STUDY OF HEAT TRANSFER IN CIRCULAR FINS
WITH VARIABLE THERMAL PARAMETERS

by

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LIST OF FIGURES

NOMENCLATURE

a	:	Starting value of the independent variable in the process.
A_1, A_2	:	Constants in the general solution.
A_b	:	Cross-sectional area at the base of the fin.
b	:	dimensionless length parameter.
B_r	:	conductivity related parameter.
B_i	:	Biot number
c	:	generalized boundary condition at the starting point of the process.
d	:	geometric constant of the fin.
f(R)	:	function describing the variation of the heat transfer coefficient.
f,g	:	functionals of the derivatives of the independent variables θ_1 and θ_2.
g_1, g_2	:	functionals of the derivatives of the independent variables X_1 and X_2.
g(b)	:	function of the parameter b.
h	:	heat transfer coefficient.
h_a	:	average value of the heat transfer coefficient.
h_{nf}	:	heat transfer coefficient at the wall in the absence of the fin.
H	:	functional of the radius, describing the heat transfer coefficient variation.
I	:	Integral function.
$\bar{J}(\bar{y})$	:	Jacobian matrix
k	:	thermal conductivity.

k_a : reference thermal conductivity.

K : function of b and m.

m : index of the heat transfer coefficient variation.

N : dimensionless fin parameter.

N_r : heat removal number.

q : heat transfer from the fin.

q_r : heat flow due to conduction.

q_{nf} : heat transfer in the absence of the fin.

Q : dimensionless heat transfer from the fin.

Q_{max} : maximum possible heat transfer from the fin.

Qh : optimized value of heat transfer from the fin.

r : radial distance.

r_b : inner (base) radius of the fin.

r_o : outer (tip) radius of the fin.

r_m : missing initial condition.

R : dimensionless radius.

t : temperature.

t_∞ : temperature of the surroundings.

T : temperature scaled with respect to t_∞.

T_b : temperature at the base of the fin.

T_{nf} : temperature at the wall surface in the absence of a fin.

U : dimensionless volume of the fin.

v : dimensionless parameter related to the base thickness of the fin.

V : Volume of the fin.

w : semi-thickness at the base of the fin.

w_o : semi-thickness at the tip of the fin.

W_r : semi-thickness of a fin of rectangular profile.

X_1, X_2 : two dependent variables.

$\bar{y}=(y_1, y_2)$: known point about which the linearization is done.

$y(r)$: y co-ordinate (thickness) of the fin.

α : thermal conductivity variation parameter.

Δ : interval size for the duration of the process.

δ : defines the grid values of c.

λ : slope parameter.

η : efficiency of fin.

Θ : dimensionless temperature

Θ_1, Θ_2 : two dependent variables.

ξ : dimensionless independent variable.

β : dimensionless thickness of the fin.

Supercripts

n-1 : previous iteration number

n : current iteration number

Subscripts

b : at the base of the fin.

c : base case. (λ = 0.5, m = 0.0, α = 0.0)

h : homogeneous solution.

p : particular solution.

o : case when either m = 0.0 or α = 0.0

op : optimum values

nf : in the absence of the fin

THIS BOOK CONTAINS NUMEROUS PAGES WITH THE ORIGINAL PRINTING BEING SKEWED DIFFERENTLY FROM THE TOP OF THE PAGE TO THE BOTTOM.

THIS IS AS RECEIVED FROM THE CUSTOMER.

PART I Conduction in Circular Fins With Variable Thermal Parameters

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Chapter I

INTRODUCTION

Fins are extended surfaces, used to reduce the thermal resistance at a surface, and thereby increase the heat transfer rate from the surface to the adjacent fluid. Fins attached to a surface, in effect, increase the total heat transfer area. The use of fins is very common in various types of heat exchange devices.

Familiar examples of the usage of fins are the circumferential or circular fins around the cylinder of a motorcycle engine and pin fins attached to the condenser tubes at the back of a domestic refrigerator. Fins have also been used in space vehicles to dissipate heat energy by radiation in space, where convection is absent. Finned surfaces are used to augment heat transfer in heat exchangers. Some other typical applications are found in vehicles, their power sources, chemical, refrigeration, cryogenic processes, electric and electronic circuitry, conventional furnaces, process heat dissipators and waste-heat boilers, and nuclear fuel modules.

Fins can be of rectangular cross-sections, like ribs attached along the length of a tube, called longitudinal fins, or concentric annular discs around a tube called circumferential or circular fins. The thickness of the fins may be uniform or variable. In this study circular fins with variable thickness are considered.

Since a fin surface protudes from the primary heat transfer surface, the temperature difference with the surrounding fluid will steadily diminish as one moves outward along the length of the fin. The design of a fin therefore requires a knowledge of the temperature distribution along the fin. The objective of the study in the Part I is to determine the temperature distribution in circular fins and to estimate the rate of heat transfer and the efficiency for circular fins of constant thickness.

The differential equation for the temperature distribution in linear fin problems, that is assuming a constant thermal conductivity and constant heat transfer coefficient has been solved in several texts on heat transfer [1-3]. However, it is seen that the thermal conductivity of pure metals tends to decrease with increasing temperature, while the thermal conductivity of alloys and insulators tends to increase with increasing temperature. In general, it does not remain a constant. Besides, another severe restriction that has been imposed in previous work, is that the heat transfer coefficient is assumed constant. It has been proved both experimentally and theoretically, that in general, the heat transfer coefficient does not remain a constant along the length of fins, [4-9].

In this study, the thermal conductivity is considered to be linearly varying with temperature, and the heat transfer coefficient to be varying along the length of the fin. Different types of variations of the heat transfer coefficient have been considered and their effects on the heat transfer rate and efficiency have been studied.

With these variations in the problem, the differential equations are nonlinear. The difficulties encountered with nonlinear boundary value problems have been explained in a later chapter. Aziz and Na [10], have studied the nonlinear fin problem with one of the boundary conditions being a periodic function of time. Other researchers who have made similar studies are Yang [11], and Eslinger and Chung, [12]. They considered the effects of radiation too. Lieblin [13] has analyzed the temperature distribution and radiant heat transfer in rectangular fins of constant thickness.

In Part I of this study, the nonlinear fin problem has been solved by using the quasilinearization technique. Quasilinearization is a method of linear approximation allied with an iterative approach. This method has been explained in full detail in Chapter 2. The fin problem has been formulated in Chapter 3, and the solution and results are presented in Chapters 4 and 5 respectively.

Chapter 2

METHOD OF QUASILINERIZATION

2.1 Introduction

The quasilinearization technique is essentially a generalized Newton-Raphson method for functional equations. It not only linearizes the nonlinear equation, but also provides a sequence of functions, which converges quadratically to the true solution of the original nonlinear equation.

In many aspects, the quasilinearization technique is similar to the generalized Newton-Raphson method. However the unknown coefficients are functionals and not the fixed roots as in the case of the Newton-Raphson method; hence both the theoretical and computational aspects are much more complicated.

Two important properties of the quasilinearization technique are quadratic convergence and monotonic convergence. Owing to the use of the Newton-Raphson type of linearization formula, the convergence is quadratic, if there is convergence at all [14]. These properties of the Newton-Raphson and quasilinearization methods will be discussed later in this chapter.

2.2 Nonlinear Boundary-Value Problems

Most of this study will be concerned with the computational solution of a nonlinear boundary value problem. The difficulties connected with this problem may be briefly summarized here. For illustrative purposes, let us consider a nonlinear second order differential equation; with one initial and one final condition known.

Since one of the initial conditions is unknown, a step by step numerical integration technique cannot be used here. This problem is known as a boundary value problem. It is much more difficult to handle both theoretically and computationally as compared to initial value problems. Theoretically, there is no general proof of the existence and uniqueness of the solutions to problems of this type. Computationally, there exists no general effective approach to obtain the numerical solutions. Most nonlinear differential equations cannot be solved analytically. Quasilinearization presents an excellent method to overcome this difficulty.

2.3 Convergence Properties of the Quasilinearization Technique

The convergence properties of the quasilinearization technique can be understood from a knowledge of the Newton-Raphson technique, and the special features which make it a powerful tool. A rigorous mathematical derivation for the convergence properties is given in [15].

Consider the single algebraic equation

$$f(u) = 0 \qquad (2\text{-}1)$$

Let us assume that f(u) is a convex function. Further, let us assume that $f'(u) < 0$. We wish to obtain r_e, the exact root of this equation. From an initial approximation U_i to the root r_e, a better approximation to r_e can be obtained by solving the following equation for u:

$$f(U_i) + (U-U_o)\, f'(U_o) = 0 \qquad (2\text{-}2)$$

Calling this approximation U_1, an even better approximation U_2, can be obtained similarly. Continuing this process, a general recurrence relationship can be set up as follows:

$$f(U_{n+1}) = f(U_n) + (U_{n+1}-U_n)\, f'(U_n) = 0 \qquad (2\text{-}3)$$

where U_n is always known, and U_{n+1} is the unknown. Note that equation (2-3) is a linear equation in the unknown U_{n+1}. Solving for U_{n+1}, equation (2-3) becomes

$$U_{n+1} = U_n - \frac{f(U_n)}{f'(U_n)} \qquad (2\text{-}4)$$

Equation (2.4) is in the Newton-Raphson form. It is clear, that (2.4) does not hold good if $f'(U_n) = 0$. Since f(u) is a convex function, by assigning successive integer values to n, we observe that

$$U_i < U_1 < U_2 < \ldots < U_n < \ldots \qquad (2.5)$$

Hence, the values of the sequence $\{U_n\}$ increase monotonically to the exact root. This property is known as monotonic convergence. This property makes quasilinearization very suitable for the computation process, because it provides an upper or lower bound for the convergent interval, and ensures automatic improvement at each iteration.

For computational purposes, the rate of convergence is another important property. For any particular iteration n, the following expression can be written by use of the mean value theorem:

$$f(r_e) - f(U_n) = (r_e - U_n)\, f'(U_n) + \frac{(r_e - U_n)^2}{2}\, f''(r_v) \qquad (2\text{-}6)$$

where r_v lies between r_e and U_n. Recalling that $f(r_e) = 0$, as r_e is the exact root, from (2.4) and (2.6) we obtain the following expression

$$r_e - U_{n+1} = -\frac{1}{2}(r_e - U_n)^2 f''(r_v) \qquad (2\text{-}7)$$

From (2-7) it is seen that $(U_{n+1} - r_e)$ is proportional to $(U_n - r_e)^2$. That is, the error in the (n+1)st iteration is proportional to the square of the error in the nth iteration.

Numerically, the quadratic convergence means that after a large number of iterations, the number of correct digits for the root r_e is approximately doubled at each iteration. It follows that as U_n approaches r_e, there is an enormous acceleration in the convergence rate.

2.4 Quasilinearization

The quasilinearization technique was developed by Bellman [16], and Kalaba [17], and applied extensively to chemical engineering problems by Lee [18] in obtaining numerical solutions of certain classes of nonlinear ordinary differential equations of the boundary value type. The governing nonlinear differential equation is first represented by a set of simultaneous first order differential equations. Each of the first order nonlinear differential equations is linearized using the Taylor series expansion, with second and higher order terms omitted. Iterative solution of the resulting linear differential equations usually converges quadratically to the solution of the original equation.

Consider the most general case of an nth order nonlinear ordinary differential equation, with n given boundary conditions. This nth order equation can be converted into n simultaneous first order nonlinear ordinary differential equations.

$$\frac{dx_i}{dt} = g_i\ (x_1,\ x_2, \ldots\ x_n, t) \tag{2.8}$$

$$i = 1,2,\ \ldots\ n. \quad 0 \le t \le t_1$$

Of the n given boundary conditions, some may be final and the remaining initial conditions. Let there be m final conditions

$$x_j\ (t_1) = x_{jt_1} \qquad j = 1,2,\ \ldots\ m \tag{2.9}$$

and (n-m) initial conditions

$$x_k\ (0) = x_{ko} \qquad k = m + 1,\ m + 2,\ \ldots\ n \tag{2.10}$$

If $g_i\ (x_1,\ x_2,\ \ldots\ x_n,\ t)$ is nonlinear, and all of the boundary conditions are of the initial type, the Runge-Kutta method can be directly employed. The nonlinear equation (2.8) is first linearized around the point $\overline{y} = (y_1,\ y_2,\ \ldots y_n)$. In vector notation this linearization can be written as

$$\frac{d\overline{x}}{dt} = \overline{g}(\overline{y},t) + \overline{J}(y)\ (\overline{x}-\overline{y}) \tag{2.11}$$

where $\overline{x}$ and $\overline{g}$ represent the vectors $(x_1,\ x_2,\ \ldots\ x_n)$ and $(g_1,\ g_2.\ \ldots\ g_n)$ respectively, and the vector $\overline{y}$ is known, [19]. The Jacobian matrix is defined as

$$\bar{J}(\bar{y}) = \begin{bmatrix} \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} & \cdots & \frac{\partial g_1}{\partial y_n} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_2} & \cdots & \frac{\partial g_2}{\partial y_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial g_n}{\partial y_1} & \frac{\partial g_n}{\partial y_2} & \cdots & \frac{\partial g_n}{\partial y_n} \end{bmatrix} \qquad (2\text{-}12)$$

in which $\partial g_i/\partial y_j$ represents partial differentiation. Equation (2.11) can be explicitly written in the matrix form as follows:

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \cdot \\ \cdot \\ \cdot \\ \frac{dx_n}{dt} \end{bmatrix} = \begin{bmatrix} g_1(y_1,y_2,\ldots y_n,t) \\ g_2(y_1,y_2,\ldots y_n,t) \\ \\ \\ \\ g_n(y_1,y_2,\ldots y_n,t) \end{bmatrix} + \begin{bmatrix} (x_1-y_1)\frac{\partial g_1}{\partial y_1}+(x_2-y_2)\frac{\partial g_1}{\partial y_2}+\ldots(x_n-y_n)\frac{\partial g_1}{\partial y_n} \\ (x_1-y_1)\frac{\partial g_2}{\partial y_1}+(x_2-y_2)\frac{\partial g_2}{\partial y_2}+\ldots(x_n-y_n)\frac{\partial g_2}{\partial y_n} \\ \\ \\ \\ (x_1-y_1)\frac{\partial g_n}{\partial y_1}+(x_2-y_2)\frac{\partial g_n}{\partial y_2}+\ldots(x_n-y_n)\frac{\partial g_n}{\partial y_n} \end{bmatrix}$$

(2.13)

From an initial approximation for $\bar{y} = \bar{x}_o$, the first approximate solution $\bar{x}_1$ can be obtained by solving (2.13), using a step by step integration method such as the Runge-Kutta method. Using $\bar{x}_1$ as $\bar{y}$ in (2.13) an improved solution, say $\bar{x}_2$ can be obtained. This procedure is continued until the desired accuracy (convergence) is achieved. The recurrence relation for the nth order system of linearized differential equations can be written as follows:

$$\frac{dx_{i,N_{rec}}}{dt} = g_i\,(x_{1,N_{rec}-1},\ x_{2,N_{rec}-1},\ \ldots\ x_{n,N_{rec}-1},t)$$

$$+ (x_{1,N_{rec}} - x_{1,N_{rec}-1})\frac{\partial g_i}{\partial x_{1,N_{rec}-1}}$$

$$+ \ . \ . \ . \tag{2.14}$$

$$+ (x_{n,N_{rec}} - x_{n,N_{rec}-1})\frac{\partial g_i}{\partial x_{n,N_{rec}-1}}$$

$$i = 1,2,\ \ldots\ n \qquad N_{rec} = 0,1,2,3\ \ldots$$

Here the first subscript i denotes the subscript of the dependent variable x_1, x_2, ... x_n and the second subscript N_{rec} denotes the N_{rec} th iteration. In (2-14), $x_{i,N_{rec}}$ is an unknown function and $x_{i,N_{rec}-1}$ a known function obtained from the previous iteration.

The solution to the system represented by (2-9), (2-10) and (2-14) can be expressed in the following form, [19]

$$x_i(t) = x_{ip}(t) + \sum_{i=1}^{n} A_L\, x_{i,Lh}(t) \quad i = 1,2,\ldots n \tag{2.15}$$

In general, we need to assume a set of n initial conditions $x_{ip}(o) = x_{ipo}$, for the particular solution, and n sets of n initial conditions $x_{i,Lh}(o) = x_{i,Lho}$ for the n sets of homogeneous equations, reduced from (2-14). The integration constants A_L are determined from the n boundary conditions, which are given and the assumed and computed boundary conditions for the particular and homogeneous solutions.

If the initial conditions both for the particular and homogeneous solutions are chosen appropriately, the number of homogeneous solutions required can be reduced, thus reducing the total computation time. It has been proved mathematically by Bellman [16], and can be seen intuitively that the general solution (2-15) can be reduced to

$$x_i(t) = x_{ip}(t) + \sum_{L=1}^{m} A_L \; x_{i,Lh} \qquad (2\text{-}16)$$

$$i = 1,2, \ldots m,\ m + 1,\ m + 2, \ldots n$$

if the given initial conditions of the original differential equations are chosen as the initial conditions of the particular solution, that is

$$x_{kp}(o) = x_k(o) = x_{ko} \qquad k = m + 1,\ m + 2, \ \ldots\ n \quad (2\text{-}17)$$

and (n-m) of the initial conditions of the homogeneous solutions satisfy the following condition,

$$\sum_{L=1}^{m} A_L \; x_{k,Lh}(0) = 0 \qquad k = m + 1,\ m + 2, \ \ldots\ n \quad (2\text{-}18)$$

$$L = 1,\ 2, \ \ldots\ m$$

This condition is imposed to make the left hand side of (2-16) compatible with the right hand side at the initial condition $i = m + 1,\ m + 2, \ \ldots\ n$ as shown below

$$x_i(o) = x_{ip}(o) + \sum_{L=1}^{m} A_L \; x_{i,Lh}(o) \qquad (2.19)$$

$$i = m + 1,\ m + 2,\ \ldots\ n$$

It can be seen that with such an appropriate selection of the initial conditions, only m sets of the n homogeneous solutions $x_{i,Lh}(t)$ $L = 1,2, \ldots m$, $i = 1,2, \ldots n$ are required. The m integration constants A_L, $L = 1,2, \ldots m$ are determined from the m given final conditions, (2.9). Obviously, the number of equations to be integrated is reduced from $(1 + n)$ sets to $(1 + \frac{n}{2})$ sets or less, since for the case $m > n/2$, the problem can be treated in the reverse direction. It should be noted that the initial conditions for the remaining particular solutions $x_{ip}(o)$ $i = 1,2, \ldots m$ and homogeneous solutions $x_{i,Lh}(o)$ $i = 1, 2, \ldots m$ $L = 1,2, \ldots m$ can be chosen arbitrarily, provided each row and column vector of the following matrix is not identically zero (19).

$$\begin{bmatrix} x_{1,1h}(o) & x_{1,2h}(o) & \cdots & x_{1,mh}(o) \\ x_{2,1h}(o) & x_{2,2h}(o) & \cdots & x_{2,mh}(o) \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ x_{n,1h}(o) & x_{n,2h}(o) & \cdots & x_{n,mh}(o) \end{bmatrix} \qquad (2.20)$$

The system of the first order linear ordinary differential equations (2-14) can be solved for the first set of approximate solutions, that is when $N_{rec} = 1$, by a step by step integration process, starting with the assumed and given initial conditions.

The initial approximate function $\bar{x}_o$ can be obtained in a variety of ways. The approximation must be chosen reasonably and by exercising engineering judgement. For a number of problems, a very rough initial approximation is enough for convergence. It should be noted that both the particular and homogeneous solutions thus obtained can be added to yield the general solution (2-16), due to the superposition (additive) property of a linear system.

Chapter 3

FORMULATION OF THE PROBLEM

3.1 Introduction

As stated earlier in this report, this part of the study deals with the calculation of the temperature profiles among the length of the fin, and the evaluation of the heat transfer rate from the fin and the efficiency of the fin.

The quasilinearization technique, described extensively in Chapter 2, is used to overcome the difficulties encountered by a *nonlinear* boundary value problem.

The derivation of the nonlinear differential equation governing the behavior of heat conduction in fins is presented in this chapter, along with the expressions for the heat transfer rate and the efficiency of the fin. The assumptions made in this derivation are:

(i) heat conduction is one-dimensional,

(ii) effects of radiation are neglected,

(iii) curvature of the fin profile is not considered,

(iv) steady state conditions prevail,

(v) the thermal conductivity and heat transfer coefficient are not constants

(vi) the fin thickness is uniform along the length of the fin

(vii) the tip of the fin is assumed to be insulated.

3.2 Derivation of the Fin Equation

Consider the fin configuration as shown in Figure 3-1. Writing the heat balance on the differential element, under steady state conditions, we have

$$\frac{-dq_r}{dr} \cdot dr - 2\cdot(2\pi r\ dr)h(t-t_\infty) = 0 \qquad (3\text{-}1)$$

where t_∞ is the temperature of the surroundings, and q_r is the heat flow due to conduction along the length of the fin.

$$q_r = -k\cdot(2\pi r)\ (2y(r))\frac{dt}{dr} \qquad (3\text{-}2)$$

Substituting (3-2) into (3-1), after simplification, we obtain

$$\frac{d}{dr}[k\cdot r\cdot y(r)\frac{dt}{dr}] - hr(t-t_\infty) = 0 \qquad (3\text{-}3)$$

For a circular fin with a rectangular profile of constant thickness W_r,

$$y(r) = W_r \qquad (3\text{-}4)$$

Let,

$$T = t-t_\infty \qquad \frac{dT}{dr} = \frac{dt}{dr} \qquad (3\text{-}5)$$

The thermal conductivity k is not a constant, but is a function of the temperature. The dependence of the thermal conductivity on the temperature is assumed as a linear function and is expressed as

$$k(T) = k_a(1 + \alpha\frac{T}{T_b}) \qquad (3\text{-}6)$$

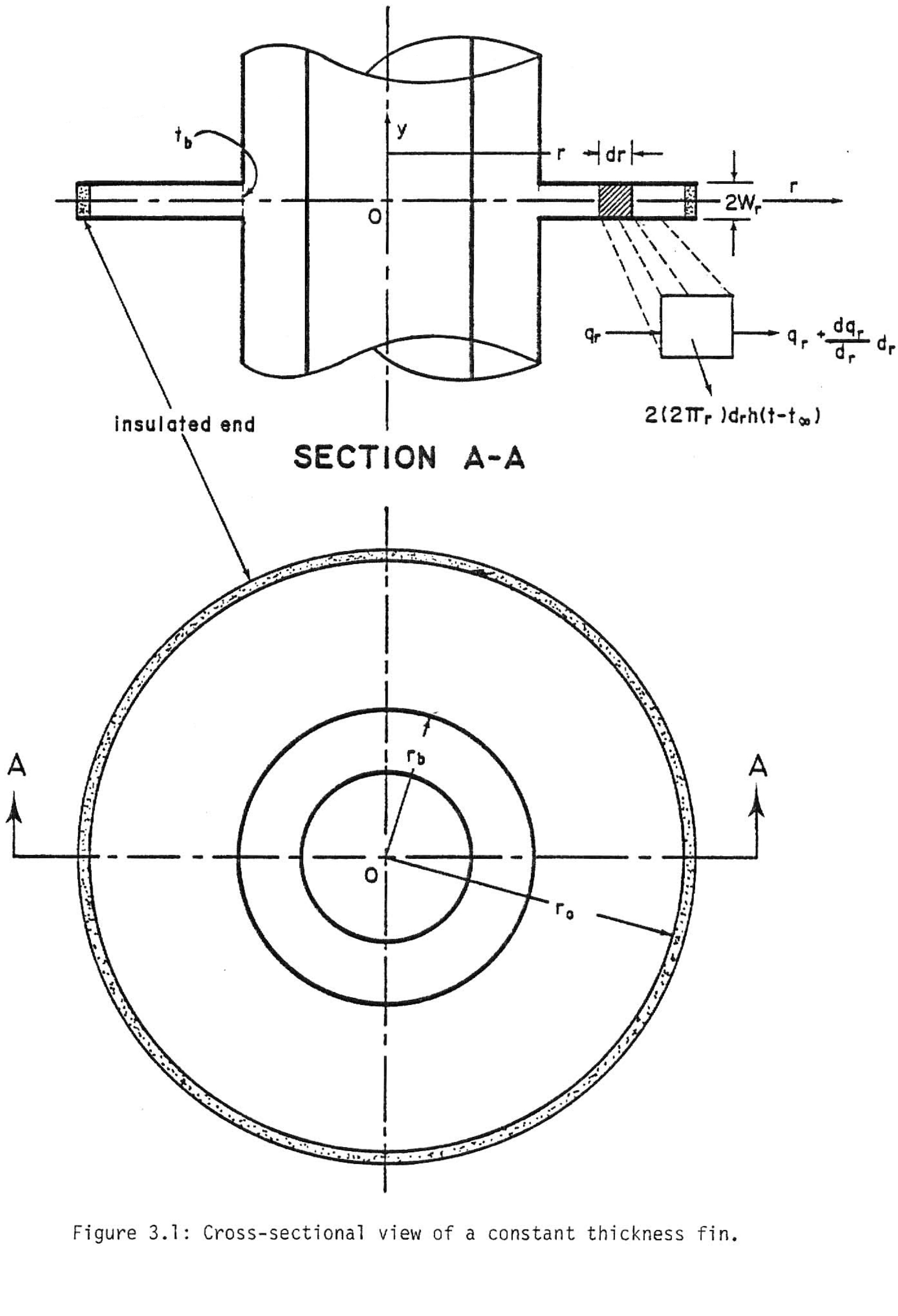

Figure 3.1: Cross-sectional view of a constant thickness fin.

where T_b is the temperature at the base of the fin, and k_a is the reference thermal conductivity. The dimensionless temperature Θ is defined as

$$\Theta = \frac{T}{T_b} \tag{3-7}$$

Therefore

$$k(T) = k_a(1 + \alpha\Theta) \tag{3-8}$$

The heat transfer coefficient is a function of the radius r, that is h = h(r). Substitution of equation (3-4), (3-5), (3-7) and (3-8) into (3-3) yields

$$\frac{1}{r}\frac{d}{dr}\left[k_a(1+\alpha\Theta)r\ T_b\frac{d\Theta}{dr}\right] - \frac{2h(r)T_b\Theta}{W_r} = 0$$

Rearranging the terms, we have

$$\frac{1}{r}\frac{d}{dr}\left[(1+\alpha\Theta)r\frac{d\Theta}{dr}\right] - \frac{2h(r)\Theta}{k_a W_r} = 0 \tag{3-9}$$

The dimensionless radius is defined as

$$R = \frac{r-r_b}{r_o-r_b} \tag{3.10a}$$

$$r = (r_o-r_b)R + r_b \qquad dr = (r_o-r_b)dR \tag{3.10b}$$

The variation of the heat transfer coefficient along the length of the fin can be expressed as a function of the dimensionless radius as follows

$$h = h(r) = h_a f(R) \tag{3.11}$$

h_a is the average value of the heat transfer coefficient. Subsitution of (3-10) and (3-11) into (3-9) yields the following equation.

$$\frac{1}{((r_o-r_b)R+r_b)}\frac{d}{dR}[(1+\alpha\Theta)\frac{((r_o-r_b)R+r_b)}{(r_o-r_b)}\frac{d\Theta}{dR}]-\frac{2h_a f(R)\Theta}{k_a W_r} = 0 \tag{3-12}$$

The fin parameter N is defined as

$$N^2 = \frac{2h_a(r_o-r_b)^2}{k_a W_r} \tag{3-13}$$

Rearranging the terms in (3-12), and substituting (3-13) into (3-12), we obtain the following dimensionless equation:

$$\frac{1}{((r_o-r_b)R+r_b)}\frac{d}{dR}[(1+\alpha\Theta)\frac{((r_o-r_b)R+r_b)}{(r_o-r_b)}\frac{d\Theta}{dR}]-N^2 f(R)\Theta = 0 \tag{3-14}$$

Rewriting equation (3-14) in the explicit form, we have

$$(1+\alpha\Theta)\frac{d^2\Theta}{dR^2} + \alpha(\frac{d\Theta}{dR})^2 + (1+\alpha\Theta)\frac{d\Theta}{dR}\frac{(r_o-r_b)}{((r_o-r_b)R+r_b)} - N^2 f(R)\Theta = 0 \tag{3-15}$$

which is a second order nonlinear ordinary differential equation. For a given fin, the dimensions r_o and r_b are given. Hence, let let

$$d = \frac{r_o}{(r_o-r_b)} \tag{3-16}$$

be a known constant for the fin. With the substitution of d, the governing equation (3-15) can be expressed as

$$(1+\alpha\Theta)\frac{d^2\Theta}{dR^2} + \alpha(\frac{d\Theta}{dR})^2 + (\frac{1+\alpha\Theta}{R+d})\frac{d\Theta}{dR} - N^2 f(R)\Theta = 0 \tag{3-17}$$

The fin tip is assumed to be insulated, and the temperature at the base of the fin is assumed to be constant, T_b. The boundary conditions can now be expressed as follows:

$$\text{At } r = r_b, \quad R = 0 \; ; \; \theta = 1 \tag{3-18a}$$

$$\text{At } r = r_o, \quad R = 1 \; ; \; \frac{d\theta}{dR} = 0 \tag{3-18b}$$

Equations (3-17) and (3-18) form a nonlinear boundary value heat conduction problem.

