A STUDY OF THE HILL-FUNCTION SOLUTION
TO PROBLEMS OF PROPAGATION IN STRATIFIED MEDIA

by

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I. INTRODUCTION

The problem of finding the electromagnetic fields due to a plane wave incident upon a plane-stratified medium reduces to the problem of solving a second-order linear differential equation with variable coefficients. There exists a limited number of specific layer variations which yield a differential equation solvable in terms of known functions [1]-[9]. If the dielectric profile is not one of these, then the differential equation may be solved by various approximate methods [10]. At high frequencies, that is, when the dielectric variation over distances of the order of a wavelength is small, the WKB (Wentzel-Kramers-Brillouin) or phase-integral solution may be used [1, Ch. 4], [2]. One may also construct an approximate differential equation to obtain a solution good at either high or low frequencies. Another technique is to approximate the dielectric profile by one or more profiles for which an exact solution is known. For example, probably the most obvious approximation is a "stepped" profile consisting of homogeneous layers of arbitrary number and size.

In a recent paper by Casey [11], it was shown that the fields in a plane-stratified layer may be expressed in terms of Hill functions. This method gives formally exact solutions for a nearly arbitrary dielectric profile in the layer. We shall in this paper be concerned with finding the fields outside the inhomogeneous region using the Hill function approach. More precisely, we are interested in the rate of convergence of certain infinite determinants associated with the Hill function method. The rate of convergence will be seen to depend upon frequency and other parameters of the problem. Additionally, we shall be interested in the effect of certain dielectric profile approximations which may conveniently be made using the
Hill-function method of solution. For this study, we shall consider a dielectric profile for which a known-function solution may be obtained and compare numerically the known-function solution and the Hill-function solution.
II. FORMULATION OF THE PROBLEM

The geometry of the problem is shown in Figure 1. A plane wave is incident upon a stratified dielectric layer of thickness d and permittivity $\varepsilon(z)$. On either side of the transition layer are homogeneous regions of permittivity $\varepsilon_1$ and $\varepsilon_2$. The permeability in all regions is $\mu_0$. We shall consider only the case of a plane wave polarized perpendicular to the plane of incidence, as shown in Fig. 1. Also, $e^{-i\omega t}$ time dependence will be assumed for all field quantities and suppressed throughout.

In region (1), $z < 0$, the total electric field is the sum of the incident and reflected fields,

$$ E_y^{(1)} = \psi_1 e^{ik_1 \sin \theta_1 x} $$

with

$$ \psi_1 = e^{ik_1 \cos \theta_1 z} + Re^{ik_1 \cos \theta_1 z}. $$

In region (2), $z > d$, the transmitted field is

$$ E_y^{(2)} = \psi_2 e^{ik_1 \sin \theta_1 x} $$

with

$$ \psi_2 = Te^{ik_2 \cos \theta_t (z-d)}. $$

$R$ and $T$ are respectively the reflection and transmission coefficients to be determined, and $k_{1,2} = \omega \sqrt{\mu_0 \varepsilon_{1,2}}$

Now, the following conditions describe the transition region:

1) linear, 2) isotropic, 3) time-invariant, 4) inhomogeneous, 5) source-free. For this case the wave equation for the electric field $\vec{E}$ is
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Fig. 1. Geometry of the problem of a plane wave incident upon an inhomogeneous dielectric layer.
\[ \nabla^2 \overline{E} + \nabla \left( \overline{E} \cdot \frac{\nabla \varepsilon}{\varepsilon} \right) + k^2 \overline{E} = 0. \quad (5) \]

We may specialize this further by noting that for our case the field will have only a \( y \) component, with no variation in \( y \), and that the permittivity is a function of \( z \) only; or \( E_x = E_z = 0 \), \( \frac{\partial E_y}{\partial y} = 0 \), and \( \varepsilon = \varepsilon(z) \). The wave equation thus becomes

\[ [\gamma_{xz}^2 + k^2(z)]E_y = 0. \quad (6) \]

Using the method of separation of variables, we seek solutions of the form

\[ E_y = \psi(z)X(x). \quad (7) \]

Substituting (7) into (6) and dividing by \( \psi X \) gives

\[ \frac{\psi''}{\psi} + k^2 = -\frac{X''}{X}. \quad (8) \]

Since \( x \) and \( z \) are independent variables, each must be constant. If this constant value is \( \gamma^2 \), then we obtain the two equations

\[ X'' + \gamma^2 X = 0 \quad (9) \]

\[ \psi'' + (k^2 - \gamma^2)\psi = 0. \quad (10) \]

Solutions to (9) are

\[ X = e^{ \pm i \gamma x}. \quad (11) \]

A boundary condition at \( z = 0 \) is the continuity of \( E_y \), so the \( x \) variation of the fields in the layer must be that of the fields in region (1). This requires that

\[ \gamma = k_1 \sin \theta_1 \quad (12) \]

and

\[ X = e^{ ik_1 \sin \theta_1 x}. \quad (13) \]

Note that the variation in \( x \) is not a function of \( k \) - since of course \( k \) is a function of \( z \) - and so does not depend upon the dielectric profile.
We have now reduced the problem to that of solving (10), a second-order, linear differential equation with a non-constant coefficient. In order to numerically investigate the resulting fields, we choose a linear dielectric profile as shown in Fig. 2. We have in the layer

\[ \varepsilon(z) = \varepsilon_1 (1 + az) \]  

(14)

where

\[ a = (\varepsilon_r - 1)/d \]  

(15)

and

\[ \varepsilon_r = \varepsilon_2/\varepsilon_1 = k_2^2/k_1^2 . \]  

(16)

The equation to be solved may now be written as

\[ \psi'' + k_1^2 (\cos^2 \theta_1 + az) \psi = 0 . \]  

(17)

Subsequent application of boundary conditions at \( z = 0 \) and \( z = d \) will complete the solution.
Fig. 2. Transition region with linear dielectric profile.
III. KNOWN-FUNCTION SOLUTION

To put (17) into a familiar form, let

\[ w = \frac{2k_1}{3a} (az + \cos^2 \theta_1)^{3/2} \]  
\[ \psi = w^{1/3} V(w) \]  
\[ \psi' = k_1 (az + \cos^2 \theta_1)^{1/2} \]  
\[ \psi'' = \frac{ak_1}{2} (az + \cos^2 \theta_1)^{-1/2} \]

\[ \psi'' = w'' \left( \frac{w^{-2/3}}{3} + w^{1/3} \psi' \right) \]
\[ + w^{2/3} [\psi''' + \frac{2}{3} w^{-2/3} \psi' + \frac{1}{9} w^{-5/3} \psi] \]  

By substituting these expressions into (17), a change of variables is accomplished and (17) becomes

\[ w'' \left( \frac{w^{-2/3}}{3} + w^{1/3} \psi' \right) + w^{2/3} [w^{1/3} \psi''' + \frac{2}{3} w^{-2/3} \psi' + \left( w^{-1/3} - \frac{2}{9} w^{-5/3} \right) \psi] = 0 \]  

Noting that \( \frac{w''}{w^{4/3}} = 3w \), we divide (23) by \( w^{4/3} \) and obtain Bessel's equation of order 1/3,

\[ \psi'' + \frac{V'}{w} + \left( 1 - \frac{1}{9 w^2} \right) V = 0 \]  

The solution to (24) may be written as

\[ V = A \ J_{1/3}(w) + B \ J_{-1/3}(w) \]  

where \( J_{\pm 1/3} \) denotes the Bessel function of the first kind of order \( \pm 1/3 \) and

A and B are constants to be evaluated. The fields in the layer are now
\[ E_y = \psi e^{ik_1 \sin \theta_1 x} \]

where

\[ \psi = w^{1/3} \left[ A J_{1/3}(w) + B J_{-1/3}(w) \right]. \]  

Now we require that the tangential components of the electric and magnetic fields be continuous across the boundaries, that is at \( z = 0 \)

\[ E_{y}^{(1)} = E_{y} \]

\[ H_{x}^{(1)} = H_{x} \]

and at \( z = d \)

\[ E_{y}^{(2)} = E_{y} \]

\[ H_{x}^{(2)} = H_{x} \].

\( H_x \) in all regions is given by the field equation

\[ H_x = \frac{i}{\omega \mu} \frac{\partial E_y}{\partial z}. \]

Substituting the expressions for the fields, given by (1), (3), (26), and (32), into (28)–(31) yields four simultaneous linear equations in \( A, B, R, T: \)

\[
\begin{bmatrix}
(-i w_o^{1/3} J_{-2/3}^o) & (i w_o^{1/3} J_{2/3}^o) & 1 & 0 \\
(w_o^{1/3} J_{1/3}^o) & (w_o J_{-1/3}^o) & -1 & 0 \\
(-i w_d^{1/3} J_{-2/3}^d) & (i w_d^{1/3} J_{2/3}^d) & 0 & -1 \\
(w_d^{1/3} J_{1/3}^d) & (w_d^{1/3} J_{-1/3}^d) & 0 & -1 \\
\end{bmatrix}
\begin{bmatrix}
A \\
B \\
R \\
T \\
\end{bmatrix}
= \begin{bmatrix}
1 \\
1 \\
0 \\
0 \\
\end{bmatrix}.
\]  

(33)
Here we have introduced the notation of subscript o and d indicating the argument \( w \) evaluated at \( z = 0 \) and \( z = d \) and superscript o and d meaning the function \( J \) evaluated at \( z = 0 \) and \( z = d \). From these equations it is found that

\[
A = \frac{2w_o^{-1/3}(iJ^d_{2/3} - J^d_{1/3})}{D} \\
B = \frac{2w_o^{-1/3}(J^d_{1/3} + iJ^d_{-2/3})}{D}
\]

\[
R = \frac{iJ^d_{2/3}J^o_{1/3} - J^d_{-1/3}J^o_{-1/3} + J^d_{1/3}J^o_{-1/3} + iJ^d_{-2/3}J^o_{-1/3}}{D}
\]

\[
T = \frac{2\sqrt[3]{w_o^{-1/3}w_d^{-2/3}}}{\pi D},
\]

where

\[
D = (iJ^d_{2/3} - J^d_{-1/3})J^o_{1/3} + (J^d_{1/3} + iJ^d_{-2/3})J^o_{-1/3}
\]

\[
+ (J^d_{2/3} + iJ^d_{-1/3})J^o_{-2/3} + (iJ^d_{1/3} - J^d_{-2/3})J^o_{2/3}
\]

This completes the solution for the fields in all regions.

