THE INFLUENCE OF INLET PRESSURE AND SHAFT SPEED ON END LEAKAGE OF A FULL JOURNAL BEARING

by

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INTRODUCTION

Journal bearings come in different sizes and shapes, but they all basically consist of a circular shaft, or journal, rotating inside a circular bearing. The diameter of a full bearing is slightly greater than that of the shaft. This allows the shaft to rotate in the bearing on a thin oil film while its lateral motion is restrained. When a load is applied to the bearing, the journal moves into an eccentric position relative to the bearing and forms a converging oil film that supports the bearing load.

In order to produce an oil film, a supply of oil must be introduced into the bearing in some manner. Oil flow directly affects the performance of the journal bearing by changing the shape of the oil film which in turn causes changes in the friction torque, and the heat generated in the oil film and the bearing.

The required rate of oil flow is affected by the bearing characteristics, such as length and diameter of the bearing and bearing clearance. Also affecting oil flow are the eccentricity and speed at which the shaft runs, the manner and pressure at which the oil is introduced into the bearing, and the viscosity of the oil.

There have been several studies done in the field of hydrodynamic lubrication as applied to journal bearings.

Several different approaches to the solution of flow rate through a journal bearing have been taken. These solutions appear in different forms and predict different flow rates. Because of this, an experiment was conducted to compare these solutions with experimental results. The influence of inlet pressure and journal velocity on the end leakage of a full journal bearing were investigated, (3) (4) (7).*

* ( ) See Bibliography

THEORETICAL REVIEW

Four works have been selected to compare with the experimental results. The authors of these works are M. Shaw and E. Macks (8), S. Needs (6). J. Boyd and B. Robertson (1), and A. Cameron (2). Their equations for end leakage are presented below, along with initial assumptions used in deriving the equations and the conditions under which the equations are applicable.

Shaw and Macks' equation for the end leakage from a full journal bearing with a single oil inlet hole is

$$Q = \frac{c^3 P_o}{3\mu} \tan^{-1}\left(\frac{2\pi r}{L}\right) (1.0 + 1.5\, n^2)$$

They derived this equation by assuming that the supply pressure has the predominant effect on end leakage (Appendix II). The equation is for a stationary eccentric journal bearing. It was assumed that the effect on flow of the pressure developed in the oil film of an operating journal bearing is neutralized by the small film thickness in the area of the developed pressure. Then the flow through a stationary eccentric journal bearing as calculated by this equation represents to a first approximation the flow expected from an actual bearing operating under load with the same eccentricity.

Needs derived his equation for end leakage from a journal bearing by solving Reynolds' three dimensional differential equation for the pressure distribution in an oil film (Appendix II, equation 24). He did this by using Kingsbury's electrical analogy method which is based on the similarity between Reynolds' equation and the equation for the flow of electric current through an electrolytic bath. This method solves Reynolds' equation for the pressure distribution in the oil film from which the remaining bearing operating characteristics are obtained by graphic integration. This results in the following equation:

$$Q = (K)\ (U)\ (r)\ (m)\ (L) \qquad \text{(eq. 2)}$$

Values for K, the end leakage constant, are tabulated in Table 5 (Appendix III) for various eccentricity and L/D ratios. Needs' work was done with a 120° journal bearing but Fuller (4) suggests that these values can be used for bearings of other arc lengths.

Boyd and Robertson developed the following equation for the flow of lubricant in a concentric, lightly loaded, high speed journal bearing:

$$Q_B = \frac{\pi Po\ C^3}{3\mu(l/D - 2\ \ln_e d/D - 4\sum_{s=1} 1/(s(1+e^{2sl/D}))} \qquad \text{(eq. 3)}$$

The lubricant is supplied under pressure from a single inlet hole. Their derivation of this equation is based on the work done by Musket and Morgan (5), who solved Reynolds' generalized equation by a successive approximation method. This method involves guessing a set of initial values for the unknowns. This set is then used to calculate a new set of values for the unknowns. This new calculated set of unknowns is then used to calculate another new set. This process continues until there is almost no change from one set to the next.

To obtain the flow from an eccentric bearing, the value of Q_B must be modified by a factor R_B. Therefore:

$$Q = Q_B \times R_B$$

$$\text{where } R_B = \frac{(1.0 + n)^3 \; (1/D - 2 \ln_e d/D)}{(1/D - 2 \ln_e d/D + 3n(1/D + 2 - \ln_e 16))} \qquad \text{(eq. 4)}$$

This equation is not intended to be used for values of eccentricity ratio, n, greater than 0.5.

Cameron assumed the inlet pressure and film thickness at the inlet hole were the dominant factors affecting end leakage. The equation derived is

$$Q = ((h_o^3 \, P_o \, d)/(12 \mu L)) \, R_d \qquad \text{(eq. 5)}$$

where h_o is the film thickness at the inlet hole and R_d is a coefficient depending on the geometry of the inlet oil opening. Values for R_d are plotted as functions of the d/L ratio in Figure 10 (Appendix III). For values of d/L less than 0.6, the empirical formula

$$R_d = 1.2 + 11.0 \, d/L \qquad \text{(eq. 6)}$$

gives close approximations for values of R_d.

All of these derived equations are based on constant oil viscosity. This is not true in an actual bearing because there is a viscosity drop due to the heating of the oil as it passes through the bearing. However, Needs showed that the viscosity change in the film can be neglected if the viscosity (μ) is taken as the average film viscosity.

The preceding equations are quite different in form and predict different results for end leakage. As a comparison of these results, Figure 1 shows these predicted flow rates as a function of inlet pressure and Figure 2 shows them as a function of journal velocity. Needs' equation indicates flow rate is independent of inlet pressure, while the other three equations indicate flow is independent of velocity.

These predicted flow rates are also compared with the experimental results in the next section of this paper.

