A STUDY OF THE HILL-FUNCTION SOLUTION TO PROBLEMS OF PROPAGATION IN STRATIFIED MEDIA

by

JAMES L. DIETRICH

B.S., Kansas State University, 1970

A MASTER'S THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Electrical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1974

Approved by:

Kendall F. Casey
Major Professor

ACKNOWLEDGEMENT

For his support and guidance throughout the preparation of this paper, I would like to express my gratitude to Dr. Kendall F. Casey, my supervising professor.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

I. INTRODUCTION

The problem of finding the electromagnetic fields due to a plane wave incident upon a plane-stratified medium reduces to the problem of solving a second-order linear differential equation with variable coefficients. There exists a limited number of specific layer variations which yield a differential equation solvable in terms of known functions [1]-[9]. If the dielectric profile is not one of these, then the differential equation may be solved by various approximate methods [10]. At high frequencies, that is, when the dielectric variation over distances of the order of a wavelength is small, the WKB (Wentzel-Kramers-Brillouin) or phase-integral solution may be used [1, Ch. 4], [2]. One may also construct an approximate differential equation to obtain a solution good at either high or low frequencies. Another technique is to approximate the dielectric profile by one or more profiles for which an exact solution is known. For example, probably the most obvious approximation is a "stepped" profile consisting of homogeneous layers of arbitrary number and size.

In a recent paper by Casey [11], it was shown that the fields in a plane-stratified layer may be expressed in terms of Hill functions. This method gives formally exact solutions for a nearly arbitrary dielectric profile in the layer. We shall in this paper be concerned with finding the fields outside the inhomogeneous region using the Hill function approach. More precisely, we are interested in the rate of convergence of certain infinite determinants associated with the Hill function method. The rate of convergence will be seen to depend upon frequency and other parameters of the problem. Additionally, we shall be interested in the effect of certain dielectric profile approximations which may conveniently be made using the

Hill-function method of solution. For this study, we shall consider a dielectric profile for which a known-function solution may be obtained and compare numerically the known-function solution and the Hill-function solution.

II. FORMULATION OF THE PROBLEM

The geometry of the problem is shown in Figure 1. A plane wave is incident upon a stratified dielectric layer of thickness d and permittivity $\varepsilon(z)$. On either side of the transition layer are homogeneous regions of permittivity ε_1 and ε_2. The permeability in all regions is μ_o. We shall consider only the case of a plane wave polarized perpendicular to the plane of incidence, as shown in Fig. 1. Also, $e^{-i\omega t}$ time dependence will be assumed for all field quantities and suppressed throughout.

In region (1), $z < 0$, the total electric field is the sum of the incident and reflected fields,

$$E_y^{(1)} = \psi_1 \, e^{ik_1 \sin \theta_i \, x} \tag{1}$$

with

$$\psi_1 = e^{ik_1 \cos \theta_i \, z} + R \, e^{-ik_1 \cos \theta_i \, z} \; . \tag{2}$$

In region (2), $z > d$, the transmitted field is

$$E_y^{(2)} = \psi_2 \, e^{ik_1 \sin \theta_i \, x} \tag{3}$$

with

$$\psi_2 = T \, e^{ik_2 \cos \theta_t \, (z-d)} \; . \tag{4}$$

R and T are respectively the reflection and transmission coefficients to be determined, and $k_{1,2} = \omega\sqrt{\mu_o \varepsilon_{1,2}}$

Now, the following conditions describe the transition region; 1) linear, 2) isotropic, 3) time-invariant, 4) inhomogeneous, 5) source-free. For this case the wave equation for the electric field $\bar{E}$ is

THIS BOOK CONTAINS NUMEROUS PAGES WITH DIAGRAMS THAT ARE CROOKED COMPARED TO THE REST OF THE INFORMATION ON THE PAGE.

THIS IS AS RECEIVED FROM CUSTOMER.

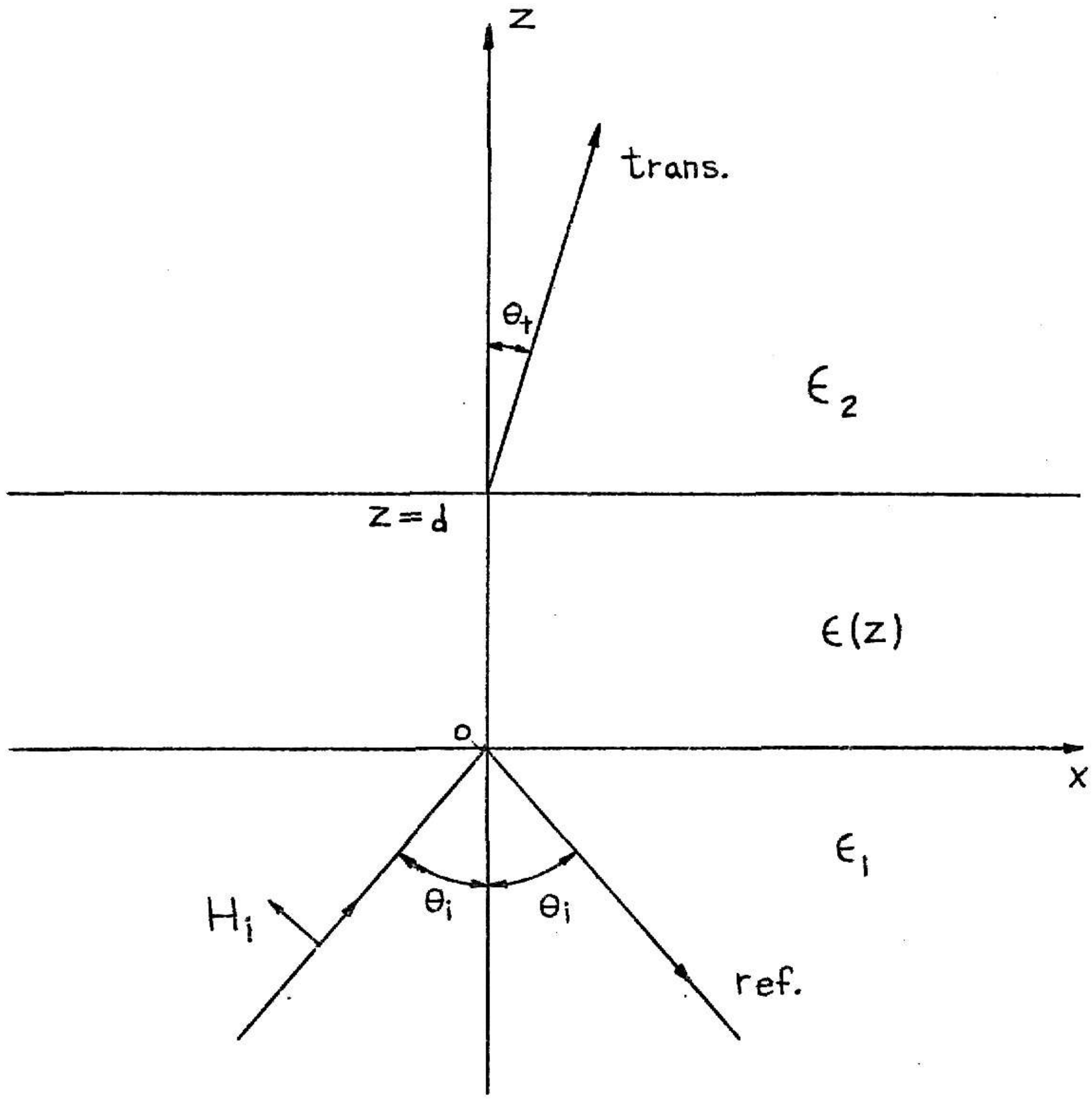

Fig. 1. Geometry of the problem of a plane wave incident upon an inhomogeneous dielectric layer.

$$\nabla^2\bar{E} + \nabla\left(\bar{E} \cdot \frac{\nabla\varepsilon}{\varepsilon}\right) + k^2\bar{E} = 0 \ . \tag{5}$$

We may specialize this further by noting that for our case the field will have only a y component, with no variation in y, and that the permittivity is a function of z only; or $E_x = E_z = 0$, $\frac{\partial E_y}{\partial y} = 0$, and $\varepsilon = \varepsilon(z)$. The wave equation thus becomes

$$[\nabla_{xz}^2 + k^2(z)]E_y = 0 \ . \tag{6}$$

Using the method of separation of variables, we seek solutions of the form

$$E_y = \psi(z)\ X(x) \ . \tag{7}$$

Substituting (7) into (6) and dividing by ψX gives

$$\frac{\psi''}{\psi} + k^2 = -\frac{X''}{X} \ . \tag{8}$$

Since x and z are independent variables, each must be constant. If this constant value is γ^2, then we obtain the two equations

$$X'' + \gamma^2 X = 0 \tag{9}$$

$$\psi'' + (k^2 - \gamma^2)\psi = 0 \ . \tag{10}$$

Solutions to (9) are

$$X = e^{\pm i\gamma x} \ . \tag{11}$$

A boundary condition at z = 0 is the continuity of E_y so the x variation of the fields in the layer must be that of the fields in region (1). This requires that

$$\gamma = k_1 \sin\theta_i \tag{12}$$

and

$$X = e^{ik_1 \sin\theta_i\ x} \ . \tag{13}$$

Note that the variation in x is not a function of k - since of course k is a function of z - and so does not depend upon the dielectric profile.

We have now reduced the problem to that of solving (10), a second-order, linear differential equation with a non-constant coefficient. In order to numerically investigate the resulting fields, we choose a linear dielectric profile as shown in Fig. 2. We have in the layer

$$\varepsilon(z) = \varepsilon_1(1 + az) \tag{14}$$

where

$$a = (\varepsilon_r - 1)/d \tag{15}$$

and

$$\varepsilon_r = \varepsilon_2/\varepsilon_1 = k_2^2/k_1^2 \ . \tag{16}$$

The equation to be solved may now be written as

$$\psi'' + k_1^2(\cos^2 \theta_i + az)\psi = 0 \ . \tag{17}$$

Subsequent application of boundary conditions at $z = 0$ and $z = d$ will complete the solution.

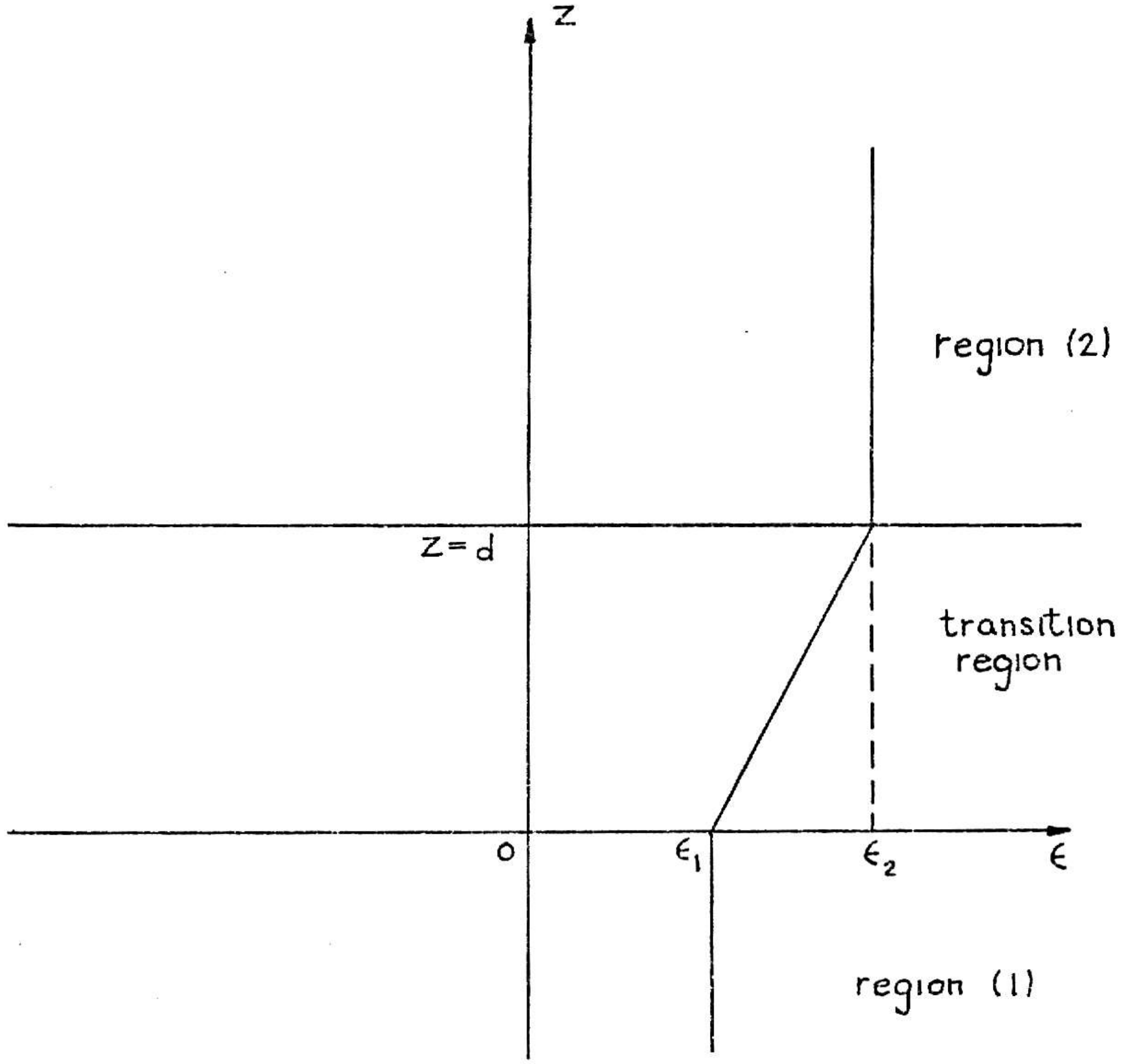

Fig. 2. Transition region with linear dielectric profile.

III. KNOWN-FUNCTION SOLUTION

To put (17) into a familiar form, let

$$w = \frac{2k_1}{3a} (az + \cos^2 \theta_i)^{3/2} \tag{18}$$

$$\psi = w^{1/3} V(w) \tag{19}$$

to obtain

$$w' = k_1 (az + \cos^2 \theta_i)^{1/2} \tag{20}$$

$$w'' = \frac{ak_1}{2} (az + \cos^2 \theta_i)^{-1/2} \tag{21}$$

$$\psi'' = w''\left(\frac{w^{-2/3}V}{3} + w^{1/3}V'\right)$$

$$+ w'^2[V''w^{1/3} + \frac{2}{3} w^{-2/3}V' - \frac{2}{9} w^{-5/3}V] \ . \tag{22}$$

By substituting these expressions into (17), a change of variables is accomplished and (17) becomes

$$w''\left(\frac{w^{-2/3}V}{3} + w^{1/3}V'\right) + w'^2[w^{1/3}V'' + \frac{2}{3} w^{-2/3}V' + \left(w^{1/3} - \frac{2}{9} w^{-5/3}\right)V] = 0 \ . \tag{23}$$

Noting that $\frac{w'^2}{w''} = 3w$, we divide (23) by $w''w^{4/3}$ and obtain Bessel's equation of order 1/3,

$$V'' + \frac{V'}{w} + (1 - \frac{1}{9w^2})V = 0 \ . \tag{24}$$

The solution to (24) may be written as

$$V = A \ J_{1/3}(w) + B \ J_{-1/3}(w) \tag{25}$$

where $J_{\pm 1/3}$ denotes the Bessel function of the first kind of order ±1/3 and A and B are constants to be evaluated. The fields in the layer are now

$$E_y = \psi\, e^{ik_1 \sin\theta_i x} \tag{26}$$

where

$$\psi = w^{1/3}[A\, J_{1/3}(w) + B\, J_{-1/3}(w)]\ . \tag{27}$$

Now we require that the tangential components of the electric and magnetic fields be continuous across the boundaries, that is at $z = 0$

$$E_y^{(1)} = E_y \tag{28}$$

$$H_x^{(1)} = H_x \tag{29}$$

and at $z = d$

$$E_y^{(2)} = E_y \tag{30}$$

$$H_x^{(2)} = H_x\ . \tag{31}$$

H_x in all regions is given by the field equation

$$H_x = \frac{i}{\omega\mu_o}\frac{\partial E_y}{\partial z}\ . \tag{32}$$

Substituting the expressions for the fields, given by (1), (3), (26), and (32), into (28)-(31) yields four simultaneous linear equations in A, B, R, T:

$$\begin{bmatrix} (-i\, w_o^{1/3}\, J^o_{-2/3}) & (i\, w_o^{1/3}\, J^o_{2/3}) & 1 & 0 \\ (w_o^{1/3}\, J^o_{1/3}) & (w_o\, J^o_{-1/3}) & -1 & 0 \\ (-i\, w_d^{1/3}\, J^d_{-2/3}) & (i\, w_d^{1/3}\, J^d_{2/3}) & 0 & -1 \\ (w_d^{1/3}\, J^d_{1/3}) & (w_d^{1/3}\, J^d_{-1/3}) & 0 & -1 \end{bmatrix} \begin{bmatrix} A \\ B \\ R \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\ . \tag{33}$$

Here we have introduced the notation of subscript o and d indicating the argument w evaluated at z = 0 and z = d and superscript o and d meaning the function J evaluated at z = 0 and z = d. From these equations it is found that

$$A = \frac{2w_o^{-1/3}(i\ J^d_{2/3} - J^d_{-1/3})}{D} \tag{34}$$

$$B = \frac{2w_o^{-1/3}(J^d_{1/3} + i\ J^d_{-2/3})}{D} \tag{35}$$

$$R = \frac{\begin{array}{c} i\ J^d_{2/3}\ J^o_{1/3} - J^d_{-1/3}\ J^o_{1/3} + J^d_{1/3}\ J^o_{-1/3} + i\ J^d_{-2/3}\ J^o_{-1/3} \\ - J^d_{2/3}\ J^o_{-2/3} - i\ J^d_{-1/3}\ J^o_{-2/3} - i\ J^d_{1/3}\ J^o_{2/3} + J^d_{-2/3}\ J^o_{2/3} \end{array}}{D} \tag{36}$$

$$T = \frac{i\ 2\sqrt{3}\ w_o^{-1/3}\ w_d^{-2/3}}{\pi D}\ , \tag{37}$$

where

$$D = (i\ J^d_{2/3} - J^d_{-1/3})J^o_{1/3} + (J^d_{1/3} + i\ J^d_{-2/3})J^o_{-1/3}$$
$$+ (J^d_{2/3} + i\ J^d_{-1/3})J^o_{-2/3} + (i\ J^d_{1/3} - J^d_{-2/3})J^o_{2/3}\ . \tag{38}$$

This completes the solution for the fields in all regions.

