ADVANCEMENTS IN ARCH ANALYSIS AND DESIGN DURING THE AGE OF ENLIGHTENMENT

by

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Abstract

Prior to the Age of Enlightenment, arches were designed by empirical rules based off of previous successes or failures. The Age of Enlightenment brought about the emergence of statics and mechanics, which scholars promptly applied to masonry arch analysis and design. Masonry was assumed to be infinitely strong, so the scholars concerned themselves mainly with arch stability. Early Age of Enlightenment scholars defined the path of the compression force in the arch, or the shape of the true arch, as a catenary, while most scholars studying arches used statics with some mechanics to idealize the behavior of arches. These scholars can be broken into two categories, those who neglected friction and those who included it. The scholars of the first half of the 18th century understood the presence of friction, but it was not able to be quantified until the second half of the century. The advancements made during the Age of Enlightenment were the foundation for structural engineering as it is known today. The statics and mechanics used by the 17th and 18th century scholars are the same used by structural engineers today with changes only in the assumptions made in order to idealize an arch. While some assumptions have proved to be incorrect, many correctly interpreted behavior and were able to formulate equations for design and analysis that were successfully used to create arches that were structurally sound and more efficient than arches designed by empirical methods. This insight into design during the 18th and 19th centuries can help modern engineers better analyze and restore arches from this era and protect our architectural and engineering history.

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Chapter 1 - Introduction

In the built environment, masonry arches have been used for a few thousand years, due to their ability to span larger distances than masonry post and beam construction. Until the age of enlightenment, arches were built based on rules of proportion made after the success or failure of previous arches. On many occasions, an arch would be constructed multiple times and adjusted after original attempts led to its collapse.

The Age of Enlightenment was the beginning of widespread study of structural mechanics. The scholars of this period laid the foundation for the principles of structural engineering used today. Their concepts of statics and material mechanics have been used and expanded on by the generations that followed them.

The scholars during the Age of Enlightenment that studied masonry arches used the rules of statics to analyze and formulate rules for stability. They made theories based on their observations of failure in existing masonry arches. They focused on the shape and stability of arches as they assumed that the masonry would be strong enough to support whatever loads were imposed upon it.

Early Age of Enlightenment scholars concerned themselves with the true arch shape, the shape that line of compression force, or thrust, takes within the arch. This line of thrust, superimposed on an arch section was used by subsequent scholars to show the failure mechanisms of various arches. The failure mechanism of arches was the main topic for debate and publication among scholars. Many of the published works dealing with masonry arches were very similar due to the numerous factors that affected analysis and how the scholars chose to approach and adjust those factors.

Methods and assumptions in analyzing friction changed significantly over the Age of Enlightenment. Early Age of Enlightenment scholars either assumed frictionless voussoirs or assumed that friction was significant enough that voussoirs could not slide along each other. Most scholars neglected the presence of mortar in the joints if it was there. Not until the second half of the 18th century was the force of friction understood to be some proportion of the force normal to the plane of contact.

The theories on masonry arches during this time show the growth of statics and material mechanics and the progression from empirical design to structural engineering. They gave subsequent scientists a solid foundation to build upon.

Scholars during the Age of Enlightenment were able to apply the advancements made during the era to analyze structures built before the time or to modify the design of new structures to be more efficient.

While the design rules of the Age of Enlightenment have been built upon and are not used in the same form today, understanding them will be beneficial in understanding how arched structures of the Age of Enlightenment and shortly after were built.

The arch theories during the age of enlightenment can be broken down three categories: the catenary, theories neglecting friction, and theories including friction. These theories are discussed in chapters 2, 3, and 4, respectively. Also included are discussions of the arch theories applied to the design and analysis of two prominent cathedrals, St. Paul’s Cathedral in London and St. Peter’s Cathedral in the Vatican.

Chapter 2 - Arch Theory and Design prior to the age of Enlightenment

The period between when the Middle Ages ended near the close of the 15th century and the start of the Age of Enlightenment during the 17th century is referred to as the Age of Discovery. During this many scientists began to study the mechanics of structures. They were curious as to why things worked, not simply satisfied with the knowledge they did. Prior to this time, as most of the arches built stood the test of time, very little thought was put into researching the mechanics of arches.

Leonardo Da Vinci (1452-1519)

Leonardo da Vinci studied arches long before the Age of Enlightenment, but his deductions and theories, while not mathematical in nature, did give a basis for the scholars of the Age of Enlightenment. His definition of an arch states:

> An arch is nothing but a strength caused by two weaknesses; that is why an arch in buildings is composed of two quarter-circles; these quarter-circles, each very weak in itself, wish to fall, and opposing each other's ruin, convert weakness into a single strength (Benvenuto, 1991, p. 309).

This definition suggests an incomplete arch is weak and unstable and only a fully complete arch is stable and capable of carrying a load.

Da Vinci's main contribution to the area of arch mechanics came from a short proposition, "the arch will not break if the outer arc chord does not touch the inner arc (Benvenuto, 1991, p. 309)" as shown in Figure 1.

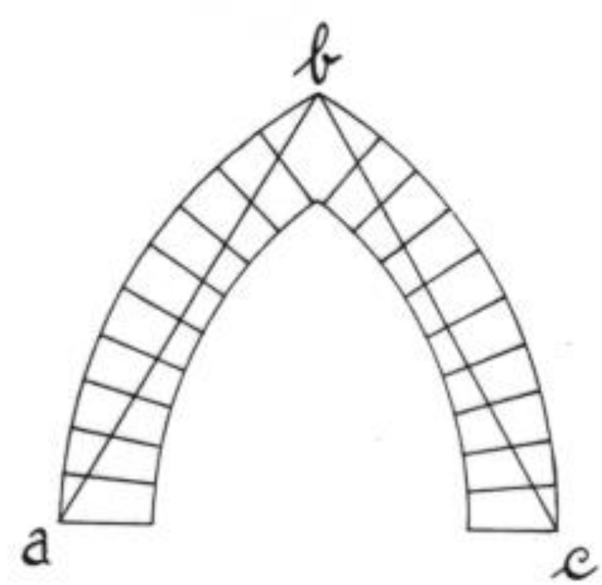

Figure 1: da Vinci's proposal of arch stability (Benvenuto, 1991, p. 310)

This method of da Vinci's shows a primitive understanding of the thrust line in an arch; the theoretical line through the arch that represents the path of the compression forces. This theory on stability was built upon by many of the scholars of the Age of Enlightenment.

1643 – Francois Derand (1588-1644)

In 1643, Francois Derand published his treatise *L'architecture des voutes*. Specifically, his specialty was stereotomy, which is the technique of cutting solids, such as stones, to specified forms and dimensions. In chapter 7 of his treatise, he outlines a rule for determining the size of the abutments to a masonry arch. Derand was the architect of the Église Saint-Paul-Saint-Louis in Paris (Montclos, 2009) and it is likely he used this rule in determine the width of the walls supporting the arches and vaults. This rule was commonly used to find abutment sizes for arches and vaults during the 17th and 18th centuries. The rule's simplicity was a likely cause of its popularity, as architects and builders could easily use it. His rule, using the geometry and notations of Figure 2, is:

> Given the circular vault *ABCD*, marked *P* at the center; divide it into three equal [parts] at points *B*, *C*; from one of these thirds, *CD*, draw the straight line *CDF*; and taking the same point *D* as center and opening [your] compass as far as the chord *CD*, make the arc *EF* below and out of the same center. And by point *F*, where the said arc cuts line *CF*, draw the plumb line *FG*; it will be the outside of the wall which will carry vault *ACD*; so the thickness of the said can be comprised between the line *EH* and *FG*, and it will be sufficient to resist the thrust of the vault, as practice and experience have shown. If the vaults are depressed [segmented or shallow], like *IKA* in figure *Q* (Figure 2), using the same construction, the wall will be thicker; this is necessary because the thrust of these vaults is stronger than that of vaults which have their full center, as in the foregoing, and much stronger than that of vaults which are raised or ogival . . . Now it is not always necessary that the aforementioned thicknesses found by practice must be maintained over the whole extension of the walls that carry the vaults; it will be sufficient to conserve them at the points of the principal arches, where they will form juts which are commonly called flying buttresses. (Benvenuto, 1991, p. 314)

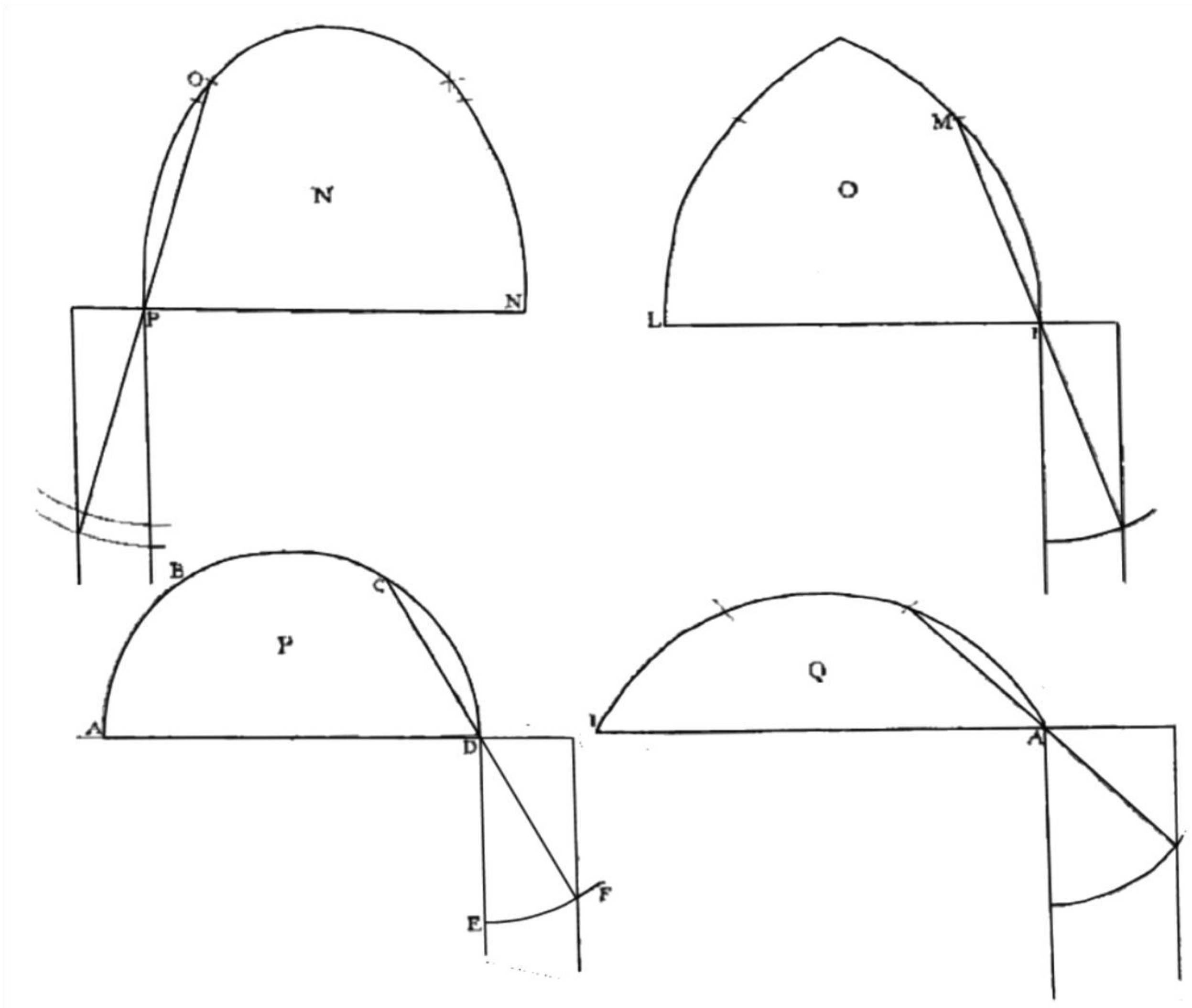

Figure 2: Derand's Rule for arches of various shapes (Benvenuto, 1991, p. 314)

The points *A*, *B*, *C*, and *D*, are on the intrados of the arch. Arch thickness does not play a part in the rule, which is wholly based on geometry, nor does the rule give a way of determining the arch thickness. Derand assumed, as did many others of the time, masonry was infinitely strong, and a sufficiently sized abutment would invoke stability on the arch.

