DISCONTINUOUS THEORY OF RELAXATION OSCILLATORS

by

-

JAI KARAN N. SHUKLA

B.S., University of Roorkee, India, 1960

_____

A MASTER'S REPORT

Submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Electrical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1965

Approved by

_____
Major Professor

TABLE OF CONTENTS

INTRODUCTION

1. General

Self oscillating systems in which one of the oscillating parameters is of secondary importance compared with damping are called relaxational. The term "relaxational" was borrowed from mechanics. In mechanics "relaxation" a gradual disappearence of elastic deformation in a medium possessing friction, is analogous to the discharge of a capacitor through resistance.

Though in general the phenomenon of relaxation oscillations was known to exist in few branches of science in the early 20th century, it is not wrong to say that study of the phenomenon in particular was initiated by van der Pol in 1922. Having discovered his famous differential equation (d. e.) $\left[y'' - \varepsilon(1 - y^2) \, y' + y = 0 \right]$, van der Pol indicated a process of graphic integration of this equation, (phase plane representation) which permited construction with enough surety. He observed that when ε is very small, the limit cycle is very close to a circle described by the representative point with a constant angular velocity: the radius of the circle is $\sqrt{2}$, that cancels well on an average $1 - y^2$. This limit cycle is deformed in proportion that ε increases, but it always defines the oscillations in a very strict manner; for each value of ε there is a determined period and amplitude of oscillation.

Upon examining the aspect of integral curve $y = f(x)$ when ε exceeded unity, van der Pol found that the system produced oscillatory forms that had escaped analysis until then. He therefore further developed this equation to the form $y'' + \varepsilon\psi(y) \, y' + y = 0$ (where $\psi(0) = -1$, and ε is very large) and named the resulting oscillations as relaxation oscillations. The

investigation of this and similar d. e. resulted in remarkable advances in the theory of oscillations, and the doctrine given by him on the subject has become classic; it was, for example, the object of an interesting account by Le Corbellier.*

It was, however, noticed later that all available analytical methods are inadequate for a rigorous treatment of van der Pol's equation when ε is very large. It is of the order of 10^5 in the case of the standard multivibrator circuit. In fact in all the analytical methods, use is made of series solutions arranged according to ascending powers of ε, and it is obvious, that if ε is not small, the series ceases to converge. A simple calculation in polar coordinates shows that at this value of ε, the isocline procedure becomes impossible, because even a very small rotation of radius vector in the neighborhood of the x axis results in a change in the direction of integral curve by nearly 90° and produces an incidental change in the velocity of the representative point from a high value to almost zero. There are thus two points of an extremely bad analyticity on the integral curve across which the analytic continuation is virtually impossible. With the combination of geometrical "gimmicks" and physical reasoning, it is possible to demonstrate what has been just said.

If one considers the basic oscillator circuit given in Figure 1 and assumes that the non-linear tube characteristic is symetrical about the bias point and cubic, the system could easily be described by van der Pol's equation.**

---

* Ph. Le Corbiller, J. Inst. Elec. Eng., London (1936).

**D. F. Lawden, Mathemetics of Engineering Systems.

Thus with a considerable amount of feedback it is possible to represent the system by equation;

$$\ddot{x} - \varepsilon(1 - x^2)\,\dot{x} + x = 0, \quad \varepsilon = 100$$

Let $x = i$

$$i = \int x dt$$

Since x is certainly expected to be periodic with no d.c. components, this is because of the nature of the original equation, one can ignore constants of integration. The result is then

$$\dot{x} - \varepsilon(x - x^3/3) + z = 0$$

or $\quad \dot{x} = \varepsilon(x - x^3/3) - z$

but as $x = \dot{z}$

$$\frac{dx}{dt} = \frac{dx}{dz} \cdot \frac{dz}{dt} = x\,\frac{dx}{dz}$$

or $\quad \dfrac{dx}{dz} = \dfrac{\varepsilon(x - x^3/3) - z}{x}$

The plot of the curve $\varepsilon(x - x^3/3) - z = 0$ with $\varepsilon = 100$ is given in Figure 2 curve A. It is simply the locus of points where $\frac{dx}{dz} = 0$. The curve divides the (x, z) plane into two regions. The area to the left of the curve is the region for which $\frac{dx}{dz}$ is positive. The area to the right of the curve is the area for which $\frac{dx}{dz}$ is negative. For any given point on the (x, z) plane, the absolute magnitude of $\frac{dx}{dz}$ is determined by the distance to that point from the curve A. Since ε is very large, it would be expected that the absolute magnitude of $\frac{dx}{dz}$ would increase rapidly as the distance from the curve increases.

Now one should see how the behaviour of the system can be represented on the (x, z) plane for any starting point on it. A typical starting point is shown as P_1. Since $\frac{dx}{dz}$ is large and negative, the solution curve will drop very rapidly to curve A at P_2 where $\frac{dx}{dz}$ is essentially zero. At P_2 $\frac{dx}{dz} = 0$, but because $\frac{dz}{dt}$ is finite and negative, the solution curve will tend directly
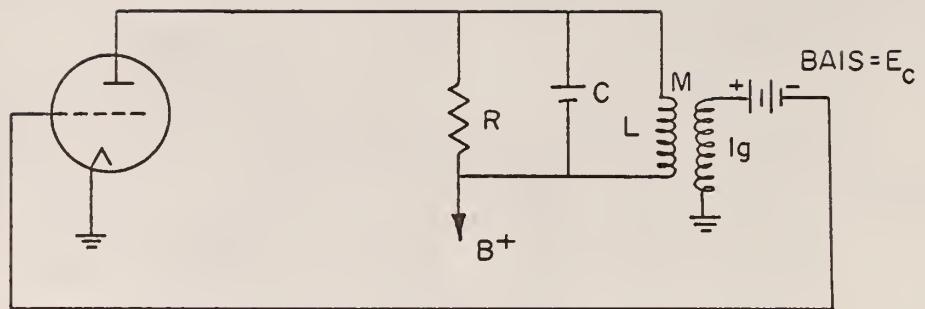
Figure 1
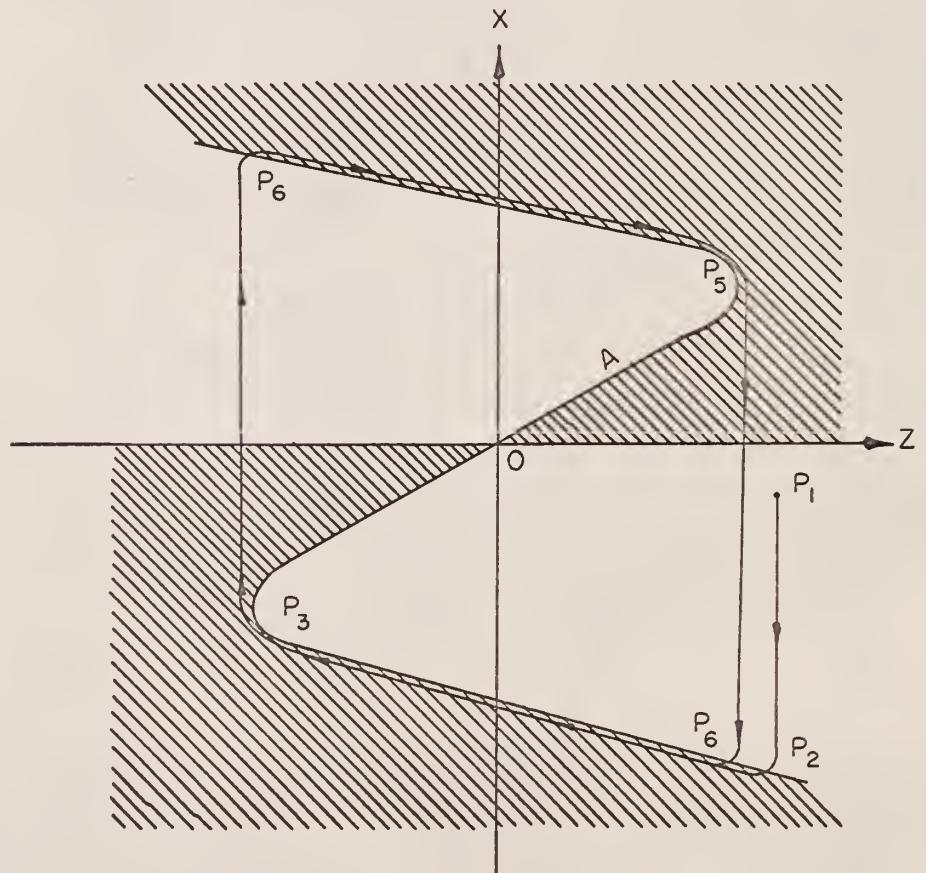
Figure 2

to the left at P_2, and cross A in a direction parallel to z axis. It cannot

now leave the neighborhood of curve A, for it is in the zone of negative

gradient and any tendency for it to move away from A would be counteracted by

a rapid increase in the magnitude of this negative gradient, bringing it back

into the neighborhood of A again. Neither can it cross the curve A, for if

it were to approach the curve, the gradient would decrease towards zero, thus

carrying it away again. At P_3 the integral curve has a large gradient and can

no longer follow A, since to do so would imply that z is increasing when x is

negative. The integral curve must accordingly proceed parallel to the x axis

until it reaches P_4 at which it commences to follow A again to P_5 for the

reasons just mentioned. At P_5 it drops almost immediately to P_6 and then

repeats the limit cycle path P_3-P_4-P_5-P_6-P_3, indefinitely. A relaxation

oscillation is therefore established which consists of alternately fast and

slow variations of x.

One should note in the proceeding example that no analyticity exists

between P_3-P_4 and P_5-P_6. It is interesting to examine the wave form which

results from an oscillation, having the solution curve in Figure 2. Obviously

the variable x would vary almost instantaneously from P_5 to P_6 and from P_3

to P_4. From a physical point of view this instantaneous variation is analogous

to the effect of a shock in machines where continuity is preserved but ana-

lyticity is lost.

Attempts have been made to extend analytic methods to oscillations when

ε is large. Liénard* succeeded in obtaining certain conclusions regarding

qualitative aspects of phase trajectories when ε was very large. N. Levinson

(1943) extended the proof of the existence of closed trajectories to cover

oscillations in which ε is not small. In 1944 J. A. Shohat indicated a form

---

* Liénard, A., Revue Générale de l' Electricité, Vol. 23, 1928.

of series expansion formerly satisfying the van der Pol equation when ε is large. These various attempts, however, did not result in any complete analytical theory in connection with oscillations in which ε is large. Moreover, not all the relaxation problems, belong to the group of van der Pol equation. More specifically it will be seen later that relaxation oscillations are frequently observed in systems which are amenable to representation by differential equations of the first order. Obviously these equations do not admit any analytic periodic solutions for the simple reason that they do not possess singularities, without which no closed analytic trajectories can exist. These difficulties led the school of physicists under the leadership of L. Mandelstam, N. Papalexi and L. Lochakov to evolve a theory (1935) called by its authors the <u>discontinuous theory of relaxation oscillations</u>. Practically the same conclusions were reached independently by T. Vogel in France (1951).

From the point of view of the discontinuous treatment, the van der Pol equation (with large ε) is not involved at all and, instead the d. e. is of the form

$$\frac{dx}{dt} = \frac{P(x, y)}{T(x, y)} \; ; \; \frac{dy}{dt} = \frac{Q(x, y)}{T(x, y)} \tag{1}$$

In this form P, Q, and T may be regarded as analytic functions of x and y; the "relaxation range" begins at the points (x_c, y_c) for which $T(x_c, y_c) = 0$. However in order to be able to reduce the d. e. to the form (1), certain idealization of physical problems of this nature are necessary. These idealizations resemble closely similar ones used in the classical theory of mechanical shocks. Once this point is clear, the formation of such d. e. does not present any difficulty. The regions of rapid transitions (similar to P_3-P_4 and P_5-P_6) on the integral curve are idealized by discontinuities.

Likewise, the intentional ignoring of the d. e. _during_ the discontinuities is compensated for by _additional information_ not contained directly in the d. e. but which appears in the form of the so-called "condition of Mandelstam" regarding the invariance of energy during a discontinuity.

The second method of approach arose from a series of important papers by Cartwright and Littlewood and concerns the van der Pol equation for large values of ϵ. The approach had a somewhat limited objective, namely, to justify analytically the graphical solution obtained by van der Pol by the isocline method. Essentially it is as follows: The graphical curve representing the solution is split into a number of characteristic stretches, each of which has definite features; for example, on some of them $\ddot{x}$ is negligible, on some others $\dot{x}$ or x are negligible, etc. This permits using easily integrable or "truncated" d. e. for each stretch, the difficulty being in the analysis of the order of magnitude of different quantities and in the ultimate "joining" of all these solutions of "truncated" equations. It should be noted that the procedure hinges on the existence of a graphical or experimental curve, and the analysis merely confirms it.

It has been shown recently by Dorodnitzin, Wason, Flanders and Stoker (U.S.A.) and Haag (France) that these difficulties can be overcome to some extent by the use of so called asymptotic expansions which by their nature do not require analyticity. However the difficulty of ultimate "junction" of these expansions still persists.

At present the whole situation seems to undergo a certain "parting of the ways" between the engineers and physicists on the one hand, and the mathematicians on the other. The former, persuing the applied problems, seem to lean more and more to purely discontinuous treatment of the relaxation

oscillations, following the pattern of the theory of shocks in classical

mechanics, whereas the latter still persist in the search of an exact solution

of the van der Pol equation as evidenced by the work of Cartwright and

Littlewood and their school.

Following are the advantages and disadvantages in the use of discontinuous

and asymptotic theories.

Discontinuous theory

(1) The discontinuous theory is purely qualitative, and uses extensively

the phase plane representation, but this phase plane is different from the

classical phase plane and hence theory appears to contradict the analytical

theory.

(2) The application procedure is very simple and reduces generally to

simple topological constructions in the phase plane. The difficult part lies

in the justification of the theory.

(3) The discontinuous theory is based on certain idealizations: for

example one assumes that the term $\mu\ddot{x} = 0$ in the equation $\mu\ddot{x} + f(x, p)\,\dot{x} + x = 0$

and deals with the degenerate d. e. of the first order. In such cases one

has to supplement intentional ignoring by certain addition information.

(4) The idealized (discontinuous) treatment of relaxation oscillators

is more convenient for qualitative appraisal of what may be expected in the

given problem. Moreover it permits reducing the investigation of a system

amenable to two d. e. of the first order to a phase plane representation.

In its final form the discontinuous theory has turned out to be eminently

successful as a practical tool of exploration of all known relaxation

phenomena, and in-spite of its certain contradictions with analytical theory,

it has acquired ever increasing importance due to the ease with which it

handles the relaxation problems of even complicated types. Very often new

phenomena have been predicted on this basis. The theory has been checked experimently, which adds a strong point in its favor. Once the appropriate variables are chosen, it becomes a simple matter to establish the connections of a cathode ray oscilloscope so as to observe the corresponding phase plane diagram directly on the screen of an oscilloscope. Once a theory reaches such a state, it cannot be easily discarded only because one is more accustomed to using analytical theories. In fact, the usual reproach, that this theory "mixes up" so to speak the analytical approach with a physical postulate (the condition of Mandelstam) is no more justified than it is in the classical theory of shocks.

Asymptotic theory

The purpose of asymptotic theory, is to avoid the short cuts offered by the discontinuous theory. It prefers to deal with the d. e. as it stands in-spite of the complications at some points of non analyticity.

(1) The theory is purely quantitative in nature, but it is impossible to start without a preliminary knowledge of the integral curve (obtained either graphically or experimentally). In fact, as was mentioned, it merely explains the curves analytically.

(2) The idea itself is simple but the difficult part is in applying the procedure.

(3) This method is often very lengthy and normally its representation on the phase plane is impossible, for the reason that in this case the d. e. representing the physical system does not undergo degeneration.

Because of its relative simplicity and effective establishment of qualitative conclusions in all known cases of relaxation oscillations, the exposition and application of the discontinuous theory of relaxation oscillations (as given by Mandelstam and Paplexi) will form the principal topic

of this report. The discontinuous theory of Vogel will not be discussed in this paper, because the principal hereditary actions, on which this theory is based, are practically absent in electronic circuits.

BASIC DISCONTINUOUS THEORY AND ITS PHYSICAL INTERPRETATION

2. Simple R L C circuit

The system of Figure 3 can be presented by the d. e.

$$L\ddot{q} + R\dot{q} + q/c = 0 \tag{1}$$

One can consider initial condition to be $q = q_0$ and $\dot{q} = \dot{q}_0$. If the coefficient L is very small compared to R and $1/c$, Equation (1) can be degenerated* to Equation (2) and the corresponding circuit is shown in Figure 4.

$$R\dot{q} + 1/c = 0 \tag{2},$$

the initial condition in this case being $q = q_0$ at $t = 0$.

If the roots of equation $(LS^2 + RS + 1/c = 0)$ are α and β where $\beta \gg \alpha$ then the solution of Equation (1) can be written as

$$q(t) = q_0\left[\{\beta/(\beta - \alpha)\}e^{-\alpha t} - \{\alpha/(\beta - \alpha)\}e^{-\beta t}\right] + \dot{q}_0/(\beta - \alpha)\left[e^{-\alpha t} - e^{-\beta t}\right] \tag{3}$$

where

$$\alpha = R/2L - \sqrt{R^2/4L^2 - 1/Lc} \quad \text{and} \quad \beta = R/2L + \sqrt{R^2/4L^2 - 1/Lc}$$

using the development of the radical

$$\sqrt{R^2/4L^2 - 1/Lc} = (R/2L)\sqrt{1 - 4L/R^2c} = R/2L\left[1 - 2L/R^2c + \cdots\right] \simeq \left[R/2L - 1/Rc\right]$$

Hence for small L,

$$\alpha = 1/Rc \quad \text{and} \quad \beta = R/L - 1/Rc \simeq R/L$$

---

* A d. e. is said to be degenerescent when coefficient of the highest order derivative of d. e. is small in comparison with coefficient of other terms in d. e. For example, equation $a\ddot{x} + b\dot{x} + kx = 0$ is degenerescent if 'a' is much smaller than b and k., and the corresponding degenerate d. e. is $b\dot{x} + kx = 0$.

In one or two particular cases general rule has been violated and d. e. has been said degenerescent when the coefficient k is very much smaller than a and b.

or approximately Equation (3) can be written as

$$q_1(t) = q_0 \left[e^{-1/Rct} - (L/cR^2)e^{-R/L\,t} \right] + \dot{q}_0\, L/R \left[e^{-1/Rc\,t} - e^{-R/L\,t} \right] \quad (4)$$

It is obvious that the approximate solution given in Equation (4) is near true solution (Equation 3) in the sense that whatever $\varepsilon > 0$, one can always find L so small that

$$\left| q_1(t) - q(t) \right| < \varepsilon \; ; \qquad \left| \dot{q}_1(t) - \dot{q}(t) \right| < \varepsilon$$

for all positive t.

The solution of Equation (2) can be written as

$$\bar{q}(t) = q_0 e^{-t/Rc} \quad (5a)$$

and

$$\dot{\bar{q}}(t) = -(1/Rc) q_0\, e^{-t/Rc} \quad (5b)$$

Comparing Equations (4) and (5), and assuming that the initial values of the cooridinates for (1) and (2) are the same, one has

$$\theta(t) = q_1(t) - q(t) = -q_0(L/R^2c)e^{-(R/L)t} + \dot{q}_0(L/R)(e^{-t/Rc} - e^{-(R/L)t})$$

As all the terms in the right hand side of the above equation contain L as a multiplier, this difference $\left[\theta(t) \right]$ can be made as small as one chooses it to be, by taking L sufficiently small. On the other hand, the situation is different for derivatives of the solutions.

$$\dot{\theta}(t) = \dot{q}_1(t) - \dot{\bar{q}}(t) = (q_0/Rc)e^{-R/L\,t} - (\dot{q}_0 L/Rc)e^{-t/Rc} + \dot{q}_0\, e^{-R/L\,t} \quad (6)$$

For small t, $\dot{\theta}(t) \simeq \dot{q}_0 + \frac{1}{RC} q_0$, a value which does not decrease with L and so cannot be made small by suitably choosing L. However, for sufficiently large t, which is supposed to be fixed, one can always find a value of L small enough so that the value of $\dot{\theta}(t)$ is smaller than a given positive number ε.

If $\dot{\theta}(t)$ as well as L is sufficiently small, the difference between currents of complete and degenerate equations will remain small for all values
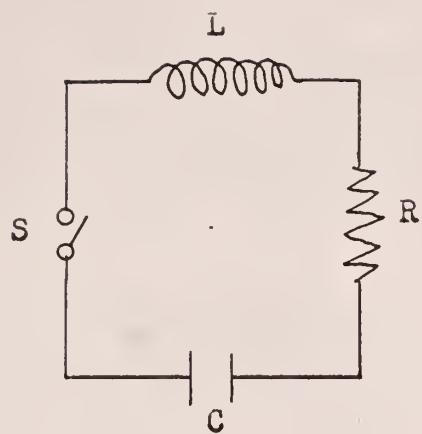
Figure 3
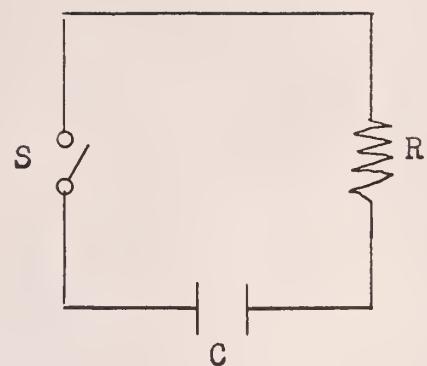
Figure 4
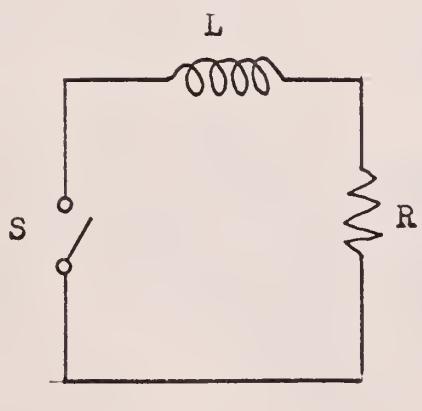
Figure 5
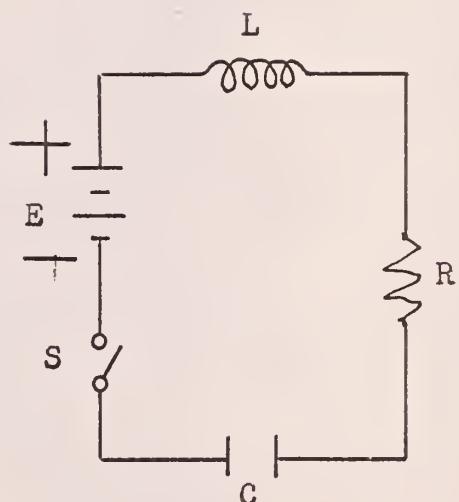
Figure 6

of t. If this difference is not small one gets the following picture: When L is sufficiently small, the current $\dot{q}$ in the complete equation of the system changes very rapidly, and after a small time say, t_2, it almost coincides with the current given by the solution of degenerate Equation (2). If this passage is sufficiently rapid, its details are often without interest. One may regard this rapid passage as an instantaneous jump and determine only the final state into which the system jumps; afterwards the behaviour of the system is determined by the equation of first order, $\left[\text{Equation (2)}\right]$. One can therefore, consider the system, free of inductance, provided we introduce the new assumption that there occurs a discontinuity. In this case it could be formulated as follows: the current $\dot{q}$ changes abruptly while the coordinate q (charge) remains constant.

