the beam. Since retaining the actual strength of the beam is more important compared to maintaining the actual area, the equivalent flanges were determined in such a way that the moment of inertia was kept a constant, Fig. 3.

The moment of inertia, I, of the beam is given as:

\[ I = \frac{t_w}{12} (d - 2t_f)^3 + 2\left(\frac{t_f^3}{12} + b_f \cdot t_f \cdot \frac{d}{2} - \frac{t_f^2}{2}\right) \]

and the moment of inertia, \( I_{eq} \), of the equivalent idealized beam is given as:

\[ I_{eq} = \frac{t_w d^3}{12} + 2\left(\Lambda_{eff} \left(\frac{d}{2}\right)^2\right) \]

If \( I_{eq} = I \), then the effective flange area, \( \Lambda_{eff} \), of the idealized beam is given as

\[ \Lambda_{eff} = \frac{2I}{d^2} - \frac{t_w d}{6} \]

where \( t_w \) is the web thickness of the actual and idealized beam sections, \( d \) is the total depth of the beam, \( t_f \) is the thickness of the flange and \( b_f \) is the width of the flange. The idealized properties of the beams are listed in Table 3.
The bar elements representing the flanges had to be increased in the corresponding length of the beam to account for the cover plates. The actual modified moment of inertia of the beam at that section is

\[ I_{\text{mod}} = I + I_{\text{cp}} \]

where \( I_{\text{cp}} = 2A_{\text{cp}} \left( \frac{d}{2} - \frac{d'}{2} \right)^2 \); being the moment of inertia of the cover plates about the x-axis of the beam,

\[ d' = \text{thickness of the cover plate.} \]

The equivalent modified moment of inertia is given as \( I_{\text{eq.mod.}} = \frac{t_w \cdot d^3}{12} + 2(A_{\text{mod}} (d/2)^2) \). If \( I_{\text{eq.mod.}} = I_{\text{mod}} \) then the modified area for the flange elements to account for cover plates is given by

\[ A_{\text{mod}} = \frac{2I}{d^2} - \frac{t_w \cdot d}{6} + A_{\text{cp}} \left( 1 - \frac{d'}{d} \right)^2 \]

The idealization using a plane stress procedure to simulate the beam behavior was similar for one and two sided reinforcement. This is based on the conclusions of other investigators (27,29), that one sided reinforcement has no significant effects different from those of two sided reinforcement. Therefore, the results of the study can represent both types of reinforcement with the same total cross-sectional area. The equivalent reinforcing elements were obtained by replacing the reinforcement by bars which have the same location and cross-sectional area as the reinforcement and shown in Fig. 3b.

**Discretization**

Though no general rule can be stated as how to best dissect a given structure, it is obvious that the accuracy improves as the size of the mesh decreases. Good results are frequently obtained with rather coarse
subdivisions Ref. (35). Consider a case of a deep rectangular beam subjected to pure bending. Two types of rectangular discretizations are shown in Fig. 4. Now, the assumptions of the beam theory states that plane cross-sections remain plane in bending though the axial fibers of the beam become curved. Thus in order to best approximate the straight transverse sections and the curved axial sections that occur during deformation, the discretization (b) in Fig. 4 is preferred over the discretization (a).

From the results of the examples considered in Ref. (35), it is found that if triangular elements are well formed, i.e. essentially equilateral, better results are expected. The poorest overall displacement patterns are produced with the discretization which contains many weak triangles.

Keeping the above mentioned points in mind, the test beams were divided into a mesh of triangular and bar elements as given in Table 3. Smaller triangular elements were used near the opening in order to get a better picture of the yield pattern near the opening. The rounded corners of the hole were approximated by the best suited triangular elements. The element configuration for the fillets at the corners of the opening were slightly different in beams 6 and 7. This change made no significant difference in terms of the yield pattern or the ultimate load.

Solid Beams

A regular beam without any opening was called a solid beam. Every beam with an opening had a corresponding solid beam. Test runs, by the finite element method only, were made on each of the solid beams in order to aid in comprehending the effects of the openings. The details of the idealization of the solid beams are given in Table 4.
PRESENTATION OF RESULTS

The solid beams were tested first, in order to get an idea of the ultimate capacity of the beams without an opening. Table 5 gives the loading details of the beams with the opening and the solid beams.

The actual experimental ultimate loads, the ultimate loads corrected for strain hardening and the ultimate loads obtained from the finite element analysis are given in Table 6.

From the computer output, the yielded elements were determined and were then plotted in order to show the yield patterns. A triangular element having any effective strain was considered yielded. With an error tolerance of 0.03 in the effective strain the effective stresses of the yielded elements were not exactly equal to the yield stress of the material but rather close to it. For a bar element to yield, its stress value must reach the yield point, however, due to the error tolerance used, it was decided to consider a bar element yielded whenever its stress was within 0.15 kip/in$^2$ of the yield stress.

The usual sign convention is used in the program, namely tensile stresses are positive and compressive stresses are negative. If the stresses in both the directions, that is $x$ and $y$, are positive then the element is considered to have been yielded in tension and likewise, in compression if both are negative. But if the stress in one direction is positive and in the other direction is negative, the element is considered to be yielded in a combination of tensile and compressive stresses. Compressive yielding is shown by horizontal lines while tensile yielding is shown by vertical lines. A combination type of yielding is shown by solid shading.

Since the elements near the opening are very small, the figures showing the yield pattern have been divided into two types. One showing the enlarged
view in the vicinity of the opening and the other showing the remaining portion of the beam. None of the stiffeners in any of the beams were close to yielding. Hence they have been omitted in order to simplify the figures.

The yield pattern for the ultimate load is shown for all the beams. Additional plots for intermediate loads are also shown for beams 1, 2 and 6.

The details of the figures are given below:

Figure 5: Yield pattern for Solid Beam 1 at ultimate load of 144 kips.

Figure 6: Yield pattern in the vicinity of its opening for Beam 1 at a load of 80 kips. No other elements in the beam have yielded at this load.

Figure 7: Yield pattern in the vicinity of its opening for Beam 1 at a load of 96 kips. No other elements being yielded in the beam.

Figure 8: Yield pattern in the vicinity of its opening for Beam 1 at a load of 112 kips. No other elements being yielded in the beam.

Figure 9: Full view of Beam 1 showing yield pattern at ultimate load of 144 kips.

Figure 10: Yield pattern in the vicinity of its opening for Beam 1 at ultimate load of 144 kips.

Figure 11: Yield pattern for Solid Beam 2 at ultimate load of 168 kips. There are cover plates at midspan and stiffeners at the supports. This being the only difference between Solid Beam 2 and Solid Beam 1.

Figure 12: Yield pattern in the vicinity of its opening for Beam 2 at a load of 96 kips. No other elements being yielded in the beam.

Figure 13: Full view of Beam 2 showing yield pattern at a load of 144 kips.

Figure 14: Yield pattern in the vicinity of its opening for Beam 2 at a load of 144 kips.

Figure 15: Full view of Beam 2 showing yield pattern at ultimate load of 152 kips.

Figure 16: Yield pattern in the vicinity of its opening for Beam 2 at ultimate load of 152 kips.

Figure 17: Yield pattern for Solid Beam 5 at ultimate load of 144 kips.
Figure 18: Full view of Beam 5 showing yield pattern at ultimate load of 128 kips.

Figure 19: Yield pattern in the vicinity of its opening for Beam 5 at ultimate load of 128 kips.

Figure 20: Yield pattern for Solid Beams 6 and 7 at ultimate load of 112 kips. Since the only difference between Beams 6 and 7 is in the size of the opening, there is no difference between the Solid Beams 6 and 7.

Figure 21: Yield pattern in the vicinity of its opening for Beam 6 at a load of 54 kips, no other elements being yielded in the Beam.

Figure 22: Yield pattern in the vicinity of its opening for Beam 6 at a load of 72 kips. No other elements in the Beam have yielded at this load.

Figure 23: Full view of Beam 6 showing yield pattern at ultimate load of 84 kips. None of the flange elements have yielded. Since a 100 iterations per step have been reached with an error tolerance of 0.03, the Beam is considered to have been yielded in a practical sense.

Figure 24: Yield pattern in the vicinity of its opening for Beam 6 at ultimate load of 84 kips.

Figure 25: Full view of Beam 7 showing yield pattern at ultimate load of 96 kips. Here again none of the flange elements have yielded. For the same reason as in Beam 6, the Beam is considered to have been yielded in a practical sense.

Figure 26: Yield pattern in the vicinity of its opening for Beam 7 at ultimate load of 96 kips.
DISCUSSION

Ultimate Loads

The ultimate loads obtained from the finite element results are given in Table 6. The accuracy of these ultimate loads depends on the size of the load increments used in the analysis. The smaller the increment size the better the accuracy, but the cost of running the computer program increases as the increment size is decreased. The ultimate load predicted by the finite element method is an upper bound value. Theoretically it should satisfy the inequality:

$$P_{true} \leq P_{FE} < (P_{true} + i)$$

where $i$ = increment size.