As stated earlier, we are interested in the field outside the transition region which requires that we know R and T. It will be seen later that the Hill-function approach requires the evaluation of the same infinite determinants to find either R or T. Therefore it will be necessary to consider only one of these quantities. We will use R.

Now, for computational purposes the reflection coefficient R will be expressed in terms of the Airy functions, Ai(•) and Bi(•). By substituting into (36) the relations [12]

$$J_{\pm 1/3}(w) = \frac{1}{2}\sqrt{3/u}\,[\sqrt{3}\,Ai(-u) \mp Bi(-u)] \tag{39}$$

$$J_{\pm 2/3}(w) = \sqrt{3}/2u\,[\pm\sqrt{3}\,Ai'(-u) + Bi'(-u)]\ , \tag{40}$$

where

$$u = (3/2\ w)^{2/3}\ , \tag{41}$$

we obtain

$$R = \frac{-(Ai^{d\prime}Bi^{o\prime} - Bi^{d\prime}Ai^{o\prime}) + \sqrt{u_d u_o}\,(Ai^d Bi^o - Bi^d Ai^o) + i[-\sqrt{u_d}(Ai^d Bi^{o\prime} - Bi^d Ai^{o\prime}) + \sqrt{u_o}(-Ai^{d\prime}Bi^o + Bi^{d\prime}Ai^o)]}{(Ai^{d\prime}Bi^{o\prime} - Bi^{d\prime}Ai^{o\prime}) + \sqrt{u_d u_o}(Ai^d Bi^o - Bi^d Ai^o) + i[\sqrt{u_o}(-Ai^{d\prime}Bi^o + Bi^{d\prime}Ai^o) + \sqrt{u_d}(Ai^d Bi^{o\prime} - Bi^d Ai^{o\prime})]}. \tag{42}$$

It should be noted that the arguments of the Airy functions in this expression are $-u_o$ and $-u_d$. If we let

$$v_o = e^{-i2\pi/3}\,u_o \tag{43}$$

$$v_d = e^{-i2\pi/3}\,u_d\ , \tag{44}$$

then it can be shown that

$$R = \frac{-(Ai^{d\prime}Bi^{o\prime} - Bi^{d\prime}Ai^{o\prime}) + \sqrt{v_o v_d}\,(Ai^d Bi^o - Bi^d Ai^o) + i[\sqrt{v_d}\,(Ai^d Bi^{o\prime} - Bi^d Ai^{o\prime}) - \sqrt{v_o}\,(-Ai^{d\prime}Bi^o + Bi^{d\prime}Ai^o)]}{(Ai^{d\prime}Bi^{o\prime} - Bi^{d\prime}Ai^{o\prime}) + \sqrt{v_o v_d}\,(Ai^d Bi^o - Bi^d Ai^o) + i[-\sqrt{v_o}\,(Bi^{d\prime}Ai^o - Ai^{d\prime}Bi^o) - \sqrt{v_d}\,(Ai^d Bi^{o\prime} - Bi^d Ai^{o\prime})]} \tag{45}$$

where the arguments are $-v_o$ and $-v_d$. In order to have the arguments of the Airy functions real, (42) is used when $\varepsilon_r > 1$ and (45) when $\varepsilon_r < 1$. This is done, again, for computational purposes.

IV. HILL-FUNCTION SOLUTION

Let the following change of variables be introduced into (10):

$$\zeta = \pi z/2d \tag{46}$$

$$u(\zeta) = \psi(z) \ . \tag{47}$$

This gives

$$u'' + \left(\frac{2d}{\pi}\right)^2 \left(k^2 - \gamma^2\right) u = 0 \ . \tag{48}$$

Now if the coefficient $g(z) = \left(\frac{2d}{\pi}\right)^2 \left(k^2 - \gamma^2\right)$ is even and periodic in π, as a function of ζ, then it may be expressed as a Fourier cosine series and (48) becomes

$$u'' + (\lambda + 2 \sum_{n=1}^{\infty} g_n \cos 2n\zeta) u = 0 \ . \tag{49}$$

This is Hill's equation. If $\sum_{n=1}^{\infty} |g_n|$ converges, then we may proceed and write the solution in terms of Hill functions [13]-[15],

$$u = Au_1 + Bu_2 \ , \tag{50}$$

where A and B are constants.

As will be seen, it is not necessary to specify the functions u_1 and u_2; that is we need not actually solve the differential equation for the fields in the transition layer if only R or T is desired.

In the layer, then,

$$E_y = Au_1 + Bu_2 \tag{51}$$

$$H_x = \frac{i\pi}{2d\omega\mu_o} [Au_1' + Bu_2'] \tag{52}$$

where the x dependence has been suppressed. Application of boundary conditions at z = 0 and z = d, as given in (28)-(31), gives

$$\begin{bmatrix} \dfrac{-i\pi}{2dk_1 \cos\theta_i} u_1'(0) & \dfrac{-i\pi\, u_2'(0)}{2dk_1 \cos\theta_i} & 1 & 0 \\ u_1(0) & U_2(0) & -1 & 0 \\ \dfrac{-i\pi\, u_1'(\pi/2)}{2dk_2 \cos\theta_t} & \dfrac{-i\pi\, u_2'(\pi/2)}{2dk_2 \cos\theta_t} & 0 & -1 \\ u_1(\pi/2) & u_2(\pi/2) & 0 & -1 \end{bmatrix} \begin{bmatrix} A \\ B \\ R \\ T \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} . \tag{53}$$

From (53), the reflection coefficient is found to be

$$R = \frac{\begin{array}{c} -(u_2^{d\prime} u_1^{o\prime} - u_1^{d\prime} u_2^{o\prime}) + a_o a_d (u_1^o u_2^d - u_2^o u_1^d) \\ + i[-a_d(u_1^d u_2^{o\prime} - u_2^d u_1^{o\prime}) + a_o(u_1^o u_2^{d\prime} - u_2^o u_1^{d\prime})] \end{array}}{\begin{array}{c} (u_2^{d\prime} u_1^{o\prime} - u_1^{d\prime} u_2^{o\prime}) + a_o a_d (u_1^o u_2^d - u_2^o u_1^d) \\ + i[a_o(u_2^{d\prime} u_1^o - u_1^{d\prime} u_2^o) + a_d(u_1^d u_2^{o\prime} - u_2^d u_1^{o\prime})] \end{array}} \tag{54}$$

where

$$a_o = L_1 \cos\theta_i \tag{55}$$

$$a_d = L_1(\varepsilon_r - \sin^2\theta_i)^{1/2} \tag{56}$$

$$L_1 = \frac{2k_1 d}{\pi} . \tag{57}$$

Now the Hill functions may be normalized by setting

$$u_1(0) = u_2'(0) = 1 \tag{58}$$

$$u_1'(0) = u_2(0) = 0 \tag{59}$$

and we have

$$R = \frac{u_1^{d\prime} + a_o a_d u_2^d - i\, a_d u_1^d + i\, a_o u_2^{d\prime}}{-u_1^{d\prime} + a_o a_d u_2^d + i\, a_d u_1^d + i\, a_o u_2^{d\prime}} . \tag{60}$$

So to find the reflection coefficient we must find $u_1(\pi/2)$, $u_2(\pi/2)$, $u_1'(\pi/2)$, $u_2'(\pi/2)$.

In principle, the Hill functions may be found if the coefficients of the Fourier series in (49) are specified. Thus, we might expect that the value at $\zeta = \pi/2$ would be expressible somehow in terms of these coefficients. This has indeed been found to be so [15, p. 34] and the results are

$$u_1(\pi/2) = \cos\left(\frac{\pi}{2}\sqrt{\lambda}\right) C_1 \tag{61}$$

$$u_2(\pi/2) = \frac{\sin\left(\frac{\pi}{2}\sqrt{\lambda}\right)}{\sqrt{\lambda}} S_0 \tag{62}$$

$$u_1'(\pi/2) = -\sqrt{\lambda}\,\sin\left(\frac{\pi}{2}\sqrt{\lambda}\right) C_0 \tag{63}$$

$$u_2'(\pi/2) = \cos\left(\frac{\pi}{2}\sqrt{\lambda}\right) S_1 \quad . \tag{64}$$

S_0, S_1, C_0, C_1 are the one-sided infinite determinants

$$S_0 = \left| \delta_{n,m} + \frac{g_{n-m} - g_{n+1}}{\lambda - 4n^2} \right|_1^\infty \tag{65}$$

$$S_1 = \left| \delta_{n,m} + \frac{g_{n-m} - g_{n+m+1}}{\lambda - (2n+1)^2} \right|_0^\infty \tag{66}$$

$$C_0 = \left| \delta_{n,m} + \frac{(g_{n-m} + g_{n+m})(1 + \operatorname{sgn} n \operatorname{sgn} m)}{\sqrt{\varepsilon_n \varepsilon_m}\,(\lambda - 4n^2)} \right|_0^\infty \tag{67}$$

$$C_1 = \left| \delta_{n,m} + \frac{g_{n-m} + g_{n+m+1}}{\lambda - (2n+1)^2} \right|_0^\infty \tag{68}$$

where

$$\varepsilon_n = \begin{cases} 1 & n = 0 \\ 2 & n > 0 \end{cases} \tag{69}$$

$$g_{-n} = \begin{cases} 0 & n = 0 \\ g_n & n > 0 \end{cases} \tag{70}$$

$$\delta_{n,m} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases} \tag{71}$$

$$\operatorname{sgn} n = \begin{cases} 0 & n = 0 \\ 1 & n > 0 \end{cases} \tag{72}$$

The Fourier coefficients λ and g_n are

$$g_n = \frac{L_1^2}{d} \int_0^d [\varepsilon_r(z) - \sin^2 \theta_i] \cos \frac{n\pi z}{d} \, dz \tag{73}$$

$$\lambda = \frac{L_1^2}{d} \int_0^d [\varepsilon_r(z) - \sin^2 \theta_i] \, dz \, , \tag{74}$$

where $\varepsilon_r(z)$ is the relative permittivity in the layer, $\varepsilon(z)/\varepsilon_1$. Some insight as to the structure of the determinants may be gained by writing them as

$$S_0 = \begin{vmatrix} 1 - \frac{g_2}{\lambda - 4} & \frac{g_1 - g_3}{\lambda - 4} & \frac{g_2 - g_4}{\lambda - 4} & \frac{g_3 - g_5}{\lambda - 4} & \cdot\cdot \\ \frac{g_1 - g_3}{\lambda - 16} & 1 - \frac{g_4}{\lambda - 16} & \frac{g_1 - g_5}{\lambda - 16} & \frac{g_2 - g_6}{\lambda - 16} & \cdot \\ \frac{g_2 - g_4}{\lambda - 36} & \frac{g_1 - g_5}{\lambda - 36} & 1 - \frac{g_6}{\lambda - 36} & \frac{g_1 - g_7}{\lambda - 36} & \cdot \\ \frac{g_3 - g_5}{\lambda - 64} & \frac{g_2 - g_6}{\lambda - 64} & \frac{g_1 - g_7}{\lambda - 64} & 1 - \frac{g_8}{\lambda - 64} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \end{vmatrix} \tag{75}$$

$$
s_1 = \begin{vmatrix}
1 - \frac{g_1}{\lambda - 1} & \frac{g_1 - g_2}{\lambda - 1} & \frac{g_2 - g_3}{\lambda - 1} & \frac{g_3 - g_4}{\lambda - 1} & \cdot \\
\frac{g_1 - g_2}{\lambda - 9} & 1 - \frac{g_3}{\lambda - 9} & \frac{g_1 - g_4}{\lambda - 9} & \frac{g_2 - g_5}{\lambda - 9} & \cdot \\
\frac{g_2 - g_3}{\lambda - 25} & \frac{g_1 - g_4}{\lambda - 25} & 1 - \frac{g_5}{\lambda - 25} & \frac{g_1 - g_6}{\lambda - 25} & \cdot \\
\frac{g_3 - g_4}{\lambda - 49} & \frac{g_2 - g_5}{\lambda - 49} & \frac{g_1 - g_6}{\lambda - 49} & 1 - \frac{g_7}{\lambda - 49} & \cdot \\
\cdot & \cdot & \cdot & \cdot &
\end{vmatrix} \tag{76}
$$

$$
c_0 = \begin{vmatrix}
1 & \frac{\sqrt{2}\, g_1}{\lambda} & \frac{\sqrt{2}\, g_2}{\lambda} & \frac{\sqrt{2}\, g_3}{\lambda} & \cdot \\
\frac{\sqrt{2}\, g_1}{\lambda - 4} & 1 + \frac{g_2}{\lambda - 4} & \frac{g_1 + g_3}{\lambda - 4} & \frac{g_2 + g_4}{\lambda - 4} & \cdot \\
\frac{\sqrt{2}\, g_2}{\lambda - 16} & \frac{g_1 + g_3}{\lambda - 16} & 1 + \frac{g_4}{\lambda - 16} & \frac{g_1 + g_5}{\lambda - 16} & \cdot \\
\frac{\sqrt{2}\, g_3}{\lambda - 36} & \frac{g_2 + g_4}{\lambda - 36} & \frac{g_1 + g_5}{\lambda - 36} & 1 + \frac{g_6}{\lambda - 36} & \cdot \\
\cdot & \cdot & \cdot & \cdot &
\end{vmatrix} \tag{77}
$$

$$
c_1 = \begin{vmatrix}
1 + \frac{g_1}{\lambda - 1} & \frac{g_1 + g_2}{\lambda - 1} & \frac{g_2 + g_3}{\lambda - 1} & \frac{g_3 + g_4}{\lambda - 1} & \cdot \\
\frac{g_1 + g_2}{\lambda - 9} & 1 + \frac{g_3}{\lambda - 9} & \frac{g_1 + g_4}{\lambda - 9} & \frac{g_2 + g_5}{\lambda - 9} & \cdot \\
\frac{g_2 + g_3}{\lambda - 25} & \frac{g_1 + g_4}{\lambda - 25} & 1 + \frac{g_5}{\lambda - 25} & \frac{g_1 + g_6}{\lambda - 25} & \cdot \\
\frac{g_3 + g_4}{\lambda - 49} & \frac{g_2 + g_5}{\lambda - 49} & \frac{g_1 + g_6}{\lambda - 49} & 1 + \frac{g_7}{\lambda - 49} & \cdot \\
\cdot & \cdot & \cdot & \cdot &
\end{vmatrix} \tag{78}
$$

One should note that approximating the dielectric profile by truncating the Fourier series does not result in finite determinants. However, truncating the determinants then requires a knowledge of only a finite number of Fourier coefficients for their evaluation. The more basic question then is what size determinants to use in finding the reflection coefficient. In practice, however, one must approximate the actual permittivity profile so we shall study independently the effect upon the reflection coefficient due to truncating the Fourier series and truncating the infinite determinants. Later, some analytical details will be presented with regard to convergence. Presently, we continue with the Hill-function solution for the linear dielectric profile.

The coefficient in (48) for the linear profile is

$$g(z) = L_1^2(az + \cos^2 \theta_i) \quad , \quad 0 \leq z \leq d \quad . \tag{79}$$

Fig. 3 shows the even periodic extension of this which is the function given by the Fourier series in Hill's equation. The two horizontal axes illustrate the change of variables in (46). The Fourier coefficients for this function are

$$\lambda = \left(\frac{\varepsilon_r - 1}{2} + \cos^2 \theta_i\right) L_1^2 \tag{80}$$

$$g_n = \begin{cases} \dfrac{-2(\varepsilon_r - 1)}{n^2 \pi^2} L_1^2 & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \tag{81}$$

Since $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is absolutely convergent, the infinite determinants converge and the reflection coefficient may be found. This completes the Hill-function solution.

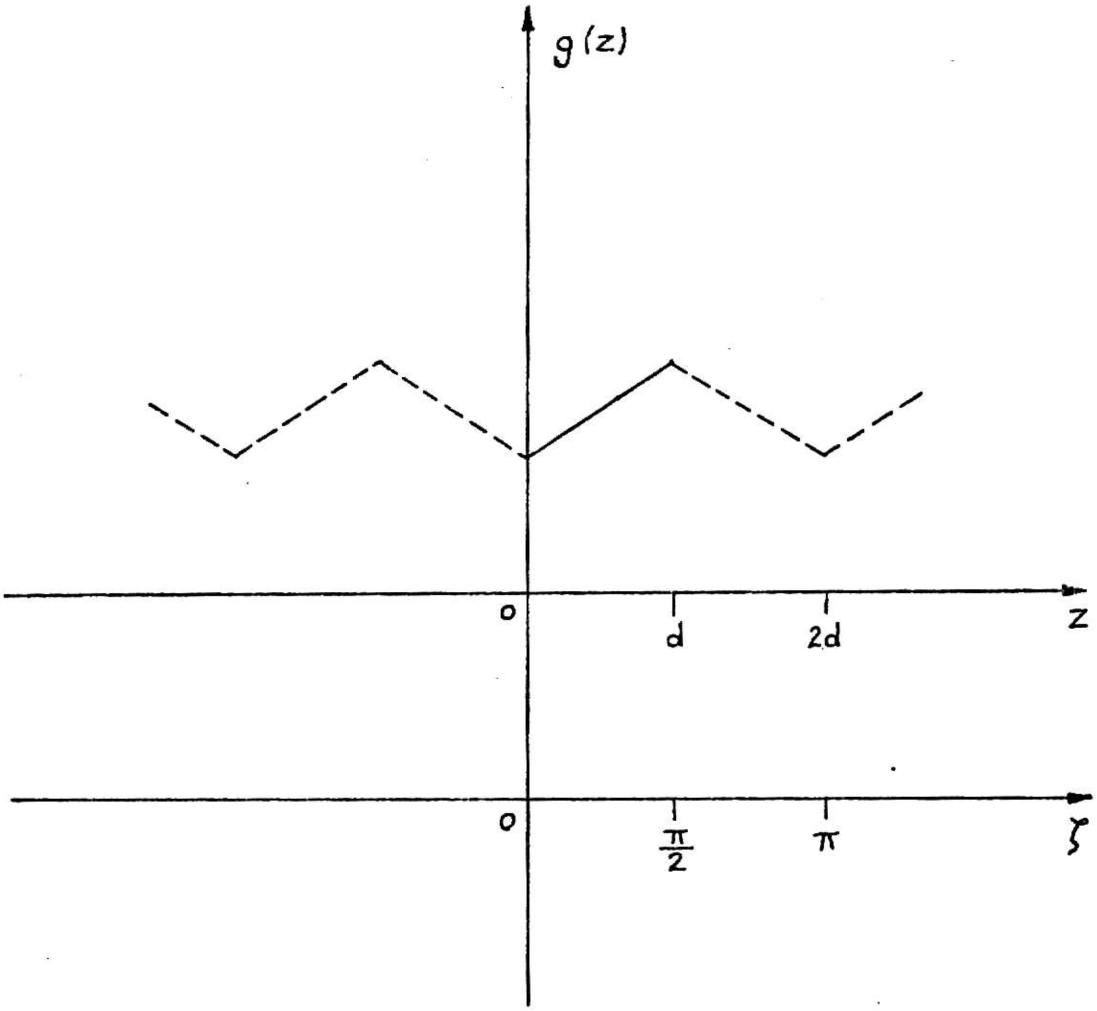

Fig. 3. Coefficient in Hill's equation represented by Fourier series.