If one, now, considers the case of R L degeneration, figure 5, in which $(\frac{1}{C})$ is very small, he finds that the complete Equation (1) degenerates to

$$L\ddot{q} + R\dot{q} = 0 \tag{7}$$

Integrating it, one obtains

$$L\dot{q} + Rq = M \tag{8}$$

where M is the constant of integration. The value of M is determined by initial conditions, namely

$$L\dot{q}_0 + Rq_0 = M \tag{9}$$

The solution of Equation (8) is then

$$q = M/R + Ae^{-(R/L)t} \tag{10}$$

From (9) and (10)

$$A = -L\dot{q}_0/R$$

Hence, from (9) and (10)

$$\bar{q}(t) = q_0 + \dot{q}_0(L/R)(1 - e^{-(R/L)t}) \tag{11}$$

If, however, one proceeds with solution of Equation (1) in the neighborhood of its degeneration, where $(\frac{1}{c})$ is very small, the approximate solution of Equation (1) is given by

$$q_1(t) = q_o\, e^{-\,t/RC} + (L/R)\, \dot{q}_o\, (1 - e^{-(R/L)t}) \tag{12}$$

Framing the functions in order to compare the solutions of Equation (1) and Equation (8), we have

$$\Phi(t) = \left|q_1(t) - \bar{q}(t)\right|, \text{ and } \dot{\Phi}(t) = \left|\dot{q}_1(t) - \dot{\bar{q}}(t)\right| \tag{13}$$

It can be seen, by the argument similar to that given in connection with R C degeneration for functions $\theta(t)$ and $\dot{\theta}(t)$, that for sufficiently small $(\frac{1}{c})$ the function $\dot{\Phi}(t)$ approaches zero when $\frac{1}{c} \to o$, uniformly in the interval $\alpha < t < \infty$, whereas $\Phi(t)$ approaches zero when $\frac{1}{c} \to o$ for all values of t except when $t \to \infty$, for which value $\Phi(t)$ approaches the value q_o.

3. Initial Conditions

Let us now return to the case of small L. In a physical system of the second order there are two aribtrary constants which appear as two initial conditions. More specifically in the circuit of Figure 4 if we assume that the charge in the capacitor is initally zero, then the initial conditions when switch S is open can be written as $q_o = \dot{q}_o = 0$. If, however, one adopts the degenerate d. e. for the description of the system, where there is only one constant, there appears the following difficulty. The state, when switch S is open, is specified by two aritrary constants and the degenerate d. e. admits only one which raised the question: what happens to the second constant when the switch is suddenly closed; i.e., an impulse is applied to the right hand side of d. e. in Equation (1).

The answer to this is that the variable $\dot{q}$ whose convergence is not uniform on the basis of theory of degeneration, will suddenly jump to its

final value beginning with which the process is determined by one single

constant (d. e. of lst order) as it should be. Thus "conflict between the

constants of integration", so to speak, has been removed, owing to the

discontinuity of the variable which can vary discontinuously on the basis of

degeneration theory. The following discussion illustrates what has just been

said.

The system of Figure 6 can be described as

$$LC\ddot{q} + RC\dot{q} + q = EC \tag{14}$$

Immediately before the application of E, when the circuit was "dead", the

conditions were obviously $q_o = \dot{q}_o = 0$.

We first consider the R C degeneration, that is when inductance L is so

small that we use the degenerate d. e. of the first order,

$$RC\dot{q} + q = EC. \tag{15}$$

There is only one constant of integration here and it is determined by the

initial condition: for $t = 0$, $A = - EC$, where A is the constant of integration.

The solution is then $q = EC(1 - e^{- t/RC})$. Differentiating this expression,

we have $\dot{q} = E/R \, e^{- t/RC}$, and for $t = 0$ this gives $\dot{q}_o = E/R$, whereas immediately

before the application of E, the current was obviously zero ($\dot{q}_o = 0$). This

means that the variable $\dot{q}$ has to change discontinuously if the degenerate d. e.

is to be used to represent a phenomena whose initial state is specified by

two initial conditions. Another conclusion is noteworthy: one has seen that

in the case of a degenerate d. e. $q = EC[1 - e^{- t/RC}]$ and $\dot{q} = (E/R)e^{- t/RC}$.

The ratio $q/\dot{q}$ in this case is a definite function of t and is not arbitrary

as in the corresponding complete equation. In other words, instead of a two

dimensional representation, (the phase plane) in the later case, we now have

a line because there is only one arbitrary constant of integration instead of

two.

The equation of R L degeneration in Figure 6 can be written as

$$L \, di/dt + Ri = E \tag{16}$$

and under the same assumed initial conditions, the solution is

$$i = (E/R) \, (1 - e^{-(R/L)t})$$

Differentiating this expression and setting $t = 0$, one finds $(di/dt)_{t=0} = -E/L$. But at the instant immediately proceeding the application of E one had $(di/dt)_{t=0} = 0$. It can therefore be concluded that the second initial condition has to jump discontinuously if the physical existence of two initial conditions just before the application of E is to be reconciled with the existence of only one initial condition imposed by the degenerate d. e. of the first order, which admits only one constant of integration.

Summing up, in both cases the situation remains the same, namely the variables in d. e. [q the charge in case of R C degeneration and i, the current in case of R L degeneration] cannot vary discontinuously and are determined directly by degenerate d. e. of first order. However, derivatives of these variables $dq/dt = i$ (current in capacitive dircuit) and di/dt [or $L \, di/dt$, the voltage across the inductance] can and, in fact, must vary discontinuously, in order to reconcile the physical existence of two initial conditions before the application of E, with the requirement of one single constant of integration - if the d. e. has to be used in degenerated form to describe the phenomena after application of E.

It is clear that what has been said about sudden application of the external impulse E, holds equally well when E is suddenly removed or generally changed. The essential point is that THE VARIABLES WHICH APPEAR IN DEGENERATE

D. E. VARY CONTINUOUSLY IN ACCORDANCE WITH THESE EQUATIONS, BUT THEIR DERIVATIVES
JUMP DISCONTINUOUSLY INTO THE VALUES WHICH THEY MUST HAVE THROUGHOUT THE
SUBSEQUENT PROCESS.

4. Graphical Representation of Discontinuity

Considering the case of R C degeneration in Section 2 and assuming the
following initial conditions

$$q = q_0, \quad \dot{q}_0 = 0 \text{ at } t = 0$$

one gets from Equation (3)

$$q(t) = q_0/(\beta - \alpha)\left[\beta e^{-\alpha t} - \alpha e^{-\beta t}\right] \tag{17}$$

and

$$\dot{q}(t) = -q_0\alpha\beta/(\beta - \alpha)\left[e^{-\alpha t} - e^{-\beta t}\right] \tag{18}$$

From (18) the maximum value of current will occur at time t_1 given by

$$t_1 = 1/(\beta - \alpha) \log (\beta/\alpha) \tag{19}$$

If L is very small; i.e., system is highly damped t_1 will be very small and
the maximum value of $\dot{q}$ designated by $\dot{q}_1$, will be very slightly smaller than
$- q_0/RC$.

In the degenerate case from Equation (5) we have

$$\bar{q}(t) = q_0 e^{-t/RC} \tag{20}$$

$$\dot{\bar{q}}(t) = - 1/RC \, q_0 e^{-t/RC} \tag{21}$$

It can be seen from Equation (21) that $\dot{q}(t)_{t=0} = - q_0/RC$ and not zero as it
should be according to assumed conditions. The nature of the graph of Equation
(17) is given in blue ink in Figure 7a, while that of Equation (20) is given in
red ink in the same figure. The nature of the graph of Equation (18) and
Equation (21) appear in Figure 7b in blue and red ink, respectively.

One should notice that both charge and current of the degenerate circuit
corresponds to the originial circuit except for a brief interval $o < t < t_2$ where

current of the degenerate circuit does not correspond to that of the original circuit. However, since inductance of the circuit is very small, the variation of current in the circuit for the brief duration t_2 is very rapid and the current will approach very rapidly to the value which will be given by the equation of the first order. If one is not interested in the details of variation of current for small t_2, one may neglect inductance and instead of studying initial stage of movement introduce the jump. As long as one considers that the circuit possesses capacitance only and no inductance one may consider that all the energy is stored in the capacitor and since the charge on the plates of the capacitor does not have time to change during the brief period t_2, the condition of jump permits an abrupt variation of current with the charge of the capacitor remaining constant.

This, however, merely confirms the result obtained in section 2 that while using degenerate d. e. (R C degeneration) one must assume, independently of the initial conditions, that the current jumps to the value defined by first order equation while charge remains essentially constant. We then have for $i = \dot{q}$ the same curve as of the first order equation. Of course a real circuit will always have some inductance, thus ruling out abrupt jumps of the current. If however, the inductance is small and the current changes rapidly, it may be assumed for many applications that it undergoes an instantaneous jump.

5. Mathematical Justification of Degeneration

In proceeding sections the discontinuous theory has been outlined with the help of a simple R L C circuit and the illustration is sufficient to give an idea of the physical meaning of the theory. In this and the following sections the theory will be generalized and some principles of its application in relaxation oscillations will be given.
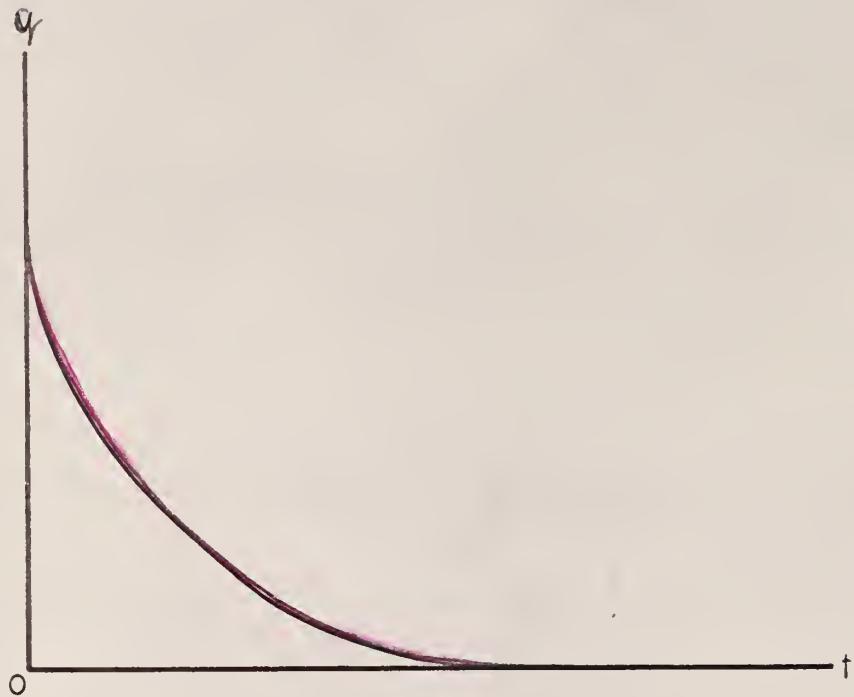
Figure 7a
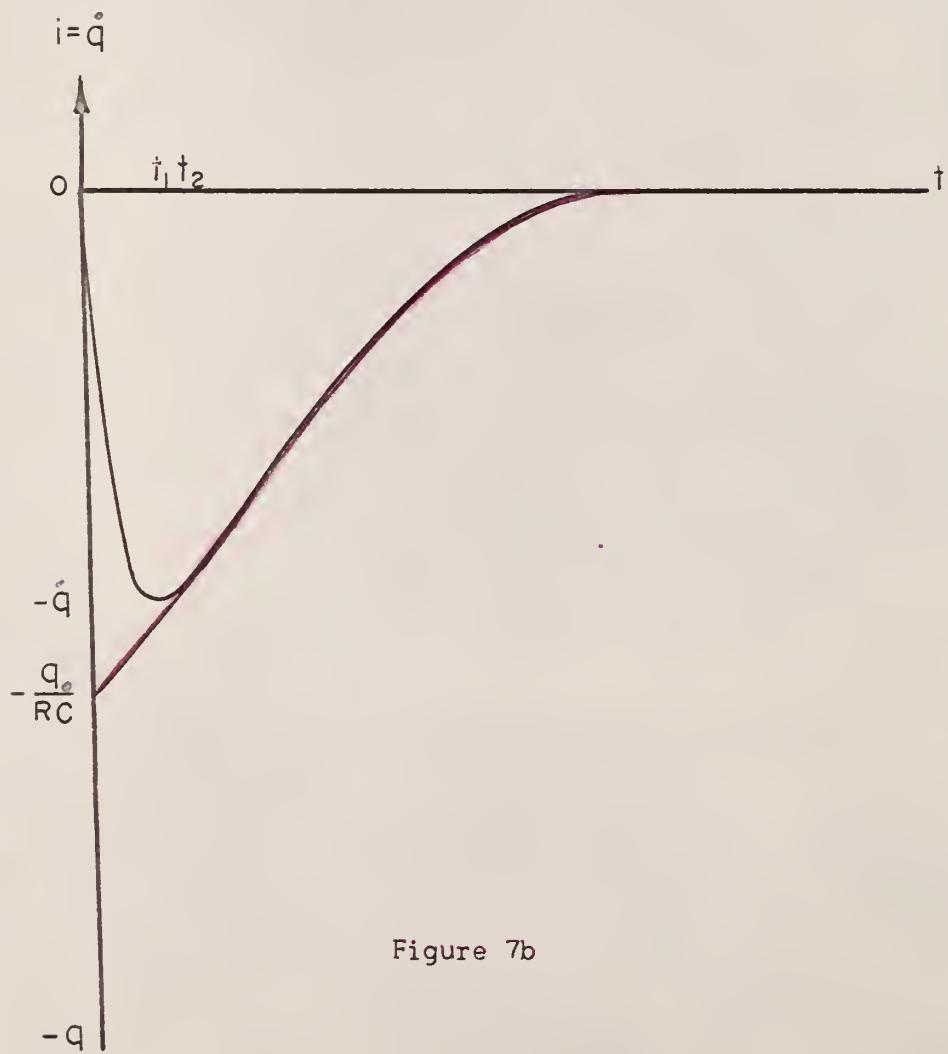
Figure 7b

Degenerate Equation: Some relaxation oscillator problems can be reduced to van der Pol's equation of the form

$$\ddot{x} + \lambda f(x)\dot{x} + g(x) = 0 \tag{22}$$

where λ (according to one of the early publications of van der Pol and later confirmed by Minorsky and by Flander and Stoker) is of the order of 10^5. The equation can then be written as

$$\varepsilon\ddot{x} + f(x)\dot{x} + \varepsilon g(x) = 0 \tag{23}$$

where $\varepsilon = 1/\lambda > o$ and is very small. For completeness a forcing term $e(t)$ will be included in the right side of d. e. and it will be assumed that $f(x)$ can change sign. Hence, Equation (23) becomes

$$\varepsilon\ddot{x} + f(x)\dot{x} + \varepsilon g(x) = e(t) \tag{24}$$

The purpose of this discussion is to demonstrate how the information about the solution of Equation (24) can be obtained from a study of the following degenerate euqation.

$$f(y)\dot{y} = e(t) \tag{25}$$

If one considers that e, f and g are continuous; then under normal circumstances for any initial values x_o, $\dot{x}_o$ and t_o, Equation (24) has the unique solution $x(t)$ such that $x(t_o) = x_o$, $\dot{x}(t_o) = \dot{x}_o$. Under this relatively mild initial assumption one can guarantee that no solution goes to infinity in a finite time; hence, every solution is continuable for all $t \geq to$. The most important extra condition is that $\int_o^x f(u)du$ shall be unbounded above and below as x varies from $-\infty$ to $+\infty$. This condition in the normalized form can be written as: $\lim\limits_{|x| \to \infty} F(x)\,Sgnx = +\infty$ where $F(x) \equiv \int_o^x f(u)du$

On the other hand, if f(x) has zeroes then Equation (25) may possess no solution for some initial values, and certain of its solutions may remain bounded, yet continuable only for values of t in a restricted interval about to. Nevertheless, Equation (25) in its integrated form

$$F(y) = F(x_o) + E(t) - E(t_o) \qquad (26)$$

where $E(t) \equiv \int_o^t e(u)du$ has solution $y = y(t)$ such that $y(t_o) = X_o$ for all values of X_o, t_o; these solutions are continuable (although perhaps not uniquely) for all $t \geq t_o$ because of the behaviour of F(x) at infinity.

In the simplest case, when f has isolated zeroes, one can select from among the solutions of Equation (26) a special class of function y(t) which approximates the solutions of Equation (22) for small positive ε. In the following paragraphs the heuristic consideration which motivates the definition of the "degenerate solutions" has been outlined.

Equation (22) can be transformed into the following equivalent pair of first order equations by the substitution $W = \varepsilon \dot{x} + F(x)$.

$$\varepsilon \dot{x} = W - F(x) \qquad (27a)$$

$$\dot{W} = \varepsilon(t) - \varepsilon g(x) \qquad (27b)$$

The solution of Equation (1) may now be thought of as trajectories $\left[x(t), W(t)\right]$ in the x, w plane. The curve Γ. $[W = F(x)]$ plays an important role in the study of the trajectories by Equation (27a). If $\left[x(t), W(t)\right]$ lies above Γ then $\dot{x}(t) > o$, while if $\left[x(t), W(t)\right]$ lies below Γ then $\dot{x}(t) < o$. Indeed for small ε, if $W(t) - F\left[x(t)\right]$ is not "very" small then $\dot{x}(t)$ is large. Equation (27b) shows that w is probably bounded as $\varepsilon \to o$.

Since f(x) has isolated zeroes, F(x) is piecewise strictly monotone. Let F_+ denote the set of values of x at which F(x) is increasing, F_o the isolated points at which F(x) has extreme, F_- the remaining points. In Figure 8,

x_1, x_2, and x_3 are in F_0, the open interval (x_1, x_2) is in F_+; the open interval (x_1, x_2) belongs to F_-. Horizontal inflectional tangents, such as x_4 are not excluded.

It seems plausible that the set of points (x, w) near to Γ with x coordinates in F_+ should be in a stable region for solutions of Equation (22). Suppose that at a certain time a trajectory is at P Figure (8) . The, since it lies above Γ, it has a large positive horizontal velocity, and hence tends to move rapidly towards Γ; its velocity decreases as it approaches Γ. Similarly a trajectory point at Q will have a large negative horizontal velocity and therefore should move towards Γ, decreasing the magnitude of x. Of course, either trajectory may cross Γ; but once near to it, it should be nearly impossible for a trajectory point to leave the trajectory so long as $x(t)$ remains in F_+.

By a similar argument it appears that the region near Γ with x in F_- will be highly unstable. Any slight tendency to leave Γ is quickly reinforced; trajectory points such as those at R and S are expected to "jump" horizontally to the first accessible increasing branch of Γ.

Assuming that the term $\epsilon g(x)$ may be neglected, and integrating Equation (27b) one obtains

$$W - W_0 = E(t) - E(t_0) \tag{28}$$

Then if $\epsilon \dot{x}$ is small we combine Equation (27a) and Equation (28) to obtain the equation.

$$F(x) = F(x_0) + E(t) - E(t_0) = W \tag{29}$$

It has also been assumed that $\epsilon \dot{x}_0$ is small.