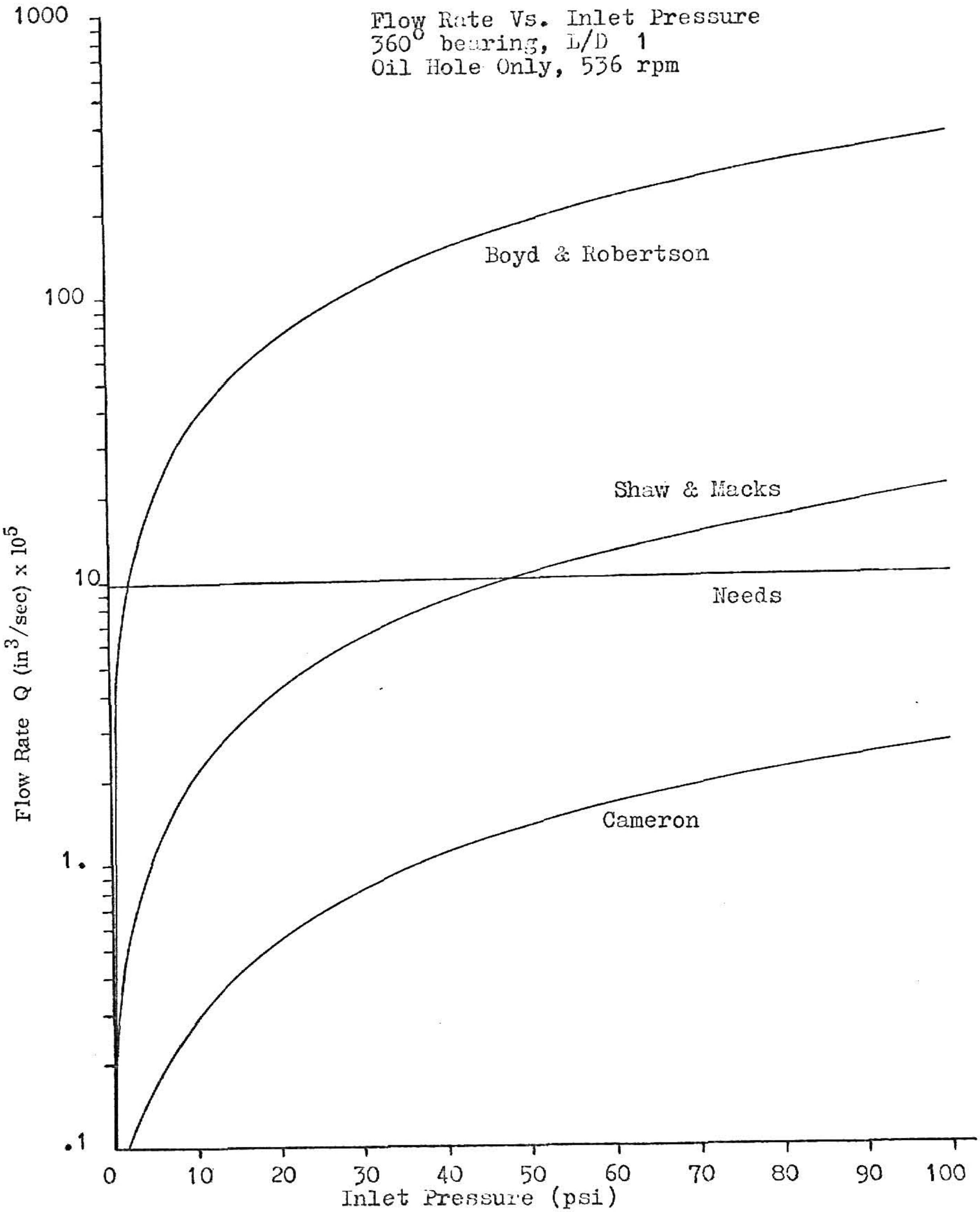

Figure 1. Predicted flow rate vs. inlet pressure

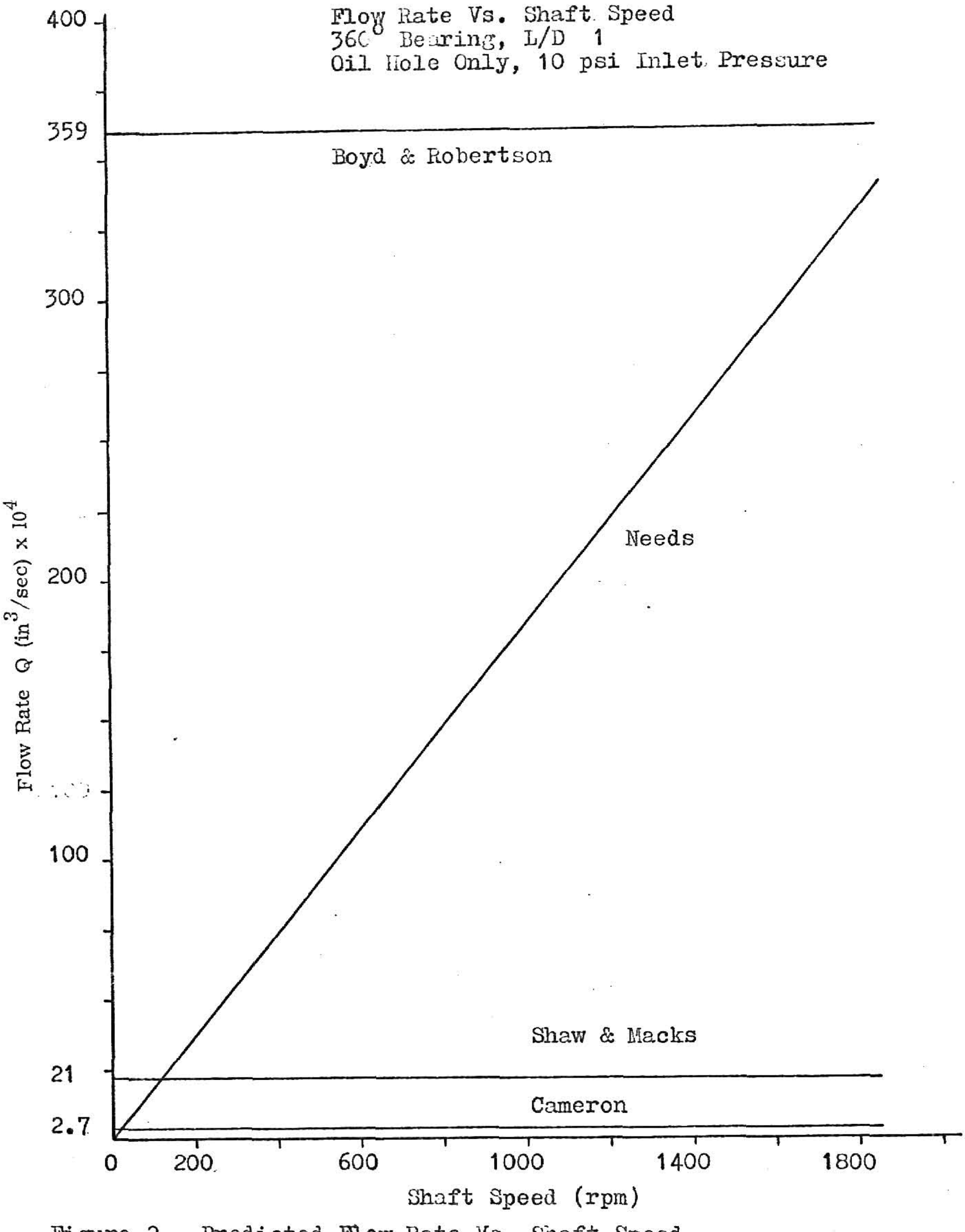

Figure 2. Predicted Flow Rate Vs. Shaft Speed

EXPERIMENTAL INVESTIGATION

Test Equipment:

The test fixture consisted of a shaft and bearing of 1-3/4 inches nominal diameter. In the usual journal bearing, the bearing is fixed and the shaft rotates inside it with eccentricity depending on the load. In this fixture the shaft position was fixed and the eccentricity was set by having the bearing movable. The eccentricity between the shaft and the bearing was measured by two pairs of dial indicators with 0.0001 inch divisions. One pair of indicators was positioned at each end of the bearing with the two indicators in each pair placed 90° apart around the bearing. Various shaft speeds were obtained by changing pulley diameters. Different oil inlet pressures were obtained by regulating the air pressure over the oil in the oil supply tank. Drawings and specifications for the test equipment are in Appendix IV.

Test Procedure:

The oil used in the test was Mobil S.A.E. 10W non-detergent. It's viscosity was measured with a Saybolt Universal Viscometer at 78° F and was found to be 6.47×10^{-6} reyns. The oil inlet pressure was measured by a 0 - 100 psi pressure gauge near the bearing inlet. Shaft speed was measured or checked during each test with a stroblight. The oil flow was determined by measuring the volume of oil required to refill the oil supply tank to a specified depth after a 15 minute test run. Two tests were run at each pressure and shaft speed to check the reproducibility of the results.

The experimental data recorded during the tests are tabulated below:

Table 1. 15 minute test for 360° bearing at 536 rpm.

Test	Run	Inlet oil pressure (psi)	Volume flow (in^3)	Flow rate (in^3/sec)
1	1	10	7.35	0.0082
	2	10	7.02	0.0078
2	1	20	10.56	0.0117
	2	20	10.40	0.0116
3	1	30	13.27	0.0147
	2	30	13.61	0.0151
4	1	40	16.66	0.0185
	2	40	17.09	0.0190
5	1	50	19.16	0.0213
	2	50	19.13	0.0213
6	1	60	23.13	0.0257
	2	60	23.55	0.0262
7	1	70	26.88	0.0298
	2	70	26.82	0.0297
8	1	80	36.98	0.0411
	2	80	36.73	0.0408

Table 2. 15 minute test for 360° bearing and 10 psi inlet pressure.

Test	Run	Shaft speed (rpm)	Volume flow (in^3)	Flow rate (in^3/sec)
1	1	0	4.70	0.0052
	2	0	4.76	0.0053
2	1	536	7.35	0.0082
	2	536	7.02	0.0078
3	1	1232	17.82	0.0198
	2	1232	17.51	0.0195
4	1	1810	21.78	0.0242
	2	1810	22.39	0.0249

DISCUSSION AND CONCLUSIONS

Figures 3 and 4 compare the experimentally obtained flow rates with those calculated from the equations. These curves were plotted from the data tabulated in Tables 3 and 4 of Appendix III.

The experimental curve of Figure 3 shows that the end leakage flow rate increases with increasing inlet pressure. The flow rate increases from 0.008 in^3/sec at 10 psi to 0.0410 in^3/sec at 80 psi inlet pressure. The dashed curve in Figure 3 is a linear estimate of the flow rate as a function of inlet pressure (9). It was obtained by applying a linear regression analysis to the experimental data (Appendix III). This resulted in the following equation for the flow rate as a function of inlet pressure:

$$Q = 0.001476 + 0.000441\ P_0 \qquad \text{(eq 7)}$$

This gives a fairly good approximation of the experimentally determined relationship between flow rate and inlet pressure for the range of pressures tested.

The experimental curve of Figure 4 shows that the end leakage flow rate also increases with increasing shaft speed. The flow rate increased 1.51 times from 0 to 536 rpm, 2.46 times from 536 to 1232 rpm, and 1.25 times from 1232 to 1810 rpm. Overall the flow rate increased 4.64 times when the shaft speed was increased from 0 to 1810 rpm. The