V. NUMERICAL CALCULATION OF THE REFLECTION COEFFICIENT

From the Fourier series for g(z) we obtain the series for the linear relative permittivity profile as

$$\varepsilon_r(z) = \frac{\varepsilon_r + 1}{2} - \frac{4}{\pi^2}(\varepsilon_r - 1)\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi z}{d} . \qquad (82)$$

Figs. (4a), (4b) show profiles approximating the linear profile obtained by truncating the series in (82). The superscript n in $\varepsilon_r^{(n)}(z)$ gives the number of terms retained. Thus, $\varepsilon_r^{(n)}(z)$ is the actual dielectric profile that is solved if one truncates the series for g(z) in n non-zero terms.

In Figs. (5)-(7) the magnitude of the reflection coefficient is given as a function of either L_1 or θ_i with ε_r and L_1 or θ_i constant. Each figure consists of a family of five curves. One is the exact reflection coefficient obtained from the Airy-function solution. The remaining four are computed from the Hill-function solution using various degrees of profile approximation, namely $\varepsilon_r^{(n)}(z)$ with n = 1, 2, 3, 4. For each curve using the Hill-function solution, the computing procedure was as follows: The infinite determinants were truncated to size 3x3 and the reflection coefficient calculated. The determinant size was the incremented by one until the computed reflection coefficient differed by less than 0.5% from the one preceding. The final values of the reflection coefficient R and the determinant size NS were recorded. For each successive data point, the initial determinant size was set at one less than the final size of the determinant for the preceding point. For Fig. (5), calculations were begun at $L_1 = 0.2$, and for Figs. (6), (7), at $\theta_i = 0$.

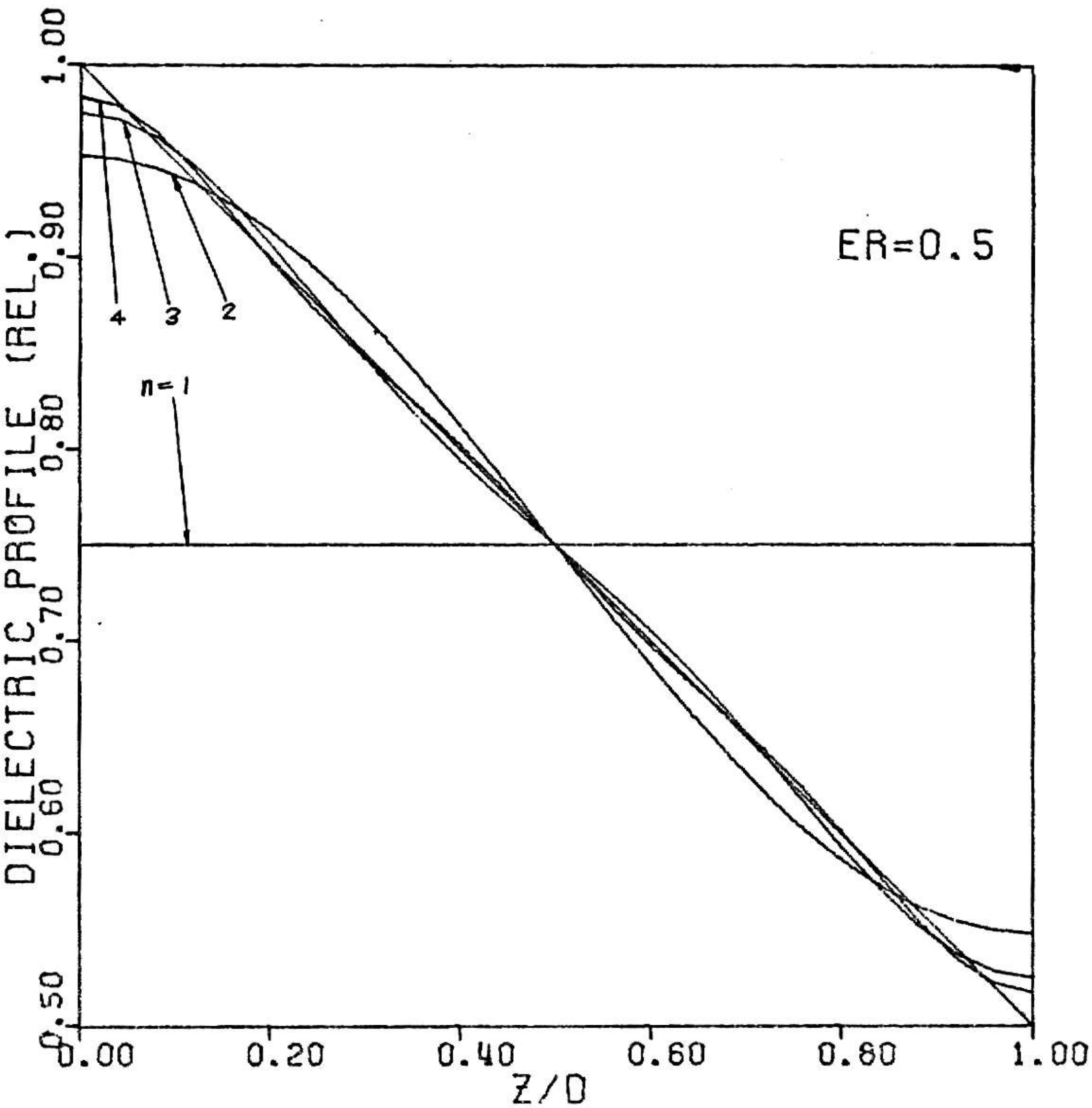

Fig. 4a. Dielectric profile represented by truncated Fourier series, $\varepsilon_r = 0.5$.

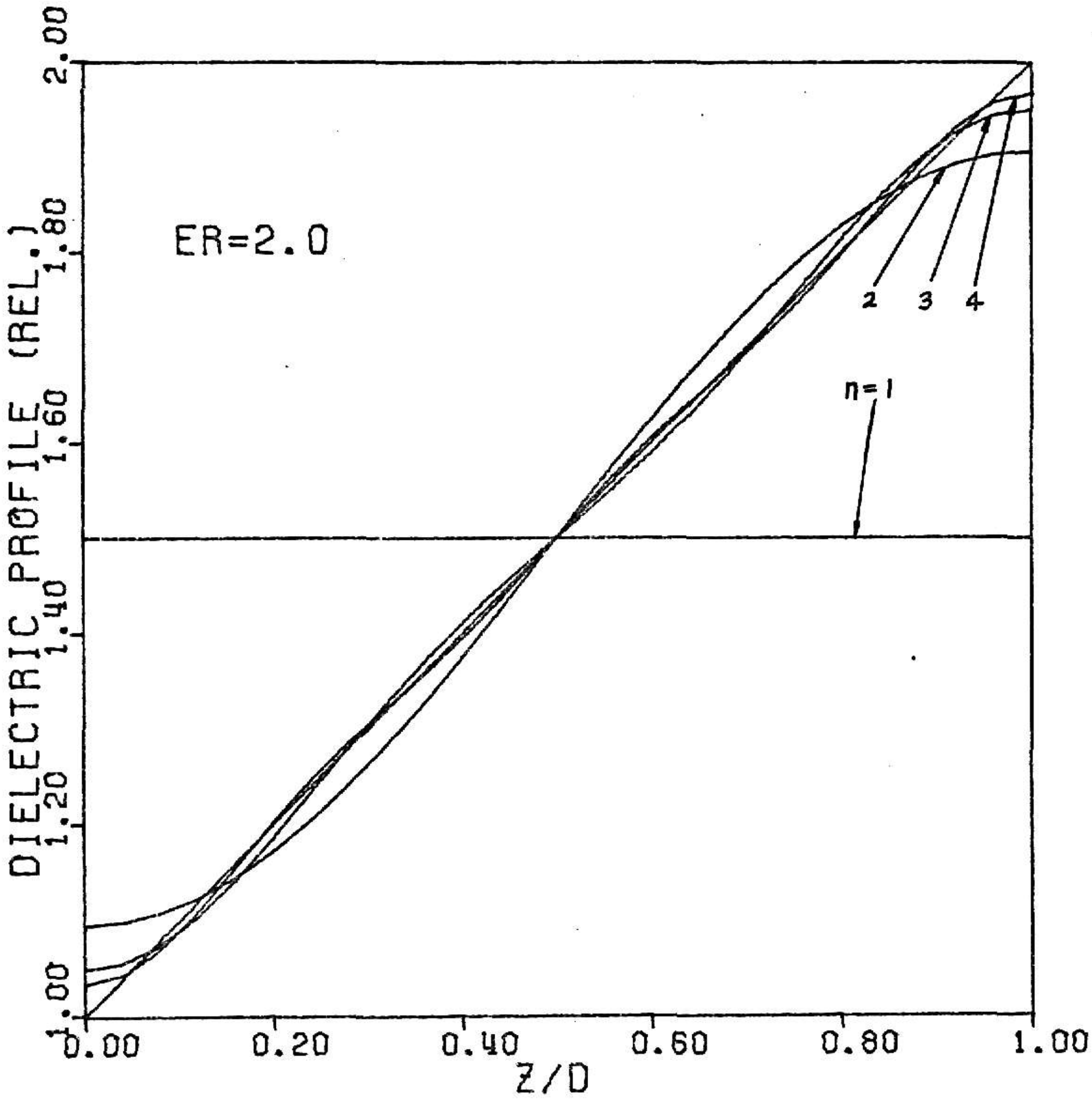

Fig. 4b. Dielectric profile represented by truncated Fourier series, $\varepsilon_r = 2.0$.

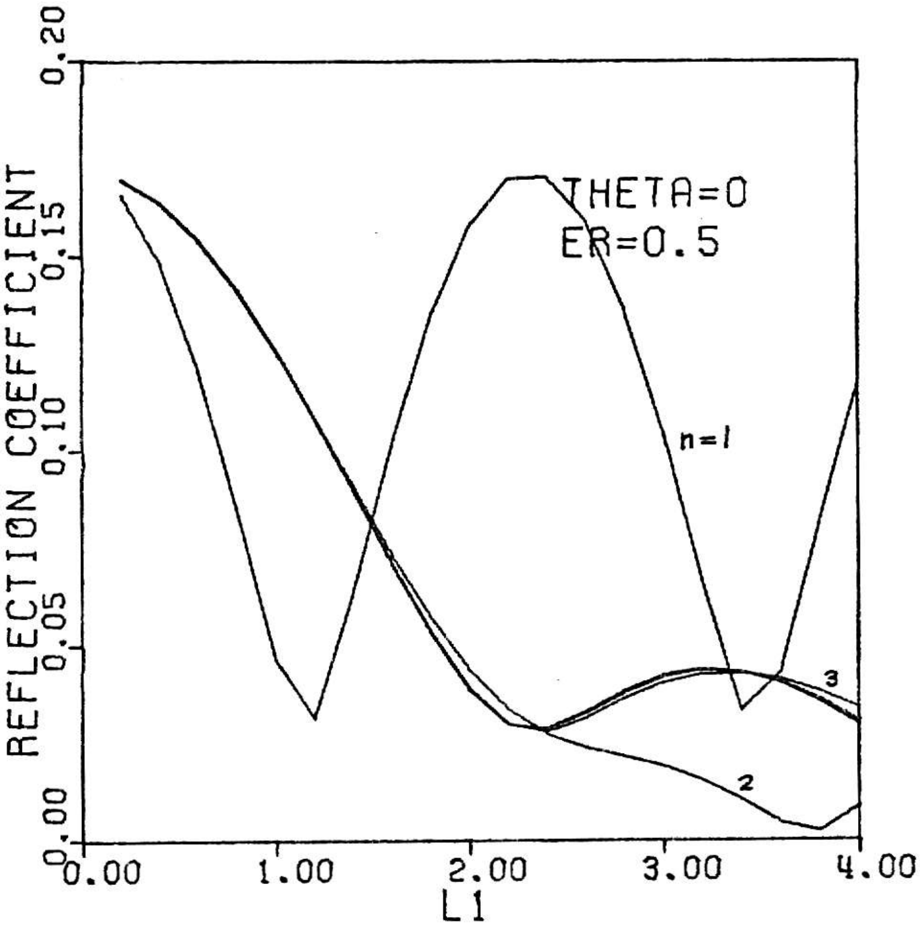

Fig. 5a. Reflection coefficient vs. L_1, $\theta_i = 0$, $\epsilon_r = 0.5$.

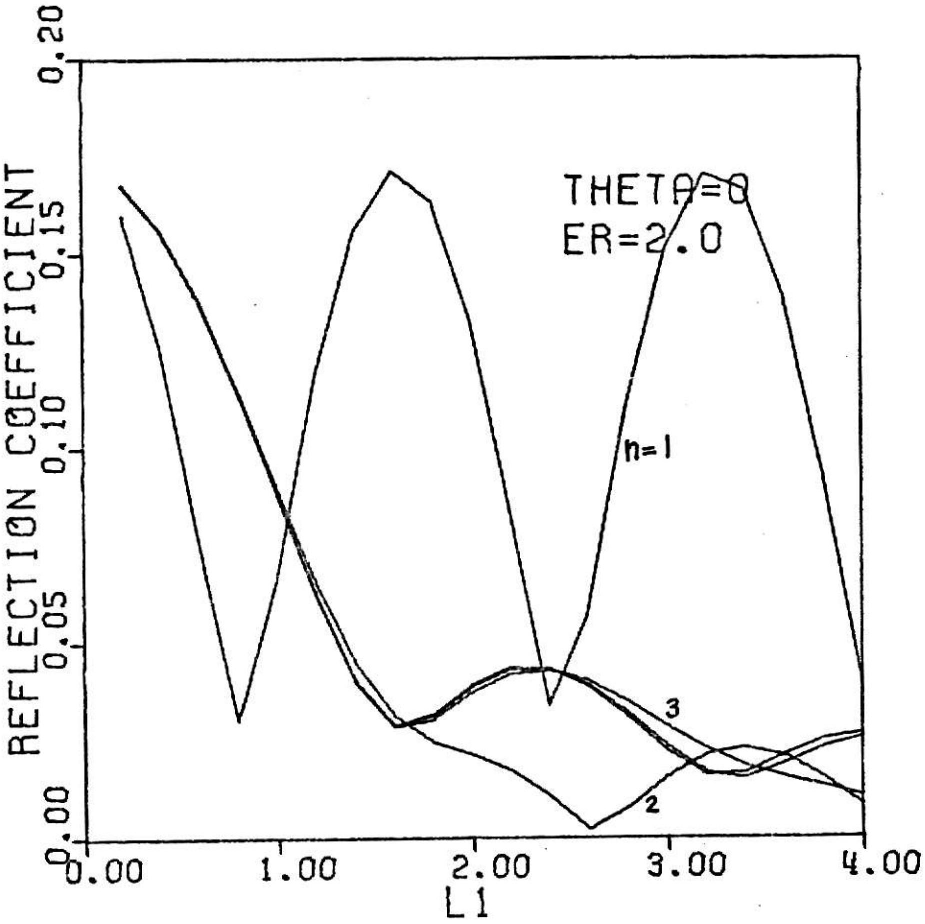

Fig. 5b. Reflection coefficient vs. L_1, $\theta_i = 0$, $\varepsilon_r = 2.0$.

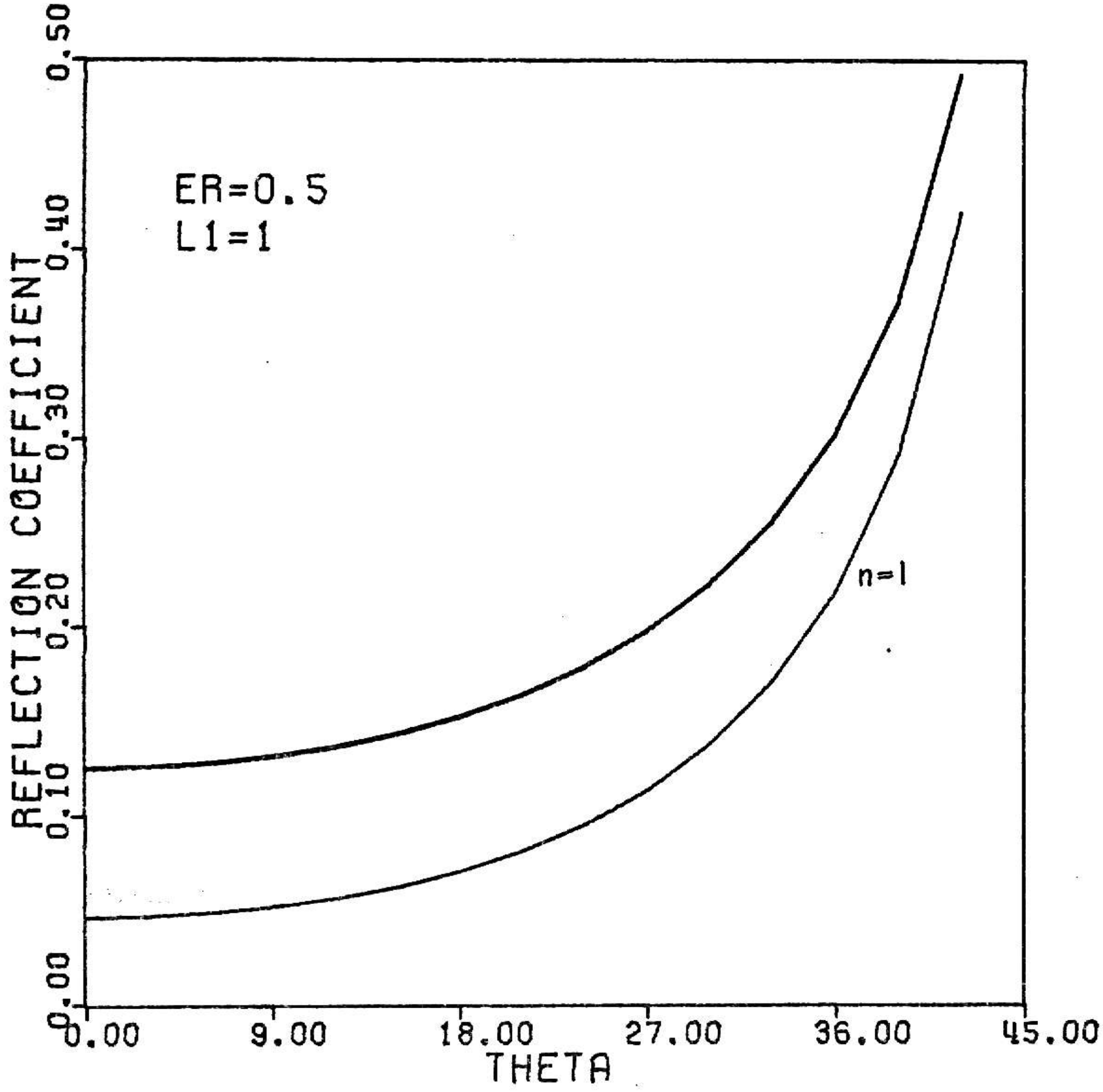

Fig. 6a. Reflection coefficient vs. θ_i, $\varepsilon_r = 0.5$, $L_1 = 1$.