3.3 Heat Dissipation from the Fin

The heat being transferred out from the fin in the steady state condition, is given by the expression

$$q = -k(T)\, A_b \frac{dT}{dr}\bigg|_{r=r_b,\; R = 0} \tag{3-19}$$

where A_b is the area of the fin at the base. Substituting (3-7), (3-8) and (3-10) into (3-19), we obtain the following expression for the heat transfer rate:

$$-q = k_a(1+\alpha\theta)\frac{T_b A_b}{(r_o - r_b)} \frac{d\theta}{dR}\bigg|_{R = 0} \tag{3-20}$$

A_b is given by the expression

$$A_b = 2\pi r_b W_r \tag{3-21}$$

The dimensionless heat transfer rate can now be defined as

$$Q = \frac{q(r_o - r_b)}{2\pi r_b W_r T_b k_a} = \frac{q}{k_a \cdot 2\pi d \cdot W_r T_b} \tag{3-22}$$

substituting (3-21) and (3-22) into (3-19), we obtain the dimensionless steady state heat transfer rate to be

$$-Q = (1 + \alpha\theta) \left. \frac{d\theta}{dR} \right|_{R = 0} \qquad (3\text{-}23)$$

3.4 Efficiency of the Fin

The fin efficiency is defined as the ratio of the actual heat transfer to the maximum possible heat transfer from the fin. The maximum possible heat transfer will occur, if the whole fin is at the same temperature, as the temperature at the base of the fin.

$$Q_{max} = \int_{r_b}^{r_o} 2\pi r \cdot dr \; h(r) \cdot 2T_b \qquad (3\text{-}24)$$

Converting this equation into the dimensionless form in terms of R, we have

$$Q_{max} = 4\pi T_b h_a \, (r_o - r_b)^2 \int_o^1 (R+d) \; f(R) \; dr$$

$$= 2\pi T_b k_a W_r N^2 \; d \int_o^1 (\frac{R}{d}+1) \; f(R) \; dr \qquad (3\text{-}25)$$

From equation (3-22), the actual heat transfer rate is given by the expression

$$q = Q \; k_a \cdot 2\pi d \cdot W_r \; T_b$$

Therefore, the efficiency η is given by the expression

$$\eta = \frac{q}{Q_{max}} = \frac{Q}{N^2 \int_o^1 (\frac{R}{d} + 1) f(R) \; dR}$$

Let

$$I = \int_{o}^{1} (\frac{R}{d} + 1) f(R)\, dR \qquad (3\text{-}26)$$

then the efficiency

$$\eta = \frac{Q}{N^2 I} \qquad (3\text{-}27)$$

Chapter 4

IMPLEMENTATION OF THE QUASILINEARIZATION TECHNIQUE

The equations that formulate the nonlinear fin problem are (3-17) and (3-18). Quasilinearization is used to calculate the temperature profile and the slope of the temperature profile along the length of the fin. Equation (3-17) can be written as

$$(1 + \alpha\Theta)\frac{d^2\Theta}{dR^2} + \alpha\left(\frac{d\Theta}{dR}\right)^2 + S(1 + \alpha\Theta)\frac{d\Theta}{dR}\frac{1}{(R+d)} - N^2 f(R)\Theta = 0 \quad (4\text{-}1)$$

where the coefficient of the third term

$S = 1$, for circular fins, and

$S = 0$, for straight fins.

This is because the equation governing the behaviour of straight fins is similar to the equation for circular fins. Hence this formulation, makes the analysis suitable for longitudinal fins also.

First of all, the second order non-linear ordinary differential equation (4-1) is converted into two first order equations as follows:

Let

$$\Theta = \Theta_1 \quad (4\text{-}2a)$$

$$\frac{d\Theta}{dR} = \frac{d\Theta_1}{dR} = \Theta_2 \quad (4\text{-}2b)$$

$$\frac{d^2\Theta}{dR^2} = \frac{d^2\Theta_1}{dR^2} = \frac{d\Theta_2}{dR} \quad (4\text{-}2c)$$

Therefore the two first order equations are

$$\frac{d\Theta_1}{dR} = \Theta_2 \qquad (4\text{-}3a)$$

$$(1+ \alpha\Theta_1)\frac{d\Theta_2}{dR} + \alpha\Theta_2^2 + S\frac{(1+ \alpha\Theta_1)}{(R+d)}\Theta_2 - N^2 f(R)\Theta_1 = 0 \qquad (4\text{-}3b)$$

The boundary conditions (3-18) would now change to the form

$$\Theta_1(0) = 1 \quad \text{at } R = 0 \qquad (4\text{-}4a)$$

$$\Theta_2(1) = 0 \quad \text{at } R = 1 \qquad (4\text{-}4b)$$

Obviously, Θ_1 represents the value of the temperature and Θ_2 the slope of the temperature profile.

Next, the two first order ordinary nonlinear differential equations need to be linearized. For this purpose, let us write the equations (4-3) into a more convenient form, as follows:

$$\frac{d\Theta_1}{dR} = \Theta_2 = g_1(R,\Theta_1,\Theta_2) \qquad (4\text{-}5a)$$

$$\frac{d\Theta_2}{dR} = \frac{N^2 f(R)\Theta_1 - \alpha\Theta_2^2}{(1+\alpha\Theta_1)} - \frac{S\Theta_2}{(R+d)} = g_2(R,\Theta_1,\Theta_2) \qquad (4\text{-}5b)$$

$$\frac{\partial g_1}{\partial\Theta_1} = 0 \qquad (4\text{-}6a)$$

$$\frac{\partial g_1}{\partial\Theta_2} = 1 \qquad (4\text{-}6b)$$

$$\frac{\partial g_2}{\partial \theta_1} = \frac{(1+\alpha\theta_1)N^2 f(R) - (N^2 f(R)\theta_1 - \alpha\theta_2^2)\alpha}{(1+\alpha\theta_1)^2}$$

$$= \frac{N^2 f(R) + \alpha^2\theta_2^2}{(1+\alpha\theta_1)^2} = \frac{N^2 f(R) + \alpha^2 y_2^2}{(1+\alpha y_1)^2} \qquad (4.7a)$$

$$\frac{\partial g_2}{\partial \theta_2} = \frac{-2\alpha\theta_2}{(1+\alpha\theta_1)} - \frac{S}{R+d} = \frac{-2\alpha y_2}{(1+\alpha y_1)} - \frac{S}{R+d} \qquad (4\text{-}7b)$$

where $\bar{y} = (y_1, y_2)$ is the point about which the governing differential equations are linearized. The Jacobian matrix, in this case, is

$$\bar{J}(\bar{y}) = \begin{bmatrix} 0 & 1 \\ \dfrac{N^2 f(R) + \alpha^2 y_2^2}{(1+\alpha y_1)^2} & \dfrac{-2\alpha y_2}{(1+\alpha y_1)} - \dfrac{S}{R+d} \end{bmatrix} \qquad (4\text{-}8)$$

In general, $\bar{y} = (y_1, y_2)$ is obtained from the earlier iteration and hence the following notation is used:

$$y_1 = \theta_1^{\,n-1} \qquad (4\text{-}9a)$$

$$y_2 = \theta_2^{\,n-1} \qquad (4\text{-}9b)$$

where superscript (n-1) denotes the previous iteration. The current iteration number is n. It follows from equation (2-11) and (2-14) that the linearized equation can be written in the form of a recurrence relationship

$$\frac{d\Theta_1^n}{dR} = \Theta_2^{n-1} + 0\cdot(\Theta_1^n - \Theta_1^{n-1}) + 1\cdot(\Theta_2^n - \Theta_2^{n-1})$$

$$= \Theta_2^n \tag{4-10a}$$

$$\frac{d\Theta_2^n}{dR} = \frac{N^2 f(R)\Theta_1^{n-1} - \alpha(\Theta_2^{n-1})^2}{(1+\alpha\Theta_1^{n-1})} - \frac{S\cdot\Theta_2^{n-1}}{R+d}$$

$$+ (\Theta_1^n - \Theta_1^{n-1})\left[\frac{N^2 f(R) + \alpha^2(\Theta_2^{n-1})^2}{(1+\alpha\Theta_1^{n-1})^2}\right]$$

$$- (\Theta_2^n - \Theta_2^{n-1})\left[\frac{2\alpha\Theta_2^{n-1}}{1+\alpha\Theta_1^{n-1}} + \frac{S}{R+d}\right]$$

$$= \frac{N^2 f(R)\Theta_1^{n-1} - \alpha(\Theta_2^{n-1})^2}{(1+\alpha\Theta_1^{n-1})}$$

$$+ (\Theta_1^n - \Theta_1^{n-1})\left[\frac{N^2 f(R) + \alpha^2(\Theta_2^{n-1})^2}{(1+\alpha\Theta_1^{n-1})^2}\right]$$

$$- (\Theta_2^n - \Theta_2^{n-1})\left[\frac{2\alpha\Theta_2^{n-1}}{1+\alpha\Theta_1^{n-1}}\right]$$

$$- \frac{S\Theta_2^n}{R+d} \tag{4-10b}$$

The given boundary conditions are now written as

$$\Theta_1^n\ (0) = 1 \tag{4-11a}$$

$$\Theta_2^n\ (1) = 0 \tag{4-11b}$$

Note that here, the value of S is 1, because this study deals with the behaviour of circular fins.

The general solution to equations (4-10) and (4-11) is of the form

$$\theta_1^n = \theta_{1p}^n + A_1 * \theta_{1,1h}^n + A_2 * \theta_{1,2h}^n \qquad (4\text{-}12a)$$

$$\theta_2^n = \theta_{2p}^n + A_1 * \theta_{2,1h}^n + A_2 * \theta_{2,2h}^n \qquad (4\text{-}12b)$$

But, as stated before in equation (2-16), the solution can be reduced to the form

$$\theta_1^n = \theta_{1p}^n + A_1 * \theta_{1,1h}^n \qquad (4\text{-}13a)$$

$$\theta_2^n = \theta_{2p}^n + A_1 * \theta_{2,1h}^n \qquad (4\text{-}13b)$$

if the initial conditions for the particular and homogeneous solutions are properly chosen. Following equations (2-17), the given initial condition of the original differential equation $\theta_1^n\,(0) = 1$, is chosen as the initial condition for the particular solution, that is,

$$\theta_{1p}^n(0) = \theta_1^n(0) = 1 \qquad (4.14)$$

Also, the initial conditions of the homogeneous solutions must satisfy $\sum_{L=1}^{m} A_L x_{k,Lh}(0) = 0$ from equation (2-18).

Therefore, $$\theta_{1,1h}^n(0) = 0 \qquad (4\text{-}15)$$

The initial condition, for the remaining particular solution, $\theta_{2p}^n(0)$, and for the remaining homogeneous solution

$\theta^n_{2,1h}(0)$ can be arbitrarily chosen, provided each row and column vectors of the following matrix

$$\begin{bmatrix} \theta_{1,1h}(0) \\ \theta_{2,1h}(0) \end{bmatrix}$$

are not zero. The following values were arbitrairly chosen during the computation,

$$\theta^n_{2p}(0) = 0 \tag{4-16}$$

$$\theta^n_{2,1h}(0) = 1 \tag{4-17}$$

The constant A_1 is evaluted from the one given final condition,

$$\theta^n_2\ (1) = 0,$$

which has not been made use of, so far.

From equation (4-13b) at the final point,

$$\theta^n_2(1) = \theta^n_{2p}\ (1) + A_1 * \theta^n_{2,1h}(1)$$

Therefore A_1 is given by the expression

$$A_1 = \frac{\theta^n_2(1) - \theta^n_{2p}(1)}{\theta^n_{2,1h}(1)} = \frac{-\theta^n_{2p}(1)}{\theta^n_{2,1h}(1)} \tag{4-18}$$

Thus from equations (4-10) and the given and assumed boundary conditions the iterative process can be started. However, we still do need an initial approximation for the functions θ^0_1 and θ^0_2 when n = 0, to start the iteration process.

One way to obtain an initial approximation would be to solve the same fin problem, with no non-linearities involved. The solution obtained to this linear problem is well known and can be expressed in the form of exponential functions or hyperbolic functions. But, it is seen that the initial approximation need only be a very rough one. The rate of convergence is not affected at all if we choose the initial approximations to be

$$\theta_1^n(R) = 1 \qquad 0 \le R \le 1 \tag{4-19a}$$

$$\theta_2^n(R) = 0 \qquad 0 \le R \le 1 \tag{4-19b}$$

Hence, these initial approximation functions were used in the computation.

The computer program, developed for this quaslinearization technique, is listed in Appendix I. The Runge-Kutta-Gill method of step-by-step numerical integration has been used in the program. A flow chart of the program has also been prepared, as a documentation of the procedure used.

Chapter 5

RESULTS AND DISCUSSION

5.1 Introduction

The quasilinearization computer program was run several times to study the effects of four important parameters. These parameters are:

(i) the thermal conductivity variation parameter α

(ii) the fin parameter N,

(iii) a geometric constant of the fin d,

(iv) the type of variation of the heat transfer coefficient along the length of the fin, f(R).

The dimensionless length of the fin has been divided into 100 intervals, with 0.01 as the step size. The temperature function was calculated at each of these points, and directly plotted as a curve on the temperature versus radial distance along the fin graph. The slope of the temperature profile was also evaluated at these points. The slope of the temperature at the base of the fin is used to calculate the heat transfer rate by using equation (3-23). The efficiency of the fin was evaluated then, by using equation (3-27).

The thermal conductivity variation parameter α was varied between -0.2 to +0.2. The values of the fin parameter N were varied from 0.25 to 3.00, and the values of the geometric parameter d were between 0.5 to 4.0. The types of variation of the heat transfer coefficient considered were

$$f(R) = 1.0, \quad \text{i.e. h remains a constant,}$$

$$f(R) = R, \quad \text{i.e. linear variation,}$$

$$f(R) = R^2, \quad \text{i.e. parabolic variation,}$$

$$f(R) = e^r, \quad \text{i.e. exponential variation}$$

A comparison can be made between these four different types of variations, because

$$e^R = 1 + R + \frac{R^2}{2} + \ . \ . \ . \ , \quad 0 \leq R \leq 1 \qquad (5\text{-}1)$$

5.2 Temperature Profile Along the Fin

Temperature profiles along the fin length have been plotted for 12 different cases, as shown in Figures 5-1 through 5-12. All the curves have the similar profile as an exponentially decaying curve. From the curves in Figures 5-2, 5-5, 5-8 and 5-11, as the value of the fin parameter N increases, the temperature profile shifts downwards. In other words the values of the temperature, obtained along the length of the fin, are lower. This is due to the fact that N is a ratio of the average heat transfer coefficient and the thermal conductivity at the base. Thus as N increases, the amount of heat leaving the fin, especially near the base, increases. Hence the temperature drops more with an increased value of N.

From Figures 5-3, 5-6, 5-9, and 5-12 it can be seen that as the value of d increases, the temperature profile shifts upwards, i.e., the values of the temperature along the dimensionless length of the fin increases. By definition

$d = r_b/(r_o - r_b)$; therefore, as d increases, the length of the fin, $(r_o - r_b)$ decreases. Hence the length available for the heat dissipation to the atmosphere is less. As a result, the fin has a higher temperature. A change in the value of d does not affect the temperature profile significantly, as can be seen from the cases when d is 4.0, 3.0 and 2.0.

Figures 5-1, 5-4, 5-7 and 5-10 show that as the thermal conductivity variation parameter α increases, the temperature profile shifts upwards. This is because the heat transfer to the surrounding medium at the base of the fin plays an important role in the total heat transfer process. Now, as α increases, the heat conduction at the base increases. As a result the heat conducted along the length of the fin increases, relative to the heat being convected, at the base. Hence the temperatures along the length of the fin increase with a higher α .

The type of heat transfer coefficient variation also varies the temperature profile considerably. The variation has been considered to be a function of the dimensionless radius R, that is f(R). It should be noted that R is a fraction, that is, $0 \leq R \leq 1$. The exponential variation e^R, dissipates the most heat. Hence the temperatures are the lowest for the exponential case. The parabolic variation R^2 dissipates the least amount of heat. As a result the temperature values are the highest for this case. The temperature profiles for the f(R) = 1.0 case and the f(R) = R case lie inbetween the profiles for the $f(R) = e^R$ and the $f(R) = R^2$ cases, with the temperatures of the f(R) = R case above the temperature for the f(R) = 1.0 case.

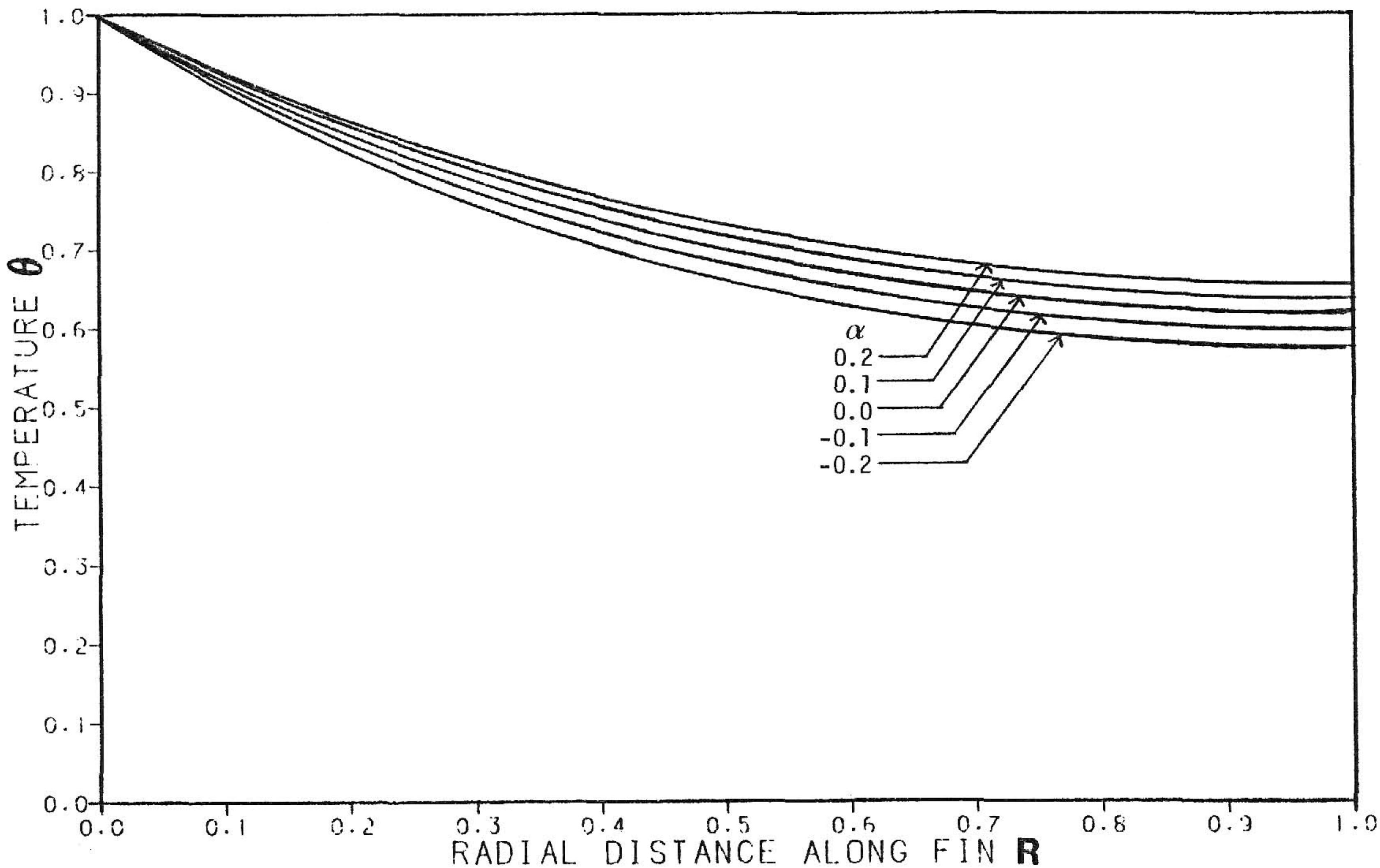

Figure 5.1: Temperature versus Radial Distance Along the Fin. f(R)=1.0, N=1.0, d=2.0

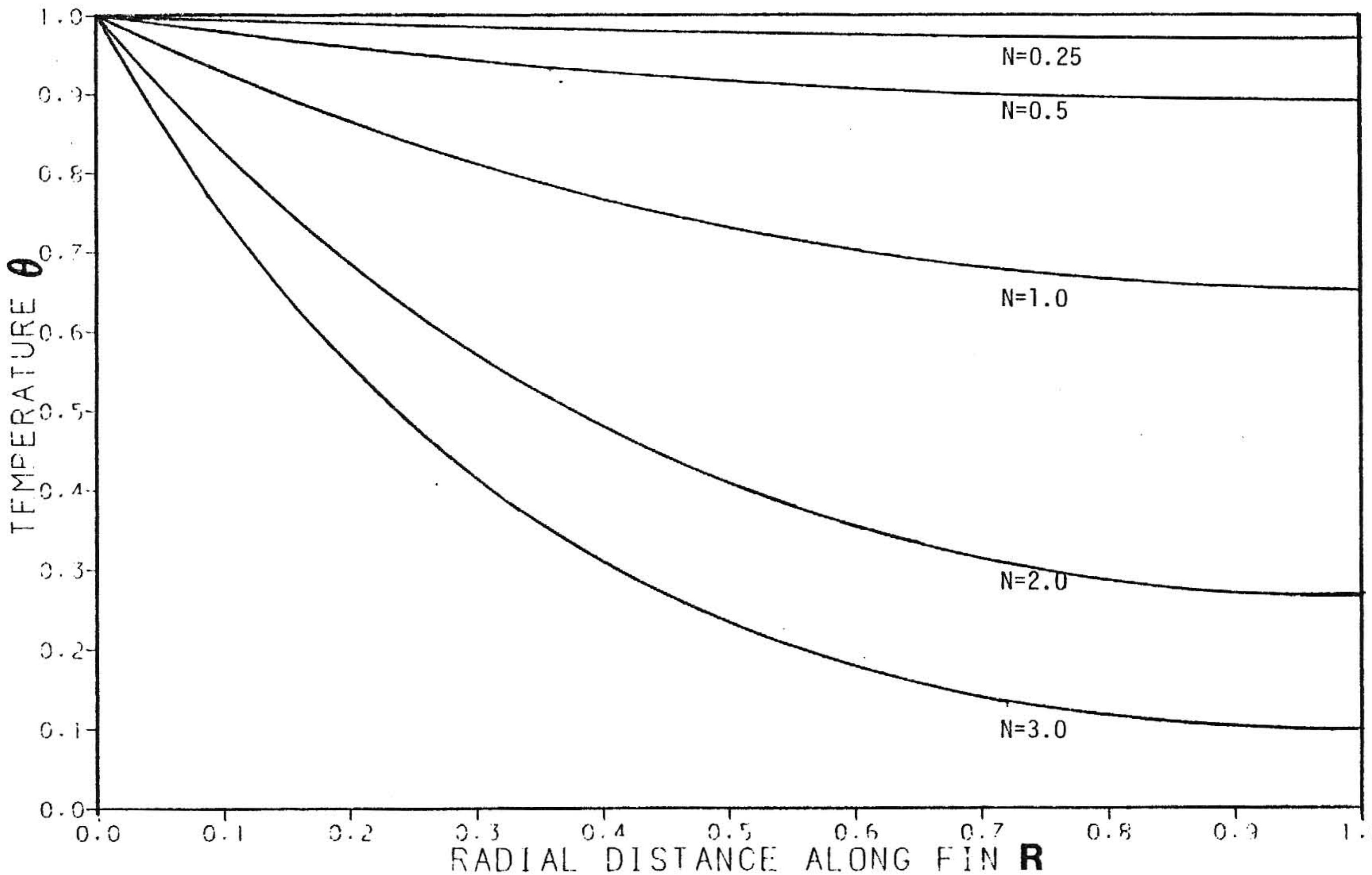

Figure 5.2: Temperature versus Radial Distance Along the Fin. $f(R)=1.0$, $d=2.0$, $\alpha=0.2$

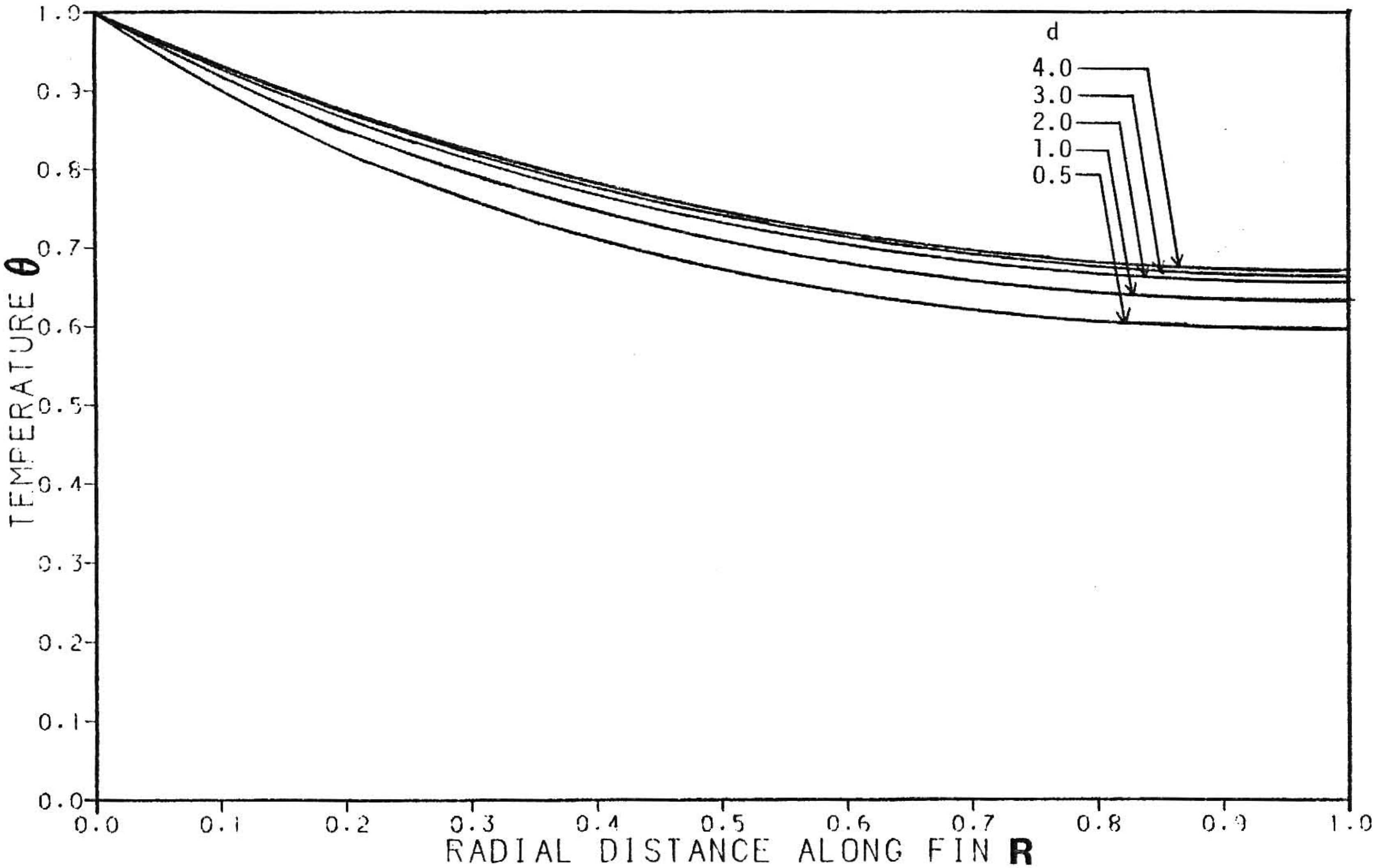

Figure 5.3: Temperature versus Radial Distance Along the Fin. $f(R)=1.0$, $N=1.0$, $\alpha=0.2$