As stated earlier, we are interested in the field outside the transition region which requires that we know \( R \) and \( T \). It will be seen later that the Hill-function approach requires the evaluation of the same infinite determinants to find either \( R \) or \( T \). Therefore it will be necessary to consider only one of these quantities. We will use \( R \).

Now, for computational purposes the reflection coefficient \( R \) will be expressed in terms of the Airy functions, \( Ai(\cdot) \) and \( Bi(\cdot) \). By substituting into (36) the relations [12]
\[ J_{\pm 1/3}(w) = \frac{1}{2} \sqrt{3/\mu} \left[ \sqrt{3} \left( A^d \left(-u\right)^2 + B^d \left(-u\right) \right) \right] \]

\[ J_{\pm 2/3}(w) = \sqrt{3/2} u \left[ \pm \sqrt{3} \left( A^d \left(-u\right) + B^d \left(-u\right) \right) \right] \]

where

\[ u = (3/2 w)^{2/3} \]

we obtain

\[ R = \frac{-\left(A^d, B^d\right) + \sqrt{u_d u_o} \left(A^d B^o - B^d A^o\right)}{\left(A^d, B^d\right) + \sqrt{u_d u_o} \left(A^d B^o - B^d A^o\right)} \]

\[ + i \left[\sqrt{u_d} \left(A^d B^o - B^d A^o\right) \right] + \sqrt{u_o} \left(-A^d B^o + B^d A^o\right) \]

It should be noted that the arguments of the Airy functions in this expression are \(-u_o\) and \(-u_d\). If we let

\[ v_o = e^{-2\pi/3} u_o \]

\[ v_d = e^{-2\pi/3} u_d \]

then it can be shown that

\[ R = \frac{-\left(A^d, B^d\right) + \sqrt{v_o v_d} \left(A^d B^o - B^d A^o\right)}{\left(A^d, B^d\right) + \sqrt{v_o v_d} \left(A^d B^o - B^d A^o\right)} \]

\[ + i \left[\sqrt{v_d} \left(A^d B^o - B^d A^o\right) \right] - \sqrt{v_o} \left(-A^d B^o + B^d A^o\right) \]

where the arguments are \(-v_o\) and \(-v_d\). In order to have the arguments of the Airy functions real, (42) is used when \(\varepsilon > 1\) and (45) when \(\varepsilon < 1\). This is done, again, for computational purposes.
IV. HILL-FUNCTION SOLUTION

Let the following change of variables be introduced into (10):

\[ \zeta = \pi z/2d \]  
(46)

\[ u(\zeta) = \psi(z) \]  
(47)

This gives

\[ u'' + \left( \frac{2d}{\pi} \right)^2 \left( k^2 - \gamma^2 \right) u = 0 \]  
(48)

Now if the coefficient \( g(z) = \left( \frac{2d}{\pi} \right)^2 \left( k^2 - \gamma^2 \right) \) is even and periodic in \( \pi \), as a function of \( \zeta \), then it may be expressed as a Fourier cosine series and (48) becomes

\[ u'' + \left( \lambda + 2 \sum_{n=1}^{\infty} g_n \cos 2n\zeta \right) u = 0 \]  
(49)

This is Hill's equation. If \( \sum_{n=1}^{\infty} |g_n| \) converges, then we may proceed and write the solution in terms of Hill functions \([13]-[15]\),

\[ u = A u_1 + B u_2 \]  
(50)

where \( A \) and \( B \) are constants.

As will be seen, it is not necessary to specify the functions \( u_1 \) and \( u_2 \); that is we need not actually solve the differential equation for the fields in the transition layer if only \( R \) or \( T \) is desired.

In the layer, then,

\[ E_y = A u_1 + B u_2 \]  
(51)

\[ H_x = \frac{1}{2d} \frac{\pi}{\omega u_0} \left[ A u_1' + B u_2' \right] \]  
(52)

where the \( x \) dependence has been suppressed. Application of boundary conditions at \( z = 0 \) and \( z = d \), as given in \((28)-(31)\), gives
\[
\begin{bmatrix}
\frac{-i\pi}{2dk_1 \cos \theta_1} u_1(0) & \frac{-i\pi}{2dk_1 \cos \theta_1} u_2(0) & 1 & 0 \\
u_1(0) & u_2(0) & -1 & 0 \\
\frac{-i\pi}{2dk_2 \cos \theta_2} u_1^{\prime}(\pi/2) & \frac{-i\pi}{2dk_2 \cos \theta_2} u_2^{\prime}(\pi/2) & 0 & -1 \\
u_1(\pi/2) & u_2(\pi/2) & 0 & -1
\end{bmatrix}
\begin{bmatrix}
A \\
B \\
R \\
T
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}.
\]

(53)

From (53), the reflection coefficient is found to be

\[
R = \frac{-(u_2^d u_1^o - u_1^d u_2^o) + a_o a_d(u_1^o u_2^d - u_2^o u_1^d) + i[a_o (u_2^d u_1^o - u_1^d u_2^o) + a_d(u_1^o u_2^d - u_2^o u_1^d)]]}{(u_2^d u_1^o - u_1^d u_2^o) + a_o a_d(u_1^o u_2^d - u_2^o u_1^d)}
\]

(54)

where

\[
a_o = L_1 \cos \theta_1
\]

(55)

\[
a_d = L_1 (\epsilon_r - \sin^2 \theta_1)^{1/2}
\]

(56)

\[
L_1 = \frac{2k_1 d}{\pi}.
\]

(57)

Now the Hill functions may be normalized by setting

\[
u_1(0) = u_2(0) = 1
\]

(58)

\[
u_1^{\prime}(0) = u_2(0) = 0
\]

(59)

and we have

\[
R = \frac{\frac{d^d}{d^d} + a_o a_d u_2^d - i a_o^d u_1^d + i a_o^d u_2^d}{-u_1^{d'} + a_o a_d u_2^{d'} + i a_d^d u_1^d + i a_d^d u_2^d}.
\]

(60)

So to find the reflection coefficient we must find \(u_1^{\prime}(\pi/2), u_2^{\prime}(\pi/2), u_1^{\prime}(\pi/2),\) and \(u_2^{\prime}(\pi/2).\)
In principle, the Hill functions may be found if the coefficients of
the Fourier series in (49) are specified. Thus, we might expect that the
value at \( \xi = \pi/2 \) would be expressible somehow in terms of these coefficients.
This has indeed been found to be so [15, p. 34] and the results are

\[
\begin{align*}
    u_1(\pi/2) &= \cos\left(\frac{\pi}{2} \sqrt{\lambda} \right) c_1 \\
    u_2(\pi/2) &= \frac{\sin\left(\frac{\pi}{2} \sqrt{\lambda} \right)}{\sqrt{\lambda}} s_0 \\
    u_1'(\pi/2) &= -\sqrt{\lambda} \sin\left(\frac{\pi}{2} \sqrt{\lambda} \right) c_0 \\
    u_2'(\pi/2) &= \cos\left(\frac{\pi}{2} \sqrt{\lambda} \right) s_1 .
\end{align*}
\]

\( s_0, s_1, c_0, c_1 \) are the one-sided infinite determinants

\[
\begin{align*}
    s_0 &= \left| \delta_{n,m} + \frac{g_{n-m} - g_{n+1}}{\lambda - 4n^2} \right| \quad (65) \\
    s_1 &= \left| \delta_{n,m} + \frac{g_{n-m} - g_{n+1}}{\lambda - (2n + 1)^2} \right| \quad (66) \\
    c_0 &= \left| \delta_{n,m} + \frac{(g_{n-m} + g_{n+m})(1 + \text{sgn } n \text{ sgn } m)}{\sqrt{\varepsilon_n \varepsilon_m} (\lambda - 4n^2)} \right| \quad (67) \\
    c_1 &= \left| \delta_{n,m} + \frac{g_{n-m} + g_{n+m+1}}{\lambda - (2n + 1)^2} \right| \quad (68)
\end{align*}
\]

where

\[
\varepsilon_n = \begin{cases} 
1 & n = 0 \\
2 & n > 0 
\end{cases}
\]
\[
g_n = \begin{cases} 
0 & n = 0 \\
\delta_n & n > 0 
\end{cases} 
\]

(70)

\[
\delta_{n,m} = \begin{cases} 
1 & n = m \\
0 & n \neq m 
\end{cases} 
\]

(71)

\[
\text{sgn } n = \begin{cases} 
0 & n = 0 \\
1 & n > 0 
\end{cases} 
\]

(72)

The Fourier coefficients \( \lambda \) and \( g_n \) are

\[
g_n = \frac{L_1^2}{d} \int_0^d \left[ \varepsilon_r(z) - \sin^2 \theta \right] \cos \frac{n \pi z}{d} \, dz 
\]

(73)

\[
\lambda = \frac{L_1^2}{d} \int_0^d \left[ \varepsilon_r(z) - \sin^2 \theta \right] \, dz 
\]

(74)

where \( \varepsilon_r(z) \) is the relative permittivity in the layer, \( \varepsilon(z)/\varepsilon_1 \). Some insight as to the structure of the determinants may be gained by writing them as

\[
S_0 = \begin{vmatrix}
1 - \frac{g_2}{\lambda - 4} & \frac{g_1 - g_3}{\lambda - 4} & \frac{g_2 - g_4}{\lambda - 4} & \frac{g_3 - g_5}{\lambda - 4} \\
\frac{g_1 - g_3}{\lambda - 16} & 1 - \frac{g_4}{\lambda - 16} & \frac{g_1 - g_5}{\lambda - 16} & \frac{g_2 - g_6}{\lambda - 16} \\
\frac{g_2 - g_4}{\lambda - 36} & \frac{g_1 - g_5}{\lambda - 36} & 1 - \frac{g_6}{\lambda - 36} & \frac{g_1 - g_7}{\lambda - 36} \\
\frac{g_3 - g_5}{\lambda - 64} & \frac{g_2 - g_6}{\lambda - 64} & \frac{g_1 - g_7}{\lambda - 64} & 1 - \frac{g_8}{\lambda - 64} \\
\vdots & \vdots & \vdots & \vdots 
\end{vmatrix} 
\]