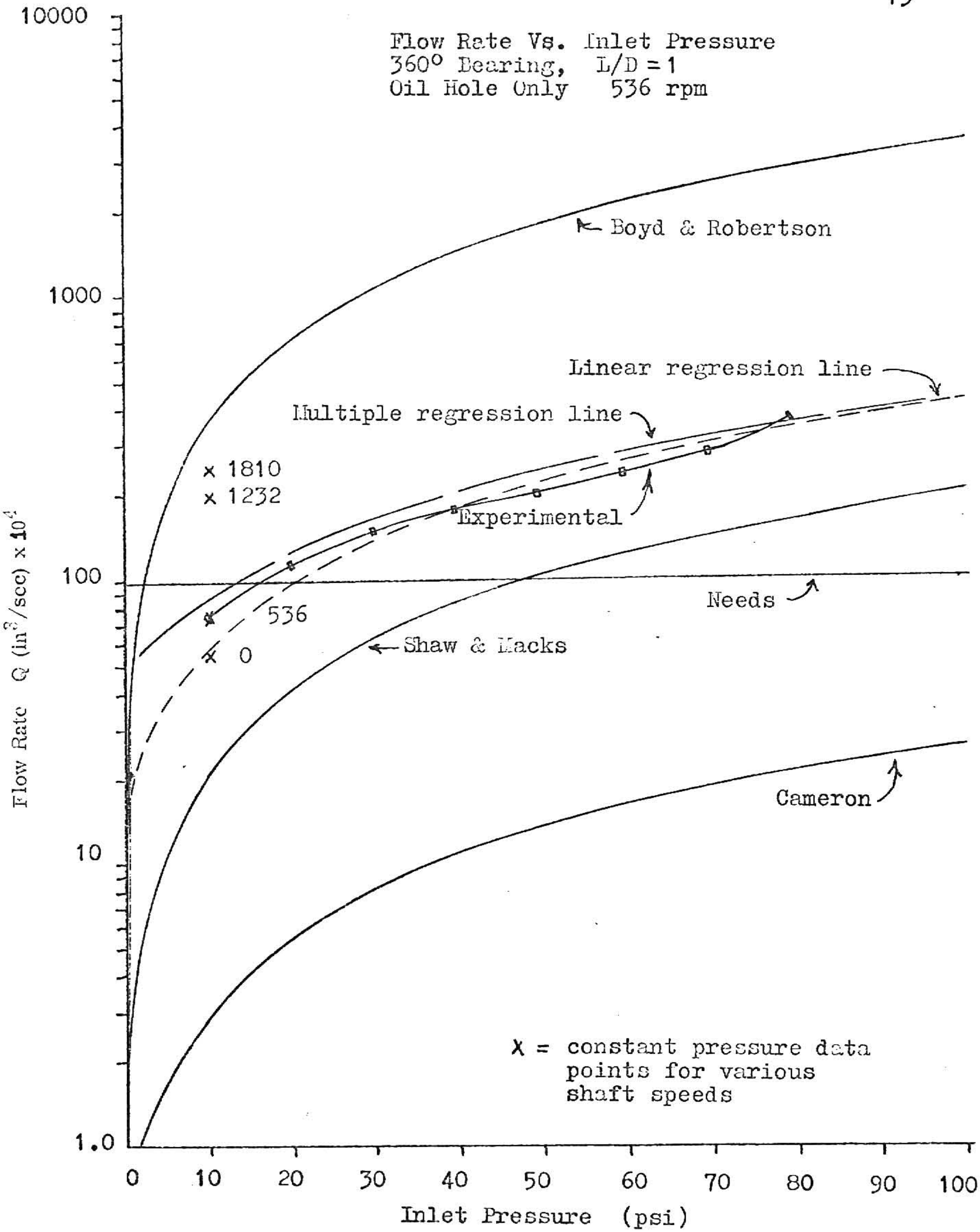

Figure 3. Flow Rate Vs. Inlet Pressure

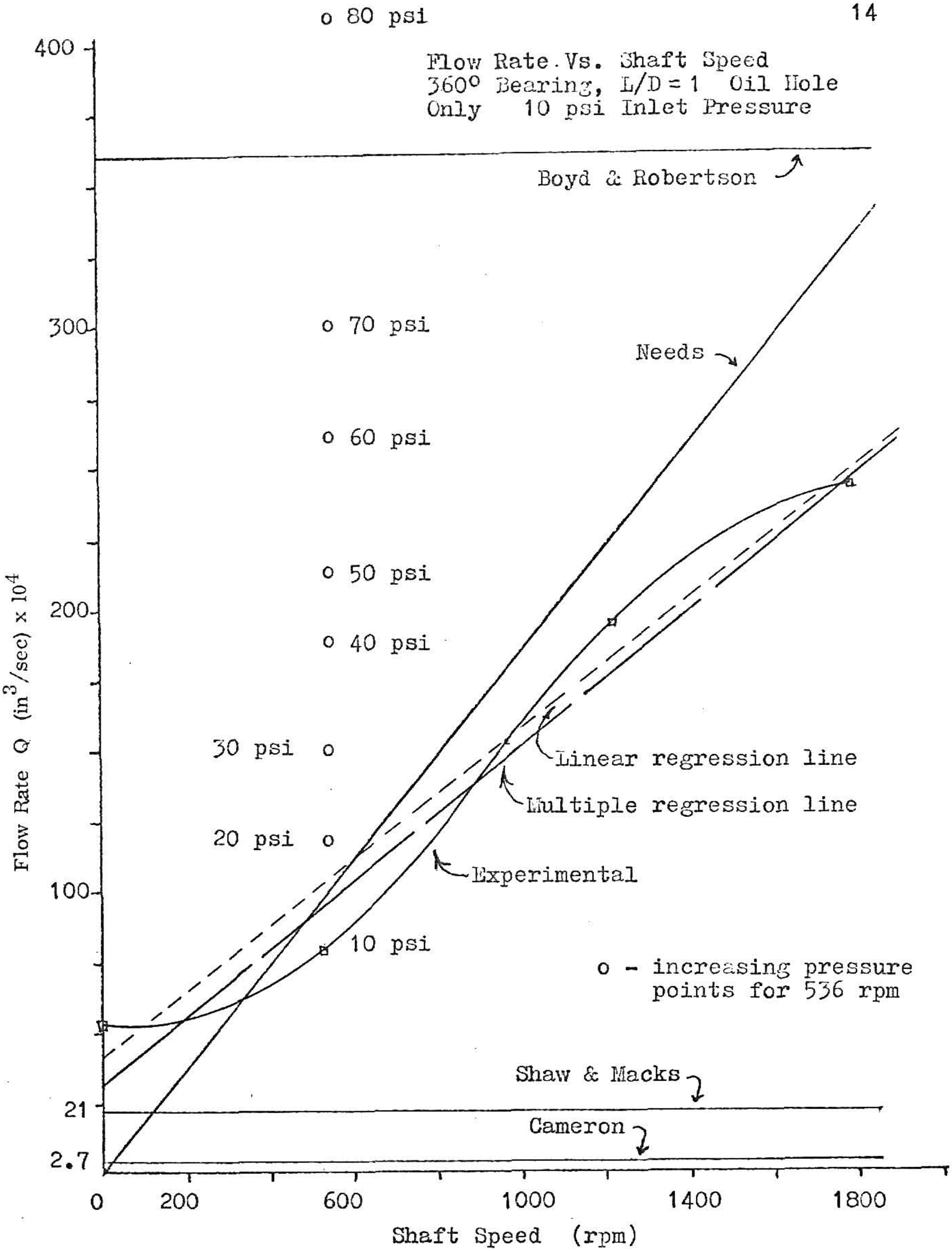

Figure 4. Flow rate vs. shaft speed

dashed line in Figure 4 is a linear estimate of the flow rate as a function of shaft speed. It was obtained in the same manner as the dashed line in Figure 3. The equation for this line is:

$$Q = 0.0042 + 0.0000114\ N \qquad \text{(eq. 8)}$$

where N is the shaft speed in rpm. This line deviates some from the experimental results but it shows well the linear trend of increasing shaft speed in the range tested. A closer fit could probably be obtained with more data points.

If the data in Tables 3 and 4 are combined, giving a table of flow rates as functions of shaft speed and inlet pressure, then a multiple regression analysis can be applied to the data (9). This gives an equation for the flow rate as a function of both the shaft speed and the inlet pressure. This analysis resulted in the following equation:

$$Q = -0.0016 + 1.23 \times 10^{-5}\ N + 3.8 \times 10^{-4} P \qquad \text{(eq 9)}$$

This equation applies for a 360° bearing with a $L/D = 1.0$, an eccentricity $= 0.23$, and a fluid viscosity as used in this experiment.

The theoretical results on the basis of Shaw and Macks' equation predict a linear increase in the flow rate with. Compared to the experimental results, their predicted values an increase in the inlet pressure. Compared to the experimental results, their predicted values are 2 to 3 times lower than those found experimentally. When the shaft speed is increased, this difference becomes even larger, which is understandable, since Shaw and Macks' equation was developed for a bearing with no shaft velocity.

Needs' theoretical equation is independent of inlet pressure and therefore predicts a constant value of end leakage for all values of inlet pressure. Compared to the experimental results, Needs' predicted flow is too high for low inlet pressures and too low for high inlet pressures, with agreement with the experimental results at about 15 psi. Needs' equation is the only one that predicts an increasing flow rate with increasing shaft speed. His predicted flow rates for increasing shaft speeds follow the experimental results fairly closely at the 10 - 20 psi level. But, as shown in Figure 4 by the "o's", which indicate flow rates for various inlet pressures for a constant rpm, Needs' results become increasingly less accurate for increasing inlet pressures.

Boyd and Robertson's theoretical equation predicted the largest flow rates. For the shaft speed of 536 rpm, their predicted values ranged from 4.5 to 8.5 times higher than the experimental values, but dropped to 1.5 times higher at a shaft speed of 1810 rpm. It can be assumed that the increase in flow rate with increase in shaft speed occurs at every inlet pressure, as indicated by the multiple regression equation and, also, the "x's" on Figure 3. Thus, Boyd and Robertson's equation gives acceptably predicted flow rates for the higher shaft speeds and lower inlet pressures.

Cameron's theoretical equation predicts the lowest flow rates. His predicted values range from 16 to 30 times less than the experimental values for low shaft speeds and up to 90 times lower for the higher shaft speeds. From this, it may be concluded that Cameron's equation is of no practical use in predicting flow rates.

It can also be concluded that for bearings with a L/D ratio equal to 1 and a low eccentricity ratio:

1. Shaw and Macks' equation would give a conservative estimate of flow rate for low shaft speeds;
2. Boyd and Robertson's equation would give satisfactory results for high shaft speeds;
3. Needs' equation would give approximate results for low inlet pressures and all shaft speeds.

SUMMARY AND RECOMMENDATIONS

An experimental investigation of the effect of inlet pressure and shaft speed on end leakage through a full journal bearing was conducted to provide data for comparison with analytical solutions. A journal bearing of 1-3/4 in. nominal diameter and length with 0.0029 in. diametral clearance was tested. The inlet pressure was varied from 10 to 80 psi with the shaft speed set at 536 rpm. The shaft speed was varied from 0.0 to 1810 rpm with the pressure held at 10 psi.

The experimental results were compared with the four following theoretical predictions:

1. Shaw and Macks' analysis of a stationary eccentric bearing where the supply pressure has the predominant effect on end leakage.
2. Needs' equation based on a solution of Reynolds' equation by means of an electrical analogy.
3. Boyd and Robertson's equation based on the work done by Muskat and Morgan who solved Reynolds' equation by a successive approximation method.
4. Cameron's equation where the inlet pressure and the local film thickness have the dominant effect on end leakage.

Comparing the theoretical predictions with the experimental results shows Shaw and Macks', Needs', and Boyd and Robertson's equations give approximate solutions to the flow rates. These solutions are acceptable only in certain inlet pressure and speed ranges.

This experimental investigation was limited in scope and certainly did not cover all the possibilities available for investigation. It is therefore recommended that the following areas be considered for further investigation:

1. For constant speed, inlet pressure, and eccentricity ratio, the effect of L/D ratio on end leakage should be studied.

2. For various L/D ratios, the effect of different shaft speeds, inlet pressures, and eccentricity ratios on end leakage should be investigated.

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APPENDICES

APPENDIX I

Nomenclature

C - diametral clearance in inches

c - radial clearance in inches

D - bearing diameter in inches

d - oil inlet hole diameter in inches

e - bearing eccentricity in inches

h, h_o - oil film thickness in inches (subscript denotes position)

K - Needs' end leakage constant

L - bearing length in inches

l - one-half bearing length in inches

m - clearance modulus (ratio of radial clearance to journal radius)

N - shaft speed in rpm

n - eccentricity ratio

P_o - oil inlet pressure in psi

Q - flow rate of oil in cubic inches per second

Q_B - flow rate with no eccentricity in cubic inches per second as used in Boyd and Robertson's equation

R_d - flow coefficient in Cameron's equation

r - shaft radius in inches

R_B - correction factor of eccentricity in Boyd and Robertson's equation

s - an integer with values 1, 2, 3, ...

U - shaft peripheral speed in inches per second

u - velocity of lubricant in x direction

v - velocity of lubricant in y direction w - velocity of lubricant in z direction

x - coordinate along the circumference of the developed surface of bearing or journal

y - coordinate along the normal to the developed surface

z - coordinate along the axis of the bearing

μ - average viscosity in reyn

τ - shear stress (subscript denotes position)

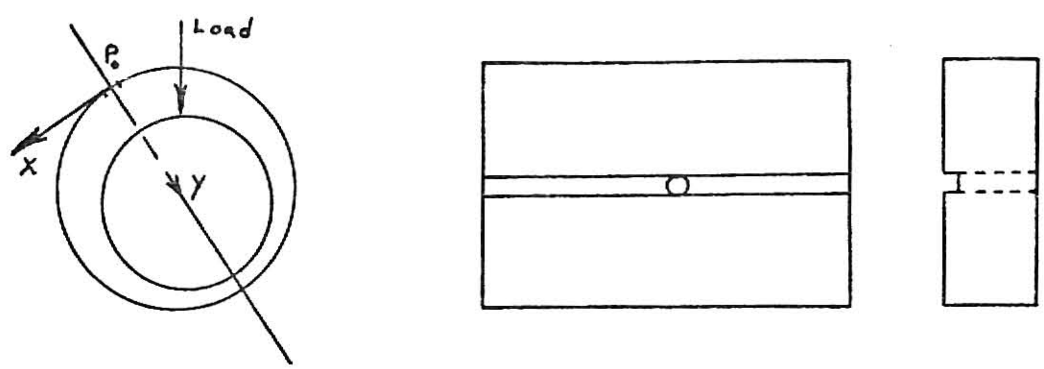
APPENDIX II

Derivations

Shaw and Macks' equation:

Shaw and Macks derived an expression for the rate of flow from the ends of a stationary journal bearing having a central oil supply groove. From this calculation they predicted the flow rate from a bearing lubricated by a single inlet hole.

Figure 5 (a) shows an eccentric journal bearing and Figure 5 (b) shows a developed view of the bearing surface.