The experimental ultimate loads corrected for the effects of strain hardening (2), along with their ratios to those obtained from the finite element method are also listed in Table 6. These ratios indicate that the values predicted by the finite element method are within 6% of those obtained experimentally in all cases.

A further comparison of the finite element results is made with those obtained from an ultimate strength analysis (2). These theoretical ultimate loads were found through the use of interaction diagrams which were obtained from a computer program developed by Wang (36). Once again the ratios of the theoretical and the finite element ultimate loads show a good correlation. Thus it can be concluded that the finite element method predicts the ultimate loads with reasonably good accuracy.

The ultimate loads of the solid beams and their ratios to the ultimate loads of the beams with the openings are also given in Table 6. These ratios show the reductions in the load carrying capacities of the beams due to the web openings. The value of this ratio for Beam 1 is 1.00, thus indicating that
the opening has very little or no effect on the ultimate load of the beam under the existing conditions. This is consistent with the experimental findings (2).

Yield Patterns and Modes of Failure

To begin with, a brief discussion of the effects of secondary moments will aid in a better understanding of the yield patterns.

Consider a section of a loaded beam at the opening. See Fig. 27a. The high moment edge of the opening is at the right, and hence primary moment \( M_2 > M_1 \). Equilibrium requires that \( 2aV = M_2 - M_1 \). The shear forces \( V_T \) and \( V_B \) are distributed to the top and bottom of the opening, Fig. 27b. To maintain equilibrium, secondary moments \( M^S_T \) and \( M^S_B \) are generated, where

\[
2M^S_T = 2aV_T
\]

and

\[
2M^S_B = 2aV_B
\]

Thus the individual sections, above and below the opening, behave as fixed ended beams. Due to the secondary moments there exists compressive normal stresses in the regions near b, d, f and h and tension in the region near a, c, e and g. Due to the primary moment effects only, the entire section above the opening is in compression while the entire section below the opening is in tension. Therefore the corners of the opening at the low moment edge, namely d and e, yield first due to the additive effect of the primary and secondary moments. At the corners c and f the effect of the primary and secondary moments is in the opposite sense and hence the yielding is not so pronounced.

Figures 7 and 12 show the plots of the yielded elements of Beams 1 and 2 at a load of 96 kips. The number of elements yielded in the reinforced case,
that is beam 2, is nearly half that of the unreinforced case, that is, Beam 1. The effect of the reinforcement is such that the yielding is limited to the area in the web between the reinforcement and the edge of the opening. In both cases there is greater yielding at the low moment edge because of the secondary moment effect. Yield patterns for the same beams at a load of 144 kips are shown in Figs. 10 and 14. While this is the ultimate load for the unreinforced case, the reinforced case is 8 kips below its ultimate load. In general there exists a greater amount of yielding in the unreinforced case with a substantial amount of yielding in the flanges, though, there is very little flange yielding at this stage in the reinforced case. This is attributed to the cover plates. The flange yielding in Beam 2 is only on the tension side, between the opening and the end of the cover plate. A closer look at Fig. 10 reveals the existence of a compression element in the tension region. This is due to the secondary moment effect. There are no stiffeners at the supports in Beam 1 and hence the elements at the supports have yielded. This is not so in the Solid Beam 1 because of larger elements used in the discretization of the Solid Beam. Solid Beam 2, Fig. 11, has a greater moment carrying capacity because of the cover plates at mid-span. The ultimate load for solid Beam 2 being higher, the number of elements yielded in Solid Beam 2 is substantially greater than in Solid Beam 1.

Figures 17 through 19 are for Beam 5 and reveal a reduction of 16 kips in the ultimate load due to the opening. Once again in this case the cover plates prevent flange element yielding at mid-span. The quantity of reinforcement used being large, there is no yielding in the reinforcement.

Looking at the yield patterns of Beams 6 and 7, Figs. 23 through 26, a great deal of similarity is observed. There is very little yielding at midspan and the failure in both cases is essentially at the opening.
In all the solid beams it can be observed that the yielding is more extensive towards one side of the centerline even though the loading and support conditions are symmetrical, see Figs. 5, 11, 17 and 20. This is due to the asymmetric element configuration. If the elements were symmetric about the centerline, then the yielding also would be symmetrical. The failure is a typical one hinged mechanism, with the plastic hinge developing at midspan under the load.

In all the beams the first sign of yielding is observed in the web adjacent to the corners of the opening. These yielded zones enlarge as the load increases. At the ultimate load the yielding takes up a pattern which can schematically be shown as in Fig. 28. This type of a pattern generally confirms that a four hinged mechanism develops at failure as assumed in the theoretical analysis (36).
CONCLUSIONS

From the comparison of the results obtained by the finite element method with the experimental and theoretical values, the conclusions reached were:

1) The ultimate loads obtained from the finite element analysis are in reasonably good agreement with those obtained from the experiments and also with those obtained theoretically from an ultimate strength analysis.

2) The yield patterns at various loads agree closely with those obtained in the experiments.

3) The failure at the opening is a four hinged mechanism as assumed in the theory, with a plastic hinge at each corner of the opening.
ACKNOWLEDGEMENTS

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APPENDIX I

Details of the Computer Program

A brief description of the Fortran IV program for the elastic-plastic analysis of plane structures composed of bar and triangular plate elements is given here.

Table AI-I. Input Data Format

| Card 1 | TITLE CARD (72H) | Col 1-72 | Any alphanumeric information |
| Card 2 | PROPERTIES.CARD (1415) | Col 1-5 | NNODE - number of nodes (maximum 450) |
|        |                  | 6-10   | NELEM - number of elements (maximum 800) |
|        |                  | 11-15  | ILAW - 1 Ramberg-Osgood Law |
|        |                  |        | - 2 Goldberg-Richard Law |
|        |                  |        | - 3 Bilinear Law |
|        |                  | 16-20  | IUNLD - 1 Unloading following loading |
|        |                  |        | - 0 Loading only |
|        |                  | 21-25  | MAT - number of materials used (maximum 10) |
|        |                  | 26-30  | MAXBND - maximum bandwidth, MAXBND = 60 for this program |
|        |                  | 31-35  | NBC - number of boundary conditions with prescribed displacement. The maximum number is 30 in this program. |
| Card 3 | MATERIAL PROPERTIES CARDS (E15.8, 3F10.5) | Col 1-15 | EE - modulus of elasticity |
|        |                  | 16-25  | EEL - secant yield stress, ultimate stress, yield stress |
|        |                  | 26-35  | PRR - Poisson's ratio |
|        |                  | 36-45  | EE2 - shape parameter, plastic modulus |
| Card 4 | CONTROL CARD (6I5, F10.0) | Col 1-5 | NDIV - number of load increments |
|        |                  | 6-10   | NIT - maximum number of iterations per step |
|        |                  | 11-15  | NPRINT - print output for each NPRINT increment. (e.g., if NPRINT = 3, for increments 3, 6, 9 etc.) |
|        |                  | 16-20  | KSTART - number of increments at which solution is to start |
|        |                  | 21-25  | KSTOP - number of increments at which solution is to stop |
|        |                  | 26-30  | NLOAD - number of nodes at which loads are specified |
|        |                  | 31-40  | TOL - error tolerance |
Table AI-1 (Continued)

Card 5  NODE CARDS (4I5, 5F10.0)

<table>
<thead>
<tr>
<th>Col</th>
<th>1-5</th>
<th>6-10</th>
<th>11-15</th>
<th>16-20</th>
<th>21-30</th>
<th>31-40</th>
<th>41-50</th>
<th>51-60</th>
<th>61-70</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Node number</td>
<td>IBCX = 1, if displacement in x-direction is specified</td>
<td>IBCY = 1, if displacement in y-direction is specified</td>
<td>IBCS = 1, if slope is specified</td>
<td>XCORD - x coordinate of the node</td>
<td>YCORD - y coordinate of this node</td>
<td>BC1 - specified displacement in x-direction</td>
<td>BC2 - specified displacement in y-direction</td>
<td>BC3 - specified slope at the node</td>
</tr>
</tbody>
</table>

Card 6  ELEMENT CARDS (5I5, F10.0)

<table>
<thead>
<tr>
<th>Col</th>
<th>1-5</th>
<th>6-10</th>
<th>11-15</th>
<th>16-20</th>
<th>21-25</th>
<th>26-35</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Element number</td>
<td>I1 - nodes defining the element</td>
<td>I2 - nodes defining the element</td>
<td>I3 - nodes defining the element</td>
<td>NTYPE - material type</td>
<td>Z - element thickness or cross-section area</td>
</tr>
</tbody>
</table>

Card 7  LOAD CARDS (I5, 2F10.0)

<table>
<thead>
<tr>
<th>Col</th>
<th>1-5</th>
<th>6-15</th>
<th>16-25</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Node number</td>
<td>x-component of force</td>
<td>y-component of force</td>
</tr>
</tbody>
</table>

The program uses a subroutine for the solution of simultaneous equations in band form written by Professor E. L. Wilson of the University of California. Great economies in storage requirements and in time required for solution are achieved in this way.