Figure 2 shows that a shallow arch requires a larger abutment. This is a result of the larger force being transferred horizontally from the arch into the abutment, which necessitates a larger abutment to support the horizontal force. Deeper arches, where the force being transferred to the abutment is largely vertical, will need a narrower abutment.

Derand's rule has been said to be both correct and incorrect by those who came after Derand (Benvenuto, 1991). Scholars of the 19th century criticized the rule to some extent for being incorrect, however the rule was on the conservative side and was not criticized for being unsafe. Jean-Baptiste Rondelet, an architect of the 18th and 19th centuries, criticized the rule for over-built walls. (Benvenuto, 1991)

This rule has been attributed to Derand, although some historians believe the rule was in place well before his time. Viollet le Duc, a French architect of the 19th century believed that evidence of application of the rule could be seen in gothic cathedrals and was therefore in place prior to Derand (Benvenuto, 1991). Derand has been given credit based on formalizing the rule in a manner that was easily applicable to design.

Chapter 3 - The Catenary

Near the end of the 17th century many scholars were trying to define the best form of an arch. Robert Hooke was the first known scholar to theorize that a catenary, or shape of a hanging chain, could be used to define an arch. Prior to this, mathematical scientists studied the shape of a hanging chain in an attempt to define its shape. Many believed that a hanging chain formed a parabola but Huygens disproved this in 1646 (Heyman, 1998). Later it would be discovered that the correct definition of catenary was that of a hyperbolic cosine (1998).

1675 – Robert Hooke (1635-1703)

In 1675, Robert Hooke published an anagram within his book on helioscopes. Translated, the anagram reads, “As hangs the flexible line, so but inverted will stand the rigid arch” (Heyman, 1972, p. 76). Hooke’s anagram describes the relationship between the shape a hanging line or chain takes when being held at each end and a rigid arch. Both the arch and the hanging chain must be in equilibrium, and the forces are simply reversed; since the chain can only carry tension, the arch therefore acts under compression. This was Hooke’s solution to “the true mathematical and mechanical form of all manner of arches for building” (1972, p. 76). This shape, the shape of the chain under its own weight, is a catenary. Hooke’s anagram did not provide means of determining this shape other than observing the shape of the hanging chain.

Hooke’s publication of his finding in anagram form was not uncommon in the competitive scientific climate of the era (Heyman, 1998). Hooke would not have wanted his findings to provide any clues to other scientists who were trying to find the solution to the correct shape of an arch. Since Hooke likely did not have any mathematical proof to his solution or a mathematical equation of the shape, he would have hidden his findings so others could not reach the solution before him because he wanted credit for discerning the correct behavior and shape.

Publishing his anagram did give him credit for his discovery: "the statics of a hanging cord in tension were the same as that of an arch in compression" (Heyman, 1998, p. 41), but his anagram was not deciphered until after his death, by Richard Waller in 1705 (1998). Hooke's idea signifies that the shape a string takes under a set of loads, if rigidified and inverted, illustrates a path of compressive forces for an arch structure to support the same set of loads. This shape of the string and arch is a funicular shape for these loads.

Until the middle of the 18th century, it was believed a hanging chain formed a parabola. While a hanging chain can form a parabola under specific loading, a parabola does not accurately describe the shape of chain of consistent weight hanging under its own weight. In Figure 3 the difference between a catenary (or chain), a parabola, and a semi-circle can be seen. The catenary falls between the parabola and semi-circle.

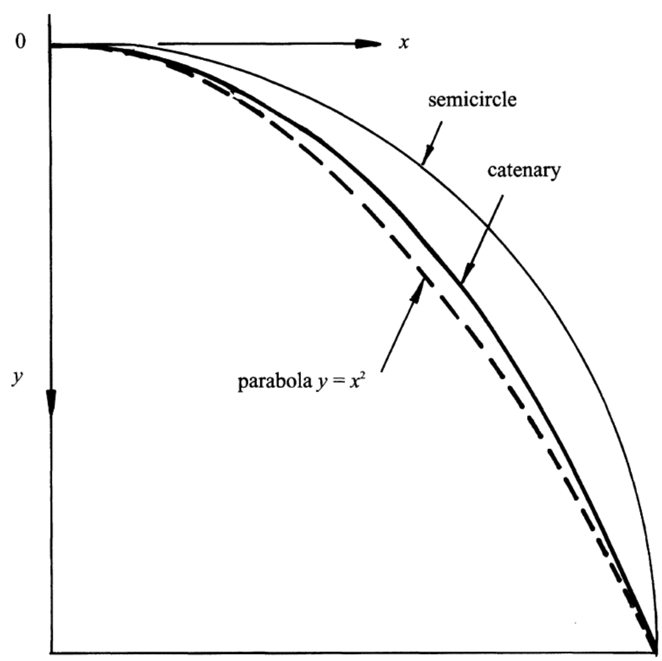

Figure 3: Catenary vs. Parabola and Semi-Circle (Heyman, 1998, p. 43)

Another factor determining the shape the chain takes is the uniformity of weight of the chain. It is possible to conclude a line can form any shape if weighted properly (Benvenuto, 1991). By adjusting the weight and uniformity of the chain, the shape of the chain will change. Similarly, by changing the loading on an arch or the weight of the arch, the thrust line within the arch will change.

An equation for the shape of the catenary was not derived until 1690. James Bernoulli challenged Gottfried Leibniz, Christiaan Huygens, and John Bernoulli to derive this equation. They found the shape of the hanging chain could be described by the equation of a hyperbolic cosine (Lockwood, 1961).

1697 – David Gregory (1659-1708)

Twenty years after Hooke's discovery David Gregory used the same principles in describing the stability of arches. In a letter Gregory wrote in 1697, he relates the catenary shape to the stability of an arch. In the letter, he writes:

> "In a vertical plane, but in an inverted situation, the chain will preserve its figure without falling, and therefore will constitute a very thin arch, or fornix; that is, infinitely small rigid and polished spheres disposed in an inverted arch of a catenaria will form an arch; no part of which will be thrust outwards or inwards by other parts, but, the lowest part remaining firm, it will support itself by means of its figure … And, on the contrary, none but the catenaria is the figure of a true legitimate arch, or fornix. *And when an arch of any other figure is supported, it is because in its thickness some catenaria is included.* Neither would it be sustained if it were very thin, and composed of slippery parts." (Heyman, 1972)

The hanging chain as a series of small spheres is shown in Figure 4. It's shape, line *BAC*, reflected across line *DE*, is the shape of the ideal arch, line *BFC*.

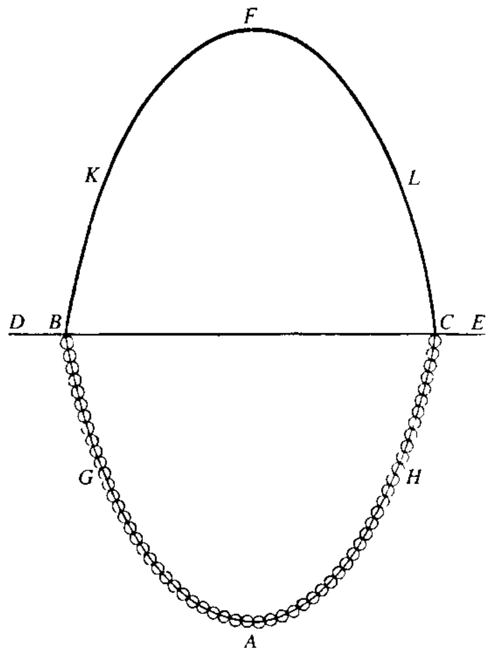

Figure 4: Hanging Chain/Arch as a series of small spheres (Poleni, 1748, p. 34)

The italicized sentence was added by Ware, another scholar on arch mechanics, when he translated the letter in 1809 (Heyman, 1972). Ware meant that if the catenary, or thrust line, of an arch with specific loading fits within the cross section of the arch then the arch is stable, but if any part of the catenary falls outside of the arch, then it is not. While the catenary may be the ideal shape for an arch, an arch does not need to be shaped as a catenary to be stable, the catenary must fit inside it. The catenary is the direction of force within the arch. Gregory also notes, while the chain represents a very thin arch that will theoretically stand up, this is only under perfect conditions and thus the thickness of the arch must be greater than the thrust line.

Chapter 4 - Theories Neglecting Friction

Many theories on the behavior of arches were published starting at the very end of the 17th century. In order to come to these theories, scholars had to simplify the arch into a system more easily analyzed. At this time, friction was not understood well and was ignored for simplicity. The theories neglecting friction laid the groundwork for future methods but were not themselves accurate.

1695 – Philippe de la Hire (1640-1718)

In his *Traite de mécanique* of 1695, de la Hire proposes a way to determine the weight of voussoirs, a wedge shape element used in building an arch, needed to stabilize an semicircular arch of a given shape. In this method, de la Hire assumes the planes of contact between the voussoirs are infinitely smooth, therefore neglecting friction. He also assumes the forces between the voussoirs act perpendicularly to the plane of contact.

His propositions give geometric rules for determining the weights of the voussoirs. He applies the weights of all the voussoirs at their respective centers of gravity, then draws a horizontal line through the center of gravity of the keystone (line *EP* in Figure 5). The lines of contact between the voussoirs are then extended to meet this horizontal line in one direction and the center point of the arch in the other.

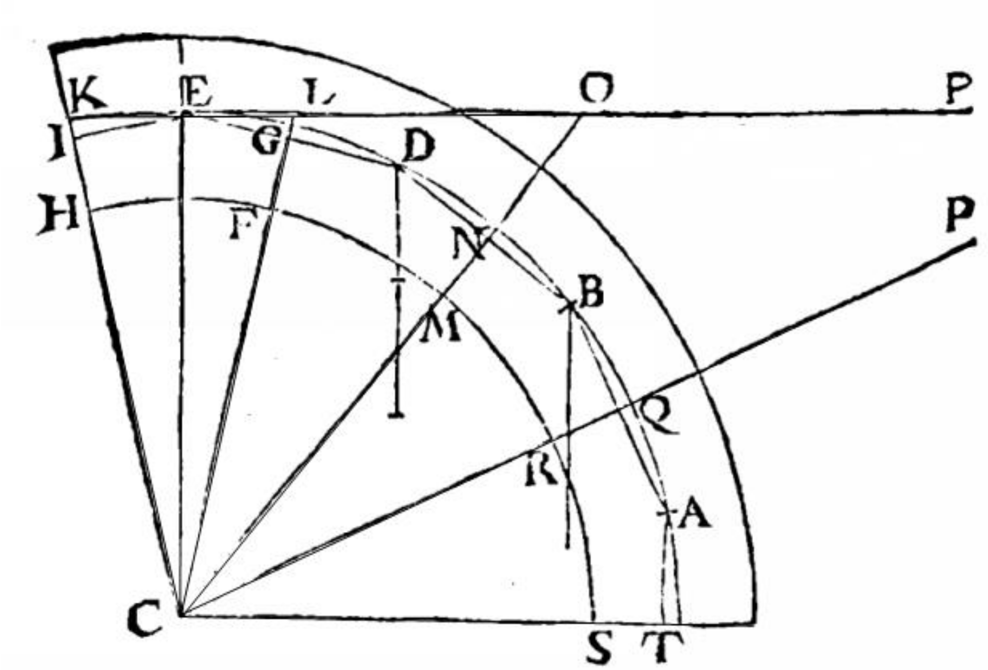

Figure 5: De la Hire's model for calculating voussoir weight (Hire, 1695, p. 466)

The triangles now drawn around each voussoir created by the extended lines of joints on either side of the voussoir and the line *EP* can be used to find the required weight of each voussoir. The ratio of lengths of each of these sides is used to find the ratios between the weight of the voussoir and the forces acting on the voussoir at each joint.