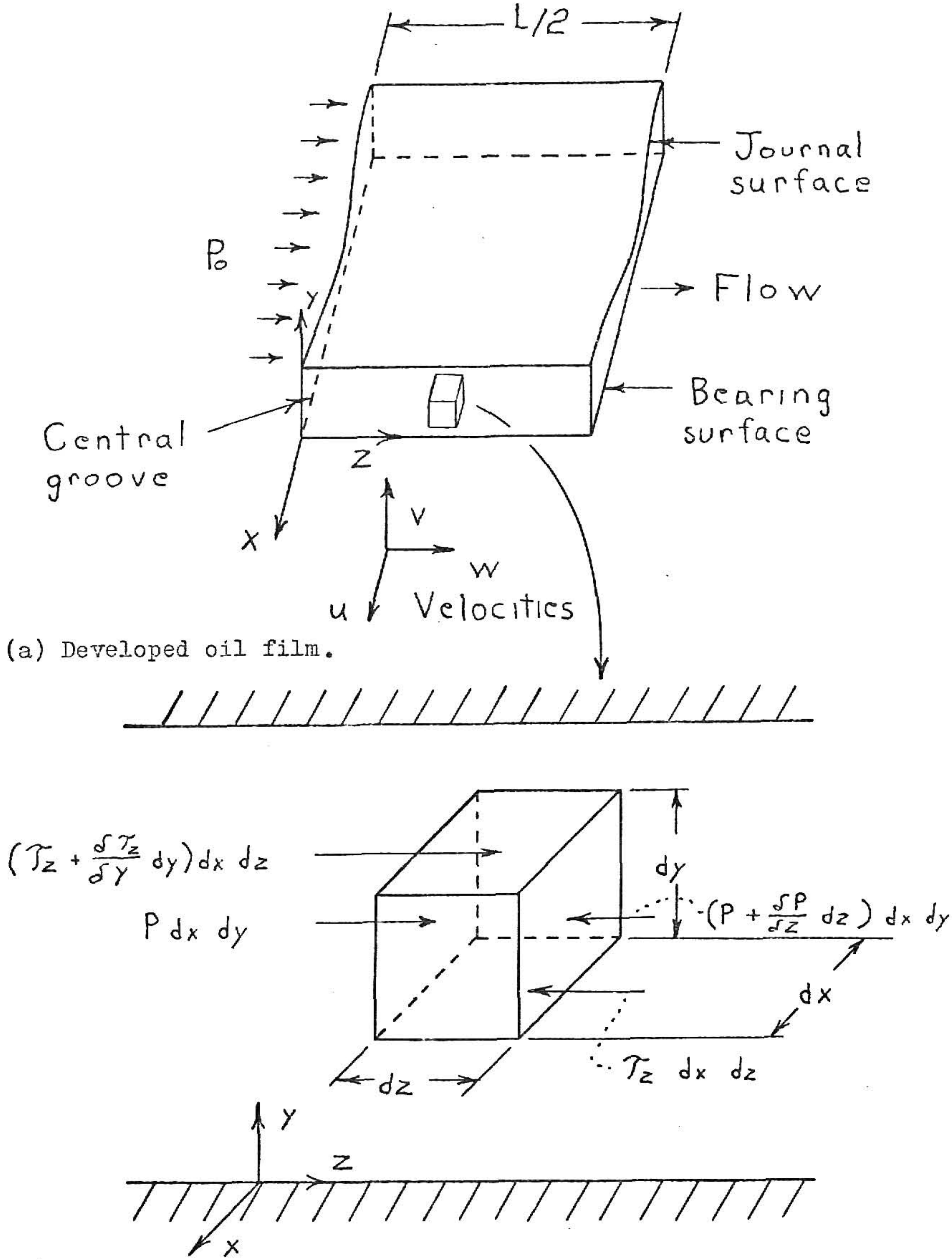


(a) Eccentric journal bearing.

(b) Developed view of bearing surface.

Figure 5. Eccentric journal bearing with central oil groove.

Figure 6 shows a developed oil film and a particle of oil with all of the axial forces acting on it. For static



(b) Forces on an elementary particle of oil between the developed bearing surfaces.

Figure 6. Freebody diagram of an oil film.

equilibrium

$$\left(\tau_z + \frac{d\tau_z}{dy} dy\right) dx\, dz - \tau_z\, dx\, dz + P dx\, dy - \left(P + \frac{dP}{dz} dz\right) dx\, dy = 0$$

which reduces to

$$\frac{d\tau_z}{dy}\, dy\, dx\, dz - \frac{dP}{dz}\, dx\, dy\, dz = 0$$

or

$$\frac{dP}{dz} = \frac{1}{\mu}\frac{dP}{dz}$$

Newton's law of viscous flow

$$\frac{dP}{dz} = \mu \frac{d^2 w}{dy^2}$$

Therefore,

$$\frac{d^2 w}{dy^2} = \frac{1}{\mu}\frac{dP}{dz}$$

Integrating twice with respect to y yields

$$\frac{dw}{dy} = \frac{1}{\mu}\frac{dP}{dz} y + C_1$$

$$w = \frac{1}{2\mu}\frac{dP}{dz} y^2 + C_1 y + C_2$$

Evaluating the two constants of integration by noting that $w = 0$ when $y = 0$ or h gives

$$C_1 = -\frac{1}{2\mu}\frac{dP}{dz} h \qquad\qquad C_2 = 0$$

Thus

$$w = \frac{1}{2\mu}\frac{dP}{dz} y(y - h) \qquad \text{(eq. 10)}$$

$$Q = -\frac{c^3 r}{12\mu}\frac{dP}{dz}\left[\theta + 3n\sin\theta + 3n^2\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)\right.$$
$$\left. + \frac{n^3 \sin\theta}{3}\left(\cos^2\theta + 2\right)\right]_0^{2\pi}$$

$$Q = -\frac{rc^3\pi}{6\mu}\frac{dP}{dz}\left(1 + 1.5\,n^2\right) \qquad \text{(eq. 11)}$$

From equation (10)

$$\frac{dP}{dz} = \frac{2\mu w}{y(y-h)}$$

Integrating with respect to z gives

$$P = \frac{2\mu w}{y(y-h)}z + C_1$$

Evaluating the constant of integration by noting $p = p_o$ when $z = 0$ yields

$$C_1 = P_o$$

and

$$P = \frac{2\mu w}{y(y-h)}z + P_o$$

Therefore

$$P - P_o = \frac{2\mu w}{y(y-h)}z$$

and since $p = 0$ at $z = L/2$

$$-P_o = \frac{2\mu w}{y(y-h)}\frac{L}{2}$$

The quantity of liquid flowing across an elementary area is

$$dQ = w\,dx\,dy$$

$$dQ = \frac{1}{2\mu}\frac{dP}{dz}\, y(y-h)\, dx\, dy$$

For a journal bearing $h = c(1 + n \cos\theta)$

and $dx = r d\theta$

Substituting these values for h and dx into the equation for dQ yields

$$dQ = \frac{r}{2\mu}\frac{dP}{dz}\left[y^2 - c y (1 + n \cos\theta)\right] dy\, d\theta$$

or

$$Q = \frac{r}{2\mu}\frac{dP}{dz}\left[\int_0^{2\pi}\int_0^{h} y^2\, dy\, d\theta - \int_0^{2\pi}\int_0^{h} c y (1 + n\cos\theta)\, dy\, d\theta\right]$$

Integrating with respect to y gives

$$Q = \frac{r}{2\mu}\frac{dP}{dz}\left[\int_0^{2\pi}\frac{h^3}{3}\, d\theta - \int_0^{2\pi}\frac{c(1 + n\cos\theta)h^2}{2}\, d\theta\right]$$

substituting in $h = c(1 + n \cos\theta)$ yields

$$Q = \frac{r}{2\mu}\frac{dP}{dz}\left[\int_0^{2\pi}\frac{c^3(1 + n\cos\theta)^3\, d\theta}{3} - \int_0^{2\pi}\frac{c(1+n\cos\theta)c^2(1+n\cos\theta)^2 d\theta}{2}\right]$$

$$= -\frac{c^3 r}{12\mu}\frac{dP}{dz}\left[\int_0^{2\pi}(1 + n\cos\theta)^3 d\theta\right]$$

$$= -\frac{c^3 r}{12\mu}\frac{dP}{dz}\left[\left(1 + 3n\cos\theta + 3n^2\cos^2\theta + n^3\cos^3\theta\right)d\theta\right]$$

or

$$-\frac{2P_o}{L} = \frac{2\mu w}{y(y-h)} = \frac{dP}{dz}$$

Substituting into equation 11 yields

$$Q = \frac{r c^3 \pi P_o}{3 \mu L} \left(1 + 1.5 n^2 \right)$$

which is the leakage from one end of the bearing. The leakage from both ends of an eccentric bearing is

$$Q = \frac{2 \pi r c^3 P_o}{3 \mu L} \left(1 + 1.5 n^2 \right) \qquad (\text{eq. } 12)$$

Shaw and Macks used the above result to predict the flow through a bearing lubricated by a single oil hole. From equation 12 the flow per unit circumferential length through a concentric bearing, $n = 0$, with a central oil groove is

$$Q = \frac{c^3 P_o}{3 \mu L}$$

From Figure 7, the flow across Δx is

$$\Delta Q_o = \frac{c^3 P_o}{3 \mu L} \Delta x$$

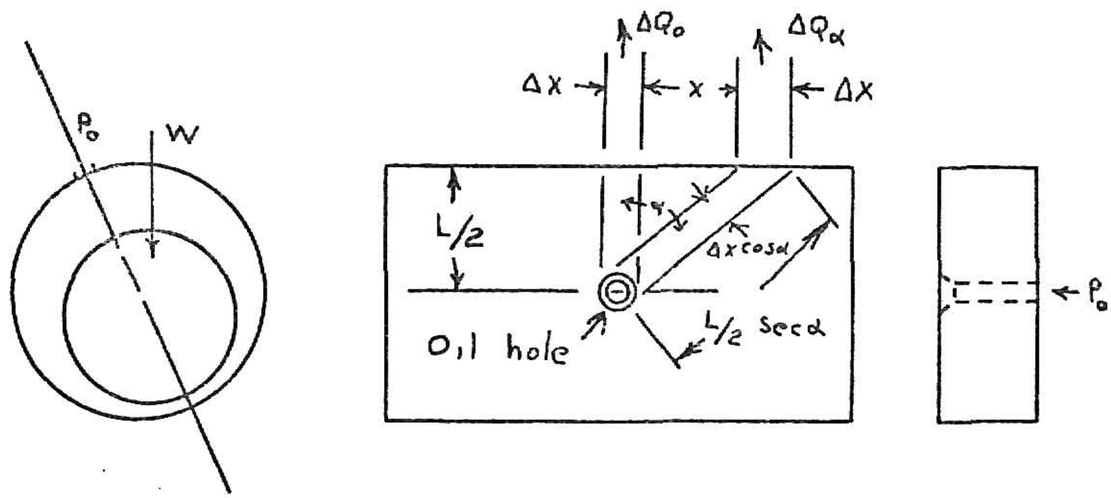


Figure 7. Journal bearing with single oil supply hole in the unloaded region.