The second equation of Equation (29) should be a good approximation to the actual motion defined by Equation (27a,b), since only the term $\epsilon g(x)$ has
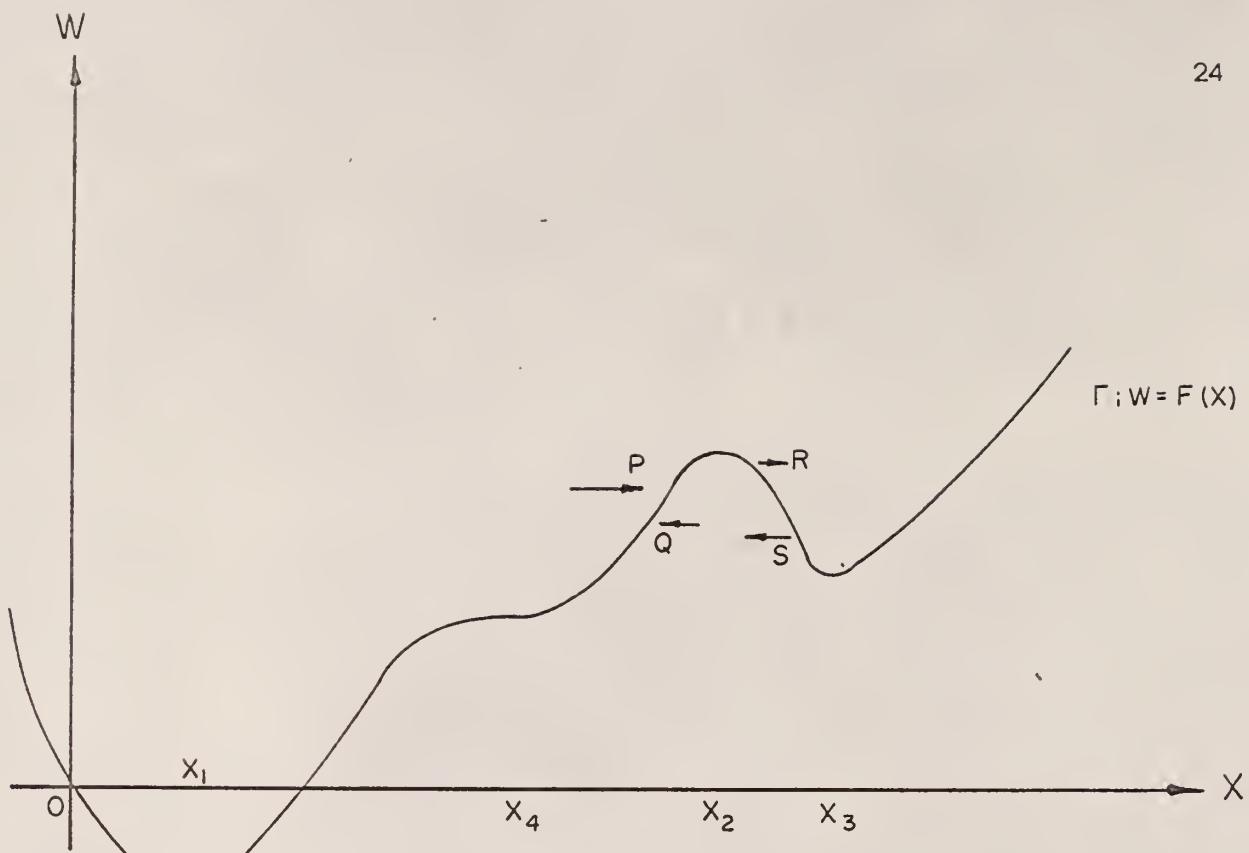
Figure 8
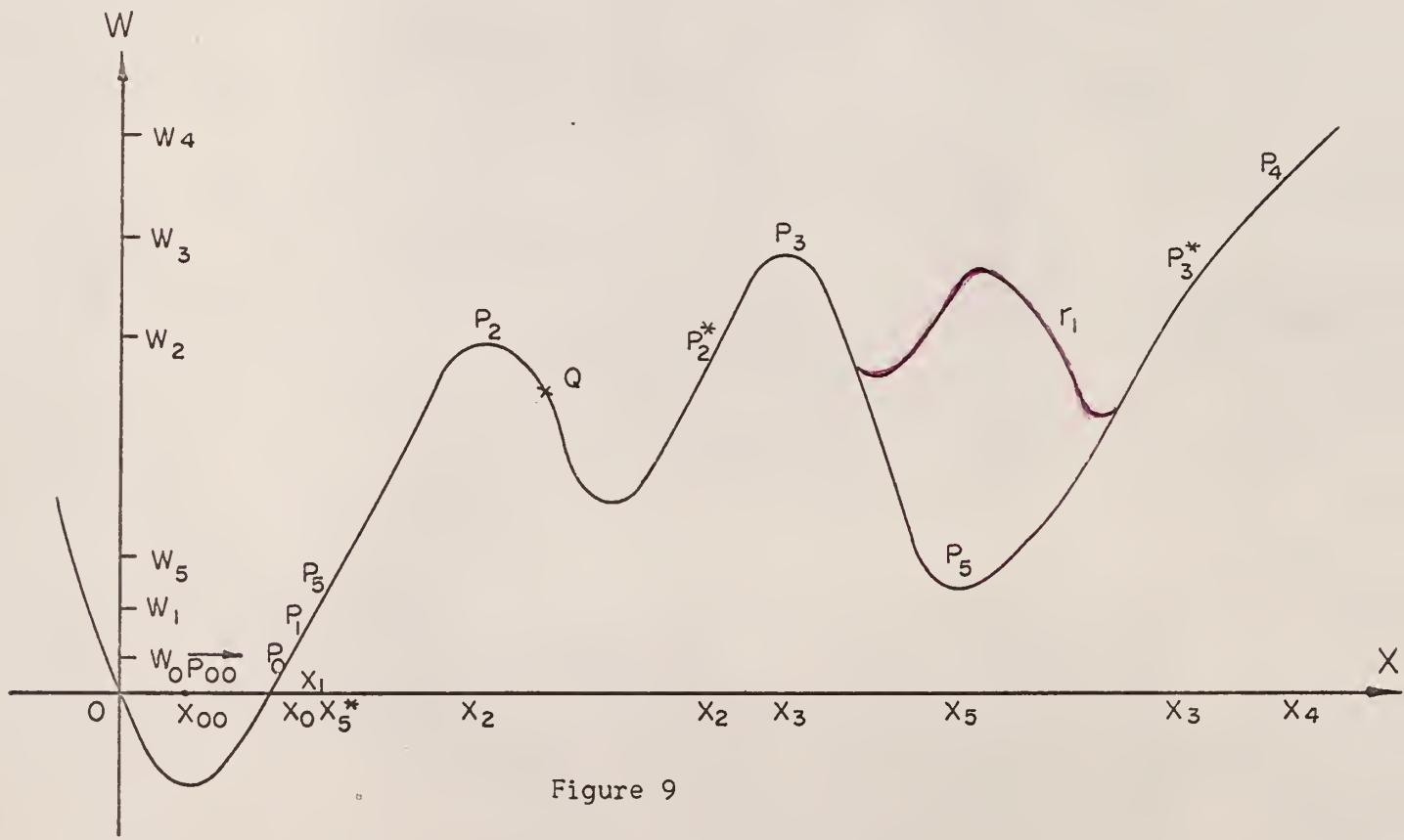
Figure 9

been neglected. The first equation of Equation (29) should be a good approximation if $\epsilon \dot{x}$ is small, which, by the stability argument above, should be the case as long as $x(t)$ stays in F_+. Thus, wherever Equation (29) is applicable, the true solution $x(t)$ should be near to an appropriate solution of Equation (26).

Let us follow the approximate motion of a trajectory beginning at $P_{oo}(X_{oo}, W_o)$ in Figure 9. (No significance is attached to the fact that Γ has been drawn for different $F(x)$ in Figures 8 and 9, nor to the fact that all of the action takes place in the first quadrant). Since P_{oo} is well above Γ, the initial velocity is positive and large. Hence, there is an almost instantaneous horizontal jump to P_{oo}: (x_o, W_o), which we may think of as a preliminary adjustment of initial conditions.

Let $e(t)$ is such that the function $W = F(x_o) + E(t) - E(t_o)$ varies between the levels W_o and W_1. The solution trajectory moves along Γ between P_o, and P_1; we expect that $x(t)$ is closely approximated by the solution $y(t)$ of the equation $F(y) = F(x_o) + E(t) - E(t_o)$, which lies between X_o and X.

Instead, if one considers that W increases steadily from W_o to W_4, then until W is near to W_2, $x(t)$ is near to the solution $Y(t)$ of $F(y) = F(x_o) + E(t) - E(t_o)$ lying between X_o and X_2. As W continues to rise, the trajectory is carried to a level considerably above Γ and thus $x(t)$ acquires a very large positive velocity. The trajectory point then jumps to the next increasing branch of Γ say to the vicinity of P_2^*; now as W rises to level W_3, $x(t)$ is approximated by the solution $y(t)$ of $F(y) = F(x_o) + E(t) - E(t_o)$ which moves from X_2^* to X_3.

At P_3, W is still rising; there is another jump to the right to the positive P_3^*. The rest of the motion is now smooth from X_3^* to X_4. The

situation would not have been different if Γ had the form of the red ink curve Γ_1, with a maximum point P_3 at the same height as P_3.

If now W falls from W_4 to W_0 then the trajectory moves smoothly from P_4 to P_5 along Γ, jumps to P_5* and returns smoothly to P_0. The corresponding solution $y(t)$ of Equation (26) moves from X_4 to X_5, jumps to X_5* and then moves to X_0. Of course, if Γ is changed to Γ_1, the number and location of the jump in downward cycle is altered.

The foregoing discussion suggests that the true solutions $X(t)$ of Equation (24) are approximated by "degenerate solutions" $y(t)$ whose essential features are:

1. $Y(t)$ satisfies $f(y) = F(x_0) + E(t) - E(t_0)$

2. $Y(t)$ lies in F_+

3. $Y(t)$ remains continuous when $Y(t)$ remains in F+, but jumps to the right or left from F_0 according as $F\left[y(t)\right]$ is a maximum or a minimum.

6. Critical Points

It has been shown in the last section that if one chooses to adopt degenerate d. e. (instead of complete d. e.) to represent a physical system, then discontinuties may appear at isolated points, as P_2, P_3 and P_5, in Figure 9. Nothing has been said so far about these isolated points except that these are the points of extrema, separating earlier defined regions F_+ and F_- of the trajectory . Determination of these critical points (points where discontinuous jump must take place) is the subject of this discussion.

In order to apply the discontinuous theory to the problems of discontinuous stationary relaxation oscillations, it becomes necessary to define the term "critical point" in a slightly different manner from that given in the previous mathematical treatment and to introduce some kind of basic assumption, the

value of which is justified by its agreement with the observed facts.

DEFINITION: CRITICAL POINTS ARE THE POINTS AT WHICH THE DIFFERENTIAL

EQUATION DESCRIBING A PHENOMENA IN A CERTAIN DOMAIN CEASES TO DESCRIBE IT.

BASIC ASSUMPTION: WHENEVER THE REPRESENTATIVE POINT FOLLOWING A TRAJECTORY

OF THE DIFFERENTIAL EQUATION DESCRIBINB A PEHNOMENON REACHES A CRITICAL

POINT, A DISCONTINUITY OCCURS IN SOME VARIABLE OF THE SYSTEM.

In what follows we will encounter three principal criteria by which the

existence of critical points can be ascertained. Criteria I and II are

largely used in relaxation oscillations, while criterion III is of immense

mathematical importance and can be used in case of extremely complicated

problems.

(I) The idea of critical points can be best explained by a general d. e. of

the form

$$dx/dt = P(x,y)/T(x,y) \; ; \quad dy/dt = Q(x,y)/T(x,y) \tag{30}$$

It is interesting to note that most of the problems reduce to the form of

Equation (30). This d. e. has nothing to do with the van der Pol equation,

because the latter uses strictly analytic theory and the question of

degeneration does not arise. It can be seen that Equation (30) becomes

meaningless or in other words ceases to describe the system at the point

x_c, y_c (critical point) for which $T(x_c, y_c) = 0$. One should note that, as

far as the trajectory is concerned, the passage through a critical point does

not in any way affect its determinanteness since T cancels out in the

expression $dy/dx = Q/P$. It is impossible, however, to determine the motion

on the trajectory in the neighborhood of the critical point. In this respect

the local properties of a critical point are opposite to that of singular

points where the trajectory is indeterminate but the motion is determinate.

In certain simple problems the reader may encounter a single point or a
number of critical points instead of a locus of critical points (critical
line). However, one should not be alarmed as the problem is to be handled
in exactly the same way.

(II) The existence of critical points or of a locus of such points can
sometimes be revealed from the study of trajectories in a certain domain of
phase plane. A typical example in which this can be done is shown in Figure
10. The trajectory can arrive at a depart from the certain threshold Γ from
both sides, as shown. IF NO SINGULAR POINTS, THAT IS POINTS OF THRESHOLD
EXIST IN THE NARROW DOMAIN SURROUNDING Γ, ONE CAN ASSERT THAT THE LINE Γ IS
A LOCUS OF CRITICAL POINTS.

It is apparent that the trajectories situated in the region M and N
belong to two different differential equations. Let us assume that the
phenomena is represented by motion of representative point P on a trajectory
W of the region N. Since the singular points are absent by an assumption, P
will reach point P on L in a finite time. Having reached this point, the
representative point finds itself in a kind of analytic impasse from which
there is no normal issue, that is, along the integral curves. In fact P
cannot pass into the trajectory W passing through P nor can it turn back on
W since, in both cases, this would be inconsistent with the differential
equations prescribing a definite direction on the trajectories of the two
regions M and N. Nor can the representative point remain at the point P which
is not a position of equilibrium. The differential equation ceases to have
any meaning at point P and therefore ceases to represent a physical phenomenon.
Hence, the point P is a critical point, and the line Γ is a locus
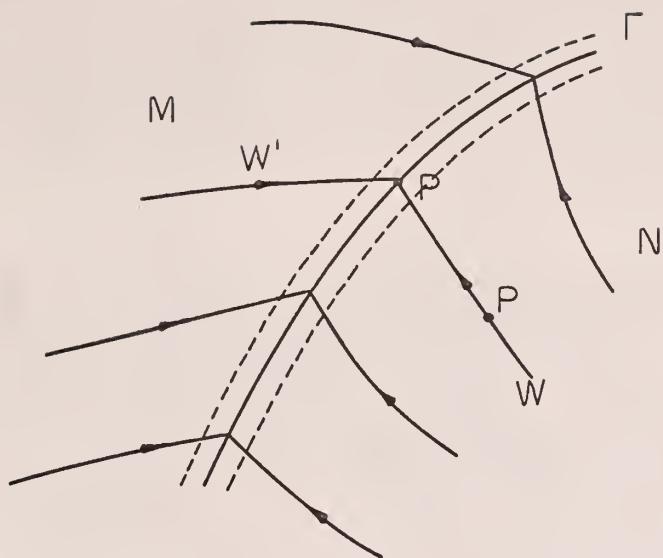
Figure 10
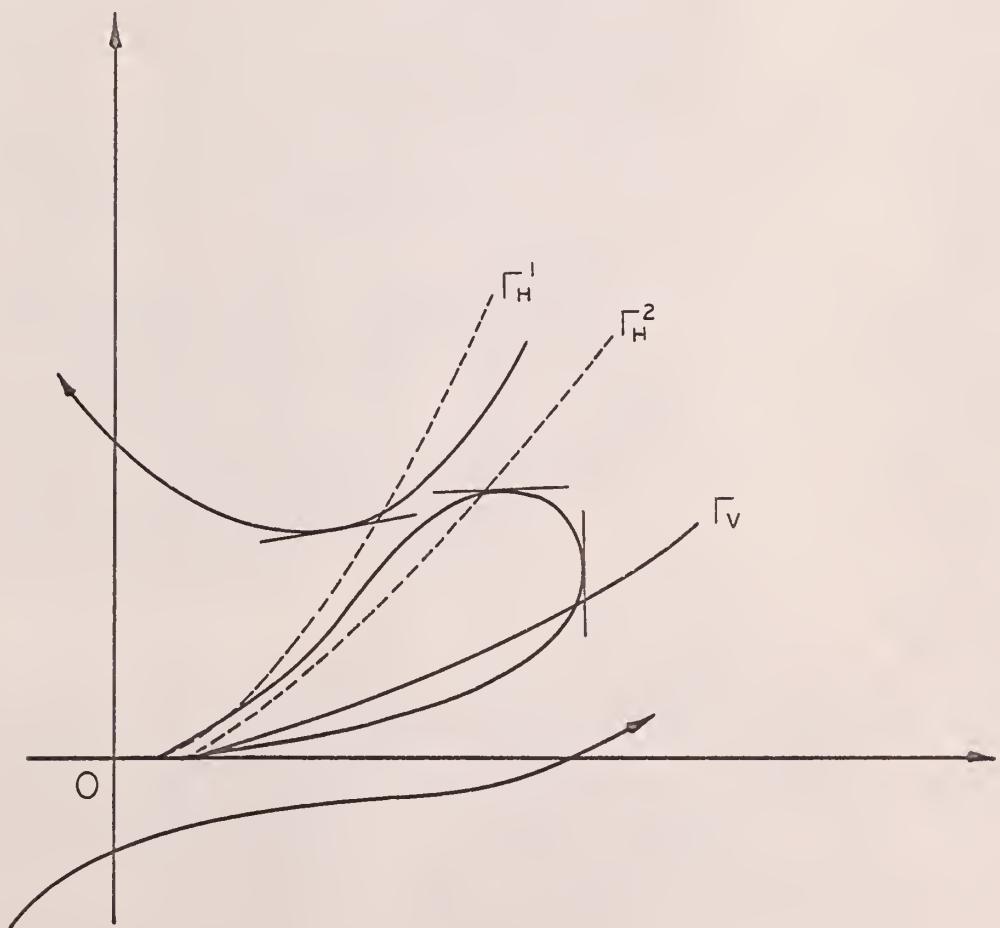
Figure 11

of such points. By our basic assumption, the discontinuities necessarily

occur once the representative point has reached some point on Γ.

Extensive use of both the assumptions mentioned earlier, will be made in

the investigation of relaxation oscillation in relatively complicated circuits

in which it is impossible to predict the nature of the phenomenon on the basis

of elementary intuitive reasoning. It will be seen in connection with the

relaxation oscillations proper that these assumptions are very handy and

useful tools in ascertaining the possibility of relaxation oscillations.

(III) The following discussion is presented here in order to demonstrate

the mathematical meaning of critical points. The presentation is the abstract

of Solomon Lefschetz's discussion on the subject in "Contribution to the

Theory of Nonlinear Oscillations, vol. IV."

The notations used are the following:

(1) $\left[X\right]_p$ and $\left[x,y\right]_p$ denote convergent power series in x or x and y

beginning in the terms of degree $\geq$p.

(2) $E(x)$ and $E(x,y)$ are convergent power series such that

$E(o) = E(o,o) = 1$

(3) <u>To-CURVE</u> denotes a path leading to or away from the origin in a

definite direction.

(4) Order of To-Curve means order of $y(x)$ on the curve.

NESTED OVALS: OVALS DESCRIBED BY REPRESENTATIVE POINT IN PHASE PLANE.

Theorem: A system with both characteristic roots zero but with first degree

terms not all zero, possesses at most a single sector of nested ovals (S.N.O.).

This single sector if it exists must be crossed by the y axis.

The following equation describes all the physical systems with both

characteristic roots zero but with terms of the first degree term not all

zero.

$$dy/dx = \left\{ \left[y^2 - 2A(x)Y + B(x) \right]^- E(x,y) \right\} \Big/ \left\{ - \left[y - c(x) \right] \right\} \tag{31}$$

$$A = \left[x \right]_1 \; ; \; B,c = \left[x \right]_2$$

We will first discuss the possible existence of S.N.O. to the right of the y axis. In the region there may exist branches issued from the origin where $dy/dt = 0$. If there are branches there will be two of them and will be denoted by Γ_H^1, Γ_H^2. In one region there exists always a branch Γ_V where $dx/dt = 0$. The branches Γ_H are jointly given by

$$y^2 - 2A(x)y + B(x) = 0 \tag{32}$$

If $\Delta = A^2 - B = [x]_2$, then the two branches are given by

$$y = A(x) \overset{+}{-} \sqrt{\Delta} \tag{33}$$

Now upon drawing various sketches corresponding to the branches to the right of y axis, one readily finds that the only disposition that might arise to an S.N.O. to the right of oy is the one of Figure 11, the Γ branches in the first quadrant and the Γ_H^1 above Γ_V.

Figure 11 has been drawn under the following convention adopted by Barocio*: the Γ_H branches are dotted lines and Γ_V is a continuous line.

Now Figure 11 is only compatible with $\Delta = \alpha^2 x^{2k} E(x)$, or else $\alpha^2 x^{2k+1} E(x)$. In order that the two branches be in the first quadrant we must have one of the following two systems of representation for our branches:-

I $\quad \Gamma_H^1 \; : \; y = ax^p E_1(x)$

$\quad\quad \Gamma_H^2 \; : \; y = bx^q E_2(x)$

$\quad\quad \Gamma_V \; : \; y = cx^r E_3(x)$

$\quad\quad p \leq q \leq r, \quad a \quad b > o, \quad c \geq o$

---

* Barocio, Universidad Nactional de Mexico

Observe that C = 0 means that Γ_v is the x axis.

$$\Gamma_H{}^1 : y = (x) + ax^{(q+1)/2}E_1(x^{\frac{1}{2}})$$

$$\Gamma_H{}^2 : y = (x) - ax^{(q+1)/2}E_2(x^{\frac{1}{2}})$$

$$\Phi = ax^p + \text{-----} + Px^q, \quad a>b, \quad a>o$$

$$\Gamma_H{}^3 : y = cx^2E_3(x)$$

$$r \geq p \text{ or else } v = p \text{ and } c \leq a$$

The general method used in finding the critical point consists in first finding possible orders of To-Curve by means of the Newton polygon. Then if μ is such an order we apply the transformation $y = x^\mu y_1$. It will turn out that μ is always an integer. The transformation replaces the given equation by a system

$$dy_1/dx = A(x,y_1)/X^r B(x,y_1) \tag{34}$$

with $x = 0$ as a solution. The images of To-Curves of the order μ can only be solutions tending to critical points P, Q -----, other than the origin and y axis. These are given by equation

$$A(o,y_1) = 0$$

and among them those corresponding to ends of an S.N.O. must be noted. The strict saddle points are thus to be eliminated at the outset.

7. Direction of Discontinuous Jump

It has been ascertained in the previous study that if a physical system is described by a degenerate d. e. and if there exists critical points in the phase plane, the representative point (P) must jump from the critical point to some other point where it encounters an analytic arc. To complete the discontinuous theory of relaxation oscillations, two questions are yet to be answered:

(1) What is the direction of discontinuous jump of P?

(2) In order to have periodic phenomona, is it necessary to have a closed integral curve with +1 as the algebraic sum of the indices of singular points in its interior?

The first question was answered by Mandelstam and is commonly known as "conditions of Mandelstam." On the basis of a few examples given earlier and many other examples, Mandelstam noticed that the variables which cannot change discontinuously in response to discontinuous changes in the forcing term are those which enter into the expression of stored energy. For example, we have seen, in the case of R C degeneration that the charge q can not change discontinuously, and at the same time we note that the stored energy in this case is purely electrostatic $E = \frac{1}{2}cv^2 = \frac{1}{2}qv$ where $q = cv$. In case of L R degeneration, the stored energy is $E = \frac{1}{2}Li^2$ and, again it was found that the variable i can not change discontinuously. On the other hand, dv/dt can change discontinuously and, therefore, also $i_c = c \, dv/dt$, where i_c is the current flowing in the capacitor circuit. Likewise di/dt can change discontinuously, which means that the voltage Ldi/dt across the inductance can change discontinuously.