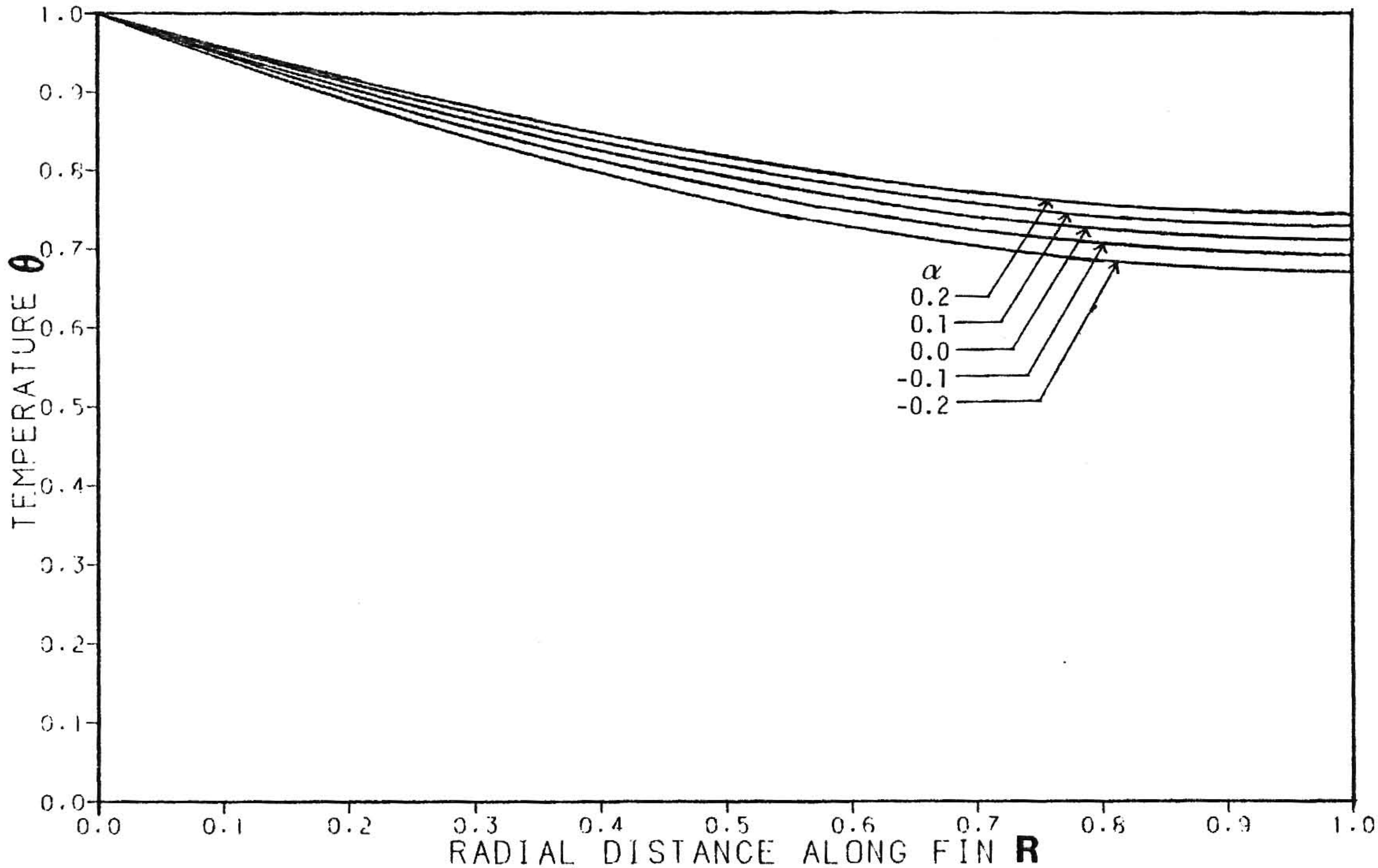

Figure 5.4: Temperature versus Radial Distance Along the Fin. f(R)=R, N=1.0, d=2.0

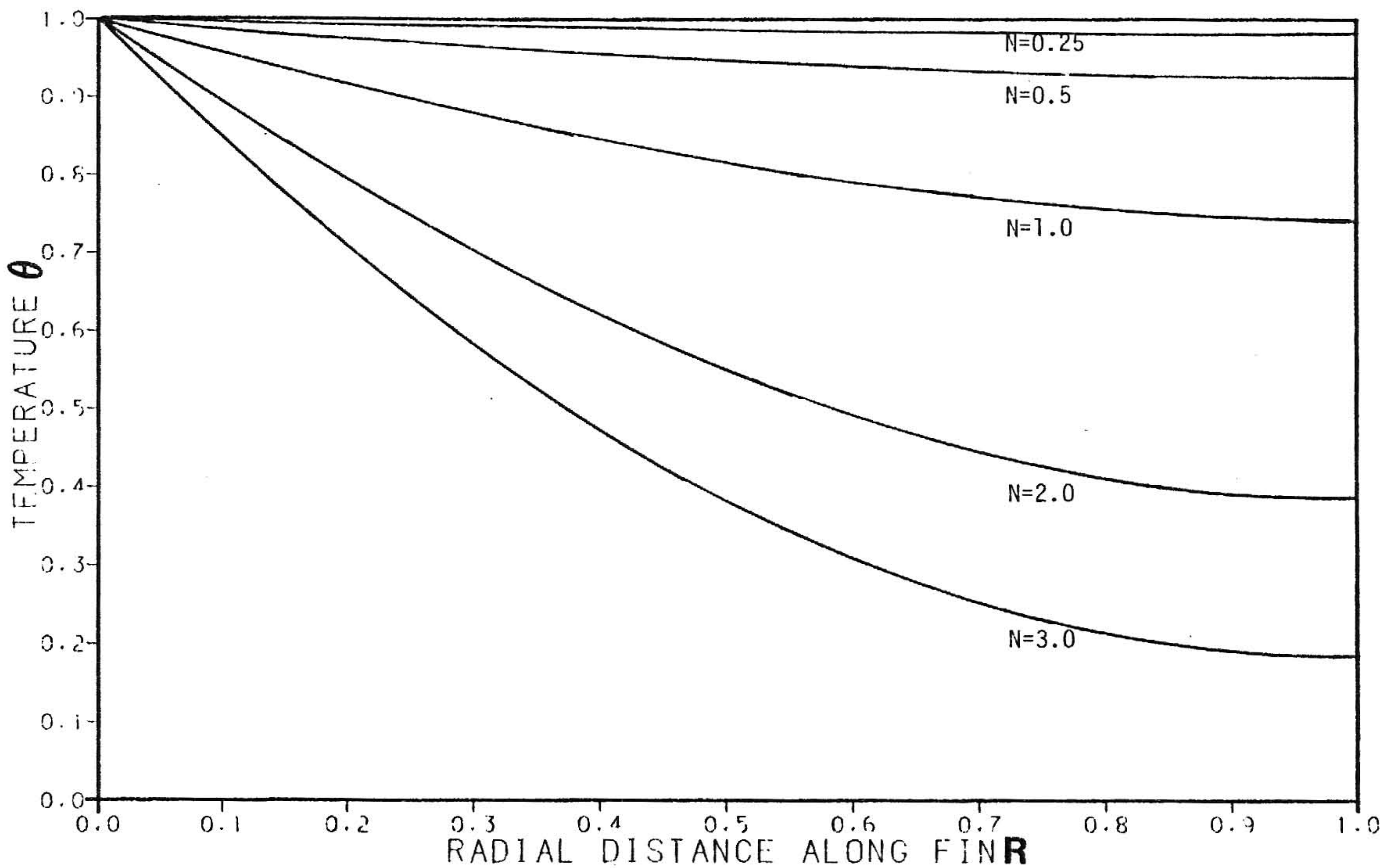

Figure 5.5: Temperature versus Radial Distance Along the Fin. $f(R)=R$, $d=2.0$, $\alpha=0.2$

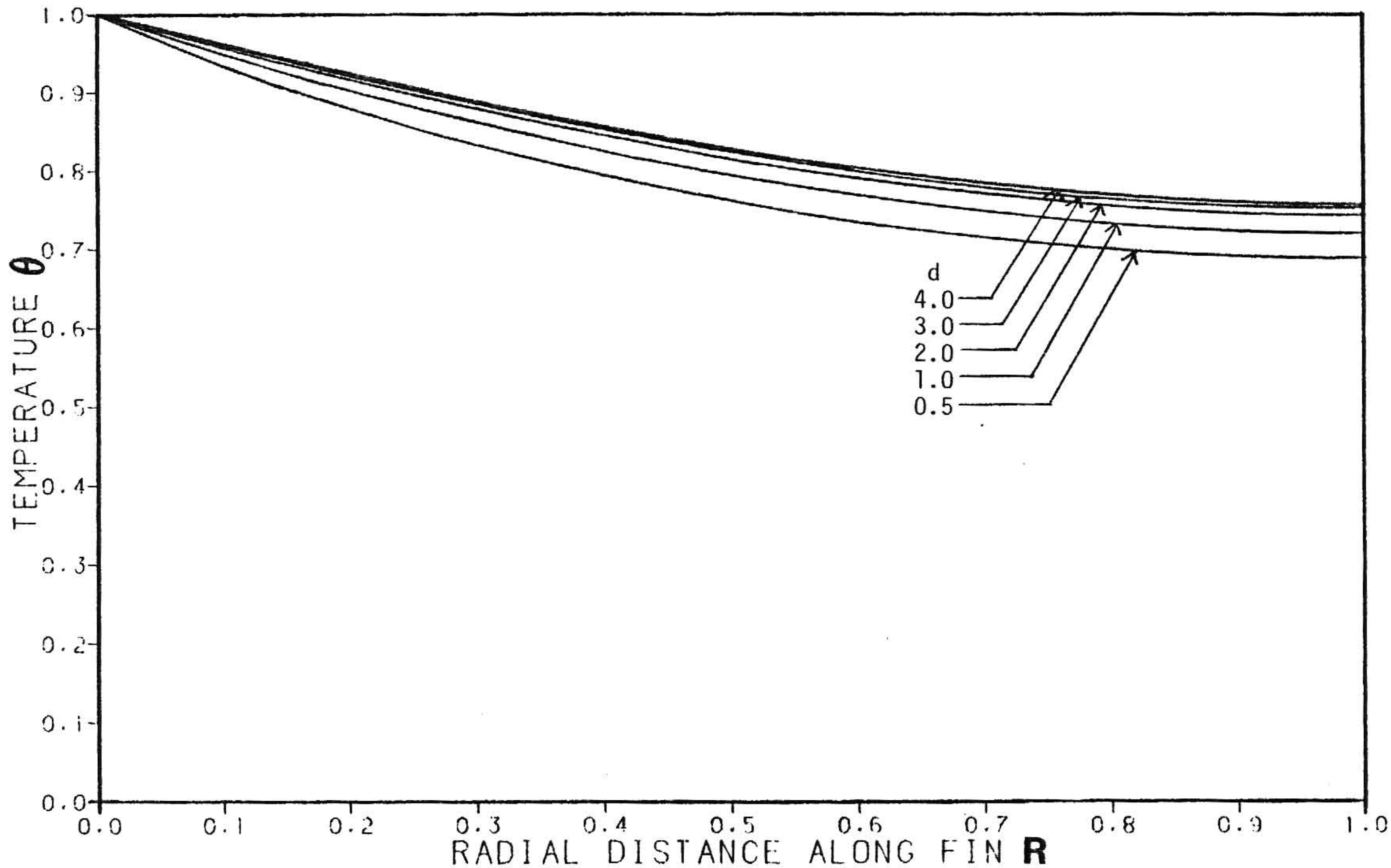

Figure 5.6: Temperature versus Radial Distance Along the Fin. $f(R)=R$, $N=1.0$, $\alpha=0.2$

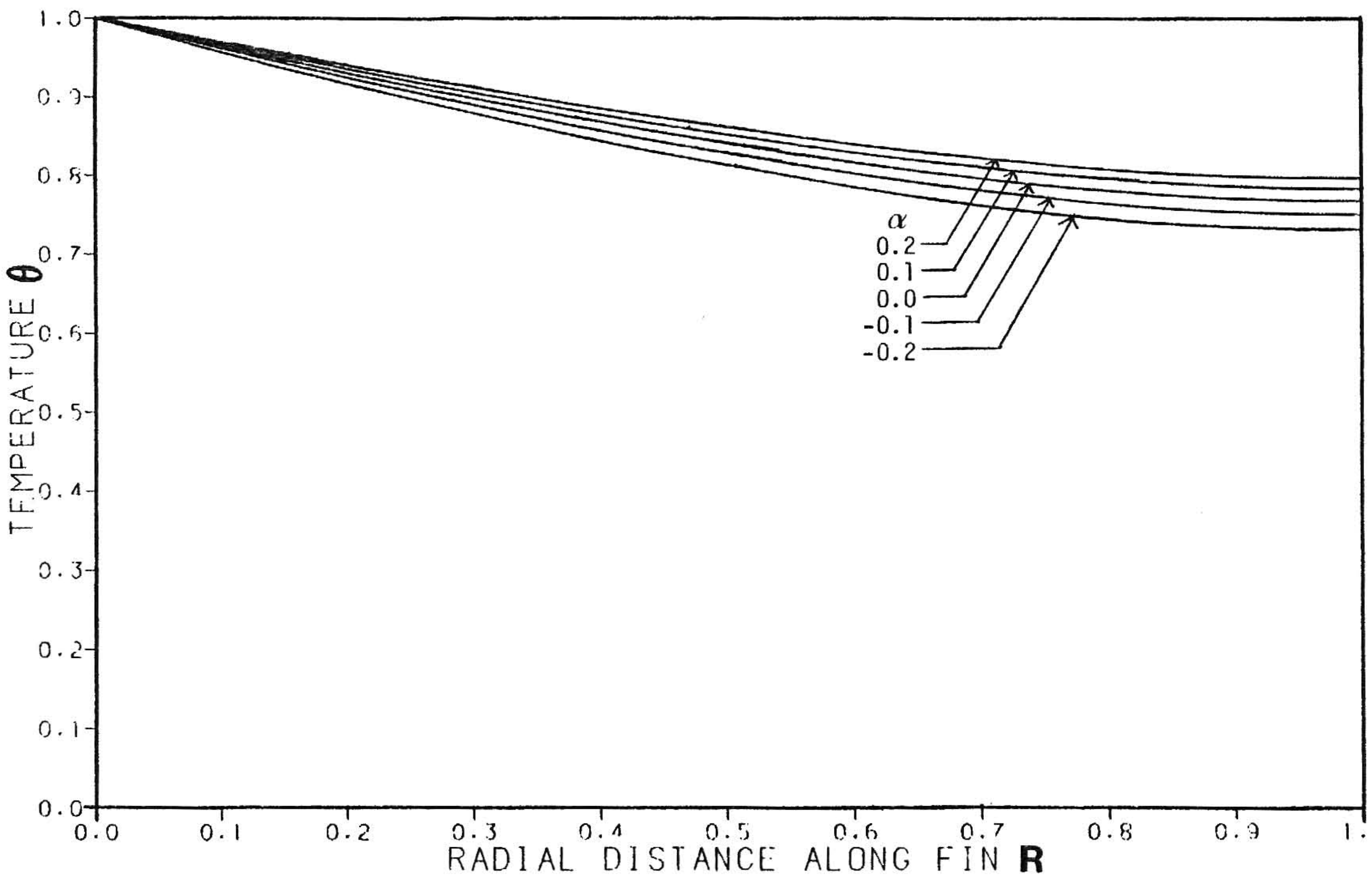

Figure 5.7: Temperature versus Radial Distance Along the Fin. $f(R)=R^2$, N=1.0, d=2.0

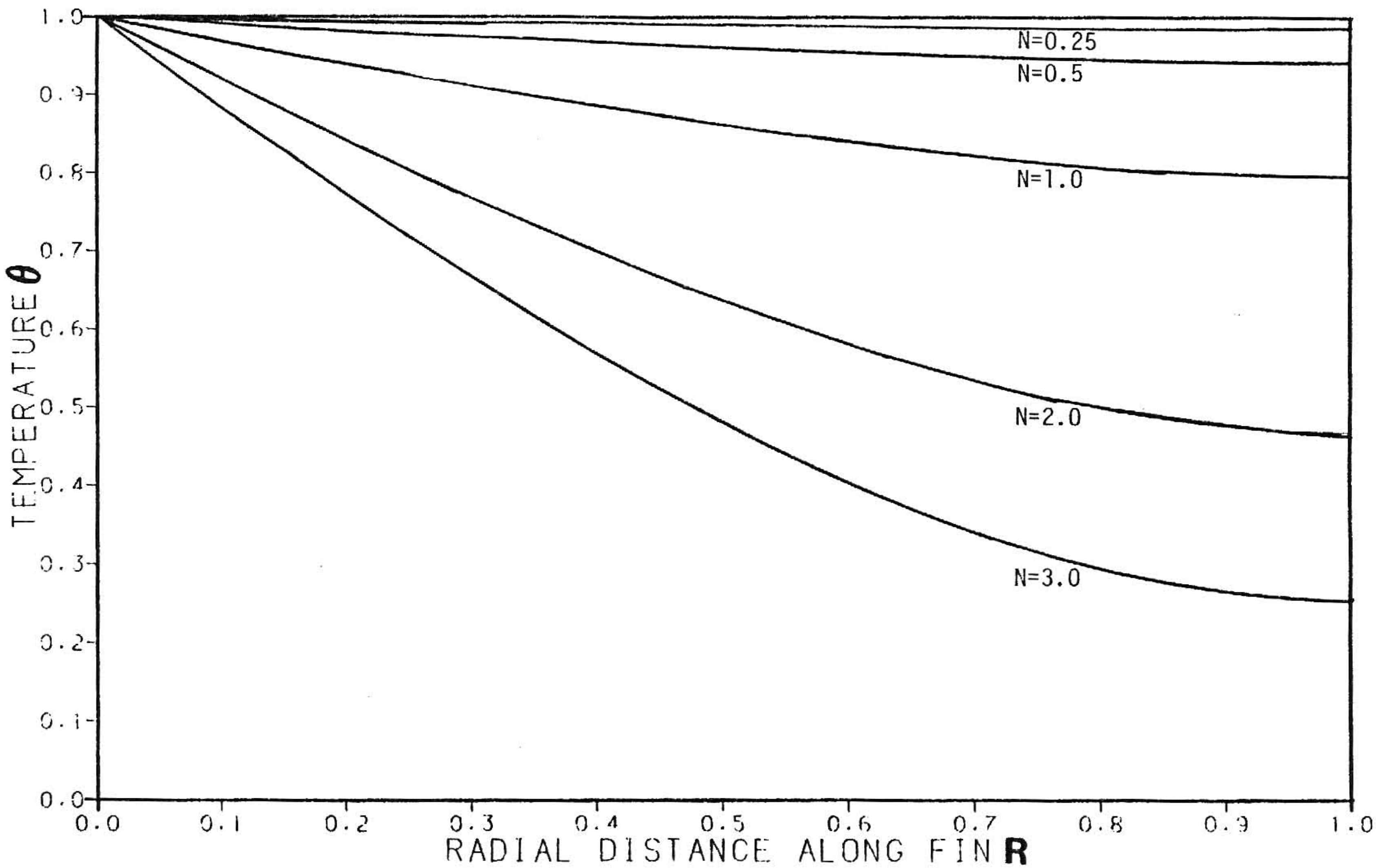

Figure 5.8: Temperature versus Radial Distance Along the Fin. $f(R)=R^2$, $d=2.0$, $\alpha=0.2$

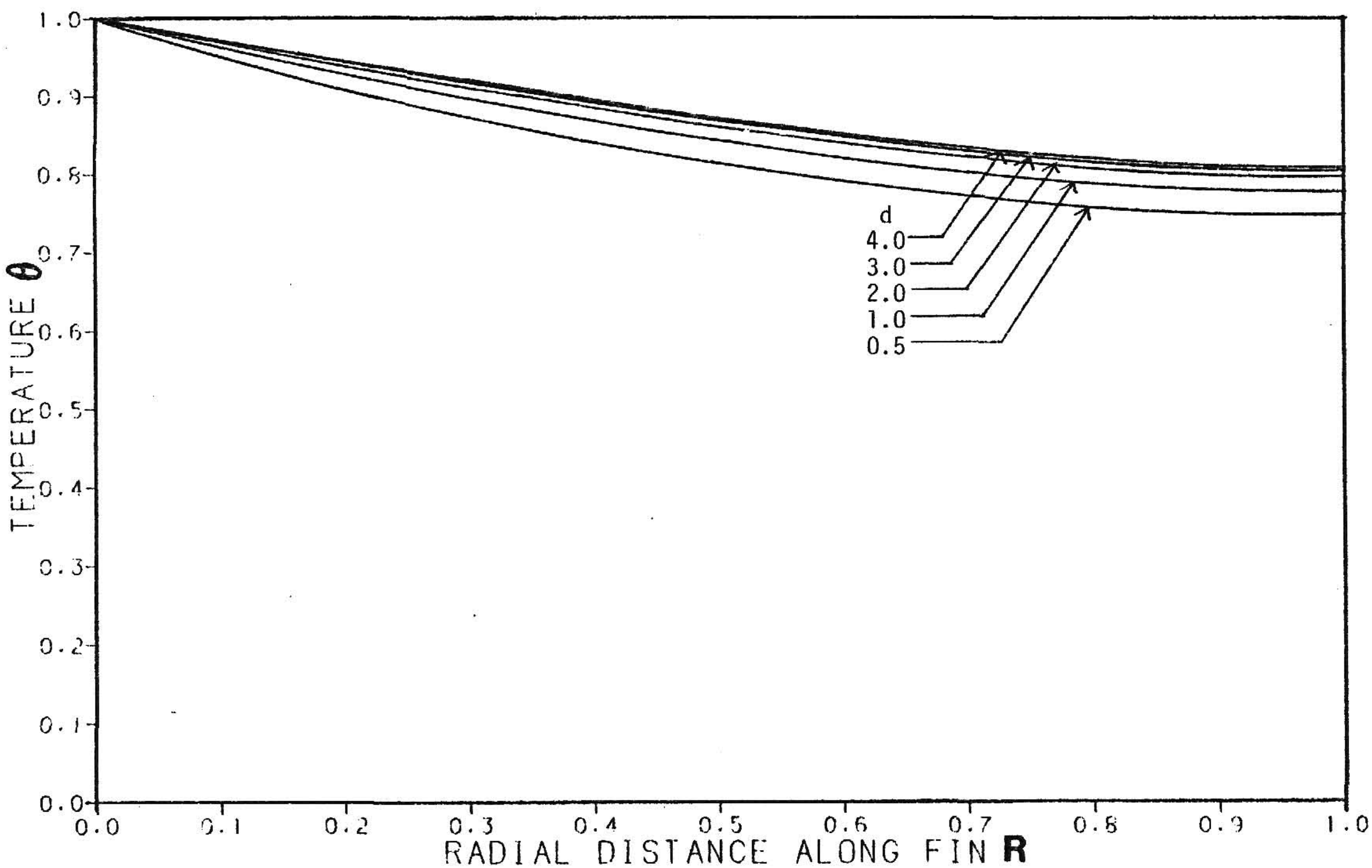

Figure 5.9: Temperature versus Radial Distance Along the Fin. $f(R)=R^2$, $N=1.0$, $\alpha=0.2$

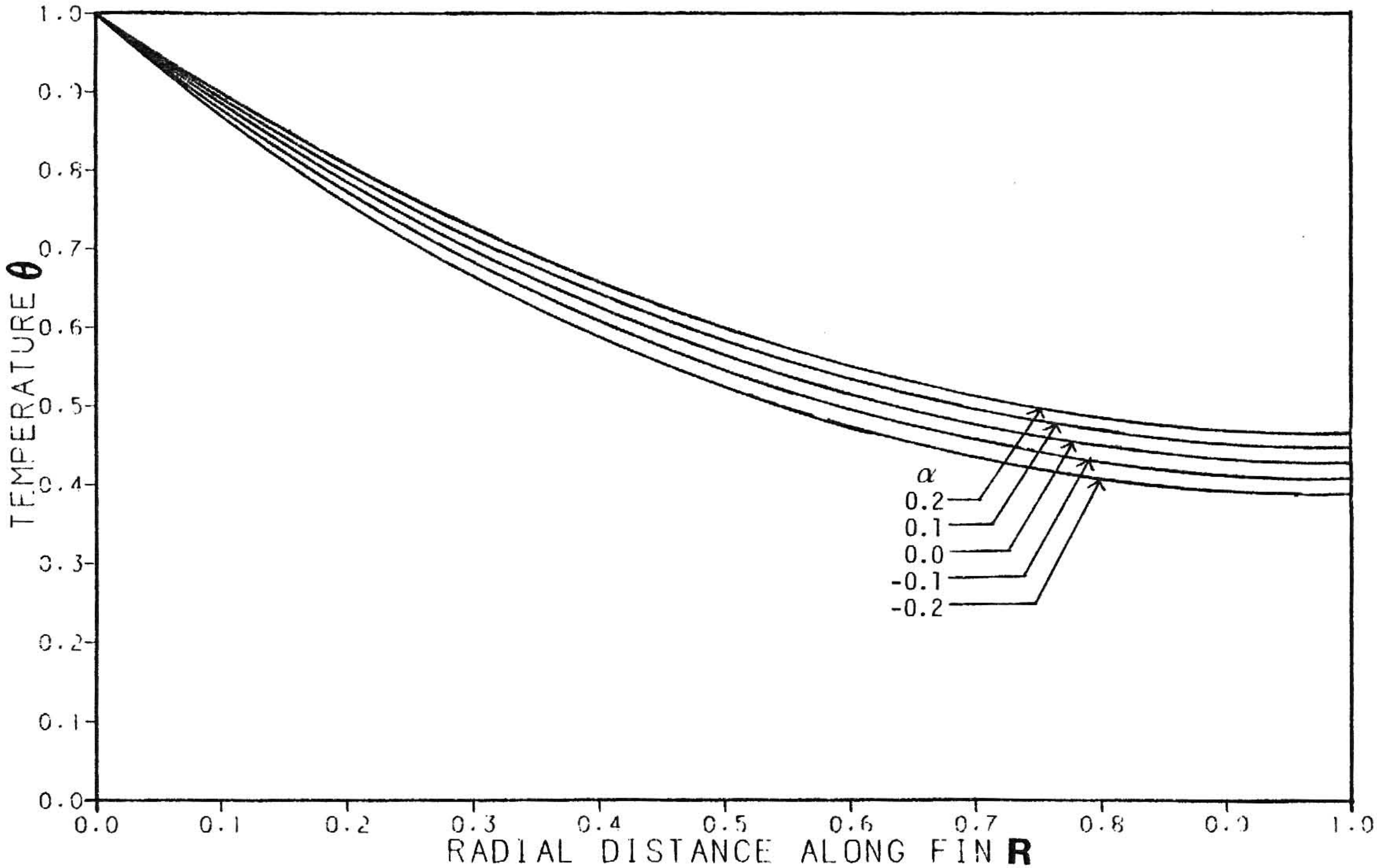

Figure 5.10: Temperature versus Radial Distance Along the Fin. $f(R)=e^R$, N=1.0, d=2.0

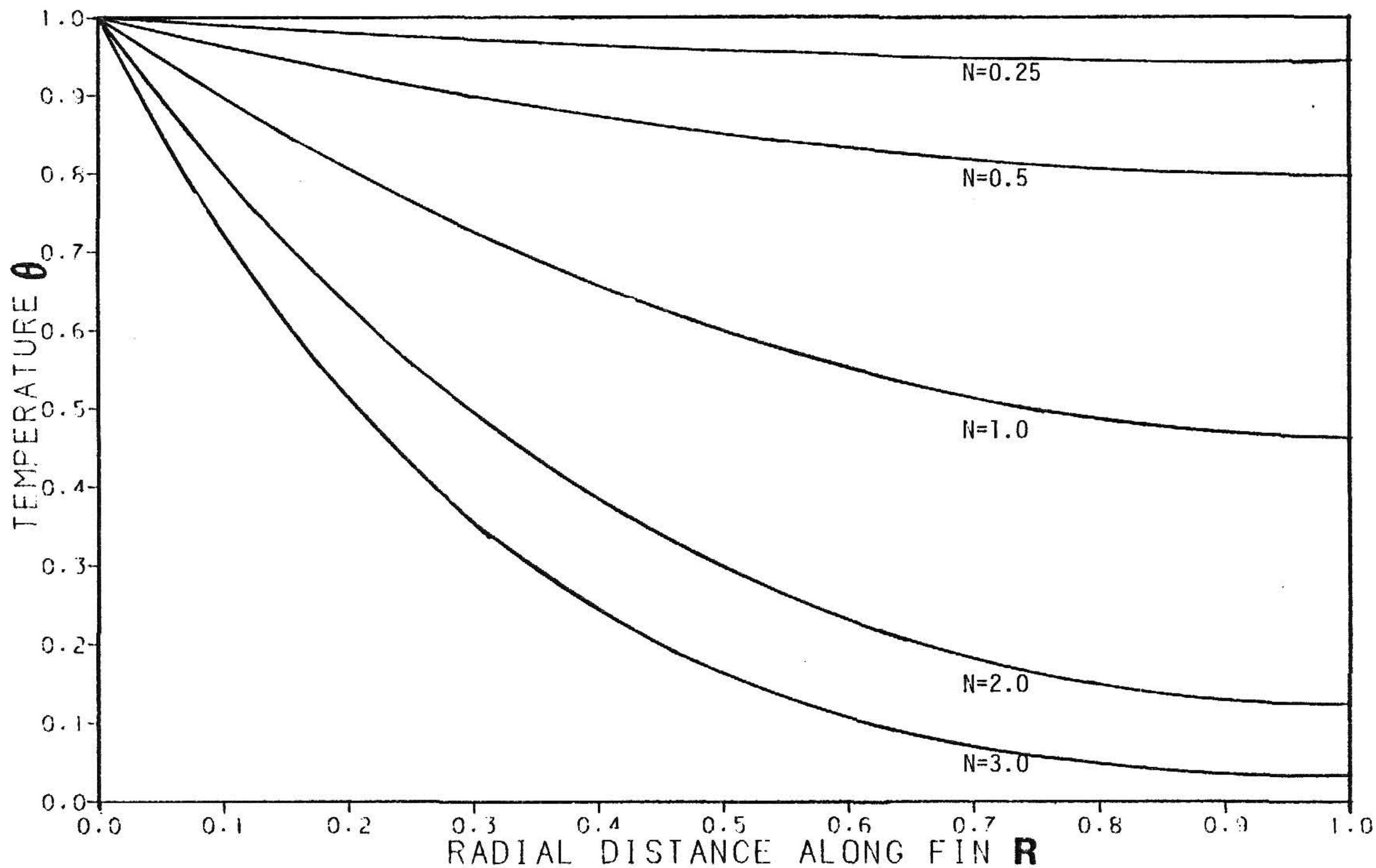

Figure 5.11: Temperature versus Radial Distance Along the Fin. $f(R)=e^{R}$, $d=2.0$, $\alpha=0.2$