and the flow at any other point along the bearing to a first approximation is

$$\Delta Q_{\alpha} = \frac{C^3 P_o}{3\mu L \sec\alpha} \cos\alpha \quad \Delta X$$

or

$$\lim_{\Delta X \to 0} \Delta Q_{\alpha} = dQ = \frac{C^3 P_o}{3\mu L} \cos^2\alpha \; dx$$

Integrating,

$$Q = 2\int_0^{\pi r} \frac{C^3 P_o}{3\mu L} \cos^2\alpha \; dx$$

and putting

$$x = \frac{L}{2} \tan\alpha$$

$$dx = \frac{L}{2} \sec^2\alpha \; d\alpha$$

and noting when

$$X = 0 \qquad \alpha = 0$$

$$x = \pi r \qquad \alpha = \tan^{-1} \frac{2\pi r}{L}$$

then

$$Q = \frac{2C^3 P_o}{3\mu L} \int_0^{\tan^{-1}\frac{2\pi r}{L}} \cos^2\alpha \; \frac{L}{2} \; d\alpha$$

$$Q = \frac{C^3 P_o}{3\mu} \tan^{-1}\left(\frac{2\pi r}{L}\right)$$

Shaw and Macks assumed the flow through a bearing fed by a single oil hole varied in the same manner with attitude as did the flow through a bearing fed by a central oil groove, then

$$Q = \frac{c^3 P_o}{3\mu} \tan^{-1}\left(\frac{2\pi r}{L}\right)\left(1 + 1.5\, n^2\right) \qquad \text{(eq. 13)}$$

Needs and Boyd and Robertson based their theoretical equations on the work done by Reynolds. By solving Reynolds' Three dimensional equation for the pressure distribution in an oil film, they were able to develop their own theoretical equations for the end leakage from a journal bearing.

Reynolds' equation:

The following assumptions are involved in developing Reynolds' three dimensional equation for the pressure distribution in an oil film:

1. Lubricant is Newtonian; i.e., shear stress is proportional to the rate of shear.
2. Film is so thin compared with the ratio of kinematic viscosity to linear velocity that motion of fluid is laminar.
3. No slip occurs between fluid and bearing surfaces.
4. Fluid inertia terms are negligible.
5. Weight of fluid is negligible.
6. Fluid is incompressible.
7. Fluid film is so thin that the pressure remains constant across its depth.
8. Fluid film is so thin that any curvature of the bearing surfaces may be ignored and the film unwrapped for analysis.
9. The viscosity of the fluid is uniform throughout the film.

Figure 8 shows a full journal bearing with both the journal and shaft rotating. Figure 9 shows an elementary cube of lubricant with all the forces acting upon it in the x and z directions. For static equilibrium the sum of the forces in the x direction must equal zero:

$$(\tau_x + \frac{d\tau_x}{dz} dz) dx\,dy - \tau_x\,dx\,dy + (\tau_x + \frac{d\tau_x}{dy} dy) dx\,dz$$

$$-\tau_x\,dx\,dz + (P - \tfrac{1}{2}\frac{dP}{dx} dx) dy\,dz - (P + \tfrac{1}{2}\frac{dP}{dx} dx) dy\,dz = 0$$

or

$$\frac{d\tau_x}{dy} dx\,dy\,dz + \frac{d\tau_x}{dz} dx\,dy\,dz - \frac{dP}{dx} dx\,dy\,dz = 0$$

$$\frac{d\tau_x}{dy} + \frac{d\tau_x}{dz} - \frac{dP}{dx} = 0 \qquad \text{(eq. 14)}$$

The summation of the forces in the z direction must also equal zero for static equilibrium:

$$(\tau_z + \frac{d\tau_z}{dx} dx) dy\,dz - \tau_z\,dy\,dz + (\tau_z + \frac{d\tau_z}{dy} dy) dx\,dz$$

$$-\tau_z\,dx\,dz + (P - \tfrac{1}{2}\frac{dP}{dz} dz) dx\,dy - (P + \tfrac{1}{2}\frac{dP}{dz} dz) dx\,dy = 0$$

which reduces to

$$\frac{d\tau_z}{dy} + \frac{d\tau_z}{dx} - \frac{dP}{dz} = 0 \qquad \text{(eq. 15)}$$

By Newton's law of viscous flow:

$$\tau_x = \mu \frac{du}{dy} \qquad \text{(eq. 16)}$$

and

$$\tau_z = \mu \frac{dw}{dy} \qquad \text{(eq. 17)}$$

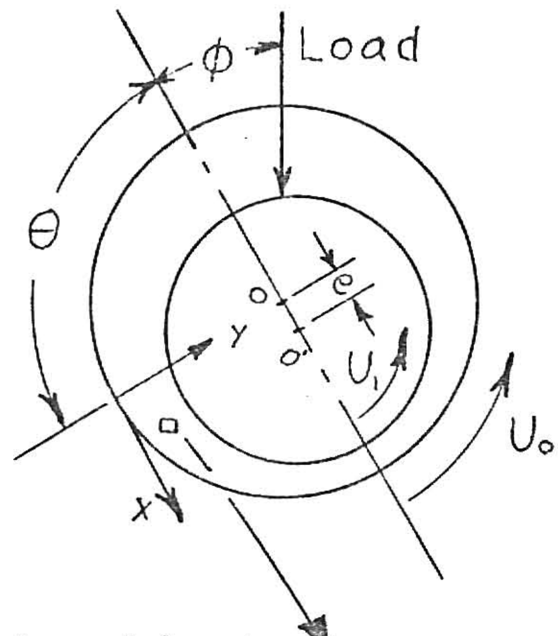


Figure 8. A full journal bearing.

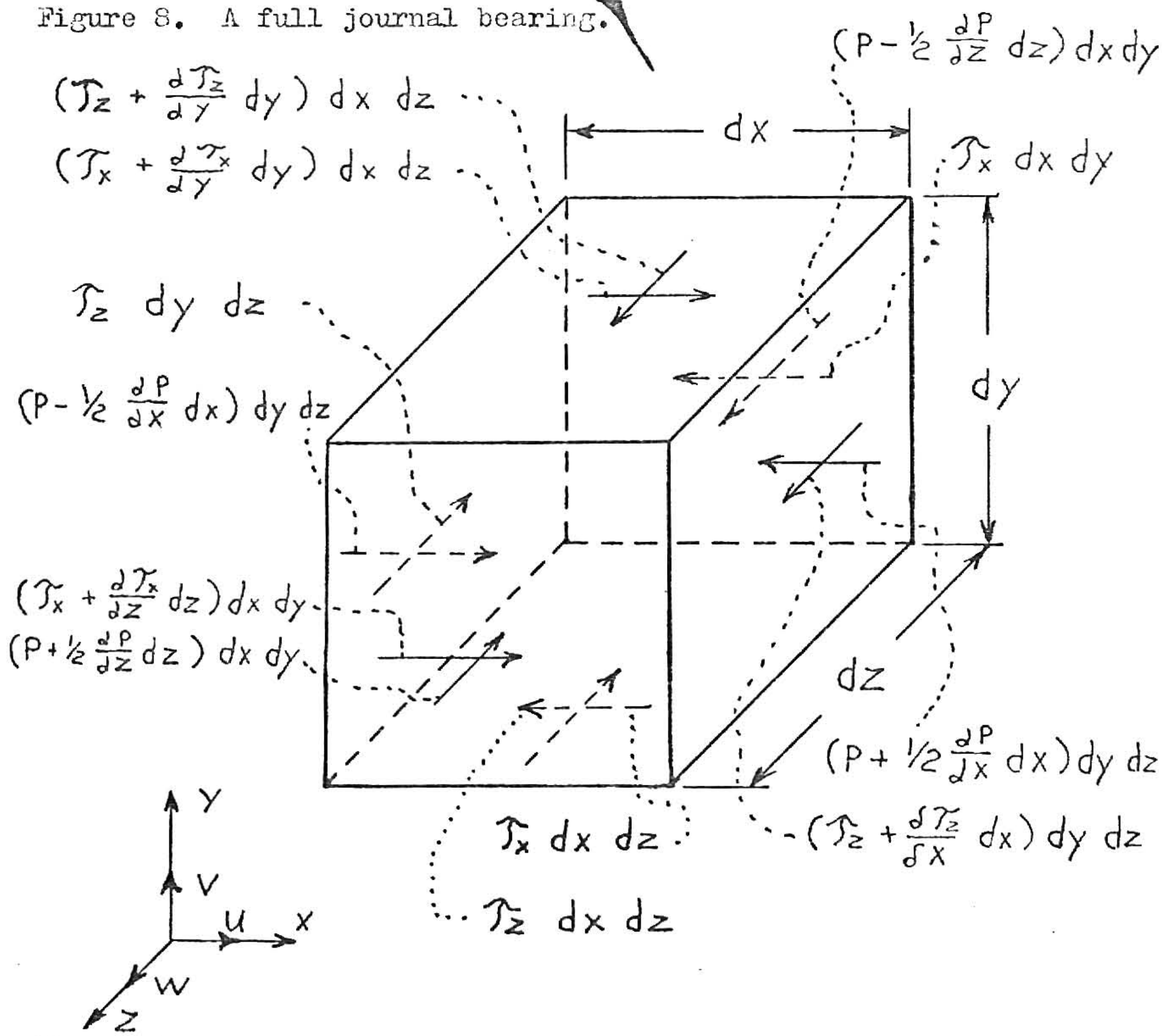


Figure 9. Forces acting upon an elementary cube of lubricant in the x and z directions.