Three types of stress-strain laws are available for use in the computer program. Each of them is a three parameter law and is given below:

1. Ramberg-Osgood Law

\[
\varepsilon_t = \frac{\varepsilon}{E} + \frac{3\varepsilon_1}{7E} \left(\frac{\sigma}{\sigma_1}\right)^n
\]
in which

\( E \) – Young's modulus

\( \sigma_1 \) – secant yield stress (stress at which the secant modulus = 0.7E)

\( n \) – shape factor

2. Goldberg-Richard Law

\[
\sigma = E \varepsilon_t \left( 1 + \frac{E \varepsilon_t}{\sigma_u} n^{-1/n} \right)
\]

in which

\( E \) – Young's modulus

\( \sigma_u \) – maximum stress

\( n \) – shape factor

3. Bilinear Law

\[
\begin{align*}
\sigma &= E \varepsilon_t & \text{for } \sigma < \sigma_y \\
\sigma &= \sigma_y + E_1 \left( \varepsilon_t - \frac{\sigma_y}{E} \right) & \text{for } \sigma \geq \sigma_y
\end{align*}
\]

in which

\( E \) – Young's Modulus

\( \sigma_y \) – yield stress

\( E_1 \) – slope of the plastic portion of the stress-strain curve.

The correspondence between the program variables and the stress-strain law parameters for each of three laws is given in Table AI-II.
Table AI-II

<table>
<thead>
<tr>
<th>Stress Strain Law</th>
<th>Program Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>ILAW</td>
</tr>
<tr>
<td>Ramberg-Osgood</td>
<td>1</td>
</tr>
<tr>
<td>Goldberg-Richard</td>
<td>2</td>
</tr>
<tr>
<td>Bilinear</td>
<td>3</td>
</tr>
</tbody>
</table>

Correspondence Between Program Variables and Stress Strain Law Parameters.

The displacement component in the x and y direction can be specified at any node or a node can be required to move along a line with a specified slope.

The x and y components of load can be specified at any node. Distributed loads must be treated as concentrated at the nodes.

The number of equal increments into which the applied loads and specified displacements are to be divided is specified as input. It is also necessary to specify the number of the increment at which the solution is to start. For example, if a number of increments \( NDIV = 20 \) is specified and a value of the starting increment \( \text{KSTART} = 5 \) is used, one quarter of the load (displacement) will be applied in the first step, the rest in 15 equal increments. If it is desired to stop the solution at an intermediate step a value of \( \text{KSTOP} \) may be specified. If the unloading solution is desired the value \( \text{IUNLD} = 1 \) is used.

An error tolerance must be specified as input. After each cycle of iteration the maximum error among all the elements is compared with the specified tolerance. If the tolerance is met the next load increment is applied, if not, the iteration is continued. If the tolerance on error is
is not met when the allowable number of iterations is reached the solution is stopped.

The nodal forces and displacements, the maximum error and the number of the element in which it occurs are printed out at the end of each increment. The cartesian components, principal values, and direction of stress and strain are printed out at the user's option by specifying a value of NPRNT as input. For example, a value of NPRNT = 3 will cause the stresses and strains to be printed out for increment numbers divisible by three. The directions of the principal axes of stress are defined by

\[ \phi = -\frac{1}{2} \tan^{-1} \frac{2\tau}{\sigma_x - \sigma_y} \quad , \quad \frac{\pi}{2} < \phi < \frac{\pi}{2} \]

The value of \( \phi \) in degrees is printed out. In the case of strain the principal directions are defined by

\[ \phi = -\frac{1}{2} \tan^{-1} \frac{\gamma}{\epsilon_x - \epsilon_y} \quad , \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2} \]

This value is also printed out since in general the principal axes of stress and total strain do not coincide when plastic flow has taken place.

The effective stress and the effective plastic strain are also given as output for each element.

An example problem of a cantilever beam has been worked out to illustrate the use of the finite element program.

The Fortran IV source program is listed along with the input data and the output of the example problem.
ILLEGIBLE DOCUMENT

THE FOLLOWING DOCUMENT(S) IS OF POOR LEGIBILITY IN THE ORIGINAL

THIS IS THE BEST COPY AVAILABLE
C  ELASTIC PLASTIC FINITE ELEMENT PROGRAM
C WITH THREE STRESS STRAIN LAW OPTIONS
COMMON /A00/ FE(10), FF1(16), FF2(10), PRR(10)
COMMON E, GC, G1, F2, PR, FPR, X2, Y2, X3, Y3, X32, Y32, ERR,
1 N2, NFLE, KEL, ILAW, MAT, NHG,
2 R(400, 60), R(30, 3), TAM(100, 20), FIX(2),
3 X(400), XCRD(450), Y(450), IC(450),
4 FF(400), F(400),
5 I1(100), I2(100), I3(100), NTYPE(100), Z(800), I4(100),
6 SFF(100), SET(100), FEP(100), EXPL(100), EYP(100), EXYP(100),
7 NNODE, MHAND
DIMENSION JX(800, 3), FE(900)
EQUIVALENCE (JX, 11)
EQUIVALENCE (IFIX(1), ICX), (IFIX(2), ICY)
C
C #### READ AND PRINT DATA ####
C
10 READ (5, 20, FMT=700)
   IOFNE=1
20 FORMAT(72H, HCD INFORMATION
   
   WRITE (6, 30)
30 FORMAT('I1H')
   WRITE (6, 20)
   READ(5, 40) NNOD, NELEM, ILAW, IUNLD, MAT, MAXRadians, NHG
40 FORMAT('I14,5')
   READ(5, 50) (FE(I), FF1(I), FF2(I), PRR(I), EE2(I), I=1, MAT)
50 FORMAT('E15.5, 3F10, 5')
   READ(5, 60) NNIV, NPRINT, KSTART, KSTOP, NLOAD, TOL
60 FORMAT('E15.5, F10, 0')
   IF(KSTOP, F0, 0) KSTOP=3DIV
   GO TO (70, 90, 110), ILAW
70 WRITE(6, 80) (I, EE(I), FF1(I), FF2(I), PRR(I), TOL, I=1, MAT)
   GO TO 130
80 FORMAT('I1H')
   WRITE(10, 90) (I, EE(I), FF1(I), FF2(I), PRR(I), TOL, I=1, MAT)
   GO TO 130
90 FORMAT('I1H')
   WRITE(10, 100) (I, EE(I), FF1(I), FF2(I), PRR(I), TOL, I=1, MAT)
   GO TO 130
100 FORMAT('I1H')
   WRITE(10, 110) (I, EE(I), FF1(I), FF2(I), PRR(I), TOL, I=1, MAT)
   GO TO 130
110 FORMAT('I1H')
   WRITE(10, 120) (I, EE(I), FF1(I), FF2(I), PRR(I), TOL, I=1, MAT)
   GO TO 130
120 FORMAT('I1H')
   WRITE(10, 130) (I, EE(I), FF1(I), FF2(I), PRR(I), TOL, I=1, MAT)
   GO TO 130
130 FORMAT('I1H')
   WRITE(10, 140) NNOD, NELEM, NNIV, NIT
   NNODE = 14/15X30HNO, OF ELEM
C

*** NODE COORDINATES AND BOUNDARY CONDITIONS ***

C

DO 200 ,I=1,NNODE
READ(5,170) K,IXCY,IUCY,IXCS,IXCIXD(K),YIK),AC1,HC2,HC3
200 FORMAT(175,F10.0)

IF(IXCY=IUCY) WRITE(6,180)K,IXCY,AC1,HC2,HC3
180 FORMAT(17,3X,3(14,1PE17.7))

ICODE(K)=IUCY=10000
IF(IXCY=IUCY) IC=1000
RC(1C,1)=RC1
RC(1C,2)=RC2
RC(1C,3)=RC3
IC=IC+1
IF(1C.LE.NAC) GO TO 200
WRITE(6,190)
190 FORMAT(54HO MORE THAN 29 NODES HAVE NON ZERO BOUNDARY CONDITIONS)
GO TO 200

200 CONTINUE

C

*** ELEMENT PROPERTIES ***

C

READ(5,210) K,11(K),12(K),13(K),NITE(K),Z(K),J=1,NELEM
210 FORMAT(53,7F10.0)

