REPORT ON MORSE THEORY ON HILBERT MANIFOLDS

by

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INTRODUCTION

The purpose of this report is to fill in more details of Palais' paper "Morse theory on Hilbert manifolds".

Palais generalized the Morse theory on finite dimensional manifolds (due to M. Morse) to a general Hilbert manifold modelled on a separable Hilbert space. (So, we can define a Riemannian metric.) To do this he needed condition (C).

> (C) If S is any subset of M on which f is bounded but on which $\|\nabla f\|$ is not bounded away from zero then there is a critical point of f adherent to S.

Using this condition he defined the so-called Morse function and from this he got the Morse theory of Hilbert manifolds. As a corollary, we have the following interesting result.

If M is a complete Riemannian manifold of class C^{k+2} $(k \geq 1)$, f is a C^{k+2} Morse function, and if f is bounded below, then M is of the homotopy type of a CW complex.

As an application, he applied the Morse theory to $\Omega(v;p,q)$ the loop space of a complete finite dimensional Riemannian manifold V. With the action integral J he derived the Morse theory of geodesics.

We are assuming most of the basic results in differential calculus in Lang's book Introduction to Differentiable Manifolds and we shall not prove or even quote them.

I. REGULAR AND CRITICAL POINTS OF FUNCTIONS

1.1. Definition. Let M be a C^1-manifold, $f : M \to R$ a C^1-function. If $p \in M$ then p is said to be a regular point of f if $df_p \neq 0$ and a critical point of f if $df_p = 0$. If $c \in R$ and $f^{-1}(c)$ contains only regular points of f then $f^{-1}(c)$ is a regular level of f and c is a regular value.

If $f^{-1}(c)$ contains at least one critical point of f then $f^{-1}(c)$ is called a critical level and c is a critical value.

1.2. Lemma. Let φ be a C^k-isomorphism of an open set V in a Banach space E onto an open set V' in Banach space E' ($k \geq 2$). Let $f : V' \to R$ be of class C^2 and let $g = f \circ \varphi : V \to R$. Then if $dg_p = 0$, $d^2g_p(v_1,v_2)$ $= d^2f_{\varphi(p)}(d\varphi_p(v_1),d\varphi_p(v_2))$.

Proof. By the chain rule we get

(1) $dg_x = df_{\varphi(x)} \circ d\varphi_x$ and by a straightforward calculation, we have

(2) $d^2g_x(v_1,v_2) = d^2f_{\varphi(x)}(d\varphi_x(v_1),d\varphi_x(v_2)) + df_{\varphi(x)}(d^2\varphi_x(v_1,v_2))$.

Let $x = p$, then (1) gives $df_{\varphi(p)} = 0$ (since $d\varphi_p$ is a linear isomorphism) and (2) becomes

$$d^2g_p(v_1,v_2) = d^2f_{\varphi(p)}(d\varphi_p(v_1),d\varphi_p(v_2)).$$

Q.E.D.

1.3. Proposition. If f is a C^2-function on a C^2-manifold M, p a critical point of f, then there is a uniquely determined continuous, symmetric, bilinear form $H(f)_p$ on M_p called the Hessian of f at p. With the following property: If φ is any chart at p, then

$$H(f)_p(v,w) = d^2(f \circ \varphi^{-1})_{\varphi(p)}(v_\varphi,w_\varphi)$$

where v_φ is the tangent vector using chart φ.

Proof. Define $H(f)_p(v,w) = d^2(f \circ \varphi^{-1})_{\varphi(p)}(v_\varphi, w_\varphi)$ where $v,w \in M_p$, φ is a chart of p. We only need to prove that this is well-defined (i.e. independent of the chart chosen).

Let ψ be another chart at p. Without loss of generality, we may assume that φ and ψ are defined on a neighborhood U of p. Let $g_1 = f \circ \varphi^{-1}$, $g_2 = f \circ \psi^{-1}$ and $\Phi = \psi \circ \varphi^{-1}$. Then Φ is C^k-isomorphism and g_2 is C^2.