Substituting equation 16 into equation 14 yields

$$\frac{d}{dy}\left(\mu \frac{du}{dy}\right) + \frac{d}{dz}\left(\mu \frac{du}{dy}\right) = \frac{dP}{dx}$$

$$\mu\left(\frac{d^2u}{dy^2} + \frac{d^2u}{dy\,dz}\right) = \frac{dP}{dx} \qquad \text{(eq. 18)}$$

Because of the very thin folm of lubricant in the y direction, the velocity gradients $\frac{du}{dx}$ and $\frac{du}{dz}$ will be negligibly small compared to the velocity gradient $\frac{du}{dy}$. Hence, the second term on the left of equation 18 can be ignored to give

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dP}{dx} \qquad \text{(eq. 19)}$$

By the same procedure for the z direction, the following equation can be derived:

$$\frac{d^2w}{dy^2} = \frac{1}{\mu}\frac{dP}{dz} \qquad \text{(eq. 20)}$$

Integrating equation 19 twice gives

$$u = \frac{1}{\mu}\frac{dP}{dx}\left(\frac{y^2}{2}\right) + C_1 y + C_2$$

Evaluating the constants of integration by use of the fact that $u = U_0$ when $y =$ zero and $u = U_1$ when $y = h$ gives

$$C_2 = U_0 \qquad C_1 = \frac{U_1 - U_0}{h} - \frac{1}{\mu}\frac{dP}{dx}\frac{h}{2}$$

and

$$u = \frac{1}{\mu}\frac{dP}{dx}\left(\frac{y^2}{2}\right) + \left[\frac{U_1 - U_0}{h} - \frac{1}{\mu}\frac{dP}{dx}\frac{h}{2}\right]y + U_0$$

or

$$u = \frac{1}{2\mu}\frac{dP}{dx}(y-h)y + \frac{h-y}{h}U_0 + \frac{y}{h}U_1 \qquad \text{(eq. 21)}$$

Applying the same procedure to equation 10 yields

$$w = \frac{1}{2\mu}\frac{dP}{dz}(y-h)y \qquad \text{(eq. 22)}$$

From the volume-continuity equation of fluid mechanics

$$\frac{dV}{dy} = -\frac{du}{dx} - \frac{dw}{dz} \qquad \text{(eq. 23)}$$

Inserting equations 21 and 22 into equation 23 gives

$$\frac{dV}{dy} = -\tfrac{1}{2}\left\{\frac{d}{dx}\left[\frac{1}{\mu}\frac{dP}{dx}(y-h)y\right] + \frac{d}{dz}\left[\frac{1}{\mu}\frac{dP}{dz}(y-h)y\right]\right\}$$

$$-\frac{d}{dx}\left[\frac{h-y}{h}U_0 + \frac{y}{h}U_1\right]$$

Integrating with respect to y and assuming constant viscosity

$$\int_0^{V_1} dV = -\frac{1}{2\mu}\int_0^h\left(\frac{d}{dx}\left[\frac{dP}{dx}(y-h)y\right] + \frac{d}{dz}\left[\frac{dP}{dz}(y-h)y\right]\right)dy$$

$$-\int_0^h\frac{d}{dx}\left(\frac{h-y}{h}\right)U_0\,dy - \int_0^h\frac{d}{dx}\left(\frac{y}{h}\right)U_1\,dy$$

Performing the integration before differentiation by the use of Leibnitz Rule gives

$$V_1 = -\frac{1}{2\mu}\left[\frac{d}{dx}\int_0^h\frac{dP}{dx}(y-h)y\,dy - \left(\frac{dP}{dx}(y-h)y\frac{dh}{dx}\right)\Big|^h\right.$$

$$\left.+\frac{d}{dz}\int_0^h\frac{dP}{dz}(y-h)y\,dy - \left(\frac{dP}{dz}(y-h)y\frac{dh}{dz}\right)\Big|^h\right]$$

$$-\left[\frac{d}{dx}\int_0^h\frac{h-y}{h}U_0\,dy - \left(\frac{h-y}{h}U_0\frac{dh}{dx}\right|^h\right]$$

$$-\left[\frac{d}{dx}\int_0^h\frac{y}{h}U_1\,dy - \left(\frac{y}{h}U_1\frac{dh}{dx}\right|^h\right]$$

$$V_1 = \frac{1}{2\mu}\left[\frac{d}{dx}\left(\frac{dP}{dx}\frac{h^3}{6}\right) + \frac{d}{dz}\left(\frac{dP}{dz}\frac{h^3}{6}\right)\right] - \frac{d}{dx}\left(\frac{h}{2}U_0\right) - \frac{d}{dx}\left(\frac{h}{2}U_1\right) + U_1\frac{dh}{dx}$$

$$= \frac{1}{2\mu}\left[\frac{d}{dx}\left(\frac{dP}{dx}\frac{h^3}{6}\right) + \frac{d}{dz}\left(\frac{dP}{dz}\frac{h^3}{6}\right)\right] - \tfrac{1}{2}\left(\frac{dh}{dx}U_0 + \frac{d(U_0)}{dx}h\right) - \tfrac{1}{2}\left(\frac{dh}{dx}U_1 + \frac{d(U_1)}{dx}h\right) + U_1\frac{dh}{dx}$$

Rearranging and canceling terms yields Reynolds' differential equation in three dimensions for the pressure distribution in an oil film as

$$\frac{1}{6\mu}\left[\frac{d}{dx}\left(h^3\frac{dP}{dx}\right) + \frac{d}{dz}\left(h^3\frac{dP}{dz}\right)\right] = (U_0 - U_1)\frac{dh}{dx} + h\frac{d(U_0 + U_1)}{dx} + 2V_1 \qquad \text{(eq. 24)}$$

It was from this equation that Needs experimentally derived his equation for the end leakage from a journal bearing. By using Kingbury's electrical analogy, Needs was able to experimentally determine the pressure distribution in the oil film. Using this, Needs obtained the remaining bearing operating characteristics by graphic integration.

Boyd and Robertson developed their equation from the works by Muskat and Morgan, who solved Reynolds' equation by a successive approximation method.

Regression analysis:

The mathematical model in linear regression is specified by the equation

$$Y = \alpha + \beta X + \varepsilon$$

Y is a function of X, but is statistics the term "regression" is used to describe this relationship. Y is called the dependent variable and X the independent variable. In this model Y is the sum of a random part, ε , and a part fixed by x. The fixed part determines the means of the populations sampled, one mean for each x. These means lie on the straight line represented by $\mu = \alpha + \beta X$, the population regression line. The parameter α is the mean of the population that corresponds to $x = 0$; thus, α specifies the height of the regression line when $X = \overline{X}$. β is the slope of the line or the change in Y per unit change in x. ε is the variable part of Y that is drawn at random from a normal population and is independent of x.

The mathematical model for the multiple regression is

$$Y = \alpha + \beta_1 X_1 + \beta_2 X_2 + \varepsilon$$

This model is the same as the linear regression model only now Y is a function of two variables x_1 and x_2. β_1 and β_2 now measure the average or expected change in Y for their respective changes in X_1 and X_2.

APPENDIX III

Sample Calculations:

Following Shaw and Mack's analysis, the flow rate can be calculated at various inlet pressures as follows:

$$Q = \frac{c^3 P_o}{3\mu} \tan^{-1}\left(\frac{2\pi r}{L}\right)(1.0 + 1.5\, n^2)$$

$c = 0.00145$ inch

$\mu = 6.47 \times 10^{-6}$ reyns.

$r = 0.8747$ inch

$L = 1.750$ inches

$n = 0.23$

$P_o = 10$ to 80 psi

$$Q = \frac{(0.00145)^3\ (10)}{(3)\ (6.47 \times 10^{-6})} \tan^{-1}(\pi)\ (1.0 + 1.5(.23)^2)$$

$Q = .0021\ \text{in}^3/\text{sec}$

The results are tabulated in Table 3. Since Shaw and Macks' analysis is independent of shaft speed, the flow rate at various shaft speeds is a constant and is tabulated in Table 4 for 10 psi inlet pressure.

Needs' formula for end leakage is independent of inlet pressure. Therefore the flow rate for various inlet pressures is a constant and is tabulated in Table 3 for a shaft speed of 536 rpm. The flow rate at various shaft speeds can be

Table 3. Flow Rates for Various Inlet Pressures at 536 rpm.

Pressure (psi)	Experimental (in^3/sec)	Shaw and Macks (in^3/sec)	Boyd and Robertson (in^3/sec)	Needs (in^3/sec)	Cameron (in^3/sec)
10	0.0080	0.0021	0.0359	0.0098	0.00027
20	0.0117	0.0042	0.0718	0.0098	0.00054
30	0.0149	0.0063	0.1077	0.0098	0.00081
40	0.0186	0.0084	0.1436	0.0098	0.00108
50	0.0213	0.0105	0.1795	0.0098	0.00135
60	0.0260	0.0126	0.2154	0.0098	0.00162
70	0.0298	0.0147	0.2513	0.0098	0.00189
80	0.0410	0.0168	0.2872	0.0098	0.00216

Table 4. Flow Rates for Various Shaft Speeds at 10 psi pressure

Shaft Speed (rpm)	Experimental (in^3/sec)	Shaw and Macks (in^3/sec)	Boyd and Robertson (in^3/sec)	Needs (in^3/sec)	Cameron (in^3/sec)
0	0.0053	0.0021	0.0359	0.0	0.00027
536	0.0080	0.0021	0.0359	0.0098	0.00027
1232	0.0197	0.0021	0.0359	0.0226	0.00027
1810	0.0246	0.0021	0.0359	0.0331	0.00027

calculated as follows:

$Q = K\ U\ r\ m\ L$

From Table 5, for $L/D = 1$ and $n = 0.23$, the value for K is 0.079.

$U = 49.09$, 112.83 and 165.77 in/sec

$r = 0.8747$ inch

$m = c/r = 0.00145/0.8747$

$L = 1.75$ inches

$$Q = (0.079)(49.09)\left(\frac{0.00145}{0.8747}\right)(0.8747)(1.75)$$

$Q = 0.0098\ in^3/sec$

These values are tabulated in Table 4 for an inlet pressure of 10 psi.

End leakage on the basis of Boyd and Robertson's equation can be calculated for a concentric journal bearing with various inlet pressures as:

$$Q_B = \frac{\pi P_o C^3}{3\mu\left[\ell/D - 2\ln d/D - 4\sum_{s=1}^{\infty} 1/s\left(1+e^{2s\ell/D}\right)\right]}$$

$C = 0.0029$ inch

$\mu = 6.47 \times 10^{-6}$ reyns

$\ell = 0.875$ inch

$D = 1.7523$ inches

$d = 0.250$ inch

S = the series 1, 2, 3, 4, ...

P_o = 10 to 80 psi

$$Q_B = \frac{(3.14)(10)(.0029)^3}{(3)(6.47 \times 10^{-6})\left[.5 - 2 \ln .1427 - 4\left(\frac{1}{1+e} + \frac{1}{2(1+e^2)} + \frac{1}{3(1+e^3)} + \frac{1}{4(1+e^4)}\right)\right]}$$

$Q_B = 0.0154$ in^3/sec.

Then for an eccentric journal bearing

$Q = Q_B \times R_B$

$$R_B = \frac{(1+n)^3 \left(l/D - 2 \ln d/D\right)}{\left[l/D - 2 \ln d/D + 3n\left(l/D + 2 - \ln 16\right)\right]}$$

n = 0.23

$$R_B = \frac{(1+.23)^3 (.5 - 2 \ln .1426)}{\left[.5 + 3.888 + (3)(0.23)(0.5 + 2.0 - \ln 16)\right]} = 2.328$$

$Q = 0.0359$ in^3/sec.

The results are tabulated in Table 3.

Boyd and Robertson's equation is also independent of shaft speed. Therefore, the end leakage for various shaft speeds is a constant and is tabulated in Table 4 for 10 psi inlet pressure.