(75)
\[ s_1 = \begin{pmatrix} 
1 - \frac{g_1}{\lambda - 1} & \frac{g_1 - g_2}{\lambda - 1} & \frac{g_2 - g_3}{\lambda - 1} & \frac{g_3 - g_4}{\lambda - 1} \\
\frac{g_1 - g_2}{\lambda - 9} & 1 - \frac{g_3}{\lambda - 9} & \frac{g_1 - g_4}{\lambda - 9} & \frac{g_2 - g_5}{\lambda - 9} \\
\frac{g_2 - g_3}{\lambda - 25} & \frac{g_1 - g_4}{\lambda - 25} & 1 - \frac{g_5}{\lambda - 25} & \frac{g_1 - g_6}{\lambda - 25} \\
\frac{g_3 - g_4}{\lambda - 49} & \frac{g_2 - g_5}{\lambda - 49} & \frac{g_1 - g_6}{\lambda - 49} & 1 - \frac{g_7}{\lambda - 49} 
\end{pmatrix} \quad (76) \]

\[ c_0 = \begin{pmatrix} 
1 & \frac{\sqrt{2} g_1}{\lambda} & \frac{\sqrt{2} g_2}{\lambda} & \frac{\sqrt{2} g_3}{\lambda} \\
\frac{\sqrt{2} g_1}{\lambda - 4} & 1 + \frac{g_2}{\lambda - 4} & \frac{g_1 + g_3}{\lambda - 4} & \frac{g_2 + g_4}{\lambda - 4} \\
\frac{\sqrt{2} g_2}{\lambda - 16} & \frac{g_1 + g_3}{\lambda - 16} & 1 + \frac{g_4}{\lambda - 16} & \frac{g_1 + g_5}{\lambda - 16} \\
\frac{\sqrt{2} g_3}{\lambda - 36} & \frac{g_2 + g_4}{\lambda - 36} & \frac{g_1 + g_5}{\lambda - 36} & 1 + \frac{g_6}{\lambda - 36} 
\end{pmatrix} \quad (77) \]

\[ c_1 = \begin{pmatrix} 
1 + \frac{g_1}{\lambda - 1} & \frac{g_1 + g_2}{\lambda - 1} & \frac{g_2 + g_3}{\lambda - 1} & \frac{g_3 + g_4}{\lambda - 1} \\
\frac{g_1 + g_2}{\lambda - 9} & 1 + \frac{g_3}{\lambda - 9} & \frac{g_1 + g_4}{\lambda - 9} & \frac{g_2 + g_5}{\lambda - 9} \\
\frac{g_2 + g_3}{\lambda - 25} & \frac{g_1 + g_4}{\lambda - 25} & 1 + \frac{g_5}{\lambda - 25} & \frac{g_1 + g_6}{\lambda - 25} \\
\frac{g_3 + g_4}{\lambda - 49} & \frac{g_2 + g_5}{\lambda - 49} & \frac{g_1 + g_6}{\lambda - 49} & 1 + \frac{g_7}{\lambda - 49} 
\end{pmatrix} \quad (78) \]
One should note that approximating the dielectric profile by truncating the Fourier series does not result in finite determinants. However, truncating the determinants then requires a knowledge of only a finite number of Fourier coefficients for their evaluation. The more basic question then is what size determinants to use in finding the reflection coefficient. In practice, however, one must approximate the actual permittivity profile so we shall study independently the effect upon the reflection coefficient due to truncating the Fourier series and truncating the infinite determinants. Later, some analytical details will be presented with regard to convergence. Presently, we continue with the Hill-function solution for the linear dielectric profile.

The coefficient in (48) for the linear profile is

$$g(z) = L_1^2(az + \cos^2 \theta_1), \quad 0 \leq z \leq d.$$  \hspace{1cm} (79)

Fig. 3 shows the even periodic extension of this which is the function given by the Fourier series in Hill's equation. The two horizontal axes illustrate the change of variables in (46). The Fourier coefficients for this function are

$$\lambda = \left( \frac{\varepsilon_r - 1}{2} + \cos^2 \theta_1 \right) L_1^2.$$ \hspace{1cm} (80)

$$g_n = \begin{cases} \frac{-2(\varepsilon_r - 1)}{n^2 \pi^2} L_1^2 & \text{n odd} \\ 0 & \text{n even} \end{cases}.$$ \hspace{1cm} (81)

Since \(\sum_{n=1}^{\infty} \frac{1}{n^2}\) is absolutely convergent, the infinite determinants converge and the reflection coefficient may be found. This completes the Hill-function solution.
Fig. 3. Coefficient in Hill's equation represented by Fourier series.
V. NUMERICAL CALCULATION OF THE REFLECTION COEFFICIENT

From the Fourier series for \( g(z) \) we obtain the series for the linear relative permittivity profile as

\[
\varepsilon_r(z) = \frac{\varepsilon_r + 1}{2} - \frac{4}{\pi^2} (\varepsilon_r - 1) \sum_{n=1}^{\infty} \frac{1}{(2n - 1)^2} \cos \left(\frac{(2n - 1)\pi z}{d}\right). \quad (82)
\]

Figs. (4a), (4b) show profiles approximating the linear profile obtained by truncating the series in (82). The superscript \( n \) in \( \varepsilon_r^{(n)}(z) \) gives the number of terms retained. Thus, \( \varepsilon_r^{(n)}(z) \) is the actual dielectric profile that is solved if one truncates the series for \( g(z) \) in \( n \) non-zero terms.

In Figs. (5)-(7) the magnitude of the reflection coefficient is given as a function of either \( L_1 \) or \( \theta_1 \) with \( \varepsilon_r \) and \( L_1 \) or \( \theta_1 \) constant. Each figure consists of a family of five curves. One is the exact reflection coefficient obtained from the Airy-function solution. The remaining four are computed from the Hill-function solution using various degrees of profile approximation, namely \( \varepsilon_r^{(n)}(z) \) with \( n = 1, 2, 3, 4 \). For each curve using the Hill-function solution, the computing procedure was as follows: The infinite determinants were truncated to size 3x3 and the reflection coefficient calculated. The determinant size was the incremented by one until the computed reflection coefficient differed by less than 0.5% from the one preceding. The final values of the reflection coefficient \( R \) and the determinant size \( NS \) were recorded. For each successive data point, the initial determinant size was set at one less than the final size of the determinant for the preceding point. For Fig. (5), calculations were begun at \( L_1 = 0.2 \), and for Figs. (6), (7), at \( \theta_1 = 0 \).
Fig. 4a. Dielectric profile represented by truncated Fourier series, $\varepsilon_r = 0.5$. 
Fig. 4b. Dielectric profile represented by truncated Fourier series, $\varepsilon_r = 2.0$. 
Fig. 5a. Reflection coefficient vs. $L_1$, $\theta_1 = 0$, $\varepsilon_r = 0.5$. 
Fig. 5b. Reflection coefficient vs. $L_1$, $\theta_i = 0$, $\epsilon_r = 2.0$. 
Fig. 6a. Reflection coefficient vs. $\theta_1$, $\varepsilon_r = 0.5$, $L_1 = 1$. 

ER = 0.5
L1 = 1
Fig. 6b. Reflection coefficient vs. $\theta_1$, $\varepsilon_r = 0.5$, $L_1 = 2$. 
Fig. 6c. Reflection coefficient vs. $\theta_i$, $\varepsilon_r = 0.5$, $L_1 = 3$
Fig. 6d. Reflection coefficient vs. $\theta_i$, $\varepsilon_r = 0.5, L_1 = 4$. 
Fig. 7a. Reflection coefficient vs. $\theta_1$, $\epsilon_r = 2.0$, $L_1 = 1$. 
ER=2.0
L1=2

Fig. 7b. Reflection coefficient vs. $\theta$, $\varepsilon_r = 2.0$, $L_1 = 2$. 
Fig. 7c. Reflection coefficient vs. $\theta_1$, $\varepsilon_r = 2.0$, $L_1 = 3$. 
Fig. 7d. Reflection coefficient vs. $\theta_1$, $\varepsilon_r = 2.0$, $L_1 = 4$. 
The criterion of 0.5% change in $R$ gives no clue as to the accuracy of the final $R$. The idea here was that if the determinants have converged close to their exact value for a given NS, then so has $R$, and therefore $R$ cannot change by more than some small amount if NS is increased. The problem is that the change in $R$ can also be small if $R$ is converging slowly, with a value perhaps far from the exact value. The error can therefore vary over a wide range, and this is evident when comparing the reflection coefficient for $n = 4$ to the exact $R$. Fortunately, however, the error was small in all but a few places, a notable one being for $\varepsilon_R = 2$, $L_1 = 4$, and $\theta_1 = 30^\circ$ (Fig. 7d). This was roughly the point of maximum error, about 10%. Here the size of the determinant would have to be made greater by one or two to reduce the error to a more typical value of 0.5%. The range of the error as obtained from reflection coefficient data (not shown) is about 0.1% - 10%.

Since the error was reasonably uniform, the behavior of convergence with respect to the parameters $\theta_1$, $L_1$, and $\varepsilon_R$ is clear. In Tables Ia, Ib, we show the range of final determinant size in the computations for each combination of $L_1$ and $\varepsilon_R^{(n)}$. It is seen that larger NS is required as $L_1$ increases. From (80), (81), if $(\varepsilon_R - 1)/2 > \cos^2 \theta_1$, then the determinants behave similarly as a function of the quantity $(\varepsilon_R - 1)L_1^2$. Since $L_1$ is the length of the transition region in quarter wavelengths of region (1), we may state that the convergence is most rapid when the change in permittivity is small and occurs through a region thin with respect to wavelength. Also, Tables Ia, Ib indicate that convergence is not strongly dependent upon $\theta_1$ since most combinations of $L_1$ and $n$ have a single value for the determinant sizes.

Now, consider an off-diagonal element of $S_o$ or $C_o$ of the form $(g_p + g_q)/(\lambda - 4n^2)$, as in (75), (77), where $n$ is the row of the element. For the linear profile this is
TABLE I. Range of final determinant size in the Hill-function computations vs. $L_1$ and $n$; a) $\epsilon_r = 0.5$, b) $\epsilon_r = 2.0$. 