C

*** LOADS ***

C

N2=2*NNODE
DO 220 K=1,N2
220 F(K)=0
IF(NLOAD,F0.0) GO TO 250
DO 230 K=1,NLOAD
230 READ(5,260)J,F(2*K-1),F(2*K)
240 FORMAT(15,2F10.0)
250 CONTINUE

WRITE(6,260)K,IXCY,IXK),YIK),F(2*K-1),F(2*K),ICODE(K),K=1,NNODE
260 FORMAT(175,F10.0)

WRITE(6,270)
270 FORMAT(10X,90H1F element: NODE 1 NODE 2 NODE 3 ELEMENT:K/)

C

*** MATERIAL TYPE ***

DO 300 K=1,NELEM
300 CONTINUE

C

*** INITIALIZATION ***

C
C
310 XDIV=NDIV
GO TO(320,340,340),ILAW
320 CONTINUE
ON 330 I=1,MAT
F=F[I]
F1=FF1[I]
F2=FF2[I]
CC=F1/F
C1=(7.0*M/3.0)**(1.0/2.0)**F**(1.0-1.0/2.0)
CALL TALF(C)
330 CONTINUE
340 CONTINUE
C
C **** DETERMINE HAND WIDTH ****
C
350 DO 350 K=1,NELEM
I4(K)=I3(K)
350 IF(I3(K),EQ,0) JX(K,3)=JX(K,1)
J=0
DO 380 N=1,NELEM
DO 380 L=1,3
DO 370 J=1,3
KK=IARS(JX(N,1),JX(N,3))
IF(KK-J)370,370,380
360 J=KK
370 CONTINUE
380 CONTINUE
MRAND=2*J+3
IF(MRAND,GT,MXRAND) WRITE(6,390) MRAND
IF(MRAND,GT,MXRAND) GO TO 10
390 FORMAT(14H0)X20,HAND WIDTH TOO LARGE END=14)
DO 400 I=1,NELEM
400 I3(I)=I4(I)
DO 410 J=1,N2
DO 410 L=1,MRAND
410 R(I,J)=0.
C
C  CALCULATION OF STIFFNESS MATRIX
C
C  CALL STIFF
C
C  **** REDUCE MATRIX ****
C
C  CALL SYMSSOL(1)
C
C  **** INCREMENT LOADS, ADD PLASTIC FORCES AND SOLVE FOR DISPLACEMENTS****
C
DO 420 I=1,NELEM
SET(I)=0
SEF(I)=0
EFF(I)=0
FXP(I)=0
420 FXP(I)=0
KN=KSTART-1
K(I)=KN
DO 430 L=1,N2
430 FP(I)=0
GO TO 490
WRITE (*,440)K1,U(1,1:4),U(I),X(I:4),I=1,NMDE

450 FORMAT(1H120X3HFORCES AND DISPLACEMENTS FOR INCREMENT,147/10X4HMDV,REF70182)
1DF5X7HX-F4RCF7X7HY-FORCE9XAHX-DISPL.7XAHY-DISPL./7X13,2F15.3,5X2F8.0,REF70183
215.4 1)
WRITE (*,440)XERR,KEL,IT

460 FORMAT(13HMAX. ERROR =FR.5,5X14HIN ELEMENT NO.,I4,5X17HNO. OF ITER.,14HITNOS,14)

470 IF(MOD(K1,NPRNT)=0)490,480,490

480 CALL OUTPUT
IF(K1,EOF,KSTOP)GO TO 620
GO TO 500

490 IF(K1,EOF,KSTOP) CALL OUTPUT
IF(K1,EOF,KSTOP) GO TO 620

500 K1=K1+1

510 IF(K1-MDIV)510,510,620

510 X(1)=X(1)
GO 520 I=1,N2

520 FF(I)=X(K1)/XDIV*F(I)
GO 530 K=1,NELEM

530 X(K)=X(K)

540 DO 570 I=1,NMDE
IF(IF(I) .NE.0) GO TO 570
CALL DCONDFHCD,1,XXCD,YXCD,YYCD,XXYCD,YYXCD,XYCD
IF(IF(I) .NE.0) GO TO 550

550 ALF=BC(1,3)
FP(I)=FP(I)+ALF*FP(I)
FP(I)=0.

560 DO 560 N=1,2
IF(IF(I) .NE.0) GO TO 560
IR=IT+N-1
FP(IR)=0.

560 CONTINUE

570 CONTINUE
C

C **** SOLVE FOR DISPLACEMENTS ****

C DO 580 I=1,N2

580 X(I)=UF(I)+FP(I)
CALL SYMSOL(I)

C

C CALCULATE TOTAL STRAINS, STRESSES AND PLASTIC
FORCES AND STRAINS FOR EACH ELEMENT

C DO 590 I=1,N2

590 FP(I)=0
XERR=0.
KEL=0
CALL STRAIN

C

C **** PICK LARGEST ERROR AND DETERMINE WHEN TO REITERATE ****

C IT=IT+1
IF(XERR-TOL)440,440,600

600 IF(IT-NIT)540,610,610

610 K1=MDIV
IUNLD=0
GO TO 460

620 IF(IUNLD .NE.0) GO TO 10


PROGRAM FORTRAN

INTEGER, PARAMETER :: N0 = 0, N0 = 1
![Code snippet with FORTRAN commands]

END

SUBROUTINE FLEM(A)

! Common variables
COMMON/ADD/ FF(10), FE1(10), FE2(10), PRR(10)
COMMON F, CC, G1, E2, PR, FEPX, X1, X2, Y1, Y2, X3, Y3, X4, Y4,
1 N2, NLEM, KE1, ILAM, MAT, NAC
2 R(900, 60), AC(30, 3), TAH(101, 20), IFIX(12),
3 X(900), XORDN(450), Y(450), ICONE(450),
4 FP(900), F(900),
5 I1(800), I2(800), I3(800), NTYPE(800), Z(800), I4(800),
6 SFX(800), SET(800), EEP(800), EXPL(800), EYP(800), EXYP(800),
7 NN000, M11NN000
![Code snippet with FORTRAN commands]

END SUBROUTINE FLEM

DIMENSION XX(450)
EQUIVALENCE (XORDN, XX)
J1 = 10(M)
J2 = 12(M)
J3 = 13(M)
X1 = XX(J1-1) - XX(J1)
Y1 = Y(J1-1) - Y(J1)
IF(J1, 0, 1) GO TO 10
Y2 = Y(J1-1) - Y(J1)
Y3 = Y(J1-1) - Y(J1)
X2 = XX(J2-1) - XX(J2)
X3 = XX(J3-1) - XX(J3)
RETURN
10 Y3 variables return
RETURN
END

SUBROUTINE GODE(ICODE, I, IRC, IRCX, IRCY, IRC, IX, IY, NAC)
DIMENSION ICOIDE(450)
IRCS = MOD(ICODE(1), 10)
IRCX = MOD(ICODE(1), 1000) / 100
IRCY = MOD(ICODE(1), 100) / 10
IC = MOD(ICODE(1), 1000) / 10
IX = 2 * I - 1
IY = IX + 1
IF(IC, 0, 1) IC = NAC
RETURN
END

SUBROUTINE DEPLA (EEP, FFT, E, E1, E2)
J = 1
X1 = FFT
X1 = 0
EEP = 0.5 * (X1 + X1)
Y = EEP + E1 / E * EEP * (1.0 / E) - FFT
Y = 1.0 + E1 / E * E2 * EEP * (1.0 / E2 - 1.0)
J = J + 1
IF(J < 50) GO TO 40, 40, 100
IF(Y < 50, 100, 100
X1 = EEP
GO TO 70
X1 = EEP
GO TO 70
RETURN
END
**CTAFL**

**STRAIN-PLASTIC STRAIN TABLE**

**SUBROUTINE TABLE**

**COMMON/ADD/ EF(10), EE1(10), EF2(10), PRR(10)**

**COMMON E,E1,E2,F2,F3,PR1,PR2,EPK,X21,Y21,X31,Y31,X32,Y32,XERR,\**

1 N2,MELFM,KFL,ILAM,MAT,NHC,

2 R(900, 60), W(30, 3), TAR(101, 20), IIFIX(2),

3 X(900), XORD(450), Y(450), ICODE(450),

4 F(900), F(900),

5 I1(800), I2(800), I3(800), I4(800), I5(800),

6 SFF(ADD), SFP(ADD), SFP(ADD), EXPL(800), EYP(800), EXYP(800),

7 NNODE, NBRAND

II2=I+K

II1=I12+1

TAR(I,II1)=0.

TAR(I1,II2)=0.