Also since p is a critical point

$$dg_{1\varphi(p)} = df_{\varphi^{-1}\varphi(p)} \circ d\varphi^{-1}_{\varphi(p)} = df_p \circ d\varphi^{-1}_{\varphi(p)} = 0.$$

Thus by the lemma and the definition of tangent vector (see L[1])

$$\begin{aligned} d^2g_{1\varphi(p)}(v,w) &= d^2g_{2\psi(p)}(d\Phi_{\psi(p)}v_\varphi, d\Phi_{\varphi(p)}w_\varphi) \\ &= d^2g_{2\psi(p)}(d\psi \circ \varphi^{-1})_{\varphi(p)}v_\varphi, d(\psi \circ \varphi^{-1})_{\varphi(p)}w_\varphi) \\ &= d^2g_{2\psi(p)}(v_\psi, w_\psi). \end{aligned}$$

Q.E.D.

1.4. <u>Definition</u>. Let B be a bounded, symmetric bilinear form on a Banach space E. Then B is <u>non-degenerate</u> if the linear map $T : E \to E^*$ defined by $T(v)(w) = B(v,w)$ for $(v,w) \in E \times E$ is a linear isomorphism of E onto E^*. Otherwise B is called <u>degenerate</u>.

1.5. <u>Definition</u>. The <u>index</u> of B is defined to be the supremum of the dimensions of subspaces F of E on which B is negative definite. The <u>coindex</u> is defined to be the index of -B.

1.6. <u>Definition</u>. If f is a C^2-function on a C^2-manifold M and p is a critical point of f, we define p to be <u>degenerate</u> or <u>non-degenerate</u> accordingly as the Hessian of f at p is degenerate or non-degenerate. The <u>index</u> and <u>coin-</u>

dex of f at p are defined respectively as the index and coindex of the Hessian of f at p.

1.7. Lemma. Let f be a C^{k+2}-real valued function defined in a neighborhood V on a Hilbert space H. Then there is a C^k-map $A : V \to L_s(H,H)$ such that

$$d^2f(x)(u,v) = \langle A(x)u,v\rangle = \langle A(x)v,u\rangle$$

i.e. A(x) is self-adjoint.

Proof. Let L(H,H) be the Banach space of continuous linear maps of H into itself, $L_s(H,H)$ the closed subspace of those A such that $\langle Au,v\rangle = \langle Av,u\rangle$ for $u,v \in V$.

Let $L_{is}(H,H)$ be the subset of L(H,H) consisting of A mapping H isometrically onto H. Then $L_{is}(H,H)$ is open in L(H,H) (see L[1]). Also $A \to A^{-1}$ is a C^∞-diffeomorphism of $L_{is}(H,H)$ onto $L_{is}(H,H)$. (L[1]). We identify $L^2(H,R)$, the Banach space of continuous bilinear functionals on H, with L(H,H) (L[1]). Then $L^2_s(H,R)$, the closed subspace of symmetric bilinear functionals, is mapped isometrically onto $L_s(H,H)$.

Now $d^2f : V \to L^2_s(H,R)$ is a C^k-map; so by the above identification we have a C^k-map $A : V \to L_s(H,H)$ defined by

$$d^2f_x(u,v) = \langle A(x)u,v\rangle = \langle u,A(x)v\rangle.$$

Q.E.D.

1.8. Lemma. Let f be a C^{k+2}-real valued function defined in a convex neighborhood V of the origin 0 in a Hilbert space H. Suppose 0 is a non-degenerate critical point of f and f(0) = 0. Then there exists a C^k-isomorphism φ at 0 such that

$$f(\varphi(x)) = \langle A(0)x,x\rangle = d^2f(0)(x,x).$$

Proof. Since 0 is a non-degenerate critical point of f, A(0) is invertible. Since A is C^k so for x sufficiently small, A(x) is invertible. Without loss of generality, we will assume that this neighborhood is V.

Define $B : V \to L(H,H)$ by $B(x) = A(0)^{-1}A(x)$. Then B is C^k and B(x) is close to the identity I if x is small. Hence $C(x) = B(x)^{1/2}$ is defined for small x. Without loss of generality, we may assume that V is so small that $C : V \to L(H,H)$ is C^k with C(x) invertible. Since A(0) and A(x) are self-adjoint and since

$$B(x) = A(x)^{-1}A(0)$$

$$A(0) = A(x)B(x).$$

Thus

$$A(x)B(x) = A(0) = A(0)^* = B^*(x)A(x).$$

Obviously, the above relation also holds for any polynomial in B(x), hence for C(x) which is a limit of such polynomials. Thus