(a) 

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
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<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>4</td>
<td>5</td>
<td>7-9</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>4</td>
<td>4-5</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4-5</td>
<td>6</td>
</tr>
</tbody>
</table>

(b) 

<table>
<thead>
<tr>
<th>$n$</th>
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<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>5</td>
<td>6-9</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>6-7</td>
</tr>
</tbody>
</table>
\[
\frac{g_p + g_q}{\lambda - 4n^2} = \frac{-2(e_r - 1) L_1^2}{\left(\frac{\frac{1}{p} + \frac{1}{q}}{2} \right) L_1^2 - 4n^2}
\]

where
\[
0 \leq \left(\frac{1}{p} + \frac{1}{q}\right) \leq \frac{5}{4}.
\]

If the frequency is such that the magnitude of the \( L_1^2 \) term in the denominator is much greater than \( 4n^2 \) then all elements of rows less than \( n \) are nearly frequency independent. As \( L_1 \) is made smaller the \( 4n^2 \) term will dominate and more elements will go as \( L_1^2 \). Those that have been going as \( L_1^2 \) will be small compared to the constant elements so that we may truncate the determinant by disregarding those elements for which
\[
\left|\left(\frac{e_r - 1}{2} + \cos^2 \theta_1\right) L_1^2\right| \ll 4n^2.
\]

With \( e_r = 2 \) we have
\[
\frac{1}{2} \leq \frac{e_r - 1}{2} + \cos^2 \theta_1 \leq \frac{3}{2}
\]
so that we write (85) as
\[
L_1^2 = \frac{4n^2}{36}.
\]

This gives the truncated determinant size as
\[
NS = 3L_1.
\]

This result is valid for \( S_1 \) and \( C_1 \) also, and could have been obtained by considering their elements. It compares well with the empirical result \( NS \approx 2L_1 \), obtained by inspection of Table Ib. It is significant that the required determinant size is directly proportional to \( L_1 \). This result may be extended to an arbitrary profile by noting that the general Fourier coefficients are proportional to \( L_1^2 \), as shown in (73), (74).
The effect upon the reflection coefficient due to truncating the Fourier series for \( g(z) \) is shown by Figs. (5)-(7). It is seen that for a given number of terms retained, more accurate values of \( R \) are obtained for smaller \( L_1 \). One should note, especially from Fig. (5), that keeping only the first term gives very poor results even for small \( L_1 \) and that a substantial improvement is effected by adding a second term. This could be expected after comparing the dielectric profile approximations in Fig. (4).

If the infinite determinants are truncated to size \( NS \), then the highest Fourier coefficient needed for their evaluations is \( g_{2NS} \). It follows then that for small \( L_1 \) the dielectric profile may be approximated with fewer terms of the Fourier series. However, the converse is not true. That is, the determinants for a profile which can accurately be represented by few terms of the series do not necessarily converge faster than if the profile requires a relatively larger number of terms. This is because keeping a finite number of series terms still gives infinite determinants whose rate of convergence depends upon the size and placement of the non-zero elements. Examples of faster convergence with more series terms may be found in Tables Ia, Ib.

Actually there is no reason to be concerned with truncating the Fourier series since one may always obtain the number of terms required to evaluate the truncated determinants. This is obvious if the profile, or the coefficient \( g(z) \), is such that the integrals in (73), (74) can be evaluated directly. If not, then graphical or numerical integration may be used. An alternate approach is the discrete Fourier transform (DFT). The DFT requires \( n+1 \) values of the function at equally-spaced points in order to obtain \( n+1 \) values approximating the first \( n+1 \) Fourier coefficients. These values converge to the actual Fourier coefficients for large \( n \). In our case then, we require the value of \( \varepsilon_r(z) \) at the points \( z = md/n, \ m = 0,1,2,\ldots,n \), to find the
Fourier coefficients $\lambda$ through $g_n$. Using the fast Fourier transform (FFT) algorithm allows one to compute the DFT coefficients very economically for large $n$.

Although the Hill-function solution for the reflection coefficient is formally exact, the accuracy of results obtained by its implementation is limited by economics. In Table IIa, we give a summary of the cost and time required for computing the Hill-function data used for this paper. It should be noted that we have tabulated the total number of reflection coefficient computations. This is slightly greater than twice the number of points used to construct Figs. (5)-(7) due to the computing procedure described earlier. In Table IIb the cost and execution time for the Airy-function computations is summarized. A comparison shows that for the accuracy obtained with the Hill-function solution and the range of $L_1$ used, the Hill-function method cost about one-fifth as much per reflection coefficient as did the Airy-function solution. At higher frequencies, or larger $L_1$, larger determinants will have to be evaluated, and since the work required for determinant evaluation goes up as the cube of the size [16], the cost will rise sharply. This limits the usefulness of the Hill-function method, as applied here, to transition regions with a thickness of no more than a wavelength or so.

A method of reducing the cost of finding $R$ that was not investigated numerically, but would appear significant, is now described. We break up the transition region into $n$ layers as shown in Fig. 8, with the solution for the $m$th layer written as

$$u^{(m)} = A_m u_1^{(m)} + B_m u_2^{(m)}.$$  \hspace{1cm} (89)

The application of boundary conditions yields 2($n$+1) equations for the unknowns $R$, $T$, $A_m$, $B_m$, $m = 1, 2, \ldots, n$, and we may solve these for $R$ in terms
### TABLE II.

Computations summary for a) Hill-function solution, b) Airy-function solution.

<table>
<thead>
<tr>
<th>NS</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of R's</td>
<td>677</td>
<td>830</td>
<td>318</td>
<td>195</td>
<td>52</td>
<td>113</td>
<td>107</td>
</tr>
<tr>
<td>Total No. of R's</td>
<td>2292</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Execution time</td>
<td>6.78 minutes</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Cost</td>
<td>20.34 dollars</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| Total No. of R's | 280 |
| Execution time | 4.08 minutes |
| Cost | 12,24 dollars |
Fig. 8. Geometry of transition region divided into n layers, with the respective Hill function for each layer.
of \( u_1^{(m)}(\pi/2), u_2^{(m)}(\pi/2), u_1^{(m)}(\pi/2), u_2^{(m)}(\pi/2), m = 1,2,\ldots,n \). We now compare
the cost for finding \( R \) by using a single transition region of thickness \( L_1 \) and
by dividing the layer into \( n \) regions of thickness \( L_1/n \). From (88), the
necessary truncated determinant size is \( 3L_1 \) so that for the single transition
region we must compute four determinants of size \( 3L_1 \). For the \( n \) layers, we
must compute \( 4n \) determinants of size \( 3L_1/n \). Remembering that the work required
for determinant computation goes as the size cubed, we find the cost ratio of
the two methods is

\[
\frac{4(3L_1)^3}{4n \left( \frac{3L_1}{n} \right)^3} = n^2.
\]

Thus, by dividing the transition layer into \( n \) equal parts, we reduce the cost
of evaluating the necessary determinants by a factor of \( 1/n^2 \). We expect a
point of diminishing returns, however, since one must solve \( 2n + 2 \) simultaneous
equations for \( R \). Also, more work is required in setting up the problem, for
example finding the Fourier coefficients for each of the \( n \) regions.

The computer subroutines will not be examined in detail, but are pre-
sented in the Appendix along with a brief description.
VI. ANALYTICAL CONSIDERATIONS

If \( L_1 \) is small, we may approximate the Bessel differential equation, (24), as

\[
V'' + \frac{V'}{W} - \frac{V}{9W^2} = 0 .
\]

Its solution is

\[
V = C \left( \frac{\omega}{2} \right)^{1/3} + D \left( \frac{\omega}{2} \right)^{-1/3} ,
\]

which gives

\[
\psi = W^{1/3} \left[ C \left( \frac{\omega}{2} \right)^{1/3} + D \left( \frac{\omega}{2} \right)^{-1/3} \right].
\]

After applying boundary conditions we obtain

\[
R = \frac{W_d^{2/3} - W_o^{2/3} + i 2/3(W_d^{-1/3} - W_o^{-1/3})}{W_d^{2/3} - W_o^{2/3} + i 2/3(W_d^{-1/3} + W_o^{-1/3})}
\]

\[
= \frac{(e_r - 1)(e_r - \sin^2 \theta_i)^{1/2} \cos \theta_i + i \frac{2(e_r - 1)}{L_1 \pi} [\cos \theta_i - (e_r - \sin^2 \theta_i)^{1/2}]}{(e_r - 1)(e_r - \sin^2 \theta_i)^{1/2} \cos \theta_i + i \frac{2(e_r - 1)}{L_1 \pi} [\cos \theta_i + (e_r - \sin^2 \theta_i)^{1/2}]} .
\]

(95)

If the frequency is low enough the higher power terms may be dropped giving

\[
R = \frac{W_d^{-1/3} - W_o^{-1/3}}{W_d^{-1/3} + W_o^{-1/3}} = \frac{\cos \theta_i - (e_r - \sin^2 \theta_i)^{1/2}}{\cos \theta_i + (e_r - \sin^2 \theta_i)^{1/2}} ,
\]

(96)

the abrupt boundary reflection coefficient. To see how good the approximation in (94) is we expand the Bessel functions and form the products in (36).

Keeping the first two terms of the series expansion gives generally,

\[
J_u^d \{ x \} = \frac{1}{u! v!} \left( \frac{d}{2} \right)^u \left( \frac{v}{2} \right)^v - \frac{1}{u! (1+v)!} \left( \frac{d}{2} \right)^u \left( \frac{v}{2} \right)^{2+v} - \frac{1}{v! (1+u)!} \left( \frac{d}{2} \right)^{2+u} .
\]

(97)
Substituting this into (36) gives all terms containing the first four powers of \( L_1 \). Keeping only those of \(-1\) and \(0\) power yields

$$
- \frac{w_d^{-1/3} w_o^{1/3} + w_d^{1/3} w_o^{-1/3} + 2 w_d w_o^{-2/3} - 1/2 w_d^{2/3} w_o^{-2/3}}{-w_d^{-1/3} w_o^{1/3} + w_d^{1/3} w_o^{-1/3} + 2 w_d w_o^{-2/3} - 1/2 w_d^{2/3} w_o^{-2/3} + 1/2 w_d w_o^{-2/3} + 1/2 w_d^{-2/3} w_o^{2/3}}
$$

$$
= \frac{- (\varepsilon_r - \sin^2 \theta_i)^{1/2} \cos^3 \theta_i + (\varepsilon_r - \sin^2 \theta_i)^{3/2} \cos \theta_i - 1/2 (\varepsilon_r - \sin^2 \theta_i)^2}{- (\varepsilon_r - \sin^2 \theta_i)^{1/2} \cos^3 \theta_i + (\varepsilon_r - \sin^2 \theta_i)^{3/2} \cos \theta_i + 1/2 (\varepsilon_r - \sin^2 \theta_i)^2} \cdot
$$

$$
+ \frac{12 (\varepsilon_r - 1)}{L_1 [\cos \theta_i - (\varepsilon_r - \sin^2 \theta_i)^{1/2}]} \cdot
$$

$$
- \frac{12 (\varepsilon_r - 1)}{L_1 [\cos \theta_i + (\varepsilon_r - \sin^2 \theta_i)^{1/2}]} \cdot
$$

(98)

If the 0 power terms are dropped here, we again have the abrupt boundary reflection coefficient, so that (98) may be considered to contain the first-order correction to the abrupt boundary coefficient. Since this expression is somewhat more complex than (94), we may assume that the reflection coefficient obtained by simplifying the differential equation directly is a rather crude approximation.