Starting with an assumed weight for the keystone, the ratios can be used to find the forces acting perpendicular to each end of the keystone which can then be used on the next voussoir to find its required weight by ratios, repeating the process to the last voussoir. The forces acting between the voussoirs act along the line between their centers of gravity. Each voussoir has three forces that require balancing; its self-weight and the force it exerts on the two voussoirs adjacent to it. Starting with the keystone, the force acting upon its neighboring voussoir is found by the following ratio:

$$Q: F_l: F_r = KL: CK: CL \qquad (1)$$

Here, Q, F_l, and F_r are all forces acting from the center of mass of the voussoir. F_l acts perpendicular to the left joint and F_r acts perpendicular to the right joint. K is the point of the intersection of the line of the left joint of the keystone and the horizontal line drawn through the center of mass of the keystone. L is the point of the intersection of the line of the right joint of the keystone and horizontal line drawn through the center of mass of the keystone. KL represents the length between points K and L. Point C is the center of the arch and CK and CL represent the distance between points C and K and C and L, respectively. In this scenario the weight of the keystone and the length of each segment are known, so the forces can be calculated. Once the forces the keystone exerts on the adjacent voussoirs are known, the same process can be applied for the voussoirs adjacent to the keystone. The forces for the voussoir adjacent to the right of the keystone are shown in Figure 6.

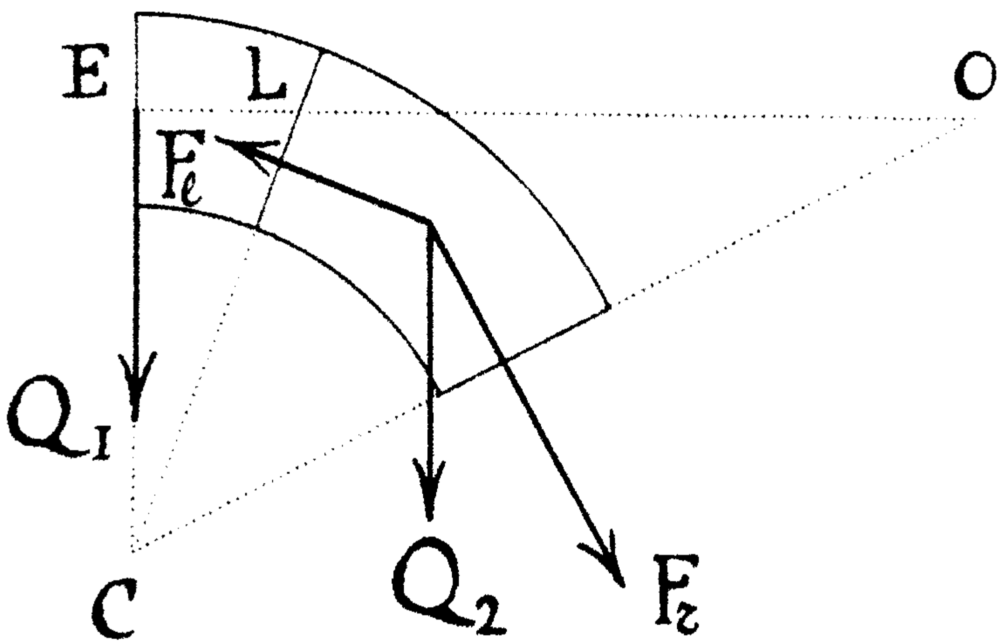

Figure 6: Forces acting on a voussoir (Benvenuto, 1991, p. 326)

In this case, F_l is known. It is F_r from the keystone calculation. A similar ratio to the keystone ratio can be used to find Q_2 and F_r.

$$Q_2 : F_l : F_r = LO : CL : CO \qquad (2)$$

LO, CL, and CO are the lengths of the line segments shown in Figure 5. The same method as was used for the keystone can be used here. This method can be repeated across the arch span, with F_r being substituted as F_l in each subsequent voussoir and the line segments measured from Figure 5.

The last voussoir is not enclosed in a triangle because the edge line of the voussoir is parallel to the horizontal line formed through the center of gravity of the keystone. This implies the weight of the last voussoir must be infinite. However, de la Hire explains friction would solve this. This makes his assumption of frictionless joints incorrect, as his model demonstrates that with this assumption no stone is heavy enough to make the arch stable. While de la Hire's process for designing arches was incorrect, his model was examined and expanded on by other scholars also looking to explain the behavior of arches.

1729 – Pierre Couplet (1670-1744)

Pierre Couplet presented a memoir to the Academiè Royale des Sciences in 1729 titled *De la poussee des voutes*. Similar to de la Hire's 1695 memoir, he considers a frictionless arch by determining the required weight of each voussoir based on the known weight of the keystone (Benvenuto, 1991).

Using Figure 7 to describe Couplet's model, the lines of the joints separating the voussoirs will, if extended, pass through point *O*. Point *O* is the point at which the lines of the joints between the abutments and arches, and the vertical line drawn from the center of the keystone, intersect. These lines, *SO*, *GO*, and *λO* are shown in Figure 7.

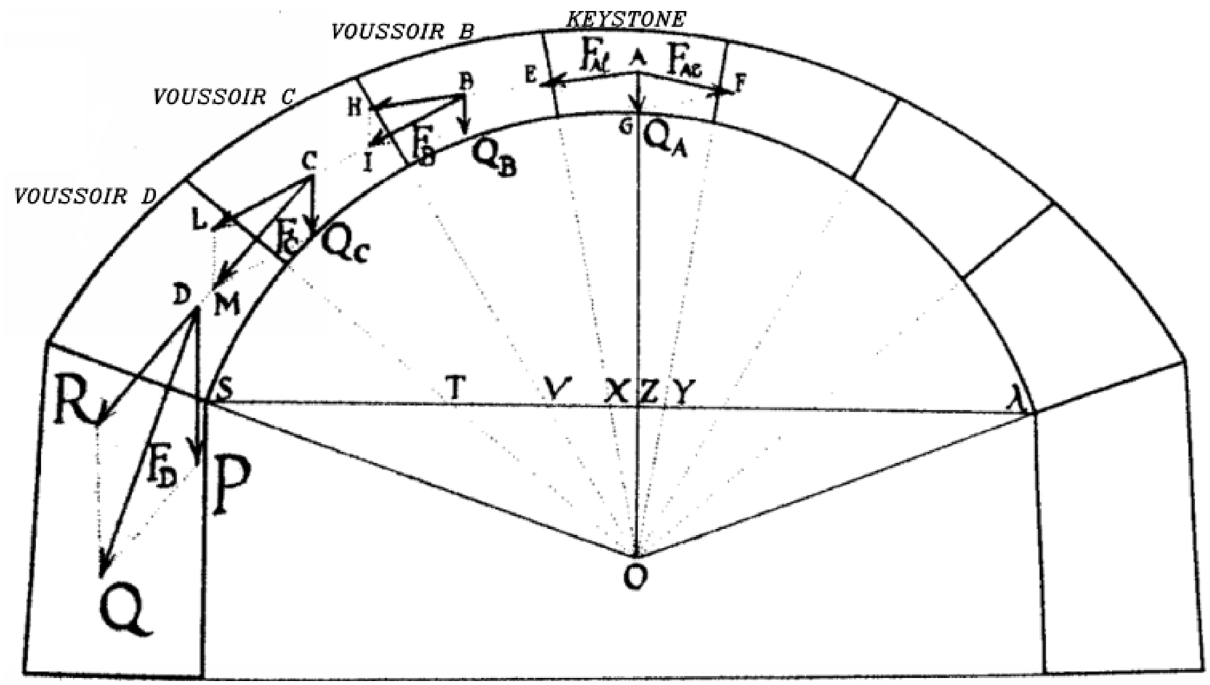

Figure 7: Couplet's frictionless arch (Benvenuto, 1991, p. 339)

An improvement over Philippe de la Hire's method, Couplet's method for determining the weight of the voussoirs can be utilized for arches of any shape and for arches of non-uniform thickness, as shown in Figure 7. He starts by finding forces F_{Al} and F_{Ar} along lines *AE* and *AF*, respectively, based on the known weight of the keystone, Q_A. F_{Al} and F_{Ar} are the forces acting perpendicular to the left and right joints of the keystone, respectively. Point *A* is the center of gravity of the keystone and lines *AE* and *AF* are perpendicular to the joints between the keystone and adjacent voussoirs. Once these forces, F_{Al} and F_{Ar}, are determined, the weight of the next voussoir can be determined. Point *B* is the center of mass of *voussoir B*. The weight of this voussoir can be applied vertically from this point, and per Couplet, the force F_{Al} is transposed from point *A* to point *B* (*AE* is equal to *BH*). To find the resultant force from *voussoir B* acting on *voussoir C*, F_B; which acts perpendicular to the joint between these two voussoirs; and the weight of the voussoir, Q_B, the force triangle, *BHI*, is balanced based on the triangle's geometry

(Benvenuto, 1991). This process continues to find the weights of all the voussoirs. This method works correctly unless the joints at the abutments are horizontal, in which case Couplet's theory has the same problem as de la Hire's, and the weight of the final voussoir is found to be infinite. Couplet concurs with de la Hire that the last stone cannot move due to friction, and therefore a voussoir of infinite weight is not needed – the loadbearing system as a whole is in equilibrium (Benvenuto, 1991).

Couplet provides a simpler method of finding voussoir weights and thrusts by examining the geometry in Figure 7. As shown, a line is drawn between point *S* and point λ. Then, segments *XY*, *VX*, *TV*, and *ST* on this line are found to be proportional to the weight of the voussoir they represent (Benvenuto, 1991). As the weight of the keystone is already known and this weight corresponds to segment *XY*, the rest can be found through simple ratios.

In the same method, the lines *OX, OV, OT,* and *OS* can be used to find the thrust of each voussoir on the block below it (Benvenuto, 1991). The thrust of the keystone on the adjacent voussoir must be found from the weight of the keystone using geometry. Then by ratios, the rest of the thrusts can be found

In this memoir, Couplet also briefly discusses centering. He states only the voussoirs within 30° of horizontal will not put force onto the centering. Rather, only the voussoirs within 60° of the crown of the arch will be supported by the centering (Benvenuto, 1991).

1773 – Charles-Augustin de Coulomb (1736-1806)

In 1773, approximately 50 years after Couplet, Coulomb submitted his memoir *Sur une application des règles de Maximis & Minimis à quelques Problemes de Statique., relatifs à l' Architecture* to the Royal Academy of Science in Paris who published it in their *Memoires de Mathematique & de Physique* in 1776. Section 27 of this memoir considers arches neglecting both friction and cohesion of the joints. The arch he considers in this problem is one of infinitesimal thickness and the thrust, P, acts perpendicular to the joint at point M, shown in Figure 8.

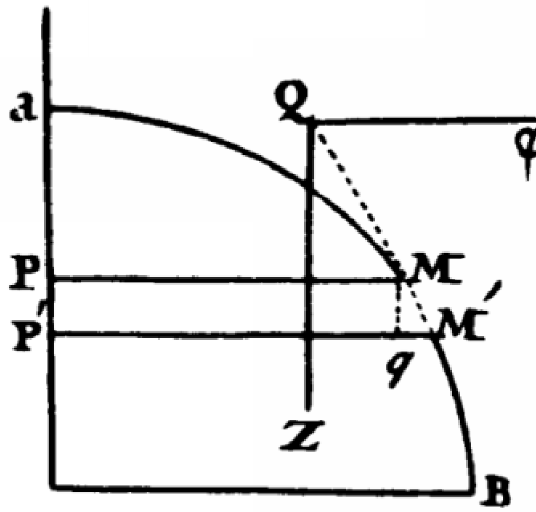

Figure 8: Geometry of Coulomb's Arch statics (Coulomb, 1776)

The thrust at this location can be represented in a horizontal component and a vertical component. Coulomb refers to the horizontal component, *Qφ*, as *π*, and the vertical component, *QZ,* as *Φ*. For equilibrium, all forces between a and M must be resolved in the thrust at M. Therefore, it is possible to solve for π and Φ. He additionally defines "*aP = y*, *PM = x*, *Mq* = dy, *qM'* = dx" (Heyman, 1972, p. 61). With these defined, Coulomb, using calculus, states:

$$P\frac{dx}{ds} = \pi \tag{3}$$

And

$$P\frac{dy}{ds} = \Phi \tag{4}$$

Dividing (3) by (4) gives the equation for the slope of an arch. This equation is:

$$\frac{dx}{dy} = \frac{\pi}{\Phi} \quad (5)$$

He gives this equation as "the shape of an arch when acted upon by any given forces" (Heyman, 1972).