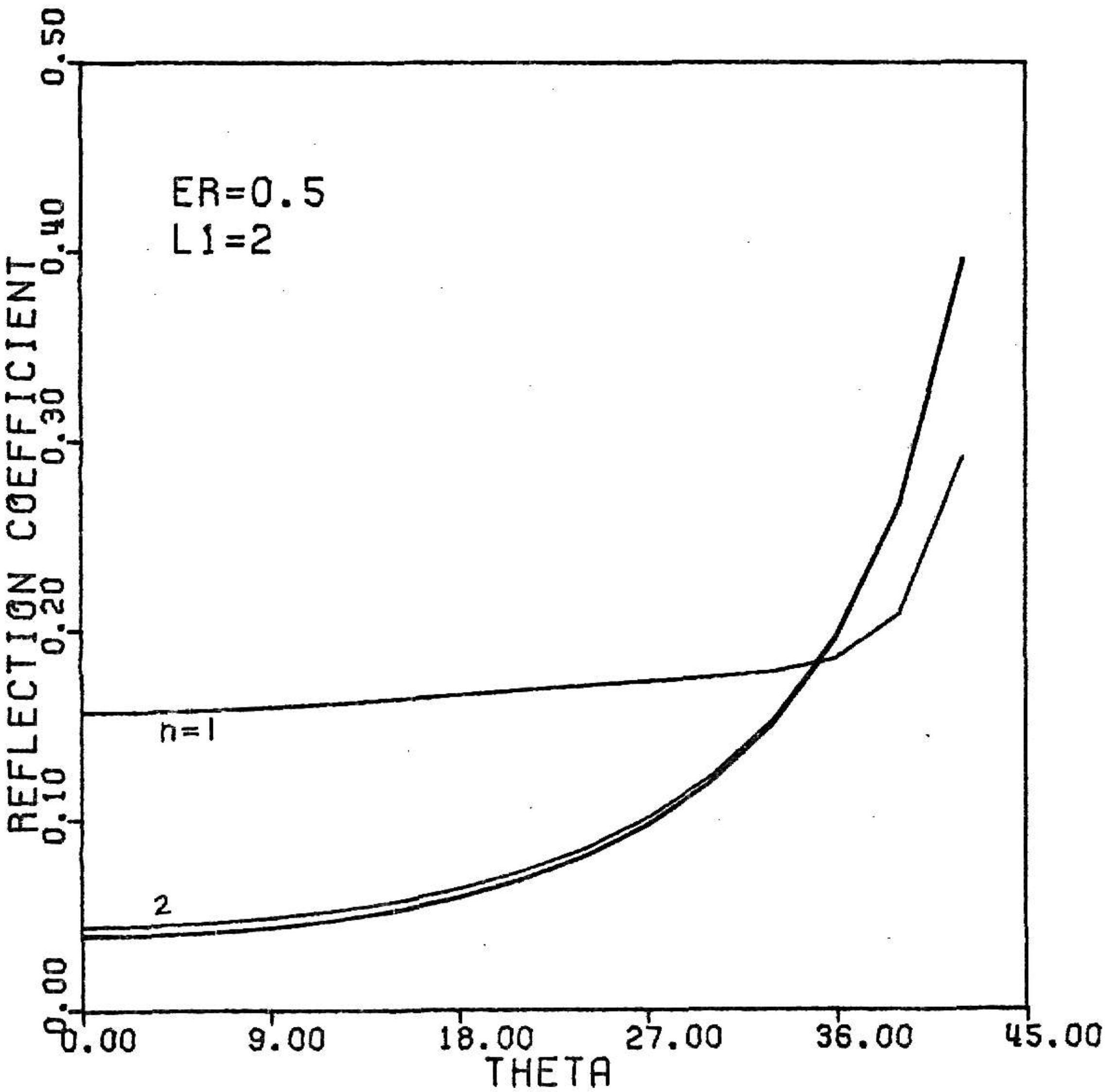

Fig. 6b. Reflection coefficient vs. θ_i, $\varepsilon_r = 0.5$, $L_1 = 2$.

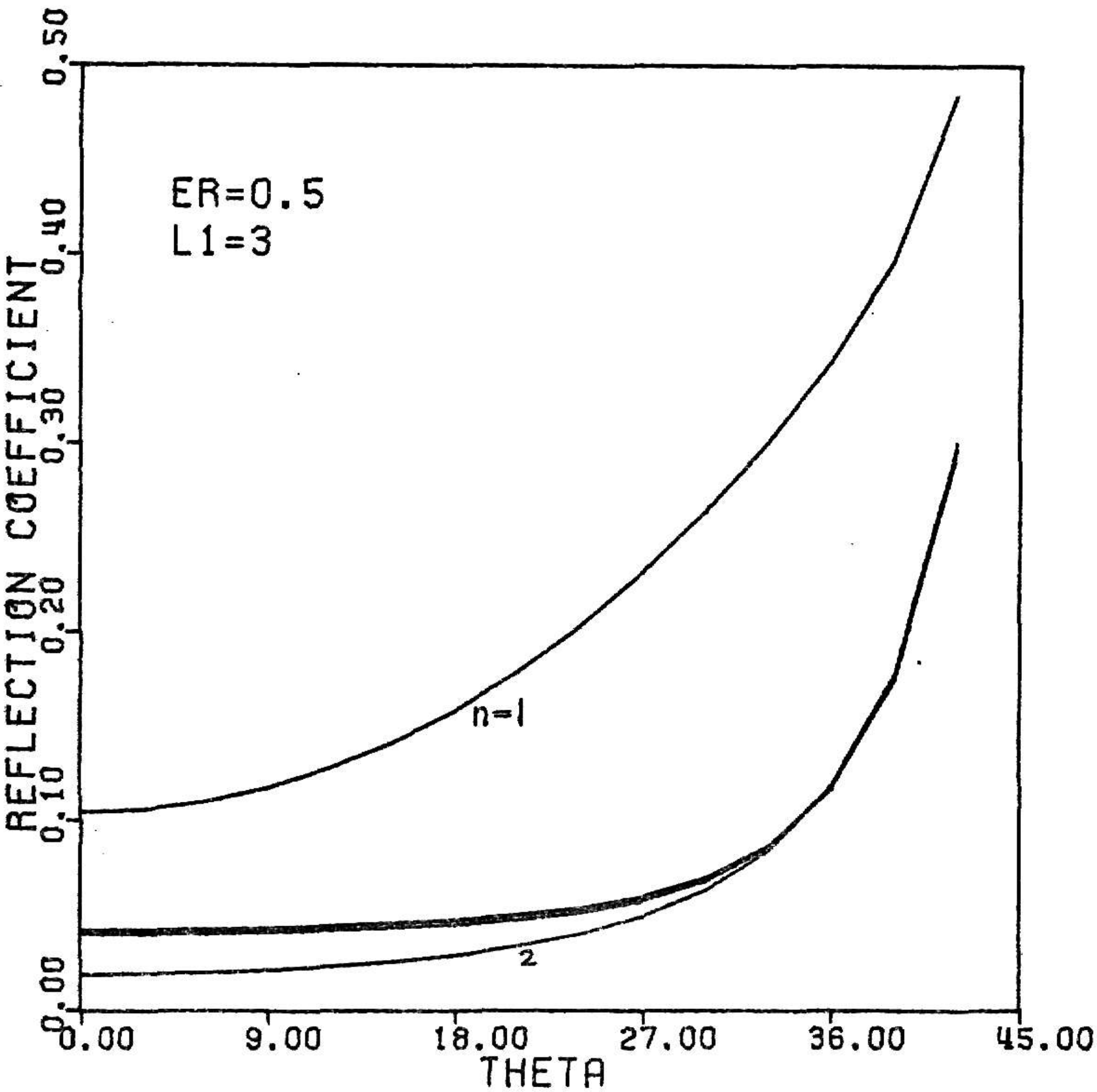

Fig. 6c. Reflection coefficient vs. θ_i, $\varepsilon_r = 0.5$, $L_1 = 3$

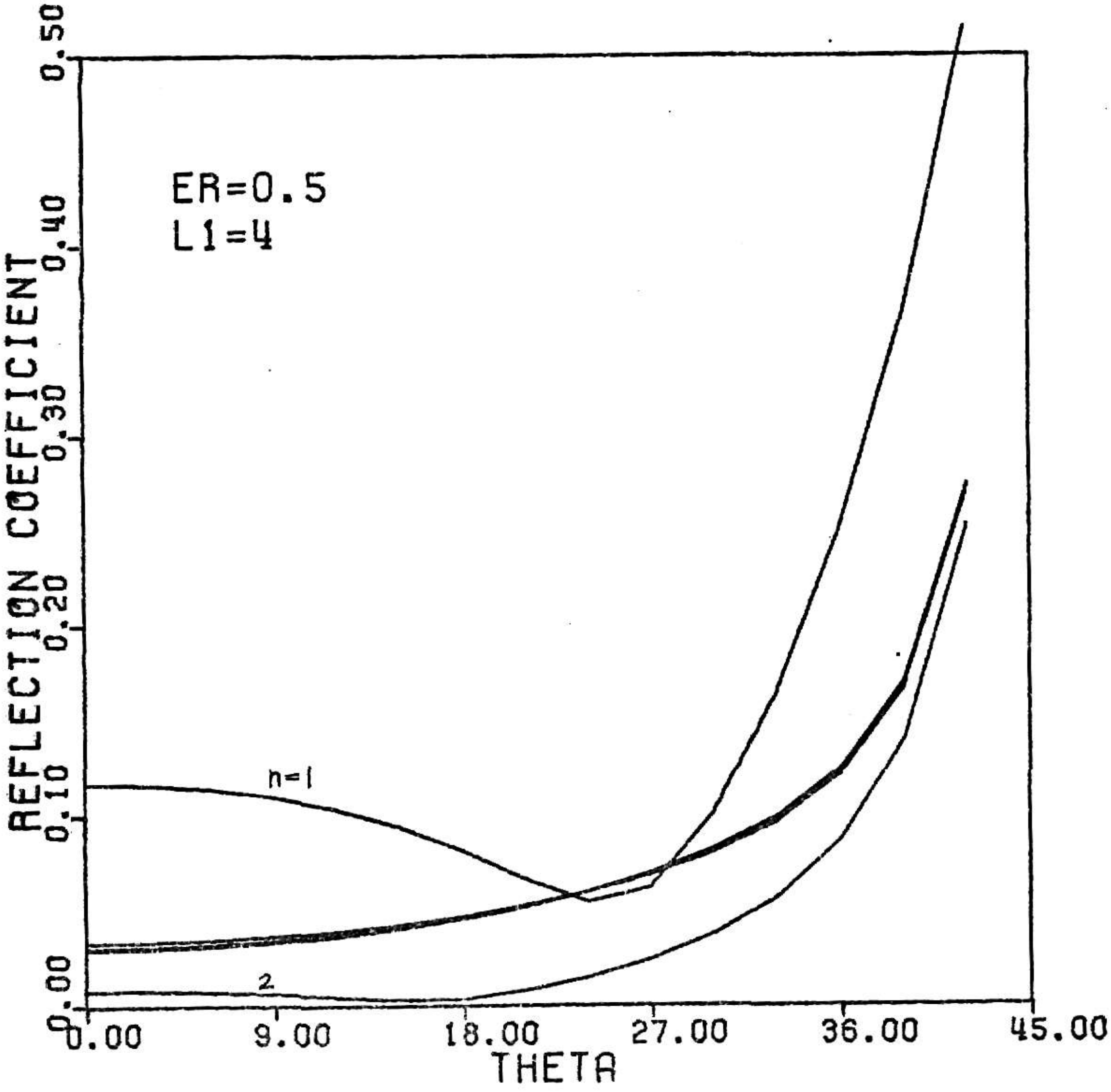

Fig. 6d. Reflection coefficient vs. θ_i, $\varepsilon_r = 0.5$, $L_1 = 4$.

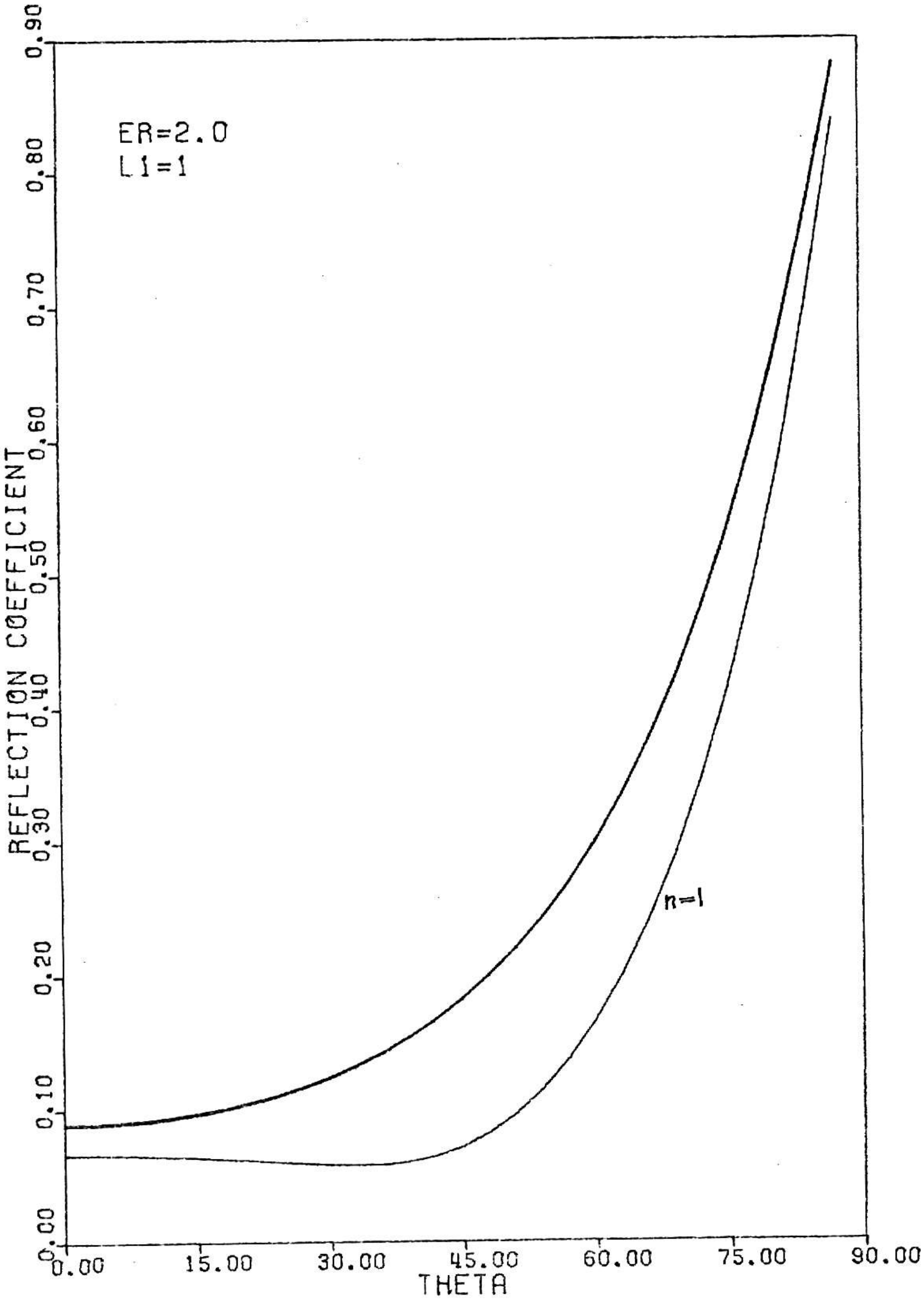

Fig. 7a. Reflection coefficient vs. θ_i, $\varepsilon_r = 2.0$, $L_1 = 1$.