The fact that the energy of the system cannot undergo a jump is a fairly plausible conclusion, because in order to produce discontinuous changes in energy, an infinite power is required, but this is ruled out on obvious physical grounds.

Thus the argument of Mandelstam is based on the continuity of the function $i(t)$, the current through inductor L, and $v(t)$, the voltage across the capacitor. Since $i(t)$ and $v(t)$ are continuous, clearly the electromagnetic energy $Li^2/2$ stored in an inductance and the electrostatic energy stored in the capacitor are also continuous functions of time. One obtains conditions

of Mandelstam by writing

$$\Delta i \begin{vmatrix} t_0+0 \\ t_0-0 \end{vmatrix} = 0 \qquad \Delta v \begin{vmatrix} t_0+0 \\ t_0-0 \end{vmatrix} = 0 \tag{35}$$

where $\left[(t_0-0), (t_0+0) \right]$ is the infinitely small time interval during which the discontinuity occurs. The important point to be noted in connection with these conditions is that they are applicable to an infinitely small time interval and to the circuits with finite dissipative parameters. It is thus clear that, if one wishes to represent a piecewise analytic phenomenon on the phase plane in the form of, say, two analytic arcs joined by discontinuous stretches, the later must correspond to variables which can vary discontinuously. Thus for instance, in case of R C degeneration, if one takes the variables $i_c = c\, dv/dt$ on the abscissa axis and V on the axis of ordinates, the discontinuous stretches are possible along lines parallel to the abscissa axis, in as much as in this direction discontinuities are possible because the condition of Mandelstam regarding the stored energy is fulfilled. Similarly in L R degenerates one can take on the abscissa axis the variable $V_L = L\, di/dt$ and on the axis of ordinate the current i through the inductance, and the representation is the same as in the previous case.

The above discussion ascertains the direction of discontinuity in the phase plane. Obviously it is this additional information (not contained in d. e. itself) which permits connecting what exists before and after discontinuity. In doing so we have intentionally ignored what happened in the rapid transition period which has been idealized by the mathematical concept of discontinuity.

The discussion now following is the answer to the second question.

If one represents the motion of a representative point of a physical system described by a degenerate d. e. on the phase plane, one gets the following picture.

A certain arc is followed until it meets a critical line at some point. At this point d. e. ceases to govern the phenomenon and a discontinuous stretch begins, being determined by condition of Mandelstam. It ends at a point where another analytic arc begins and d. e. again takes charge of the phenomenon until another critical point is reached which results in another discontinuity which brings P to the first arc, etc. If this point is that at which the process started, the periodic process is established at once. One has thus a piecewise analytic cycle which has no limit cycle feature. If, however, the process approaches the ultimate piecewise analytic cycle only after a series of rotations of radius vectors one has a kind of piecewise analytic limit cycle.

On the basis of observed facts it has been found that in order to have a periodic phenomenon it is not necessary to have the sum of indices of singular points inside the piecewise limit cycle as +1. In fact one may not have any singular point inside the piecewise bounded curve. An asymmetrical multivibrator is an example of this.

8. Summary

In Equation (30) if $T(x,y) \neq 0$, the system described by it is a normal one and classical theory is applicable. Thus, for instance, in Figure 12, if a point A of a phse plane is given (which means certain initial conditions), a trajectory, represented by an analytic arc A B will begin at this point and will continue up to the point B of coordinates x_B, y_B for which $T = 0$. As B is a critical point, at this point the d. e. lose their meaning and the analytic continuation of solution is impossible. If, however, one takes into account the condition of Mandelstam, a physical continuation is still possible. In fact, the point B in this theory is the "beginning" of the
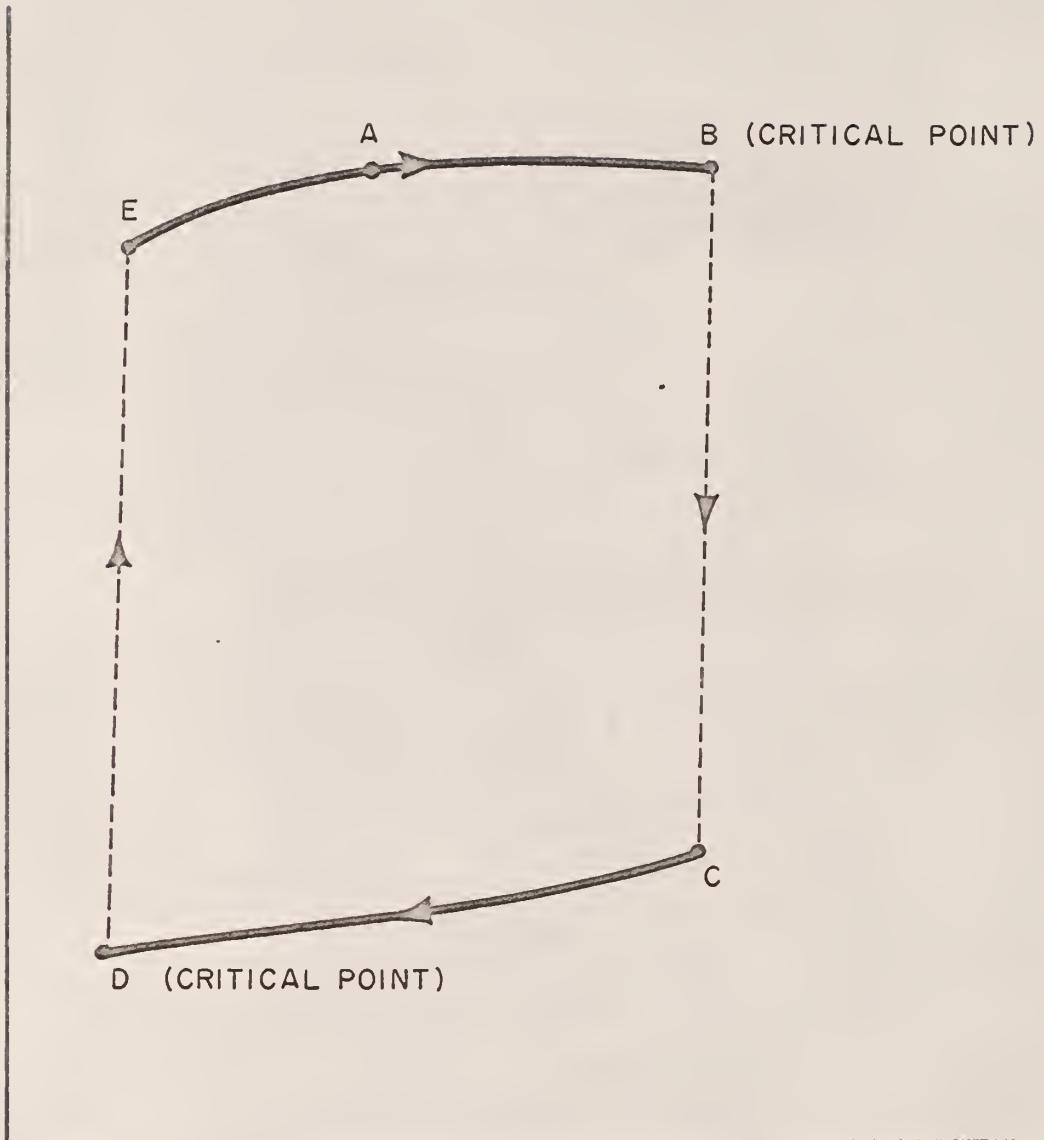
Figure 12

discontinuity B C traversed in no time, provided C is on another analytic arc C D representing the solution of the d. e.

Assume that arc D C ends at the point D for which T = 0 again. This determines another discontinuity D E which ends at point E which is on the arc E B and so on. The cycle consists thus of two analytic arcs E B and C D on which the motion of P occurs with finite velocity, joined by discontinuous stretches B C and D E traversed instantaniously.

It is useful to note the following points:

(1) The form of d. e., Equation (30), appears in practically all relaxation problems and it is generally impossible to reduce it to van der Pol's equation with a large parameter value. In fact, the parameter does not figure at all in these equations and the "critical points" B and D appear when T vanishes.

(2) The oscillatory phenomenon is governed by the d. e. as long as P moves continuously on the analytic arc but, on arriving at the critical point, the phenomenon ceases to be governed by d. e. during its rapid (instantaneous) transition until another analytic arc is encountered on which the motion takes place again in accordance with d. e. The instantaneous transition occurs in accordance with the conditions of Mandelstam.

The discontinuous theory of relaxation oscillations is now more or less completely established and in what follows we will encounter its application in some typical examples.

DEGENERATE SYSTEMS OF FIRST ORDER

9. Thyratron Relaxation Oscillator

An equation of the first order

$$\dot{x} = f(x) \tag{1}$$

obviously does not posses continuous analytic periodic solutions. Moreover

one can assert that if the function $f(x)$ is single - valued, no continuous,

although not necessarily analytic, periodic solutions are possible. In fact,

in order that some periodicity may exist, it is necessary that the system

traverse the same line $x = x_1$, with two oppositely directed velocities; this

however, is impossible if $f(x)$ is single valued.

As discussed earlier, the change from one branch of the function $f(x)$

to the other one generally occurs at critical points and is discontinuous.

Very frequently this is equivalent to saying, that the phenomenon is governed

by two distinct differential equations during its cycle. During one fraction

of the cycle the phenomenon is described by one d. e. and during the other

fraction by the other equation. The change from one d. e. to the other occurs

at the critical points.

Consider the degenerate equation of the form given in Equation (2)

$$f(y)y' + y = 0 \tag{2}$$

On the basis of the above discussion, the following condition must be

satisfied for the existence of relaxation oscillations

(1) $f(y)$ is a double valued function of y in some interval $y_1 < y < y_2$.

One of the branches of $f(y)$ is prolonged to form the curve in the interval

$y < y_1$, (Branch 1) while the other branch forms the curve in the interval $y > y_2$

(Branch 2)

(2) For the establishment of oscillatory regime it is essential that on

branch 1, f(y) is negative, while on branch 2, f(y) is positive.

(3) The energy contained by the system in the initial state of the second regime must be the same as that contained by the system in the final state of the first regime.

As an example let us consider the circuit given in Figure 13. Following notations have been used:

V_1 = Firing voltage

V_2 = Extinction voltage

i = Current in the neon tube

$i = \Phi(v)$ is the neon tube characteristic

Applying Kirchhoff's laws

$$R(I + i) + V = E, \quad c\dot{v} = 1$$

Hence

$$R\left[c\dot{v} + \Phi(v)\right] + v = E$$

or

$$\dot{v} = f(v) = 1/Rc \left[E - v - R\Phi(v)\right] \tag{3}$$

This d. e. is valid only when the discharge exists. During extinction, from the knowledge of neon tube performance, we know that $i = \Phi(v) = 0$ and hence from Equation (3)

$$\dot{v} = 1/Rc \left[E - v\right] \tag{4}$$

The equilibrium will be given by $f(v) = 0$ or $E - v/R = \Phi(v)$. To find the roots of this equation we construct the graphs $i = \Phi(v)$ and $Z = (E-v)/R$ and find their intersection. [Figure (14)] It is obvious that one can place the equilibrium point in upper or lower portions of the characteristic $i = \Phi(v)$ by changing E or R. We will set such a value of R that the equilibrium point ("0") lies on the lower portion of the characteristic.
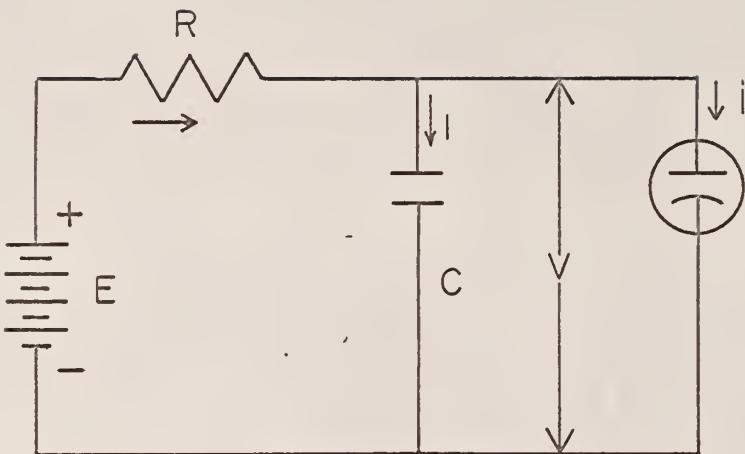
Figure 13
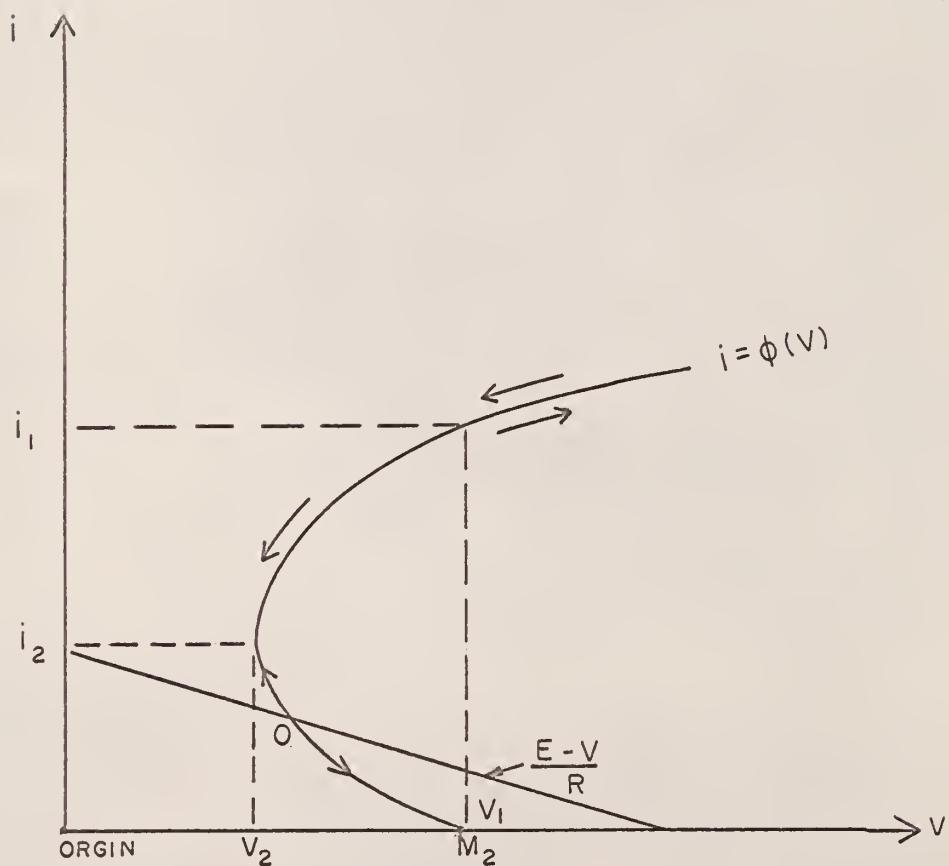
Figure 14

From Equation (3) $f(v) = (1/Rc) \left[E - v - R\phi(v) \right]$

$df(v)/dt = 1/Rc \left[- dv/dt - R\phi'(v) \ dv/dt \right]$

where

$\phi'(v) = d \ (v)/dv$

or

$df(v)/dt = -(1/c)dv/dt \left[1/R + \phi'(v) \right]$ (5)

Obviously for the upper portion of the curve, slope $\phi(v)$ is positive and

since $1/R$ is always positive, $df(v)/dt$ is negative and the upper portion of

the curve is stable. Also if R is sufficiently large, for the lower portion

of the curve, $\phi'(v)$ $>1/R$ and slope is negative, hence the lower portion of

the curve is unstable. Thus in Figure 14 if the equilibrium point lies above

M_1 on the curve $i = \phi(v)$ it is stable, and it is unstable otherwise. The

stability has been marked by arrowheads in Figure 14.

As this is the case of R C degeneration, current can change abruptly

and hence critical points can be found as follows:

$i = \phi(v)$

$i = \phi'(v) \ dv/dt = \phi'(v) \ \dot{v}$

$= \phi'(v) \ f(v)$ (6)

We know that critical points can occur only when $\dot{i}$ goes to infinity. As

$f(v)$ remains finite, hence $\dot{i}$ can be infinity only when $\phi'(v)$ becomes infinite.

This can happen only when either (v) does not exist or ceases to be continu-

ous; i. e., at the points M_1 and M_2 of the characteristic. Hence M_1 and M_2 are

the critical points.

Beginning at the origin if we start charging the capacitor the represen-

tative point (P) in the phase plane Figure (15) will move from the origin to

$D(M_2)$. At D the tube will fire and P will jump discontinuously to A, according

to conditions of Mandelstam. Here d. e., Equation (3), will take care of
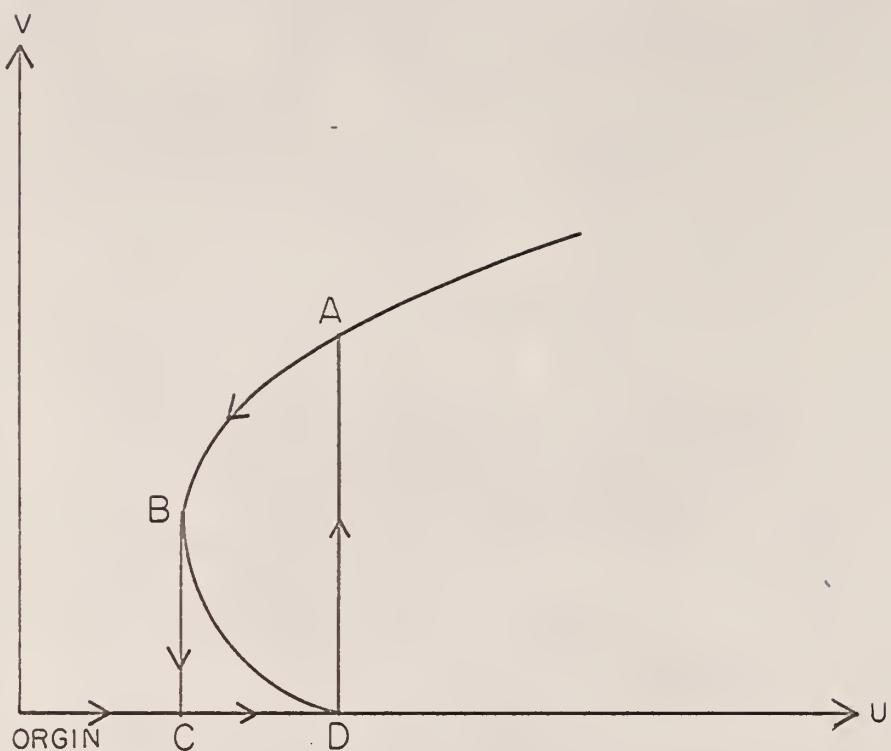
Figure 15
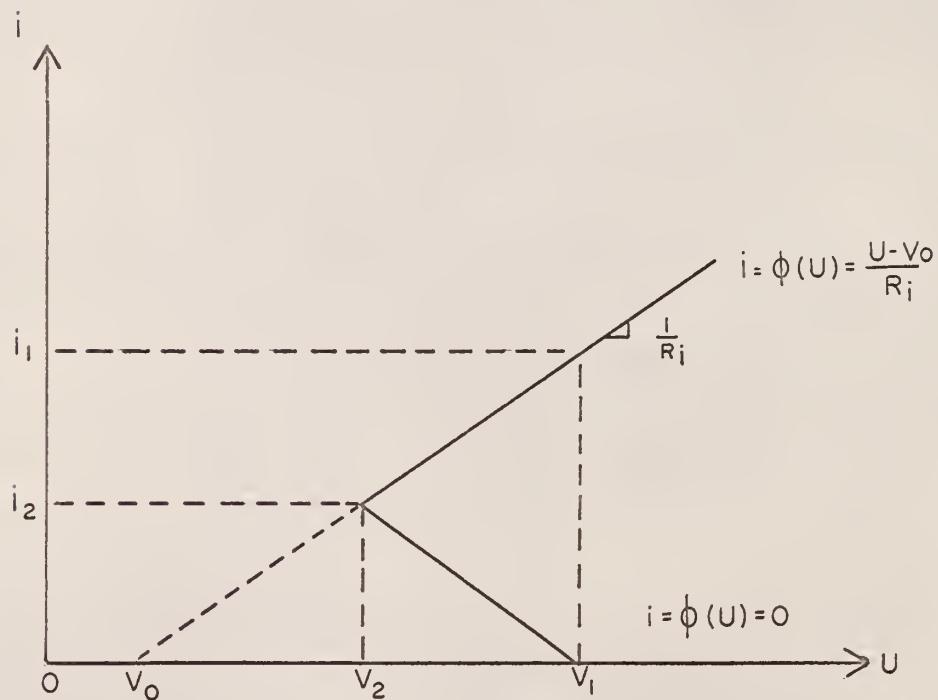
Figure 16

the phenomenon and P will move to B (B corresponds to critical point M_1).