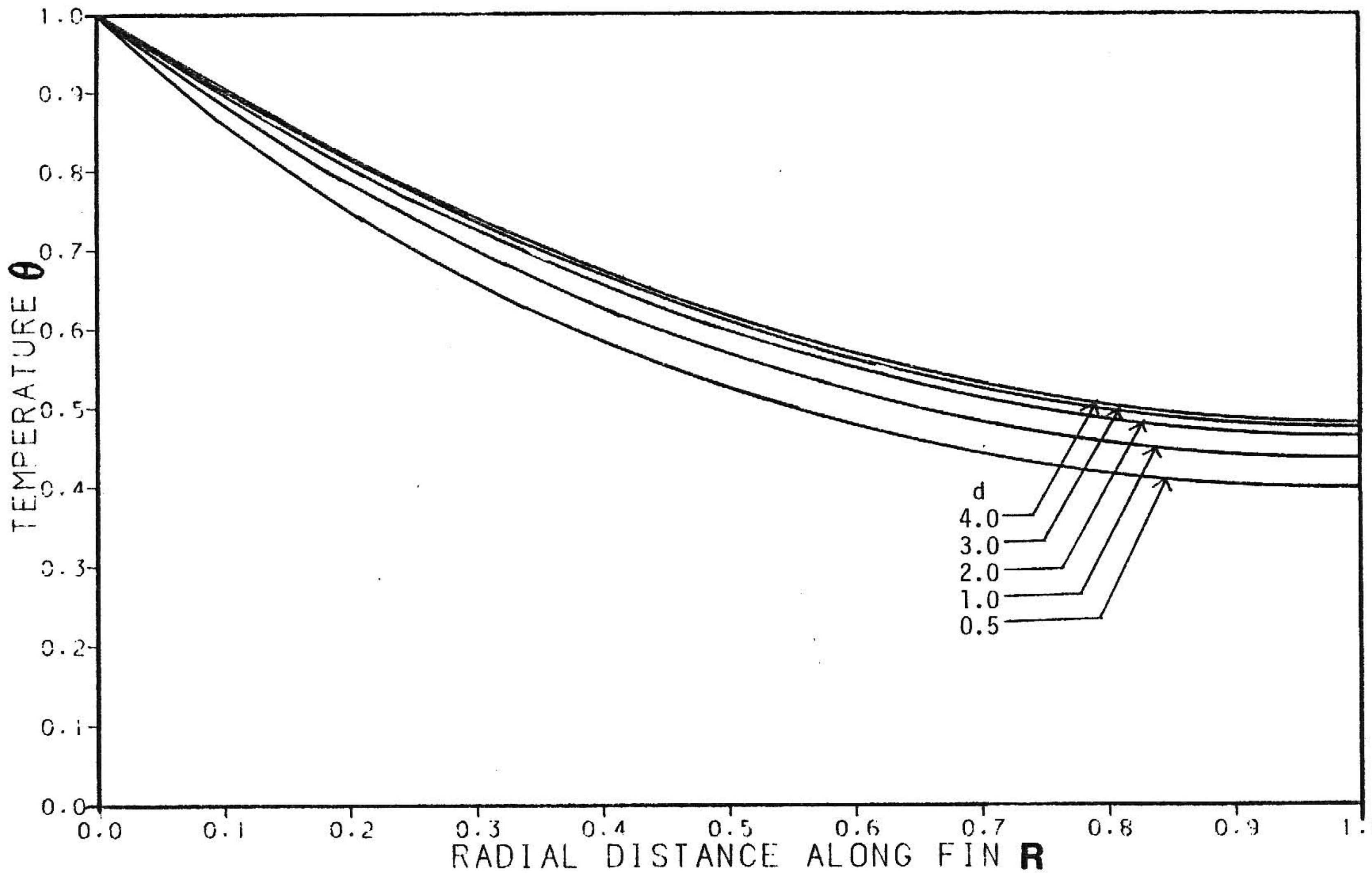

Figure 5.12: Temperature versus Radial Distance Along the Fin. $f(R)=e^R$, $N=1.0$, $\alpha=0.2$

5.3 Heat Transfer Rate

Figure 5-13 shows the variation of the heat transfer rate with respect to the fin parameter N, for the four different types of the heat transfer coefficient variation. In studying this variation, the values of d and α were kept constant at $d = 2.0$ and $\alpha = 0.2$. As N increases, the heat transfer rate increases, and in the range of N values used, the increase is almost linear. The reason for this is that, as N increases, the amount of heat being convected out of the fin increases. Also note that for the exponential variation of the heat transfer coefficient, the heat being transferred out is the most, and for the parabolic $(f(R)=R^2)$, the heat transfer rate is minimum. The linear $(f(R)=R)$ and constant $(f(R)=1.0)$ cases lie inbetween.

Figure (5-14) shows the variation of the heat transfer rate with respect to the conductivity variation parameter α. In studying this variation, the values of N and d were kept constants at $N = 1.0$ and $d = 2.0$. It was noted in Figure (5-13), that the heat transfer rates were a maximum for the exponential variation of the heat transfer coefficient and a minimum for the parabolic variation, with the linear variation and constant cases lying inbetween. This dependence on the type of the heat transfer coefficient variation is also observed in Figure 5-14. The heat transfer rate varies almost linearly with respect to the conductivity variation parameter α, showing only a slight increase as α increases. The slope of the curves increases as we go from the $f(R) = R^2$ case to

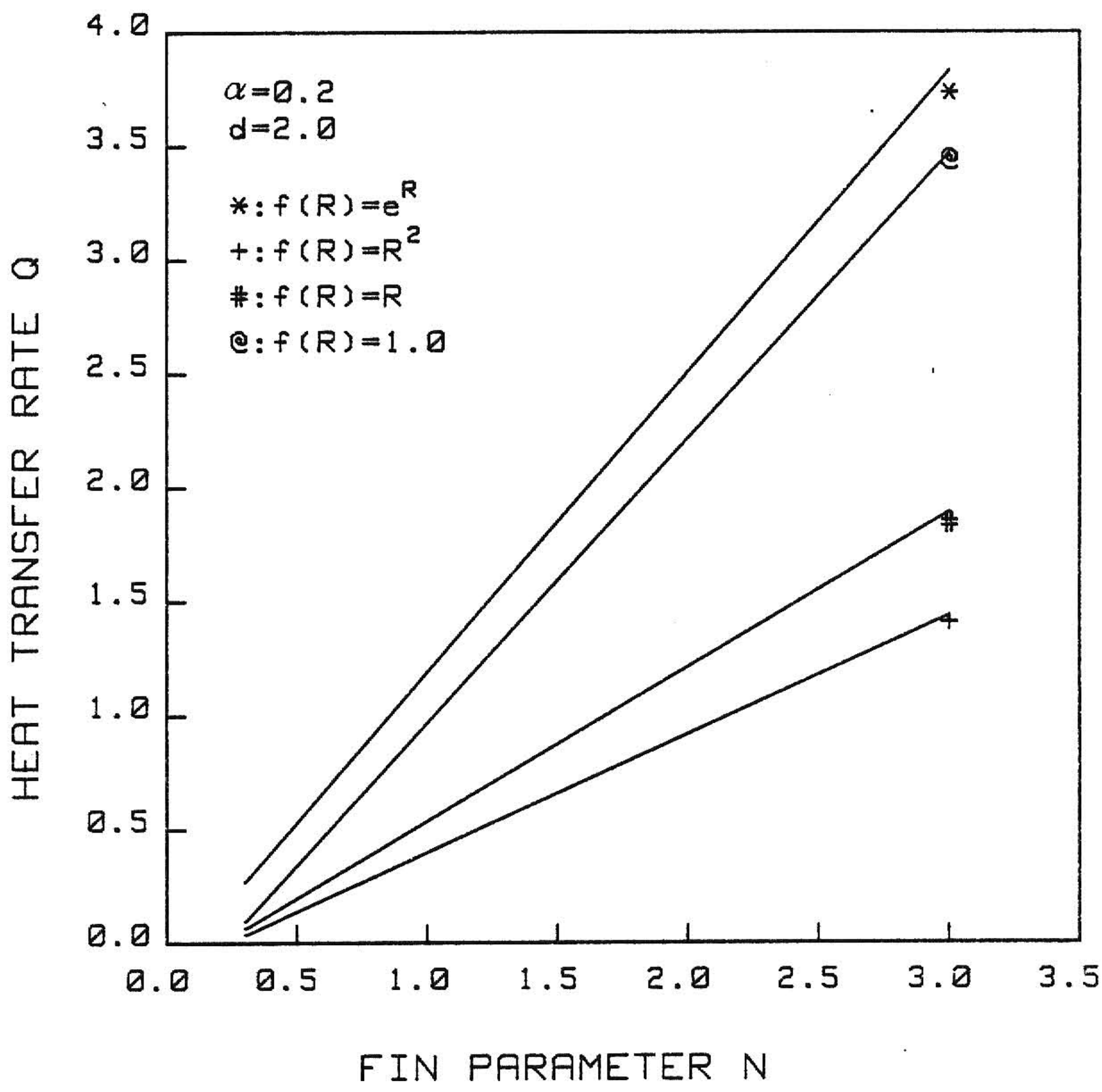

Figure 5.13: Heat Transfer Rate versus Fin Parameter.

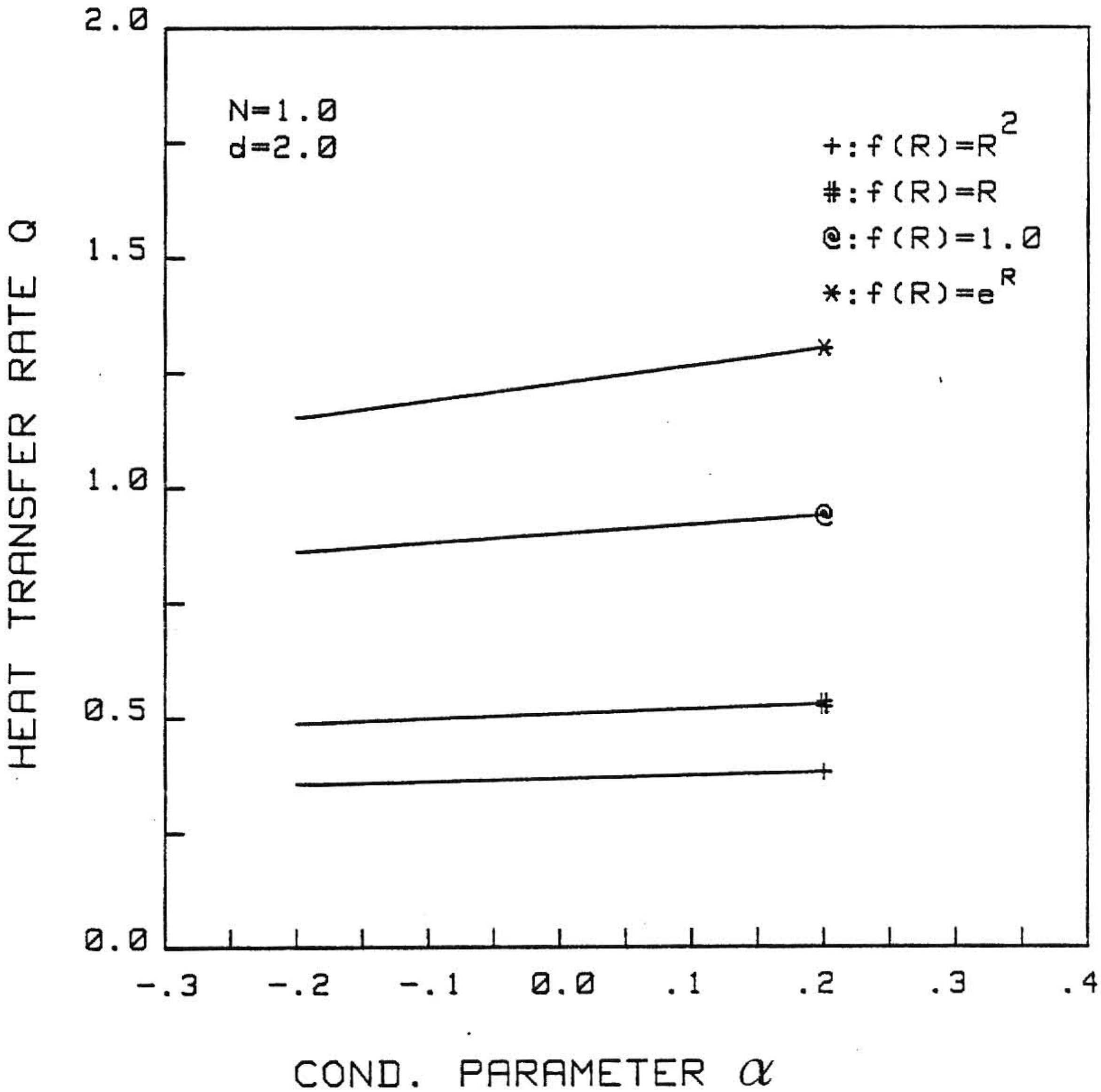

Figure 5.14: Heat Transfer Rate versus Conductivity Parameter.

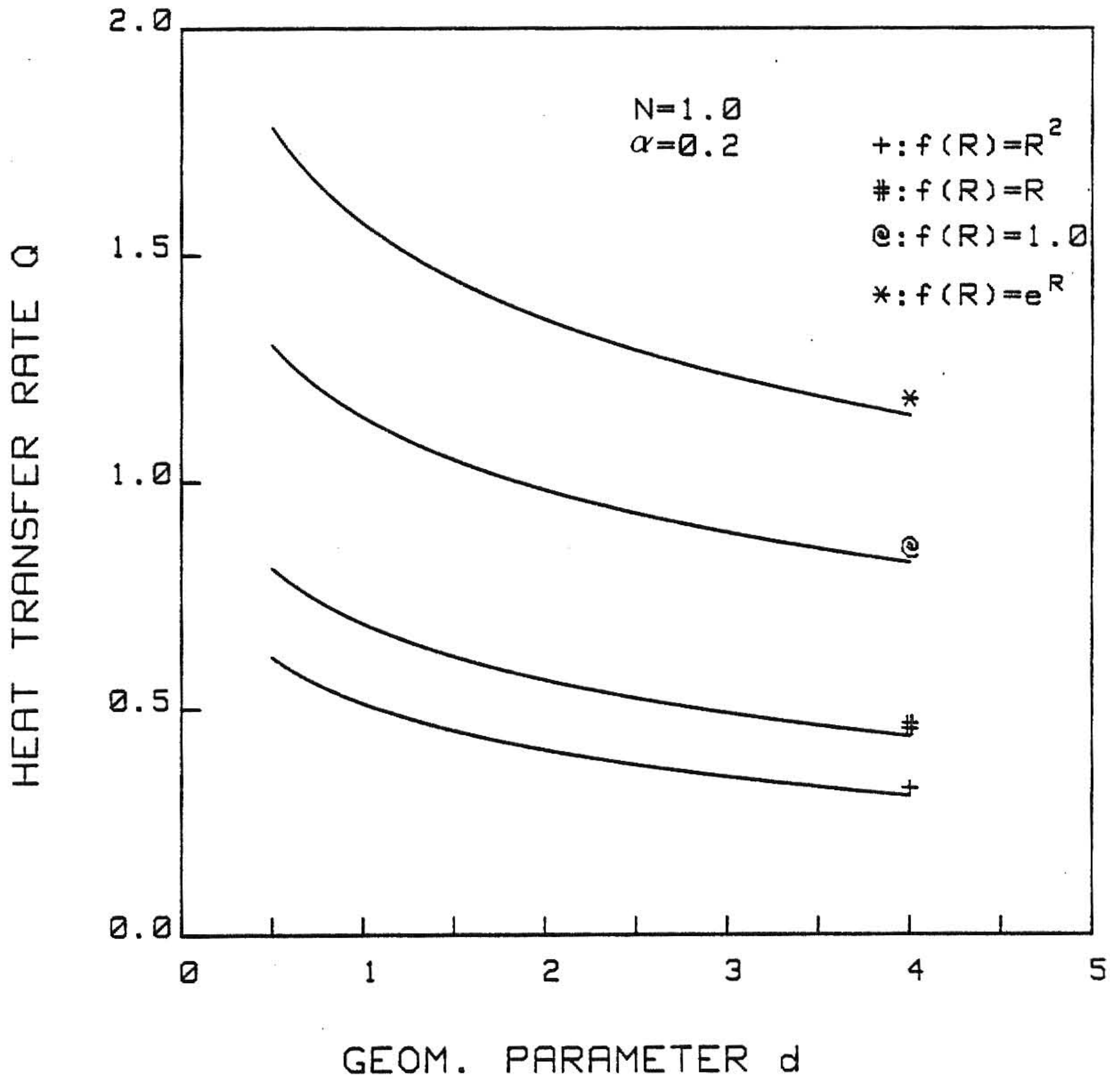

Figure 5.15: Heat Transfer Rate versus Geometric Parameter.

the $f(R) = e^R$ case, thereby showing that the effect of variable thermal conductivity is more prominent for more severe variations in the heat transfer coefficient.

Figure 5-15 shows the variation in the heat transfer rate with respect to the geometric parameter d. In studying this variation, the values of N and α were kept constants at $N = 1.0$ and $\alpha = 0.2$. It is seen that as the value of d increases, there is a decrease in the heat transfer rate. This decrease has been found to be an exponential decrease in the range of the d values studied. The reason for this is, that as d increases, the length of the fin available for transferring heat to the surroundings decreases. Once again, as noted in Figure 5-14, the decrease in the heat transfer rate is more prominent for more severe variations in the heat transfer coefficient.

5.4 Efficiency of the Fin

Before analyzing the variations in the efficiency of the fin, let us review the expression for the efficiency of the fin, equations (3-26) and (3-27).

$$\eta = \frac{Q}{N^2 I} ,$$

where $$I = \int_0^1 \left(\frac{R}{d} + 1\right) f(R)\,dR$$

I has been evaluated for each of the four types of variations of the heat transfer coefficients, f(R).

$$f(R) = 1.0 \qquad I = (1 + 2d)/2d \tag{5.2a}$$

$$f(R) = R \qquad I = (2 + 3d)/6d \tag{5.2b}$$

$$f(R) = R^2 \qquad I = (3 + 4d)/12d \tag{5.2c}$$

$$f(R) = e^R \qquad I = (1.7183 + d)/d \tag{5.2d}$$

Figure 5-16 shows the variations in the efficiency with respect to the fin parameter N. The efficiency decreases sharply as N increases in the range of N values studied, because of the N^2 term in the denominator of equation (3-27). The efficiency is also dependent on the type of the variation of the heat transfer coefficient. The efficiency is the highest for the $f(R) = R^2$ case and lowest for the $f(R) = e^R$ case, with the $f(R) = R$ and $f(R) = 1.0$ cases in between. This may not be very obvious from Figures 5-13, 5-14, and 5-15 in which the heat transfer rate is a maximum for the $f(R) = e^R$ case and lowest for the $f(R) = R^2$ case. The explanation for this is that the value of I is highest for the $f(R) = e^R$ case and lowest for the $f(R) = R^2$ case. The denominator for the $f(R) = e^R$ case is the highest, and hence the efficiency is a minimum. This trend of the change in the efficiency, with respect to the type of variations of the heat transfer coefficient, is also seen in the results obtained by Aziz and Na [10].

Figures 5-17 and 5-18 illustrate the same change in the efficiency as we go from the $f(R) = e^R$ to the $f(R) = R^2$ case.

Figure 5-17 shows the effect of the conductivity variation parameter α on the efficiency. The efficiency increases almost linearly with an increase in α , but the increase is less than 10 percent in the range of α values studied.

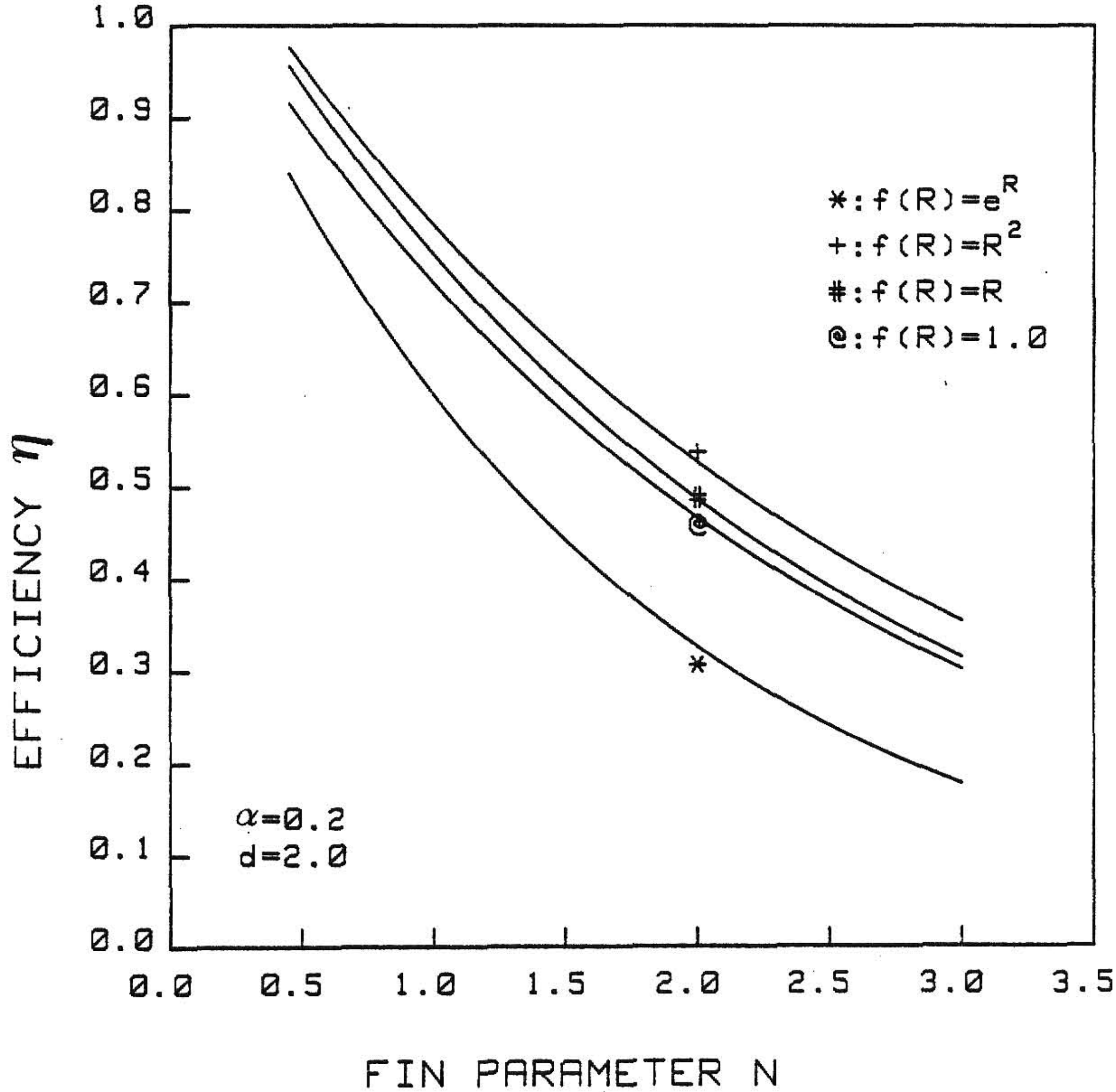

Figure 5.16: Efficiency versus Fin Parameter.

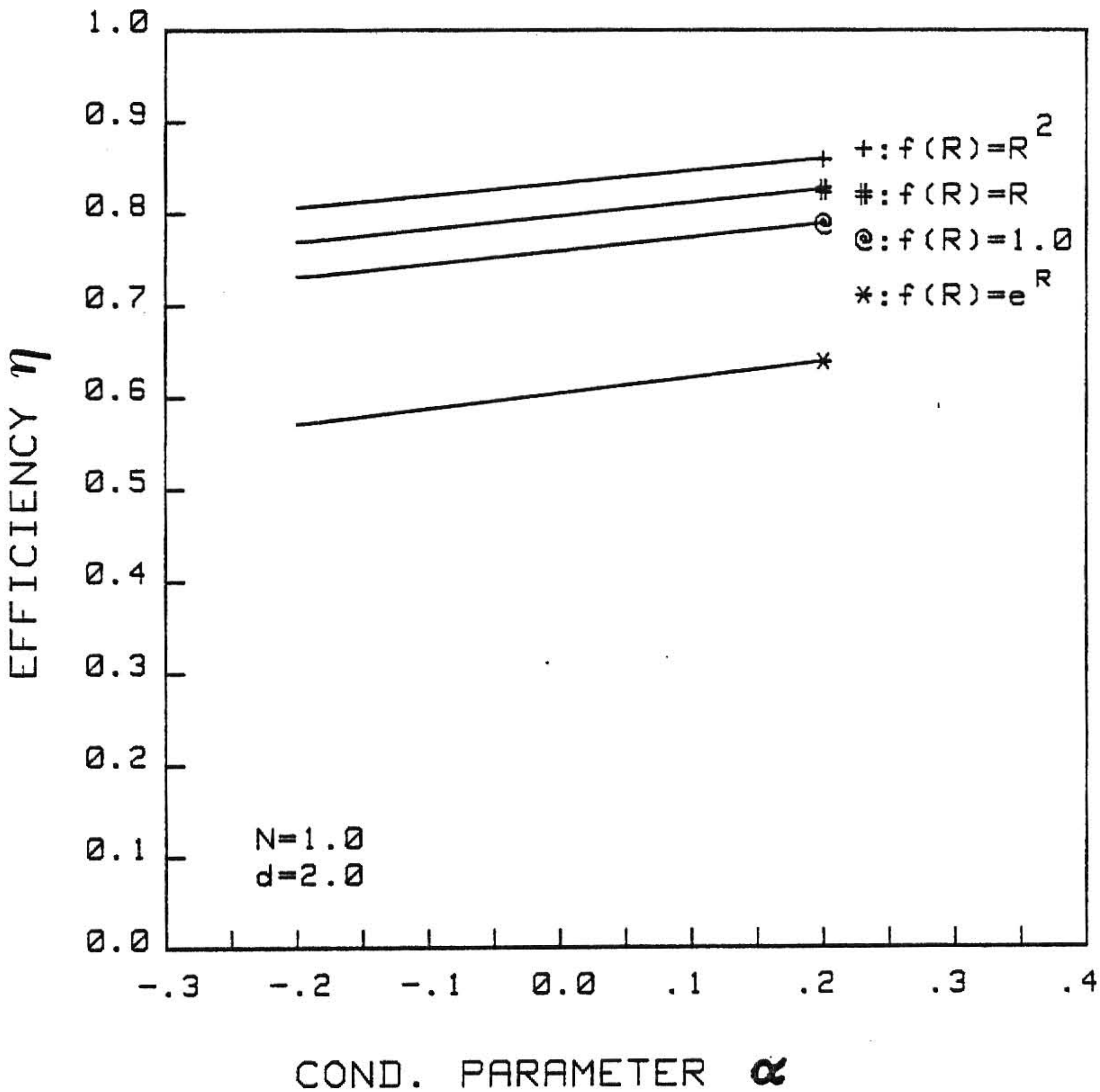

Figure 5.17: Efficiency versus Conductivity Parameter.