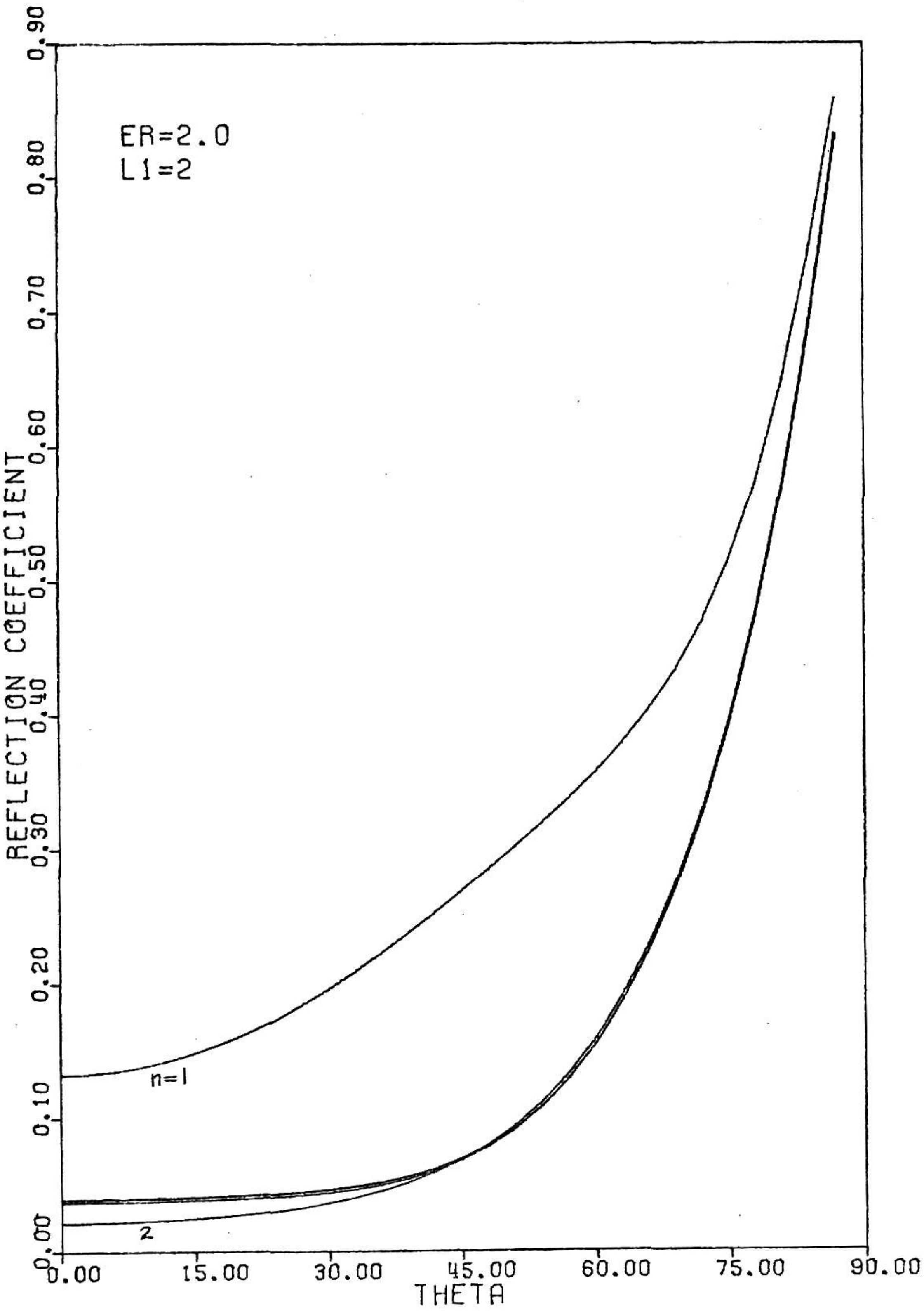

Fig. 7b. Reflection coefficient vs. θ_i, $\varepsilon_r = 2.0$, $L_1 = 2$.

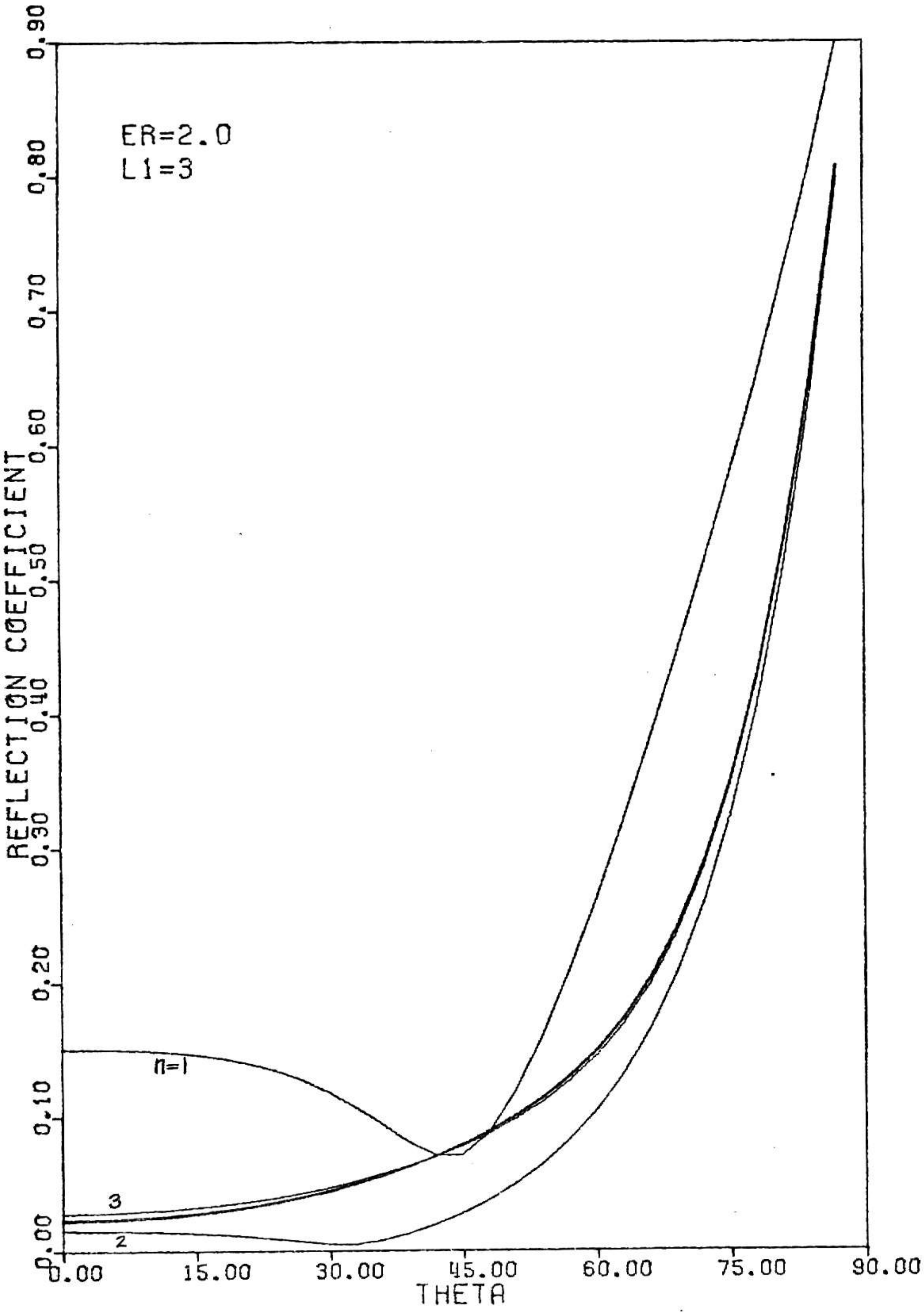

Fig. 7c. Reflection coefficient vs. θ_i, ε_r = 2.0, L_1 = 3.

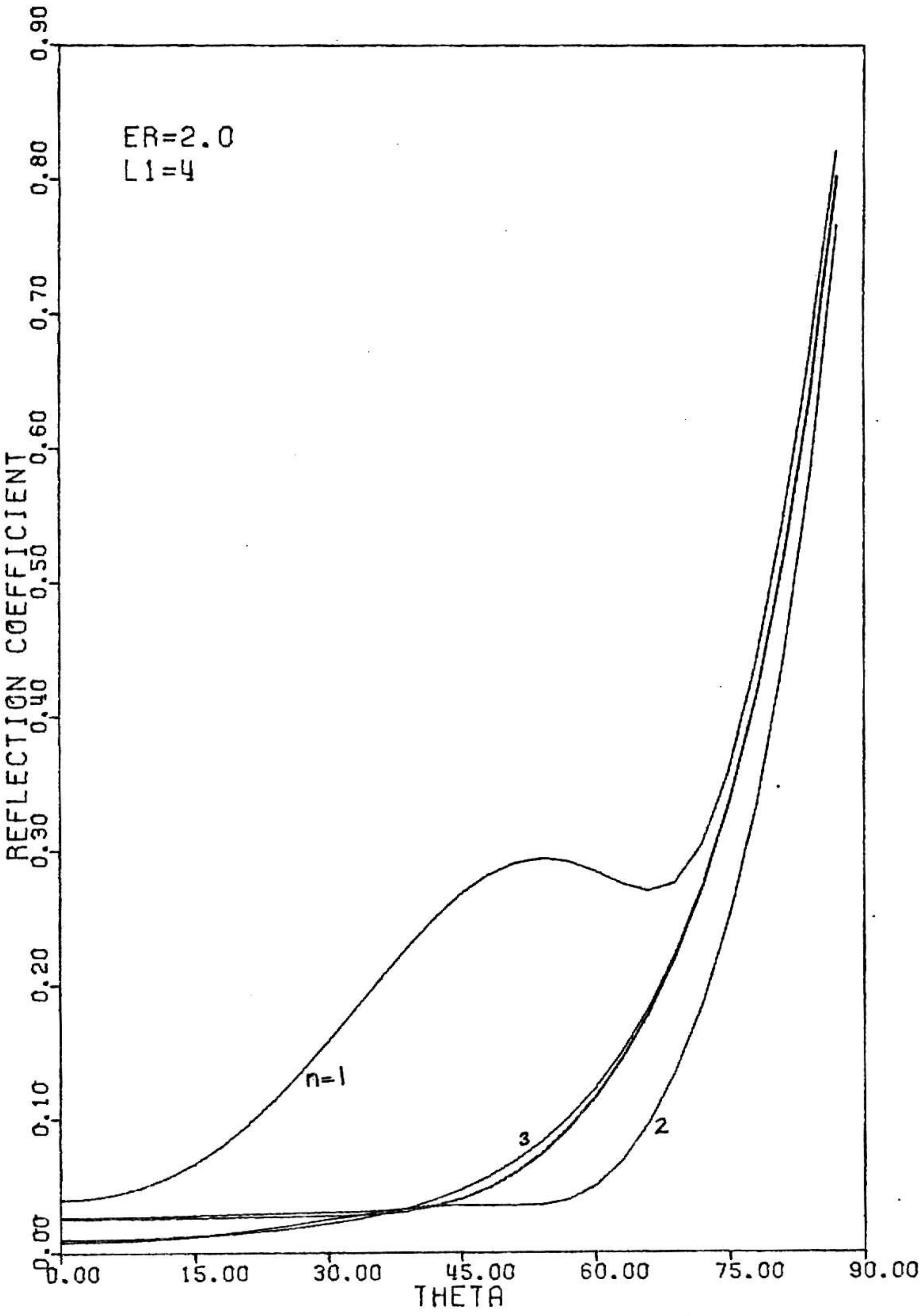

Fig. 7d. Reflection coefficient vs. θ_i, $\varepsilon_r = 2.0$, $L_1 = 4$.

The criterion of 0.5% change in R gives no clue as to the accuracy of the final R. The idea here was that if the determinants have converged close to their exact value for a given NS, then so has R, and therefore R cannot change by more than some small amount if NS is increased. The problem is that the change in R can also be small if R is converging slowly, with a value perhaps far from the exact value. The error can therefore vary over a wide range, and this is evident when comparing the reflection coefficient for n = 4 to the exact R. Fortunately, however, the error was small in all but a few places, a notable one being for $\varepsilon_r = 2$, $L_1 = 4$, and $\theta_i = 30°$ (Fig. 7d). This was roughly the point of maximum error, about 10%. Here the size of the determinant would have to be made greater by one or two to reduce the error to a more typical value of 0.5%. The range of the error as obtained from reflection coefficient data (not shown) is about 0.1% - 10%.

Since the error was reasonably uniform, the behavior of convergence with respect to the parameters θ_i, L_1, and ε_r is clear. In Tables Ia, Ib, we show the range of final determinant size in the computations for each combination of L_1 and $\varepsilon_r^{(n)}$. It is seen that larger NS is required as L_1 increases. From (80), (81), if $(\varepsilon_r - 1)/2 >> \cos^2 \theta_i$, then the determinants behave similarly as a function of the quantity $(\varepsilon_r - 1)L_1^2$. Since L_1 is the length of the transition region in quarter wavelengths of region (1), we may state that the convergence is most rapid when the change in permittivity is small and occurs through a region thin with respect to wavelength. Also, Tables Ia, Ib indicate that convergence is not strongly dependent upon θ_i since most combinations of L_1 and n have a single value for the determinant sizes.

Now, consider an off-diagonal element of S_o or C_o of the form $(g_p + g_q)/(\lambda - 4n^2)$, as in (75), (77), where n is the row of the element. For the linear profile this is

n \ L_1	1	2	3	4
1	4	4	4	4
2	4	4	5	7–9
3	4	4	4–5	6
4	4	4	4–5	6

(a)

n \ L_1	1	2	3	4
1	4	4	4	4
2	4	5	6–9	9
3	4	4	6	9
4	4	5	6	6–7

(b)

TABLE I. Range of final determinant size in the Hill-function computations vs. L_1 and n; a) $\varepsilon_r = 0.5$, b) $\varepsilon_r = 2.0$.

$$\frac{g_p + g_q}{\lambda - 4n^2} = \frac{-2(\varepsilon_r - 1)L_1^2 \left(\frac{1}{p^2} + \frac{1}{q^2}\right)}{\left(\frac{(\varepsilon_r - 1)}{2} + \cos^2 \theta_i\right)L_1^2 - 4n^2} \tag{83}$$

where

$$0 \leq \left(\frac{1}{p^2} + \frac{1}{q^2}\right) \leq \frac{5}{4} \, . \tag{84}$$

If the frequency is such that the magnitude of the L_1^2 term in the denominator is much greater than $4n^2$ then all elements of rows less than n are nearly frequency independent. As L_1 is made smaller the $4n^2$ term will dominate and more elements will go as L_1^2. Those that have been going as L_1^2 will be small compared to the constant elements so that we may truncate the determinant by disregarding those elements for which

$$\left|\left(\frac{(\varepsilon_r - 1)}{2} + \cos^2 \theta_i\right)L_1^2\right| << 4n^2 \quad . \tag{85}$$

With $\varepsilon_r = 2$ we have

$$\frac{1}{2} \leq \frac{(\varepsilon_r - 1)}{2} + \cos^2 \theta_i \leq \frac{3}{2} \tag{86}$$

so that we write (85) as

$$L_1^2 = \frac{4n^2}{36} \quad . \tag{87}$$

This gives the truncated determinant size as

$$NS = 3L_1 \, . \tag{88}$$

This result is valid for S_1 and C_1 also, and could have been obtained by considering their elements. It compares well with the empirical result $NS \sim 2L_1$, obtained by inspection of Table Ib. It is significant that the required determinant size is directly proportional to L_1. This result may be extended to an arbitrary profile by noting that the general Fourier coefficients are proportional to L_1^2, as shown in (73), (74).

The effect upon the reflection coefficient due to truncating the Fourier series for $g(z)$ is shown by Figs. (5)-(7). It is seen that for a given number of terms retained, more accurate values of R are obtained for smaller L_1. One should note, especially from Fig. (5), that keeping only the first term gives very poor results even for small L_1 and that a substantial improvement is effected by adding a second term. This could be expected after comparing the dielectric profile approximations in Fig. (4).

If the infinite determinants are truncated to size NS, then the highest Fourier coefficient needed for their evaluations is g_{2NS}. It follows then that for small L_1 the dielectric profile may be approximated with fewer terms of the Fourier series. However, the converse is not true. That is, the determinants for a profile which can accurately be represented by few terms of the series do not necessarily converge faster than if the profile requires a relatively larger number of terms. This is because keeping a finite number of series terms still gives infinite determinants whose rate of convergence depends upon the size and placement of the non-zero elements. Examples of faster convergence with more series terms may be found in Tables Ia, Ib.

Actually there is no reason to be concerned with truncating the Fourier series since one may always obtain the number of terms required to evaluate the truncated determinants. This is obvious if the profile, or the coefficient $g(z)$, is such that the integrals in (73), (74) can be evaluated directly. If not, then graphical or numerical integration may be used. An alternate approach is the discrete Fourier transform (DFT). The DFT requires n+1 values of the function at equally-spaced points in order to obtain n+1 values approximating the first n+1 Fourier coefficients. These values converge to the actual Fourier coefficients for large n. In our case then, we require the value of $\varepsilon_r(z)$ at the points $z = md/n$, $m = 0,1,2,\ldots,n$, to find the

Fourier coefficients λ through g_n. Using the fast Fourier transform (FFT) algorithm allows one to compute the DFT coefficients very economically for large n.

Although the Hill-function solution for the reflection coefficient is formally exact, the accuracy of results obtained by its implementation is limited by economics. In Table IIa, we give a summary of the cost and time required for computing the Hill-function data used for this paper. It should be noted that we have tabulated the total number of reflection coefficient computations. This is slightly greater than twice the number of points used to construct Figs. (5)-(7) due to the computing procedure described earlier. In Table IIb the cost and execution time for the Airy-function computations is summarized. A comparison shows that for the accuracy obtained with the Hill-function solution and the range of L_1 used, the Hill-function method cost about one-fifth as much per reflection coefficient as did the Airy-function solution. At higher frequencies, or larger L_1, larger determinants will have to be evaluated, and since the work required for determinant evaluation goes up as the cube of the size [16], the cost will rise sharply. This limits the usefulness of the Hill-function method, as applied here, to transition regions with a thickness of no more than a wavelength or so.

A method of reducing the cost of finding R that was not investigated numerically, but would appear significant, is now described. We break up the transition region into n layers as shown in Fig. 8, with the solution for the m^{th} layer written as

$$u^{(m)} = A_m u_1^{(m)} + B_m u_2^{(m)} \; . \tag{89}$$

The application of boundary conditions yields 2(n+1) equations for the unknowns R, T, A_m, B_m, m = 1,2,...,n, and we may solve these for R in terms

NS	3	4	5	6	7	8	9
No. of R's	677	830	318	195	52	113	107
Total No. of R's	2292						
Execution time	6.78 minutes						
Cost	20.34 dollars						

(a)

Total No. of R's	280
Execution time	4.08 minutes
Cost	12.24 dollars

(b)

TABLE II. Computations summary for a) Hill-function solution, b) Airy-function solution.