Once P has reached B, it cannot move along the characteristic, since the

d. e. prescribes on it an opposite direction. At the same time B is not

the position of equilibrium and hence P has to jump discontinuously to C. At

C, P is again on the analytic curve $\left[i = \Phi(v) = 0 \right]$ and d. e., Equation (4)

takes charge of the phenomenon. Thus the piecewise analytic cycle ABCDA

consists of two analytic branches on which either of the d. e. takes charge

alternately, closed by two discontinuous stretches. $\Big[$DA (firing) and BC

(extinction).$\Big]$

If it is possible to idealize the $i = \Phi(v)$ curve as shown in Figure 16,

computation of wave form and frequency can be done as follows. The investi-

gation of the drooping portion of the characteristic can be omitted because

it does not come in the path of P. Two other branches of the characteristic

can be represented by

    (1) $i = \Phi(v) = 0$ when the tube is not conducting

and

    (2) $i = \Phi(v) = (V - V_0)/R$ when the tube is conducting.   (7)

Considering that initially the tube is not conducting and $v = v_2$, one has

from Equation (4)

$$Rc\dot{v} = E - v$$

or

$$v = A\,e^{-t/Rc} + E$$

As at $t = 0$, $v = v_2$, hence

$$A = v_2 - E$$

or

$$v = E - (E - v_2)\,e^{-t/Rc} \tag{8}$$

When the tube fires at D $\left[\text{Figure (15)}\right]$ the initial condition becomes $v = v_1$ at $t = 0$ and Equations (3) and (7-2) take charge of the phenomenon. Hence, from the two equations

$$Rc\dot{v} = E - v - R(V - V_o)/Ri$$

or $\quad Rc\dot{v} + (R/Ri)(V - V_o) + V = E$

Rearranging

$$c\dot{v} + v/Ri + v/R = E/R + V_o/Ri$$

or

$$c\dot{v} + v/\lambda = E/R + V_o/Ri$$

$$v = A e^{-t/c\lambda} + E\lambda/R + V_o\lambda/Ri$$

As at $t = 0$, $v = v_1$

$$A = v_1 - E\lambda/R - V_o\lambda/Ri$$

$$v = E\lambda/R + V_o\lambda/Ri + (v_1 - E\lambda/R - V_o\lambda/Ri) e^{-t/\lambda c}$$

or

$$v/\lambda = E/R + V_o/Ri + (v_1/\lambda - E/R - V_o/Ri) e^{-t/\lambda c} \qquad (9)$$

The nature of various wave forms is shown in Figure 17.

Time Period

From Equation (8) the time taken by the capacitor to charge from v_2 to v_1 designated by T_1 can be represented as

$$(E - v_2) e^{-T_1/Rc} = (E - v_1)$$

or

$$T_1 = 1/Rc \log \left[E - v_2/E - v_1\right] \qquad (10)$$

Similarly from Equation (9) the time taken by the capacitor to discharge from v_1 to v_2 designated by T_2 can be represented as

$$v_2/\lambda = E/R + V_o/Ri + (v_1/\lambda - E/R - V_o/Ri) e^{-T_2/\lambda c}$$
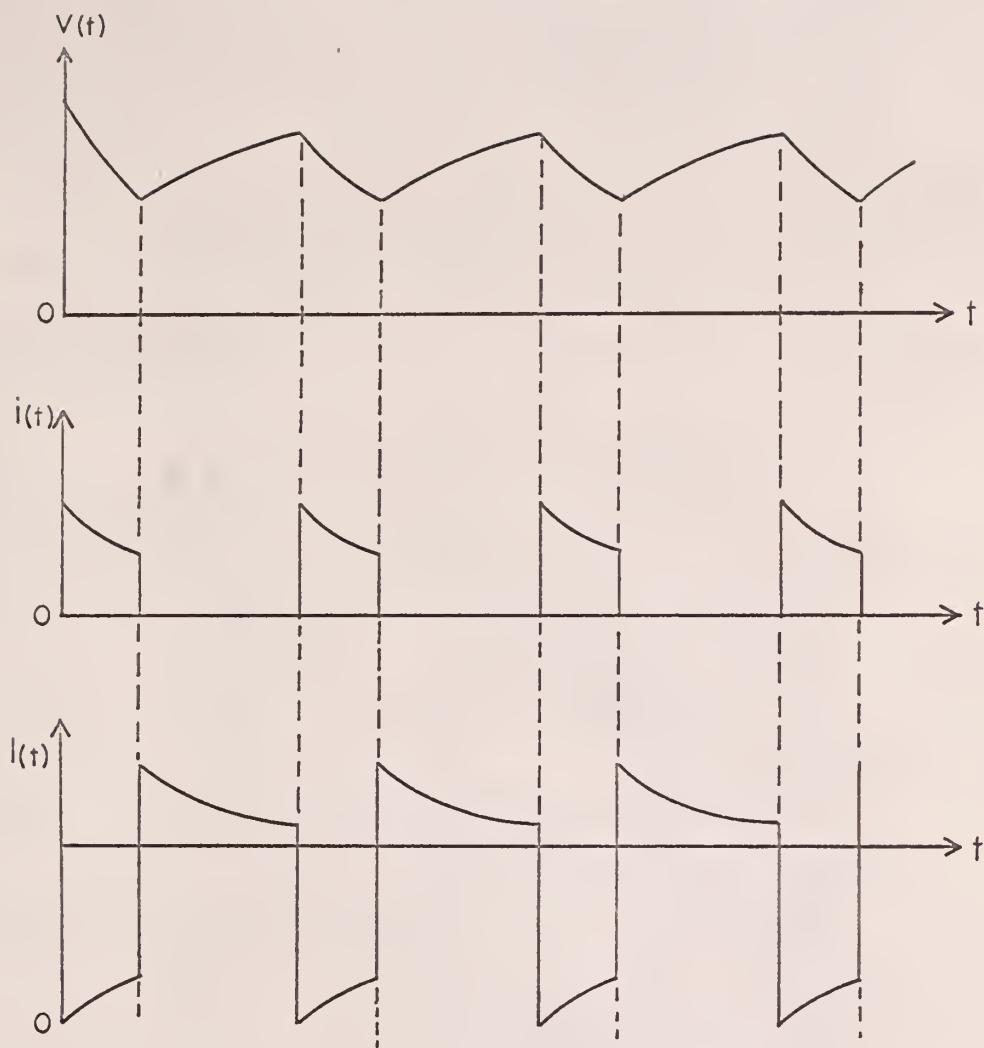
Figure 17

Figure 18

or

$$(v_2/\lambda - E/R - V_0/Ri) \; e^{\,T_2/\lambda c} = v_1/\lambda - E/R - V_0/Ri$$

or
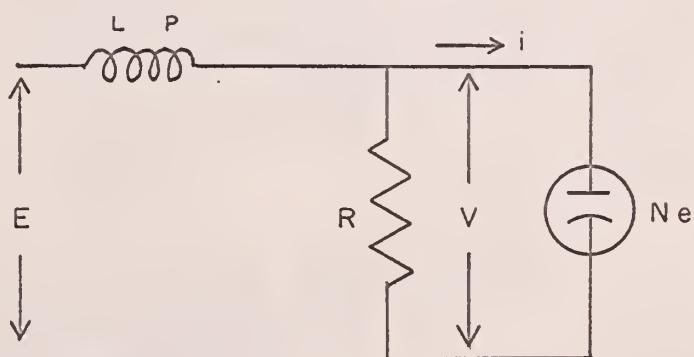
$$T_2 = \lambda c \left\{ \log \; (v_1 - V_0) \, R - (E - v_1) \, Ri \Big/ (v_2 - V_0) \, R - (E - v_2) \, Ri \right\} \quad (11)$$

Thus the time period is

$$T = (T_1 + T_2) \tag{12}$$

Once T has been determined, one may use Equation (8) from 0 to T_1, and Equation (9) from T_1 to T to expand $v(t)$ in a Fourier series expansion and get the structural composition of oscillations.

10. Neon Lamp Circuit Containing L and R

It can also be shown that if all three conditions given on pages 38 and 39 are not satisfied relaxation oscillations will not exist. As an example let us consider the circuit in Figure 18. Applying the Kirchhoff's laws we have

$$E = L \, d\big[i + V/R \big] / dv \cdot dv/dt + P(i + V/R) + v$$

or

$$L/R(1 + R \, di/dv) \, dv/dt = E - v - P(1 + V/R) \tag{13}$$

From Equation (13) it is clear that $f(v)$ is a double valued function and the system may satisfy the condtion (2) also on page 38 provided

$$v_1(1 + P/R) < E < v_2(1 + P/R) + Ri_0 \tag{14}$$

where i_0 is the value of current in the tube corresponding to extinction voltage. One may therefore, conceive the presence of relaxation oscillation in the circuit of Figure 18, that has however never been observed. The reason being that all the energy in this case has been assumed to be stored in inductance, and condition of Mandelstam implies that current should remain constant during the jump. From Figure 18

$$I \overset{..}{=} i + V/R$$

we know that current and voltage just before firing are $v = v_1$, $i = 0$ and those just after jump are given by $R i + v = v_1$. Also just before extinction the following relations must be satisfied $R i + v > R i_0 + v_2$. Combining the two statements gives

$$v_1 > R i_0 + v_2$$

or

$$1/R > i_0/v_1 - v_2 \tag{15}$$

Obviously this inequality cannot be satisfied if Equation (14) is, because this will mean

$$1/R + 1/P < i_0/(v_1 - v_2) \tag{16}$$

But if this is the case the equilibrium point will lie in the stable region and oscillations are out of question. The result will remain the same if one interchanges the position of circuit elements L and R. No doubt the relaxation oscillation can be expected from the circuit if one has voltage as a double valued function of current.

11. Dynatron Oscillator

The circuit of a Dynatron Oscillator is shown in Figure 19. In this case it has been assumed that plate potential is lower than the grid potential (300 - 400V). Obviously when plate potential V increases from zero the plate current i will first increase; it diminishes then because of emission of secondary electrons that are absorbed by the grid, but it begins to increase again in proportion as the potential, V, of the plate continues to increase, the plate re-absorbs itself more and more of the secondary electrons that it had emitted. One will thus get the negative resistance characteristic shown in Figure 20.

The plate voltage is related to the plate current by the relation

$$V = E - Ri - L \, di/dt \tag{17}$$
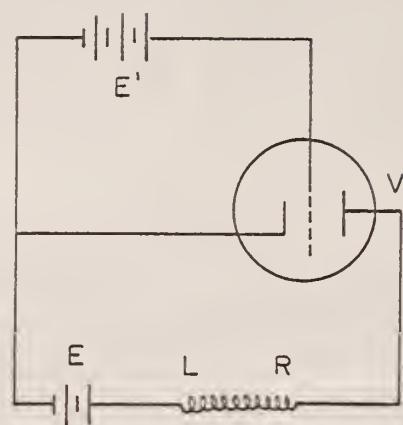
Figure 19
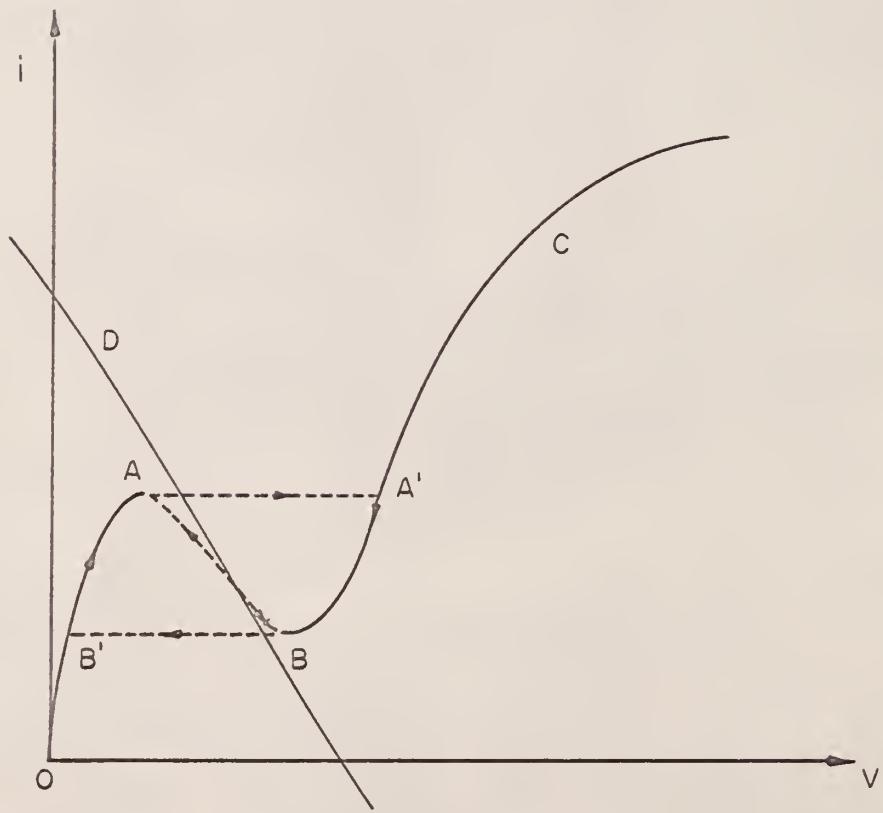
Figure 20

Let us draw the line $(D)\left[V = E - Ri\right]$ on the graph of Figure 20; such that the equilibrium point lies on the unstable negative resistence region.*

From Equaltion (17)

$$L \, di/dv \cdot dv/dt = E - V - Ri$$

or

$$dv/dt = E - V - Ri/ \, di/dt \tag{18}$$

From Equation (18) it is clear that the critical points will occur where $di/dv = 0$; i.e., points A and B on the characteristic. Thus the representative point (P) will move continuously from B' to A, at A it will jump to A' in accordance with the condition of Mandelstam (current being the invarient). The motion will be governed by d. e. and P will travel continuously to B where another jump will take place, bringing P back to B'. It establishes the piecewise limit cycle A A' B B' A.

12. Degenerate R C Multivibrator

The circuit of Degenerate R C Multivibrator is given in Figure 21. The fundamental assumption here is that the effect of small parastitic inductance is negligible. This means that from the very beginning one places oneself under the condition of R C Degeneration, in terms of discontinuous theory. Other assumptions are:

(1) The tube V_1, is a linear amplifier with amplification factor K, amplifying voltage between B D and provides the necessary 180° phase reversal so that tube V_2 may work as an oscillator if the total loop gain is equal to or more than unity. One has then

$$eg = Kri$$

(2) The grid current and reaction on the plate is neglected.

---

* Stability can be ascertained by Liaponnoff's conditions. A. A. Andronow and C. E. Chaikin. Theory of Oscillations (1949) pp - 147.
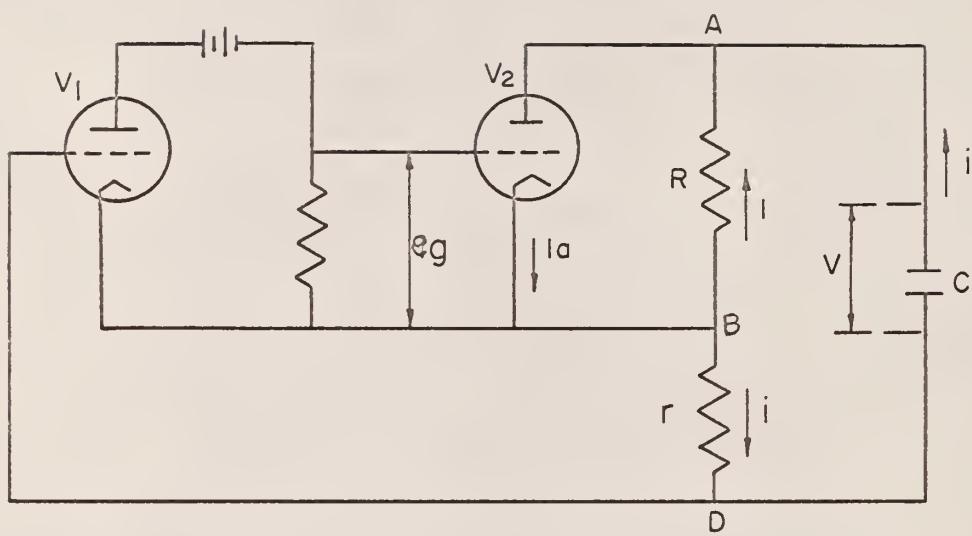
Figure 21

(3) Tube V_2 is a non-linear conductor whose characteristic is given by $I_a = \Phi(eg)$

Applying Kirchoff's law to the circuit we have

$RI = ri + v$

$I = I_a - i$

$I_a = \Phi(eg)$ according to assumption.

Combining first two equations

$$R\left[I_a - i\right] = ri + v$$

Rearranging and substituting $I_a = \Phi(eg)$ gives

$$(R + r)\ i + v = RI_a$$
$$= R\Phi(eg)$$

or

$$(R + r)\ i + v = R\Phi(Kri) \tag{19}$$

Also

$$v = 1/c\int i dt$$
$$i = c\ dv/dt$$

or

$$i = c\dot{v} \tag{20}$$

Differentiating Equation (19), gives

$$(R + r)\ \dot{i} + \dot{v} = RKr\Phi'(Kri)\ \dot{i}$$

Substituting i/c for v from Equation (20),

$$\left[RKr\Phi'(Kri) - (R + r)\right]\dot{i} = i/c \tag{21}$$

or

$$di/dt = i\Big/\left\{c\left[RKr\Phi'(Kri) - (R + r)\right]\right\} \tag{22}$$
$$= i/cT(i) \tag{23}$$

where

$$T(i) = \left[RKr\Phi'(Kri) - (R + r)\right] \tag{24}$$

and

$$\Phi'(Kri) = d\left[\Phi(Kri)\right]/di \tag{25}$$

The root i of the curve $T(i) = 0$ will give the critical points. One can proceed either analytically if polynominal approximation of $\Phi(Kri)$ is given, or graphically.

We shall adopt the graphical procedure and assume the idealised nature of characteristic $\left[I_a = \Phi(eg)\right]$ of tube V_2. This characteristic has been represented by $C\left[R\Phi(Kri)\right]$ in Figure 22a. C_1 represents curve $(R + r)i$. The difference of ordinates of C and C_1 is curve $V(i)$ and is represented by C_2. In Figure 22b, C_3 represents slope of C. It is clear that if we subtract from this slope curve (C_3) the constant slope $(R + r)$ of line $(R + r)i$, which implies shifting axis M' N' to M N, we get the roots of equation $T(i) = 0$. Hence, by difinition the points P and Q will be the critical points. It should be noted that by virtue of Equation (19), curve C_3, when referred to M N axis represents the slope of curve C_2. Because the slope of curve C_2 is positive between points B and D and negative everywhere else, critical points P and Q, when transferred to phase plane curve C_2 (plot of differential equation of first order in i and v or $\dot{i}$) must correspond to points B and D.

We note that C_3 when referred to M N (slope of curve C_2) is positive inside the interval $i_3 < i < i_1$, and negative outside; hence, according to Liaponnoff's criteria,* the system is unstable in this interval and stable outside. From Equation (1) the only equilibrium point on the curve C_2 is $(v = 0, i = 0)$, the origin, and hence the origin is the point of equilibrium and one can mark the stability as shown in the curve C_2. It can be seen that the representative point moves towards B and D from both sides, but

---

* Andronow, A. A. and Chaikin, C. E., Theory of Oscillations (1949) pp - 147.
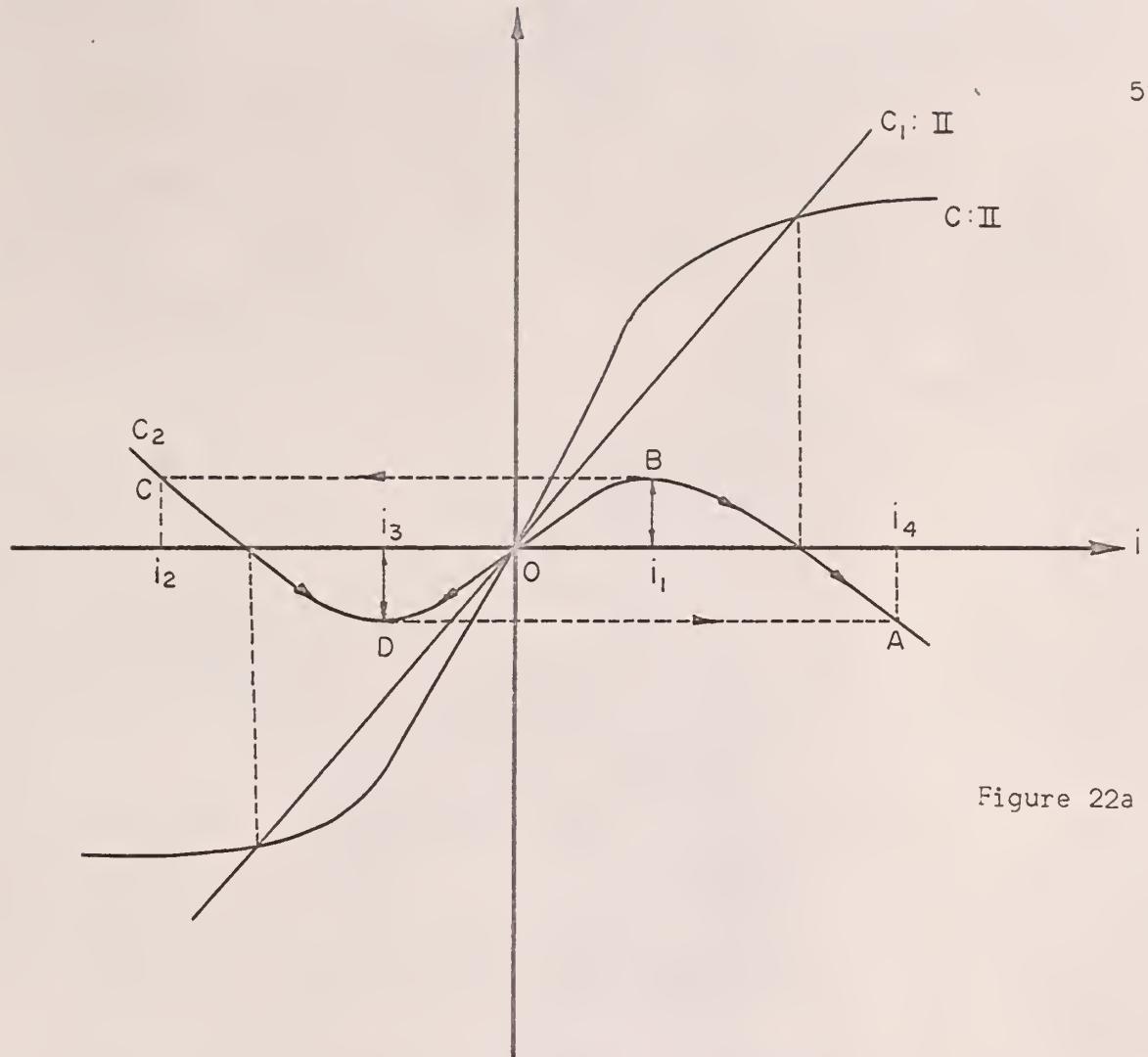
Figure 22a
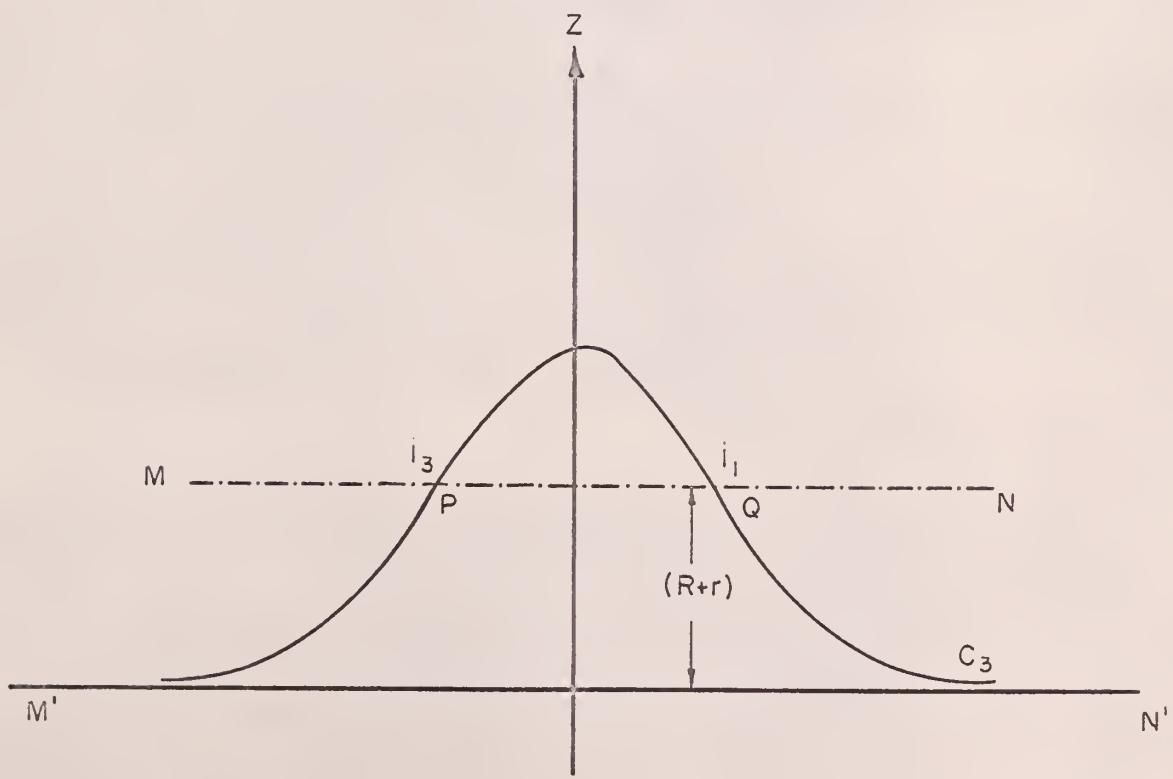
Figure 22b

B and D are not the points of equilibrium because at these points $T(i) = 0$ and hence according to discontinuous theory a jump must take place.