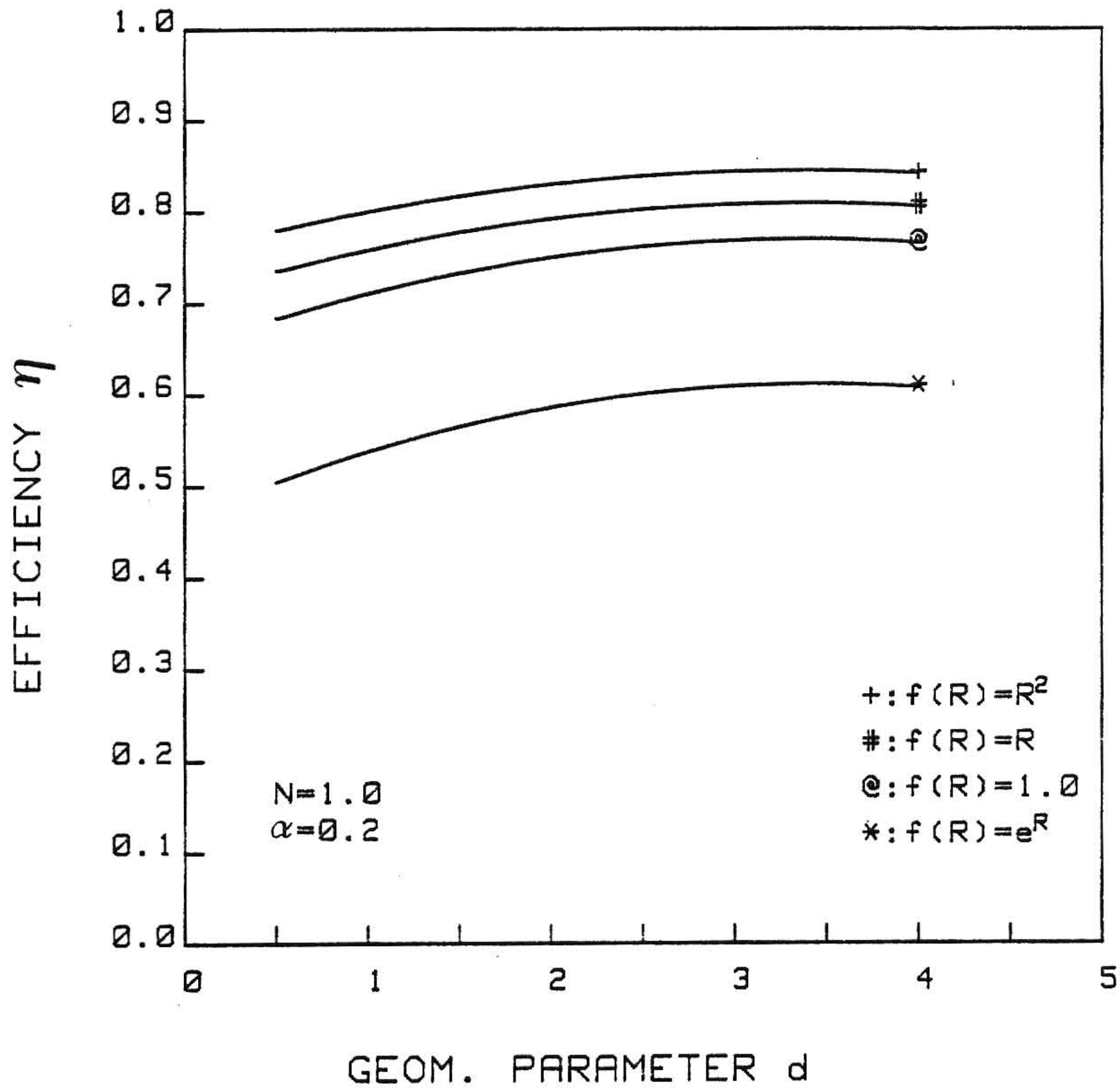

Figure 5.18: Efficiency versus Geometric Parameter.

Figure 5-18 shows the effect of the geometric parameter d on the efficiency. As d increases, the efficiency increases, but beyond a value of d = 2.5, the efficiency remains almost a constant. As d increases, the value of I decreases. But as d increases, the value of the heat transfer rate decreases. The ratio of Q and I increases initially, because the decrease in I plays a more significant role in the evaluation of the efficiency. But beyond a value of d = 2.5, the two effects combine to produce an almost constant value of efficiency.

In conclusion, it must be mentioned that the efficiency is merely a number, which shows us how much heat is being transferred from the fin, when compared to the heat that would have been transferred, under the condition that the whole fin were at the same temperature as the temperature at the base of the fin. Hence, in the selection or design of a fin, the heat transfer rate, and not the efficiency, should be considered.

PART II Optimization of Circular Fins With Variable Thermal Parameters

Chapter 6

INTRODUCTION TO THE OPTIMIZATION PROBLEM

The importance of fins in heat exchange devices has been discussed in Part I. The nonlinearities arise in the problem from the variation of the thermal conductivity and the heat transfer coefficient, which have been discussed previously. This part will study the optimization of circular fins with variable thermal parameters.

Invariant imbedding is used in the optimization process. It will be discussed in detail in Chapter 7, and the advantages of using the invariant imbedding approach will be discussed in Chapter 9.

The optimization of fins has been a classical problem. The design of the optimum cooling fin has been considered by many researchers since the mid-1900s. In 1926 Schmidt [20] suggested that the minimum weight cooling fin has a linear temperature distribution along its length, that is each and every cross-section of the fin carries the same heat flux. Schmidt was the first to set forward a criteria for an optimum fin. The optimization of straight fins has been considered in detail in a number of references, [21-23].

Duffin [24] has verified Schmidt's intuitive arguments by an analytical method, and placed the problem on a firm mathematical foundation. Wilkins [25,26] has studied minimum mass straight fins, which transfer heat only the by mechanism of radiation to the surroundings. In 1964 Brown [27] derived an equation relating the optimum dimensions of uniform annular fins to the heat transfer, thermal properties of the fin, and the heat transfer coefficient. Cobble [28], studied the problem of nonlinear fin heat transfer, and later he extended his previous work to the problems of fin optimization.

The fin optimization has been investigated by many researchers in the last decade. Cobble [29] investigated the problem of the optimum fin shape for steady-state nonlinear heat transfer of a straight fin, with the surroundings, by both convection and radiation. Later in 1974, Maday [30], using Pontryagin's minimum principle, obtained the minimum weight one-dimensional straight cooling fin. Güceri and Maday [31] followed up Maday's work with a paper on a least weight circular cooling fin. Arora and Dhar [32] have described methods of carrying out the minimum weight design of finned surfaces of various types.

More recently, Razelos [33] has obtained the solution of the optimization problem for longitudinal convective fins of constant thickness, triangular or parabolic profile, and uniform internal heat generation. He presented an analytical method, and obtained a transcedental equation for determining the optimum dimensions of straight fins from quantities which are specified, namely the fin profile, the desired heat dissipation, the thermal parameters and the heat generation. Razelos and Imre in 1980, published a paper [34], on the optimum dimensions of circular fins with variable thermal parameters. Optimum dimensions of circular fins of trapezoidal profile, with variable thermal conductivity and heat transfer coefficients, were obtained. They used a quasi-Newton algorithm to solve the nonlinear differential equation.

From the preceding discussion, we can conclude that the fin optimization problems can be classified either as the least volume fin or the maximum heat dissipation fin. The least volume, also known as the minimum weight fin problem, is to define the shape of the fin, which minimizes the fin volume for a given heat dissipation. Solutions obtained to these problems have resulted in desirable fin profiles. However, this type of optimization is only of academic interest with little practical usage, because the manufacture of fins with such complex profiles is exceedingly difficult and costly.

The maximum heat dissipation problem is more practical in engineering design. We select a suitable and simple profile, such as a trapezoidal or triangular fin, and then determine the dimensions of the fin, such that the fin with a given volume yields maximum heat dissipation. Due to the behaviour of the nonlinearities being considered, it is not possible to obtain a solution analytically. In the absence of such a solution, one must resort to numerical methods in evaluating optimal values. Invariant imbedding has been used in this study. The results obtained from the study, have been compared with the results obtained by Razelos and Imre [34], who used a quasi-Newton algorithm to solve this problem.

The optimization problem has been formulated in Chapter 8, and the solution and the results are prosented in Chapters 9 and 10 respectively.

Chapter 7

METHOD OF INVARIANT IMBEDDING

7.1 Introduction

In Part I of this study, quasilinearization has been used to solve the nonlinear boundary value problem. In Part 2 of this study, which deals with the optimization problem, a completely different approach has been used.

The principles of invariance, which are now known as invariant imbedding, were introduced by Ambarzumian [35], in his studies on transport phenomenon. Later, problems in radiative transfer, were solved by Chandrasekhar using this concept [36]. Bellman, Kalaba and Wing developed this technique in the study of neutron transport phenomenon, [37-41]. Others who made contributions in employing this technique include Priesendorfer [42], Bailey [43], and Lee [14].

This chapter will deal with the numerical aspects of the invariant imbedding approach. Invariant imbedding has been proven useful in treating boundary value problems, [41]. In this chapter the basic concept of the invariant imbedding will be emphasized, because this concept of imbedding is very useful and frequently gives new insights to the problem being studied, which may have been treated by other classical approaches. Rigorous derivations of the invariant imbedding method can be found in Bailey's paper [43].

The difficulties encountered with the solution of non-linear boundary value problems have already been discussed in Section (2.2), Chapter 2 of Part I. The optimization of fins is also governed by a nonlinear differential equation, and the advantages in using invariant imbedding will be discussed in a later chapter.

7.2 The Invariant Imbedding Approach

The invariant imbedding approach can be best illustated by an example. Consider the problem

$$\frac{dx}{dt} = f(x,y,t) \tag{7.1a}$$

$$\frac{dy}{dt} = g(x,y,t), \tag{7.1b}$$

where f and g are nonlinear functions.

The boundary conditions are

$$x(0) = c \tag{7.2a}$$

$$y(t_f) = 0 \tag{7.2b}$$

with the independent variable t defined in the range $0 \leq t \leq t_f$.

Equations (7.1) and (7.2) represent a nonlinear, two point, boundary value problem. In order to avoid the various computational difficulties in solving the above boundary value problem, we shall convert it to an initial-value problem. That is, we shall obtain the missing initial condition y(0), by using the invariant imbedding concept.

To do this, we first convert this problem, to a problem with a more general set of boundary conditions as follows:

$$x(a) = c \tag{7.3a}$$

$$y(t_f) = 0 \tag{7.3b}$$

where $a \leq t \leq t_f$, and a is the starting value of the independent variable t. By changing the value of a, we can change the duration of the process, that is, a denotes the control of the duration of the process. A whole family of problems can be generated by assigning different values to a, say $a = 0, \Delta, 2\Delta, \ldots t_f$, where Δ is an incremental value. If $a = 0$, then we get the original boundary conditions (7.2), and the duration of the process will be t_f. If $a = t_f$, then the duration of the process will be zero. For any intermediate value of a, we will get a duration between 0 and t_f. The basic equations of a typical problem of this family are represented by equations (7-1) and boundary conditions (7-3).

To convert the problem into an initial value problem, we need to obtain the missing initial value y(a). y(a) will be obtained in a general form for the whole family of problems. The basic idea in imbedding is that neighbouring processes within this family of problems, are related to each other. Neighbouring processes are those, for which the duration changes only by a small incremental value Δ. It is possible to obtain the missing initial condition y(0) by examining the relationships between neighbouring processes.

The most important step in the imbedding process, is to recognize the fact, that the missing initial condition y(a) for the family of problems, is not only a function of the starting point of the process a, but also a function of the starting state of the system, that is, the given initial condition c, [14]. The functional of the missing initial condition r_m can therefore be defined as

$$r_m = r_m(c,a) = \text{the missing initial condition for the system represented by equations (7-1) and (7-3), where } a \leq t \leq t_f \text{ and } x(a) = c \qquad (7.4a)$$

In other words

$$y(a) = r_m(c,a) \qquad (7\text{-}4b)$$

Remember, that x(a) and y(a) represent the initial state of a typical system.

The r_m is considered to be the dependent variable, and c and a are considered to be the independent variables. An expression for r_m, in terms of c and a will be obtained as follows. Consider a neighbouring process having the starting point $(a + \Delta)$. The missing initial condition for this neighbouring process can be related to the initial value of the previous process y(a), by the use of Taylor's series expansion. We neglect all terms with powers of Δ higher than the first order and obtain the following expression,

$$y(a+\Delta) = y(a) + y'(a)\Delta + O(\Delta), \qquad (7.5)$$

where $O(\Delta)$ represents higher order terms

Rewriting equations (7-1), at the starting point of the process t = a, we obtain,

$$\frac{dx}{dt}\bigg|_{x=a} = f(x(a),\ y(a),\ a) \tag{7.6a}$$

$$\frac{dy}{dt}\bigg|_{x=a} = g(x(a),\ y(a),\ a) \tag{7.6b}$$

Here, we do already know that $x(a) = c$ and $y(a) = r_m(c,a)$ substituting these values into equation (7-6), we obtain,

$$\frac{dx}{dt}\bigg|_{x=a} = f(c, r_m(c,a),\ a) \tag{7.7a}$$

$$\frac{dy}{dt}\bigg|_{x=a} = g(c, r_m(c,a),\ a) \tag{7.7b}$$

Substituting equations (7-7b) and (7-4b) into equation (7-5), we obtain the following expression for the missing initial conditions of a process having the starting state at $t = a + \Delta$:

$$y(a+\Delta) = r_m(c,a) + g(c,r_m(c,a),a)\Delta + O(\Delta) \tag{7.8}$$

From equation (7-4) we know that $r_m = r_m(c,a)$, which can be rewritten as follows:

$$r_m = r_m(c,a) = r_m(x(a),a) \tag{7-9}$$

From equation (7-9) the function $y(a+\Delta)$ can be obtained by simply replacing a by $a + \Delta$ in the equation (7-9). Therefore

$$y(a + \Delta) = r_m(x(a + \Delta),\ a + \Delta) \tag{7-10}$$

Again, we need to express $x(a + \Delta)$ in terms of $x(a)$, by using Taylor's series expansion with terms involving powers of Δ higher than the first being neglected.

$$x(a + \Delta) = x(a) + x'(a)\Delta + O(\Delta) \qquad (7\text{-}11)$$

We know that $x(a) = c$. Substituting equation (7-7a) into equation (7-11), we arrive at the following equation.

$$x(a + \Delta) = c + f(c, r_m(c,a), a)\Delta + O(\Delta) \qquad (7\text{-}12)$$

Substituting the expression (7-12) into equation (7-10), we have,

$$y(a + \Delta) = r_m(c + f(c, r_m(c,a), a)\Delta + O(\Delta),\ a + \Delta) \qquad (7\text{-}13)$$

Equating (7-8) and (7-13) we obtain the desired relation

$$\begin{aligned} &r_m(c,a) + g(c, r_m(c,a), a)\Delta \\ &\quad = r_m(c + f(c, r_m(c,a), a)\Delta\ ,\ a + \Delta) \qquad (7\text{-}14) \end{aligned}$$

We should note that in equation (7-14), we omit all terms involving powers of Δ higher than the first.

Equation (7-14) governs the whole imbedding process. It can directly be used to obtain the missing initial condition $r_m(c,a)$.

Nevertheless, it is also possible to adopt a fully analytical approach from this point. A partial differential equation can be formulated from equation (7-14). Expanding the right hand side of (7-14) by Taylor's series expansion we obtain

$$r_m(c+f(c,r_m(c,a),a)\Delta,\ a + \Delta) = r_m(c,a)$$

$$+ f(c,r_m(c,a),a)\Delta\ (\frac{\partial r_m(c,a)}{\partial c})$$

$$+ \Delta(\frac{\partial r_m(c,a)}{\partial a}) + O(\Delta) \qquad (7\text{-}15)$$

In the limit, Δ tends to zero, as neighbouring processes approach one another. The following first order quasilinear partial differential equation can be obtained from expressions (7-14) and (7-15).

$$f(c,r_m(c,a),\ a)\ \frac{\partial r_{m(c,a)}}{\partial c} + \frac{\partial r_{m(c,a)}}{\partial a}$$

$$= g(c,r_m(c,a),\ a) \qquad (7\text{-}16)$$

From (7-3b) and (7-4), we know that when $a = t_f$, we have a zero duration process

$$r_m(c,t_f) = 0 \qquad (7\text{-}17)$$

Thus, the missing initial conditions $r_m(c,a)$ for the family of processes, with the starting values of the independent variable a from zero to t_f, can be obtained by solving the system (7-16) and (7-17).

7.3 Considerations in Using the Invariant Imbedding Approach

Though equations (7-16) and (7-17) can be solved by various techniques, the finite difference equation (7-14) is often used. It is advantageous to use the difference equation (7-14), whose limiting value yields the differential equation (7-16), rather than constructing finite difference equations from the differential equation (7-16),[14]. The original difference equation (7-14) preserves the physical characteristics of the process, and thus yields more insight into the problem being studied.

It should be noted in equation (7-14), that we are attempting to obtain $r_m(c,a)$. Therfore, we can rewrite (7-14) as follows

$$r_m(c,a) = r_m(c+f(c,r_m(c,a), a)\Delta, a + \Delta) - g(c,r_m(c,a), a)\Delta \qquad (7\text{-}18)$$

We note that $r_m(c,a)$ occurs an both sides of this equation. Since we are considering neighbouring processes, we can replace $r_m(c,a)$ on the right hand side of equation (7-18) by the following approximation

$$r_m(c,a) \cong r_m(c,a + \Delta) \qquad (7\text{-}19)$$

in both f and g functions, where Δ must be small. Substituting (7-19) into (7-18), we get

$$r_m(c,a) = r_m(c+f(c,r_m(c,a + \Delta)a, a)\Delta, a + \Delta) - g(c,r_m(c, a + \Delta), a)\Delta \qquad (7\text{-}20)$$

Equation (7-20) can be solved in a backward recursive fashion by starting with the condition.equation (7-17), at t_f. Clearly we cannot evaluate r_m at all values of c. Hence, some discrete values must be chosen, say

$$c = 0, \delta, 2\delta, \ldots \qquad (7\text{-}21)$$

and r_m evaluated at only these discrete values of c. Thus there are grid points in both dimensions (the duration t_f and the initial condition c) of the problem. The initial state of the system, c, has been divided into c/δ grid points and the duration of the process t_f has been divided into t_f/δ grid points. There is very little sophistication involved in choosing the grid values of the independent variables. Experience, computer memory capacity and accuracy requirements play a major role in the selection of Δ and δ values used, [14].

Since the function r_m is represented by a set of grid values, some type of interpolation scheme is necessarily used to create a general value from the values at the grid points, during the calculations. From equation (7-20), adopting a backward recursive approach, for the first step let $a = t_f - \Delta$. Then,

$$r_m(c,t_f-\Delta) = r_m(c+f(c,r_m(c,t_f),\ a)\Delta,\ t_f) - g(c,r_m(c,t_f),\ a)\Delta \tag{7-21a}$$

We know that for any value of c, $r_m(c,t_f) = 0$ from equation (7-17) . Therefore (7-21a) can be written as

$$r_m(c,t_f-\Delta) = -g(c,o,a)\Delta \tag{7-21b}$$

Equation (7-21b) can be solved for $r_m(c,t_f-\Delta)$ for all the grid values of c, that is 0, δ, 2δ, ... A table can then be set up from all the values of r_m obtained at this particular value of $a = t_f - \Delta$.

For the second step, the duration of the process is increased by a small value, by taking $a = t_f - 2\Delta$. Substituting this value of a in (7-20) we get,

$$r_m(c,t_f-2\Delta) = r_m(c+f(c,r_m(c,t_f-\Delta),\ a)\Delta,\ t_f-\Delta) - g(c,r_m(c,t_f-\Delta),\ a)\Delta \qquad (7\text{-}22)$$

So to obtain the values of $r_m(c,t_f-\Delta)$ for different values of c, we need to know the values of $r_m(c,t_f-\Delta)$ on the right hand side. These values have been stored in a table as mentioned above. By interpolating within this table, a general value of r_m can be obtained for any particular value of c.

This recursive process is continued in the backward direction till the terminating value of a is reached, that is $a = 0$. At this point, the value of r_m we obtain will be the missing initial condition y(0) which we have been searching for.

7.4 General Discussion of the Imbedding Approach

The imbedding process is hinged on the fact that any individual process can be considered as a member of a family of related processes. The sizes of these individual processes are represented by the duration of the processes or the intervals of interest ($a \leq t \leq t_f$). The basic idea is that although we do not know the value of the missing initial condition of the original process with duration t_f, we do know the value of this missing initial condition if the process has zero duration [14]. Thus starting with a process of zero duration, we gradually increase the duration of the process by imbedding the present unknown process into the previously known process with a shorter duration.

The invariant imbedding approach is a concept or an idea. It is not a rigorous technique. Thus the invariant imbedding equations can be obtained by various different formulations or derivations. These equations can be obtained not only from the usual equations representing the process, but also from an analysis of the original physical process, provided that the physical picture of the process is fairly simple and clear. Since 1960, several schemes have been developed to obtain the invariant imbedding equations, without considering the physical process. The invariant imbedding can also be applied to solve a general n-dimensional system of nonlinear differential equations.

Chapter 8

ANALYTICAL FORMULATION OF THE OPTIMIZATION PROBLEM

8.1 Introduction

As stated in Chapter 6, this part of the study deals with the optimization or maximization of heat transfer rate from a fin of given profile. The variation in the optimum conditions with respect to thermal and geometric parameters has also been considered. Invariant imbedding has been used to ovércome the difficulties of the nonlinearities involved in the optimization process.

The derivation of the nonlinear differential equation governing the behaviour of the fins has already been presented in section 3.2 of Part I. Reviewing the assumptions then made, most of them do hold even in the optimization problem. In brief, there is steady-state one-dimensional heat conduction, radiation is neglected, fin profile curvature is neglected, fin tip is insulated, and the thermal conductivity and heat transfer coefficient are not constants.

8.2 The Optimization Problem

Under the assumptions stated above, the governing differential equation has been obtained in equation (3-3) of Part I which is being repeated here.

$$\frac{d}{dr}\left[k(t)ry(r)\frac{dt}{dr}\right] - h(r)r(t-t_\infty) = 0$$

If $T = t - t_\infty$, then the above equation can be rewritten as

$$\frac{d}{dr}\left[k(T)ry(r)\frac{dT}{dr}\right] = h(r)rT \qquad (8\text{-}1)$$

The difference between this formulation and the one in Chapter 3 of the first part, is that here an even more generalized case has been considered, in the sense, that the thickness of the fin is not a constant, and could vary between a triangular profile and a constant thickness profile.

The boundary conditions are

$$T|_{r = r_b} = \text{constant},\ T_b \qquad (8\text{-}2a)$$

$$\frac{dT}{dr}\Big|_{r = r_o} = 0 \qquad (8\text{-}2b)$$

$$T|_{r = r_o},\ \text{bounded if } y(r_o) = 0 \qquad (8\text{-}2c)$$

Now the optimization problem can be stated as follows:
For a given volume of the fin

$$V = \int_{r_b}^{r_o} 4\pi y(r)\ r\ dr, \qquad (8\text{-}3)$$

we need to determine the dimensions of the fin, (that is, the semi-thickness at the base of the fin w, and the length of the fin, (r_o-r_b)), which will maximize the heat being dissipated by the fin under steady state condition, in other words, the following quantity will be maximized:

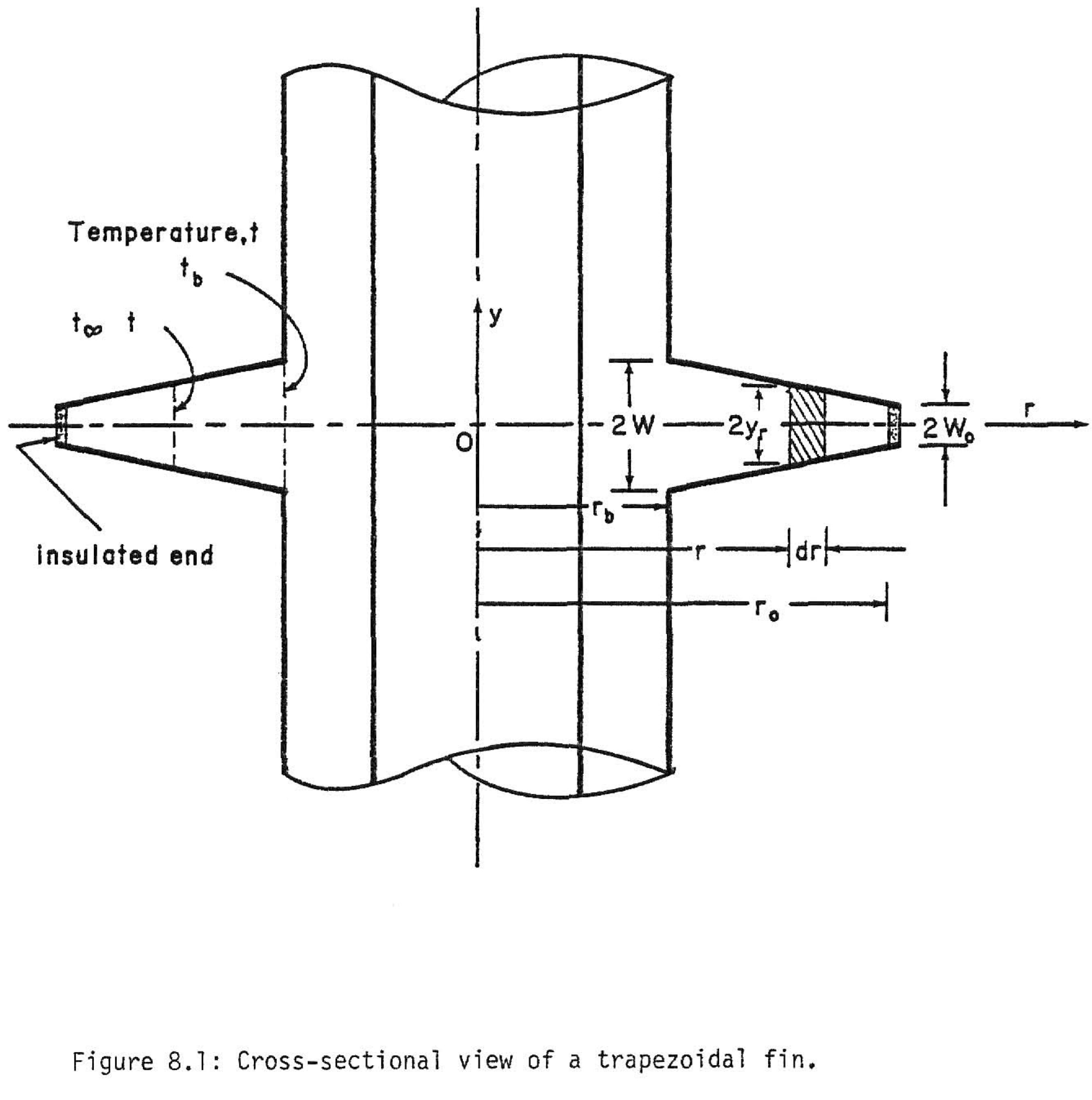

Figure 8.1: Cross-sectional view of a trapezoidal fin.

$$q = -4\pi r_b \bar{w} k(T) \frac{dT}{dr}\Big|_{r = r_b} \tag{8-4}$$

A constraint condition requires that

$$q > q_{nf}, \tag{8-5a}$$

where $$q_{nf} = 4\pi r_b \bar{w} h_{nf} T_{nf} \tag{8-5b}$$

q_{nf} is the heat transfer that would occur from the surface in the absence of a fin, and h_{nf} and T_{nf} are the corresponding heat transfer coefficient and temperature at the base.

Once again, the thermal conductivity has been assumed to be linearly dependent on temperature, and the heat transfer coefficient has been assumed to vary with the radius. Therefore,

$$k = k_a(1 + \alpha\frac{T}{T_b}) = k_a(1 + \alpha\Theta) \tag{8-6}$$

and $$\frac{h}{h_a} = K\left[\frac{r-r_b}{r_o-r_b}\right]^m = H(\frac{r}{r_b}) \tag{8-7}$$

where K is a function of the slope of the fin profile λ, the index m and the length of the fin b, and h_a is the average heat transfer coefficient. Θ is the dimensionless temperature, $\Theta = T/T_b$, and k_a is the reference thermal conductivity.