A first-order approximation identical to (98) may be easily obtained from the Hill-function formulation. Using (61)-(64) we have for the reflection coefficient of (60)

$$
\tan \left( \frac{\pi}{2} \sqrt{\frac{L_1}{\lambda}} \right) \left[ S \frac{a_o a_d}{\sqrt{\lambda} C_0} \right] + \frac{1}{L_1} \left[ \cos \theta_i S_1 - (\varepsilon_r - \sin^2 \theta_i)^{1/2} C_1 \right] \cdot
$$

$$
\tan \left( \frac{\pi}{2} \sqrt{\frac{L_1}{\lambda}} \right) \left[ S \frac{a_o a_d}{\sqrt{\lambda} C_0} + \sqrt{\lambda} C_0 \right] + \frac{1}{L_1} \left[ \cos \theta_i S_1 + (\varepsilon_r - \sin^2 \theta_i)^{1/2} C_1 \right] \cdot
$$

(100)
Let \( \tan \left( \frac{\pi}{2} \sqrt{\lambda} \right) = \frac{\pi}{2} \sqrt{\lambda} \) and recall that \( a_0, a_d, \sqrt{\lambda} \propto L_1 \). In order to avoid higher powers of \( L_1 \) in \( R \), we set the four infinite determinants equal to one, their limit as \( L_1 \to 0 \). Under these conditions, (100) may be shown to be identical to the first-order Bessel function approximation in (98).

To get a better approximation with higher powers of \( L_1 \), we may expand (75)-(78) and obtain

\[
S_0 = 1 - \sum_{n=1}^{\infty} \frac{g_{2n}}{\lambda - 4n^2} + \ldots \tag{101}
\]

\[
S_1 = 1 - \sum_{n=1}^{\infty} \frac{g_{2n-1}}{\lambda - (2n - 1)^2} + \ldots \tag{102}
\]

\[
C_0 = 1 + \sum_{n=1}^{\infty} \frac{g_{2n}}{\lambda - 4n^2} - \frac{2}{\lambda} \sum_{n=1}^{\infty} \frac{g_n^2}{\lambda - 4n^2} + \ldots \tag{103}
\]

\[
C_1 = 1 + \sum_{n=1}^{\infty} \frac{g_{2n-1}}{\lambda - (2n - 1)^2} + \ldots \tag{104}
\]

At low frequencies \( \lambda \) is small so \( L_1^2 \) may be factored from the numerator of the series shown explicitly. Terms with higher powers of \( L_1 \) may be found by factoring subsequent infinite-series terms in (101)-(104). These further terms are products of \( g_n \)'s and are complicated due to the many combinations possible.
VII. SUMMARY

The problem of finding the electromagnetic fields outside an inhomogeneous, stratified layer for TE plane-wave incidence has been solved in terms of Hill functions. For a region with a linear dielectric profile we have demonstrated the utility of the Hill-function formulation by direct comparison with the Bessel function solution. For a transition region with thickness on the order of a wavelength or less, the Hill-function method was substantially cheaper for obtaining accurate numerical results for the reflection coefficient. At higher frequencies, convergence of infinite determinants becomes slower and limits the usefulness of the Hill-function solution. However, it was shown that breaking up the transition region into subregions can reduce the cost of finding $R$ at higher frequencies. At low frequencies analytical approximations for $R$ were readily obtained by simply expanding the suitably truncated determinants.
APPENDIX

The subroutines used for numerical evaluation of R by the Hill-function method are now presented. The subroutines HY1, HY1P, HY2, HY2P are used to compute \( u_1(\pi/2), u'_1(\pi/2), u_2(\pi/2), u'_2(\pi/2), u'_2(\pi/2) \) respectively. These quantities are denoted by the variables Y1, Y1P, Y2, Y2P. ALAM is the Fourier coefficient \( \lambda \), NS is the determinant size, and \( G \) is a vector consisting of the Fourier coefficients \( g_1, g_2, \ldots, g_{60} \). The subroutines CDET and CBIG are called by the above subroutines and actually perform the determinant evaluation by the Gauss elimination method.

For evaluating the known-function solution, three subroutines are used. The subroutine AIRY computes \( Ai(z), Ai'(z), Bi(z), Bi'(z) \) for the real argument \( z \). In the I1 list for this subroutine we have the argument \( Z \) for the above functions, respectively, \( Ai, AiP, Bi, BiP \). The subroutines BESI, BESJ are called by AIRY.
ILLEGIBLE DOCUMENT

THE FOLLOWING DOCUMENT(S) IS OF POOR LEGIBILITY IN THE ORIGINAL

THIS IS THE BEST COPY AVAILABLE
SUBROUTINE HYL(ALG,AUS,Y)
C     THIS SUBROUTINE CONSTRUCTS THE ARRAY CAF
C     CORRESPONDING TO THE DETERMINANT C1.
C     COMPLEX CAF, SORT, CSRL, ALG
C     COMPLEX CAF1, GFACT, GINT
C     DIMENSION CAF(51,51), DINT(5)
C     MCA=0
C     61M,14M3
C     C MINE THE N-TH ROW OF THE DETERMINANT.
C     ON 1 Y=1,NS
C     62M,12M2
C     ELEMENTS BE THE N-TH ROW HAVE A CURVAIN POLE.
C     CHECK TO SEE IF THE ELEMENTS ARE NEAR THE POLE.
C     IF SO, MULTIPLY THIS ROW BY THE APPROPRIATE
C     FUNCTION, WHICH WILL HAVE A ZER0 HERE.
C     (££ANSSELF, E1, LSEM=0) GO TO 2
C     DEFINE THE ELEMENT IN THE N-TH COLUMN
C     OF THE N-TH ROW.
C     ON 5 NS=1,NS
C     63M,10M4
C     IF (M1.5, CSRL(M1,M1)=1.0D0) GOTO 1
C     IF (M1.5, CSRL(M1,M1)=C(M1+M1-1))/EN
C     GOTO 1
C     5 CONTINUE
C     5 CONTINUE
C     GOTO 1
C     5 CONTINUE
C     CALL SUBROUTINE GINT TO EVALUATE THE DETERMINANT.
C     CALL COEF(Y, CSRL, CAF)
C     IF AN ELEMENTS HERE BEAR A POLE, NOW MULTIPLY THE
C     DETERMINANT BY THE APPROPRIATE FUNCTION TO OBTAIN Y1.
C     IF (M1.5, LSEM) GO TO 3
C     SETI(0)
C     3 CSRL=CSRL(CAFLYAAL.+0.)
C     CAF=COEF(PI*CSRL/2.1)
C     Y1=I*REAL(CAF)
RETURN
END
ENTRY
$STOP
/*
SUBROUTINE HYP(ALAM, N, YIP)
C THIS SUBROUTINE COMPUTES THE DETERMINANT CORRESPONDING TO THE
C DETERMINANT COFACTORS OF THE INTEGER MATRIX. COMPLEX CVALUE.
C UPLO = CMAT, CSRL, FACT
C DIMENSION CMAT(31, 31), G(31)
C CHECK
M = 31, N = 31
IF(M.NE.14) GO TO 1
IF(M.NE.31) GO TO 2
IF(M.NE.4) GO TO 3
IF(M.NE.11) GO TO 4
IF(M.NE.14) GO TO 5

C CONSIDER THE M-TH ROW OF THE DETERMINANT.
DO 1 N = 1, NS

C ELEMENTS OF THE M-TH ROW HAVE A COMMON RISE.
C CHECK TO SEE IF THE ELEMENTS ARE NEAR THE RISE.
C IF SO, MULTIPLY THIS ROW BY THE APPROPRIATE
C FUNCTION, WHICH WILL HAVE A ZERO HERE.
IF(M .GT. 1) GO TO 2
IF(M .GT. 5) GO TO 7

C DEFINE THE ELEMENT IN THE M-TH COLUMN OF THE M-TH ROW.
DO 5 N = 1, NS

C CALL SUBROUTINE FACT TO EVALUATE THE DETERMINANT.
CALL FACT(YIP, CMAT, NS)
C IF NO ELEMENTS WERE NEAR A RISE, NOW MULTIPLY THE
C DETERMINANT BY THE APPROPRIATE FUNCTION TO OBTAIN YIP.
C (END CHECK, END) GO TO 3
RETURN