$$C(x)^*A(x)C(x) = A(x)C(x)^2 = A(x)B(x) = A(0)$$

or $A(x) = C_1(x)^*A(0)C_1(x)$ where $C_1(x) = C(x)^{-1}$. Write $\psi(x) = C_1(x)x$; then ψ is C^k in V and $f(x) = \langle C_1(x)^*A(0)C_1(x)x,x\rangle = \langle A(0)\psi(x),\psi(x)\rangle$. So it suffices to show only that $d\psi_0$ maps H isometrically and hence, by the inverse function theorem that ψ is C^k-isomorphism on a neighborhood of the origin. Since

$$|\psi(v + x) - \psi(v) - C_1(v)x - d(C_1)_v(v)x|$$

$$= |C_1(v + x)(v + x) - C_1(v)v - C_1(v)x - d(C_1)_v(v)x|$$

$$\le |C_1(v + x)v - C_1(v)v - d(C_1)_v(v)x| + |C_1(v + x)x| + |C_1(v)x|$$

$\to 0$ as $|x| \to 0$,

$$d\psi_v = C_1(v) + d(C_1)_v(v).$$

In particular, $d\psi_0 = C_1(0) = C(0)^{-1} = B(0)^{-1/2} = I$. Thus $\varphi = \psi^{-1}$ is C^k-isomorphism and

$$f(\varphi(x)) = f(\psi^{-1}(x)) = \langle A(0)\psi(\psi^{-1}(x)),\psi(\psi^{-1}(x))\rangle = \langle A(0)x,x\rangle.$$

Q.E.D.

1.9. Morse Lemma. Let H be a Hilbert space, V a convex neighborhood of the origin in H, $f : V \to R$ a C^{k+2}-function ($k \geq 1$) having the origin as a non-degenerate critical point and $f(0) = 0$. Then there is a neighborhood U of the origin and a C^k-diffeomorphism $\varphi : U \to V$ with $\varphi(0) = 0$ and

$$f(\varphi(x)) = \|Px\|^2 - \|(1 - P)x\|^2$$

where P is an orthogonal projection in H.

Proof. Let A be as in 1.7. Let h be the characteristic function of $[0,\infty)$. Then $P = h(A)$ is an orthogonal projection. Let $g(\lambda) = |\lambda|^{-1/2}$.

Since A is invertible, zero is not in the spectrum of A; and since g is continuous except at zero, $T = g(A)$ is a non-singular self-adjoint operator which commutes with A. Now $\lambda g(\lambda)^2 = \mathrm{sgn}(\lambda) = h(\lambda) - (1 - h(\lambda))$ so $AT^2 = P - (I - P)$. Then $f(\varphi Tx) = \langle ATx,Tx\rangle = \langle AT^2x,x\rangle = \|Px\|^2 - \|(1 - P)x\|^2$. So if we write $\varphi \circ T$ as φ we have the desired form. Q.E.D.

1.10. Corollary. The index of f at the origin is the dimension of the range of $(1 - P)$ and the coindex of f at the origin is the dimension of the range of P.

Proof. Let W be the space on which d^2f_0 is negative definite. If $w \in W$ and $(1 - P)w = 0$, then by 1.8 and 1.9

$$d^2f_0(w,w) = f(\varphi(w)) = \|Pw\|^2 - \|(1 - P)w\|^2$$

$$= \|Pw\|^2 \geq 0 \text{ so } w = 0.$$

Thus $(I - P)$ is non-singular on W, hence

$$\dim W \leq \dim \text{range } (I - P).$$

On the other hand, we have

$$\dim W = \dim ([P + (I - P)]W)$$

$$\geq \dim \text{range } (I - P).$$

So the index of f at $0 = \dim \text{range } (I - P)$. Q.E.D.

It is easy to see that if p is any non-degenerate critical point of f, then the Morse Lemma can be stated as:

1.11. Let f be a C^{k+2}-real valued function $(k \geq 1)$ defined in a convex neighborhood V of p in a Hilbert space H. Suppose that p is a non-degenerate critical point of f. Then there is an origin preserving C^k-isomorphism φ of a neighborhood of the origin into H such that $f(\varphi(v) + p) = f(p) + \|Pv\|^2 - \|(1 - P)v\|^2$ where P is an orthogonal projection in H.

1.12. Corollary. A non-degenerate critical point of a C^{k+2}-function on a Hilbert manifold is isolated.

Proof. Without loss of generality, assume 0 is a non-degenerate critical point and $f(0) = 0$. Then $f(\varphi(v)) = \|Pv\|^2 - \|(1 - P)v\|^2$. Thus $df_{\varphi(v)}(w) = 2\langle Pv,w\rangle - 2\langle (1 - P)v,w\rangle$. Hence, if $df_{\varphi(v)} = 0$, in particular put $w = Pv$ in the above formula. We have $0 = 2\langle Pv,Pv\rangle$, i.e. $Pv = 0$. On the other hand, if we let $w = (1 - P)v$, then we have $(1 - P)v = 0$. So $v = Pv + (1 - P)v = 0$. That is, the only critical point in the neighborhood $\varphi(v)$ of 0 is just 0 itself.