8.3 <u>Dimensionless parameters</u>

The nondimensional independent variable is defined as

$$\xi = r/r_b \tag{8-8}$$

The thickness of the fin is made dimensionless with respect to the base semi-thickness w.

$$\beta = y/w \tag{8-9}$$

Therefore,

$$\frac{dy}{dr} = \frac{w}{r_b}\frac{d\beta}{d\xi} \tag{8-10}$$

Substituting (8-6), (8-7), (8-8) and (8-9) into (8-1), we have

$$\frac{1}{r_b}\frac{d}{d\xi}\,[k(T)r_b \xi w \beta \frac{1}{r_b}\frac{d\Theta}{d\xi}\,T_b] = h_a\, H\, \xi\, r_b\, \Theta\, T_b$$

which on simplification leads to

$$\frac{d}{d\xi}\,[(1+\alpha\Theta)\xi\beta\frac{d\Theta}{d\xi}] = (\frac{h_a\, r_b^2}{k_a w})H\xi\Theta \tag{8-11}$$

Let v be defined as

$$v = r_b\,(\frac{h_a}{k_a w})^{1/2} \tag{8-12}$$

Therefore,

$$\frac{d}{d\xi}\,[(1+\alpha\Theta)\xi\beta\frac{d\Theta}{d\xi}] = v^2 H\xi\Theta \tag{8-13}$$

The boundary conditions (8-2) in terms of the non-dimensional parameters are

$$r = r_b,\ \xi = 1,\ \Theta = 1 \tag{8-14a}$$

$$r = r_o,\ \xi = b,\ \frac{d\Theta}{d\xi} = 0 \tag{8-14b}$$

$$r = r_o, \quad \xi = b, \quad \theta \text{ must be bounded, if } \beta(b) = 0, \tag{8-14c}$$

where b is a dimensionless length parameter defined as

$$b = r_o/r_b \tag{8-15}$$

The dimensionless volume U is defined as

$$U = \frac{k_a V}{4\pi r_b^4 \; h_a} \tag{8-16}$$

The total given volume is

$$V = 4\pi \int_{r_b}^{r_o} y(r)\; r\; dr = 4\int_1^b \pi w r_b^2 \;\beta\xi d\xi$$

Let g(b) be defined as

$$g(b) = \int_1^b \beta\xi d\xi \tag{8-17}$$

Then,

$$V = 4\pi w r_b^2 g(b)$$

Substituting this expression into (8-16), we have

$$U = \frac{k_a \; 4\pi \; w r_b^2}{4\pi \; r_b^4 \; h_a} \; g(b)$$

$$= \frac{g(b)}{v^2} \tag{8-18}$$

8-4 <u>Heat dissipation</u>

The heat dissipated under steady state conditions is given by (8-4)

$$q = 4\pi r_b \; k(T) \left.\frac{dT}{dr}\right|_{r = r_b}$$

The dimensionless heat dissipation Qh is defined as

$$Qh = \frac{q}{4\pi r_b^2 \; h_a T_b} \qquad (8\text{-}19)$$

Substituting q in terms of α and Θ, equation (8-19) yields

$$Qh = \frac{(1+\alpha)\Theta'(1)}{-v^2} \qquad (8\text{-}20)$$

where $\Theta'(1) = \left.\frac{d\Theta}{d\xi}\right|_{\xi = 1}$

The problem now is to determine the values of parameters v^2 and b, which maximize the heat dissipation Qh for a given dimensionless volume U, under the constraint condition (8-5), i.e.,

$$N_r = \frac{-q}{q_{nf}} > 1, \qquad (8\text{-}21)$$

where N_r is called the heat removal number [34].

In concluding this chapter, it should be mentioned that the main equations we are interested in are (8-13), (8-14), (8-18) and (8-20).

Chapter 9

IMPLEMENTATION OF THE INVARIANT IMBEDDING APPROACH

9-1 General Considerations

The governing differential equation (8-13) is a nonlinear second order equation, which can be converted to two first order equations as follows

$$\frac{dX_1}{d\xi} = \frac{X_2}{(\beta\xi(1+\alpha X_1))} \qquad (9\text{-}1a)$$

$$\frac{dX_2}{d\xi} = v^2 H\xi X_1, \qquad (9\text{-}1b)$$

where

$$X_1 = \Theta \qquad (9\text{-}2a)$$

$$X_2 = (1+\alpha\Theta)\beta\xi\frac{d\Theta}{d\xi} \qquad (9\text{-}2b)$$

The boundary conditions are

$$X_1|_{\xi = 1} = 1 \qquad (9\text{-}3a)$$

$$X_2|_{\xi = b} = 0 \qquad (9\text{-}3b)$$

Before proceeding, we need to note what kinds of functions β, g(b) and K are.

For a trapezoidal profile, the y coordinate can be expressed as

$$y = w_o + (w-w_o)\frac{(r_o-r)}{(r_o-r_b)}$$

$$\beta = \frac{y}{w} = \frac{w_o}{w} + (1-\frac{w_o}{w})\frac{(\frac{r_o}{r_b} - \frac{r}{r_b})}{(\frac{r_o}{r_b} - 1)} \qquad (9.4)$$

We define a slope parameter λ as

$$\lambda = \frac{w_o}{w} \qquad (9-5)$$

Equation (9-4) now becomes

$$\beta = \lambda + (1-\lambda)\frac{(b-\xi)}{(b-1)} \qquad (9-6)$$

Note that when $\lambda = 0$, we have the case of a triangular profile fin, and when $\lambda = 1$, we have the case of a straight fin. For $0 < \lambda < 1$, we have a trapezoidal fin.

From equation (8-7) we know that,

$$\frac{h}{h_a} = H = K\left[\frac{r-r_b}{r_o-r_b}\right]^{-m}$$

$$= K\left[\frac{\xi-1}{b-1}\right]^{m}$$

K can be any arbitrary function of b and m, that is $K = K(b,m)$. In this study $K(b,m)$ has been taken to be

$$K(b,m) = \frac{(b+1)(m+1)(m+2)}{2[(m+1)b+1]} \qquad (9-7)$$

This variation has been taken from the paper by Razelos and Imre [34] and has been used for a qualitative study of the effects of m on the optimum conditions. It should be mentioned that for design purposes, a complete knowledge of the h variation must be observed from the physical process and the flow of coolant around the fin.

Knowing the expression for β equation (9-6), we can obtain g(b) from its definition (8-17)

$$g(b) = \int_1^b \beta\xi d\xi$$

$$= \int_1^b \xi[\lambda + (1-\lambda)\frac{(b-\xi)}{(b-1)}]d\xi$$

$$= \frac{(b-1)}{6}[(1-\lambda)(b+2) + 3\lambda(b+1)] \qquad (9\text{-}8)$$

A new expression for the heat dissipation Qh can be obtained from the new independent variables X_1 and X_2 defined in (9-2). Equation (8-20) defines Qh as

$$Qh = \frac{(1+\alpha)\theta'(1)}{-v^2}$$

From equation (9-2b), at $\xi = 1$,

$$X_2(1) = (1+\alpha\theta(1))\beta\Big|_{\xi=1} \xi\,\theta'(1)$$

$$= (1+\alpha)\theta'(1)$$

Substituting this value into equation (8-20) we get

$$Qh = \frac{-X_{2(1)}}{v^2} \qquad (9.9)$$

Expressions (9.6), (9.7) and (9.8) define β, K, and g(b). Substituting these values into equations (9.1), we obtain

$$\frac{dX_1}{d\xi} = \frac{X_2(b-1)}{\xi(1+\alpha X_1)\;[\lambda(b-1)+(1-\lambda)(b-\xi)]} \tag{9-10a}$$

$$\frac{dX_2}{d\xi} = v^2\left(\frac{(b+1)(m+1)(m+2)}{2[(m+1)b+1]}\right)\xi\; X_1 \tag{9.10b}$$

Further, substituting (9.8) into (8-18), we get

$$v^2 = \frac{(b-1)}{6U}[(1-\lambda)(b+2) + 3\lambda(b+1)] \tag{9-11}$$

Substituting (9-11) into (9-10), we get the final form of the differential equations as follows

$$\frac{dX_1}{d\xi} = \frac{X_2(b-1)}{\xi(1+\alpha X_1)[\lambda(b-1) + (1-\lambda)(b-\xi)]} \tag{9-12a}$$

$$\frac{dX_2}{d\xi} = \frac{(b-1)}{6U}[(1-\lambda)(b+2)+3\lambda(b+1)]\left[\frac{(b+1)(m+1)(m+2)}{2((m+1)b+1)}\left(\frac{\xi-1}{b-1}\right)^m\right]\xi\; X_1 \tag{9.12b}$$

9-2 <u>Application of the Imbedding Approach</u>

Equation (7-20) is rewritten here again, for reference,

$$r_m(c,a) = r_m(c+f(c,r_m(c,a+\Delta),\; a)\Delta,\; a+\Delta)$$

$$- g(c,r_m(c,a+\Delta),\; a)\Delta \tag{7-20}$$

The functions f and g in this case are given in the equations (9-12a) and (9-12b), that is

$$f(X_1,X_2,\xi) = \frac{X_2(b-1)}{\xi(1+\alpha X_1)[\lambda(b-1)+(1-\lambda)(b-\xi)]} \tag{9.13a}$$

$$g(X_1,X_2,\xi) = \left(\frac{b-1}{6U}\right)[(1-\lambda)(b+2)+3\lambda(b+1)]\left[\frac{(b+1)(m+1)(m+2)}{2(m+1)b+1)}\left(\frac{\xi-1}{b-1}\right)^m\right]\xi X_1 \tag{9-13b}$$

The boundary conditions (9-3) will now be converted to a more general form as follows

$$X_1(a) = c \tag{9-14a}$$

$$X_2(b) = 0 \tag{9-14b}$$

Applying equation (7-20) to the system (9-13) and (9-14), we obtain

$$r_m(c,a) = r_m\left(c + \frac{r_m(c,a+\Delta)(b-1)\Delta}{a(1+\alpha c)[\lambda(b-1)+(1-\lambda)(b-a)]},\ a+\Delta\right)$$

$$-\left(\frac{b-1}{6U}\right)[(1-\lambda)(b+2)+3\lambda(b+1)]\left[\frac{(b+1)(m+1)(m+2)}{2((m+1)b+1)}\left(\frac{a-1}{b-1}\right)^m\right]ac\Delta \tag{9-15}$$

To solve equation (9-15) in the backward recursive fashion, we first need an initial condition, which will allow us to start the process. This condition can be written from equation (7-17), and in this case, happens to be

$$r_m(c,b) = X_2(b) = 0 \tag{9-16}$$

for all values of c, at the final point b, when the process has zero duration.

The first step in the recursive process is to obtain $r_m(c,b-\Delta)$, that is when $a = b - \Delta$. From equation (9-15) we obtain

$$r_m(c,b-\Delta) = r_m\left(c + \frac{r_m(c,b)(b-1)\Delta}{(b-\Delta)(1+\alpha c)[\lambda(b-1)+(1-\lambda)\Delta]},\ b\right)$$

$$-\left(\frac{b-1}{6U}\right)[(1-\lambda)(b+2)+3\lambda(b+1)]\left[\frac{(b+1)(m+1)(m+2)}{2((m+1)b+1)}\left(\frac{b-\Delta-1}{b-1}\right)^m\right](b-\Delta)c\Delta \tag{9.17}$$

The first term on the right hand side of (9-17), $r_m(\ldots,b)$ is zero from equation (9-16). Thus $r_m(c,b-\Delta)$ is calcuated for different values of c. In this study, the values of c used were 0, 0.1, 0.2, ... 1.0, that is $\delta = 0.1$. The value of Δ used was $\Delta = 0.01$. All the values of r_m obtained for different values of c, were stored in the computer memory in the form of a table, to be used in the next step of the recursive process.

The next step is to obtain the value of $r_m(c,b-2\Delta)$, that is when $a = b - 2\Delta$. From equation (9-15) we obtain,

$$r_m(c,b-2\Delta) = r_m\left(c + \frac{r_m(c,b-\Delta).(b-1)\Delta}{(b-2\Delta)(1+\alpha c)[\lambda(b-1)+(1-\lambda)2\Delta]},\ b-\Delta\right)$$

$$-\left(\frac{b-1}{6U}\right)[(1-\lambda)(b+2)+3\lambda(b+1)]\left[\frac{(b+1)(m+1)(m+2)}{2((m+1)b+1)}\left(\frac{b-2\Delta-1}{b-1}\right)^m\right](b-2\Delta)c\Delta$$

(9-18)

The first term on the right hand side of (9-18) needs $r_m(\ldots,$ $b-\Delta)$ to be evaluated. This value of $r_m(\ldots,b-\Delta)$ is obtained by interpolation, using the values from the table of $r_m(\ldots,b-\Delta)$, obtained at the end of the previous step. All the values of $r_m(\ldots,b-2\Delta)$ obtained for different values of c are now stored in the computer memory in the form of a table, to be used when calculating $r_m(\ldots,b-3\Delta)$ in the future step.

This recursive process is continued in the backward fashion, reducing the value of a by Δ in each step. This process is stopped when $a = 1$, since we are interested in $r_m(c,1)$. Of all the c values at this point, from the boundary conditions (9-3), we know that the only one we are interested in is $c = 1$. In other words, we are interested in obtaining

$r_m(1,1) = X_2(1)$, which is the missing initial condition.

9.3 Summary of the Optimization Process

All the optimization principles and procedures have been described in the foregoing discussions. But, it would be appropriate to organize all the important features, and present the optimization process in a summarized form.

A value of the dimensionless volume U, equation (8-16) is selected, and the values of the dimensionless variables v^2 (8-12) and b (8-15) which maximize the heat dissipation from the fin Qh (9-9), are calculated. v^2 and b are not independent, but are related by equation (8-18).

The optimization process starts by assuming a value of the dimensionless length parameter b (which is the control variable) and calculating v^2 from equation (8-18), as we already have a known value of U.

By the imbedding approach the missing initial value $X_2(1)$ is calculated, and the objective function Qh evaluated from (9-9). Next the value of b is incremented and Qh calculated again. The current calculated value of Qh is compared to the earlier value of Qh. Initially the value of Qh keeps increasing monotonically, until the optimum value of b is reached. Beyond this optimum value of b, the heat dissipation Qh begins to decrease. In other words, the value of the parameter b is obtained, for which the heat dissipation Qh reaches the maximum value. From the optimum value of b, the optimum value of v^2 is then calculated, and hence the optimum dimensions of the fin for maximum heat dissitation are determined. With some intuition and experience, the value of b close to the optimum can be selected, hence the computation time can be reduced considerably. The computer program developed for this invariant imbedding approach is listed in Appendix II. A flow chart has also been prepared, as a documentation of the procedure used.

9.4 Advantages of the Invariant Imbedding Approach

It may be argued that quasilinearization, which had been used to calculate temperature profiles and heat transfer rates in the first part, could also have been used in the optimization process. But, the invariant imbedding approach has three distinct advantages over quasilinearization, which are described below.

It is believed, that, for this optimization problem, the invariant imbedding approach would require much less computation time, than the time required by the quasilinearization technique. This is because the quasilinearization involves an integration process, and an iteration process till convergence is reached.

The invariant imbedding approach is much more stable in the numerical computation than the quasilinearization technique. This is because quasilinearization involves a numerical intergration, and even a slight error in selecting the appropriate initial conditions could lead to a totally erroneous result.

Finally, the invariant imbedding approach automatically takes care of any possible mathematical singularities that may occur near the tip of the fin, in case of triangular fins. This singularity has been noted in equations (8-2c) and (8-14c). By using the invariant imbedding approach in the backward recursive fashion, no singularity problems were encountered in this study, for the triangular fin ($\lambda = 0$) case.

Chapter 10

RESULTS AND DISCUSSION

10-1 General Consideration

The invariant imbedding computer program was run to obtain the optimum fin dimensions and to study the effects of the three important parameters - thermal conductivity variation parameter α, the index of the heat transfer coefficient m, and the slope parameter of the fin profile λ - on the optimum dimensions.

The value of α was varied between -0.4 and +0.4. Five values of λ, starting at $\lambda = 0$ for a triangular profile to $\lambda = 1$ for a constant thickness fin, were selected. Values of m selected were 0, 0.5, 1.0, 2.0. When α is 0, we have the case of constant thermal conductivity, and when m = 0, we have the case of constant heat transfer coefficient. It must once again be emphasized, that the variation of heat transfer coefficient selected here was only for a qualitative examination of its influence upon the optimum dimensions.

10.2 The Base Case

The program was run first for the case when $\lambda = 0.5$, $\alpha = 0.0$ and m = 0.0. This was the base case and the effects of varying λ, α and m were compared to this base case.

The maximum Qh was obtained for various values of U between 0.01 and 400. The results obtained ($U_{op}^{1/2}$, v_{op}, b_{op},

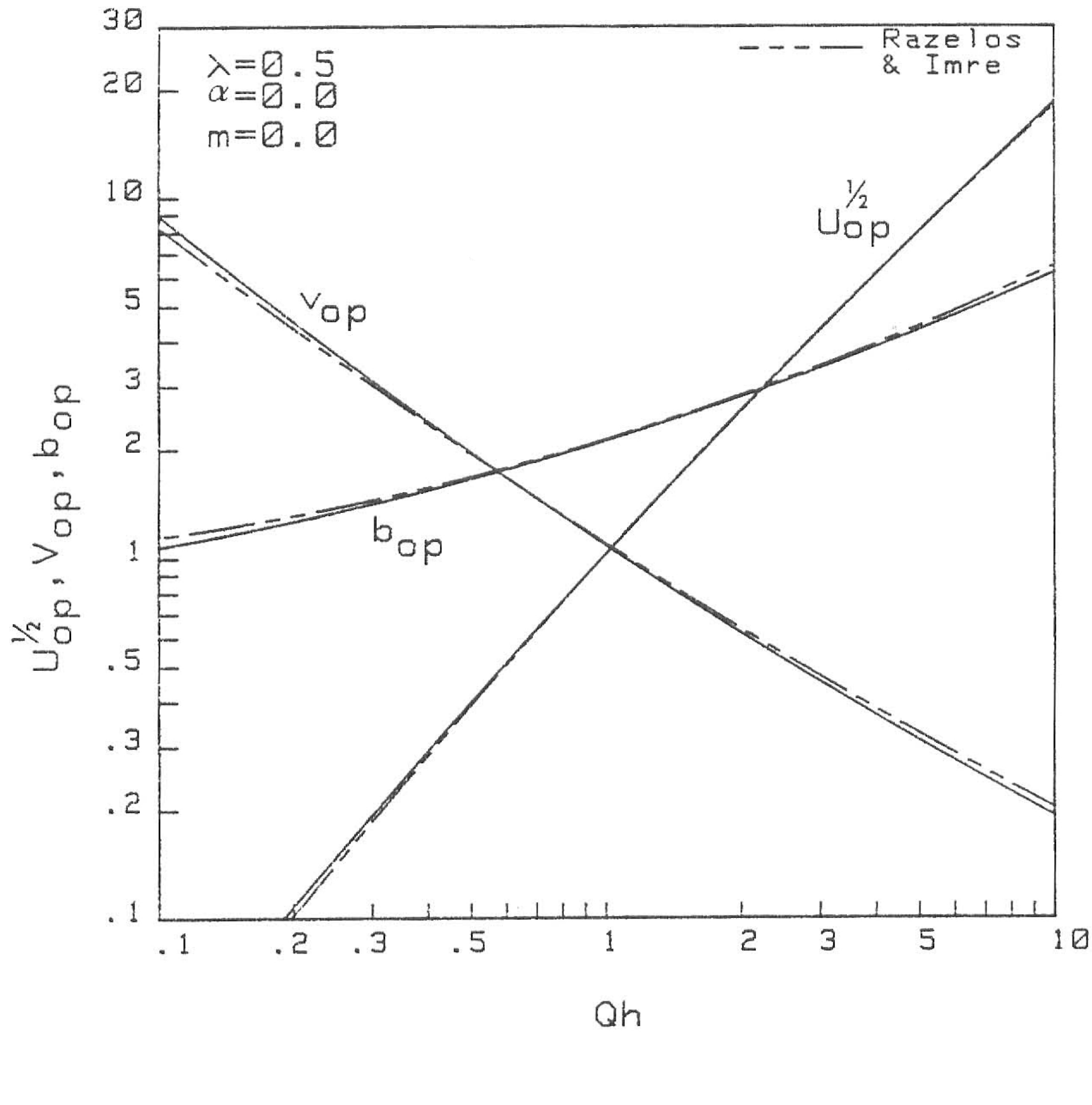

Figure 10.1: Optimum Volume, Optimum Length, Optimum Base Thickness versus Heat Transfer Rate, for the base case.

Qh) have been plotted in Figure 10-1. The dotted line shows the results obtained by Razelos and Imre [34], by a method using a quasi-Newton algorithm. It can be seen that both results agree very well. Qh has been plotted on the horizontal axis allowing for a better interpretation of the results. From Figure 10-1, we can see that as Qh increases, the volume required increases. Also note that v_{op} decreases as Qh increases. A decrease in v_{op} implies an increase in w. This is due to the fact, that a bulk of the heat transfer takes place near the base of the fin. Hence the variation in the width of the fin at the base plays an important role in the heat transfer process. The length also increases as Qh increases. But the change in the magnitude of v_{op} is more significant than the change in b_{op}, for a given change in the heat dissipation Qh.

10-2 <u>Effects of the Slope Parameter λ</u>

Figures 10-2 and 10-3 show the effects of the slope parameter λ on the optimum conditions. In Figure 10-2, (U/U_c) has been plotted versus Qh for different values of λ, where the subscript c denotes the base case and U_c is obtained from Figure 10-1. $\alpha = 0.0$ and $m = 0.0$ were held constant. Hence, for the $\lambda = 0.5$ case, we have a straight line at $U/U_c = 1.0$. Note that as λ decreases, the ratio U/U_c decreases, that is as we go from a constant thickness fin to a triangular fin, less volume is required for the same heat dissipation. This variation can be understood by comparison with the shape of the fin for a minimum volume or weight fin design. The

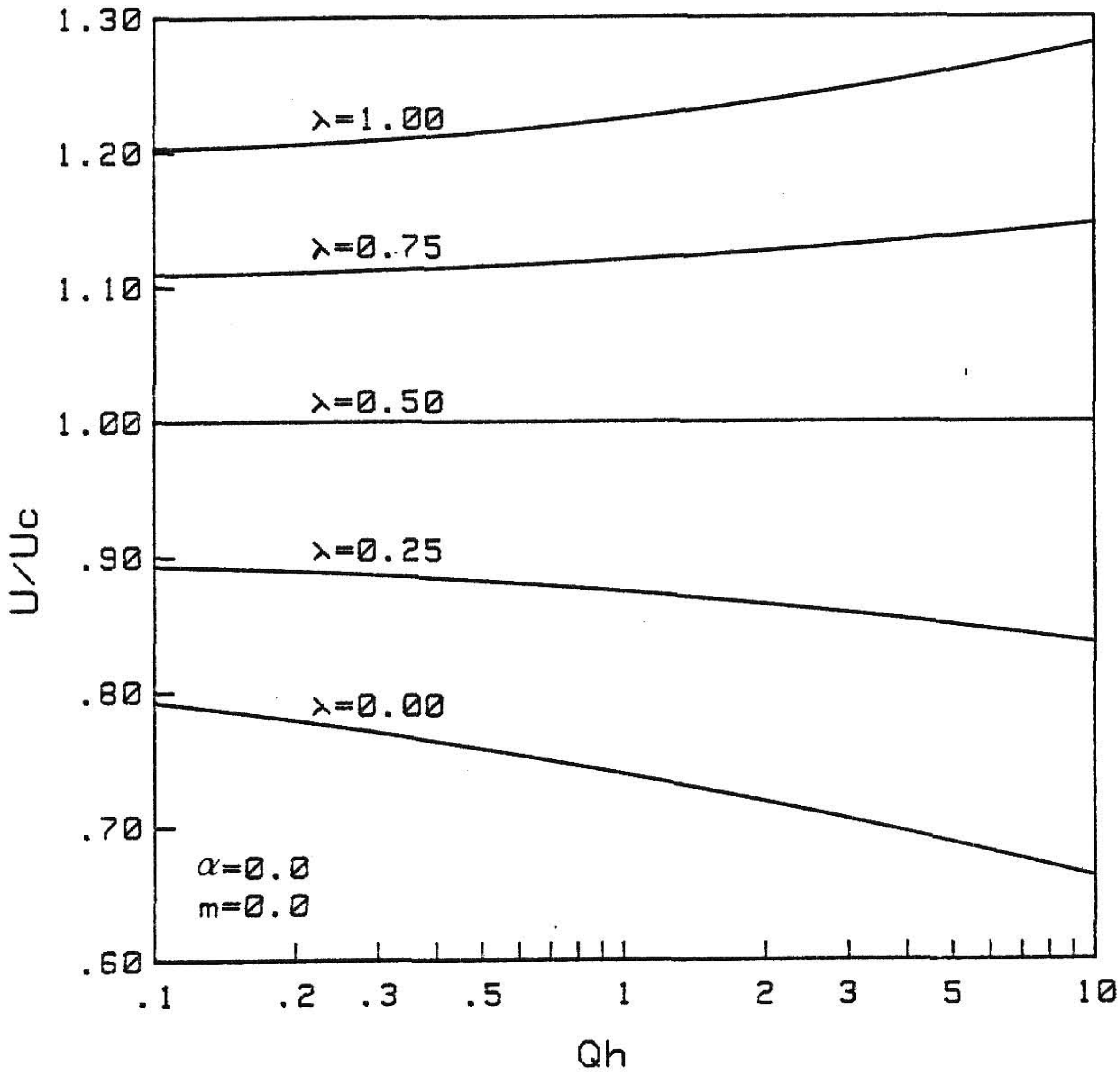

Figure 10.2: Effect of the Slope Parameter on the Optimum Volume.

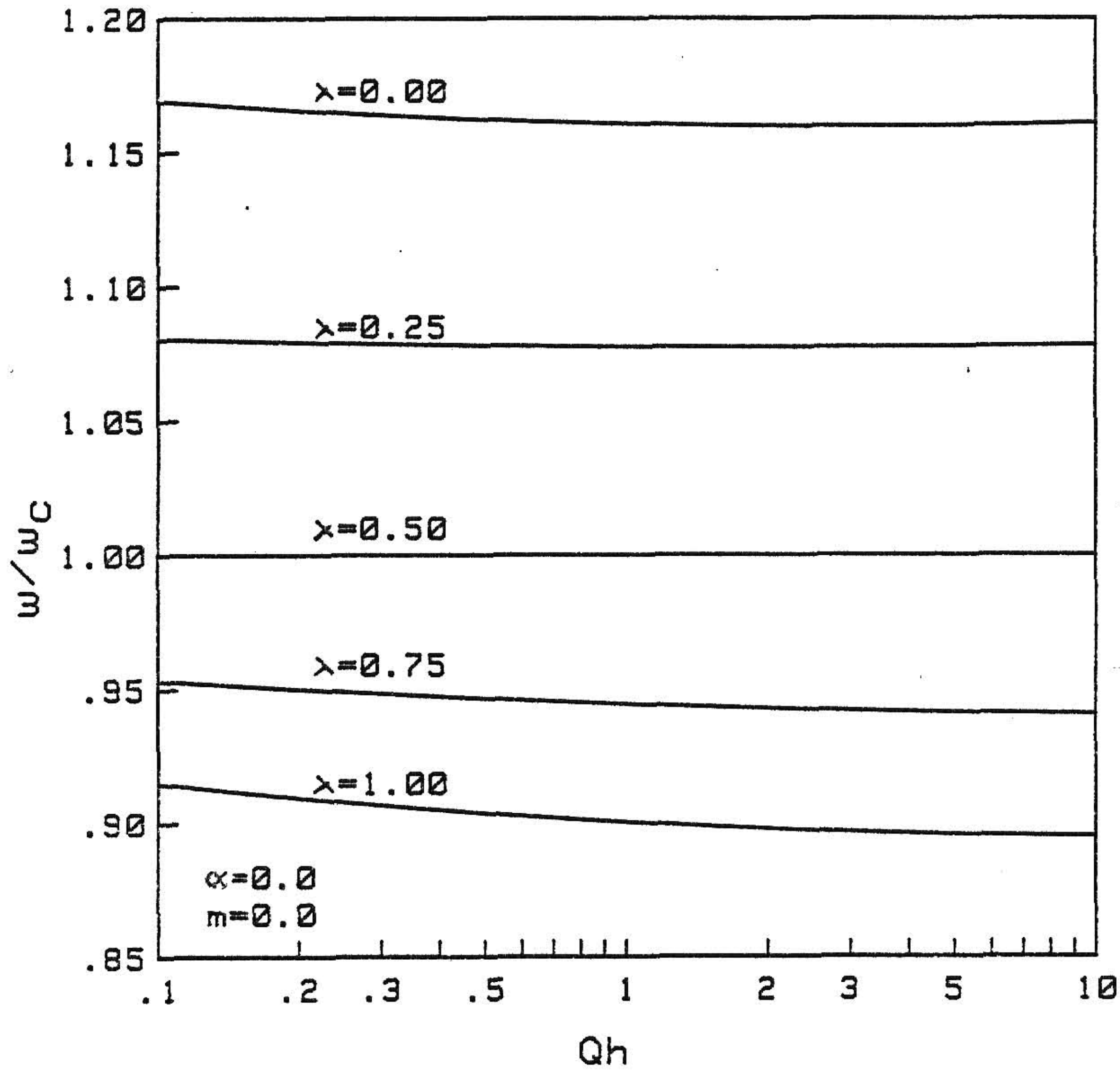

Figure 10.3: Effect of the Slope Parameter on the Optimum Base Thickness.

minimum weight fin has a profile similar to a parabola, with zero width at the tip of the fin, that is $\lambda = 0$. Also note that as Qh increases, for λ less than 0.5, the U/U_c curve has a negative slope, and for λ greater than 0.5, the U/U_c curve has a positive slope. The effects of λ variation are more significant at higher values of Qh.