DD 20 I=1,101

IF(I-I1)20,20,10

10 TAR(I,II1)=FLNT(I-1)*CC5.

EF=EF(I,II1)

CALL PNEW(FPK,EET,E,F1,F2)

TAR(I1,II2)=EPK

20 CONTINUE

RETURN

END

**SUBROUTINE STIFF**

**COMMON/ADD/ EF(10), EE1(10), EE2(10), PRR(10)**

**COMMON E, E1, E2, F2, F3, PR1, PR2, PR3, XR1, XR2, X31, Y31, X32, Y32, XERR,\**

1 N2, MELF, KFL, ILAM, MAT, NHC,

2 R(900, 60), W(30, 3), TAR(101, 20), IIFIX(2),

3 X(900), XORD(450), Y(450), ICODE(450),

4 F(900), F(900),

5 I1(800), I2(800), I3(800), I4(800), I5(800),

6 SFP(ADD), SFP(ADD), EEP(ADD), EXPL(800), EYP(800), EXYP(800),

7 NNODE, NBRAND

**DIMENSION EFP(2), INode(3), LNode(6), NSK(6, 6)**

**EQUIVALENCE (IIFIX(1), IICX1), (IIFIX(2), IICX2)**

**EQUIVALENCE (INODE(1), J1), (INODE(2), J2), (INODE(3), J3)**

DD 80 M=1, NELEM

J1=11(M)

J2=12(M)

J3=13(M)

CALL FLEM(M)

NTY=NTYPE(M)

F=FF(NTY)

PR=PRR(NTY)

IF(J3.EQ.0) GO TO 30

IF(NTY.LE.MAT) GO TO 10

GO TO 160

C

**** STIFFNESS MATRIX CALCULATIONS FOR TRIANGULAR ELEMENTS ****

C

10 MAT=6
AF=E*7(M)
A123=X21*Y31-X31*Y21
A123=ABS(A123)
FT1=AF*(2.0*A123*(1.0-PR*2))
FT2=AF*(4.0*A123*(1.0+PR))

NSK(1,1)=FT1+Y31*2
NSK(2,1)=FT1+PR*Y32*2
NSK(3,1)=FT1+Y31*2
NSK(1,2)=FT1+PR*Y31*2
NSK(2,2)=FT1+Y32*2
NSK(3,2)=FT1+Y32*2
NSK(1,3)=FT1+Y31*2
NSK(2,3)=FT1+Y32*2
NSK(3,3)=FT1+Y31*2

NSK(4,1)=FT1+PR*1*3*2
NSK(4,2)=FT1+PR*X31*2
NSK(4,3)=FT1+PR*Y31*2
NSK(4,4)=FT1+X31*2
NSK(5,1)=FT1+PR*X31*2
NSK(5,2)=FT1+PR*Y31*2
NSK(5,3)=FT1+Y31*2
NSK(5,4)=FT1+PR*2
NSK(5,5)=FT1+Y21*2
NSK(6,2)=FT1+PR*Y31*2
NSK(6,3)=FT1+PR*Y31*2
NSK(6,4)=FT1+PR*Y31*2
NSK(6,5)=FT1+PR*Y31*2
NSK(6,6)=FT1+PR*Y31*2

DO 20 I=1,JMAT
DO 20 J=1,JMAT
20 NSK(I,J)=NSK(J,I)

GO TO 50

C

C *** STIFFNESS MATRIX CALCULATIONS FOR BARS ***

C
30 ET1=2(M)*E/Y32**2
FFP(I)=X21
FFP(2)=Y21
DO 40 I=1,2
DO 40 J=1,2
NSK(I,J)=FT1+PR*FFP(I)*FFP(J)
NSK(I+2,J)=NSK(I,J)
NSK(I,J+2)=NSK(I,J)

40 NSK(I+2,J+2)=NSK(I,J)
JMAT=4

C

C *** INTEGRATION OF ELEMENT MATRICES INTO COMPLETE STIFFNESS MATRIX ***

C
50 JMAT2=JMAT/2
K=0
DO 60 J=1,JMAT2
DO 60 I=1,2
K=K+1
60 LNODE(K)=2+I*NODE(K)-2+J
DO 80 I=1,JMAT
KI=LNODE(I)
DO 80 J=1,JMAT
KJ=LNODE(J)
IF(KJ=KI)RAO,70,71
70 K=KJ-KI+1
R(K1,K)=R(K1,K)+NSK(I,J)
CONTINUE

*** DISPLACEMENT BOUNDARY CONDITIONS ***

DO 150 I=1, MNODE
    IF (ICode(I), IODE, 0) GO TO 150
    CALL NOCODE (ICode(I), ICX, ICY, IX, IY, IAC, NAC)
    IF (ICX, ICY, 1) GO TO 110
    A(LF, RC, IC, 3)
    K(IX, 1) = R(IY, 1) + ALF * (ALF * (R(IY, 1) + 1) + 2. * R(IY, 2))
    R(IY, 2) = ALF
    R(IY, 1) = 1.
    F(IY) = ALF * F(IY) + F(IY)
    F(IY) = 0
    K(IY) = IX - MRAND + 2
    KU = IX + MRAND - 1
    IF (KU, K, 1) KU = 1
    IF (KU, GT, N2) KU = N2
    GO TO 100
    K = KL, KU
    IF (K, FO, IX, NR, K, FO, IY) GO TO 100
    IF (K, GT, IY) GO TO 90
    L = IX - K + 1
    R(K, L) = R(K, L) + ALF * R(K, L + 1)
    R(K, L + 1) = 0.
    GO TO 100
   90  L = K - IX + 1
    R(IY, L) = R(IY, L) + ALF * R(IY, L - 1)
    R(IY, L - 1) = 0.
    100 CONTINUE

110 DO 140 N=1, 2
    IF (IX(N), NE, 1) GO TO 140
    TR = IX + N - 1
    ML = IR - MRAND + 1
    MU = IR + MRAND - 1
    IF (ML, LT, 1) ML = 1
    IF (MU, GT, N2) MU = N2
    GO TO 130
    M = ML, MU
    L = IR - M + 1
    IF (M, LE, 1) GO TO 120
    R(M) = R(M) - R(M, L) * RC(IC, N)
    R(M, L) = 0.
    GO TO 130
  120  L = M - IR + 1
    R(M) = R(M) - R(M, L) * RC(IC, N)
    R(IR, L) = 0.
  130 CONTINUE

  130 CONTINUE

  140 CONTINUE

  150 CONTINUE

RETURN

WRITE(A, 170) M

170 FORMAT(1X, 'INVALID ELEMENT CODE ELEMENT NO: ', N4, 1X)
STOP
END

SUBROUTINE SYMOSOL(KKK)

COMMON /AND/ EE1(10), EE2(10), PRR(10)
COMMON F, C, G, P, PR, FPR, X21, Y21, X31, Y31, X32, Y32, XERW,
        N2, RF, KEL, ILAM, MAT, NAC

SYM50001
SYM50002
SYM50003
SYM50004
NN=NN+1
MM=MM+NN
GO TO (10,40), KKK

C REDUCE MATRIX

10 DD 50 N=1, MM
20 DD 40 L=2, MM
30 C=R(N,L)/R(N,1)
40 I=N+L-1
50 IF (NN-I) 40, 20, 20
60 J=0
70 DD 30 K=L+1, MM
80 J=J+1
90 R(I,J)=R(I,J)-C*R(N,K)
100 R(N,L)=C
110 CONTINUE
GO TO 130

C REDUCE VECTOR

60 DD 80 N=1, NN
70 DD 70 L=2, MM
80 I=N+L-1
90 IF (NN-I) 80, 70, 70
100 X(I)=X(I)-R(N,L)*X(N)
110 X(N)=X(N)/R(N,1)

C BACK SUBSTITUTION

90 N=NN-1
100 IF (N) 100, 130, 100
110 DD 120 K=2, MM
120 L=N+K-1
130 IF (NN-L) 120, 110, 110
140 X(N)=X(N)-R(N,K)*X(L)
150 CONTINUE
GO TO 90

C 130 RETURN

END

SUBROUTINE STRAIN

COMMON/ADD/ FF1(10), FE1(10), FE2(10), PRR(10)

COMMON F, CC, G, F2, PR, EPR, X21, X22, X31, Y31, X32, Y32, XERR,
1 Z, N2, NFILFM, KFL, LAN, MAT, NRC,
2 R(900, 60), H(30, 3), TAB(120, 20), IFIX(2),
3 X(900), XORD(450), Y(450), ICODE(450),
4 FP(900), F(900),
5 I1(800), I2(800), I3(800), NTYPE(800), Z(800), I4(800),
6 SFF(800), SET(800), EEP(800), EXPL(800), EYP(800), EXYP(800),
7 NNODE, MRAND