Q.E.D.

1.13. Canonical Form Theorem for a Regular Point. Let f be a C^k-real valued function defined in a neighborhood U of the origin of a Banach space E $(k \geq 1)$. Suppose that the origin is a regular point of f and f vanishes there. Then there is a non-zero linear functional ℓ on E and an origin preserving C^k-isomorphism φ of a neighborhood of the origin E into E such that $f(\varphi(v)) = \ell(v)$.

Proof. Clearly $\ell = df_0 \neq 0$.

Choose $x \in E$ such that $\ell(x) = 1$ and let $W = \ell^{-1}(0)$. Then $T : E \to W \times R$ by $T(v) = (v - \ell(v)x, \ell(v))$ is a linear isomorphism onto. Define $\psi : U \to W \times R$ by $\psi(v) = (v - \ell(v)x, f(v))$. Then ψ is C^k and $d\psi_u(v) = (v - \ell(v)x, df_u(v))$ and $d\psi_0 = T$. By the inverse function theorem, ψ is a C^k-isomorphism which obviously preserves the origin. If $v' = \psi^{-1}Tv$ then $(v' - \ell(v')x, f(v')) = \psi(v') = T(v) = (v - \ell(v)x, \ell(v))$. Hence $f(v') = f(\psi^{-1}Tv) = \ell(v)$. Let $\varphi = \psi^{-1}T$. Q.E.D.

1.14. Let f be a C^k-real valued function on a C^k-manifold M $(k \geq 1)$. Let $a \in R$ be a regular value of f and assume $f^{-1}(a)$ does not meet the boundary of M. Then $M_a = \{x \in M \mid f(x) \leq a\}$ and $f^{-1}(a)$ are closed C^k-submanifolds of M and ∂M_a is the disjoint union of $M_a \cap \partial M$ and $f^{-1}(a)$.

Proof. For each $x \in F^{-1}(a)$, choose a chart (U, ψ) at x in M. Without loss of generality that we may assume $a = 0$ and $\psi(x) = 0$. Define $\bar{f} : \psi(U) \to R$ by $\bar{f} = f \circ \psi^{-1}$, then $\bar{f}(0) = f \circ \psi^{-1}(0) = 0$. Also $d\bar{f}_0 = df_x \circ d\psi_0^{-1} \neq 0$.

By the canonical theorem, there is a C^k-isomorphism φ on a neighborhood v of the origin and a linear functional $\ell : E \to R$ such that $\bar{f} \circ \varphi(v) = \ell(v)$ for all v in v. Put $\Phi = \varphi^{-1} \circ \psi : U \to V$. Then Φ is a C^k-isomorphism and f $= \ell \circ \Phi$. Hence $f^{-1}(0) \cap U = f^{-1}(0) \cap \Phi^{-1}(v) = \Phi^{-1}(\ell^{-1}(0) \cap V)$. Since $\ell^{-1}(0)$ is a half space in some Banach space, $(f^{-1}(0) \cap U, \Phi_{|f^{-1}(0)})$ is a chart at

$x \in f^{-1}(0)$. Thus $f^{-1}(0)$ is a closed C^k-submanifold with boundary. Q.E.D.

II. THE STRONG TRANSVERSALITY THEOREM

2.1. <u>Proposition</u>. Let M be a C^{k+1}-manifold without boundary ($k \geq 1$), X a C^k-vector field on M and φ_t the maximum local one parameter group generated by X. If $f : M \to R$ is C^k define a real valued function Xf on M by $Xf(p) = df_p(X_p)$. If $Xf \equiv 1$, then $f(\varphi_t(p)) = f(p) + t$.

Proof. Let $h(t) = f(\varphi_t(p)) = f(\sigma_p(t))$. Then $h'(t) = df\sigma_p(t)(\sigma'_p(t)) = df\sigma_p(t)(X\sigma_p(t)) = Xf(\varphi_t(p)) \equiv 1$. Thus $h(t) = t + f(p)$. Q.E.D.