He then uses this equation for the shape of an arch in three corollaries. The first corollary is the case of an arch without thickness loaded under its own weight. For this case, the only forces acting on section *mM* are the horizontal thrust at a and the weight of the arch from a to *M*. Equation (5) becomes:

$$\frac{dx}{dy} = \frac{\mathrm{A}}{\int p * ds} \quad (6)$$

Here *A* is the horizontal thrust and *p* is weight per unit length of the arch (Heyman, 1972). Using this equation, the uniform weight of an arch needed for a given curve is found, or inversely, the curve needed for a given weight of arch is found. As any built arch will have thickness, this method is only theoretical.

Coulomb's second corollary now considers the arch having a thickness. This thickness, which Coulomb defines as the variable *z*, is the distance between *M* and *m* in Figure 9: .

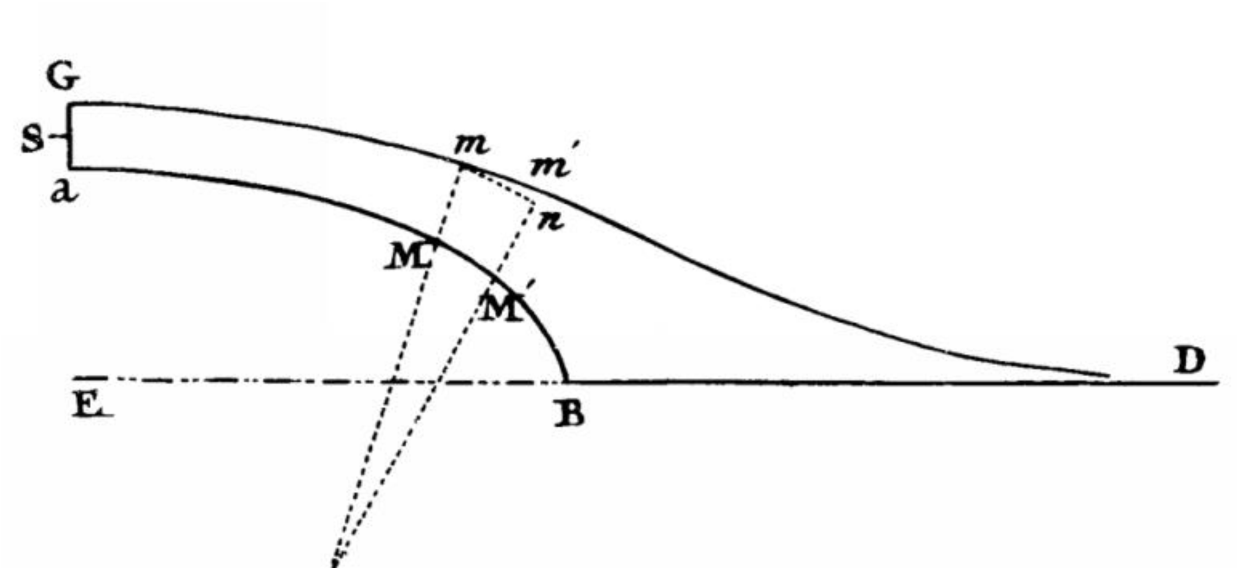

Figure 9: Coulombs geometry for an arch of any shape (Coulomb, 1776)

For calculation purposes, Coulomb defines the area of section *MM'mm'* as:

$$MM'mm' = \frac{zds(2R + z)}{2R} \quad (7)$$

This equation for area can be used in equation (6) as a substitute for weight per unit length of the arch. In corollary 1, Coulomb found the equation for an arch of an infinitesimal thickness and *p* was the weight of that arch per unit length. With an arch of finite thickness, the arch area, equation (7), is substituted for arch weight in equation (6), and, assuming the arch is homogeneous and of constant thickness the equation becomes:

$$\frac{\mathrm{dx}}{\mathrm{dy}} = \frac{\mathrm{A}}{\int \frac{\mathrm{zds(2R + z)}}{\mathrm{2R}}} \quad (8)$$

This can be rearranged to:

$$\mathrm{Ad}\left(\frac{\mathrm{dy}}{\mathrm{dx}}\right) = \frac{\mathrm{zds(2R + z)}}{\mathrm{2R}} \quad (9)$$

Coulomb states the radius can be written in terms of the change in ds, dx, and dy:

$$R = \frac{ds^3}{d(dy)dx} \quad (10)$$

Which rearranged gives:

$$d\left(\frac{dy}{dx}\right) = \frac{ds^3}{Rdx^2} \quad (11)$$

Combining equations (9) and (11):

$$\frac{A}{R}\left(\frac{ds}{dx}\right)^2 = \frac{z(2R + z)}{2R} \quad (12)$$

This equation can also be rearranged to give:

$$R + z = \left[R^2 + 2A\left(\frac{ds}{dx}\right)^2\right]^{1/2} \quad (13)$$

Which Coulomb gives as the equation for any arch under gravity loading (Heyman, 1972).

Chapter 5 - Theories Including Friction

The scholars studying structural mechanics and arches eventually realized the need to include friction in the behavior and stability of arches. Some early theories, such as that of de la Hire in 1712, include friction as an infinite force while theories from the second half of the 18th century were aware of friction's proportional relationship with the contact force.

1712 – Philippe de la Hire (1640-1718)

In 1712, de la Hire submitted another memoir, *Sur la construction des voûtes dans les édifices,* discussing a theory of the behavior of semi-circular arches and a method of design. Unlike his memoir of 1695, he did not neglect friction but rather assumed friction was great enough to prevent sliding between the voussoirs. In this memoir he focuses on abutments in addition to the arch. De la Hire wrote, "when the piers of an arch are too weak to carry the thrust, that arch breaks at a section somewhere between the springing and the keystone" (Heyman, 1972, p. 83). His theory is based on his observations of vaulted structures and their faults or failures. He states a semi-circular arch will break at 45° between the impost and keystone, creating a hinge in the arch. After the arch breaks, the top section drops due to its weight and pushes the two other sections of the arch and abutments apart and they will rotate around the outer base of the abutments (Benvenuto, 1991, p. 332). This movement is shown in Figure 10.

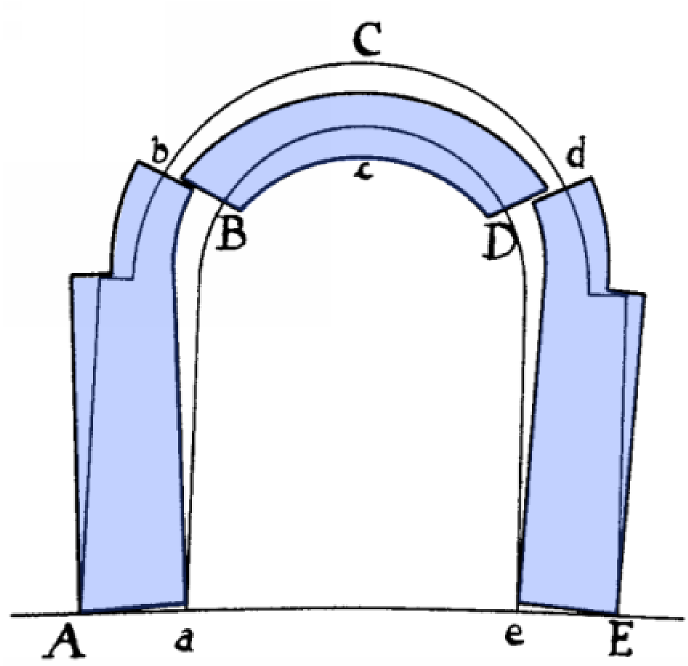

Figure 10: De la Hire's theory on arch movement (Benvenuto, 1991, p. 333)

As the two portions of the arch, including the abutments, rotate around their bases, only the intrados of the lower arch portion is in contact with the top arch portion. The thrust of the arch is transferred through this point. In order to determine the stability of the structure, de la Hire sums the moments about the outer edge of the abutment (Point *A* in Figure 10, Point H in Figure 11) (Heyman, 1972, p. 83). De la Hire then derives a method to determine the minimum width of the abutment needed to stabilize the arch. His geometric method uses the diagram in Figure 11.

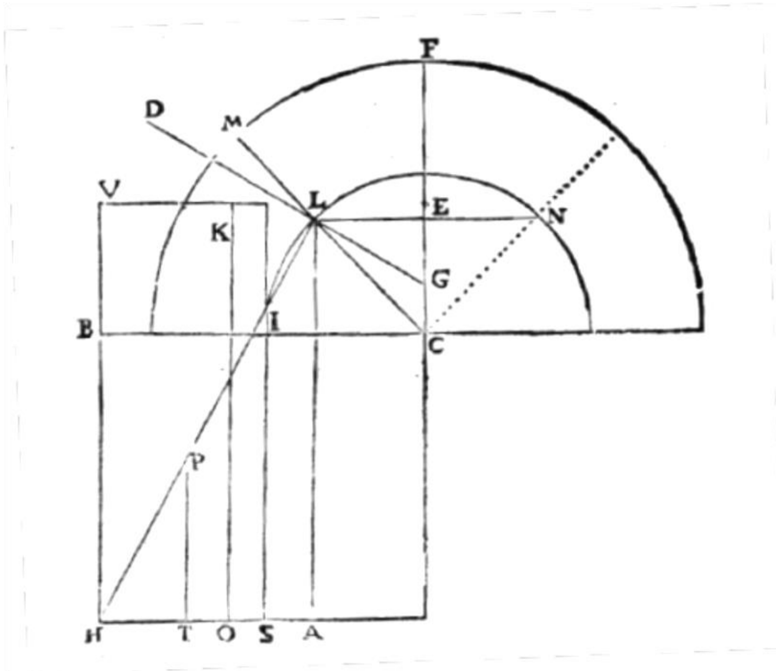

Figure 11: De la Hire's geometry for determining abutment width (de la Hire, 1713, p. 71)

If the weight of the top portion of the arch is Q_c, then the force of the thrust acting perpendicular to the 45° break at the point *L* is $Q_c * \sqrt{2}$. A line is then drawn from point L where the arch breaks to point *H*, the outer edge of the abutment and another line perpendicular to this line at point *L*. The force of the thrust is broken down into components parallel to line *LH* and *LD*. He then applies the weights of the abutment and lower arch to a point *T*, the horizontal center of mass of the abutment. These forces are seen in Figure 12, where *D* is the portion of $Q_c * \sqrt{2}$ perpendicular to the lever arm *LH* (Benvenuto, 1991, pp. 332-333).