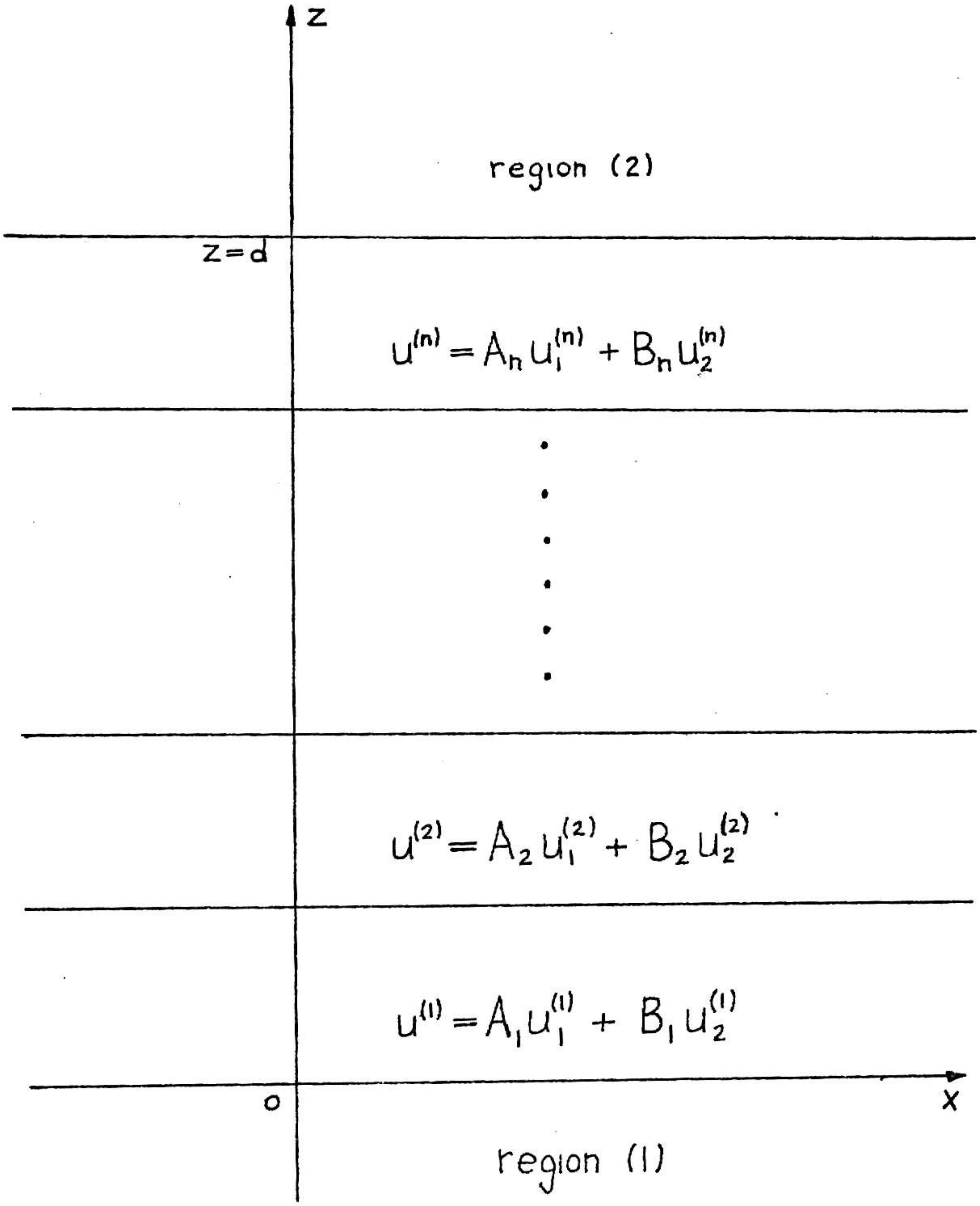

Fig. 8. Geometry of transition region divided into n layers, with the respective Hill function for each layer.

of $u_1^{(m)}(\pi/2)$, $u_2^{(m)}(\pi/2)$, $u_1'^{(m)}(\pi/2)$, $u_2'^{(m)}(\pi/2)$, $m = 1,2,\ldots,n$. We now compare the cost for finding R by using a single transition region of thickness L_1 and by dividing the layer into n regions of thickness L_1/n. From (88), the necessary truncated determinant size is $3L_1$ so that for the single transition region we must compute four determinants of size $3L_1$. For the n layers, we must compute 4n determinants of size $3L_1/n$. Remembering that the work required for determinant computation goes as the size cubed, we find the cost ratio of the two methods is

$$\frac{4(3L_1)^3}{4n\left(\frac{3L_1}{n}\right)^3} = n^2 \; . \tag{90}$$

Thus, by dividing the transition layer into n equal parts, we reduce the cost of evaluating the necessary determinants by a factor of $1/n^2$. We expect a point of diminishing returns, however, since one must solve $2n + 2$ simultaneous equations for R. Also, more work is required in setting up the problem, for example finding the Fourier coefficients for each of the n regions.

The computer subroutines will not be examined in detail, but are presented in the Appendix along with a brief description.

VI. ANALYTICAL CONSIDERATIONS

If L_1 is small, we may approximate the Bessel differential equation, (24), as

$$V'' + \frac{V'}{w} - \frac{V}{9w^2} = 0 \ . \tag{91}$$

Its solution is

$$V = C\left(\frac{w}{2}\right)^{1/3} + D\left(\frac{w}{2}\right)^{-1/3} , \tag{92}$$

which gives

$$\psi = w^{1/3} \left[C\left(\frac{w}{2}\right)^{1/3} + D\left(\frac{w}{2}\right)^{-1/3}\right] . \tag{93}$$

After applying boundary conditions we obtain

$$R = \frac{w_d^{2/3} - w_o^{2/3} + i\ 2/3(w_d^{-1/3} - w_o^{-1/3})}{w_d^{2/3} - w_o^{2/3} + i\ 2/3(w_d^{-1/3} + w_o^{-1/3})} \tag{94}$$

$$= \frac{(\varepsilon_r-1)(\varepsilon_r-\sin^2\theta_i)^{1/2}\cos\theta_i + i\,\frac{2(\varepsilon_r-1)}{L_1\pi}\left[\cos\theta_i - (\varepsilon_r-\sin^2\theta_i)^{1/2}\right]}{(\varepsilon_r-1)(\varepsilon_r-\sin^2\theta_i)^{1/2}\cos\theta_i + i\,\frac{2(\varepsilon_r-1)}{L_1\pi}\left[\cos\theta_i + (\varepsilon_r-\sin^2\theta_i)^{1/2}\right]} \ . \tag{95}$$

If the frequency is low enough the higher power terms may be dropped giving

$$R = \frac{w_d^{-1/3} - w_o^{-1/3}}{w_d^{-1/3} + w_o^{-1/3}} = \frac{\cos\theta_i - (\varepsilon_r - \sin^2\theta_i)^{1/2}}{\cos\theta_i + (\varepsilon_r - \sin^2\theta_i)^{1/2}} , \tag{96}$$

the abrupt boundary reflection coefficient. To see how good the approximation in (94) is we expand the Bessel functions and form the products in (36). Keeping the first two terms of the series expansion gives generally,

$$J_u^d J_v^o = \frac{1}{u!v!}\left(\frac{w_d}{2}\right)^u\left(\frac{w_o}{2}\right)^v - \frac{1}{u!(1+v)!}\left(\frac{w_d}{2}\right)^u\left(\frac{w_o}{2}\right)^{2+v} - \frac{1}{v!(1+u)!}\left(\frac{w_o}{2}\right)^v\left(\frac{w_d}{2}\right)^{2+u} . \tag{97}$$

Substituting this into (36) gives all terms containing the first four powers of L_1. Keeping only those of -1 and 0 power yields

$$R = \frac{-w_d^{-1/3}w_o^{1/3} + w_d^{1/3}w_o^{-1/3} + i\ 2/3\ w_d^{-2/3}w_o^{-1/3} - 1/2\ w_d^{2/3}w_o^{-2/3} - i\ 2/3\ w_d^{-1/3}w_o^{-2/3} + 1/2\ w_d^{-2/3}w_o^{2/3}}{-w_d^{-1/3}w_o^{1/3} + w_d^{1/3}w_o^{-1/3} + i\ 2/3\ w_d^{-2/3}w_o^{-1/3} + 1/2\ w_d^{2/3}w_o^{-2/3} + i\ 2/3\ w_d^{-1/3}w_o^{-2/3} - 1/2\ w_d^{-2/3}w_o^{2/3}} \tag{98}$$

$$= \frac{-(\varepsilon_r-\sin^2\theta_i)^{1/2}\cos^3\theta_i + (\varepsilon_r-\sin^2\theta_i)^{3/2}\cos\theta_i - 1/2(\varepsilon_r-\sin^2\theta_i)^2 + 1/2\cos^4\theta_i + \frac{i2(\varepsilon_r-1)}{L_1\pi}[\cos\theta_i - (\varepsilon_r-\sin^2\theta_i)^{1/2}]}{-(\varepsilon_r-\sin^2\theta_i)^{1/2}\cos^3\theta_i + (\varepsilon_r-\sin^2\theta_i)^{3/2}\cos\theta_i + 1/2(\varepsilon_r-\sin^2\theta_i)^2 - 1/2\cos^4\theta_i + \frac{i2(\varepsilon_r-1)}{L_1\pi}[\cos\theta_i + (\varepsilon_r-\sin^2\theta_i)^{1/2}]} \ . \tag{99}$$

If the 0 power terms are dropped here, we again have the abrupt boundary reflection coefficient, so that (98) may be considered to contain the first-order correction to the abrupt boundary coefficient. Since this expression is somewhat more complex than (94), we may assume that the reflection coefficient obtained by simplifying the differential equation directly is a rather crude approximation.

A first-order approximation identical to (98) may be easily obtained from the Hill-function formulation. Using (61)-(64) we have for the reflection coefficient of (60)

$$R = \frac{\tan\left(\frac{\pi}{2}\sqrt{\lambda}\right)\left(\frac{S_o}{\sqrt{\lambda}}a_o a_d - \sqrt{\lambda}\,C_0\right) + i\,L_1[\cos\theta_i\,S_1 - (\varepsilon_r-\sin^2\theta_i)^{1/2}\,C_1]}{\tan\left(\frac{\pi}{2}\sqrt{\lambda}\right)\left(\frac{S_o}{\sqrt{\lambda}}a_o a_d + \sqrt{\lambda}\,C_0\right) + i\,L_1[\cos\theta_i\,S_1 + (\varepsilon_r-\sin^2\theta_i)^{1/2}\,C_1]} \ . \tag{100}$$

Let $\tan\left(\frac{\pi}{2}\sqrt{\lambda}\right) \simeq \frac{\pi}{2}\sqrt{\lambda}$ and recall that a_o, a_d, $\sqrt{\lambda} \propto L_1$. In order to avoid higher powers of L_1 in R, we set the four infinite determinants equal to one, their limit as $L_1 \to 0$. Under these conditions, (100) may be shown to be identical to the first-order Bessel function approximation in (98).

To get a better approximation with higher powers of L_1, we may expand (75)-(78) and obtain

$$S_0 = 1 - \sum_{n=1}^{\infty} \frac{g_{2n}}{\lambda - 4n^2} + \ldots \tag{101}$$

$$S_1 = 1 - \sum_{n=1}^{\infty} \frac{g_{2n-1}}{\lambda - (2n-1)^2} + \ldots \tag{102}$$

$$C_0 = 1 + \sum_{n=1}^{\infty} \frac{g_{2n}}{\lambda - 4n^2} - \frac{2}{\lambda} \sum_{n=1}^{\infty} \frac{g_n^2}{\lambda - 4n^2} + \ldots \tag{103}$$

$$C_1 = 1 + \sum_{n=1}^{\infty} \frac{g_{2n-1}}{\lambda - (2n-1)^2} + \ldots \tag{104}$$

At low frequencies λ is small so L_1^2 may be factored from the numerator of the series shown explicitly. Terms with higher powers of L_1 may be found by factoring subsequent infinite-series terms in (101)-(104). These further terms are products of g_n's and are complicated due to the many combinations possible.

VII. SUMMARY

The problem of finding the electromagnetic fields outside an inhomogeneous, stratified layer for TE plane-wave incidence has been solved in terms of Hill functions. For a region with a linear dielectric profile we have demonstrated the utility of the Hill-function formulation by direct comparison with the Bessel function solution. For a transition region with thickness on the order of a wavelength or less, the Hill-function method was substantially cheaper for obtaining accurate numerical results for the reflection coefficient. At higher frequencies, convergence of infinite determinants becomes slower and limits the usefulness of the Hill-function solution. However, it was shown that breaking up the transition region into subregions can reduce the cost of finding R at higher frequencies. At low frequencies analytical approximations for R were readily obtained by simply expanding the suitably truncated determinants.

APPENDIX

The subroutines used for numerical evaluation of R by the Hill-function method are now presented. The subroutines HY1, HY1P, HY2, HY2P are used to compute $u_1(\pi/2)$, $u_1'(\pi/2)$, $u_2(\pi/2)$, $u_2(\pi/2)$, $u_2'(\pi/2)$ respectively. These quantities are denoted by the variables Y1, Y1P, Y2, Y2P. ALAM is the Fourier coefficient λ, NS is the determinant size, and G is a vector consisting of the Fourier coefficients $g_1, g_2, \ldots, g_{60}$. The subroutines CDET and CBIG are called by the above subroutines and actually perform the determinant evaluation by the Gauss elimination method.

For evaluating the known-function solution, three subroutines are used. The subroutine AIRY computes Ai(z), Ai'(z), Bi(z), Bi'(z) for the real argument z. In the IO list for this subroutine we have the argument Z for the above functions, respectively, AI, AIP, BI, BIP. The subroutines BESI, BESJ are called by AIRY.