2.2. <u>Proposition</u>. If $Xf \equiv 1$, $f(M) = (-\epsilon,\epsilon)$ for some $\epsilon > 0$, and $\varphi_t(x)$ is defined for $|t + f(x)| < \epsilon$, then $W = f^{-1}(0)$ is a closed C^k-submanifold of M and the map $F : W \times (-\epsilon,\epsilon) \to M$ defined by $F(w,t) = \varphi_t(w)$ is a C^k-isomorphism of $W \times (-\epsilon,\epsilon)$ onto M which for each $c \in (-\epsilon,\epsilon)$ maps $W \times \{c\}$ C^k-isomorphically onto $f^{-1}(c)$.

Proof. Since $Xf \equiv 1$, f cannot have any critical values. Thus by the smoothness theorem $f^{-1}(c)$ and W are closed C^k-submanifolds of M.

F is 1 - 1. Since if $F(w,t) = F(w',t')$, then

$$t = f(w) + t = f(\varphi_t(w)) = f(\varphi_{t'}(w')) = f(w') + t' = t'$$

by the proposition 2.1. Thus $t = t'$ and hence $\varphi_{t'}(w')$. Set $t = 0$, then we have $w = w'$.

F is also onto. If $m \in M$, then $|-f(m) + f(m)| < \epsilon$ so $w = \varphi_{-f(m)}(m)$ is defined. Thus $f(w) = f(m) - f(m) = 0$ so $w \in W$. Also note that $F(w,f(m)) = \varphi_{f(m)}(\varphi_{-f(m)}(m)) = \varphi_{f(m)}\varphi_{-f(m)}(m) = m$. This proves that F is onto. By an easy calculation, we see $F^{-1}(m) = (\varphi_{-f(m)}(m), f(m))$ which is obviously C^k. Thus F is a C^k-isomorphism. Also $f(F(w,c)) = f(\varphi_c(w)) = f(w) + c = c$. Thus F maps $W \times \{c\}$ C^k-isomorphically onto $f^{-1}(c)$. Q.E.D.

2.3. <u>Corollary</u>. *$W = f^{-1}(0)$ is C^k-isomorphic to $f^{-1}(c)$ for any c*

$\in (-\epsilon,\epsilon)$.

2.4. Definition. A C^k-vector field x on a C^{k+1}-manifold without boundary M ($k \geq 1$) *will be said to be* C^k-strongly transverse to a C^k-function $f : M \to R$ on a closed interval $[a,b]$ if for some $\delta > 0$ the following two conditions are true for $V = f^{-1}(a - \delta, b + \delta)$.

(1) Xf is C^k and $Xf \neq 0$ on V.

(2) If $p \in V$ and σ_p is the maximum solution curve of X with initial condition p then $\sigma_p(t)$ is defined and not in V for some positive t and also for some negative t.

2.5. Lemma. Let X be a C^k-vector field on a C^{k+1}-manifold without boundary M ($k \geq 1$) and be C^k-strongly transverse to a C^k-function $f : M \to R$ on $[a,b]$.

Let $Y = X/Xf$, $V = f^{-1}(a - \delta, b + \delta)$, $g = f1_V - \frac{a + b}{2}$ and $\epsilon = \frac{b - a}{2} + \delta$, then the triple (V,g,Y) satisfies proposition 2.2.

Proof. Clearly, V is an open submanifold of M and $Yf \equiv 1$ on V. If σ is an integral curve of X, then

$$Y(\sigma(t)) = \frac{X(\sigma(t))}{Xf(\sigma(t))} = \frac{\sigma'(t)}{Xf(\sigma(t))} .$$

Since $Xf(\sigma(t))$ is a scalar function, this means on $\sigma(t)$, Y has the same direction as X. Since $Yf \equiv 1$ on V, the integral curves of Y are just the integral curves of X reparametrized so that $f(\sigma(t)) = f(\sigma(0) + t$. By condition (2) of 2.4 we know that if ψ_t is the maximum local one parameter group generated by Y on V, then $\psi_t(p)$ is defined on V. That is

$$a - \delta < f(p) + t < b + \delta.$$

Then $g(v) = (f_{1v} - \frac{a + b}{2})(v) = (a - \delta, b + \delta) - \frac{a + b}{2} = (a - \delta - \frac{a + b}{2}, b + \delta - \frac{a + b}{2}) = (-\epsilon,\epsilon)$. Also for $x \in V$, $t + g(x) = t + f(x) - \frac{a + b}{2}$.