5 CONTINUE
GO TO 1

2 FACT = FACT(CMATS(N+1, N))/4.

4 CONTINUE
GO TO 1

3 FACT = FACT(CMATS(N+1, N))/2.

RETURN

ENTRY
STOP
/
SUBROUTINE HY2(ALAM,G,NS,Y2)
C THIS SUBROUTINE CONSTRUCTS THE ARRAY SMAT
C CORRESPONDING TO THE DETERMINANT SO.
C DOUBLE COMPLEX,CSORT,CSIN,COS
C DOUBLE CSRL,CFAC
DIMENSION SMAT(31,31),G(60)
NCHK=1
PI=3.14159
C CONSIDER THE $M$-TH ROW OF THE DETERMINANT.
DO 1 $M=1,NS
  $\delta_a=ALAM-4.,\pi.M
C ELEMENTS OF THE $M$-TH ROW HAVE A COMMON POLE.
C CHECK TO SEE IF THE ELEMENTS ARE NEAR THE POLE.
C IF SO, MULTIPLY THIS ROW BY THE APPROPRIATE
C FUNCTION, WHICH WILL HAVE A ZERO HERE.
1 IF(ALM(FN).LE.1.EM(-04)) GOTO 2
C DEFINE THE ELEMENT IN THE $M$-TH COLUMN
C OF THE $M$-TH ROW.
DO 5 $N=1,NS
  SMAT=1.EM(-N)
  IF($M+FN,N,1)=1.EM(-2.N)/FN
  IF($M+FN,N.1)=GM(N).EM(-G(N+1)/FN))FN
5 CONTINUE
GOTO 1
2 FACT=PI*COS(PIM(N))/16./N.N
NCHK=1
DO 4 $N=1,NS
  SMAT=1.EM(-N)
  IF($M+FN,N,1)=FACT*GM(N)
  IF($M+FN,N.1)=FACT*GM(N).GM(1)EM(-G(N+1))
4 CONTINUE
1 CONTINUE
C CALL SUBROUTINE COST TO EVALUATE THE DETERMINANT.
CALL COST(Y2,SMAT,NS)
C IF NO ELEMENTS WERE NEAR A POLE, NOW MULTIPLY THE
C DETERMINANT BY THE APPROPRIATE FUNCTION TO OBTAIN Y2.
IF(NCHK.LT.1) GOTO 3
RETURN
3 IF(ALM(ALAM).LE.1.EM(-04)) GOTO 6
C CSRL=CSORT(COMPLEX(ALAM,0.))
CFAC=CSIN(PIM*CSRL/2.1/CSRL
Y2=Y2*FACT(CFAC)
RETURN
6 Y2=Y2*PI/2.
RETURN
END
SUBROUTINE HY2P(ALAMG,NS,Y2P)
C THIS SUBROUTINE CONSTRUCTS THE ARRAY SMAT
C CORRESPONDING TO THE DETERMINANT S1.
C COMPLEX CAXP,CSORT,CSIN,CSUS
C COMPLEX CSRL,CFACT
C DIMENSION SMAT(31,31),G(n)
NCHK=0
P1=3.141593
C CONSIDER THE N-TH ROW OF THE DETERMINANT.
DO 1 J=1,NS
FN=ALAMG(2*(J-1))
C ELEMENTS OF THE N-TH ROW HAVE A COMMON POLE.
C CHECK TO SEE IF THE ELEMENTS ARE NEAR THE POLE.
C IF SO, MULTIPLY THIS ROW BY THE APPROPRIATE
C FUNCTION, WHICH WILL HAVE A ZERO HERE.
1 IF(ABS(FN).LE.1.0D-04) GO TO 2
C DEFINE THE ELEMENT IN THE N-TH COLUMN
C OF THE N-TH ROW.
DO 5 M=1,NS
MM=IAS(N-M)
IF(MM,M0.0) SMAT(M,M)=1.-G(MM-1)/FN
IF(MM,MM0.0) SMAT(M,M)=(G(MM)-G(MM-1))/FN
5 CONTINUE
GO TO 1
C N=1
2 FACT=PI*COS(P1*N)/(N*N-4)
NCHK=1
DO 4 M=1,NS
MM=IAS(N-M)
IF(MM,M0.0) SMAT(N,M)=-FACT*(2*M-1)
IF(MM,MM0.0) SMAT(N,M)=FACT*(G(MM)-G(MM-1))
4 CONTINUE
1 CONTINUE
C CALL SUBROUTINE CDOT TO EVALUATE THE DETERMINANT.
CALL CDOT(Y2P,SMAT,NS)
C IF NO ELEMENTS WERE NEAR A POLE, NOW MULTIPLY THE
C DETERMINANT BY THE APPROPRIATE FUNCTION TO OBTAIN Y2P.
C IF(NCHK.LT.1) GO TO 3
RETURN
3 CSRL=CSORT(CMPLX(ALAMG,M1))
CFACT=CDOT(PI*CSRL/2.)
Y2P=Y2P*REAL(CFACT)
RETURN
END

END
SUBROUTINE CDOT(A,N)
DIMENSION A(31,31)
OUT=1.
L=1
9 CALL CRIG(A,N,IR,JR,L)
10 IF(IR.EQ.L.AND.JR.EQ.L) GO TO 11
11 OUT=OUT
12 DO 1 J=L,N
13 B=A(IR,J)
14 A(IR,J)=A(L,J)
15 A(L,J)=B
16 A(L+J)=A(L,J)
17 2 A(L+J)=B
18 10 IF(A(RA(A(L,J))) .LT. 12) GO TO 11
19 OUT=0.
20 RETURN
21 20 DO 5 I=M,N
22 5 DO 5 J=M,N
23 A(I,J)=A(I,J)-A(L,J)*A(L,J)/A(L,L)
24 OUT=OUT*A(L,L)
25 IF(N-L-1)8 7
26 7 L=L+1
27 GO TO 9
28 OUT=OUT*A(N,N)
29 RETURN
30 END
SUBROUTINE CRIG(A,N,IBIG,J0,BIG,L)
DIMENSION A(31,31)
BIG=A(L,L)
IBIG=L
J0=J0
DO 5 I=L,N
DO 5 J=L,N
IF(ABS(A(I,J))-ABS(IBIG))5,5,2
BIG=A(I,J)
IBIG=I
J0=J
5 CONTINUE
RETURN
END
SUBROUTINE ATRY(Z, A1, AIP, B1, BIP)

IMPLICIT REAL*8(A-H,O-Z)

REAL*8 ARR(100)

GNU1 = 3.33333333333333

GNU2 = 4.666666666666667

ZETA = GNU2 * DABS(Z) / 1.5D0

SR3 = DSORT(3, 0)

SRZ = DSORT(DABS(Z))

PI = 3.141592653589793

IF(Z) 1, 3, 2

2 DZETP = DEXP(ZETA)

CALL BESI(ZETA, GNU1, 0, ARR, 100)

A1 = ARR(1) / DZETP

CALL BESI(ZETA, GNU2, 0, ARR, 100)

A2 = ARR(1) / DZETP

CALL BESK(ZETA, GNU1, 0, ARR, 100)

AK1 = ARR(1) / DZETP

CALL BESK(ZETA, GNU2, 0, ARR, 100)

AK2 = ARR(1) / DZETP

AJ1 = SRZ * AK1 / (SR3 * PI)

AJP = -2 * AK2 / (SR3 * PI)

BJ1 = 2 * D0 * SR2 * AK1 / (SR3 * PI)

BJP = -2 * D0 * Z * A12 / (SR3 + Z * AK2 / PI)

RETURN

1 CALL BESJ(ZETA, GNU1, 0, ARR, 100)

A1 = ARR(1)

CALL BESJ(ZETA, GNU2, 0, ARR, 100)

A2 = ARR(1)

CALL BESY(ZETA, GNU1, 0, ARR, 100)

AY1 = ARR(1)

CALL BESY(ZETA, GNU2, 0, ARR, 100)

AY2 = ARR(1)

AJ1 = (SRZ / 2, D0) * (AJ1 - AY1 / SR3)

AJP = (-Z / 2, D0) * (AJ2 + AY2 / SR3)

BJ1 = (SRZ / 2, D0) * (AJ1 / SR3 + AYL)

BJP = (-Z / 2, D0) * (AJ2 / SR3 - AY2)

RETURN

3 AJ1 = -355028053837817

BJ1 = A1 * SR3

AIP = -258319403792807

RETURN

END
SUBROUTINE RESI (X,Y,N,A,LDIMX)

IMPLICIT REAL*8 (A-H,O-Z)
REAL*8 X,Y,A
DIMENSION A(1), GA(1), GB(1)

DATA EPS/1.0D-14/
DATA ACM/Z409DOOOO00000000001/
DATA THRSHX/2.0D0/
DATA THRSHY/7.0D-3/
DATA SD720P /741978676F6564F6/
DATA PI /7413243F6A89B9A31/
DATA S2 /7411A51A6250703/
DATA S3 /74113A0D04FD9621/
DATA S4 /7411A5122AC7D948/
DATA S5 /7411097418ECA7CCE/
DATA EC /74363C46E3TD96C8/
DATA TWDN3 /240000000000000000000000/
DATA TWDN15 /240000000000000000000000/
DATA GA(1) /740868F3AEECC38B/
DATA GA(2) /7417F3D3A835552/
DATA GA(3) /7413A07B9393309/
DATA GA(4) /7418456348R2EAC07/
DATA GA(5) /7413A8390F09F565E7/
DATA GA(6) /741211111F357A2/
DATA GA(7) /74225637952E7502C/
DATA GA(8) /7416423A52A6A87D/
DATA GR(1) /74185FPA40A558A29/
DATA GR(2) /7422FC4698A114F/
DATA GR(3) /74161666EFC29673/
DATA GR(4) /742698FE7F6E26F/
DATA GR(5) /74136F9E38AB8B56C/
DATA GR(6) /74167695A73281AC4/
DATA GR(7) /7407550306D234C9C/
DATA QND3/Z409DOOOO000000000000000000/
COSH(EXP)=.5D0*(EXPX+1./EXPX)
VKLY=EXPX*(1.0D0-COSH(EXP))
WRNSK=AVP1*AV=(*5D0+TWDX-AVPI+AKV)/AV
LMN=N
ASSIGN 450 TO IK1
KV=1
IF (V NE.0) KV=2

ENTRY FOR RESI CALCULATES (THE 1-BESSL FUNCTION)
ENTRY FOR ORDERS V+1,V+2,V+1,...,N+1 EXP(-X)
F(K)=2K+1.X)/2K+12K+1. X) IS STORED IN AVK+2)
TEMPORARILY. THIS ELIMINATES OVERFLOW PROBLEMS

DO = 1, 100 CONTINUE
IF (X LE.0.0 OR V LT.0.0 OR V.GE.1.0 OR V.NE.1.0)
X = X GO TO 995
TWDX = 2.0D0/X
FK=U+00
FKPI=1.00
K=LMN
TEMP=(DFL(DAT149)+V)
TEMP=TEMP+TWDX
FKM1=FK
FK=FKPI
FKM1=FK
0052 TEMP = TEMP - TWO
0053 FKP1 = TEMP * FKP * FKP1
0054 IF ((FKP1 + 1, D01, KV) * FKP1) GO TO 120

THE POINT OF RECURSION K IS NOW WELL DETERMINED.