If one considers that the representative point is in the interval $i_3 < i < i_1$, and moving towards B (the interval being the region of instability), at B the jump must take place in the direction shown (V being the invarient). At C the representative point is again on C_2 (the analytic arc) and the analytic stretch C D is traversed with finite velocity. The discontinuous transition again takes place between D A, followed by analytic stretch A B. A piecewise analytic cycle A B C D A thus results.

The oscillations thus established have two continuous motions, from i_2 to i_3 and from i_4 to i_1. The form of oscillations or the form of function $i = \phi(t)$ is shown in Figure 23. It is a simple matter to determine the amplitude of oscillations which is determined by i_4 and i_2 as is clear from Figure 23.

The period of oscillation can be calculated in the following manner. We will idealize the curve C in Figure 22a by curve I in Figure 24 and assume that it is symetrical; i.e., $i_1 = i_3$ and $i_2 = i_4$.

From Equation (23)

$$dt = c \left[T(i)/i \right] di \tag{26}$$

The time period of oscillation is the time required by the representative point to describe the complete limit cycle and because, according to discontinuous theory the jumps are instantaneous, we can integrate Equation (26) between limits i_2, i_3 and i_4, i_1 to get the time period.

Hence

$$T = C \int_{i_2}^{i_3} T(i)/i \ di + c \int_{i_4}^{i_1} T(i)/i \ di \tag{27}$$

Because $T(i)$ is nothing but the slope of curve I minus the slope of curve II

and since in the region of interest $(i_2 < i < i_3)$ and $(i_1 < i < i_4)$ curve I is constant. Hence

$$T(i) = -(R + r) \tag{28}$$

It has been assumed that $i_4 = i_2$ and $i_1 = i_3$ hence Equation (27) reduces to

$$T = 2c(R + r) \int_{i_4}^{i_1} di/i$$
$$= -2c(R + r) \log i_1/i_4$$
$$= 2c(R + r) \log i_4/i_1 \tag{29}$$

If V_s is the saturation voltage and I_s the saturation current then obviously $i_1 = i_3 = V_s/2r$ (as given in Figure also). It is possible to see from the geometry of the figure that

$$i_2 = i_4 = \left[RI_s/(R + r) - V_s/2r \right]$$

Substituting these values of i_1 and i_4 in Equation (29), we have

$$T = 2c(R + r) \log \left[2RrI_s/V_s(R + r) - 1 \right] \tag{30}$$

13. <u>System Described by Two Degenerate Equations of First Order</u>

We have seen that neglecting the oscillatory parameter that plays a secondary role generally lowers the order of the oscillations. It may well happen, however, that the disregarding of certain parameters may result in a discontinuous solution but the order of the equation remains unchanged. This can be demonstrated by the multivibrator of Figure 21, if R is replaced by L. The resulting circuit is shown in Figure 25. Making the same assumptions which we have made in case of R C multivibrator and applying Kirchhoff's laws to the circuit of Figure 25, the following equations can be written:

$$I = \Phi(Kri) - i \tag{31}$$
$$L\dot{I} = ri + 1/c \int i \, dt$$

where

$$I_a = \Phi(Kri)$$
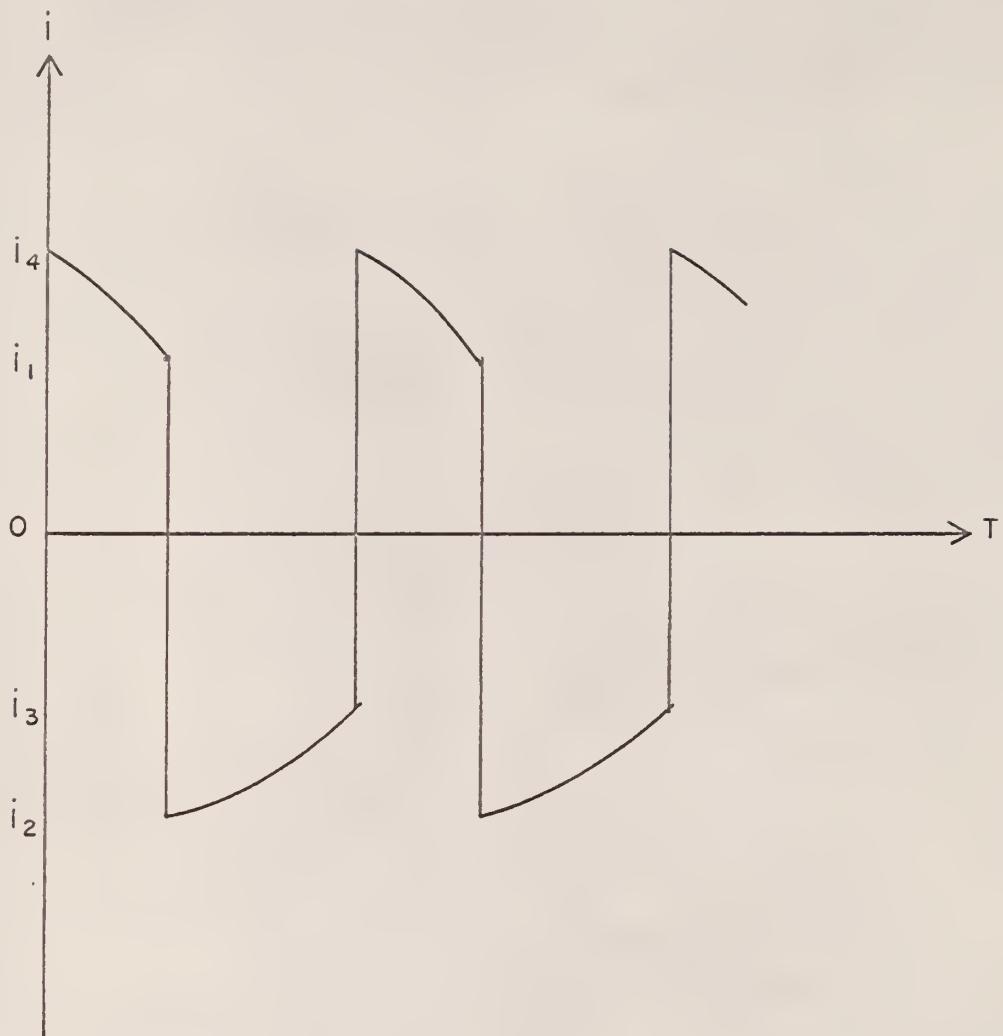
Figure 23
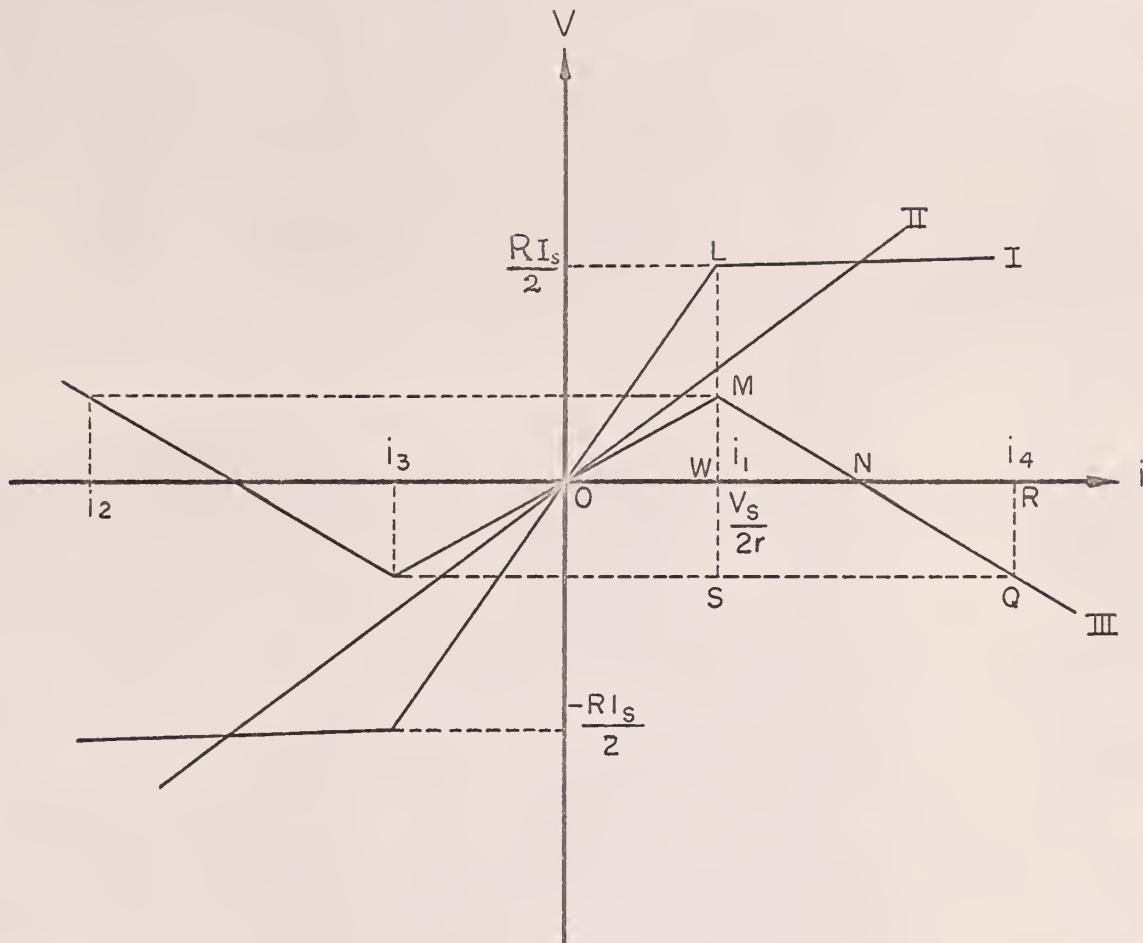
Figure 24

Calculations of i_2 and i_4:

I is given by

$v = R\Phi(Kri)$, but $\Phi(Kri) = I_s/2$ at L

Hence $WL = RI_s/2$

$WM = RI_s/2 - V_s/2r \, (R + r)$

$-i_2 = i_4 = OR = OW + WR$

$= V_s/2r + SQ$

$= V_s/2r + 2MW/(R + r)$

$= V_s/2r + 2\left[RI_s/2 - V_s/2r \, (R + r)\right]1/(R + r)$

$= \left[R/(R + r) \quad I_s - V_s/2r\right]$

$K = \mu Z/(Z + r_p)$ (as assumed previously)

Z = Load impedence

r_p = Plate resistence of Tube v_1

and Φ is the transfer characteristic of Tube v_2.

Let us assume that

$Kri = x, \quad \dot{I} = y$

Then from Equation (31)

$$Ly = x/K - 1/Krc \int x\,dt = 0$$
$$\Phi(x) - x/Kr - \int y\,dt = 0 \tag{32}$$

Differentiating Equation (32) one has

$$x = y/\left[\Phi'(x) - 1/Kr\right]$$
$$y = 2/KrLC + 1/KL \quad y/\left[\Phi'(x) - 1/Kr\right] \tag{33}$$

The transfer characteristic $\Phi(x)$ is a bounded monotonic odd function and $\Phi'(x)$ will then be an even function monotonic for $x > 0$ and decreases monotonically from the maximum to both sides of zero. Thus if $\Phi'(o) > 1/Kr$, there will be two values $\pm x_1 (x_1 > 0)$ such that

$$\Phi'(x_1) = \Phi'(-x_1) = 1/Kr$$

In both the Equations of (33) one has

$$T = \left[\Phi'(x) - 1/Kr\right]$$

hence the root of $T = 0$; i.e., $\Phi'(x) = 1/Kr$, will give critical lines. These lines are $x = x_1$, as will be shown later.

Now if x_1 and y_1 are the coordinates of the representative point before jump and x_2, y_2 that after jump then applying condition of Mandelstam; i.e.,

$$\Delta I \Big|_{t_o - 0}^{t_o + 0} = \int_{t_o - 0}^{t_o + 0} y\,dt = 0,$$
$$\dot{\Delta V} \Big|_{t_o - 0}^{t_o + 0} = 1/Krc \int_{t_o - 0}^{t_o + 0} x\,dt = 0 \tag{34}$$
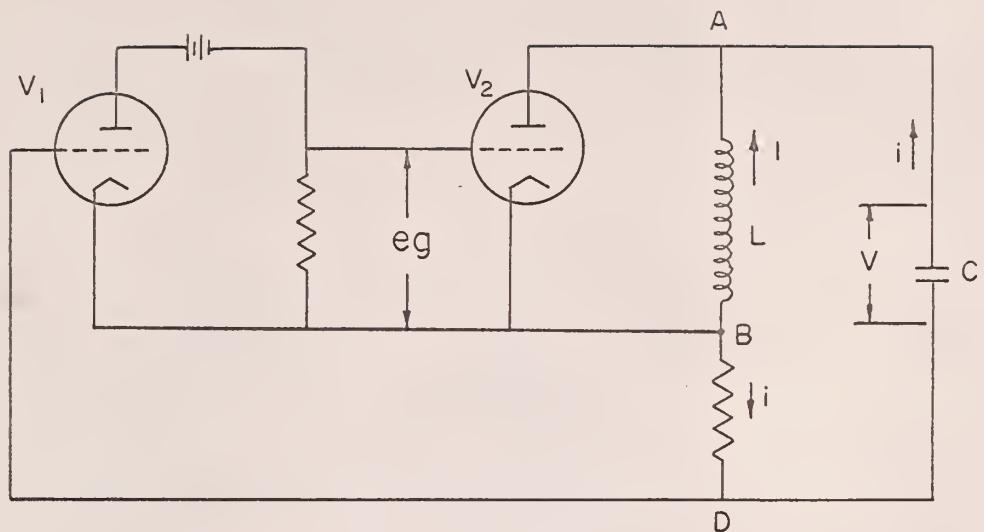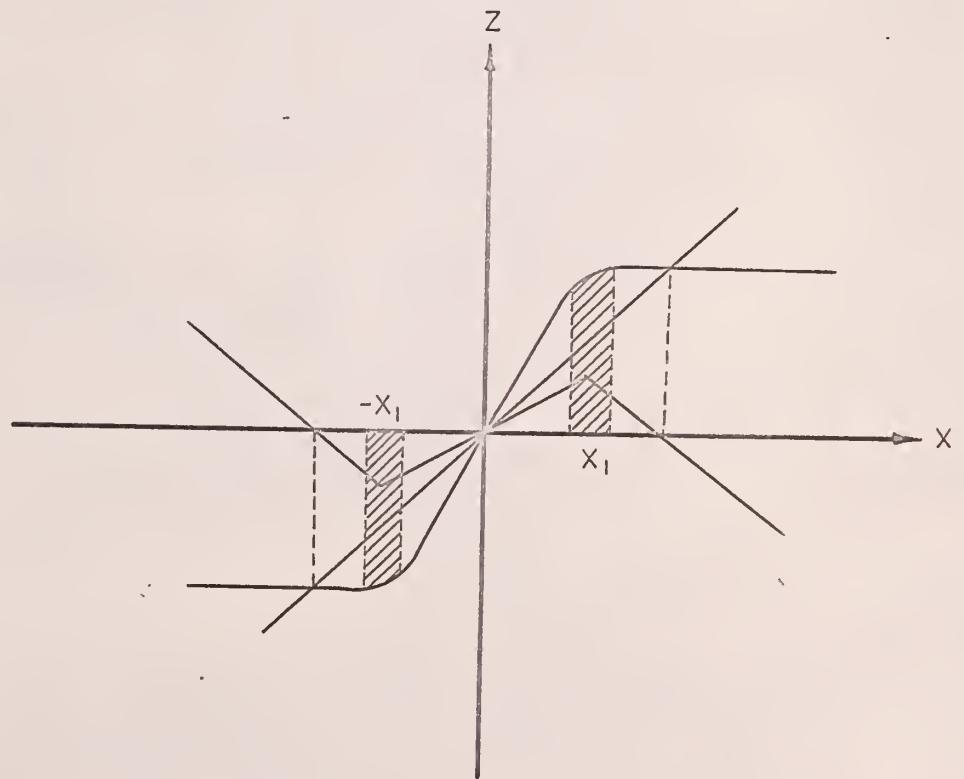
Figure 25

Figure 26

one obtains

$$\Phi(x_1) - x_1/Kr = \Phi(x_2) - x_2/Kr \tag{35}$$

$$Ly_1 - x_1/K = Ly_2 - x_2/K$$

Equation (35) is thus the condition of jump. If one assumes that the tube

characteristic is linear, $\Phi'(x)$ can be approximated to g (standard g_m of the

tube). Equation (33) then becomes

$$\dot{x} = y/(g - 1/Kr) \ , \quad \dot{y} = x/KrLc + (1/KL)y/(g - 1/Kr) \tag{36}$$

Setting $r/Krg - 1 = P$ the characteristic equation can be written as

$$\lambda^2 - P/L \ \lambda - P/rLC = 0 \tag{37}$$

The following results are thus obvious:

    (1) $P>0$

      or   ----------   origin is saddle point

       $Krg>1$

    (2) $Krg<1$ ----------   stable node or focus

      (a) $P^2rc> -4PL$ ----   stable node

      (b) $P^2rc< -4PL$ ----   stable focus

From (a) and (b) above it is clear that the origin is a stable node when

L is small and a stable focus when L is large. We will, however, not entertain

the second case because, discontinuous jumps can occur only when the origin

is unstable. In all the following discussions it will be assumed that the

origin is a saddle point $(Krg>1)$.

Figure 26 shows the graphical construction of the tube characteristic,

the load line and the resulting curve, under the assumption that tube character-

istic is linear over almost the entire region from zero to maximum

$\left[\Phi'(x) = \text{constant} = g>1/Kr \right]$ and $\Phi'(x) = 0$ outside the linear range. The

characteristic will thus be curvilinear only over two small strips of width

Δx, containing $x = x_1$, and $x = -x_1$. It is thus at $x = \pm x_1$ that $\Phi'(x) = 1/Kr$

or in other words $x = \pm x_1$ are critical lines.

Region Between the Strips

The equation of the system is Equation (36) and the origin is a saddle point; hence, paths in this region must be concentric hyperbolas with the slope of the asymptote as

$$1/KrLC \left[rc/2 \pm \sqrt{(rc/2)^2 - LC(Krg - 1)} \right]$$

The direction of paths can be determined with reference to the initial system, Equation (36). The nature of the paths is shown in Figure 27.