But $a - \delta < f(p) + t < b + \delta$. Thus, $-\epsilon = a - \delta - \frac{a+b}{2} < t + g(p) = f(p) + t - \frac{a+b}{2} < b + \delta - \frac{a+b}{2} = \epsilon$; i.e. $-\epsilon < t + g(p) < \epsilon$ or $|t + g(p)| < \epsilon$. Q.E.D.

2.6. <u>Strong Transversality Theorem</u>. Let f be a C^k-real valued function on a C^{k+1}-manifold without boundary M $(k \geq 1)$. If there exists a C^k-vector field X on M which is C^k-strongly transverse to f on $[a,b]$, then $W = f^{-1}(a)$ is a closed C^k-submanifold of M; and for some $\delta > 0$ there is a C^k-isomorphism F of $W \times (a - \delta, b + \delta)$ onto an open submanifold of M such that F maps $W \times \{c\}$ C^k-isometrically onto $f^{-1}(c)$ for all $c \in (a - \delta, b + \delta)$.

Proof. Use the same notation as in 2.5. Then (V,g,Y) satisfies 2.2. Since $\frac{a-b}{2} \in (-\epsilon,\epsilon)$ is a regular value of g then $g^{-1}(\frac{a-b}{2})$ is a closed C^k-submanifold of M. But $g(x) = \frac{a-b}{2}$ iff $g(x) + \frac{a+b}{2} = a$ iff $f(x) = a$. So $f^{-1}(a)$ is a C^k-closed submanifold of M. By 2.2, there is a C^k-isomorphism $G : g^{-1}(0) \times (-\epsilon,\epsilon)$ onto V which maps $g^{-1}(0) \times \{c\}$ C^k-isomorphically onto $g^{-1}(c)$. By 2.3 $g^{-1}(0)$ is C^k-isomorphic to $g^{-1}(\frac{a-b}{2}) = f^{-1}(a) = W$, and note that there is a C^∞-isomorphic function which maps the interval $(-\epsilon,\epsilon)$ onto $(-\epsilon,\epsilon) + \frac{a+b}{2}$ which is equal to $(a - \delta, b + \delta)$. So we have a C^k-isomorphism $F : W \times (a - \delta, b + \delta) \to W$. Q.E.D.

2.7. <u>Corollary</u>. There is a C^k-map $H : M \times I \to M$ such that if we put $H_s(p) = H(p,s)$ then

(1) H_s is a C^k-isomorphism of M onto itself for all $s \in I$.

(2) $H_s(m) = m$ if $m \notin f^{-1}(a - \delta/2, b + \delta/2)$.

(3) H_0 = identity.

(4) $H_1(f^{-1}(-\infty,a) = f^{-1}(-\infty,b)$.

Proof. Let $h : R \to R$ be a C^∞-function with strictly positive derivative such that $h(t) = t$ if $t \notin (a - \delta/2, b + \delta/2)$ and $h(a) = b$. This is possible by rotating a certain bell shaped function. Define H_s as follows: $H_s(x) = x$

if $x \notin f^{-1}(a - \delta/2, b + \delta/2)$. $H_s(F(w,t)) = F(w,(1 - s)t + sh(t))$ for $x = F(w,t) \in f^{-1}(a - \delta, b + \delta)$. Then

(1) H_s is well-defined. If $x \in f^{-1}(a - \delta, a - \delta/2] \subset f^{-1}(a - \delta, b + \delta)$ then $x = F(w,t)$ for some $w \in W$ and $t \in (a - \delta, b + \delta)$, since $W \times (a - \delta, b + \delta)$ is C^k-isomorphic to $f^{-1}(a - \delta, b + \delta)$ under F by 2.6 and t in fact is equal to $f(x) \in (a - \delta, a - \delta/2]$. Thus $H_s(x) = H_s(F(w,t)) = F(w,(1 - s)t + sh(t)) = F(w,(1 - s)t + st) = F(w,t) = x$. Thus H_s is well-defined on $f^{-1}(a - \delta, a - \delta/2]$. Similarly we can prove for $f^{-1}[b + \delta/2, b + \delta)$.

(2) H_s is a C^k-isomorphism of M onto itself for all $s \in I$.

(3) If $m \notin f^{-1}(a - \delta/2, b + \delta/2)$, then $H_s(m) = m$ by the construction of H_s.

(4) $H_0(F(w,t)) = F(w,t)$ by definition.