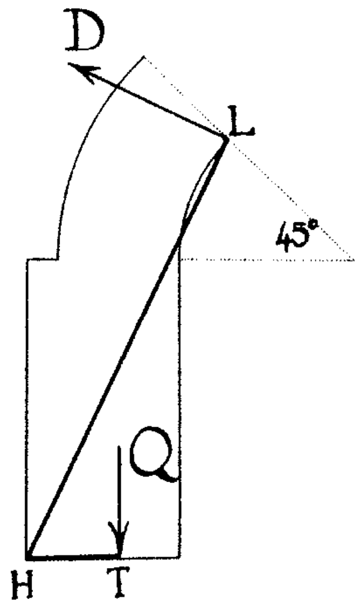

Figure 12: Abutment Forces (Benvenuto, 1991, p. 333)

Benvenuto describes the method de la Hire uses from here as intricate and almost obscure (Benvenuto, 1991, p. 332). Indeed, de la Hire's process for determining the required weight of abutment requires significant manipulation of variables. He notes the mechanics determines "the ratio between Q_c and D is equal to the ratio between LG and CG" (Benvenuto, 1991, p. 334). CG and LG can be determined based on the following equations:

$$CG = CE - \left(\frac{LE}{LA}\right)(HS - SA) \tag{14}$$

$$LG = \sqrt{LE^2 + \frac{LE^2}{LA^2}(HS + SA)^2} \tag{15}$$

Combining these equations with the known ratios and having a known Q_c; weight of 45° of the arch; the force D is determined.

$$D = Q_c \frac{(CE * LA) - (LE * HS) - (LE * SA)}{LE * \sqrt{LA^2 + (HS + SA)^2}} \qquad (16)$$

Another force needed is *Q*. De la Hire assumes the arch and abutment are equivalent densities and thicknesses and therefore can be determined based on area. The weight *Q* can be found by the following equation.

$$Q = (HB * HS) + \frac{\left(\frac{1}{2} HS + TO\right) Q'_c}{\frac{1}{2} HS} \qquad (17)$$

Here the first portion of equation (17) is the weight, or area, of the abutment and the second portion is the lower portion of the arch adjusted to act through the center of mass of the abutment. Q'_c is the weight of the lower portion of the arch. If the lower portion of the arch has the same density as the upper portion, Q'_c will equal Q_c.

Now de la Hire states based on the law of the lever:

$$D * HL = Q * HT \qquad (18)$$

at the limit state of stability for the arch. To ensure stability the right side of the equation must be greater than or equal to the left. The distance *HL* is found by equation (19) and by substituting all known quantities into equation (18), equation (20) is derived.

$$HL = \sqrt{(HS + SA)^2 + LA^2} \qquad (19)$$

$$\frac{1}{2}(HB * LE * HS^2) + \frac{1}{2}\left(Q'_c * LE * HS\right) + \left(Q'_c * LE * TO\right) = (Q_c * CE * LA) - (Q_c * LE * HS) - (Q_c * LE * SA) \qquad (20)$$

All the variables in this equation except *HS*, the abutment thickness, are predetermined, the abutment thickness needed to stabilize the arch is found.

While de la Hire gives these equations in relation to a semi-circular arch, he states the equations can also be used for a variety of arch shapes. All of the variables can also be applied to an arch of any shape so the process of how it was derived will work for arches of other shapes. However, his assumption of the hinging location on the arch is incorrect so therefore the process as it relates to any arch is incorrect.

1729 – Bernard Forest de Belidor (1698-1761)

Belidor's *La science des ingenieurs dans la conduit des travaux de fortification et d'architecture civile* focuses, similar to de la Hire's 1712 memoir, on determining the abutment thickness needed to stabilize an arch. Belidor assumes, as does de la Hire, a hinge will form in an arch at 45° between the keystone and the impost. Unlike de la Hire, Belidor assumes the force acts at the center of the arch and perpendicular to the joint (Benvenuto, 1991, p. 336). This change in the location of the thrust makes the force diagram in Figure 12 change to look like the diagram in Figure 13.

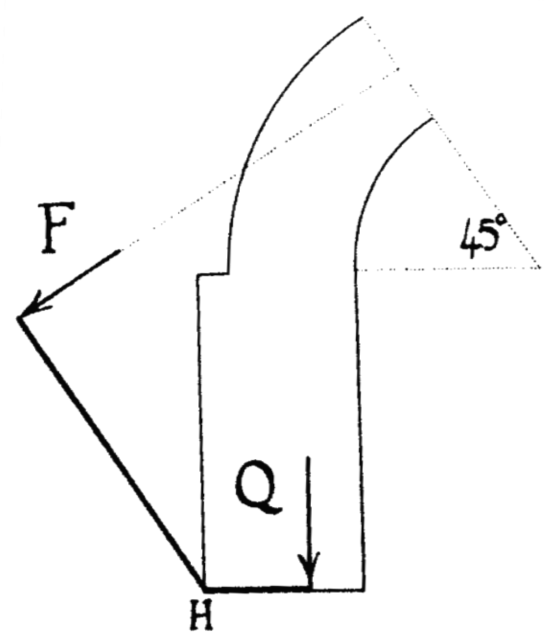

Figure 13: Belidor's Location of Forces in an arch (Benvenuto, 1991, p. 337)

Belidor assumes the same behavior as de la Hire; when the hinge forms between the impost and keystone, the top arch segment will try to drop and cause the abutment and lower arch segment to rotate about point *H*. Finding the force tangent to the arch at 45° is the same as in de la Hire's method and is equal to $Q_c * \sqrt{2}$ where Q_c is the weight of the arch above the breaking joint to section *aA* as shown in Figure 14. Where the arch is of consistent thickness and of homogenous material, Q_c can also express the area. *F* is equal to $Q_c * \sqrt{2}$ in Figure 13.

Thus, Belidor uses the geometry shown in Figure 14 to determine the forces and lever arms needed to balance moments about point *H* and find the minimum abutment thickness. All dimensions except the width of the abutment are set prior to solving for the abutment width and therefore the abutment width is the only variable.

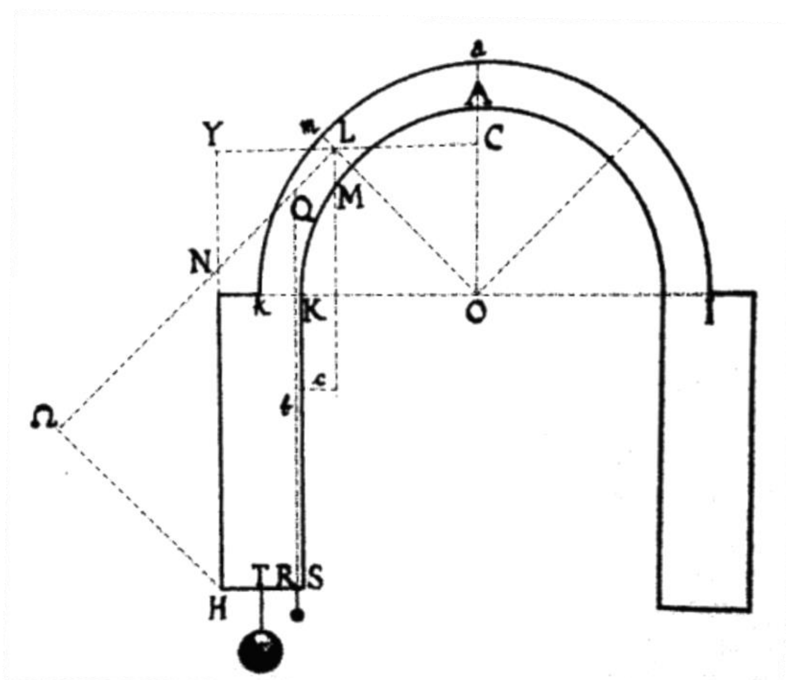

Figure 14: Belidor's arch geometry (Benvenuto, 1991, p. 338)

Belidor sets simple variables in order to simplify the equations. He sets the abutment width, *HS*, equal to *y*; the abutment height, *SK*, equal to *l*; the horizontal distance between the abutment intrados and the lower arch segment center of mass equal, *RS*, equal to *b*; and the horizontal distance between the intrados of the abutment and point *L*, *KO-OC*, equal to *c* (Benvenuto, 1991, p. 338). The balancing moment equation can be written as:

$$F * H\Omega = Q_{abutment}\left(\frac{y}{2}\right) + Q'_c(y - b) \quad (21)$$

F has already been determined as equal to $Q_c * \sqrt{2}$, and Q'_c can be set equal to Q_c as it is assumed the arch is constant thickness and made of homogenous material. He then sets other dimensions in terms of his chosen variables. It is interpreted from the diagram:

$$H\Omega = \frac{\sqrt{2}HN}{2} \quad (22)$$

And

$$HN = HY - NY = HY - YL \quad (23)$$

Where

$$YL = y + c \quad (24)$$

$$HY = l + OC \quad (25)$$

Now by substituting equations (24) and (25) into equation (23) and by setting a new variable f equal to OC+l-c to simplify the equation::

$$HN = f - y \quad (26)$$

And

$$H\Omega = \frac{\sqrt{2}(f - y)}{2} \quad (27)$$

The last variable needed is $Q_{abutment}$. As area is assumed to be proportional to weight, $Q_{abutment}$ is equal to ly. Substituting all known variables into equation (21) gives:

$$Q_c\sqrt{2} * \frac{\sqrt{2}(f - y)}{2} = \frac{ly^2}{2} + Q_c(y - b) \quad (28)$$

After simplifying and solving for y using the quadratic formula:

$$y = \frac{-2Q_c}{l} + \sqrt{\frac{4{Q_c}^2 + 2lQ_c(b+f)}{l^2}} \qquad (29)$$

This method does not make any advancements on the work of de la Hire, as the only difference is the location of the force at the breaking joint. Belidor's assumption that the force will act at the center of the arch cross section contradicts his assumption of the hinging collapse of the arch. At the time of collapse, only the intrados of the arch will be in contact, therefore the force must pass through the intrados at the hinge location. Therefore, de la Hire's model will be more accurate as the hinge indicates the force is through the intrados.

1730 – Pierre Couplet (1670-1744)

In his second memoir *Seconde partie de l'examen de la poussee des voûtes*, Couplet reiterates Da Vinci's theorem on the stability of an arch, saying an arch will be stable "if the chord of half the extrados does not cut the intrados, but passes anywhere within the thickness of vault" (Benvenuto, 1991, p. 342). The limiting case for this theorem is shown in Figure 15. For this theorem, Couplet assumes the voussoirs of the semi-circular arch have infinite compressive strength and friction between the voussoirs prevents sliding failure.

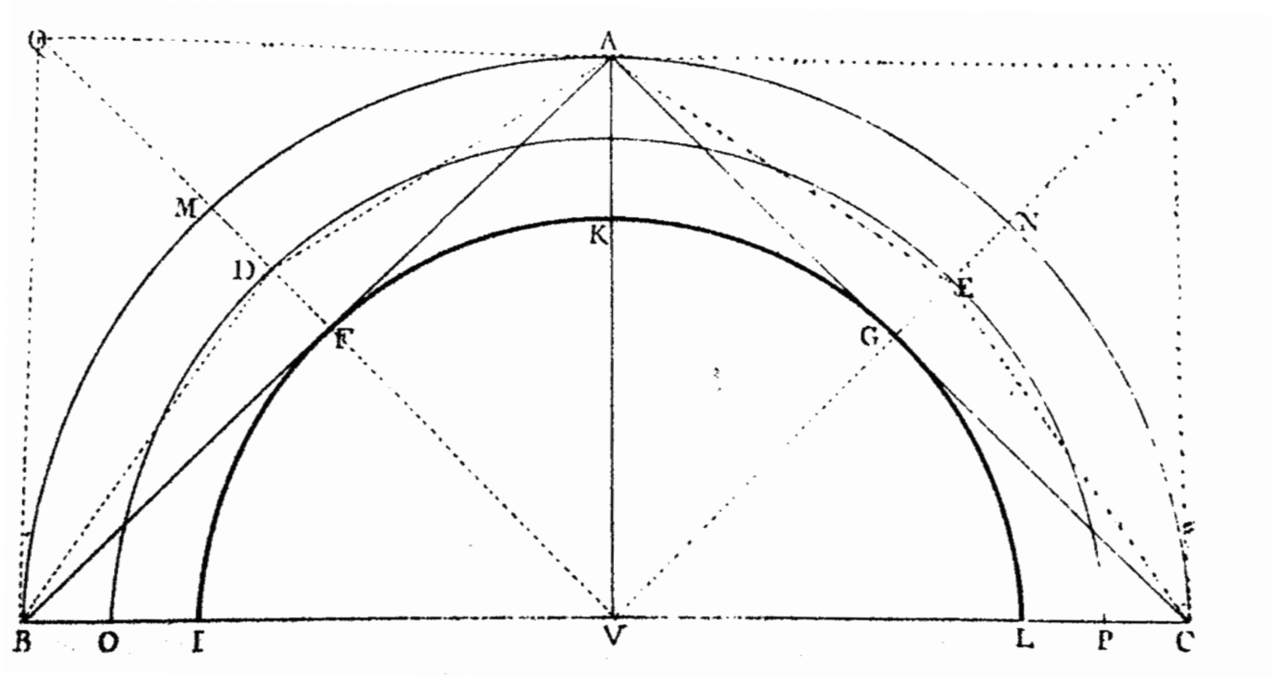

Figure 15: Couplet's Stability of an Arch based on Leonardo's Theorem (Couplet, 1730)

While this method provides the thickness of an arch needed for stability, an arch of smaller thickness will not automatically be unstable. In this memoir, Couplet proceeds to explain a theorem to determine the smallest uniform thickness of a semi-circular arch under only its own weight, based on his understanding of the failure of an arch. For this theorem he assumes friction will be sufficient to prevent a sliding failure and therefore the arch will break at points of rotation about the extrados or intrados, which he calls charnières, or hinges (Benvenuto, 1991). Couplet says these hinges will occur at 45° from the horizontal and vertically at the crown of the arch (Heyman, 1972). Couplet then derives an equation relating the thickness of the arch, t, to its average radius, R. This relationship is:

$$\frac{t}{R} = 10.1\% \qquad (30)$$

Couplet's assumption of the 45° hinge location was an assumption common at that time. This assumption greatly impacts the minimum thickness that Couplet is solving for. He assumes that the force acts through the hinge location and tangential to the intrados. This is a correct assumption of behavior however centuries later Jacques Heyman showed that the force is not parallel to the intrados at 45° (Heyman, 2009).