0055 IF ((K + 2) * KV.GT.LDIMA) GO TO 995
0056 KP = K + 2
0057 A(KP) = 0, 0, 0
0058 DR = DR/DR(K) * V
0059 DO 150 M = 1, K
0060 KD = KP - 1
0061 A(KP) = XV / (D0 + XV * A(KP + 1))
0062 DR = DR - 2.0
0063 CONTINUE

NORMALIZE I'S BY SUMMATION FORMULA

0065 BGAM = 1.000
0066 Q20XPV = 1.000
0067 IF (KV.EQ.1) GO TO 200
0068 Q20XPV = (TWO * V) * V
0069 TEMP = V * 2.000
0070 BGAM = GA(1) + GB(1) / (TEMP +

1. GA(2) + GB(2) / (TEMP +
2. GA(3) + GB(3) / (TEMP +
3. GA(4) + GB(4) / (TEMP +
4. GA(5) + GB(5) / (TEMP +
5. GA(6) + GB(6) / (TEMP +
6. GA(7) + GB(7) / (TEMP + GA(8) + GA(9)))
7.))

0071 BGAM = RGAM / (1.000 + V)
0072 CONTINUE

200 TEMP = Q20XPV * RGAM

SIG = TEMP

0073 A(1) = 1.000
0074 TEMP = TEMP * EN1 + EN1

0075 SIG = TEMP

0076 EN1 = V * 1.000

0077 TEMP = TEMP * (EN1 + EN1)
0078 EN2 = V * V

0079 D1 = 1.000

0080 D2 = V

0081 DD 250 M = 1, K

0082 A(M + 1) = A(M) * A(M + 1)

0083 SIG = SIG + TEMP * A(M + 1)

0084 EN1 = EN1 + 1.000

0085 EN2 = EN2 + 1.000

0086 D1 = D1 + 1.000

0087 D2 = D2 + 1.000

0088 TEMP = TEMP * (EN1 * D1) * (EN2 / D2)

0089 CONTINUE

250 A(1) = 1.000 / SIG

0090 DD 300 M = 1, K

0091 A(M + 1) = A(M + 1) * A(1)

0092 CONTINUE

0093 GO TO 1K1, (45.0, 1100)

0094 450 RETURN

0096 ENTRY BESK(X, V, N, A, LDIMA)

0097 LEB = 1
ENTRY RBEK1 CALLS EXP(X)*THE K-BESSEL
FUNCTIONS FOR ORDERS V, V+1, V+2, ..., V+N.

ASSIGN 1100 TO IK1
VX=V
KV=1
IF (V.EQ.0.00) GO TO 100
IF (VLT.1000.00) GO TO 1020
TEMP=V-1.00
IF ((TEMP+1000.00).GT.0.00) GO TO 1030
KV=3
GO TO 100
1020 CONTINUE
KV=2
GO TO 100
GO TO 100
1030 CONTINUE
VX=TEMP
KV=4
GO TO 100
1100 CONTINUE
IF (VLT.1000.00) GO TO 1500
C COMPUTE K(V,X) BY INTEGRATION
C
H=DLOG((X+9.7)/(X*.500-1.00))
1110 CONTINUE
VKM=VX(H)
IF (KV+-1.00) VKM=VKM*COSH(DEXP(V*H))
IF (VKM-.100) GO TO 1150
H=.500-H
1150 GO TO 1110
1100 CONTINUE
LMP=2
SIG=500+VKM
S=SIG+H
1190 CONTINUE
HP=500-H
X=HP
SIG=0.00
DO 1200 M=1,LMP
X=XP+H
VKM=VX(XP)
IF ((KV++.100) VKM=VKM*COSH(DEXP(V*XP))
SIG=SIG+VKM
1200 CONTINUE
SIG=SIG+SIG1
SP=S
S=SIG+HP
IF (DABS(SP-S).LE.(EPS*SP)) GO TO 197
H=HP
LMB=LMP+1
1497 CONTINUE
GO TO 197
1500 CONTINUE
EXPX=DEXP(X)
C COMPUTE K(VX,X) BY SUMMATION OF I'S
LMH=3
IF (KV .LE. 0.3) GO TO 1525
AA = 5C - 0LOG(TWDX)
GO TO 1520
CONTINUE
DZ = AA
GO TO 1550
CONTINUE
DZ = (BGA + Q20XPV)**2 - PI**2/V/DSSN(PI*V)**2.5/V
GO TO 1550
CONTINUE

FUNCTIONS OF V MUST BE TRANSFORMED TO FUNCTIONS
OF VX = V - 1.

PGAM = RGAM/V
Q20XPV = Q20XPV/TWDX
LM3 = 2
CONTINUE

COMPUTE DZ BY SERIES APPROXIMATION

AA2 = AA*AA
AA3 = AA2*AA
AA4 = AA3*AA
AA5 = AA4*AA
S = VX
DZ = (TWD15 + AA5 + S5 + TWD3 + (AA3 + S2 + AA2 + S3) + 0.5D0 + AA +
X (S4 + S2**2 + 0.0N3 - S2 - S3 + X +
2500*(S4 + S2**2 + 1) + TWD3 + AA + S2 +
X 0.0N3 + AA - 50.72 +
DZ = 10DZ**5 + 10D3 - S3 + AA + S2 + TWD3 + AA +
X AA2**5 + AA
DZ = -DZ
IF (KV .EQ. 2) GO TO 1550
SIG1 = A(2) + TWD3 = V + A(1)
SIG = DZ*SIG1*EXPDX
GO TO 1560
CONTINUE
SIG = DZ**A(1)*EXPDX
LM3 = 3
CONTINUE

COMPUTE D1

TEMP = (BGA + Q20XPV)**2 - (VX + 2 + D) / (1.0D0 - VX)**2
EN1 = VX + 2 + D
EN2 = VX + VX
EN3 = VX
D2 = EN3
D3 = 1.0D0 - D
D1 = 1.0D0
DO 171 Y = LM3, 1.2
SIG = SIG1*TEM
EN1 = EN1 + 2.0D0
EN2 = EN2 + 1.0D0
EN3 = EN3 + 1.0D0
D1 = D1 + 1.0D0
D2 = D2 + 2.0D0
0103      N3=O3+1,ND
0104      TEMP=TEMP*(EN1/01)*(EN2/02)*(EN3/03)
0105      1570 CONTINUE

C       SIG CONTAINS KIVX,X)
C
0106      S=EXP0X*SIG
0107      IF (KIVX,NF=4) GO TO 1800

C       USE WRONSKIAN TO GET K(V,X)*EXP(X)
C
0108      S=WRONSKIAN(1),SIG1,S)
0109      1800 CONTINUE

C       NOW USE WRONSKIAN TO GET K(V+1,X)*EXP(X)
C       S CONTAINS K(V,X)=EXPX

0200      SP=WRONSKIAN(2),A(1),S)
0201      A(2)=SP
0202      A(1)=S
0203      DR=TWODX*V
0204      DO 1900 M=2,N
0205      DR=DR+TWODX
0206      A(M+1)=DR*A(M)+A(M-1)
0207      1900 CONTINUE
0208      RETURN
0209      WRITE(*,995)
0210      995 FORMAT(*ERROR IN RES I/K X=',E14.5,' V=',E14.5,' N=',I5,' K=',I5,' X=',F15.5)
0211      RETURN
0212      END
SUBROUTINE BESJ(X,V,N,A,LOIMA)

ENTRY RESJ OF SUBROUTINE CALCULATES THE J-BESSEL
FUNCTION OF X FOR ORDERS V,1+V,2+V,...,NV
RIK IS STORED IN A(K+2) (TEMPORARILY)
J(V+K) IS STORED INTO A(K+1)
X>0
0<X<1.0D0
N>0

DIMENSION OF ARRAY A MUST BE AT LEAST MAX(X,N)+1A
LOIMA IS THE DIMENSION OF A SUPPLIED BY THE
USER.

**********************************************************************

ERROR RETURNS

FOR X, V, OR N OUT OF RANGE OR IF DIMENSION
OF ARRAY FURNISHED IS TOO SMALL.

ERROR RETURN INFORMATION INCLUDES
X, V, N, WU, AND LOIMA WHERE MU IS THE
SIZE-2 OF THE ARRAY A NEEDED (MU IS
MEANINGLESS IF ANOTHER PARAMETER IS
OUT OF RANGE, E.G., X=5.1)

DIMENSION A(1)
DIMENSION GA(8),GQ(7)

DATA P1/3.141592653598697805D0/
DATA GA/0.5636944117267350-1.154970280472704941,
X-3.52683982596126598D0,-1.1271344629330988D1,
X6.7016417889220691D0,-2.6910415854447927D0,
X1.3413193873241110D1,-1.6260629618306871D0/
DATA G5/1.1373935797962849D1,4.7985452207286753D0,
X6.037646011100565400,1.355547915497223102,
X-3.62350891965661200,6.4632210253382896D0,
X4.5169233959659386D-1/

DATA P12/1.5707963267948966D0/
EQUIVALENCE (GO,GQGO,GO5MLV)

DATA THRSLV/.789D-3/
DATA EC/2.4093C467E17920C8/
DATA S2/1.64493436684822641,
1.320250/74242N2B546CE272C1/
2.55/1.0362755143370000/
3.53374026856343653141/
4.54/1.59232323711138190D0/

DATA P14D72/74115A5E57E579C55/
DATA THDSJ/.1200/

DIMENSION GOV5'M(4)
ASSIGN 510 TO JY1
ASSIGN 110 TO JY4

CONTINUE

IF(X.LE.0.OR.V.LT.0.OR.V.GE.1.0.OR.V.LT.0) GO TO 995

C
C SET OVERFLOW INDICATOR OFF

C 87 CALL OVERFL(WH)
VP1=V+1.000
0022  IX=IFIX(SNGL(X)).
0023  LMAX=IX+1
0024  MU=LMAX
0025  GO TO JY4,(11C,12C)
0026  110 CONTINUE
0027  IF(LM4,LT,N) MU=N
0028  120 CONTINUE

C LMAX IS THE VALUE OF I FOR WHICH WE ARE ASSURED
C THAT J(I+1,X) IS ON THE TAIL OF THE FUNCTION.
C I.E., LMAX=IFIX(X+1)
C LET LM4=MAX(LMAX,N).