Figure 10-3 show the variation in the base thickness w/w_c with respect to λ for a given heat dissipation Qh, where w_c denotes the base thickness of the base case, that is when $\lambda = 0.5$. Note that as λ decreases, the base width increases, and is a maximum for the $\lambda = 0.0$ case. As mentioned in the preceding paragraph, the minimum weight fin has a profile similar to a parabola, with a large base width. This is because each of the cross-sections of the fin is equally burdened in the heat conduction process. This explains the reason for the increase in the base width, as λ decreases. Also note that the change in the width, for a given heat dissipation Qh, decreases as λ increases. In the $\lambda = 1.0$ case, w represents not only the width at the base of the fin, but the width everywhere along the length of the constant thickness fin.

10.3 Effects of the Conduction Parameter α

Figure 10-4 illustrates the effects of α on the optimum conditions. U_o and w_o represent the case when $\alpha = 0.0$. In Figure 10-4, U/U_o and w/w_o have been plotted versus the heat dissipation Qh. U_o and w_o have been obtained from Figure 10-1.

It can be seen from Figure 10-4, that as α increases, both curves, for the volume required, U/U_o and the base

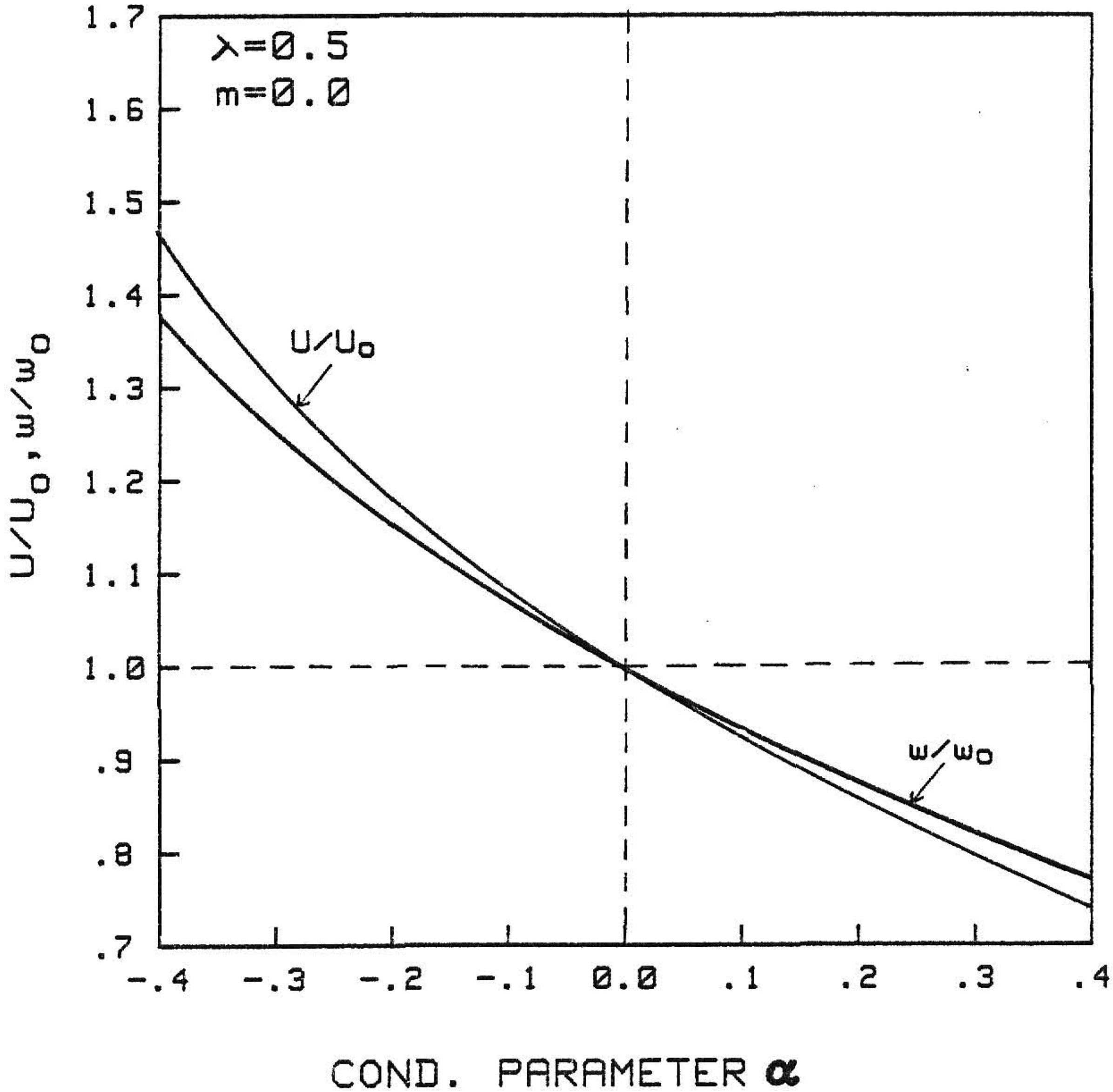

Figure 10.4: Effect of the Conductivity Parameter on the Optimum Volume and Optimum Base Thickness.

thickness of the fin, w/w_o have a negative slope, and these curves pass through the (0,1) point. Note that for a given heat dissipation, we require larger volumes for the case of $\alpha < 0.0$, and smaller volumes for $\alpha > 0.0$. The same also holds good for the base thickness of the fin.

10.4 Effects of the Heat Transfer Coefficient Variation Index m

Figures 10-5 and 10-6 illustrate the effects of the index m on the optimum conditions. To study this effect, α and λ have been kept constant (at $\alpha = 0.0$ and $\lambda = 0.5$). It should be noted that as m increases, the value of the heat transfer coefficient decreases, because the index m is a power of a fraction.

Figure 10-5 shows the effects of m on the volume ratio U/U_o, where U_o is the optimum volume for the m = 0.0 case. U_o is obtained from Figure 10-1. As m increases, for the same heat dissipation, the volume increases, because the heat transfer ceofficient has decreased. Figure 10-6 shows the change in w/w_o, with a change in m where w_o is the optimum base thickness for the m = 0.0 case. w_o is obtained from Figure 10-1. It can be seen that the base thickness ratio increases as m increases, and in the range of values of m studied, the ratio reached values upto 2.5. Therefore, we can conclude that the optimum base width of the fin changes considerably with an increase in m.

From Figures 10-5 and 10-6, it can be observed that the effect of a variable heat transfer coefficient decreases as

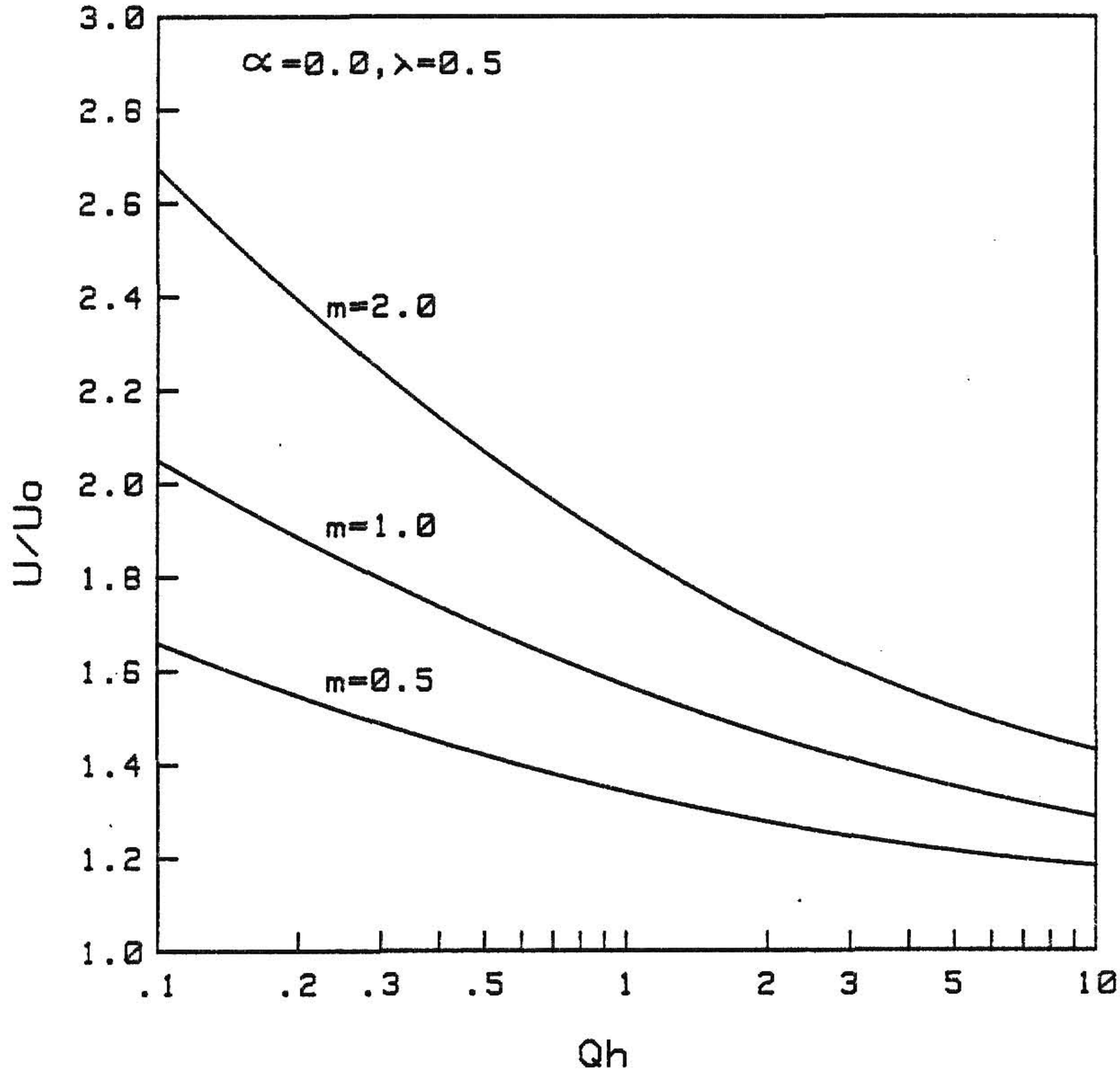

Figure 10.5: Effect of the Heat Transfer Coefficient Variation Index m, on the Optimum Volume.

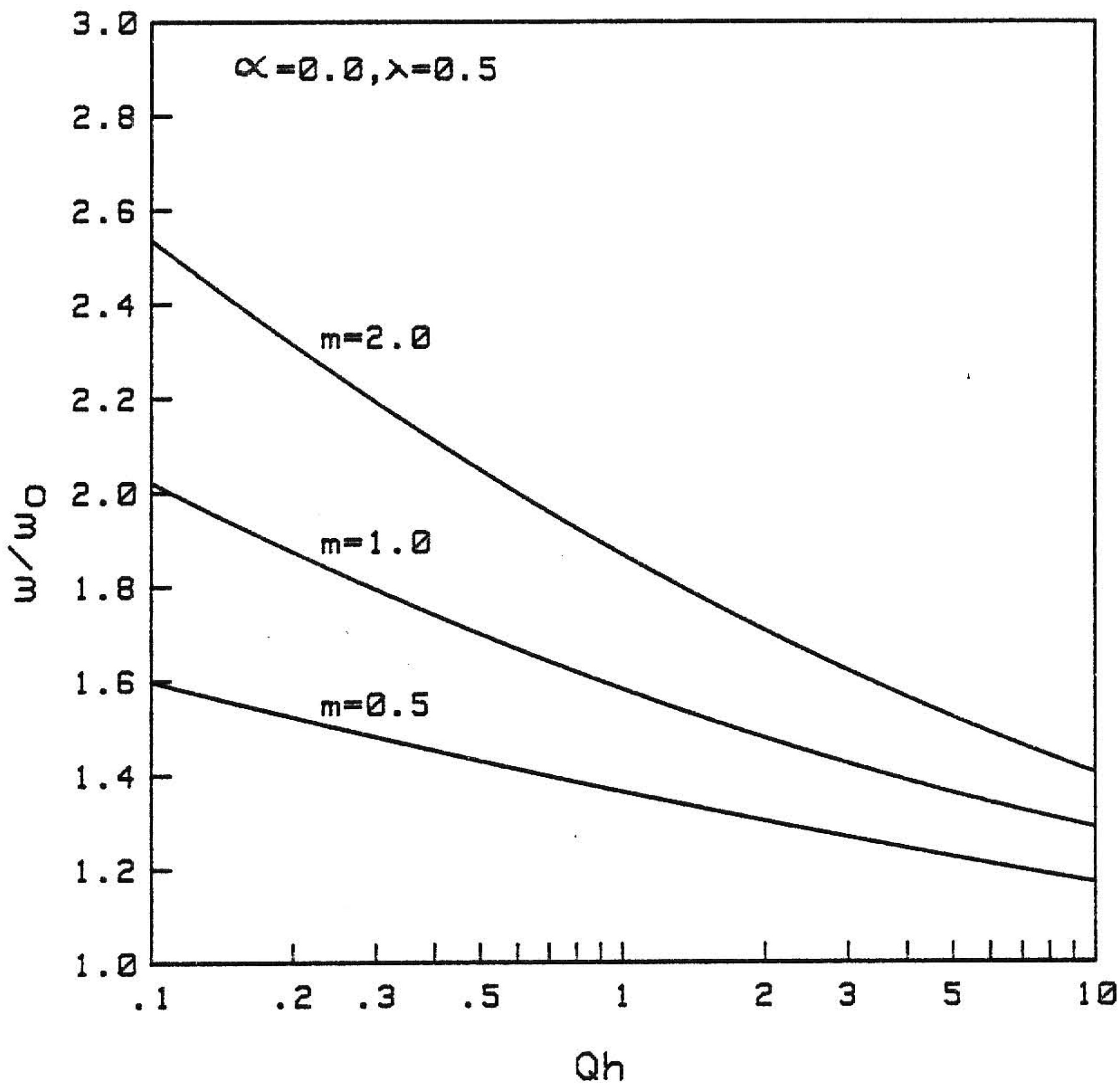

Figure 10.6: Effect of the Heat Transfer Coefficient Variation Index m, on the Optimum Base Thickness.

the heat dissipation increases. At the low values of heat dissipation, a change in the heat transfer coefficient variation causes a considerable change in the optimum conditions. However, at higher heat dissipation values, the effect of the variable heat transfer coefficient diminishes. At the high values of heat dissipation Qh and volume U_o, the ratio U/U_o is smaller than that for the low values of Qh and U_o.

10.5 Variation in the Length Parameter b

It was found that in all the cases considered (the base case, the variable λ cases, the variable α cases and the variable m cases), the length of the fin does not vary by more than 4 percent. Thus the optimum length parameter of the fin is not significantly affected by a change in λ, α, m.

10.6 The Constraint Condition

The constraint condition $q > q_{nf}$, has already been stated in equations (8-5). This condition can be rewritten as

$$N_r = \frac{q}{q_{nf}} > 1 \qquad (10\text{-}1)$$

where N_r has been referred to as the heat removal number. From (8.5b) and (8.19), (10-1) can be expressed as

$$N_r = Qh \cdot \frac{4\pi r_b^2 \, h_a \, T_b}{4\pi r_b \, wh_{nf} \, T_{nf}}$$

This can be simplified to the following form

$$N_r = \frac{Qh \cdot v^2}{B_r} \left(\frac{h_a T_b}{h_{nf} T_{nf}}\right) > 1 \qquad (10\text{-}2)$$

where B_r is a parameter defined as

$$B_r = \frac{h_a r_b}{k_a} \qquad (10-3)$$

The optimum values of v and b must satisfy equation (10-2) to demonstrate the usefulness of the fin. The ratio $(h_a T_b)/(h_{nf} T_{nf})$ in (10-2) needs to be evaluated for this purpose. An assumption could have been made that the temperature of the wall near the base of the fin remains the same both in the presence and the absence of the fin. However, Sparrow, et al. [44, 45], have shown that in the presence of the fin there is a temperature depression near the interface of the fin and the wall. This depression phenomenon of the base temperature can be physically understood. If the presence of the fin augments the heat transfer, then heat must be brought by conduction to the base from more remote regions of the wall, and this conductive transport necessitates the temperature drop, [44]. Physical reasoning suggests that the presence of the fin will act to depress the level of the base temperature, and this needs to be considered in evaluating the usefulness of the fin.

It has been shown [44], that the ratio (T_b/T_{nf}) varies between 0.77 and 0.92, for length to thickness ratios of the fin varying between 2 and 20. An intermediate value of 0.8 is used in this study. The value of 0.8 also assures us of more conservative results.

To evaluate h_{nf}, we need to obtain the heat transfer coefficient for the flow a fluid perpendicular to cylinder.

This can be done using the co-relation in [23]. To do this, we need to know the Reynold's number of the flow around the cylinder, and the values of the radius of the cylinder, and the thermal conductivity. The co-relation is given in terms of the Nusselt number.

It should be recognized that h_a and h_{nf} have different values. However, since we are only interested in proving the inequality (10-2), the ratio (h_a/h_{nf}) has been assumed to be 1 here.

Equation (10-2) is an inequality and the exact value of this ratio is not critical. Hence a value of 1 has been assumed, that is, let

$$N_r = \frac{Qh \cdot v^2}{B_r} (0.8) = 1 \qquad (10\text{-}3)$$

Therefore

$$B_r = Qh \cdot v^2 (0.8) \qquad (10\text{-}4)$$

can be evaluated for a particular value of Qh. B_r is directly dependent on the heat transfer coefficient and the radius r_b, and is inversely proportional to the thermal conductivity. Specific values of these quantities need to be given in any problem. In a more general sense, a plot of Qh versus B_r will give us the limiting values of Qh for any value of B_r. Actually, N_r must be greater than 1. Hence, in Figure 10-7, the usefulness or the operating values of Qh should lie to the left of these curves. It can be seen from Figure 10-7, that as B_r decreases (i.e. the thermal conductivity k_a

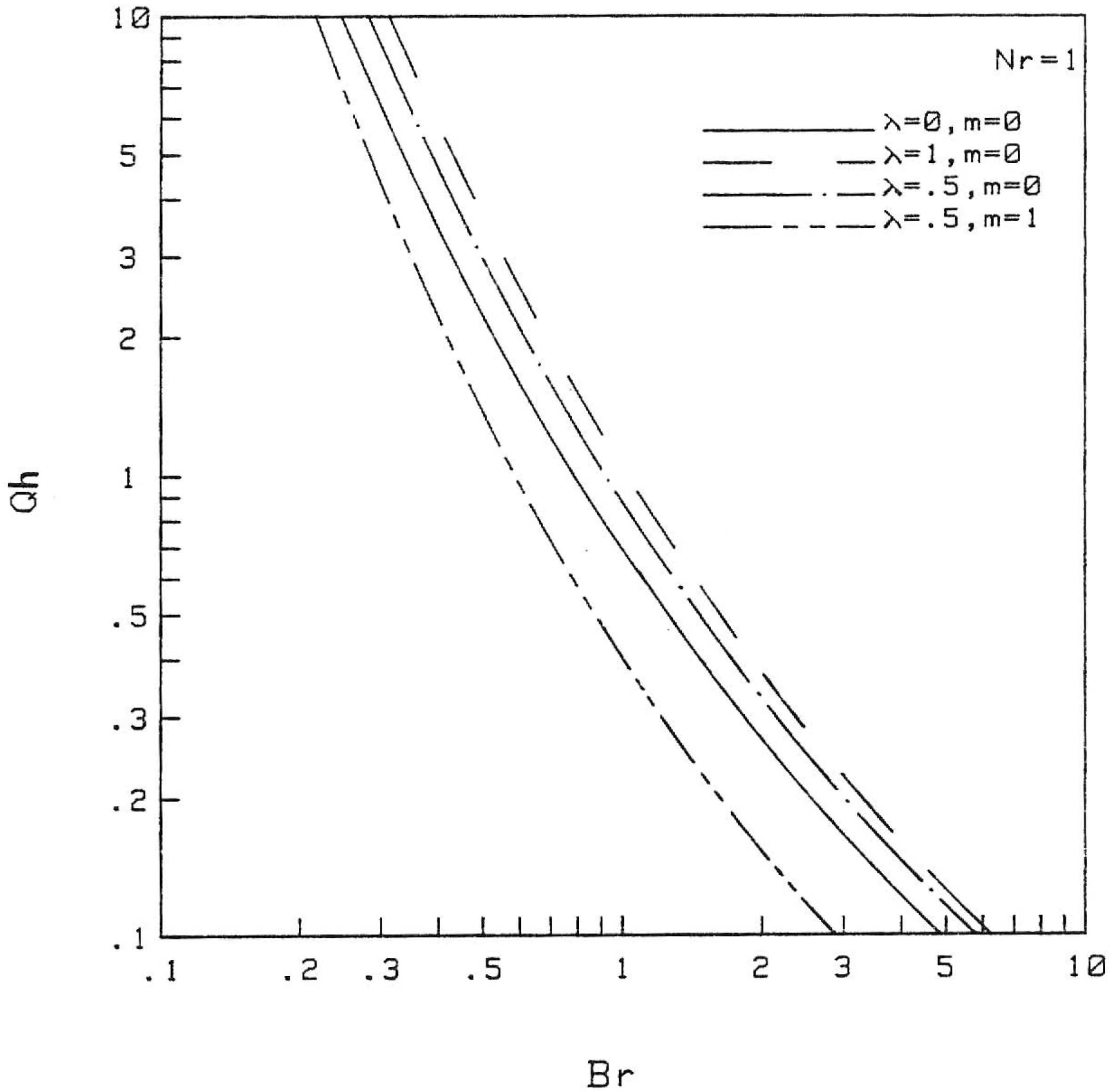

Figure 10.7: Limiting Value of the Heat Transfer Rate versus the Conductivity Related Parameter Br.

increases), the heat dissipated Qh increases. Figure 10-7 has been plotted for the $\lambda = 0.5$ case, for two different values of m, and for the $\lambda = 0.0$ and $\lambda = 1.0$ cases.

10.7 The One-Dimensional Assumption

The justification of the one-dimensional approximation is based on the criterion of a large fin length to thickness ratio. This approach does not take into consideration the ratio of interior (conductive) to exterior (convective) resistances. The Biot number represents this ratio. It has been shown by Irey [46], that only for small Biot numbers, the one-dimensional approximation is satisfactory. Other works [47, 48] have shown analytically that the sole requirement for reducing the exact solution to the one-dimensional approximation is that the Biot number is much less than one.

In equation (10-2), the term $(h_a T_b)/(h_{nf} T_{nf})$ is a fraction, due to the base temperature depression. Substituting the values for B_r and v^2, we have the following inequality

$$N_r < Qh \cdot r_b^2 \frac{h_a}{k_a w} * \frac{k_a}{h_a r_b}$$

This can be simplified to the form

$$N_r < \frac{Qh \cdot v}{B_i^{1/2}} \qquad (10\text{-}5)$$

where B_i is the Biot number defined as follows

$$B_i = \frac{h_a w}{k_a} \qquad (10\text{-}6)$$

The product $(Qh \cdot v)$ is found to have a maximum value of about

2 in all the cases considered. If we choose a value of N_r of about 6, which assures us that the use of fins is economically feasible, then the Biot number has a maximum value of about 0.11. This statement can be seen from equation (10-6). It has been shown in [49] that for Biot numbers of this order, the error in the heat dissipation with a one-dimensional approach is about 1 percent. Hence, the one-dimensional approximation is justified here.

10-8 An Example for Design

Let us consider a circular fin of bore radius 0.05 m, which will dissipate 500w, when the average heat transfer coefficient is $200 w/m^2$-$^o k$, and the temperature difference between the base of the fin and the coolant is $100^o k$. Three different materials are studied, namely copper, aluminum and cast iron. This problem has been abstracted from [34]. The solution to the problem demonstrates the practical usage of the results obtained here. The problem has been studied, for three different cases, a constant thickness fin, a triangular fin, and a trapezoidal fin with $\lambda = 0.5$. Results are presented in Table 10-1.

Note that in all the cases, the maximum value of B_i is less than 0.05, which will justify the assumption of a one-dimensional approximation. Also note that in most cases, the heat removal number for the fin of cast iron material is less than 6. An assumption has been made in section 10.6, that for economical reasons, we need an N_r value of about 6.

TABLE 10-1

$q = 500w$ $r_b = 0.05m$ $h_a = 200\ w/m^2\text{-}k$ $T_b = 100^{o}k$ $Qh = 0.7958$

1. Constant thickness fin. $\lambda = 1.0$

Material	k_a (w/m-k)	Br	v_{op}	$U_{op}^{1/2}$	b_{op}	2w (cm)	V (cm^3)	Bi	Nr
[1.1: m=0.0]									
Copper	386	0.0259	1.3774	0.8361	1.9502	0.1365	28.45	0.0004	46.63
Aluminum	204	0.0490	1.3774	0.8361	1.9502	0.2584	53.83	0.0013	30.81
Cast Iron	52	0.1923	1.3774	0.8361	1.9502	1.0136	211.17	0.0195	6.28
[1.2: m=0.5]									
Copper	386	0.0259	1.1725	0.9804	1.9133	0.1884	39.11	0.0005	33.79
Aluminum	204	0.0490	1.1725	0.9804	1.9133	0.3566	74.01	0.0017	17.86
Cast Iron	52	0.1923	1.1725	0.9804	1.9133	1.3988	290.35	0.0269	4.55
[1.3: m=2.0]									
Copper	386	0.0259	0.9902	1.1637	1.9170	0.2642	55.05	0.0007	24.10
Aluminum	204	0.0490	0.9902	1.1631	1.9170	0.4999	104.17	0.0025	12.74
Cast Iron	52	0.1923	0.9902	1.1731	1.9170	1.9613	408.65	0.0377	3.25

2. Trapezoidal fin $\lambda = 0.5$

Material	k_a (w/m-k)	Br	v_{op}	$U_{op}^{1/2}$	b_{op}	2w (cm)	V (cm^3)	Bi	Nr
[2.1: m=0.0]									
Copper	386	0.0259	1.3067	0.7570	1.9502	0.1517	23.32	0.0004	41.97
Aluminum	204	0.0490	1.3067	0.7570	1.9502	0.2871	44.12	0.0014	22.18
Cast Iron	52	0.1923	1.3067	0.7570	1.9502	1.1263	173.10	0.0217	5.65
[2.2: m=0.5]									
Copper	386	0.0259	1.1144	0.8893	1.9133	0.2086	32.18	0.0005	30.53
Aluminum	204	0.0490	1.1144	0.8893	1.9133	0.3947	60.90	0.0019	16.14
Cast Iron	52	0.1923	1.1144	0.8893	1.9133	1.5485	238.88	0.0298	4.11

Table 10-1, continued

Material	k_a (w/m-k)	B_r	v_{op}	$U_{op}^{1/2}$	b_{op}	2w (cm)	V (cm^3)	Bi	Nr
[2.3: m=2.0]									
Copper	386	0.0259	0.9394	1.0530	1.9170	0.2936	45.12	0.0008	21.69
Aluminum	204	0.0490	0.9394	1.0530	1.9170	0.5555	85.37	0.0027	11.47
Cast Iron	52	0.1923	0.9394	1.0530	1.9170	2.1792	334.95	0.0419	2.92
3. Triangular fin. λ = 0.0									
[3.1: m=0.0]									
Copper	386	0.0259	1.2132	0.6556	1.9502	0.1760	17.49	0.0005	36.18
Aluminum	204	0.0490	1.2132	0.6556	1.9502	0.3330	33.09	0.0016	19.12
Cast Iron	52	0.1923	1.2132	0.6556	1.9502	1.3066	129.84	0.0251	4.87
[3.2: m=0.5]									
Copper	386	0.0259	1.0365	0.7674	1.9133	0.2411	23.96	0.0006	26.41
Aluminum	204	0.0490	1.0365	0.7674	1.9133	0.4563	45.35	0.0022	13.96
Cast Iron	52	0.1923	1.0365	0.7674	1.9133	1.7900	177.89	0.0344	3.56
[3.3: m=2.0]									
Copper	386	0.0259	0.8710	0.9131	1.9170	0.3415	33.93	0.0009	18.65
Aluminum	204	0.0490	0.8710	0.9131	1.9170	0.6462	64.20	0.0032	9.86
Cast Iron	52	0.1923	0.8710	0.9131	1.9170	2.5349	251.86	0.0487	2.51

Cast iron has a very low thermal conductivity as compared to copper and aluminum, and thus gives a low value of N_r.

Observing the cases when m = 0.0 and m = 0.5, we conclude that the base thickness and volume have both increased by about 38 percent. A comparison of the cases when m = 0.0 and m = 2.0 shows that the base thickness and volume increase by about 94 percent. In all the cases mentioned, the optimum length remains practically unchanged. The trapezoidal ($\lambda = 0.5$) requires about 33 percent greater volume than the triangular fin. The constant thickness fin needs a 62 percent higher volume than that of the triangular fin. The triangular fin has the maximum thickness at the base. The trapezoidal ($\lambda = 0.5$) and the constant thickness fin have 14 percent and 22 percent smaller base thicknesses than that of the triangular fin, respectively.

The consideration of different materials reveals that for a given heat dissipation Qh, the optimum base thickness and volume of the fin increase with a decrease in the thermal conductivity of the materials, that is, the optimum base thickness and volume are inversely proportional to the thermal conductivity. However, the length of the fin is independent of the thermal conductivity.