(5)
$$\begin{aligned} H_1(f^{-1}(-\infty,a]) &= H_1(f^{-1}((-\infty, a - \delta/2]) \cup f^{-1}((a - \delta/2, a]))] \\ &= H_1(f^{-1}((-\infty, a - \delta/2])) \cup H_1[f^{-1}((a - \delta/2, a]))] \\ &= (-\infty, a - \delta/2] \cup H_1[f^{-1}((a - \delta/2, a]))]. \end{aligned}$$

Claim that $H_1(f^{-1}(a - \delta/2, a]) = f^{-1}(a - \delta/2, b]$.

If $x \in f^{-1}((a - \delta/2, a])$, then $x \in f^{-1}(f(x))$ by the C^k-isomorphism F of theorem 2.6 that $x = F(w,f(x))$ for some $w \in W = f^{-1}(a)$. Hence $H_1(x) = H_1(F(w,f(x))) = F(w,h(f(x)))$. But F maps $W \times \{h(f(x))\}$ C^k-isomorphically to $f^{-1}(h(f(x)))$ or $f \circ F$ maps $W \times \{h(f(x))\}$ onto $h(f(x))$. Thus $f(H_1(x)) = h(f(x))$. Hence as x varies in $f^{-1}(a - \delta/2, a]$, $f(x)$ varies in $(a - \delta/2, a]$ but as h is strictly increasing $hf(x)$ varies in $(h(a - \delta/2),h(a)] = (a - \delta/2, b]$ (by definition of h). Therefore

$$H_1(f^{-1}(a - \delta/2, a]) = f^{-1}((h(a - \delta/2),h(a)])$$

$$= f^{-1}(a - \delta/2,\ b].$$

Hence $H_1(f^{-1}(-\infty,a]) = f^{-1}(-\infty,b]$. Q.E.D.

III. RIEMANNIAN MANIFOLD

3.1. Definition. If M is a C^{k+1}-manifold and for each $p \in M$ M_p is a separable Hilbert space, then we say that M is a C^{k+1}-Hilbert manifold ($k \geq 0$). For each $p \in M$, denote $< , >$ to be an admissible inner product in M_p, i.e. a positive definite symmetric, bilinear form on M_p such that the norm $\|v\|_p = \langle v,v\rangle^{1/2}$ defines the topology of M_p.

Let $(D(\varphi),\varphi)$ be a chart in M with image in a Hilbert space $(H, < , >)$. Define $G^{\varphi} : D(\varphi) \to L^2_{s,p}(H)$ by $\langle G^{\varphi}(x)u,v\rangle = \langle d\varphi_x^{-1}(u), d\varphi_x^{-1}(v)\rangle_x$ where $H^2_{s,p}(H)$ is the space of positive definite symmetric operators on H.

If $(D(\psi),\psi)$ is another chart in M modelled on H, then let $U = D(\varphi) \cap D(\psi)$, $f = \varphi \circ \psi^{-1} : \psi(U) \to \varphi(V)$. Then $df_{\psi(x)} = d\varphi_x \circ d\psi^{-1}_{\varphi(x)}$ or $d\psi_x^{-1} = d\varphi_x^{-1} \circ df_{\psi(x)}$ for $x \in U$. Then

$$\langle G^{\psi}(x)u,v\rangle = \langle d\psi_x^{-1}(u), d\psi_x^{-1}(v)\rangle = \langle d\varphi_x^{-1}(df_{\psi(x)}(u)), d\varphi_x^{-1}(df_{\psi(x)}(v))\rangle$$

$$= \langle G^{\varphi}(x)df_{\psi(x)}(u), df_{\psi(x)}(v)\rangle = \langle df^{*}_{\psi(x)}G^{\varphi}_{(x)}df_{\psi(x)}(u),v\rangle$$

for all $u,v \in H$, thus $G^{\psi}(x) = df^{*}_{\psi(x)}G^{\varphi}_{(x)}df_{\psi(x)}$. Hence G^{φ} is C^k iff G^{ψ} is C^k. Thus it makes sense to say that G^{φ} is C^k if φ is. We call $x \to < , >_x$ a C^k-Riemannian structure for M and M is a C^{k+1}-Riemannian manifold.

3.2. Lemma. If M is a connected C^k-Banach manifold $x,y \in M$ then there is a C^k-path $\sigma : [a,b] \to M$ such that $\sigma(a) = x$ and $\sigma(b) = y$.

Proof. Define a relation $x \sim y$ if such a σ exists. Then it is easy to see that $\sim$ is an equivalence relation.