C DEFINE MU TO BE THE POINT FROM WHICH WE
C MUST RECURL TO ASSURE THAT J(I+LL,X)
C IS ACCURATELY DETERMINED

C FOR I=LL,LL+1,...,MU
C WE STORE R(I)=J(I+1,X)/J(I,X) INTO A(I+2).
C THIS AVOIDS THE PROBLEM OF OVERFLOW
C INHERENT IN THE GROWTH OF THE J-FUNCTION
C WHEN RECURLING BACKWARD ON ITS TAIL
C NOTE THAT R(I-1)=X/(2*(I+1)-X*R(I))

0029  TWODX=2.0D0/X
0030  DR=TWODX*(Y+DFLOAT(MU))
0031  FKP1=1.0
0032  FK=0.0
0033  180 CONTINUE

C ITERATE UNTIL FKP1 IS GREATER THAN REGISTER ACCURACY

0034  MU=I+MU
0035  DR=DR+TWODX
0036  FKM1=FK
0037  FK=FKP1
0038  FKP1=DR+FK-FKM1
0039  IF ((FKP1+1.0D0).NE.FKP1) GO TO 180

C THE VALUE OF MU IS NOW WELL DETERMINED

0040  MU=MU+1
0041  M=MU
0042  IF((M+2).GT.LDIMA) GO TO 995
0043  A(M+2)=0.0D0
0044  200 CONTINUE
0045  IF (M.EQ.LMAX) GO TO 250
0046  DR=2*M
0047  M=M-1
0048  A(M+2)=X/(DR-X*A(M+3))
0049  GO TO 200

C STORE JBARI(I) INTO A(I+1) FOR I=0,1,2,...,LMAX+1

0050  250 CONTINUE
0051  A(M+1)=1.0D0/A(M+2)
0052  A(M+2)=1.0D0

C RECURL BACKWARD TO GET JBARI'S
FORTAN IV Level 21

0053  280 CONTINUE
0054   DR=2*(M+1)/X
0055   A(M)=DA*A(M+1)-A(M+2)
0056       M=M-1
0057   IF (M.GT.0) GO TO 280
0058      CLEAR R'S UNDERFLOW MAY OCCUR HERE
0059      LMA=LMB+1
0060      DO 200 M=LMB+1
0061      A(M+2)=A(M+2)*A(M+1)
0062      287 CALL OVERFLL
0063      298 IF (1.EQ.3) A(M+2)=0.00
0064      299 CONTINUE
0065      NORMALIZE "SEQUENCE OF JBAR'S" BY SUMMATION
0066      IF (V.EQ.0.00) GO TO 305
0067      VX=V+2.000
0068      300 CONTINUE
0069      VX=V+2.000
0070      305 CONTINUE
0071      IF (V.EQ.0.00) GO TO 329
0072      DI=1.000
0073      D2=V
0074      EN2=V
0075      EN1=V+2.000
0076      PHI=QD0XPV*RA
0077      ALPHA=PHI+DI
0078      320 CONTINUE
0079      DO 350 M=1, M+2
0080      IF (V.EQ.0.00) GO TO 330
0081      PHI=PHI*(EN2/D2)*(EN1/D1)
0082      D2=D2+2.000
0083      EN1=EN1+2.000
0084      D1=1.000+D1
0085      EN2=EN2+1.000
0086      330 CONTINUE
0087      350 CONTINUE
0088      ALPHA=ALPHA+PHI*A(M+2)
0089      A(M)=A(M)*A(M+1)
0090      A(M+2)=A(M+2)*A(M+1)
0091      CONTAIN JBAR(D1), JBAR(D1), ..., JBAR(LMB+1)
0092      A(M+3), ..., A(M+2) CONTAIN JBAR(LMB+1), ..., JBAR(MU)
0093      NORMALIZE JBAR'S
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0099  M=M+1
0100  DD 460 I=1,M
0101  IF(A1)=A1/ALPHA
0102  460 CONTINUE
0103  500 CONTINUE
0104  GO TO JY1,(S10+110J)
0105  510 CONTINUE
0106  RETURN
0107  995 PRINT 1,X,V+NN+MV+LDIA
C 996 RETURN
C 997 1 FORMAT(5GERROR IN BESL X=1.E14,5,V=1.E14,5,H=1.15,M=1.15,
C X'LDIA'=1.15)
C 1000 RETURN
C 1001 ENTRY BESY(X,V+NN+MV+LDIA)
C 1002 ASSIGN 110J TO JY1
C 1003 ASSIGN 120 TO JY4
C 1004 GO TO 90
C 1005 1100 CONTINUE
C 1006 VX=V
C 1007 JT=DATA(A(11))-THSJ
C 1008 COMPUTE GO AND GI
C 1009 IF(VL+JT+THSJ+1) GO TO 1200
C 1010 VX=V-1*DD
C 1011 IF(VL+JT+THSJ+1) GO TO 1190
C 1011 VX=V
C 1012 GO TO 1200
C 1013 VX=V
C 1014 1100 CONTINUE
C 1015 FUNCTIONS OF V USED IN COMPUTING GO AND GI
C 1016 MUST BE TRANSFORMED TO FUNCTIONS OF VX=1-V
C 1017 Q2DXPV=Q2DXPV/TWODX
C 1018 BGAMSQ=BGAMSQ/MM2
C 1019 1200 CONTINUE
C 1020 COMPUTE GO USING EXPANSION
C 1021 Z=EC+LOG(X/2.000)
C 1022 GO TO 1210
C 1023 G1=2.000/P12
C 1024 BGAMSQ=1.000
C 1025 Q2DXPV=Q2DXPV/MM2
C 1026 GO TO 1230
C 1027 1210 CONTINUE
C 1028 ACSZ=SZ
C 1029 AAFORTH=Z+ACSZ
C 1030 AFIETH=AFORTH/Z/15,DO
C 1031 G1=VX
C 1032 GOSMLV=
C 1033 1111(1.000)*((AFIETH+ACURŽ)+SZ3+2.00+ASQ+S33+ASQ+S2D250)
C 1034 +Z*SZ+2.00+ASQ+SZ3+SZ5)*G1+
C 1035 2.000*Z*SZ3+S2D250+ASQ*SZ2
C 1036 +AFORETH*Z/1.000+P14724/G1+
C 1037 S33*SZ2+ASQ1*G1+Z/P12
```fortran
0132 1220 CONTINUE C COMPUTE G1
0133 G1 = (22DXPV**2/P12)**(GAMSQ**2*(2-UDQ+VX))/(1.0D-J-UX)
0134 CONTINUE C COMPUTE YO FROM SUM(J'S) FORM
0135 C EN3 = VX + 1.0D0
0136 EN2 = VX + EN3
0137 EN1 = VX + 4.0D0
0138 D1 = 2.0D0
0139 D2 = D1 - VX
0140 D3 = D1 + VX
0141 TJE = 0
0142 IF (TJ + GE.0, DO AND VX + GE.0, DO) GO TO 1232
0143 TJE = 1

C Y(VX+1,X) MUST ALSO BE COMPUTED BY A SUM
0144 C THVDX = 3.0D0 - VX/X
0145 PSIZ = -8GAMSQ*2DXPV**2/(P12*X)
0146 PSI1 = GO - 500*G1
0147 CONTINUE 1232
0148 IF (VX + LT. 3.0D0) GO TO 1233
0149 M = 3
0150 YV = GO * A(1)
0151 IF (TJE + GE.0) GO TO 1238
0152 YVPI = PSI1* A(1) + PSI1* A(2)
0153 GO TO 1238
0154 CONTINUE 1233
0155 Z = THVDX*V*X(1) + A(2)
0156 YV = GO * Z
0157 M = 2
0158 YVPI = PSIZ + PSI1*A(1)
0159 CONTINUE 1238
0160 DO 1250 I = M, MU + 2
0161 YV = G1 * A(1) + YV
0162 G = G1
0163 G1 = 2.0D0 + (EN1/D1)*(EN2/D2)*(EN3/D3)
0164 EN1 = EN1 + 2.0D0
0165 EN2 = EN2 + 1.0D0
0166 EN3 = EN3 + 1.0D0
0167 D1 = D1 + 4.0D0
0168 D2 = D1 + 4.0D0 + D2
0169 D3 = D3 + 2.0D0
0170 IF (TJE) 1240, 1250, 1240
0171 CONTINUE 1240
0172 YVPI = YVPI + THVDX*G*X(1) + 5*(G - G1)*A(I+1)
0173 CONTINUE 1250
0174 IF (VX + GE.0, DO) GO TO 1260
0175 Z = YVPI
0176 YVPI = YV = Z + THVDX - YV
0177 YV = 7
0178 GO TO 1400
0179 CONTINUE 1260
0180 IF (TJE + GE.0, DO) GO TO 1410
0181 CONTINUE C NOW COMPUTE Y(V+1).
```

C  WRONSKIAN PROVIDED NOT NEAR A ZERO OF J
C
0182  YVP1=(YV*A(1)-1*LDO/(X*D1))/A(1)
0183  1400  CONTINUE
C     RECURL FORWARD TO GET Y'S (WISE?)
C
0184  A(1)=YV
0185  A(2)=YVP1
0186  G=Y*TWODX
0187  DO 1500  I=2,N
C
0188  G=G+TWODX
C OVERFLOW MAY OCCUR HERE
0189  A(I+1)=G*A(I)-A(I-1)
C1487  CALL OVERFL(LMB)
C1488  IF (LMB.EQ.1) A(I+1)=1.078
0190  1500  CONTINUE
0191  RETURN
0192  END
REFERENCES


A STUDY OF THE HILL-FUNCTION SOLUTION
TO PROBLEMS OF PROPAGATION IN STRATIFIED MEDIA

by

JAMES L. DIETRICH

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AN ABSTRACT OF A MASTER'S THESIS

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ABSTRACT

The problem of determining the reflection and transmission coefficients for a plane wave incident upon an inhomogeneous, stratified region of finite thickness is solved in terms of Hill functions. The effect that parameters of the problem have upon cost and accuracy of the solution is discussed from both analytical considerations and numerical results obtained from solving the problem for a region with a linear dielectric profile. Expressions for the reflection coefficient at low frequencies are obtained using the Hill-function formulation, as well as the familiar Airy-function formulation.