From Cameron's analysis the end leakage for various inlet pressures can be calculated as:

$$Q = \left(\frac{h_o^3}{12}\frac{P_o}{L}\frac{d}{}\right) R_d$$

A value for R_d can be taken from Figure 10 for $d/L = .143$ or it can be calculated from the empirical formula

$R_d = 1.2 + 11.0\ d/L$

$R_d = 2.77$

$h_o = 0.00175$

$d = 0.25$ inch

$L = 1.75$ inches

$\mu = 6.47 \times 10^{-6}$ reyns.

$P_o = 10$ to 80 psi

$$Q = \frac{(0.00175)^3(10)(0.25)(2.77)}{(12)(6.47)(10^{-6})(1.75)} = 0.00027 \text{ in}^3/\text{sec.}$$

These values are tabulated in Table 3. The value for end leakage for different shaft speeds is also tabulated in Table 4 for 10 psi inlet pressure for Cameron's equation.

Table 5. Values of K for Reed's Equation for End Leakage.

n L/D	0.9	0.8	0.6	0.4	0.2
0.250	0.150	0.220	0.234	0.192	0.096
0.500	0.180	0.225	0.195	0.186	0.104
0.750	0.130	0.153	0.162	0.133	0.087
1.000	0.108	0.126	0.132	0.104	0.075
1.250	0.090	0.108	0.110	0.086	0.063
1.500	0.078	0.093	0.096	0.072	0.054
2.000	0.054	0.069	0.078	0.054	0.042
2.500	0.040	0.051	0.060	0.040	0.030
3.000	0.027	0.034	0.039	0.026	0.021
4.000	0.0	0.0	0.0	0.0	0.0

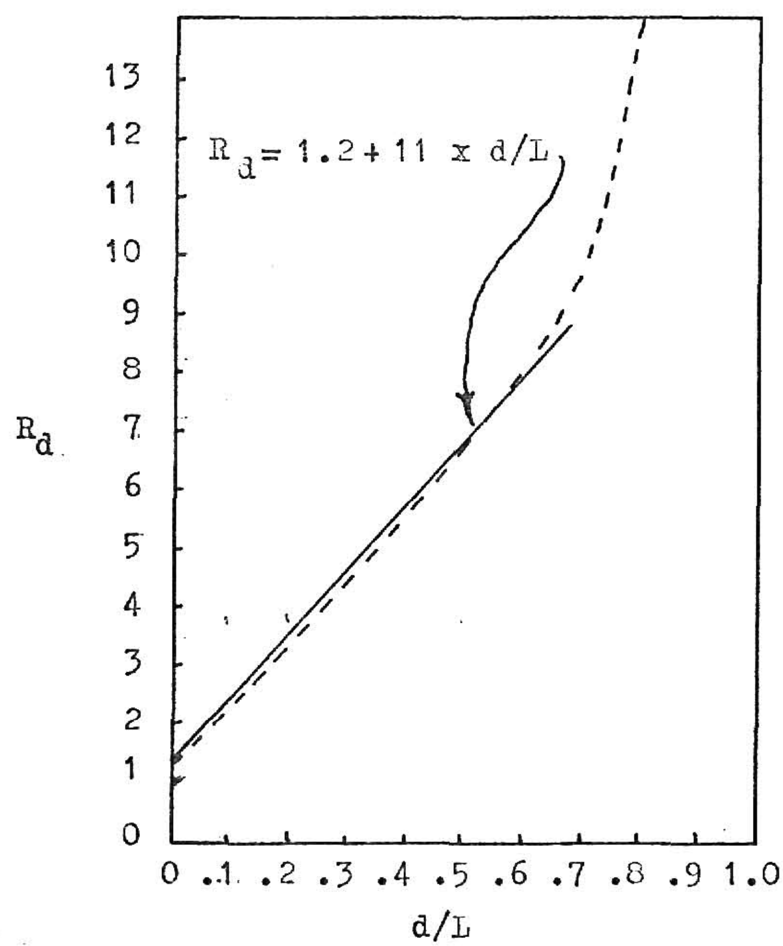


Figure 10. Cameron's coefficient for end leakage.

Table 6. Linear Regression Analysis of Flow Rate vs. Inlet Pressure.

Pressure (P) (psi)	Flow Rate (Q) (in^3/sec)	Deviations p	q	Square p^2	Product pq
0	0	-40	-.0191	1600	0.764
10	0.0080	-30	-0.0111	900	0.333
20	0.0117	-20	-0.0074	400	0.148
30	0.0149	-10	-0.0042	100	0.042
40	0.0188	0	-0.0003	0	0.0
50	0.0213	10	0.0022	100	0.022
60	0.0260	20	0.0069	400	0.138
70	0.0298	30	0.0107	900	0.321
80	0.0410	40	0.0219	1600	0.876
sum-360	0.1715			6000	2.644
mean-40	0.0191				

b = sample regression coefficient

$$b = \frac{\text{sum of products of deviations, p and q}}{\text{sum of squares of deviations, p}}$$

b = 2.644/6000 = 0.000441

$Q = \overline{Q} + b(P - \overline{P})$ (sample regression equation)

$\overline{Q}$ = mean flow rate

$\overline{P}$ = mean inlet pressure

Q = 0.0191 + 0.000441 (P-40)

Q = 0.001476 + 0.000441

Table 7. Linear Regression Analysis of Flow Rate vs. Shaft Speed.

Shaft Speed (N) (RPM)	Flow Rate (Q) (in^3/sec)	Deviations n	q	Squares n^2	Product nq
0	0.0053	-894.5	-0.0091	800130	8.14
536	0.0080	-358.5	-0.0064	128522	2.29
1232	0.0197	337.5	0.0053	113906	1.79
1810	0.0246	915.5	0.0102	838140	9.34
sum-3578	0.0576			1880698	21.56
mean-894.5	0.0144				

$$b = 21.56/1880698 = 0.0000114$$

$$Q = \overline{Q} + b(N-\overline{N})$$

$\overline{N}$ = mean speed

$$Q = 0.0144 + 0.0000114\ (N-894.5)$$

$$Q = 0.0042 + 0.0000114\ N$$

Table 8. Multiple Regression Analysis of Flow vs. Shaft Speed and Inlet Pressure.

Flow Rate $Qx10^4$ (in^3/sec)	Speed N (rpm)	Press. P (psi)	Q^2 x 10^6	N^2	P^2	NQ	PQ	NP
53	0	10	28	0	100	0.0	0.053	0
80	536	10	64	287296	100	4.288	0.083	5360
117	536	20	137	287296	400	6.271	0.234	10720
149	536	30	222	287296	900	7.986	0.477	16080
188	536	40	353	287296	1600	10.077	0.752	21440
197	1232	10	388	1517824	100	24.270	0.197	12320
213	536	50	454	287296	2500	11.417	1.065	26800
246	1810	10	605	3276100	100	44.526	0.246	18100
260	536	60	676	287296	3600	13.936	1.560	32160
298	536	70	888	287296	4900	15.973	2.086	37520
410	536	80	1681	287296	6400	21.976	3.280	42880
2211	7330	390	5496	7092292	20700	160.72	10.00	223380
201	660	35.5						

The next to the last last are sums.

The last line are means.

The multiple regression equation is:

$$Q = a + b N + b P$$

$$b = \frac{(\Sigma p^2)\ (\Sigma nq) - (\Sigma np)\ (\Sigma pq)}{D}$$

$$b = \frac{(\Sigma n^2)\ (\Sigma pq) - (\Sigma np)\ (\Sigma nq)}{D}$$

$$a = \overline{Q} - b_1 \overline{N} - b_2 \overline{P}$$

$$D = (\Sigma n^2)\ (\Sigma p^2) - (\Sigma np)^2$$

$$\Sigma n^2 = \Sigma N^2 - (\Sigma N)^2/11 = 7092292 - (7330)^2/11 = 2207847$$

$$\Sigma p^2 = \Sigma P^2 - (\Sigma P)^2/11 = 20700 - (390)^2/11 = 6873$$

$\Sigma q^2 = \Sigma Q^2 - (\Sigma Q)^2/11 = 0.005496 - (o.2211)^2/11 = 0.00105$

$\Sigma np = \Sigma NP - (\Sigma N)\ (\Sigma P)/11 = 223380 - (7330)\ (390)/11 = -36501$

$\Sigma nq = \Sigma NQ - (\Sigma N)\ (\Sigma Q)/11 = 160.72 - (7330)\ (0.2211)/11 = 13.387$

$\Sigma pq = \Sigma PQ - (\Sigma PQ) - (\Sigma P)(\Sigma Q)/11 = 10.0 - (390)(0.2211)/11 = 2.161$

$D = (2207847)(13.387) - (-36501)^2 = 1.3842 \times 10^{10}$

$$b_1 = \frac{(6873)(13.387) - (-36501)(2.161)}{1.3842 \times 10^{10}} = 1.23 \times 10^{-5}$$

$$b_2 = \frac{(2207847)(2.161) - (-36501)(13.387)}{1.3842 \times 10^{10}} = 3.8 \times 10^{-4}$$

$a = 0.0201 - 1.23 \times 10^{-5}\ (666) - 3.8 \times 10^{-4}\ (35.5) = -0.0016$

$Q = -0.0016 \quad 1.23 \times 10^{-5}\ N \quad 3.8 \times 10^{-4}\ P$

APPENDIX IV

Test equipment drawings and specifications:

Shaft diameter - 1.7494 inches

Bearing diameter - 1.7523 inches

Eccentricity - 0.00033 inches

Eccentricity ratio - 0.23

Bearing length - 1.75 inches

Oil inlet pressure - 10 to 80 psi

Shaft speed - 0, 536, 1232, 1810 rpm

Oil brand - Mobile SAE 10W, nondetergent

Oil viscosity - 6.47×10^{-6} reyns

Local film thickness - 0.00175 inch

Oil inlet hole diameter - 0.25 inch

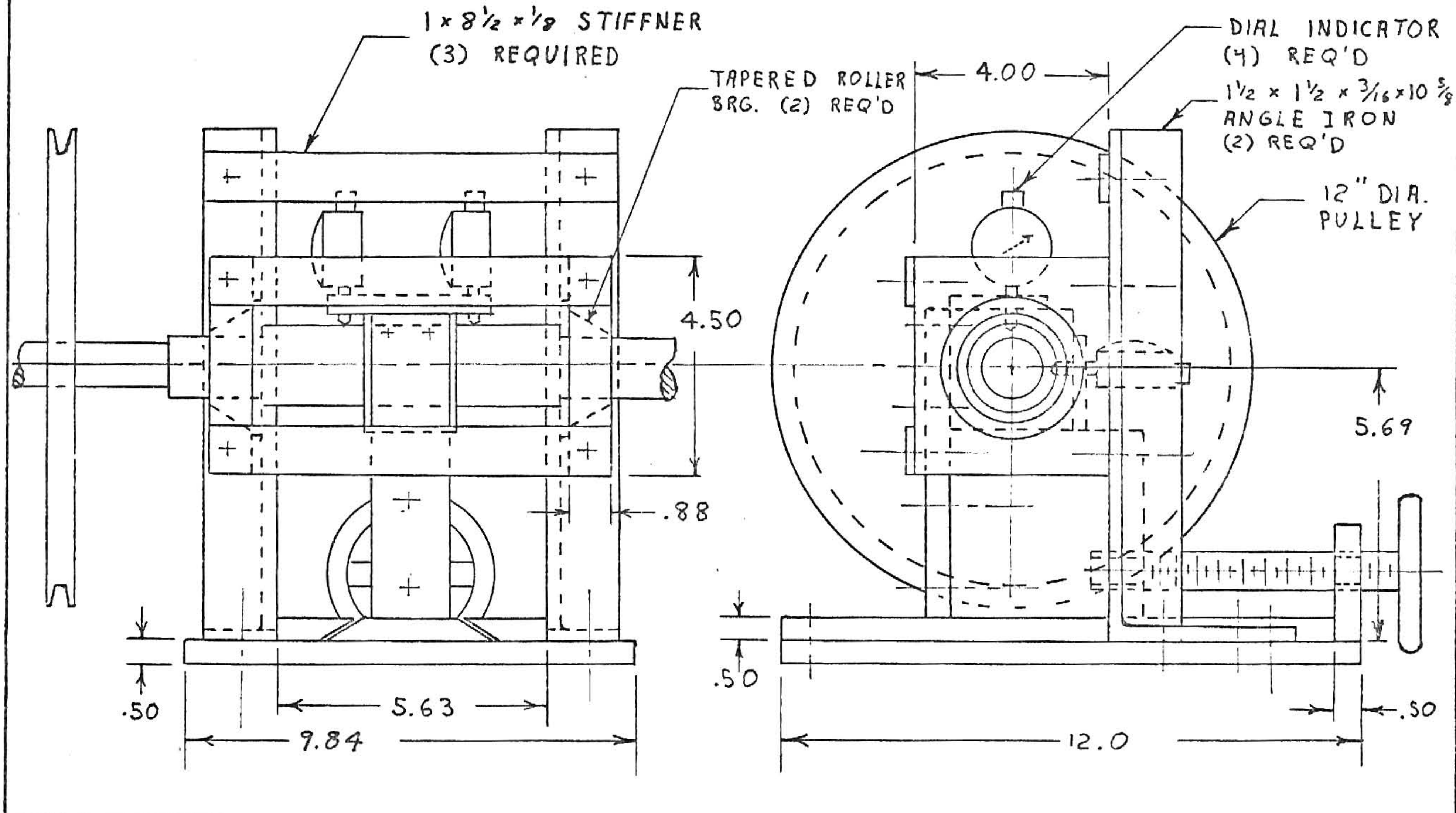


Figure 11. Test apparatus assembly

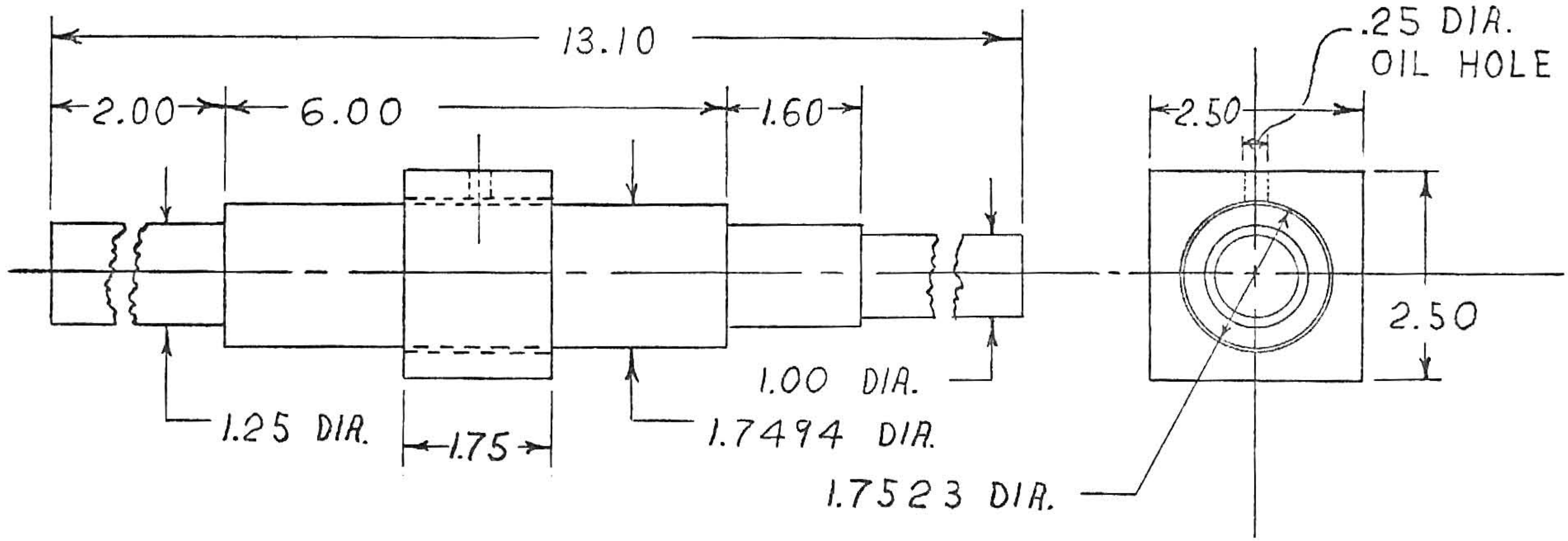


Figure 12. Shaft and bearing assembly.

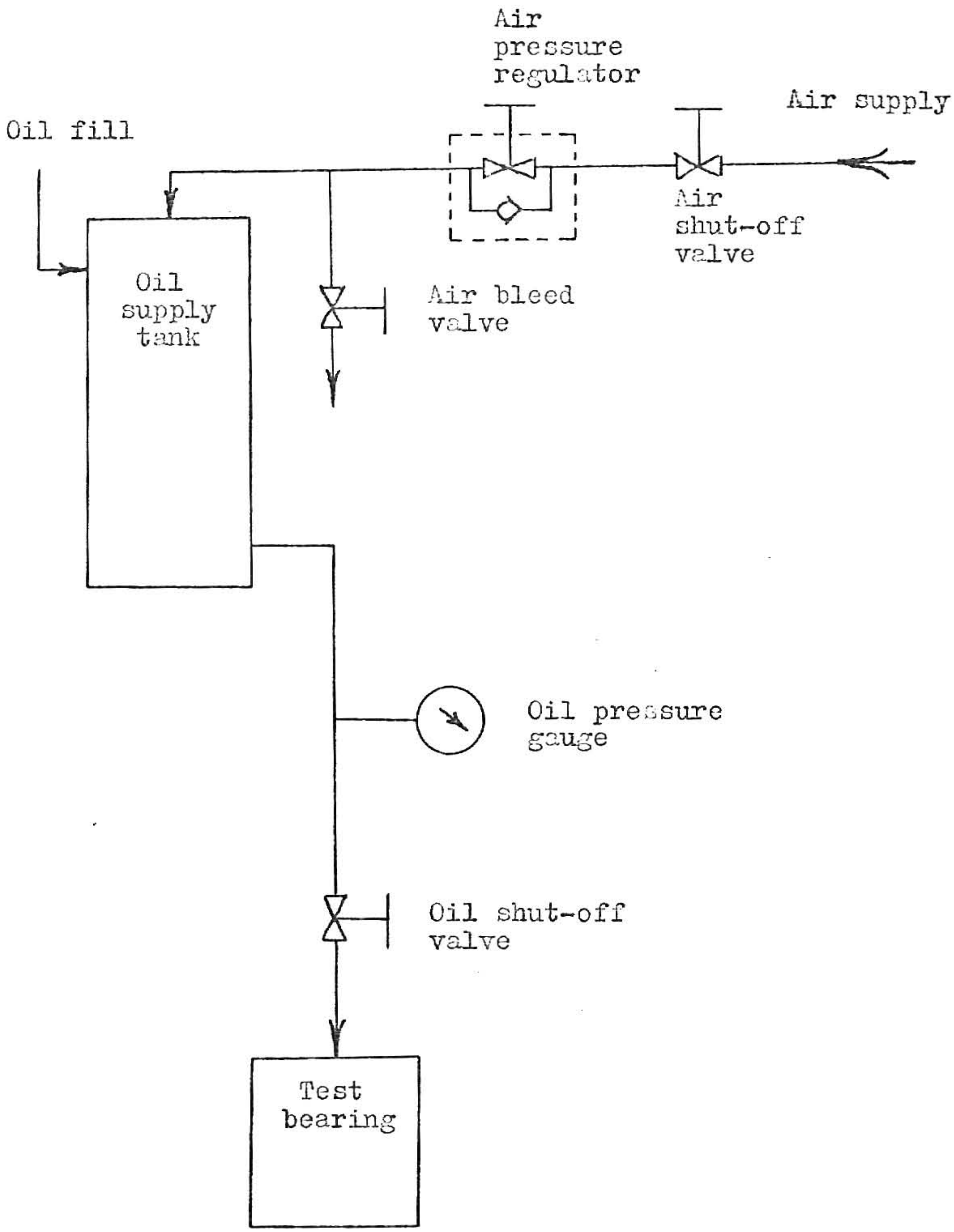


Figure 13. Test apparatus piping diagram.

THE INFLUENCE OF INLET PRESSURE AND SHAFT SPEED ON END LEAKAGE OF A FULL JOURNAL BEARING

by

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B.S., Kansas State University, 1971

AN ABSTRACT OF A MASTER'S THESIS

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1975

An experiment was conducted to determine the influence of inlet pressure and shaft speed on the end leakage from a full journal bearing. The results of the experimental investigation were then compared with four theoretical predictions for the end leakage.

The experiment was performed on a journal and bearing of 1-3/4 inch nominal diameter with a diametral clearance of 0.0029 inch, eccentricity ratio equal to 0.23, and a L/D ratio equal to 1.0. The oil used was Mobil SAE 10W nondetergent with a viscosity of 6.47×10^{-6} reyns at 78° F. The oil was introduced into the bearing through a single inlet hole of 1/4 inch diameter. The inlet pressure was varied from 10 to 80 psi with a shaft speed of 536 RPM. The shaft speed was varied from 0 to 1810 RPM with the inlet pressure at 10 psi.

The results showed that Shaw and Macks', Needs', and Boyd and Robertson's equations gave approximate solutions to the flow rate. Each solution compared to the experimental results in different speed and pressure ranges. Shaw and Macks' predicted results were the closest to the experimental results for low shaft speeds for the inlet pressures tested. Needs' predicted results agreed within reason with the experimental results for the shaft speeds tested with the inlet pressure between 10 to 20 psi. Boyd and Robertson's results agreed with the experimental results best for 70-80 psi inlet pressure and low rpm and 10-20 psi inlet pressure and the higher shaft rpms. Cameron's solution did not agree with any of the above predicted values or with the experimental results.