ILLEGIBLE DOCUMENT

THE FOLLOWING DOCUMENT(S) IS OF POOR LEGIBILITY IN THE ORIGINAL

THIS IS THE BEST COPY AVAILABLE

P-474

```
$JOB                              HY1
      SUBROUTINE HY1(ALAM,G,NS,Y1)
C      THIS SUBROUTINE CONSTRUCTS THE ARRAY CMAT
C      CORRESPONDING TO THE DETERMINANT C1.
      COMPLEX CMPLX,CSQRT,CSIN,CCOS
      COMPLEX CSRL,CFACT
      DIMENSION CMAT(31,31),G(60)
      NCHK=0
      PI=3.141593
C      CONSIDER THE N-TH ROW OF THE DETERMINANT.
      DO 1 N=1,NS
      FN=ALAM-(2*N-1)**2
C      ELEMENTS OF THE N-TH ROW HAVE A COMMON POLE.
C      CHECK TO SEE IF THE ELEMENTS ARE NEAR THE POLE.
C      IF SO, MULTIPLY THIS ROW BY THE APPROPRIATE
C      FUNCTION, WHICH WILL HAVE A ZERO HERE.
      IF(ABS(FN).LE.1.0E-04) GO TO 2
C      DEFINE THE ELEMENT IN THE M-TH COLUMN
C      OF THE N-TH ROW.
      DO 5 M=1,NS
      MM=IABS(M-N)
      IF(MM.EQ.0)CMAT(N,M)=1.+G(2*N-1)/FN
      IF(MM.NE.0)CMAT(N,M)=(G(MM)+G(N+M-1))/FN
    5 CONTINUE
      GO TO 1
    2 FACN=PI*COS(PI*N)/(8*N-4)
      NCHK=1
      DO 4 M=1,NS
      MM=IABS(N-M)
      IF(MM.EQ.0)CMAT(N,M)=FACN*G(2*N-1)
      IF(MM.NE.0)CMAT(N,M)=FACN*(G(MM)+G(N+M-1))
    4 CONTINUE
    1 CONTINUE
C      CALL SUBROUTINE CDET TO EVALUATE THE DETERMINANT.
      CALL CDET(Y1,CMAT,NS)
C      IF NO ELEMENTS WERE NEAR A POLE, NOW MULTIPLY THE
C      DETERMINANT BY THE APPROPRIATE FUNCTION TO OBTAIN Y1.
      IF(NCHK.LT.1) GO TO 3
      RETURN
    3 CSRL=CSQRT(CMPLX(ALAM,0.))
      CFACT=CCOS(PI*CSRL/2.)
      Y1=Y1*REAL(CFACT)
      RETURN
      END
$ENTRY
$STOP
/*
```

```
$JOB                              HY1P
      SUBROUTINE HY1P(ALAM,G,NS,Y1P)
C        THIS SUBROUTINE CONSTRUCTS THE ARRAY CMAT
C        CORRESPONDING TO THE DETERMINANT C0.
      COMPLEX CMPLX,CSORT,CSIN,CCOS
      COMPLEX CSRL,CFACT
      DIMENSION CMAT(31,31),G(60)
      NCHK=0
      PI=3.141593
C        CONSIDER THE N-TH ROW OF THE DETERMINANT.
      DO 1 N=1,NS
      FN=ALAM-4.*(N-1)**2
C        ELEMENTS OF THE N-TH ROW HAVE A COMMON POLE.
C        CHECK TO SEE IF THE ELEMENTS ARE NEAR THE POLE.
C        IF SO, MULTIPLY THIS ROW BY THE APPROPRIATE
C        FUNCTION, WHICH WILL HAVE A ZERO HERE.
      IF(ABS(FN).LE.1.0E-04) GO TO 2
C        DEFINE THE ELEMENT IN THE M-TH COLUMN
C        OF THE N-TH ROW.
      DO 5 M=1,NS
      MM=IABS(N-M)
      IF(M.EQ.1.AND.N.EQ.1)CMAT(N,M)=1.
      IF(M.EQ.1.AND.N.NE.1)CMAT(N,M)=1.414214*G(N-1)/FN
      IF(M.NE.1.AND.N.EQ.1)CMAT(N,M)=1.414214*G(M-1)/FN
      IF(M.EQ.M.AND.N.NE.1)CMAT(N,M)=1.+G(2*N-2)/FN
      IF(M.NE.N.AND.N.NE.1.AND.M.NE.1)CMAT(N,M)=(G(MM)+G(N+M-2))/FN
    5 CONTINUE
      GO TO 1
    2 FACN=PI*COS(N*PI)/4.
      NCHK=1
      DO 4 M=1,NS
      MM=IABS(N-M)
      IF(M.EQ.1.AND.N.EQ.1)CMAT(N,M)=0.
      IF(M.EQ.1.AND.N.NE.1)CMAT(N,M)=1.414214*G(N-1)*FACN
      IF(M.NE.1.AND.N.EQ.1)CMAT(N,M)=1.414214*G(M-1)*(-PI/2.)
      IF(M.EQ.N.AND.N.NE.1)CMAT(N,M)=FACN*G(2*N-2)
      IF(M.NE.N.AND.N.NE.1.AND.M.NE.1)CMAT(N,M)=(G(MM)+G(N+M-2))*FACN
    4 CONTINUE
    1 CONTINUE
C        CALL SUBROUTINE CDET TO EVALUATE THE DETERMINANT.
      CALL CDET(Y1P,CMAT,NS)
C        IF NO ELEMENTS WERE NEAR A POLE, NOW MULTIPLY THE
C        DETERMINANT BY THE APPROPRIATE FUNCTION TO OBTAIN Y1P.
      IF(NCHK.LT.1) GO TO 3
      RETURN
    3 CSRL=CSORT(CMPLX(ALAM,0.))
      CFACT=-CSRL*CSIN(PI*CSRL/2.)
      Y1P=Y1P*REAL(CFACT)
      RETURN
      END
$ENTRY
$STOP
/*
```

```
$JOB                              HY2
      SUBROUTINE HY2(ALAM,G,NS,Y2)
C       THIS SUBROUTINE CONSTRUCTS THE ARRAY SMAT
C       CORRESPONDING TO THE DETERMINANT SO.
      COMPLEX CMPLX,CSQRT,CSIN,CCOS
      COMPLEX CSRL,CFACT
      DIMENSION SMAT(31,31),G(60)
      NCHK=0
      PI=3.141593
C       CONSIDER THE N-TH ROW OF THE DETERMINANT.
      DO 1 N=1,NS
      FN=ALAM-4.*N*N
C       ELEMENTS OF THE N-TH ROW HAVE A COMMON POLE.
C       CHECK TO SEE IF THE ELEMENTS ARE NEAR THE POLE.
C       IF SO, MULTIPLY THIS ROW BY THE APPROPRIATE
C       FUNCTION, WHICH WILL HAVE A ZERO HERE.
      IF(ABS(FN).LE.1.0E-04) GO TO 2
C       DEFINE THE ELEMENT IN THE M-TH COLUMN
C       OF THE N-TH ROW.
      DO 5 M=1,NS
      MM=IABS(M-N)
      IF(MM.EQ.0)SMAT(N,M)=1.-G(2*N)/FN
      IF(MM.NE.0)SMAT(N,M)=(G(MM)-G(N+M))/FN
    5 CONTINUE
      GO TO 1
    2 FACN=PI*COS(PI*N)/(16.*N*N)
      NCHK=1
      DO 4 M=1,NS
      MM=IABS(M-N)
      IF(MM.EQ.0.) SMAT(N,M)=-FACN*G(2*N)
      IF(MM.NE.0) SMAT(N,M)=FACN*(G(MM)-G(M+N))
    4 CONTINUE
    1 CONTINUE
C       CALL SUBROUTINE CDET TO EVALUATE THE DETERMINANT.
      CALL CDET(Y2,SMAT,NS)
C       IF NO ELEMENTS WERE NEAR A POLE, NOW MULTIPLY THE
C       DETERMINANT BY THE APPROPRIATE FUNCTION TO OBTAIN Y2.
      IF(NCHK.LT.1) GO TO 3
      RETURN
    3 IF(ABS(ALAM).LE.1.0E-04) GO TO 6
      CSRL=CSQRT(CMPLX(ALAM,0.))
      CFACT=CSIN(PI*CSRL/2.)/CSRL
      Y2=Y2*REAL(CFACT)
      RETURN
    6 Y2=Y2*PI/2.
      RETURN
      END
$ENTRY
$STOP
/*
```

```
$JOB                                HY2P
      SUBROUTINE HY2P(ALAM,G,NS,Y2P)
C      THIS SUBROUTINE CONSTRUCTS THE ARRAY SMAT
C      CORRESPONDING TO THE DETERMINANT S1.
      COMPLEX CMPLX,CSQRT,CSIN,CCOS
      COMPLEX CSRL,CFACT
      DIMENSION SMAT(31,31),G(60)
      NCHK=0
      PI=3.141593
C      CONSIDER THE N-TH ROW OF THE DETERMINANT.
      DO 1 N=1,NS
      FN=ALAM-(2*N-1)**2
C      ELEMENTS OF THE N-TH ROW HAVE A COMMON POLE.
C      CHECK TO SEE IF THE ELEMENTS ARE NEAR THE POLE.
C      IF SO, MULTIPLY THIS ROW BY THE APPROPRIATE
C      FUNCTION, WHICH WILL HAVE A ZERO HERE.
      IF(ABS(FN).LE.1.0E-04) GO TO 2
C      DEFINE THE ELEMENT IN THE M-TH COLUMN
C      OF THE N-TH ROW.
      DO 5 M=1,NS
      MM=IABS(N-M)
      IF(MM.EQ.0) SMAT(N,M)=1.-G(2*N-1)/FN
      IF(MM.NE.0) SMAT(N,M)=(G(MM)-G(N+M-1))/FN
    5 CONTINUE
      GO TO 1
    2 FACN=PI*COS(PI*N)/(8*N-4)
      NCHK=1
      DO 4 M=1,NS
      MM=IABS(N-M)
      IF(MM.EQ.0)SMAT(N,M)=-FACN*G(2*N-1)
      IF(MM.NE.0)SMAT(N,M)=FACN*(G(MM)-G(N+M-1))
    4 CONTINUE
    1 CONTINUE
C      CALL SUBROUTINE CDET TO EVALUATE THE DETERMINANT.
      CALL CDET(Y2P,SMAT,NS)
C      IF NO ELEMENTS WERE NEAR A POLE, NOW MULTIPLY THE
C      DETERMINANT BY THE APPROPRIATE FUNCTION TO OBTAIN Y2P.
      IF(NCHK.LT.1) GO TO 3
      RETURN
    3 CSRL=CSQRT(CMPLX(ALAM,0.))
      CFACT=CCOS(PI*CSRL/2.)
      Y2P=Y2P*REAL(CFACT)
      RETURN
      END
$ENTRY
$STOP
/*
```

```
      SUBROUTINE CDET(OUT,A,N)
      DIMENSION A(31,31)
      OUT=1.
      L=1
    9 CALL CBIG(A,N,IB,JB,L)
      IF(IB.EQ.L.AND.JB.EQ.L) GO TO 10
      IF(IB.EQ.L.OR.JB.EQ.L) GO TO 11
      GO TO 12
   11 OUT=-OUT
   12 DO 1 J=L,N
      B=A(IB,J)
      A(IB,J)=A(L,J)
    1 A(L,J)=B
      DO 2 I=L,N
      B=A(I,JB)
      A(I,JB)=A(I,L)
    2 A(I,L)=B
   10 IF(ABS(A(L,L)))20,21,20
   21 OUT=0.
      RETURN
   20 M=L+1
      DO 5 I=M,N
      DO 5 J=M,N
    5 A(I,J)=A(I,J)-A(L,J)*A(I,L)/A(L,L)
      OUT=OUT*A(L,L)
      IF(N-L-1)8,8,7
    7 L=L+1
      GO TO 9
    8 OUT=OUT*A(N,N)
      RETURN
      END
```

```
      SUBROUTINE CBIG(A,N,IBIG,JBIG,L)
      DIMENSION A(31,31)
      BIG=A(L,L)
      IBIG=L
      JBIG=L
      DO 5 I=L,N
      DO 5 J=L,N
      IF(ABS(A(I,J))-ABS(BIG))5,5,2
    2 BIG=A(I,J)
      IBIG=I
      JBIG=J
    5 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE AIRY(Z,AI,AIP,BI,BIP)
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*8 ARR(100)
      GNU1=.3333333333333333
      GNU2=.6666666666666667
      ZETA=GNU2*DABS(Z)**1.5D0
      SR3=DSQRT(3.D0)
      SRZ=DSQRT(DABS(Z))
      PI=3.141592653589793
      IF(Z)1,3,2
    2 DZETP=DEXP(ZETA)
      CALL BESI(ZETA,GNU1,0,ARR,100)
      AI1=ARR(1)*DZETP
      CALL BESI(ZETA,GNU2,0,ARR,100)
      AI2=ARR(1)*DZETP
      CALL BESK(ZETA,GNU1,0,ARR,100)
      AK1=ARR(1)/DZETP
      CALL BESK(ZETA,GNU2,0,ARR,100)
      AK2=ARR(1)/DZETP
      AI=SRZ*AK1/(SR3*PI)
      AIP=-Z*AK2/(SR3*PI)
      BI=2.D0*SRZ*AI1/SR3+SRZ*AK1/PI
      BIP=2.D0*Z*AI2/SR3+Z*AK2/PI
      RETURN
    1 CALL BESJ(ZETA,GNU1,0,ARR,100)
      AJ1=ARR(1)
      CALL BESJ(ZETA,GNU2,0,ARR,100)
      AJ2=ARR(1)
      CALL BESY(ZETA,GNU1,0,ARR,100)
      AY1=ARR(1)
      CALL BESY(ZETA,GNU2,0,ARR,100)
      AY2=ARR(1)
      AI=(SRZ/2.D0)*(AJ1-AY1/SR3)
      AIP=(-Z/2.D0)*(AJ2+AY2/SR3)
      BI=(-SRZ/2.D0)*(AJ1/SR3+AY1)
      BIP=(-Z/2.D0)*(AJ2/SR3-AY2)
      RETURN
    3 AI=.355028053887817
      BI=AI*SR3
      AIP=-.258819403792807
      BIP=-AIP*SR3
      RETURN
      END
```

```
      SUBROUTINE BESI (X,V,N,A,LDIMA)
      IMPLICIT REAL*8 (A-H,O-Z)
      REAL*8 X,V,A
      DIMENSION A(1),GA(8),GB(7)
      DATA EPPS/1.D-14/
      DATA ACM/Z4000000000000001/
      DATA THRSHX/2.01D0/
      DATA THRSHV/.78D-3/
      DATA SD720P   /Z41978676F65D64F6/
      DATA PI       /Z413243F6A8885A31/
      DATA S2       /Z411A51A6625307D3/
      DATA S3       /Z41133BA004F00621/
      DATA S4       /Z411151322AC7D848/
      DATA S5       /Z411097418ECA7CCE/
      DATA EC       /Z4093C467E37DB0C8/
      DATA TWOD3    /Z40AAAAAAAAAAAAAB/
      DATA TWOD15   /Z402222222222222222/
      DATA GA(1)    /Z408E6F3DAEECC38B/
      DATA GA(2)    /ZC1F7F3D3AE838B52/
      DATA GA(3)    /ZC13A078933933909/
      DATA GA(4)    /ZC1B456348B2EBC07/
      DATA GA(5)    /Z41BB39ECBD8ED5E7/
      DATA GA(6)    /ZC12B1211E1F357A2/
      DATA GA(7)    /Z4022567952E7902C/
      DATA GA(8)    /ZC1642B8A52A6BA7D/
      DATA GB(1)    /Z41B5FBA40A55B259/
      DATA GB(2)    /Z422FFC46988A114F/
      DATA GB(3)    /Z416166FEFAC29673/
      DATA GB(4)    /Z42698F4E7FBEA26F/
      DATA GB(5)    /ZC139F9E38A8B156C/
      DATA GB(6)    /Z4167695A733B1AC4/
      DATA GB(7)    /Z407B50306D234C9C/
      DATA OND3/Z405555555555555555/
      COSH(EXPX)=.5D0*(EXPX+1./EXPX)
      VKL(Y)=DEXP(X*(1.D0-COSH(DEXP(Y))))
      WRONSK(AIVP1,AIV,AKV)=(.5D0*TWODX-AIVP1*AKV)/AIV
      LMB=N
      ASSIGN 450 TO IK1
      KV=1
      IF (V.NE.0) KV=2
C
C     ENTRY FOR BESI CALCULATES (THE I-BESSEL FUNCTION
C            OF X FOR ORDERS V,1+V,2+V,...,N+V)*EXP(-X)
C            R(K)=I(K+1,X)/I(K,X) IS STORED IN A(K+2)
C            TEMPORARILY.  THIS ELIMINATES OVERFLOW PROBLEMS
C
  100 CONTINUE
      IF (X.LE.0.D0.OR.V.LT.0.D0.OR.V.GE.1.D0.OR.N.LT.0)
     X     GO TO 995
      TWODX=2.D0/X
      FK=0.D0
      FKP1=1.D0
      K=LMB
      TEMP=-(DFLOAT(LMB)+V)
      TEMP=TEMP*TWODX
  120 K=K+1
      FKM1=FK
      FK=FKP1
```

```
                TEMP=TEMP-TWODX
                FKP1=TEMP*FK+FKM1
                IF ((FKP1+1.D0).NE.FKP1) GO TO 120
C
C               THE POINT OF RECURSION K IS NOW WELL DETERMINED.
C
                IF ((K+2).GT.LDIMA) GO TO 995
                KP=K+2
                A(KP)=0.D0
                DR=DFLOAT(K )+V
                DR=DR+DR
                DO 150 M=1,K
                KP=KP-1
                A(KP)=X/(DR+X*A(KP+1))
                DR=DR-2.0D0
          150   CONTINUE
C
C               NORMALIZE I'S BY SUMMATION FORMULA
C
                BGAM=1.0D0
                Q2DXPV=1.0D0
                IF (KV.EQ.1) GO TO 200
                Q2DXPV=(TWODX)**V
                TEMP=V+2.0D0
                BGAM=GA(1)+GB(1)/(TEMP+
               2       GA(2)+GB(2)/(TEMP+
               3       GA(3)+GB(3)/(TEMP+
               4       GA(4)+GB(4)/(TEMP+
               5       GA(5)+GB(5)/(TEMP+
               6       GA(6)+GB(6)/(TEMP+
               7       GA(7)+GB(7)/(TEMP+GA(8)))))))
                BGAM=BGAM/(1.0D0+V)
          200   CONTINUE
                A(1)=1.0D0
                TEMP=Q2DXPV*BGAM
                SIG=TEMP
                EN1=V+1.0D0
                TEMP=TEMP*(EN1+EN1)
                EN2=V+V
                D1=1.0D0
                D2=V
                DO 250 M=1,K
                A(M+1)=A(M)*A(M+1)
                SIG=SIG+TEMP*A(M+1)
                EN1=EN1+1.0D0
                EN2=EN2+1.0D0
                D1=D1+1.0D0
                D2=D2+1.0D0
                TEMP=TEMP*(EN1/D1)*(EN2/D2)
          250   CONTINUE
                A(1)=1.D0/SIG
                DO 300 M=1,K
                A(M+1)=A(M+1)*A(1)
          300   CONTINUE
                GO TO IK1,(450,1100)
          450   RETURN
                ENTRY BESK(X,V,N,A,LDIMA)
                LMB=1
```

```
C     ENTRY BESK CALCULATES EXP(X)*THE K-BESSEL
C           FUNCTIONS FOR ORDERS V,V+1,V+2,...,V+N.
      ASSIGN 1100 TO IK1
      VX=V
      KV=1
      IF (V.EQ.0.D0) GO TO 100
      IF (V.LT.THRSHV) GO TO 1020
      TEMP=V-1.D0
      IF ((TEMP+THRSHV).GT.0.D0) GO TO 1030
      KV=3
      GO TO 100
 1020 CONTINUE
      KV=2
      GO TO 100
 1030 CONTINUE
      VX=TEMP
      KV=4
      GO TO 100
 1100 CONTINUE
      IF (X.LT.THRSHX) GO TO 1500
C
C     COMPUTE K(V,X) BY INTEGRATION
C
      H=DLOG((X+38.7)/(X*.5D0-1.D0))
 1110 CONTINUE
      VKM=VKL(H)
      IF (KV.NE.1) VKM=VKM*COSH(DEXP(V*H))
      IF (VKM.GT.ACM) GO TO 1150
      H=.5D0*H
      GO TO 1110
 1150 CONTINUE
      LMB=2
      SIG=.5D0+VKM
      S=SIG*H
 1190 CONTINUE
      HP=.5D0*H
      XP=-HP
      SIG1=0.D0
      DO 1200 M=1,LMB
      XP=XP+H
      VKM=VKL(XP)
      IF (KV.NE.1) VKM=VKM*COSH(DEXP(V*XP))
      SIG1=SIG1+VKM
 1200 CONTINUE
      SIG=SIG+SIG1
      SP=S
      S=SIG*HP
      IF (DABS(SP-S).LE.(EPPS*S)) GO TO 1497
      H=HP
      LMB=LMB+LMB
      GO TO 1190
 1497 CONTINUE
      GO TO 1800
 1500 CONTINUE
      EXPOX=DEXP(X)
C
C     COMPUTE K(VX,X) BY SUMMATION OF I'S
      LMB=3
```

```
0149            IF(KV.EQ.3) GO TO 1525
0150            AA=EC-DLOG(TWODX)
0151       1510 GO TO (1520,1540,1510,1530),KV
0152       1520 CONTINUE
0153            DZ=-AA
0154            GO TO 1550
0155       1525 CONTINUE
0156            DZ=((BGAM*Q2DXPV)**2-PI*V/DSIN(PI*V))*.5/V
0157            GO TO 1550
0158       1530 CONTINUE
          C
          C     FUNCTIONS OF V MUST BE TRANSFORMED TO FUNCTIONS
          C          OF VX=V-1.
          C
0159            BGAM=BGAM/V
0160            Q2DXPV=Q2DXPV/TWODX
0161            LMB=2
0162       1540 CONTINUE
          C
          C     COMPUTE DZ BY SERIES APPROXIMATION
          C
0163            AA2=AA*AA
0164            AA3=AA2*AA
0165            AA4=AA3*AA
0166            AA5=AA4*AA
0167            S=-VX
0168            DZ=(TWOD15*AA5+S5+TWOD3*(AA3*S2+AA2*S3)+.5D0*AA *
               X   (S4+S2**2)+OND3*S2*S3)*S+
               X   .25D0*(S4+S2**2)+TWOD3*AA*S3+AA2*S2+
               X   OND3*AA4-SD72?P
0169            DZ=((DZ*S+(OND3*S3+AA*S2+TWOD3*AA3))*S+
               X   AA2)*S+AA
0170            DZ=-DZ
0171            IF (KV.EQ.2) GO TO 1550
0172            SIG1=A(2)+TWODX*V*A(1)
0173            SIG=DZ*SIG1*EXPOX
0174            GO TO 1560
0175       1550 CONTINUE
0176            SIG=DZ*A(1)*EXPOX
0177            LMB=3
0178       1560 CONTINUE
          C
          C     COMPUTE D1
          C
0179            TEMP=((BGAM*Q2DXPV)**2*(VX+2.0D0)/(1.0D0-VX))*EXPOX
0180            EN1=VX+2.0D0
0181            EN2=VX+VX
0182            EN3=V X
0183            D2=EN3
0184            D3=1.0D0-D2
0185            D1=1.0D0
0186            DO 1570 M=LMB,K,2
0187            SIG=SIG+A(M)*TEMP
0188            EN1=EN1+2.0D0
0189            EN2=EN2+1.D0
0190            EN3=EN3+1.D0
0191            D1=D1+1.D0
0192            D2=D2+2.D0
```

```
                 D3=D3+1.D0
                 TEMP=TEMP*(EN1/D1)*(EN2/D2)*(EN3/D3)
          1570 CONTINUE
      C
      C          SIG CONTAINS K(VX,X)
      C
                 S=EXPDX*SIG
                 IF (KV.NE.4) GO TO 1800
      C
      C          USE WRONSKIAN TO GET K(V,X)*EXP(X)
      C
                 S=WRONSK(A(1),SIG1,S)
          1800 CONTINUE
      C
      C          NOW USE WRONSKIAN TO GET K(V+1,X)*EXP(X)
      C          S CONTAINS K(V,X)*EXPX
                 SP=WRONSK(A(2),A(1),S)
                 A(2)=SP
                 A(1)=S
                 DR=TWODX*V
                 DO 1900 M=2,N
                 DR=DR+TWODX
                 A(M+1)=DR*A(M)+A(M-1)
          1900 CONTINUE
                 RETURN
           995 PRINT 1,X,V,N,K,LDIMA
             1 FORMAT('0ERROR IN BES I/K X=',E14.5,' V=',E14.5,' N=',I5,' K=',I5,
              X' LDIMA=',I5)
                 RETURN
                 END
```

```
0001           SUBROUTINE BESJ(X,V,N,A,LDIMA)
0002           IMPLICIT REAL*8 (A-H,O-Z)
0003           REAL*8 X,A,V
          C
          C    ENTRY BESJ OF SUBROUTINE CALCULATES THE J-BESSEL
          C    FUNCTION OF X FOR ORDERS V,1+V,2+V,....,N+V
          C         R(K) IS STORED IN A(K+2) (TEMPORARILY)
          C         J(V+K) IS STORED INTO A(K+1)
          C         X>0
          C         0<=V<1.D0
          C         N>=0
          C    DIMENSION OF ARRAY A MUST BE AT LEAST MAX(X,N)+16
          C    LDIMA IS THE DIMENSION OF A SUPPLIED BY THE
          C    USER.
          C
          C    ************************************************************
          C              E R R O R   R E T U R N S
          C    FOR X, V, OR N OUT OF RANGE OR IF DIMENSION
          C         OF ARRAY FURNISHED IS TOO SMALL.
          C
          C    ERROR RETURN INFORMATION INCLUDES
          C         X, V, N, MU, AND LDIMA WHERE MU IS THE
          C         SIZE-2 OF THE ARRAY A NEEDED (MU IS
          C         MEANINGLESS IF ANOTHER PARAMETER IS
          C         OUT OF RANGE, E.G.,X=-5.)
          C
0004           DIMENSION A(1)
0005           DIMENSION GA(8),GB(7)
0006           DATA PI/3.14159265358979320D0/
0007           DATA GA/5.5636964340807375D-1,-1.5497028047270494D1,
              X-3.626839829961285 9D0,-1.1271046203309880D1,
              X8.7016417889220691D0,-2.6919115854447927D0,
              X1.3413198732411108D-1,-6.2606299618306871D0/
0008           DATA GB/1.1373935737962844D1,4.7985452207286753D1,
              X6.0876455111805654D0,1.0555979154972231D2,
              X-3.6235080158088192D0,6.4632210255382896D0,
              X4.8169233955859386D-1   /
0009           DATA PI2/1.5707963267948966D0/
0010           EQUIVALENCE (GO,GOLRGV,GOSMLV)
0011           DATA THRSHV/.789D-3/
0012           DATA EC      /Z4093C467E37DB0C8/
0013           DATA S2/1.6449340668482264/,
              1     S2D2SQ/Z40AD2BF5ABCE72C3/,
              2     S5/1.0369277551433700D0/,
              3     S33/.4006856343865314/,
              4     S4/1.0823232337111381 9D0/
0014           DATA PI4D72/Z4115A57EB579CE55/
0015           DATA THSJ/.12D0/
0016           DIMENSION GOVSMC(4)
0017           ASSIGN 510 TO JY1
0018           ASSIGN 110 TO JY4
0019        80 CONTINUE
0020           IF(X.LE.0.OR.V.LT.0.OR.V.GE.1.0.OR.N.LT.0) GO TO 995
          C
          C    SET OVERFLOW INDICATOR OFF
          C
          C 87 CALL OVERFL(MU)
0021           VP1=V+1.D0
```

```
0022                  IX=IFIX(SNGL(X))
0023                  LMB=IX+1
0024                  MU=LMB
0025                  GO TO JY4,(110,120)
0026              110 CONTINUE
0027                  IF(LMB.LT.N) MU=N
0028              120 CONTINUE
              C
              C       LMB IS THE VALUE OF I FOR WHICH WE ARE ASSURED
              C            THAT J(V+I,X) IS ON THE TAIL OF THE FUNCTION,
              C            I.E.,LMB=IFIX(X+1)
              C       LET LL=MAX(LMB,N)...
              C            DEFINE MU TO BE THE POINT FROM WHICH WE
              C            MUST RECUR TO ASSURE THAT J(V+LL,X)
              C            IS ACCURATELY DETERMINED
              C
              C       FOR I=LL,LL+1,....,MU
              C            WE STORE R(I)=J(I+1,X)/J(I,X) INTO A(I+2).
              C            THIS AVOIDS THE PROBLEM OF OVERFLOW
              C            INHERENT IN THE GROWTH OF THE J-FUNCTION
              C            WHEN RECURRING BACKWARD ON ITS TAIL
              C       NOTE THAT R(I-1)=X/(2*(V+I)-X*R(I))
0029                  TWODX=2.0D0/X
0030                  DR=TWODX*(V+DFLOAT(MU))
0031                  FKP1=1.D0
0032                  FK=0.D0
0033              180 CONTINUE
              C
              C       ITERATE UNTIL FKP1 IS GREATER THAN REGISTER ACCURACY
              C
0034                  MU=1+MU
0035                  DR=DR+TWODX
0036                  FKM1=FK
0037                  FK=FKP1
0038                  FKP1=DR*FK-FKM1
0039                  IF ((FKP1+1.D0).NE.FKP1) GO TO 180
              C
              C       THE VALUE OF MU IS NOW WELL DETERMINED
              C
0040                  MU=MU+1
0041                  M=MU
0042                  IF((M+2).GT.LDIMA) GO TO 995
0043                  A(M+2)=0.D0
0044              200 CONTINUE
0045                  IF (M.EQ.LMB) GO TO 250
0046                  DR=2*(M+V)
0047                  M=M-1
0048                  A(M+2)=X/(DR-X*A(M+3))
0049                  GO TO 200
              C
              C        STORE JBAR(I) INTO A(I+1) FOR I=0,1,2,....,LMB+1
              C
0050              250 CONTINUE
0051                  A( M +1)=1.0D0/A( M +2)
0052                  A( M +2)=1.0D0
              C
              C       RECUR BACKWARD TO GET JBAR'S
              C
```

```
0053           280 CONTINUE
0054               DR=2*(M+V)/X
0055               A(M)=DR*A(M+1)-A(M+2)
0056               M=M-1
0057               IF (M.GT.0) GO TO 280
             C
             C     CLEAR R'S   UNDERFLOW MAY OCCUR HERE
             C
0058               LMB=LMB+1
0059               DO 290 M=LMB,MU
0060               A(M+2)=A(M+2)*A(M+1)
             C
             C 287 CALL OVERFL(I)
             C 288 IF (I.EQ.3) A(M+2)=0.D0
0061           290 CONTINUE
             C
             C     NORMALIZE SEQUENCE OF JBARS BY SUMMATION
             C
0062               IF (V.EQ.0.D0) GO TO 305
0063               VX=V+2.0D0
0064               BGAM=GA(1)+GB(1)/(VX+
                  X      GA(2)+GB(2)/(VX+
                  X      GA(3)+GB(3)/(VX+
                  X      GA(4)+GB(4)/(VX+
                  X      GA(5)+GB(5)/(VX+
                  X      GA(6)+GB(6)/(VX+
                  X      GA(7)+GB(7)/(VX+GA(8)))))))
0065               BGAM=BGAM/VP1
0066               BGAMSQ=BGAM*BGAM
0067               Q2DXPV=(2.0D0/X)**V
0068           305 CONTINUE
             C
             C     SUMMATION
0069               ALPHA=A(1)
0070               PHI=2.0D0
0071               IF (V.EQ.0.0D0) GO TO 320
0072               D1=1.0D0
0073               D2=V
0074               EN2=V
0075               EN1=V+2.0D0
0076               PHI=Q2DXPV*BGAM
0077               ALPHA=PHI*ALPHA
0078           320 CONTINUE
0079               DO 350 M=1, MU,2
0080               IF (V.EQ.0.D0) GO TO 330
0081               PHI=PHI*(EN2/D2)*(EN1/D1)
0082               D2=D2+2.0D0
0083               EN1=EN1+2.0D0
0084               D1=1.0D0+D1
0085               EN2=EN2+1.0D0
0086           330 CONTINUE
0087               ALPHA=ALPHA+PHI*A(M+2)
0088           350 CONTINUE
             C     A(1),A(2),....,A(LMB+2) CONTAIN JBAR(0),JBAR(1),....,JBAR(LMB+1)
             C     A(LMB+3),....,A(MU+2) CONTAIN R(LMB+1),....,R(MU)
             C
             C     NORMALIZE JBARS
             C
```

```
              M=MU+2
              DO 460 I=1,M
          457 A(I)=A(I)/ALPHA
          460 CONTINUE
          500 CONTINUE
              GO TO JY1,(510,1100)
          510 CONTINUE
              RETURN
          995 PRINT 1,X,V,N,MU,LDIMA
              RETURN
            1 FORMAT('0ERROR IN BESL X=',E14.5,' V=',E14.5,' N=',I5,' MU=',I5,
             X' LDIMA=',I5)
              RETURN
              ENTRY BESY(X,V,N,A,LDIMA)
              ASSIGN 1100 TO JY1
              ASSIGN 120 TO JY4
              GO TO 80
         1100 CONTINUE
              VX=V
              TJ=DABS(A(1))-THSJ
C
C              COMPUTE G0 AND G1
              IF (V.LT.THRSHV) GO TO 1200
              VX=V-1.0D0
              IF ((VX+THRSHV).GT.0.D0) GO TO 1190
              VX=V
              G0LRGV=DCOTAN(V*PI)-(Q2DXPV**2/PI)*BGAMSQ/V
              GO TO 1220
         1190 CONTINUE
C
C             FUNCTIONS OF V USED IN COMPUTING G0 AND G1
C             MUST BE TRANSFORMED TO FUNCTIONS OF VX=1-V
C
              Q2DXPV=Q2DXPV/TWODX
              BGAMSQ=BGAMSQ/V**2
         1200 CONTINUE
C
C             COMPUTE G0 USING EXPANSION
              Z=EC+DLOG(X/2.0D0)
              G0= Z/PI2
              IF (V.NE.0.0D0) GO TO 1210
              G1=2.0D0/PI2
              BGAMSQ=1.0D0
              Q2DXPV=BGAMSQ
              GO TO 1230
         1210 CONTINUE
              ASQ=Z*Z
              ACUB=Z*ASQ
              AFORTH=Z*ACUB
              AFIFTH=AFORTH*Z/15.D0
              G1=-VX
              G0SMLV=
             X       (((((2.0D0*(AFIFTH+ACUB*S2/3.D0+ASQ*S33+Z*S2D2SQ)
             X         +Z*S4/2.D0+S2*S33+S5)*G1+
             X                2.0D0*Z*S33+S2D2SQ+ASQ*S2
             X        +AFORTH/3.0D0+PI4D72)*G1+
             X               S33+Z*S2+ACUB/1.5D0 )*G1+
             X                1.5D0*S2+ASQ  )*G1+Z)/PI2
```

```
0132      1220 CONTINUE
          C    COMPUTE G1
0133           G1=(Q2DXPV**2/PI2)*BGAMSQ*(2.0D0+VX)/(1.0D0-VX)
0134      1230 CONTINUE
          C    COMPUTE YO FROM SUM(J*S) FORM
          C
0135           EN3=VX+1.0D0
0136           EN2=VX+EN3
0137           EN1=VX+4.0D0
0138           D1=2.0D0
0139           D2=D1-VX
0140           D3=D1+VX
0141           TJE =0
0142            IF(TJ.GE.0.D0.AND.VX.GE.0.D0) GO TO 1232
0143           TJE=1
          C
          C    Y(VX+1,X) MUST ALSO BE COMPUTED BY A SUM
          C
0144           THVDX=3.0D0*VX/X
0145           PSIZ=-BGAMSQ*Q2DXPV**2/(PI2*X)
0146           PSI1=G0-.5D0*G1
0147      1232 CONTINUE
0148           IF (VX.LT.0.D0) GO TO 1233
0149           M=3
0150           YV=G0*A(1)
0151           IF (TJ.GE.0) GO TO 1238
0152           YVP1=PSIZ*A(1)+PSI1*A(2)
0153           GO TO 1238
0154      1233 CONTINUE
0155           Z=TWODX*V*A(1)-A(2)
0156           YV=G0*Z
0157           M=2
0158           YVP1=PSIZ*Z+PSI1*A(1)
0159      1238 CONTINUE
0160           DO 1250 I=M,MU,2
0161           YV=G1*A(I)+YV
0162           G=G1
0163           G1=-G1*(EN1/D1)*(EN2/D2)*(EN3/D3)
0164           EN1=EN1+2.0D0
0165           EN2=EN2+1.0D0
0166           EN3=EN3+1.0D0
0167           D1=D1+1.0D0
0168           D2=1.0D0+D2
0169           D3=D3+2.0D0
0170           IF (TJE) 1240,1250,1240
0171      1240 CONTINUE
0172           YVP1=YVP1+THVDX*G*A(I)+.5*(G-G1)*A(I+1)
0173      1250 CONTINUE
0174           IF (VX.GE.0.D0) GO TO 1260
0175           Z=YVP1
0176           YVP1=V*Z*TWODX-YV
0177           YV=Z
0178           GO TO 1400
0179      1260 CONTINUE
0180           IF (TJ.LT.0.D0) GO TO 1400
0181      1270 CONTINUE
          C
          C    NOW COMPUTE Y(V+1)
```

```
                C      WRONSKIAN PROVIDED NOT NEAR A ZERO OF J
                C
0182                   YVP1=(YV*A(2)-1.0D0/(X*PI2))/A(1)
0183             1400  CONTINUE
                C
                C      RECUR FORWARD TO GET Y'S (WISE?)
                C
0184                   A(1)=YV
0185                   A(2)=YVP1
0186                   G=V*TWODX
0187                   DO 1500 I=2,N
                C
0188                   G=G+TWODX
                C      OVERFLOW MAY OCCUR HERE
0189                   A(I+1)=G*A(I)-A(I-1)
                C1487  CALL OVERFL(LMB)
                C1488  IF (LMB.EQ.1) A(I+1)=1.D78
0190             1500  CONTINUE
0191                   RETURN
0192                   END
```