10-9 Future Study

The invariant imbedding approach has been effectively applied to the problem of fin optimization, with variable thermal parameters. The program presented in Appendix II, with minor modifications, can be utilized to solve a family of fin optimization problems.

It is therefore suggested that further research should study the following problems: (1) optimization of fins with the consideration of radiation effects along with the variable thermal parameters, (ii) optimization of fins, whose base temperature is a periodic function of time, (iii) extension of the results of (i) and (ii) to include other heat exchange enhancing devices, namely spines, pin fins, fin arrays on walls and in internally finned tubes.

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APPENDIX I

Computer Program For Quasilinearization

(Used in Part I)

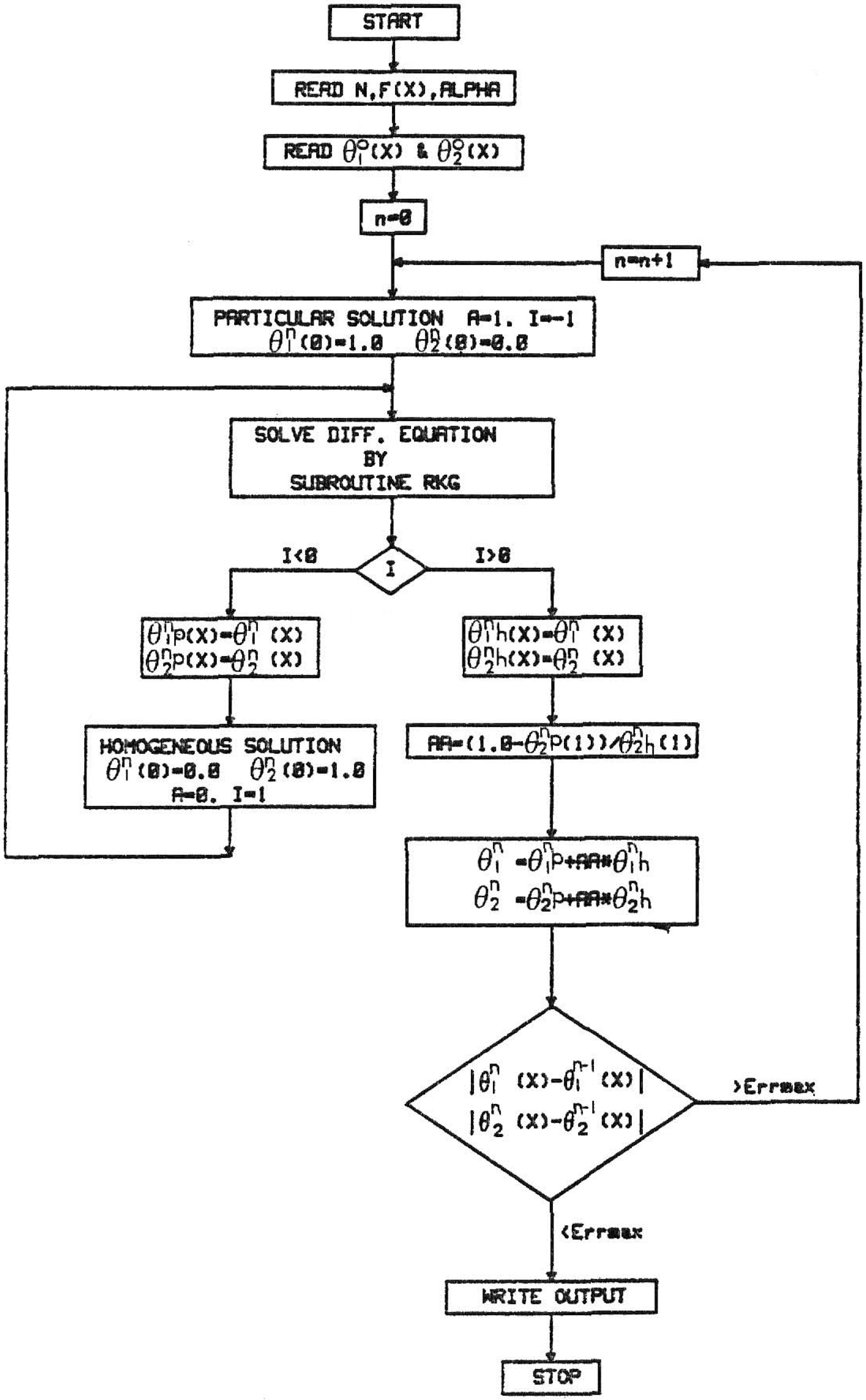

Figure Appendix I: Flow Chart for the Quasilinearization program.

```
C      * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C      ****COMPUTER PROGRAM FOR QUASILINEARIZATION****                *
C      * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *

C      ***THIS PROGRAM COMPUTES THE TEMPERATURES AND THE SLOPE
C      ***OF THE TEMPERATURE PROFILE ALONG THE LENGTH OF THE FIN
C      ***USING QUASILINEARIZATICN. THE VALUES OF HEAT TRANSFER
C      ***FROM THE FIN AND THE EFFICIENCY OF THE FIN ARE ALSO
C      ***EVALUATED.


C      * * * * * * * *
C      * MAIN PROGRAM*
C      * * * * * * * *

       DIMENSION YNIT(2),DYNIT(2),YPNIT1(100),YPNIT2(100),
      XYHNIT1(100),YHNIT2(100),YMIT1(100),YMIT2(100),S1(100),
      XS2(100),ERR1(100),ERR2(100),XK(103),T(103),TS(103),
      XXLIN(103),TLIN(103),ALPH(5),EN(5),DD(5)
       REAL N

C      ****START INPUT FORMATS********************
    98 FORMAT(F10.2)
C      ****END INPUT FORMATS**********************

C      ****START OUTPUT FORMATS*******************
   112 FORMAT('-',9X,'AA=',F10.7)
   114 FORMAT(////)
   116 FORMAT((9X,'HEAT TRANSFER RATE=',F10.6,9X,'EFFICIENCY=',
      XF10.6)////////////)
   296 FORMAT((12X,'XK',16X,'T',17X,'TS')//)
   297 FORMAT(9X,F6.3,9X,F10.5,9X,F10.5)
   299 FORMAT('-',9X,'F(X)=EXP(X)')
C      ***THIS PARTICULAR RUN WAS MADE FOR THE CASE
C      ***F(X)=EXP(X),ALPHA=0.2,D=2.0  THE PROGRAM WAS RUN
C      ***FOR FIVE DIFFERENT VALUES OF N
   300 FORMAT('1')
   301 FORMAT('-',9X,'INTERVAL SIZE=',F6.3,9X,'NUMBER OF STEPS=',
      XI4)
   302 FORMAT('-',9X,'N=',F6.3,9X,'ALPHA=',F6.3,9X,'D=',F6.3)
   304 FORMAT('-',9X,'NUMBER OF ITERATIONS=',I2)
C      ****END OUTPUT FORMATS*********************

C      * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C      *** START PLOTTING SUBPROGRAMS*******                          *
C      * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *

       CALL PLOT (2.0,1.0,23)
       CALL ASPECT (0.75)
       CALL DAXIS (0.0,0.0,8.0,0.0,0.8,-1)
       CALL SAXIS (0.0,0.1,1,1,-1,'RADIAL DISTANCE ALONG FIN',-25)
       CALL DAXIS (0.0,0.0,5.0,90.0,0.5,+1)
```

```
      CALL SAXIS (0.0,0.1,1,3,1,'TEMPERATURE',-11)
      CALL GRID (0.0,0.0,8.0,5.0,1,1)
      CALL SYMBOL (14.0,1.0,0.10,38HTEMPERATURE DISTRIBUTION
     XALONG THE FIN,90.0,38)
      CALL SYMBOL (14.5,1.5,0.1,24HF(X)=EXP(X)     ALPHA=0.2,
     X90.0,24)
C     ****END PLOTTING SUBPROGRAMS **************

C     ***NITMAX SPECIFIES THE MAXIMUM NUMBER OF ITERATIONS
C     ***BEFORE THE ITERATION PROCESS IS TERMINATED
C     ***MIT IS THE PREVIOUS ITERATION NUMBER.
C     ***H IS THE STEP SIZE AND NS THE NUMBER OF STEPS.
C     ***NIT IS THE CURRENT ITERATION NUMBER.

      NITMAX=5
      MIT=0
      H=0.01
      NS=100
      NEQ=2
      READ(5,98) ALPHA
      READ (5,98) D
      DO 500 II=1,5
      READ(5,98) EN(II)
      N=EN(II)
      NIT=1
      DO 120 K=1,NS
      YMIT1(K)=1.0
      YMIT2(K)=0.0
  120 CONTINUE

C     ***THE ARRAY XK STORES THE VALUES OF THE LENGTHS ALONG
C     ***THE FIN, WHERE THE TEMPERATURE AND TEMPERATURE PROFILE
C     ***SLOPE ARE CALCULATED.

      XK(1)=0.0
      DO 211 K=2,101
      XK(K)=(K-1.)/100.
  211 CONTINUE
      WRITE(6,300)
      WRITE(6,301) H,NS
      WRITE(6,299)
      WRITE(6,302) N, ALPHA, D
      GO TO 251
  250 IF(NIT. GT. NITMAX) GO TO 444
      NIT=NIT+1
      MIT=NIT-1

C     ***THE VALUES OF THE TEMPERATURE AND THE THE TEMPERATURE
C     ***PROFILE SLOPE ARE STORED IN ARRAYS S1 AND S2.

      DO 260 K=1,NS
      YMIT1(K)=S1(K)
```

```
      YMIT2(K)=S2(K)
  260 CONTINUE

C     * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C     TO OBTAIN THE PARTICULAR SOLUTION                         *
C     * * * * * * * * * * * * * * * * * * * * * * * * * * * * *

  251 P=1.0
      YNIT(1)=1.0
      YNIT(2)=0.0
      X=0.0
      DO 101 I=1,NEQ
  101 Q(I)=0.0
      K1=1
      DO 200 K=1,NS
      CALL RKG(NEQ,H,X,YNIT,DYNIT,YMIT1,YMIT2,Q,P,N,ALPHA,K1,D)
      K1=K1+1
C     ***THE ARRAYS YPNIT1 AND YPNIT2 STORE THE VALUES FROM
C     ***THE PARTICULAR SOLUTION.
      YPNIT1(K)=YNIT(1)
      YPNIT2(K)=YNIT(2)
  200 CONTINUE

C     * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C     TO OBTAIN THE HOMOGENEOUS SOLUTION                        *
C     * * * * * * * * * * * * * * * * * * * * * * * * * * * * *

      P=0.0
      YNIT(1)=0.0
      YNIT(2)=1.0
      X=0.0
      DO 102 I=1,NEQ
  102 Q(I)=0.0
      K1=1
      DO 206 K=1,NS
      CALL RKG(NEQ,H,X,YNIT,DYNIT,YMIT1,YMIT2,Q,P,N,ALPHA,K1,D)
      K1=K1+1
C     ***THE ARRAYS YHNIT1 AND YHNIT2 STORE THE VALUES FROM
C     ***FROM THE HOMOGENEOUS SOLUTION.
      YHNIT1(K)=YNIT(1)
      YHNIT2(K)=YNIT(2)
  206 CONTINUE

C     ***THE SOLUTION IS IN THE FORM
C     ***               YNIT=YPNIT+AA*YHNIT                       ***
C     ***THE CONSTANT AA IS EVALUATED FROM THE END CONDITIONS

      AA=-YPNIT2(100)/YHNIT2(100)
      DO 208 K=1,NS
      YNIT(1)=YPNIT1(K)+AA*YHNIT1(K)
      YNIT(2)=YPNIT2(K)+AA*YHNIT2(K)
      S1(K)=YNIT(1)
```

```
      S2(K)=YNIT(2)
      T(1)=1.00000
      TS(1)=AA
      L=K+1
      T(L)=YNIT(1)
      TS(L)=YNIT(2)
  208 CONTINUE
C     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C     TEST FOR CONVERGENCE                                        *
C     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
      ERRMAX=0.0001
C     ***ERRMAX SPECIFIES THE MAXIMUM PERMISSIBLE DIFFERENCE
C     ***IN THE VALUES OBTAINED FROM TWO SUCCESSIVE ITERATIONS.
      DO 209 K=1,NS
      ERR1(K)=YMIT1(K)-S1(K)
      ERR2(K)=YMIT2(K)-S2(K)
  209 CONTINUE
      DO 210 K=1,NS
      IF (ABS(ERR1(K)). GT. ERRMAX) GO TO 250
      IF(ABS(ERR2(K)). GT. ERRMAX) GO TO 250
  210 CONTINUE
  444 CONTINUE
C     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C     TO CALCULATE HEAT TRANSFER RATE QH                          *
C     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
      QH=-(1.0+ALPHA*1.00000)*AA
C     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C     TO CALCULATE FIN EFFICIENCY ETA                             *
C     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
      ETA=QH*D/(N*N*(1.0+1.7182818*D))
C     ***PRINT OUT RESULTS
      WRITE(6,114)
      WRITE(6,304) NIT
      WRITE(6,112) AA
      WRITE(6,114)
      WRITE(6,116) QH,ETA
      WRITE(6,300)
      WRITE(6,296)
      DO 213 M=1,101
      WRITE(6,297) XK(M),T(M),TS(M)
  213 CONTINUE
C     ***START PLOT EXECUTION****
      XK(102)=0.0
      XK(103)=0.125
```

```
      T(102)=0.0
      T(103)=0.2
      CALL FLINE (XK,T,101,1,+0,1)
      WRITE(6,300)
  500 CONTINUE
      CALL PLOT (0.0,0.0,999)
C     ***END PLOT EXECUTION*****

      STOP
      END
```

```
C     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
      SUBROUTINE RKG(NEQ,H,X,Y,DY,YM1,YM2,Q,P,N,ALPHA,K1,D)   *
C     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *

C     ***THE INDEPENDENT VARIABLE X IS INCREMENTED IN THIS
C     ***PROGRAM.
C     ***Y(I) AND DY(I) ARE THE DEPENDENT VARIABLE AND ITS
C     ***DERIVATIVE.
C     ***ALL THE Q(I) MUST BE INITIALLY SET TO ZERO IN THE MAIN
C     ***PROGRAM.
C     ***NEQ=NUMBER OF FIRST ORDER EQUATIONS
C     ***H=INTERVAL SIZE
C     ***A SUBROUTINE DERIV (NEQ,X,Y,DY,YM1,YM2,P,N,ALPHA,K1,D)
C     ***MUST BE PROVIDED.
C     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *

      DIMENSION  A(2)
      DIMENSION Y(NEQ),DY(NEQ),YM1(100),YM2(100),Q(NEQ)
      REAL N
      A(1)=0.29289321881345
      A(2)=1.70710678118655
      H2=.5*H
      CALL DERIV(NEQ,X,Y,DY,YM1,YM2,P,N,ALPHA,K1,D)
      DO 13 I=1,NEQ
      B=H2*DY(I)-Q(I)
      Y(I)=Y(I)+B
   13 Q(I)=Q(I)+3.*B-H2*DY(I)
      X=X+H2
      DO 20 J=1,2
      CALL DERIV(NEQ,X,Y,DY,YM1,YM2,P,N,ALPHA,K1,D)
      DO 20 I=1,NEQ
      B=A(J)*(H*DY(I)-Q(I))
      Y(I)=Y(I)+B
   20 Q(I)=Q(I)+3.*B-A(J)*H*DY(I)
      X=X+H2
      CALL DERIV(NEQ,X,Y,DY,YM1,YM2,P,N,ALPHA,K1,D)
      DO 26 I=1,NEQ
      B=0.16666666666666*(H*DY(I)-2.*Q(I))
      Y(I)=Y(I)+B
   26 Q(I)=Q(I)+3.*B-H2*DY(I)

      RETURN
      END
```

```
C     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
      SUBROUTINE DERIV(NEQ,X,YNIT,DYNIT,YMIT1,YMIT2,P,N,                    *
     XALPHA,K1,D)                                                          *
C     * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
      DIMENSION YNIT(NEQ),DYNIT(NEQ),YMIT1(100),YMIT2(100)
      REAL N
      K=K1
      XKK=K/100.
      FUNX=EXP(XKK)

C     ***DERIVATES FOR THE CIRCULAR FIN********

      DYNIT(1)=YNIT(2)
      DYNIT(2)=YNIT(1)*(N*N*FUNX+ALPHA*ALPHA*YMIT2(K)*YMIT2(K)
     X)/((1.0+ALPHA*YMIT1(K))**2.0)
     X-YNIT(2)*2.0*ALPHA*YMIT2(K)/(1.0+ALPHA*YMIT1(K))
     X-YNIT(2)/(XKK+D)
     X+P*(N*N*FUNX*YMIT1(K)-ALPHA*YMIT2(K)*YMIT2(K))/
     X(1.0+ALPHA*YMIT1(K))
     X-P*YMIT1(K)*(N*N*FUNX+ALPHA*ALPHA*YMIT2(K)*YMIT2(K))/
     X((1.0+ALPHA*YMIT1(K))**2.0)
     X+P*YMIT2(K)*2.0*ALPHA*YMIT2(K)/(1.0+ALPHA*YMIT1(K))

      RETURN
      END
```

APPENDIX II

Computer Program For Invariant Imbedding

(Used in Part II)

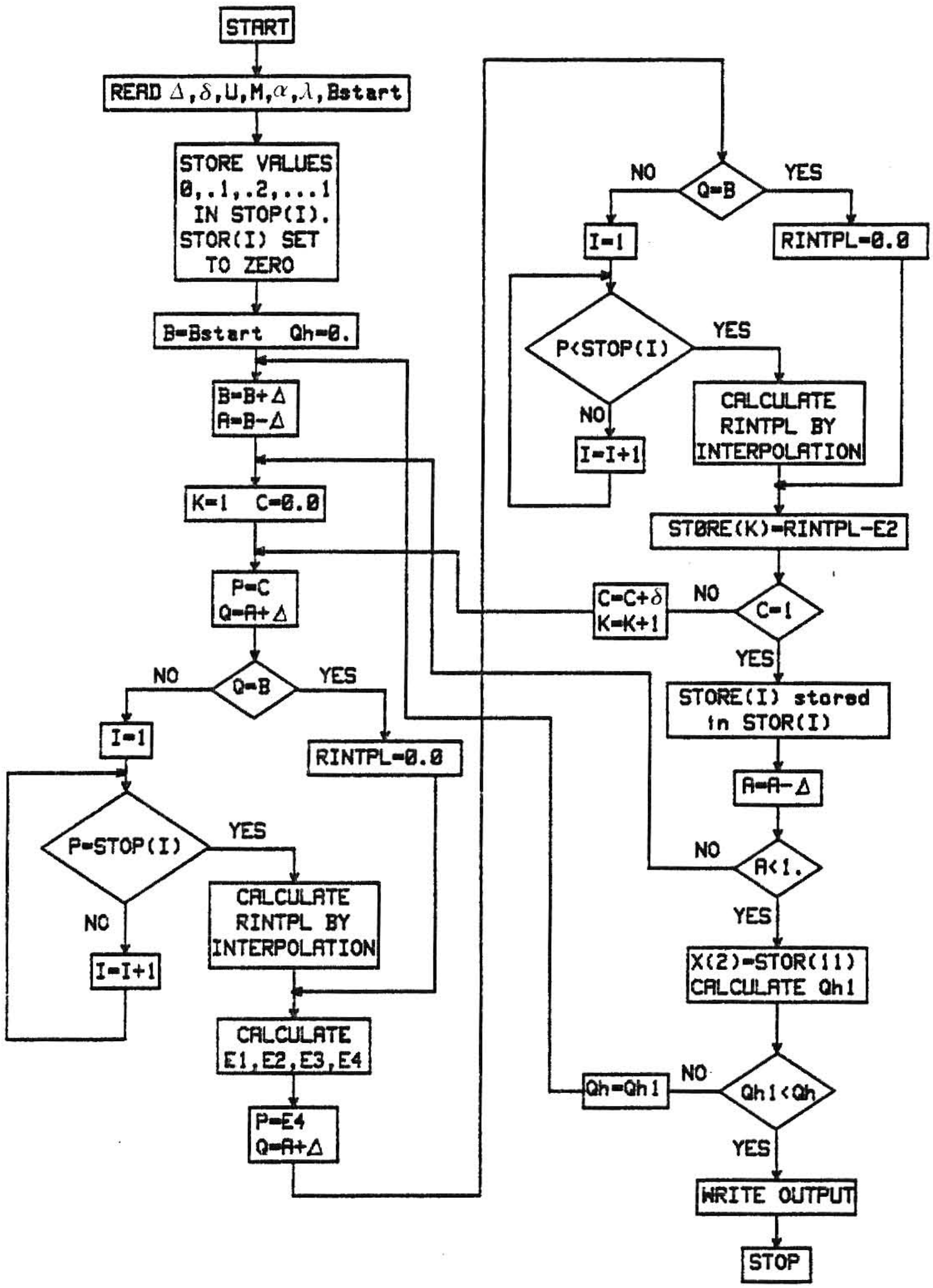

Figure Appendix II: Flow Chart for the Invariant Imbedding program.

```
C      * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *
C      ****COMPUTER PROGRAM FOR INVARIANT IMBEDDING****               *
C      * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * *

C      ****THIS PROGRAM COMPUTES THE OPTIMUM DIMENSIONS OF
C      ****CIRCULAR FINS. THE PROGRAM CALCULATES THE MAXIMUM
C      ****HEAT DISSIPATION FROM A FIN FOR A GIVEN VOLUME,
C      ****BY OPTIMISING THE HEAT TRANSFER WITH RESPECT TO
C      ****THE DIMENSIONLESS LENGTH PARAMETER.


       DIMENSION STOP(11),STOR(11),STORE(11),X(2)
       REAL LAMDA,M

C      ****START INPUT FORMATS**********************
   100 FORMAT(2F10.3)
   101 FORMAT(4F10.3)
   102 FORMAT(F10.3)
C      ****END INPUT FORMATS************************

C      ****START OUTPUT FORMATS*********************
   300 FORMAT('1')
   301 FORMAT(//(9X,'DELTA=',F6.3,9X,'DEL=',F6.3))
   302 FORMAT((9X,'LAMDA=',F6.3,9X,'U=',F6.3,9X,'M=',F6.3,9X,
      X'ALPHA=',F6.3)//)
   303 FORMAT(//(9X,'B=',F6.3))
   304 FORMAT(9X,'VSQ=',F6.3,9X,'X(2)=',F9.6)
   305 FORMAT(9X,'VSQ=',F7.4,9X,'V1=',F7.4,9X,'QH1=',F7.4)
   306 FORMAT(///(9X,'OPT.B=',F6.2,9X,'OPT.V=',F9.4,9X,
      X'OPT.QH=',F9.4))
C      ****END OUTPUT FORMATS***********************

       READ(5,100) DELTA,DEL
       READ(5,101) LAMDA,U,M,ALPHA
       READ(5,102) BSTART

C      ****LAMDA IS THE GIVEN SLOPE OF THE FIN PROFILE,
C      ****U IS THE DIMENSIONLESS VOLUME, M IS THE INDEX OF
C      ****THE HEAT TRANSFER COEFFICIENT VARIATION, ALPHA IS
C      ****THERMAL CONDUCTIVITY VARIATION PARAMETER.
C      ****BSTART IS THE STARTING VALUE OF THE DIMENSIONLESS
C      ****LENGTH PARAMETER B, IN THE OPTIMISATION PROCESS.

       A2=0.0
       DO 10 I=1,11
       STOP(I)=A2*DEL
       A2=A2+1.0
    10 CONTINUE

C      ****THE STOP(I) ARRAY CONTAINS THE ELEMENTS 0.1,0.2,...1.0

       WRITE(6,300)
```

```
      WRITE(6,301) DELTA,DEL
      WRITE(6,302) LAMDA,U,M,ALPHA
      DIFF1=0.0
      DIFF2=10.0
      QH=0.0

      DC 11 I=1,11
      STCR(I)=0.0
   11 CONTINUE

C     ****ALL THE STOR(I) ELEMENTS ARE INITIALLY SET TO ZERO.

      B=BSTART
  160 CCNTINUE
      B=B+DELTA
      WRITE(6,303) B
      A=B-DELTA

C     ****A CONTROLS THE DURATION OF THE PROCESS AND VARIES
C     ****FRCM B TO 1.0

  170 CCNTINUE
      K=1
      C=0.0

C     ****C CAN TAKE VALUES FRCM 0.0 TC 1.0, AND IS USED TO
C     ****SET UP TABLES FOR THE INTERPOLATION ROUTINE.

  180 CCNTINUE

      E1=(B-1.0)*DELTA/(((1.0-LAMDA)*(B-A)+LAMCA*(B-1.0))*A*
     X(1.C+ALPHA*C))
      E2=(B-1.0)*((1.0-LAMDA)*(B+2.0)+3.0*LAMDA*(B+1.0))*
     X((B+1.C)*(M+1.C)*(M+2.0))*A*C*DELTA/(12.0*U*((M+1.0)
     X*B+1.0))*(((A-1.0)/(B-1.C))**M)

C     ****START FIRST INTERPOLATION RCUTINE*******
      P=C
      C=A+DELTA
      IF(ABS(Q-B).LT.0.00001) GC TO 202
      I=1
  200 IF(ABS(P-STOP(I)).LT.0.00001) GC TC 201
      I=I+1
      GC TC 200
  201 RINTPL=STOR(I)
      GC TC 203
  202 RINTPL=0.0
C     ****END FIRST INTERPOLATICN ROUTINE*********

  203 E3=RINTPL
      E4=C+E3*E1
```

```
C     ****START SECOND INTERPCLATICN ROUTINE******
      P=E4
      C=A+DELTA
      IF(ABS(Q-B).LT.0.00001) GC TO 220
      IF((P-1.0).LT.0.0) GO TO 215
      I=11
      II=10
      RINTPL=STOR(I)+(STOR(I)-STOR(II))*(P-STOP(I))
     X/(STOP(I)-STOP(II))
      GC TO 225
  215 I=1
  216 IF(P.LT.STOP(I)) GO TC 217
      I=I+1
      GC TC 216
  217 II=I-1
      RINTPL=STOR(II)+(P-STOP(II))*(STOR(I)-STOR(II))
     X/(STOP(I)-STOP(II))
      GO TC 225
  220 RINTPL=0.0
C     ****END SECOND INTERPOLATION RCUTINE********

  225 E6=RINTPL
      R=E6-E2
      STCRE(K)=R

C     ****ALL THE INTERMEDIATE VALUES OF THE MISSING INITIAL
C     ****CONDITION ARE STCREC IN THE ARRAY STCRE(I) AND
C     ****LATER TRANSFERRED TC THE STCR(I) ARRAY, TO BE USED
C     ****IN THE INTERPCLATICN ROUTINES.

      C=C+DEL
      K=K+1
      IF(C.GT.1.0) GO TO 240
      GC TC 180
  240 DO 15 I=1,11
      STOR(I)=STORE(I)
   15 CCNTINUE
      A=A-DELTA

C     ****THE VALUE OF A IS DECREMENTED TILL A=1.0

      IF(A.LT.1.0) GO TO 250
      GO TC 170
  250 CONTINUE
      X(2)=STOR(11)

C     ****STCR(11) IS THE MISSING INITIAL CONDITION WE ARE
C     ****INTERESTED IN. IT IS THE SLCPE OF THE TEMPERATURE
C     ****PRCFILE AT THE BASE CF THE FIN.

      VSQ=(B-1.0)*((1.0-LAMDA)*(B+2.0)+3.0*LAMDA*(B+1.0))
     X/(6.0*U)
```

```
      V1=SQRT(VSQ)
C     ****V1 CORRESPONDS TC THE DIMENSIONLESS WIDTH AT THE
C     ****FIN BASE.
      QH1=-X(2)/VSQ
      WRITE(6,304) A,X(2)
      WRITE(6,305) VSQ,V1,QH1
      IF(QH1.LT.QH) GO TO 260
C     ****THE ABOVE 'IF' STATEMENT IS THE CPTIMISING CONDITION.
      DIFF1=QH1-QH
      DIFF2=DIFF1
      QH=QH1
      B1=B
      V=V1
      GC TC 160
  260 CONTINUE
      WRITE(6,306) B1,V,QH
C     ****PRINTS OUT THE OPTIMUM CONDITICNS.
      STOP
      END
```

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by

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ABSTRACT

This study presents a numerical solution for the problem of a circular fin, whose thermal conductivity is a linear function of a temperature. The heat transfer coefficient is variable, and different types of variations are considered. The governing differential equation is nonlinear, with one initial and one final condition known.

Part I of this study deals with obtaining the temperature profile along the length of the fin, and qualitatively studying the effects of the conductivity parameter α, the geometric parameter d, the fin parameter N and the type of the heat transfer coefficient variation on the heat transfer rate and the efficiency of the fin. Quasilinearization has been used in this part.

Part II of the study deals with the optimization of the circular fin with variable thermal parameters. The effects of α, the index of the heat transfer coefficient variation m, and the slope parameter λ on the optimum dimensions have been studied. Invariant imbedding has been used in Part II of the study.