Region Exterior to the Strips

The equation of the system is again Equation (36) but with $\Phi'(x) = g = 0$ or

$$\dot{x} = -Kry \quad , \quad \dot{y} = x/KrLC - (r/L) y \tag{38}$$

The characteristic equation is then

$$LC\lambda^2 - rc\lambda + 1 = 0$$

and the roots are

$$\lambda = [-rc \ \sqrt{(rc)^2 - 4LC}]/2LC$$

It has been assumed here that $r < 2\sqrt{L/c}$; i.e., the system represented by Equation (38) is a focus. In Figure 28 the shaded region shows the portion between the strips and only the unshaded portion is to be considered. The direction of motion is again determined by the initial system, Equation (38). It can be seen from the figure that whatever the initial position, the representative point must sometime reach $x = \pm x_1$ and hence jump must occur and during the jump Equation (35) must be satisfied. The first equation of Equation (35) asserts that $Z(x) = \tilde{\Phi}(x) - (x/Kr)$ must not change. The graph in Figure 29 shows that to x_1, there corresponds a unique value $-x_2(x_2 > 0)$ such that $Z(x_1) = Z(-x_2)$. Since $Z(-x) = -Z(x)$, similarly $Z(-x_1) = Z(x_2)$ so that $-x_1$ will correspond similarly to x_2. As far as y is concerned the second

jump condition, Equation (35), asserts that (x_1, y_1) goes to a point on the line of slope $1/LK$ through (x_1, y_1). This is to say that the segment from new position to the old has the fixed slope $1/LK$. Thus to find the new position of jump one must merely draw a parallel to this direction, and if the position before the jump is on $x = -x_1$ find its intersection with $x = +x_2$ while if the position of P before the jump is on $x = +x_1$ find its inter- section with $x = -x_2$, and the position after the jump will be determined. If we contract the small widths Δx at $x = \pm x_1$, to a point or, in other words, consider that the tube characteristic is essentially rectilinear, we will see the piecewise limit cycle as shown in Figure 30. If we assume that $r \ll 2\sqrt{L/c}$ the slope $1/KL$ will almost be horizontal. The discontinuity is shown dotted. Considerations of continuity show that there must exist a pair of portions of spirals whose extremities are closed by a jump, thus producing a closed path (shown by heavy line) to which corresponds the periodic motion. It can be seen that motion across this closed path is stable. In fact the representative point moves along one of the internal curls of the spiral, it "jumps out" for outside and as a result oscillations grow and approach the closed path while if it is inside the closed path it "jumps out" and remains after the jump inside of the spiral, within which it was situated before the jump, and as a result, oscillations dampen out. For some inter- mediate position (closed path) there is a comp-ensation, and steady discontin- uous oscillations are produced.

The experiment corroborates these conclusions. If the connections of a cathode-ray oscilloscope are made to represent variables x and y, two arcs of spirals with an empty space in between them will be observed as shown in Figure 31. This indicates that in this inner interval the motion of the
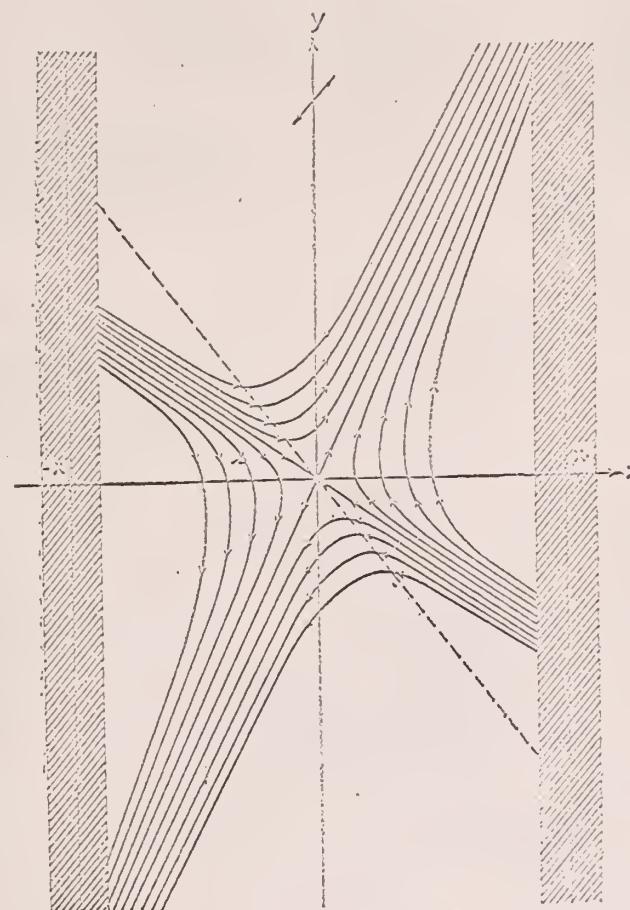
Figure 27
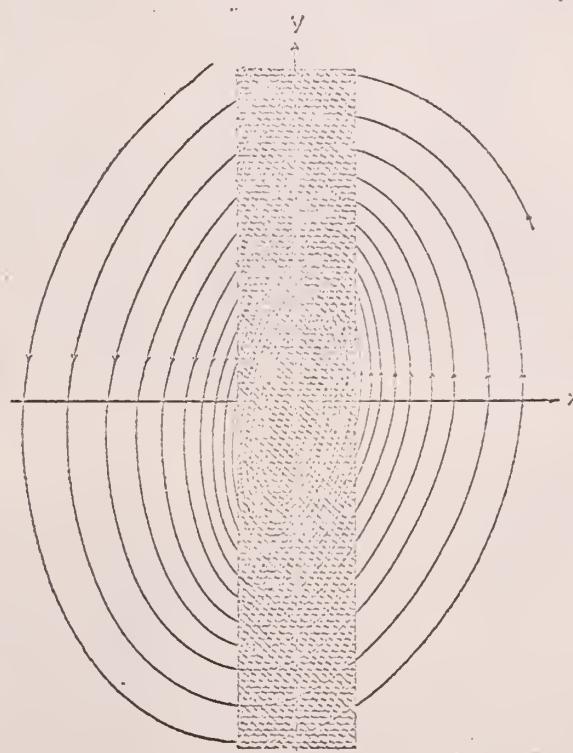
Figure 28
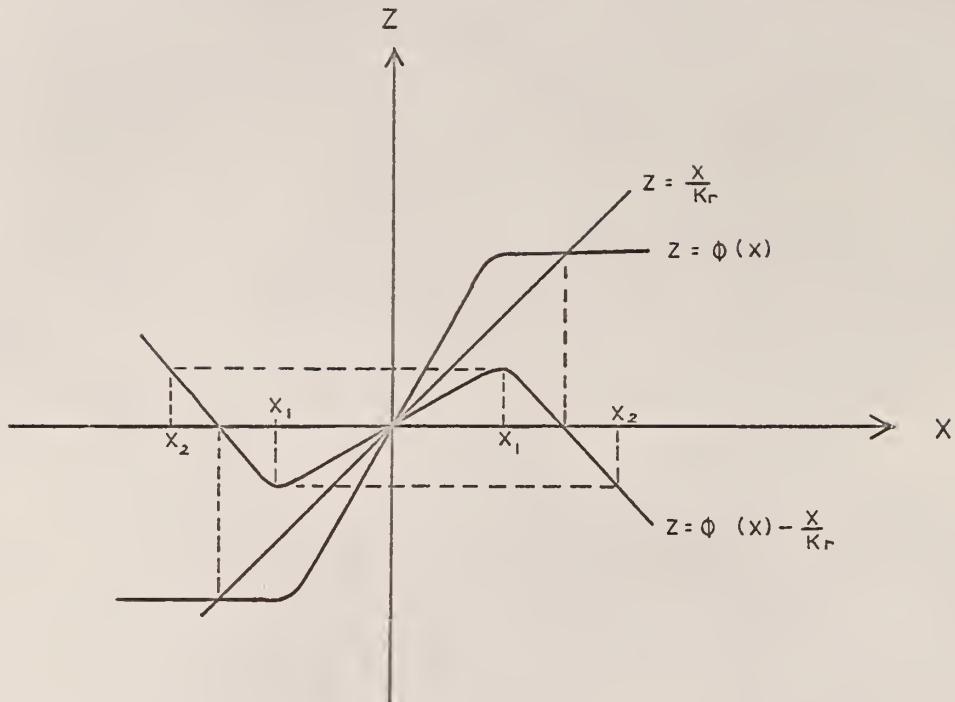
Figure 29
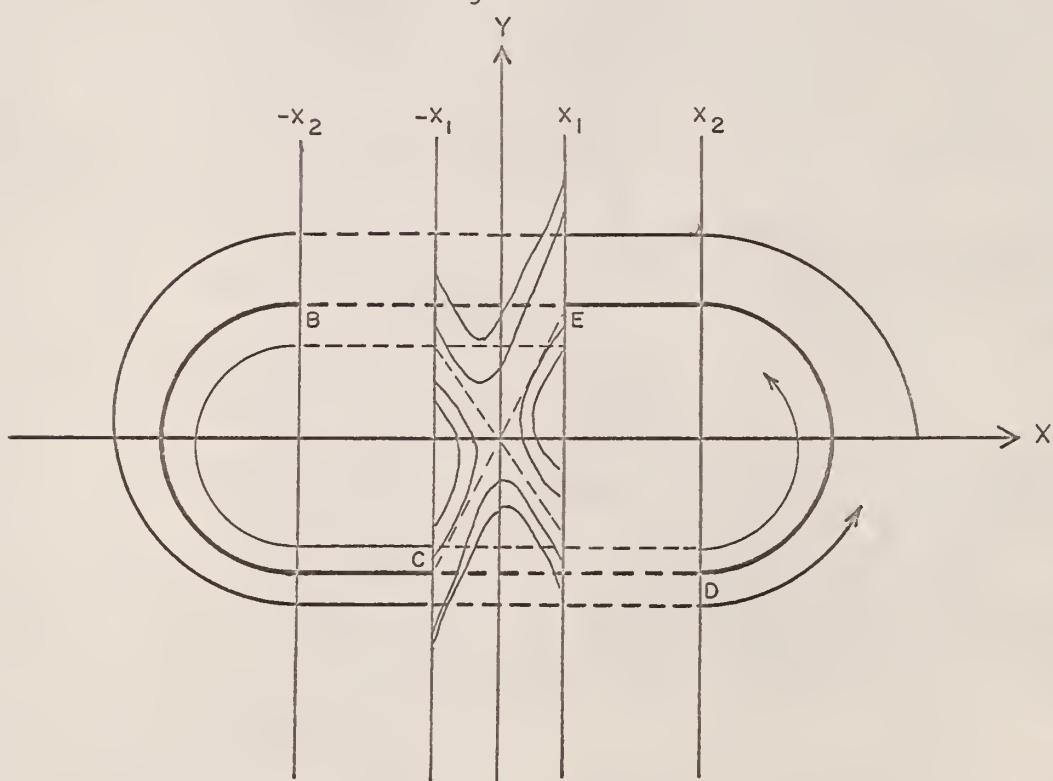
Figure 30

electron beam is so fast that the fluorescent material of the screen has no time to respond to the passage of the beam. It is interesting to note that, although the inner interval corresponds to the existence of saddle point in the d. e., the hyperbolic trajectories of this point have nothing to do with the actual motion of the representative point which is governed in the region by condition of Mandelstam and not by the d. e.

A remark here will be suitable. This system has two degrees of freedom and hence there are two degenerate equations. It is clear that without degeneration procedure, the oscillatory system in this case would be amenable to a differential system of fourth order and its representation on the phase plane would be impossible.

If we assume that oscillations are not too heavily relaxational (near to sinusoidal), period and amplitude can be easily computed. The tube characteristics are chosen to be essentially rectilinear and are shown in Figure 32. Obviously

$$x'/Kr = i' = I_s/2$$

and

$$x_1/Kr = V_s/2Kr = i_1$$

where V_s and I_s are the saturation voltage and current respecively of the tube V_2. Then

$$x_2/Kr = i_2 = i_1 + 2(i' - i_1)$$
$$x_2 = V_s/2 + 2Kr(I_s/2 - V_s/2Kr)$$

consequently

$$x_1 = V_s/2 \ , \ x_2 = V_s/2 - V_s + KrI_s$$
$$= V_s \left[KrI_s/V_s - 1/2 \right]$$

Defining

$$x = x_1 + x_2 = KrI_s$$

Amplitude

Let δ be the logarithmic decrement of the linear oscillations, Δy_1, the loss in y through a half oscillation, Δy_2 the variation in y through a jump. In the periodic motion $\Delta y_1 = \Delta y_2$

On the other hand

$$\Delta y_1 = y_o(1 - e^{-\delta/2}), \quad \Delta y_2 = 1/KL(x_1 + x_2)$$
$$= x/KL = rI_s/L$$

Hence the steady state amplitude E of the voltage at the terminals of the inductance

$$E = Ly_o = x/K(1 - e^{-\delta/2}) = rI_s/(1 - e^{-\delta/2})$$

when

$$Kr(I_s/V_s) \gg 1 \quad \text{and} \quad \delta \ll 1$$
$$E \simeq 2rI_s/\delta = (2I_s/\pi\sqrt{L/c}) \tag{39}$$

Period

Along the spirals the representative point would pass, without jumping, from y_o to y_o' in time

$$T = 2\pi/\sqrt{W_o^2 - h^2}$$

where

$$W_o^2 = 1/Lc \quad \text{and} \quad h = r/2L$$

It takes, however, less time since the jump is "instantaneous." The time saved may be calculated as follows. The region across which the system jumps, would be crossed by a linear system with almost constant velocity. This velocity can be found by Equation (38) as $\dot{x} = -Kry_o$. Consequently, the time necessary is approximately the correction for the period:

$$T = 2x/Kry_o = (2L/r)(1 - e^{-\delta/2})$$
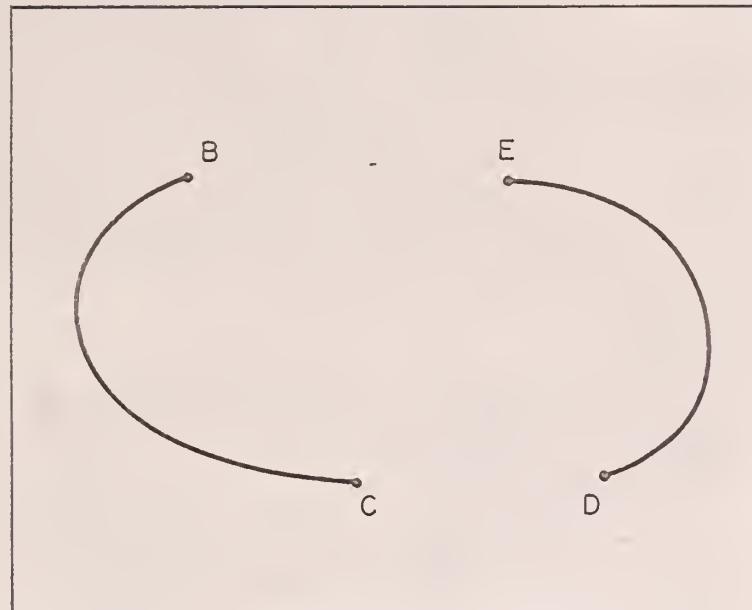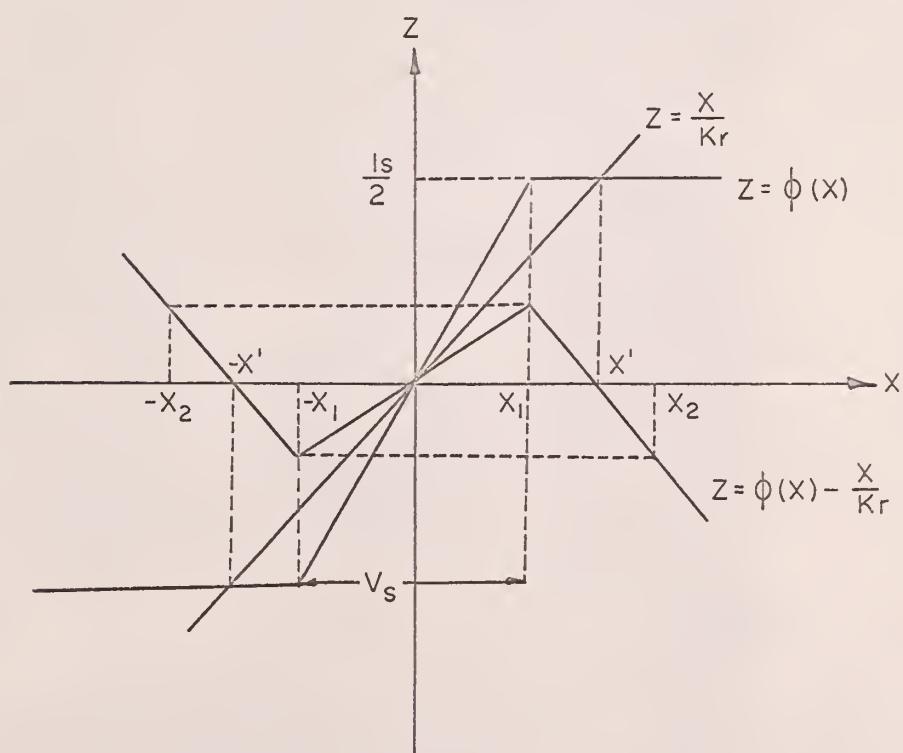
Figure 31

Figure 32

When damping is rather small, $(\delta \ll 1)$

One has

$$T = (2L/r) \; \delta/2 = (L/r) \; rT_o/2L = T_o/2$$

Where T_o is the period of oscillations of a friction-less linear system.

When L is rather large; i.e., damping is small, $T_o = T$ (T = duration of period

of damped oscillations) and hence the period of self oscillation is

approximately $T_1 = T_o/2$.

DOUBLY DEGENERATE SYSTEMS

A system of two differential equations of the second order can generally be reduced to a system of four differential equations of the first order, which means a system of the fourth order. If, however, each of the original equations of the second order degenerates into one equation of the first order, the system of the fourth order reduces to one equation of the second order, and its solution can be represented by trajectories in the phase plane. This resultant equation, however, represents the result of degeneration of the system of the fourth order. We can express this by saying that we have a doubly degenerate system. Since each of the two differential equations of the first order admits discontinuous solutions, the doubly degenerate system of the second order will also possess certain discontinuous stretches in the phase plane so that its trajectories, in general, will be composed of certain analytic arcs joined by these stretches. The free-running plate-coupled multivibrator of Abraham and Bloch forms a good example of a doubly degenerate system. Though Heegner's circuit (Figure 35) is also doubly degenerate system, it would not be discussed under this section because oscillations in this system are continuous. It would, however, serve as a good example when one tries to establish connection between continuous and discontinuous systems.

14. Free-Running Plate-Coupled Multivibrator

The circuit of a free-running plate-coupled multivibrator shown in Figure 33. The following assumptions have been made

(1) Circuit in symmetrical

(2) Effect of grid current and plate reaction is negligible.

Applying Kirchhoff's laws

$$I_1 = I_{o1} + i_1 \quad ; \quad I_2 = I_{o2} + i_2$$

$$RI_1 + 1/c \int i_1 dt + ri_1 = E \quad ; \quad RI_2 + 1/c \int i_2 dt + ri_2 = E$$

$$I_{o1} = \mathfrak{D}(eg_1) = \mathfrak{D}(ri_2) \quad ; \quad I_{o2} = \Phi(eg_2) = \Phi(ri_2) \tag{1}$$

where $I_o = \mathfrak{D}(eg)$ is the non-linear characteristic of electron tubes V_1 and V_2.

From above equation

$$(R + r)i_1 + 1/c \int i_1 dt + R\mathfrak{D}(ri_2) = E$$

and

$$(R + r)i_2 + 1/c \int i_2 dt + R\mathfrak{D}(ri_1) = E$$

Differentiating above equations we have

$$(R + r)di_1/dt + 1/c \; i_1 + Rr\mathfrak{D}'(ri_2)di_2/dt = 0$$

$$Rr\mathfrak{D}'(ri_1)di_1/dt + (R + r)di_2/dt + 1/c \; i_2 = 0 \tag{2}$$

Solving the above equation for di_1/dt and di_2/dt one gets

$$di_1/dt = \left\{(R + r)i_1/c - Rr\mathfrak{D}'(ri_2)i_2/c\right\} / \left\{R^2 r^2 \mathfrak{D}'(ri_2) - (R + r)^2\right\}$$

$$di_2/dt = \left\{(R + r)i_2/c - Rr\mathfrak{D}'(ri_1)i_1/c\right\} / \left\{R^2 r^2 \mathfrak{D}'(ri_1) - (R + r)^2\right\} \tag{3}$$

Equation (3) is of the form

$$di_1/dt = P(i_1, i_2)/T(i_1, i_2) \quad ; \quad di_2/dt = Q(i_1, i_2)/T(i_1, i_2) \tag{4}$$

The phase trajectories in the (i_1, i_2) plane are given by

$$di_2/dt = Q(i_1, i_2)/P(i_1, i_2) \tag{5}$$

From Equations (3) and (5) one can see that the only singular point is the origin $(i_1 = i_2 = 0)$

Applying Bendixson's negative criterion* to Equation (4) we see that

$$\partial P/\partial i_1 + \partial Q/\partial i_1 = 2(R + r)/c = \text{constant} \tag{6}$$

---

* If the equation of motion is represented by x = P(x,y)/T(x,y),
y = Q(x,y)/T(x,y), then no periodic solution can exist in domain D of
the phase plane if ∂P/∂x + ∂Q/∂y does not change sign in the domain.
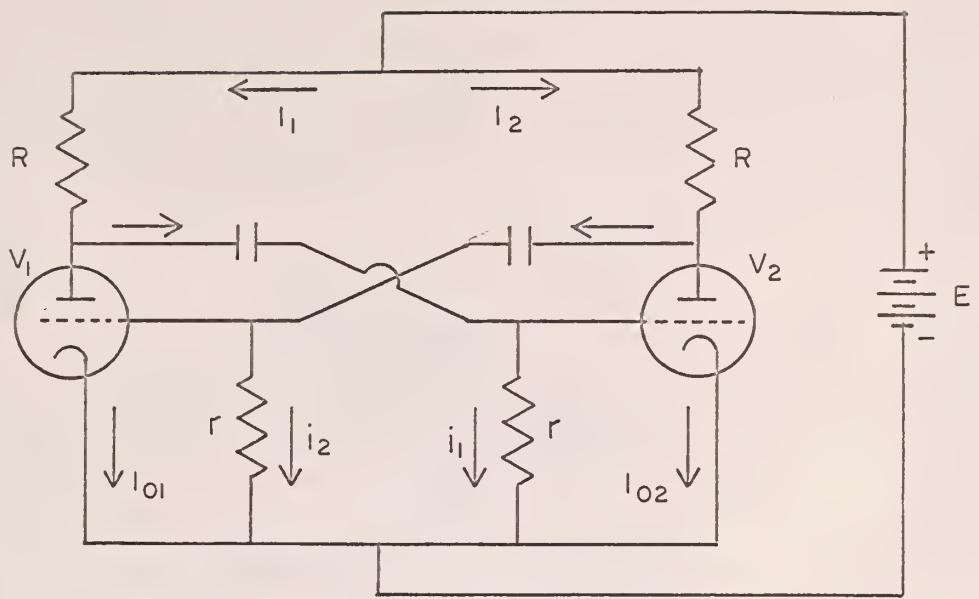
Figure 33
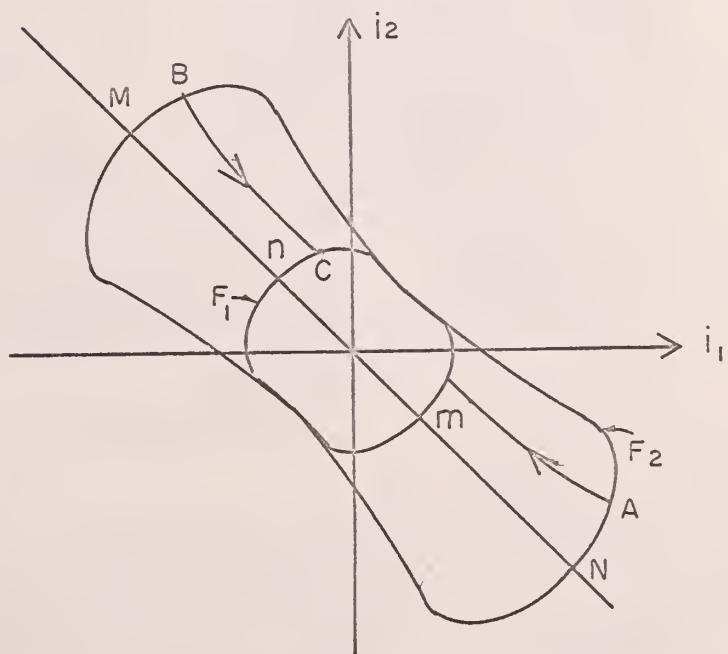
Figure 34

and hence no closed analytic trajectories are possible. The nature of the singular point can be determined as follows. If