REFERENCES

[1] J. R. Wait, Electromagnetic Waves in Stratified Media. Oxford, England: Pergamon, 1970, Chs. 3, 4.

[2] G. Tyras, Radiation and Propagation of Electromagnetic Waves. New York: Academic Press, 1969, Ch. 4.

[3] R. Burman and R. N. Gould, J. Atmos. Terr. Phys., Vol. 25, pp. 543-549, 1963.

[4] ___________, J. Atmos. Terr. Phys., Vol. 26, pp. 1127-1129, 1964.

[5] R. N. Gould and R. Burman, J. Atmos. Terr. Phys., Vol. 26, pp. 335-340, 1964.

[6] J. Heading, "Refractive index profiles based on the hypergeometric equation and the confluent hypergeometric equation," Proc. Camb. Phil. Soc., Vol. 61, pp. 897-913, 1965.

[7] ___________, "Investigations into a new stratified hyperbolic profile," Proc. Camb. Phil. Soc., Vol. 63, pp. 439-450, 1967.

[8] B. S. Westcott, "Exact solutions for vertically polarized electromagnetic waves in horizontally stratified isotropic media," Proc. Camb. Phil. Soc., Vol. 66, pp. 675-684, 1969.

[9] ___________, "Oblique reflexion of electromagnetic waves from isotropic media with limiting Epstein-type stratifications," Proc. Camb. Phil. Soc., Vol. 68, pp. 765-771, 1970.

[10] L. M. Brekhovskikh, Waves in Layered Media. New York: Academic Press, 1960.

[11] K. F. Casey, "Application of Hill's Functions to Problems of Propagation in Stratified Media," IEEE Trans. Ant. Prop., Vol. AP-20, pp. 368-374, May 1972.

[12] M. Abramowitz and I. A. Segun, Handbook of Mathematical Functions. New York: Dover, 1965, pp. 446-447.

[13] N. W. McLachlan, Theory and Application of Mathieu Functions. New York: Dover, 1964, Ch. 6.

[14] E. T. Whittaker and G. N. Watson, A Course of Modern Analysis. New York: Cambridge Univ. Press, 1963, pp. 406, 413-417.

[15] W. Magnus and S. Winkler, Hill's Equation. New York: Interscience, 1966.

[16] L. Fox, An Introduction to Numerical Linear Algebra. New York: Oxford University Press, 1965, Ch. 7.

A STUDY OF THE HILL-FUNCTION SOLUTION
TO PROBLEMS OF PROPAGATION IN STRATIFIED MEDIA

by

JAMES L. DIETRICH

B.S., Kansas State University, 1970

AN ABSTRACT OF A MASTER'S THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Electrical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1974

ABSTRACT

The problem of determining the reflection and transmission coefficients for a plane wave incident upon an inhomogeneous, stratified region of finite thickness is solved in terms of Hill functions. The effect that parameters of the problem have upon cost and accuracy of the solution is discussed from both analytical considerations and numerical results obtained from solving the problem for a region with a linear dielectric profile. Expressions for the reflection coefficient at low frequencies are obtained using the Hill-function formulation, as well as the familiar Airy-function formulation.