In the 20th century, Heyman was able to form a transcendental equation for the exact location of the hinges in a semicircular arch based on the location where the force, or line of thrust, in an arch is tangential to the intrados. This equation is (Heyman, 2009):

$$\frac{\pi}{2} = \beta \cos\beta \left[\frac{2\beta \cos\beta + \sin\beta cos^2\beta + \sin\beta}{2\beta \cos\beta + \sin\beta cos^2\beta - \sin\beta \cos\beta}\right] \qquad (31)$$

β, the angle between the crown and hinge, is 58° 49' by this equation, significantly different from the common assumption in the early 1700's of a 45° hinge location. Heyman (1972) uses this angle in his equation for minimum safe thickness:

$$\frac{t}{R} = 2\frac{(\beta - sin\beta)(1 - cos\beta)}{\beta(1 + cos\beta)} \qquad (32)$$

He finds the minimum thickness to radius ratio equal to 10.6%, which while very close to Couplets ratio, makes Couplet's ratio un-conservative.

1773 – Charles-Augustin de Coulomb (1736-1806)

Also included in Coulomb's 1773 memoir was an arch theory considering friction and cohesion between the voussoirs due to mortar. Section 28 of his memoir does not give any practical methods for designing arches, but gives the required horizontal thrust to ensure stability of an arch under its own weight. His rule is given based on the geometry in Figure 16, with the horizontal force at the crown being applied at point *f*.

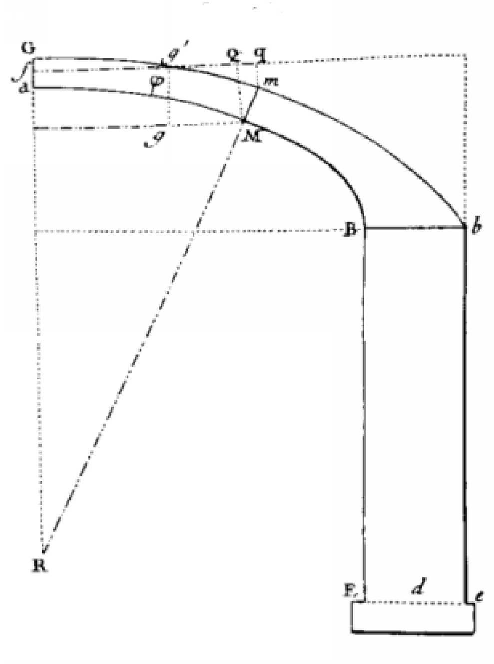

Figure 16: Coulomb's Arch Geometry including friction (Coulomb, 1776, p. 382)

Coulomb defines the horizontal thrust at *f* as *A*. *GaMm* is the portion of the arch above the joint *Mm*, *R* is the point at which the lines extended from *Ga* and *mM* meet and the angle between them is h. *φ* is the weight of the section *GaMm*. The portion of the horizontal thrust along *Mm* is *A*sin(h)*. The portion of the weight along *Mm* is *φ**cos(h). The portion of the horizontal thrust perpendicular to *Mm* is *A*cos(h)*. The portion of the weight perpendicular to *Mm* is *φ**sin(h).

Coulomb finds the horizontal thrust required at *f* based on two mechanisms of failure. The first failure mechanism is the sliding of the voussoirs at joint *Mm*. Coulomb writes an equilibrium equation for the joint based on its tendency to slide along joint *Mm* taking into account both friction along the joint and cohesion of the joint due to the mortar. This equation is:

$$\phi \cos h - A \sin h - \frac{\phi \sin h + A \cos h}{n} = \delta Mm \tag{33}$$

Here *1/n* is the coefficient of friction and *δ* is the maximum tangential (or shear) stress the mortar between the joints can support.

He rearranges this equation to write the equation for the thrust *A* required to resist a sliding failure. This equation is:

$$A = \frac{\varphi\left(\cos h - \frac{1}{n}\sin h\right) - \delta Mm}{\sin h + \frac{1}{n}\cos h} \tag{34}$$

Coulomb states this equation must be solved for multiple locations of joint *Mm* in order to find the values of *A* needed for stability against sliding failure at all joint locations. The highest required value of *A* he calls this value A_1. And the thrust *A* at point *f* must be greater than this value.

This method ensures stability against the voussoir above joint *Mm* sliding down the joint, but sliding failure can also occur in the other direction. For this case the signs of the forces from the thrust and cohesion in equation (34) are switched, giving the equation:

$$A = \frac{\varphi\left(\cos h + \frac{1}{n}\sin h\right) + \delta Mm}{\sin h - \frac{1}{n}\cos h} \tag{35}$$

The forces from the weight of the arch remain in the same direction. This equation is solved for multiple locations of joint *Mm* to find the minimum value of A for equilibrium. This value, distinguished as A_2, is the maximum value of *A* allowed before the arch slides out or up at joint Mm. The value of *A* must be between A_1 and A_2. If values of A_1 and A_2 are found and A_1 is greater than A_2, equilibrium cannot be achieved.

The second failure mechanism is the rotation of the arch around point *M* or *m*. For stability, the force must pass through a point along the line between points *M* and *m*. For this failure mechanism, *B* is defined as the horizontal thrust at point *f*, instead of *A*. Coulomb writes the equation of equilibrium as:

$$B * MQ = \phi * GM - \delta'(Mm)^2 \tag{36}$$

Here, δ' is the maximum tensile stress the mortar between the voussoirs can carry. The equation rearranged to solve for the horizontal thrust *B* is:

$$B = \frac{\phi * GM - \delta'(Mm)^2}{MQ} \tag{37}$$

For any location of *Mm*, *B* from equation (37) is the minimum magnitude for the resultant force that will pass at or above point *M*, thus providing stability. It is necessary to solve for all joint locations to find the location with the largest value of *B*, which will be the minimum required value of *B* for the arch. Coulomb calls this value B_1 (Heyman, 1972).

Equation (37) defines the thrust from the equilibrium equation with respect to rotation around the intrados, or point *M*. To discover the maximum thrust, the equilibrium equation must be found with respect to rotation around the extrados, or point *m*. This equation for the horizontal thrust is:

$$B = \frac{\phi * g'q + \delta'(Mm)^2}{mq} \tag{38}$$

This gives the maximum thrust magnitude which will support a stable arch (Heyman, 1972). The largest permissible B magnitude for a stable arch is the minimum value found by solving equation (38) for all joint locations. This value Coulomb calls B_2 (Heyman, 1972).

As with the case of sliding failure, the magnitude of the horizontal thrust exerted at *f* must be between B_1 and B_2 to ensure stability of the arch. If B_2 is less than B_1, then the arch is not stable. As either collapse mechanism can occur, the minimum permissible value of the thrust at *f* is the larger of A_1 and B_1, and the maximum is the smallest of A_2 and B_2.

Coulomb comments often that it can be assumed friction is large enough sliding friction cannot occur and the case of the thrust *A* can be neglected. He adds when an arch is first constructed it should be assumed there is no cohesion between joints. The *B* thrusts become:

$$B_1 = \frac{\phi * GM}{MQ} \tag{39}$$

$$B_2 = \frac{\phi * g'q}{mq} \tag{40}$$

The values of B_1 and B_2 are found by the same method as previously outlined. This method of calculating the thrust is most easily solved by calculating the thrust at each joint and using the required maximum or minimum.

This method of Coulomb's is more theoretical than practical as it cannot be used to design the arch but rather to analyze the forces within it. In addition, Coulomb deduces the single condition necessary for an arch to be stable – the line of thrust should be within the masonry arch at all locations along the arch.

1785 – Lorenzo Mascheroni (1750-1800)

In 1785 Lorenzo Mascheroni's *Nuove ricerche sull'equilibrio delle volte* was published. In this publication, Maschheroni describes his methods for determining the necessary abutment size very similarly to de la Hire and Couplet. He assumes infinite compressive strength of the arch, no tensile strength between voussoirs, and sliding cannot occur between voussoirs. However, while both de la Hire and Couplet base their models on resisting overturning of the abutment, Mascheroni also considers resisting sliding of the base of the abutment (Benvenuto, 1991). He also considers two separate collapse mechanisms. The first collapse mechanism is the mechanism first outlined by de la Hire. It is a sliding mechanism where the center portion of the arch slides downwards without friction at the imminent rupture of joints and infinite friction at the springings of the arch, shown in Figure 17. The second collapse mechanism corresponds to the five hinge mechanism identified by Couplet or Coulomb as shown in Figure 18, although he made no mention of Coulomb in his work.

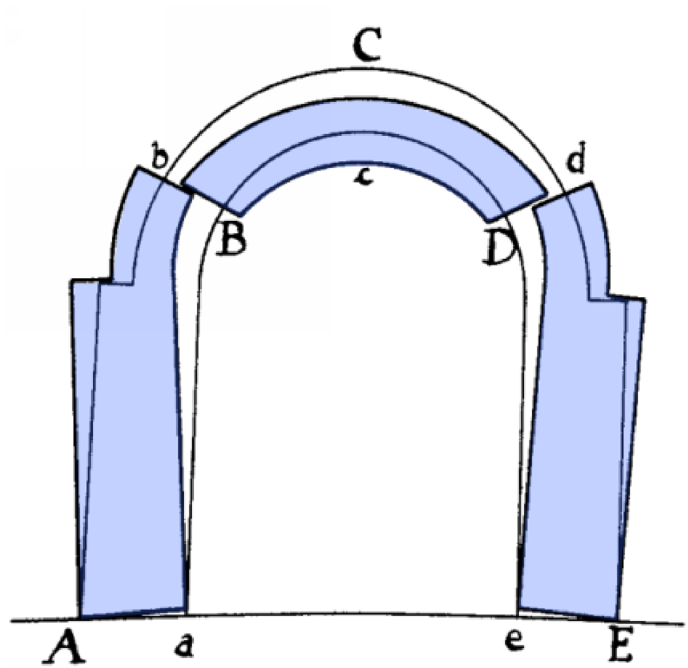

Figure 17: de la Hire Collapse Mechanism (Benvenuto, 1991, p. 416)

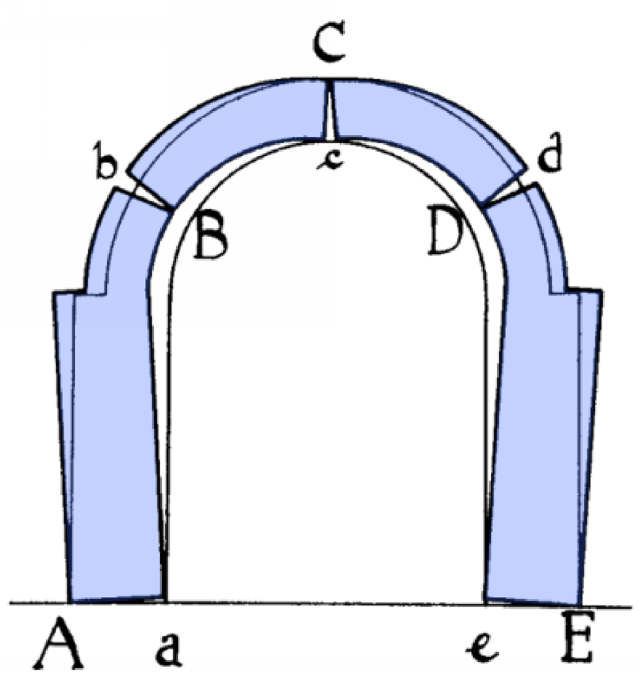

Figure 18: Couplet/Coulomb Collapse Mechanism (Benvenuto, 1991, p. 418)

In addition to the abutment size being variable, he also assumes the locations of *B* and *D* in Figure 17 and Figure 18 are also variable. After deriving equations for the stability of the arch, Mascheroni notes the equations must be solved for various locations of *B* to determine the weakest joint location and therefore the location of the hinge.