(1) $\Phi'(ri_1)i_{1=0} = \Phi'(ri_2)i_{2=0} = S$

and (2) $M = c\left[R^2r^2S^2 - (R + r)^2\right] > 0$

then Equation (3) can be written as

$$di_1/dt = [(R + r)/M]i_1 - [RrS/M]i_2 \quad ; \quad di_2/dt = -(RrS/M)i_1 + [(R + r)/M]i_2 \quad (7)$$

The characteristic equation of the system from Equation (7) is

$$\lambda^2 - [2(R + r)/M]\lambda + \left\{\left[(R + r) + RrS\right]\left[(R + r) - RrS\right]\right\}/ML = 0 \qquad (8)$$

We will assume that $RrS > (R + r)$ and hence the characteristic roots will be of opposite sign. Also since the roots of this equation are always real, the origin is a saddle point. Since, initially, $RrS > (R + r)$ and the origin is unstable, the variables i_1 and i_2 begin to increase. On the other hand, from the form of the characteristic $I_0 = \Phi(ri)$, we know, that $|\Phi'(ri)| \to 0$ when $i \to \infty$. The function

$$T(i_1, i_2) = c\left[R^2r^2\Phi'(ri_1)\,\Phi'(ri_2) - (R + r)^2\right] \qquad (9)$$

which is initially positive, decreases monotonically when i_1 and i_2 increases and is negative when i_1 and i_2 are very large and equal. Hence there are certainly some values of i_1 and i_2 for which $T = 0$. This means that the system has critical points and hence by virtue of the basic assumption discontinuities must occur at these points. The locus of critical points (i_1', i_2') will be given by

$$T(i_1', i_2') = c\left[R^2r^2\Phi'(ri_1')\,\Phi'(ri_2') - (R + r)^2\right] = 0 \qquad (10)$$

In as much as $\Phi'(o)rR = RrS(R + r)$ and $\Phi'(ri)$ decreases monotonically with i increasing, the curve F_1 described by Equation (10) is a closed curved symmetrical with respect to the origin. (Figure 34)

The point (i_1'', i_2'') into which the representative point P jumps, once it has reached the critical point (i_1', i_2'), is determined by the condition of Mandelstam. As the only form of stored energy here is electrostatic, the voltage v across the capacitor remain invarient during the jump, which results in the relations

$$v_1 = E - R\wp(ri_2) - (R + r)i_1$$
$$v_2 = E - R\wp(ri_1) - (R + r)i_2$$

The conditions of invarience of v during discontinuity are thus

$$R\wp(ri_2') + (R + r)i_1' = R\wp(ri_2'') + (R + r)i_1''$$
$$R\wp(ri_1') + (R + r)i_2' = R\wp(ri_1'') + (R + r)i_2'' \tag{11}$$

There exists thus a one-to-one correspondence between (i_1', i_2') before the discontinuity and (i_1'', i_2'') after it.

The piecewise analytic phenomenon thus takes place in the following manner. From some point 'a' on F_1, the point P jumps into the corresponding point A on F_2. From this point there begins a continuous motion on the stretch Ab. At b begins another jump which transfers P to the point B on F_2, from which begins continuous stretch Bc, and so on.

On account of the symmetry the motion should be symmetrical and hence after a series of jumps, the motion approaches the bisector line M N so that ultimately the stationary state consists of a continuous motion Mn followed by a jump nN followed again by a continuous motion Nm, etc. In the steady state if $i_1 = -i_2 = i$ and $\wp'(ri_1) = -\wp'(ri_2) = \wp'(ri)$ we then have from Equation (3)

$$di/dt = \left\{ (R + r) + rR\wp'(ri) \right\} \Big/ \left[R^2r^2\wp'^2(ri) - (R + r)^2 \right] c$$
$$= \left\{ 1 / \left[Rr\wp'(ri) - (R + r) \right] \right\} \left\{ i/c \right\} \tag{12}$$

Equation (12) is the same as Equation (22) on page 51 and hence the nature of oscillations of the R C multivibrator are similar to those of free running multivibrator in the steady state.

By virtue of Equation (12) it can be said that a free running multivibrator in the steady state is a triply degenerate system.

CRITICAL POINTS AND DISCONTINUOUS SOLUTIONS

It has been shown earlier that the zeros of the function T result in the appearance of critical points and later, in turn, appear as the criterion for the existence of discontinuous solutions. Conversely it can also be shown that if $T(x,y)$ [Equation (30) on page 27] does not go through zero, the piecewise analytic character of oscillations disappear. This was first demonstrated experimentally with the help of Heegner's circuit.

15. Heegner's Circuit

Heegner's circuit, which is only a slight modification of the R C multivibrator (Page 49), is shown in Figure 35. Modification consists in shunting the resistence R by an additional capacitor C_1. We will now see, how the addition of this capacitor radically modifies the behavior of the circuit.

Applying Kirchhoff's laws to the circuit gives

$$I_a = I + I_1 + i$$

$$i = c \, d/dt \left[RI - ri \right]$$

$$I_1 = c_1 \, d(RI)/dt = c_1 R \, di/dt \tag{1}$$

It will be again assumed that the transfer characteristic of tube V_2 is given by $I_a = \Phi(eg) = \Phi(Kri)$

Hence

$$i = cR \, di/dt - cr \, di/dt$$

$$= c/c_1 \, I_1 - cr \, di/dt \text{ because } R \, di/dt = I_1/c$$

or

$$di/dt = I_1/c_1 r - i/cr \tag{2}$$

$$I_1 = c_1 R \, d/dt(I_a - I_1 - i)$$

$$= c_1 R \left[Kr \Phi'(Kri)di/dt - dI_1/dt - di/dt \right]$$

or $\quad dI_1/dt = \left[Kr\Phi'(Kri) - 1 \right] di/dt - I_1/c_1 R$
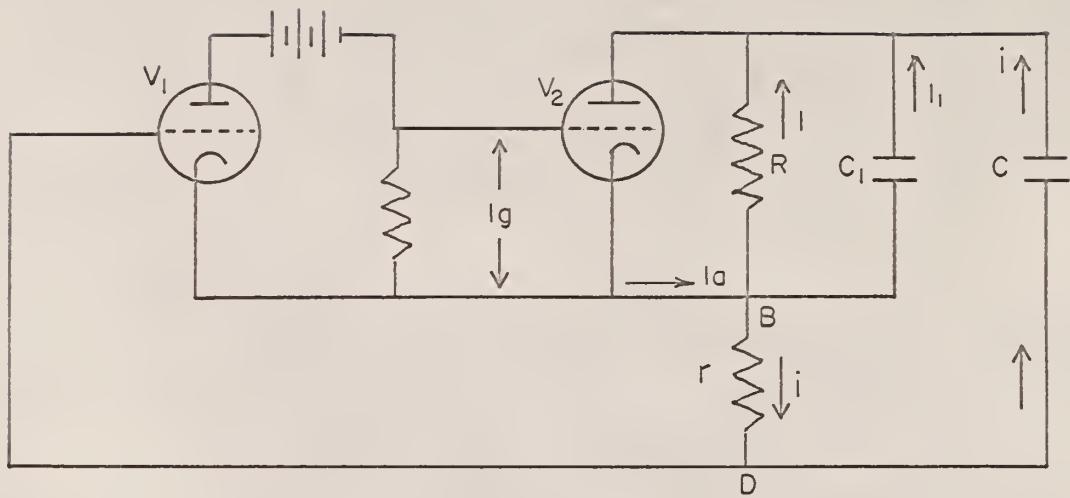
Figure 35
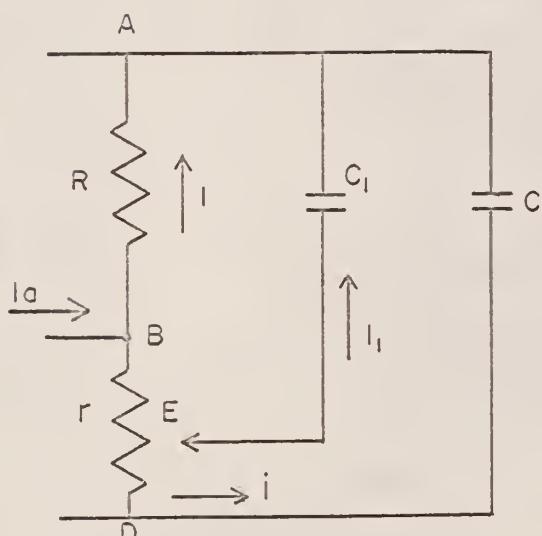
Figure 36
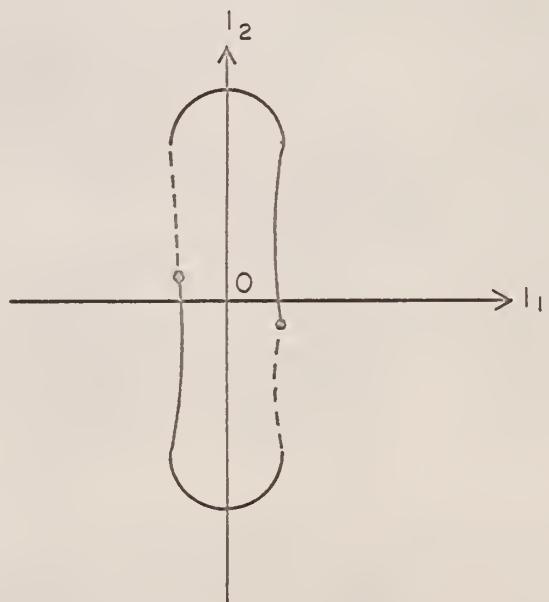
Figure 37

Hence from Equation (2)

$$dI_1/dt = \left[Kr\vartheta'(Kri) - 1\right]\left[I_1/c_1r - i/cr\right] - I_1/c_1R$$

$$= i\left[1 - Kr\vartheta'(Kri)\right]/cr - I_1/c_1R - I_1/c_1r + Kr\vartheta'(Kri)I_1/c_1r$$

or

$$dI_1/dt = i\left[1 - Kr\vartheta'(Kri)\right]/cr - \left[(R + r) - RKr\vartheta'(Kri)\right]I_1/c_1Rr \tag{3}$$

From Equations (2) and (3) it is clear that the system has no critical

points, and hence no discontinuous solution is to be expected. The only

singular point is $I_1 = i = 0$. Note that $\vartheta'(o) = S$ is a maximum and the

function $|\vartheta'(KrI_1)| \to o$ when $I_1 \to \infty$. The characteristic equation of the

system is

$$\lambda^2 + \left\{\left[c_1/c + (R + r - KRrS/R\right] 1/rc_1\right\} \lambda + 1/Rrcc_1 = 0 \tag{4}$$

It should be noted that the singularity here is not a saddle point. Hence,

it is either a nodal point, if the roots λ_1 and λ_2 are real, or a focul

point, if they are conjugate complex. In both cases the singularity is

unstable if

$$(R + r - RrKS)/R \geq c_1/c \tag{5}$$

It is well known that in Heegner's circuit if Equation (5) is satisfied,

self exitation from rest is possible and a stable limit cycle does exist,

implying the existence of continuous self-excited oscillations. This proves

that the converse of the previously established rule $\left[\text{if function T has zero's}\right.$

(critical points) a discontinuous solution should exist$\left.\right]$ is also true.

16. Relation Between Continuous and Discontinuous Solutions

In view of the fact that Heegner's circuit is a modification of the R C

multivibrator and Heegner's circuit has continuous solutions while the R C

multivibrator has only discontinuous solutions, one might ask whether a gradual

modification of an electric circuit, might cause a transition from continuous performance to a discontinuous performance, or vice versa. The answer to this question is yes. It will now be seen that such a transition does exist and it depends directly on the appearance or disappearance, of critical points as a result of variation of certain parameters in the differential equations. One should now consider a slightly modified Heegner's circuit as shown in Figure 36. As the rest of the circuit remains the same, we indicate in this figure only the modified part of the circuit shown in Figure 35. The capacitor C_1, instead of being connected directly to B is now connected by an adjustable sliding contact to some point E along the resistance r. Let r_1 be the resistance between B and E and r_2 be that between E and D, where $r_1 + r_2 = r$ and $r_1/r = \beta$. Proceeding as before one gets the following equations instead of Equations (2) and (3)

$$di/dt = I_1/(1 - \beta)rc_1 - i/(1 - \beta)rc \tag{4}$$

$$dI_1/dt = \frac{\left[\beta r + R - rR\partial'Kr(i - \beta I_1)\right]i/c - \left[r + R - rR\partial'Kr(i + \beta I_1)\right]I_1/c_1}{(1 - \beta)\left[R + \beta r - \beta rR\partial'Kr(i + \beta I_1)\right]} \tag{5}$$

It is obvious that for $\beta = 1$, $\left[r_1 = r\right]$ the circuit reduces to the R C multivibrator where only a discontinuous performance occurs. For $\beta = 0$ one has Heegner's circuit which has only continuous oscillations. This implies that for some intermediate value of β, the co-factor of $(1 - \beta)$ in Equation (5) may vanish, which means that continuous oscillations will undergo a discontinuous jump parallel to the I_1 axis as shown in Figure 37. This generally occurs when the system $\left[\text{Equation (4) and (5)}\right]$ is characterised by a saddle point, and the transition takes place where an unstable focul point degenerates into a saddle point.

This has been demonstrated experimentally by a cathode-ray oscilloscope which shows that the continuous closed curve of Heegner's circuit begins to be

interrupted by a small discontinuity which gradually grows as β approaches

unity.

CONCLUSION

It can be concluded that the present theory of relaxation oscillations is less satisfactory than the theory of linear oscillations where all the known phenomena are logically connected with analytic theory. The existence of two different trends in these studies - the discontinuous and the analytic - reflects the difficulty of this problem.

From the convenience point of view, it is evident that the discontinuous theory is more convenient, in as much as it is nearer to the real quasidiscontinuous character of the problem. In this connection it is interesting to note the comments of Boussinesq. He said,

"Si La continuite' simplifie les choses quand ele en relie plusiewis qui suivent la meme Loi, elle les complique, an contraire, le plus souvent, Lorsequ'elle et blit la transition entre deux categories d'objects ou de faits regis par deux lois simples diffe'rentes; et c'est alors une discontinuite' fictive, un passage brusque de la premiere cate'gorie a' la seconde, qui rend les questions abordables".*

It is to be admitted, however, that this method appears somewhat disappointing when compared with purely analytical methods used in the "nearly linear" domain. It may be possible that in the future a purely analytic approach may be extended also to the d. e. connected to the relaxation phenomenon, [Equation (30) on page 27] but no such attempt has been successful so far. Even if one succeeds in doing so, one can always question whether an analytic approach can be extended to an oscillatory phenomenon which by its very nature

---

* "If the continuity simplifies the matter when it connects several phenomena following similar laws, it complicates, on contrary, the relations when it is used for the purpose of connecting phenomena following different laws. It is precisely here that there is an idealized discontinuous passage from one law to the other which renders the study possibe".

exhibits essentially non-analytic features, at least at some point of its cycle.

It is recalled that at one time Hertz tried to explain the mechanism of shocks on the basis of a continuous theory by considering two different d. e. - one governing the motion before and after the separation of colliding bodies, and the other during the (short) time when these bodies are in contact with each other. It is sufficient to assume the continuity of solutions at the cost of loss of analyticity at points where one d. e. replaces the other. In spite of the possibility of accomplishing this result, this theory was ultimately given up in favor of the present discontinuous idealization which is now classical in theoretical mechanics. It was thus 'convenience' in the sense of Poincare', which gave the preference to the ultimate discontinuous theory of shocks.

It is quite probable that similar considerations may eventually be a deciding factor in formation of the ultimate theory of relaxation oscillations, but one has to admit that the last word in this difficult field has not yet been said.

ACKNOWLEDGEMENTS

BIBLIOGRAPHY

1. Andronow, A. A. and Chaikin, C. E.
        Theory of Oscillations (1949)

2. Bogolinbov, N. N. and Mitropolsky, Y. A.
        Asymptotic Methods in the Theory of Non-linear
            Oscillations (1961)

3. Davis, Harold T.
        Introduction to Nonlinear Differential and Integral
            Equations (1960)

4. Hale, Jack K.
        Oscillations in Nonlinear Systems (1963)

5. Hughes, William L.
      Nonlinear Electrical Networks (1960)

6. Hayashi, Chihiro
        Forced Oscillations in Nonlinear Systems (1953)

7. Jean, P.
            (a) Compt. Rend. Acad. Sci. (Paris) May 1954, 1955
            (b) Joun. de Phynique et le Radium, January 1951 and
                    Oct. 1953. (In French)

8. Ku, Yu. H.
        Analysis and Control of Nonlinear Systems (1958)

9. Lefschetz, S.
        Contributions to the Theory of Nonlinear Oscillations (1958)

10. Le Corbeiller, Philippe
        Ler Systemes Autoentretenus Et les Oscillations de
            Relaxation (In French) (1931)

11. Lawden, Derek F.
        Mathematics of Engineering Systems (1954)

12. Minorsky, Nicholas
            (a) Nonlinear Oscillations (1962)
            (b) Dynamics and Nonlinear Mechanisc (1947)
            (c) Introduction to Nonlinear Mechanics (1947)

13. Melachlan, N. W.
        Ordinary Nonlinear Differential Equations in Engineering and
            Physical Sciences (1956)

14. Reich, Herbert J.
        Functional Circuits and Oscillators (1961)

15. Sideriades, L
       (a) C. R. Acad. Sci. (Paris) December 1956
       (b) Joun. Phys. Radium. May 1957 (In French)

16. Wendel, James G.
       Bull. Amer. Math. Soc. 1948. p. 836

17. Zheleztsov, N. A. and Feigin M. I.
       Radio Tekh Elektron. June 1957
         (English translation from New York University Press,
         original paper in Russian.)

DISCONTINUOUS THEORY OF RELAXATION OSCILLATORS

by

JAI KARAN N. SHUKLA

AN ABSTRACT OF A MASTER'S REPORT

Submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Electrical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1965

This report presents a logical development of the "Discontinuous Theory of Relaxation Oscillations." Section 1 gives a brief historical background of the subject of relaxation oscillations and indicates the presence of two different theories to solve the relaxation oscillation problems, which are fundamentally quasidiscontinuous in nature. The two theories have been compared and it has been indicated that developments in the asymptotic theory unfortunately have not yet reached the stage where the theory can be easily applied to various practical problems.

Sections 2, 3 and 4 explain the physical meaning of discontinuous theory with the help of a series R L C circuit. Both R C and R L degenerations have been discussed and it has been shown graphically that while using the degenerate d. e., one must assume, independently of the initial conditions, that in case of R C degeneration, current jumps to the value defined by d. e. of first order while charge remains essentially constant. It has been explained that in R C as well as in R L degeneration, the situation remains the same, namely, the variable in the d. e. cannot vary discontinuously and are determined directly by the degenerate d. e. of the first order while derivatives of these variables must vary discontinuously in order to reconcile with the physical existence of two initial conditions in the complete equation describing the system.

In the following five sections, the discontinuous theory has been discussed elaborately and all the underlying principles of the theory have been presented in detail. The theory deals with a broad class of d. e.

$$dx/dt = P(x,y)/T(x,y) \quad , \quad dy/dt = Q(x,y)/T(x,y)$$

in which the van der Pol's equation does not figure at all. As the theory is based on the existence of critical points, enough space has been devoted to discuss all the aspects of the topic. Another topic which really is the

backbone of the theory is the "Condition of Mandelstam" and this has been dealt with in considerable detail.

In order to provide a clear understanding of theory, approximately half of the space has been devoted to the solution of various relaxation oscillation problems. Effort has been made to deal with only basic types of problems under the following broad classifications:

(1) Systems with one degree of freedom described by single degenerate differential equations of the first order.

(2) Systems with one degree of freedom described by two degenerate differential equations of the first order.

(3) Systems with two degrees of freedom described by two degenerate equations of the first order.

Mention has also been made about triply and multiply degenerate systems. A neon lamp, containing R L circuit has been used to explain that if the "Condition of Mandelstam" is not satisfied the relaxation oscillations cannot exist. The fact that appearance of the critical points is the necessary criterion for the existence of discontinuous solutions has been explained with the help of Heegner's circuit.

The report has been concluded with the mention of facts which may appear as deciding factors in the formation of the ultimate theory of relaxation oscillators. As the situation now exists, it is clear that the discontinuous theory has a definite edge over the asymptotic theory.