SYMS0010
SYMS0011
SYMS0012
SYMS0013
SYMS0014
SYMS0015
SYMS0016
SYMS0017
SYMS0018
SYMS0019
SYMS0020
SYMS0021
SYMS0022
SYMS0023
SYMS0024
SYMS0025
SYMS0026
SYMS0027
SYMS0028
SYMS0029
SYMS0030
SYMS0031
SYMS0032
SYMS0033
SYMS0034
SYMS0035
SYMS0036
SYMS0037
SYMS0038
SYMS0039
SYMS0040
SYMS0041
SYMS0042
SYMS0043
SYMS0044
SYMS0045
SYMS0046
SYMS0047
SYMS0048
SYMS0049
SYMS0050
SYMS0051
SYMS0052
SYMS0053
SYMS0054
STRN0001
STRN0002
STRN0003
STRN0004
STRN0005
STRN0010
GO TO 110

*** RAR CALCULATIONS ***

60 FFT=(X(2)XJ3)-X(1))+Y21=(X(4)-X(2))/Y2*Y2
STRK=ABS(FFT-FXPL(K))
SIGN=(FFT-FXPL(K))/STRK
SF=S*STRK
CRIT=SF-SFF(K)
FFT=STRK+FFPK(K)
IF(CRIT)=.90,.90,.70
70 CONTINUE
CALL STRTSTN(FET,FFPK,SEFK,NTY)
DEFP=EPF*EFP(K)
80 FFPK=FFPK
SEFK=SEFK
EXPL(K)=FXPL(K)+SIGN*DEEP
FRR=F+DEFP/SE
ERR=ABS(FRR)
GO TO 100
90 FRR=0.0
100 CONTINUE
EXPT=FXPL(K)
61=F*Z(K)/Y32
FPF(J1)=FPF(J1)-01*X21*EXPT
FPF(J2)=FPF(J2)-01*Y21*EXPT
FPF(J3)=FPF(J3)+01*X21*EXPT
FPF(J4)=FPF(J4)+01*Y21*EXPT
110 IF(ERR=XERR)=130,130,120
120 XERR=ERR
KEL=K
130 CONTINUE
RETURN
END

SUBROUTINE STRTSTN(FET,FFPK,SEFK,NTY)
COMMON/AADD/ FF(10),FF1(10),FF2(10),PRR(10)
COMMON E,CC,GI,G1,F2,PR,EPK,X21,Y21,X31,Y31,X32,Y32,SEPK,
1 NPZ,FLFM,KEL,ILAW,MAK,WHG,
2 H9000,60,AC303,TAB(101,20),IFIX(2),
3 X9000,XCND(450),Y450,ICNDF450,
4 FP9000,F9000,
5 11(800),12(800),13(800),NTYPE(800),Z8000,14(800),
6 SEPF(800),SEF(800),EPF(800),EPK(800),EXPL(800),EYP(800),EXYP(800),
7 NMFred,MBAND
GO TO(10,50,60),ILAY
10 J=5,0,FFET/CC+1.0
20 IF(J=101)=20,30,30
30 EPF=TEAR(J,NT2)+[TAB(1+J,NT2)-TAB(J,NT2)]*[(FFT-FXPL(J,NT1))
40 IF(TAB(J+1,NT1)-TAB(J,NT1))
GO TO 40
30 CALL PNPW1(FFPK,FET,E,G1,F2)
40 SEFK=EPK*FFPK(1.0/F2)
RETURN
50 FPK=ABS(FFPK(1.0+F2))/E2(1.0/F2)
RETURN
60 FC=G1/F
100 IF(FF-FC)=70,70,80
70     FFPK=0,
       SETK=F*FFT
       RETURN
80     FFPK=FFT*FC
       SETK=G*E2*FFPK
       RETURN
END

C12345    SUBROUTINE OUTPUT (INITPT
SUBROUTINE INITPT (COMMON/ADD/ FF(10),FE1(10),FE2(10),PRT(10),
               COMMON F,CC,FC,E2,PR,FPX,YP1,Y21,X31,Y31,X32,Y32,XERR,
1               MN2,MELEM,KFL,II,AW,MAT,NK1,
2               R(900,40),AC130,31,TAR(10,20),IFIX(2),
3               Y(900),XC(ND(450),Y(450),ICNG(450),
4               F(900),F(900),
5               II(RO1),12(RO1),13(RO1),NTYPE(800),Z(RO1),14(RO1),
6               SEF(RO1),SET(RO1),EFP(RO1),EXP(800),EYP(800),EXPY(800),
7               NMADE,PMAND)
L=0
DO 70 K=1,NELEM
C
C     **** TRIANGULAR ELEMENT CALCULATIONS ****
C
NTY=NTYPE(K)
E=EE(NTY)
PR=PRR(NTY)
EPR=E/(1.-PR*PR)
IF((13(K)-E0,0) GT 0) GO TO 70
10    CALL ELEM(K)
     A123=X21*Y31-X31*Y21
     J1=2*11(K)-1
     J2=2*11(K)
     J3=2*12(K)-1
     J4=2*12(K)
     J5=2*13(K)-1
     J6=2*13(K)
     EX=1*(Y32*X(J1)-Y31*X(J3)-Y21*X(J5))/A123
     FY=1*(X32*Y(J2)-X31*Y(J4)+X21*Y(J6))/A123
     EXY=(X32*Y(J1)-Y32*X(J2)-X31*Y(J3)+Y31*X(J4))/A123
1     EXF=EX-EXPL(K)
     FYF=FY-FYP(K)
     EYF=EXY-FYYP(K)
     SX=FPX*(E+PR*EFP)
     SY=EPR*(FY+PR*FYP)
     SXY=E/(1.+PR)*EXYE/2.*N
     PEP2=SORI((5.*(SX-SY))**2+5.*SY)**2)
     PH1=.5*ATAN2((-2.,0.,SY),((SX-SY))*57.29578)
     PH2=.5*ATAN2((-EXYT),(EX-EXYT))*57.29578
     PSL=.5*(SX+SY)
     SIGF1=PS1+PEP2
     SIGF2=PS1-PEP2
     PST1=.5*(EX+FY)
     PFT2=SORI((1.5*(EX-EXYT))**2+EXYT**2/4.,0)
     STRE=PST1*PFT2
     STRE=PST1-PFT2
     PFT2=2.0*PFT2
     N1=I1(K)
     N2=I2(K)
     N3=I3(K)
XC = XCOR(N) + (X2 + X31) / 3.0

YC = Y(N) + (Y2 + Y31) / 3.0

EPF = 2.0 * SORT((EXPL(K) + EXPL(K) + FY(K) + FY(K) + EXYP(K)) / 3, 1)

SF = SORT(SX*2 - SX*SY + SY*2 + 3, 3, SXY*2)

L = L + 1

IF (MND(L-1, 14)) 30, 20, 30

20 WRITE (4, 50)

30 WRITE (6, 60) XC, YC, SX, SY, SXY, SIG1, SIG2, PE2, PHI, PH2, EXT, EY1, EXIT

40 FORMAT (9H)FLL. NO. /5X11HCOORDINATES28X33HSTRESSES /STRAIN

11 NS / RH PH1X1HXX1HY6X6Y9H TAU-XXX6X6Y9H TAU-YY6X9H TAU-XY1X1P61P6

2Y1X7MAXIMUM7MINIMUM7HMAXIMUM7MINIMUM7HMAX SHEAR

50 FORMAT (1H017, OFRA, 3, F9.3, 1P6E15.4/1H OFP7.2, F8.2, 9X1P6E15.4)

WRITE (6, 60) SF, EPE

60 FORMAT (1H23H EFFECTIVE STRESS=E12.5, 23H EFFECTIVE STRAIN

IN=1E12.5)

70 CONTINUE

C

**** BAR CALCULATIONS ****

C

J=0

DO 120 K=1, NELEM

NTY = NTY(K)

E = E(F(NTY))

PR = RPR(NTY)

J1 = 2*K1(K)-1

J2 = 2*K2(K)-1

J3 = 2*K3(K)-1

IJ = 2*K4(K)

IF (J1 .NE. 0) Go To 120

C

CALL ELEM(K)

EEF = (X21 * (X(J3) - X(J1)) + Y21 * (Y(J4) - Y(J2))) / Y32**2

SF = EEF + EXPL(K)

SMK = SF*Z(K)

K1 = 11(K)

K2 = 12(K)

IF (.NOT. J100, 90, 100)

90 WRITE (6, 130)

100 CONTINUE

110 WRITE (6, 140) K, K1, K2, SF, EEF, SMK

120 CONTINUE

130 FORMAT (9HBAR NO. 6X10HMONO NOS. 8X7HSTRESS 8X7HSTRAIN 4X, 11HMEMBER FORCES)

140 FORMAT (1H031A, 2E15.5, 2X, 2E15.5)

RETURN

END
**Example Problem No. 1: Cantilever Beam Ultimate Load**

<table>
<thead>
<tr>
<th>Linear Law</th>
<th>Material</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modulus of Elasticity</td>
<td>0.29E0</td>
<td>05</td>
</tr>
<tr>
<td>Secant Yield Stress</td>
<td>0.26E0</td>
<td>02</td>
</tr>
<tr>
<td>Shape Parameter</td>
<td>0.0</td>
<td></td>
</tr>
<tr>
<td>Poisson's Ratio</td>
<td>0.30E0</td>
<td></td>
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<td>-1.114E+01</td>
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<td>34.21</td>
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<td>-1.514E+01</td>
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<table>
<thead>
<tr>
<th>BAR NO.</th>
<th>AIME NO.</th>
<th>STRESS</th>
<th>STRAIN</th>
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<tr>
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<td>0.565E+02</td>
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<tr>
<td>10</td>
<td>5</td>
<td>0.346CCE</td>
<td>0.594E+02</td>
</tr>
<tr>
<td>11</td>
<td>6</td>
<td>0.333CCE</td>
<td>0.594E+02</td>
</tr>
<tr>
<td>20</td>
<td>10</td>
<td>0.333CCE</td>
<td>0.594E+02</td>
</tr>
</tbody>
</table>
Example Problem

To illustrate the use of the program a simple example of a cantilever beam is worked out. The beam, a W16 x 40, is 36 inches long with a load at its cantilevered tip.