For Mascheroni's model, based off the collapse mechanism of de la Hire, the geometry is in Figure 19.

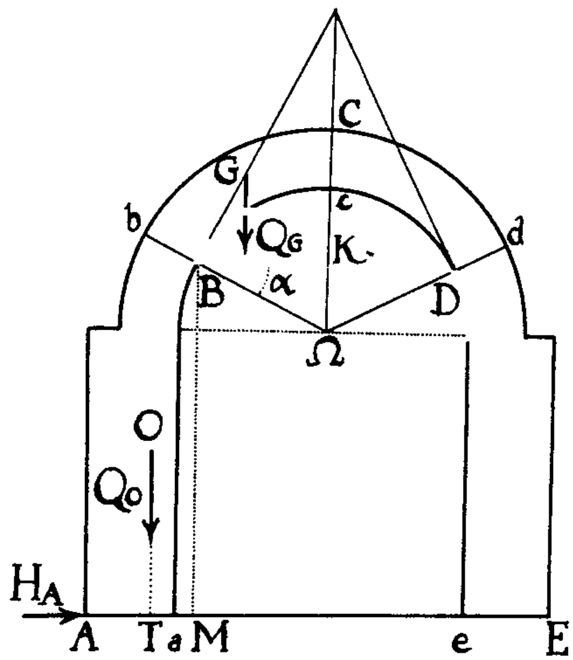

Figure 19: Three-Segment Geometry (Benvenuto, 1991, p. 417)

He gives equilibrium equations for sliding and overturning of the abutments. H_A, the horizontal force at the base of the abutment, which is the same magnitude as the horizontal component of the thrust at point *B*, although opposite in direction. Therefore:

$$H_A = H_B = Q_G \tan \alpha \tag{41}$$

Where Q_G is the weight of the arch from joint *Bb* to the crown and α is the angle between joint *Bb* and horizontal. The force resisting this horizontal reaction is the force of friction which is the vertical force being exerted at the base multiplied by a friction factor. The vertical force at the base V_A is:

$$V_A = Q_G + Q_O \tag{42}$$

Where Q_O is the weight of the abutment and portion of the arch below joint *Bb*. The equilibrium equation for sliding of the abutment is:

$$f_s V_A = H_A \tag{43}$$

Substituting equations (41) and (42) into equation (43) yields:

$$f_s(Q_G + Q_O) = Q_G \tan\alpha \tag{44}$$

The moment about the base, *A*, due to the horizontal component of the thrust at *B* is:

$$M = H_B * BM \tag{45}$$

And the moment resisting the overturning is:

$$M_O = Q_G * AM + Q_O * AT \tag{46}$$

For equilibrium:

$$M_O = M \tag{47}$$

So by equations (41), (45), (46), and (47):

$$Q_G * AM + Q_O * AT = Q_G \tan\alpha * BM \tag{48}$$

For stability, the left side of equations (44) and (48) must be greater than the right side and both equations must be solved for multiple locations of *B* to find the weakest section.

Mascheroni uses a similar method to solve for stability with Coulomb's collapse mechanism (Figure 20).

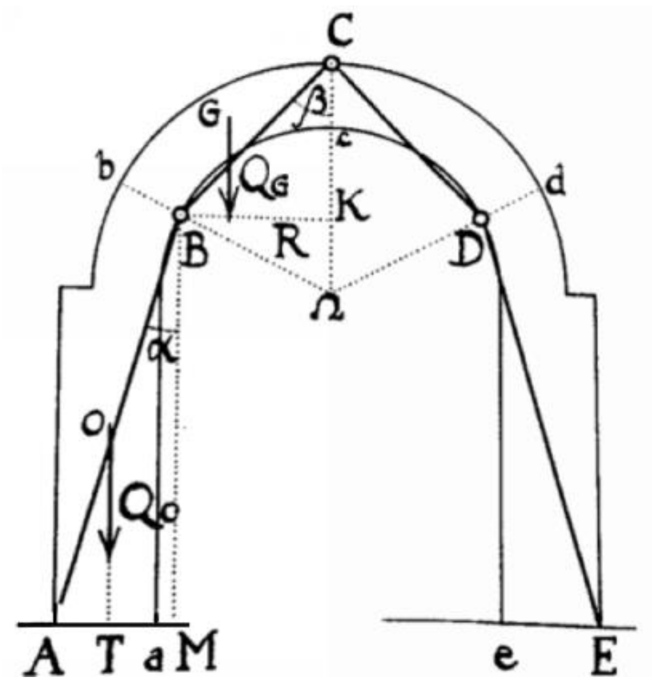

Figure 20: Four Segment Geometry (Benvenuto, 1991, p. 418)

He starts by idealizing the weight of the arch as acting at the hinge locations. He sets Q_G as the weight of the arch segment *BbCc* and Q_O as the weight of the segment *aAbB*. He then sets the vertical force at points *A*, *B*, and *C* based on the following ratios:

$$Q_A = Q_O \frac{TM}{AM} \quad (49)$$

$$Q_B = Q_O \frac{AT}{AM} + Q_G \frac{RK}{BK} \quad (50)$$

$$Q_C = Q_G \frac{BR}{BK} \quad (51)$$

From these equations he determines the horizontal and vertical resultant forces at *A*, H_A and V_A.

$$H_A = (Q_B + Q_C)\ \tan\alpha \quad (52)$$

$$V_A = Q_A + Q_B + Q_C \quad (53)$$

Due to equilibrium, and balancing overturning, the horizontal force at *B* must be balanced so:

$$Q_C \tan\beta = (Q_B + Q_C)\ \tan\alpha \tag{54}$$

α and β can be expressed in terms of ratios of length:

$$\alpha = \frac{AM}{BM} \tag{55}$$

$$\beta = \frac{BK}{CK} \tag{56}$$

Now equations (49) through (53), (55) and (56), are substituted into equations (43) and (54) to find the equilibrium conditions for sliding and overturning, respectively. These simplify to:

$$f_s Q_o = Q_G\left(\frac{BR}{CK} - f_s\right) \tag{57}$$

For sliding equilibrium, and:

$$Q_o \frac{AT}{BM} = Q_G\left(\frac{BR}{CK} - \frac{AM}{BM}\right) \tag{58}$$

Similar to the de la Hire model, the left side of equations (57) and (58) must be greater than or equal to the right in order for the arch to be stable and the equations must be solved for multiple locations of *B* to find the critical section.

Chapter 6 - Applications of Age of Enlightenment Theories

While the previously explained theories and advancements of arch mechanics did not have an immediate impact on arch design, it's trace is visible on a few structures during the age of enlightenment. The domes of St. Paul's Cathedral and St. Peter's cathedral were either designed or analyzed by methods founded during the era.

St. Paul's Cathedral

On the 27th August, 1666, Sir Christopher Wren's designs for a new dome of St. Paul's Cathedral in London were approved. Six days later the Great Fire of 1666 broke out and the plan for the new dome was halted. Following the fire, Wren and Hooke were appointed to positions in charge of the rebuilding of the city. While Wren was in charge of the building of churches throughout the city, he worked very closely with Hooke on many of them. Following the fire, Wren submitted three designs of a new St. Paul's Cathedral with the final design finally being approved in 1675. The final design, the Warrant Design, was changed significantly from its original approved design to the built design, specifically in the design of the dome (Heyman, 1998).

Wren used Hooke's analogy of an arch to a hanging chain to adjust the shape of the dome. In addition to changing the shape of the dome, he changed the orientation of the base of the dome to be angled outward, as Hooke's experiments demonstrated the force would never be transferred only vertically, in addition to changing the shape of the dome. The final dome section is shown in Figure 21.

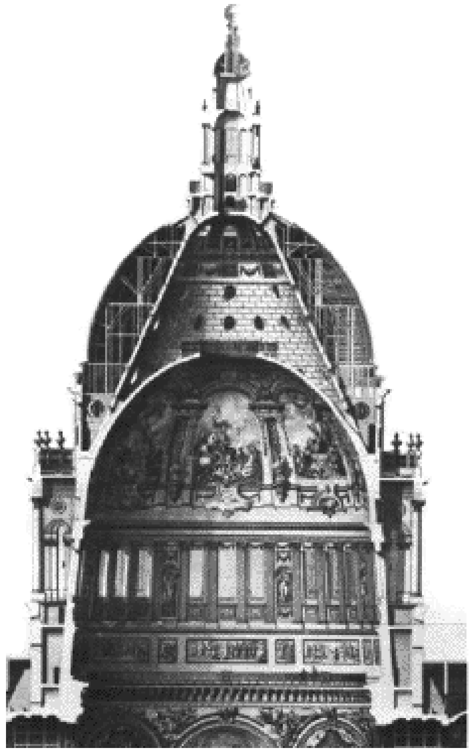

Figure 21: Cross Section of the dome of St Paul's Cathedral (Gwynn, 1755)

As seen the interior support walls for the dome are inclined. In no catenary but a weighted catenary that forms a semi-circular arch, will the catenary be vertical at its springing. The most efficient transfer of forces will occur when the forces in the arch are able to transfer straight through the abutments. Inclining the abutment walls inward helped to counteract the horizontal force exerted by the dome.

The exterior dome structure is separate from the interior structure. The exterior dome is shown in Figure 22.

Figure 22: Dome of St. Paul's Cathedral (Gagnon, 2008)

By having multiple walls, Wren could incline the exterior walls while orienting the interior walls for maximum stability and strength. He accomplished a similar feat with the dome. He built a semi-circular dome-like structure over the interior dome to achieve the desired architecture. The outer dome is supported on framing supported by slanted walls eventually meeting the less slanted walls of the interior dome.

By using Hooke's theorem on the ideal shape of an arched structure, the dome of St. Pauls Cathedral was the first dome built with inclined supporting walls for the purpose of minimizing thrust.

St. Peter's Cathedral

The masonry dome of St. Peter's Church in the Vatican City is one of the largest masonry domes in existence. In the 18th century the Vatican worried about the structural integrity of the dome of St. Peters Basilica (Figure 23). Many scholars and scientists studied the dome and made recommendations as to its repair but no one was able to fully answer the question of its integrity. In 1743 Pope Benedict XIV summoned the scholar Giovanni Poleni to the Vatican to examine the dome. His analysis and recommendations of repair were submitted to the Vatican later that year and restoration began almost immediately.

Figure 23: The Dome of St. Peter's Basilica (Stuck, 2004)

In 1748 Poleni's treatise, *Memorie istoriche della gran cupola del tempio Vaticano e de' danni di essa, e de' ristoramenti loro* (Historical memoirs of the great dome of the Vatican church, it's damage and restoration), was published. In this treatise he outlined his analysis of the dome and the theories of scholars that were used in the dome's evaluation.