The properties of the cross-section are listed below: \( d = 16.0" \); \( tw = 0.307" \); \( t_f = 0.503" \); \( b_f = 7.0" \); \( Z = 72.8 \text{ in}^3 \); \( S = 64.6 \text{ in}^3 \).

For a yield stress of 36 ksi, the theoretical ultimate load can be calculated as follows: The plastic moment of the section \( = M_p = F_y Z \). Equating it to the externally applied load: \( M_p = P_u L \)

\[
P_u = \frac{F_y Z}{L} = 72.8k
\]

Similarly, the load at first yield can be given by:

\[
P_y = \frac{F_y S}{L} = 64.6 k
\]

Solution by the Finite Element Method

To work out the example by the use of the finite element program, the three dimensional beam is idealized into a two dimensional plane stress problem, see Fig. AI-1a. Having established 21 nodes in the cartisian coordinates, the web is comprised of 26 triangular elements of thickness 0.307 in. and the flanges are simulated by 8 bar elements having a cross-sectional area of 3.22 sq. in.

The beam is fixed at the left end by specifying zero displacements in the \( x \) and \( y \) directions for nodes 1 through 5.

The Bilinear law was made use of with the material properties as follows:

- Modulus of Elasticity = \( 29 \times 10^3 \) ksi
- Yield Stress = 36 ksi
- Poisson's Ratio = 0.3
- Plastic Modulus = 0.0
THIS BOOK CONTAINS NUMEROUS PAGES WITH DIAGRAMS THAT ARE CROOKED COMPARED TO THE REST OF THE INFORMATION ON THE PAGE.

THIS IS AS RECEIVED FROM CUSTOMER.
Fig. AI-1a. Idealized Cantilever Beam.

Fig. AI-1b. Yield pattern at ultimate load (76 kips).
A total load of 80 kips was applied in 20 increments of 4 kips each. KSTART was specified as 15 and therefore the solution started with a load of 60 kips.

A maximum of 100 iterations per step and an error tolerance of 0.03 were specified. At increment number 19, that is a load of 76 kips, the error in element number 9 could not be reduced to less than 0.03 in 100 iterations and therefore, the solution was stopped at that increment. The load of 76 kips is considered as the ultimate load with an error of 0.03 in the effective plastic strain. A plot of the yielded elements at the ultimate load is shown in Fig. AI-Ib.

The choice of the maximum number of iterations per step and the error tolerance is left to the individual and depends on the accuracy desired and the computer time available. For the purpose of this project the above values were selected based on a few trial runs.

**Trial Runs**

Two sets of trial runs were made. One on Beam 1 and the other on Solid Beam 1. Table AI-III gives the details of these runs. The variables were, number of nodes at which the load was applied, error tolerance and the maximum number of iterations per step.

In the actual testing of the beam the load was applied through a plate 1" x 6" x 7". To simulate this condition a three point loading was tried in a few runs, with one half of the load applied at the center and one quarter of the load at the other two points. As can be observed in the tables, the three point loading had no significant effect on the maximum number of load increments reached, the idea of 3 point loading was therefore abandoned.
With the overall computer time available for the project as a limiting constraint, a maximum of 100 iterations per step and an error tolerance of 0.03 gave a reasonably good prediction of the ultimate load and were selected for the rest of the beams.

The execution time required to test a beam with the opening on the IBM 370 model 158 ranged from 7 to 9 minutes.

<table>
<thead>
<tr>
<th>Run No.</th>
<th>No. of Nodes at which Load is Specified</th>
<th>No. of Increments at which Solution Stopped</th>
<th>Specified Error Tolerance</th>
<th>Specified Maximum No. of Iterations Per Step</th>
<th>Maximum Residual Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>15</td>
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<td>20</td>
<td>0.0244</td>
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<td>25</td>
<td>0.0457</td>
</tr>
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<td>1</td>
<td>17</td>
<td>0.03</td>
<td>50</td>
<td>0.0514</td>
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<tr>
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<td>3</td>
<td>17</td>
<td>0.03</td>
<td>50</td>
<td>0.0528</td>
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<tr>
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<td>1</td>
<td>18</td>
<td>0.03</td>
<td>100</td>
<td>0.1057</td>
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<tr>
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<td>1</td>
<td>18</td>
<td>0.10</td>
<td>25</td>
<td>0.1640</td>
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</table>

**Beam 1**

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<th>No. of Nodes at which Load is Specified</th>
<th>No. of Increments at which Solution Stopped</th>
<th>Specified Error Tolerance</th>
<th>Specified Maximum No. of Iterations Per Step</th>
<th>Maximum Residual Error</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>18</td>
<td>0.02</td>
<td>100</td>
<td>0.0708</td>
</tr>
<tr>
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<td>3</td>
<td>18</td>
<td>0.02</td>
<td>100</td>
<td>0.0288</td>
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<td>100</td>
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<tr>
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<td>3</td>
<td>20</td>
<td>0.10</td>
<td>100</td>
<td>0.1354</td>
</tr>
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</table>

* with cover plates and stiffeners at supports

Table AI-III. Trial Runs on Beam 1 and Solid Beam 1.
APPENDIX II - REFERENCES


### APPENDIX III - NOTATIONS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>$A_{cp}$</td>
<td>cross sectional area of cover plate</td>
</tr>
<tr>
<td>$A_{eff}$</td>
<td>effective flange area of idealized beam</td>
</tr>
<tr>
<td>$A_{mod}$</td>
<td>modified area of flange element with cover plates</td>
</tr>
<tr>
<td>$A_{Reff}$</td>
<td>effective reinforcement area of idealized beam</td>
</tr>
<tr>
<td>$a$</td>
<td>half the opening length</td>
</tr>
<tr>
<td>$b_f$</td>
<td>width of flange</td>
</tr>
<tr>
<td>$d$</td>
<td>overall depth of beam</td>
</tr>
<tr>
<td>$d'$</td>
<td>thickness of cover plate</td>
</tr>
<tr>
<td>$F$</td>
<td>nodal force vector</td>
</tr>
<tr>
<td>$F^p$</td>
<td>vector of plastic forces corresponding to plastic strains</td>
</tr>
<tr>
<td>$h$</td>
<td>half the opening depth</td>
</tr>
<tr>
<td>$I$</td>
<td>moment of inertia of actual beam</td>
</tr>
<tr>
<td>$I_{cp}$</td>
<td>moment of inertia of cover plates about the axis of beam</td>
</tr>
<tr>
<td>$I_{eq}$</td>
<td>moment of inertia of equivalent idealized beam</td>
</tr>
<tr>
<td>$I_{eq,mod.}$</td>
<td>moment of inertia of equivalent idealized beam at section with cover plates</td>
</tr>
<tr>
<td>$I_{mod}$</td>
<td>modified moment of inertia of beam at section with cover plates</td>
</tr>
<tr>
<td>$i$</td>
<td>size of load increment</td>
</tr>
<tr>
<td>$K$</td>
<td>stiffness matrix</td>
</tr>
<tr>
<td>$M_1, M_2$</td>
<td>primary bending moments</td>
</tr>
<tr>
<td>$M^S_B$</td>
<td>secondary bending moment in the bottom section at the opening</td>
</tr>
<tr>
<td>$M^S_T$</td>
<td>secondary bending moment in the top section at the opening</td>
</tr>
<tr>
<td>$P_{u}^{exp}$</td>
<td>experimental ultimate load corrected for strain hardening</td>
</tr>
<tr>
<td>$P_{u}^{FE}$</td>
<td>ultimate load obtained from finite element analysis</td>
</tr>
<tr>
<td>$P_{u}^{Theo}$</td>
<td>theoretical ultimate load</td>
</tr>
</tbody>
</table>
\( P_{u_{\text{true}}} \) true ultimate load
\( P_{u_{\text{solid}}} \) ultimate load of solid beam
\( r \) radius of opening corners
\( t_f \) thickness of flange
\( t_w \) thickness of web
\( V \) shear force at any section
\( V_B \) vertical shear force in the bottom section at the opening
\( V_T \) vertical shear force in the top section at the opening
\( X \) distance between center of opening and centerline of beam
\( \bar{X} \) element displacement vector
\( \gamma_{xy}^p \) plastic angular strain
\( \varepsilon \) total element strain vector
\( \varepsilon^p \) effective plastic strain
\( \varepsilon_t \) total strain
\( \varepsilon_x^p \) plastic strain in x-direction
\( \varepsilon_y^p \) plastic strain in y-direction
\( \sigma \) element stress vector
\( \sigma \) effective stress
\( \sigma_x \) normal stress in x-direction
\( \sigma_y \) normal stress in y-direction
\( \tau_{xy} \) shear stress
<table>
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<tr>
<th>Beam</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<td>X</td>
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<td>25&quot;</td>
<td>26&quot;</td>
<td>26&quot;</td>
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<td>4.5&quot;</td>
<td>4.5&quot;</td>
<td>6&quot;</td>
<td>8&quot;</td>
<td>6&quot;</td>
</tr>
<tr>
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<td>3&quot;</td>
<td>3&quot;</td>
<td>4&quot;</td>
<td>4&quot;</td>
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<tr>
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<td>3/4&quot;</td>
<td>17/32&quot;</td>
<td>1/2&quot;</td>
<td>1/2&quot;</td>
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<td>Both Sides</td>
<td>One Side</td>
<td>One Side</td>
</tr>
<tr>
<td>Cover Plates</td>
<td>None</td>
<td>PL $\frac{5}{16}$ x 4 x 24</td>
<td>PL $\frac{5}{16}$ x 4 x 31</td>
<td>None</td>
<td>None</td>
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</table>