Poleni used the work of many of the scholars discussed previously in this report to analyze the dome of St. Peter's. All of the methods could not be readily utilized for the dome, as most were for an arch of uniform thickness and a segment of a dome; a cupola, does not possess uniform thickness (Figure 24).

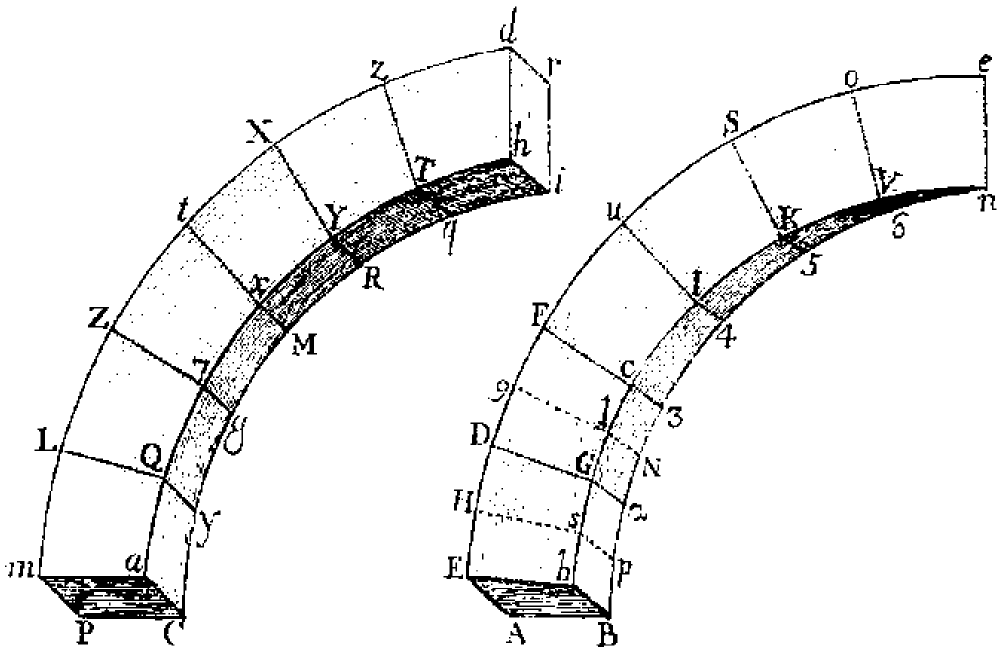

Figure 24: Arch Segment vs. Cupola (Poleni, 1748)

Poleni's work is based largely off the idea of the catenary as a series of spheres, a concept taken from David Gregory and James Stirling, another scholar whose work paralleled that of Gregory. As the width of each section of the dome tapered to a point at the top, the spheres were required to be of non-uniform weight to correctly represent the shape of the catenary, decreasing in weight from the abutment to the crown. Here Poleni states this decreasing weight of the dome as it nears the crown is "admirable form . . . in which the parts from the impost on the pilaster up to the keystone were becoming smaller" (Benvenuto, 1991, p. 360). He bases this hypothesis on the work of Couplet and his own work which shows this decreasing weight from top to bottom.

To begin his analysis, Poleni divides the cupola into wedge segments and records their length and weight. This data is seen in Figure 25.

COLONNA I.	COLONNA II.	COLONNA III.	COLONNA IV.
NVMERI DE' CVNEI.	LVNGHEZZE DE' RAGGI.	PESI DE' CVNEI.	PROPORZIONE DE' PESI IN NVMERI MINORI.
A. 1.	200.	88976.	89.
B. 2.	198.	88086.	88.
C. 3.	195.	86752.	87.
D. 4.	191.	84972.	85.
E. 5.	184.	81858.	82.
F. 6.	177.	78743.	79.
G. 7.	168.	74740.	75.
H. 8.	159.	70735.	71.
K. 9.	148.	65842.	66.
L. 10.	135.	60059.	60.
M. 11.	122.	54275.	54.
N. 12.	108.	48047.	48.
P. 13.	92.	40929.	41.
Q. 14.	76.	33811.	34.
R. 15.	60.	26693.	27.
S. 16.	41.	18240.	18.
	2254.	1002758.	
S. 16.	43.	18240. Cupolino. 81629.	
		99869.	100.

Figure 25: Properties of Poleni's Wedges of St. Peter's Dome (Poleni, 1748, p. 47)

The first column identifies the wedge identification letter and number, the second column identifies the length of each wedge, and the third column the weight of each wedge. Poleni finally modeled the cupola by proportioning down the weight of each wedge (column 4) and created lead spheres of these masses. Linking these together he discovered the catenary showing the direction of forces within the dome. The chain of the lead spheres is shown in Figure 26. This figure also illustrates this shape transposed onto the dome geometry (smaller dashed line).

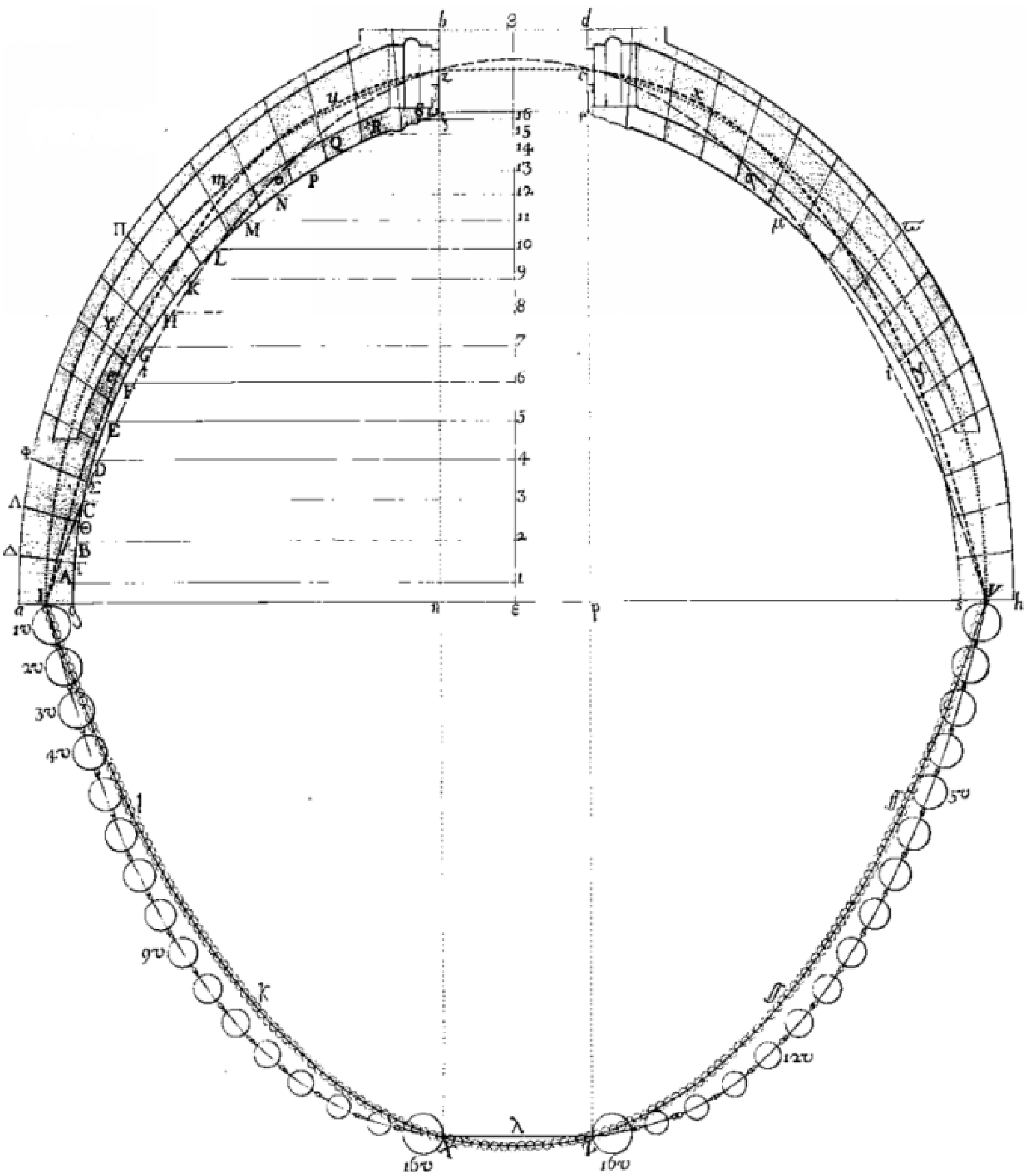

Figure 26: Weighted Catenary of St. Peter's Dome compared to its geometry (Poleni, 1748)

Poleni concludes because the weighted catenary fits entirely within the geometry of the dome, then the dome is sound. This was the first time a safe theorem of limit analysis was applied to a masonry dome. (Lopez, 2006) This conclusion, based off the work of age of enlightenment scholars was able to relieve the worries of Vatican leaders that the dome was in immediate danger of collapsing.

Chapter 7 - Conclusion

The first significant growth of the study and application of structural statics and mechanics occurred during the Age of Enlightenment. This beginning of research and understanding of behavior of materials and structures was the beginning of the field of structural engineering. Scholars during the Age of Enlightenment sought to design structures which supported the loads they were expected to resist. Prior to the advancements made during this time period, lack of understanding of material behavior led to significant overdesign of structures.

During the Age of Enlightenment many scholars spent significant time studying the behavior of masonry arches. These scholars discovered and understood the importance of knowing the path of force within an arch, known as the thrust line. They applied, then revolutionary, rules of statics to arches in order to determine stability. Robert Hooke defined this shape as a catenary, or the shape of a hanging chain. David Gregory explained that for an arch to be stable the thrust line must remain entirely between the intrados and extrados of the arch.

Other scholars sought to create methods of analysis and design of masonry arches. These theories are able to be broken into two groups based on how they approached friction. Scholars such as de la Hire, Couplet, and Coulomb published methods of design neglecting friction as well as methods including friction. Their methods disregard friction and consequently all run into the same problem of requiring a voussoir of infinite weight at the base of the arch. The scholars knew the presence of friction meant that this infinite weight voussoir was not needed. Perhaps for this reason, all three scholars went on to formulate new methods that included friction.

Scholars included friction in their methods in two ways. Most assumed that friction completely inhibited sliding between voussoirs but Coulomb used the actual force of friction in his calculations. Friction was not fully understood until the middle of the 18th century therefore scholars prior to this time understood friction but could not quantify it. These scholars focused largely on the collapse mechanisms of the arches. They used statics and in some cases mechanics to balance moments and forces. Their statics depended on their assumptions about how the arch collapsed, and were updated by subsequent scholars as the location of the weakest section became known.

The new understanding of arches was used in the design of St. Paul's Cathedral in London and in the conditions assessment of St. Peter's Cathedral in the Vatican. Both used the model of the hanging chain to demonstrate the proper shape of the arch. At the time, Vatican officials were worried that the dome of St. Peter's was unstable and would possibly need to be taken down. Poleni's assessment of the dome proved it was stable and minimal repairs were needed. Due to Poleni's assessment, the dome is still standing today.

The methods formulated during the Age of Enlightenment are the basis of most structural analyses used today. They were the first methods based off of an understanding of structural behavior. As such, significant assumptions were made that have been changed and refined over time as more advancements were made and the breadth of understanding grew. While these rules are no longer used, they provide insight into the inception of scientific arch design. In the words of Edmund Burke (1860, p. 518), "In history a great volume is unrolled for our instruction, drawing the materials of future wisdom from the past errors and infirmities of mankind"

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Appendix A - Glossary

Abutment – the support an arch sits on

Catenary – The shape of a hanging chain held at both ends

Extrados – the exterior line of an arch

Intrados – the interior line of an arch

Impost – the location where the arch meets the abutment

Pilaster – another term for abutment

Keystone – the voussoir at the center of an arch

Springing – The joint where the arch meets the support

Voussoir – an individual block of a masonry or stone arch

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Place of Publication:	London
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