Table 1. Test Variables.
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<tr>
<th>Beam</th>
<th>Average $F_y$ (ksi)</th>
<th>Maximum Deviation from Average $F_y$ (%)</th>
<th>Reinforcing bar</th>
<th>Average $F_y$ (ksi)</th>
<th>Maximum Deviation from Average $F_y$ (%)</th>
</tr>
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<td>2.82</td>
<td>30.96</td>
<td>0.16</td>
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<td>40.80</td>
<td>2.82</td>
<td>30.96</td>
<td>0.16</td>
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Table 2. Static Yield Stresses.
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<th>7</th>
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<td>341</td>
</tr>
<tr>
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<td>608</td>
<td>596</td>
<td>684</td>
<td>684</td>
</tr>
<tr>
<td>Number of Elements</td>
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<td>0.346</td>
<td>0.307</td>
<td>0.307</td>
<td>0.307</td>
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<tr>
<td>Web Thickness (in.)</td>
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<td>3.64</td>
<td>3.22</td>
<td>3.22</td>
<td>3.22</td>
</tr>
<tr>
<td>Flange Area (in²)</td>
<td>4.96</td>
<td>4.52</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>Modified Flange Area for Cover Plates (in²)</td>
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<td>1.0</td>
<td>0.5</td>
<td>0.5</td>
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<tr>
<td>Area of Reinforcement (in²)</td>
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<td>3.0</td>
<td>3.0</td>
<td>3.0</td>
<td>3.0</td>
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<tr>
<td>Area of each stiffener (in²)</td>
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<td>---</td>
<td>---</td>
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Table 3. Idealized Properties of Beams.
<table>
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<th>6 &amp; 7</th>
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<td>51</td>
<td>51</td>
<td>51</td>
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<td>102</td>
<td>102</td>
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<td>Web Thickness (in.)</td>
<td>0.346</td>
<td>0.346</td>
<td>0.307</td>
<td>0.307</td>
</tr>
<tr>
<td>Flange Area (in²)</td>
<td>3.64</td>
<td>3.64</td>
<td>3.22</td>
<td>3.22</td>
</tr>
<tr>
<td>Modified Flange Area for Cover Plates (in²)</td>
<td>---</td>
<td>4.96</td>
<td>4.52</td>
<td>---</td>
</tr>
<tr>
<td>Area of each stiffener (in²)</td>
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<td>3.0</td>
<td>3.0</td>
<td>3.0</td>
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Table 4. Idealized Properties of Solid Beams.
<table>
<thead>
<tr>
<th>Beam</th>
<th>Increment Size (kips)</th>
<th>Increment Number at Start of Solution</th>
<th>Increment Number at First Yield</th>
<th>Increment Number at which Solution Stopped</th>
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<tbody>
<tr>
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Beams with Opening

<table>
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<th>Increment Size (kips)</th>
<th>Increment Number at Start of Solution</th>
<th>Increment Number at First Yield</th>
<th>Increment Number at which Solution Stopped</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>6</td>
<td>13</td>
<td>18</td>
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<tr>
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<td>18</td>
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<td>667</td>
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Solid Beams

Table 5. Loading Details for Beams with Opening and Solid Beams.
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<th>7</th>
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<td>FE Load Increment (kips)</td>
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<td>8</td>
<td>6</td>
<td>6</td>
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</tbody>
</table>

| Pu$^{FE}$ (kips) | 144 | 152 | 128 | 84  | 96  |
| Pu$^{exp}$ (kips) | 136 | 155 | 124 | 84  | 101 |
| Pu$^{FE}/Pu^{exp}$ | 1.06 | 0.98 | 1.03 | 1.00 | 0.95 |
| Pu$^{Theo}$ (kips) | 129 | 152 | 129 | 83  | 96  |
| Pu$^{FE}/Pu^{Theo}$ | 1.12 | 1.00 | 0.99 | 1.01 | 1.00 |
| Pu$^{Solid}$ (kips) | 144 | 168 | 144 | 112 | 112 |
| Pu$^{FE}/Pu^{Solid}$ | 1.00 | 0.90 | 0.89 | 0.75 | 0.86 |

*These values are obtained from reference 2.

Table 6. Ultimate Loads.
Bar Element

Triangular Plate Element

Figure 1. Bar and Plate Element.
(a) Test Setup

(b) Opening and Reinforcement Details

Figure 2. Test Setup and Reinforcement Details.
Figure 3. Actual and Idealized Beam Sections.
Figure 4. Discretization of a Deep Beam.
KEY TO FIGURES 5 THROUGH 26

Horizontal lines indicate compressive yielding.

Verticles lines indicate tensile yielding.

Solid shading indicates a combination of compressive and tensile yielding.

In figures showing a full view of a beam, the portion near the opening has been omitted and is shown enlarged in the following figure.
Figure 6. Yield Pattern in Vicinity of Opening for Beam 1 at 80 k.
Figure 8. Yield Pattern in Vicinity of Opening for Beam 1 at 112°.
Figure 10. Yield Pattern in Vicinity of Opening for Beam 1 at Ultimate Load (144 k).
Figure 15. Yield Pattern for Beam 2 at Ultimate Load (152').
Figure 17. Yield Pattern for Solid Beam 5 at Ultimate Load (144 ksi).
Figure 18. Yield Pattern for Beam 5 at Ultimate Load ($128^k$).
Figure 20. Yield Pattern for Solid Beam 6 and 7 at Ultimate Load ($112^k$).
Figure 21. Yield Pattern in Vicinity of Opening for Beam 6 at 5 kN.
Figure 22: Yield Pattern in Vicinity of Opening for Beam 6 at 72 k.
Figure 23. Yield Pattern for Beam 6 at Ultimate Load (84%)
Figure 27. Secondary Moments.
indicates location of plastic hinge

Figure 28. Four Hinged Mechanism at the Opening.
ULTIMATE LOAD CAPACITY OF STEEL BEAMS WITH WEB OPENINGS
BY THE FINITE ELEMENT METHOD

by

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AN ABSTRACT OF A MASTER'S THESIS

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ABSTRACT

A finite element program was used to carry out ultimate load analyses on five A36 steel beams with varying sizes of rectangular web openings. Two of the beams were W16 x 45 shapes and three were W16 x 40 shapes. In all cases the opening had an eccentricity of 2 inches and the moment to shear ratio at the centerline of the opening was 30 inches. The openings in all but one beam were reinforced.

The ultimate loads based on the finite element analysis indicated good agreement with those obtained experimentally and with those obtained from an ultimate strength analysis. The yield patterns at various loads also agree closely with those observed in the experiments. It was confirmed that the failure at the opening is a four hinged mechanism as assumed in the theory, with a plastic hinge at each corner of the opening.