To see the transitivity, let

$$\sigma : [a,b] \to M$$

$$\tau : [c,d] \to M$$

be two C^k-paths with $\sigma(b) = \tau(c)$. Without loss of generality assume $a = 0$, $b = c = 1/2$, $d = 1$. Let $\varphi : R \to R$ be a non-decreasing C^∞-function with $\varphi(\sigma) = 0$ and $\varphi(t) \approx 1/2$ if $1/4 \le t \le 3/4$, and $\varphi(1) = 1$. Define $\gamma : [0,] \to M$ by

$$\begin{aligned} \gamma(t) &= \sigma(\varphi(t)) \qquad 0 \le t \le 3/4 \\ &= \tau(\varphi(t)) \qquad 1/4 \le t \le 1. \end{aligned}$$

Thus $\sigma(0) \sim \tau(1)$. Q.E.D.

<u>Claim</u>. The equivalence class of each point φx is open. Since if $\varphi : U \to V$ is a chart at x, then every point $y \in U$ can be joined to x by a straight line so $x \sim y$. Thus M can have only one equivalence class, that is, M is C^k-path connected.

3.3. <u>Definition</u>. If $\sigma : [a,b] \to M$ is a C^1-map then define the length $L(\sigma)$ of σ by

$$L(\sigma) = \int_a^b \|\sigma(t)\| \, dt.$$

For $x,y \in M$, define

$$\rho(x,y) = \inf \{L(\sigma) : \sigma \text{ is a } C^1\text{-path joining } x \text{ and } y\}.$$

This is well-defined by lemma 3.2. It is easy to see that ρ thus defined is a pseudo metric. To see ρ is a compatible metric we need the following lemma.

3.4. <u>Lemma</u>. Let H be a Hilbert space, $f : [a,b] \to H$ a C^1-map. Then $\int_a^b \|f'(t)\| \, dt \ge \|f(b) - f(a)\|$.

Proof. Assume $f(a) \ne f(b)$. Let $g(t)(f(b) - f(a))$ be the orthogonal projection of $f(t) - f(a)$ on the one-dimensional space spanned by $f(b) - f(a)$. Then $g : [a,b] \to R$ is C^1. $g(a) = 0$, $g(b) = 1$ and $f(t) - f(a) = g(t)(f(b) - f(a)) + h(t)$ where $h(t)$ is in orthogonal complement of $f(b) - f(c)$ and is C^1.

Then $f'(t) = g'(t)(f(b) - f(a)) + h'(t)$ where $h'(t) + (f(b) - f(a))$. This is obvious since $(h(t), f(b) - f(a)) = 0$. So

$$\|f'(t)\|^2 = \|f(b) - f(a)\|^2 \cdot |g'(t)|^2 + \|h'(t)\|^2$$

$$\geq \|f(b) - f(a)\|^2 \cdot |g'(t)|^2.$$

So $\int_a^b \|f'(t)\| \, dt \geq \|f(b) - f(a)\| \cdot \int_a^b |g'(t)| \, dt \geq \|f(b) - f(a)\|$. Since $\int_a^b |g'(t)| \, dt \geq \int_a^b g'(t)dt = g(b) - g(a) = 1$. Q.E.D.

3.5. <u>Theorem</u>. ρ is a metric and is compatible with the original metric.

Proof. Let $x,y \in$ chart $D(\varphi)$. Let $\sigma : [a,b] \to M$ be a C^1-map joining x and y, i.e. $\sigma(a) = x$, $\sigma(b) = y$, $x \neq y$. Thus $f = \varphi \circ \sigma$ is a C^1-map : $[a,b] \to H$. By lemma 3.4 we have

$$\|\varphi(y) - \varphi(x)\| = \|\varphi \circ \sigma(b) - \varphi \circ \sigma(a)\|$$

$$= \|f(b) - f(a)\| \leq \int_a^b \|(\varphi \circ \sigma)'(t)\| \, dt$$

$$= \int_a^b \|d\varphi_{\sigma(t)} \circ \sigma'(t)\| \, dt$$

$$\leq \int_a^b \|d\varphi_{\sigma(t)}\| \, \|\sigma'(t)\| \, dt$$

$$\leq M \int_a^b \|\sigma'(t)\| \, dt$$

where $M = \sup_{t \in [a,b]} \|d\varphi_{\sigma(t)}\| < \infty$. Clearly $M > 0$. Thus we have $L(\sigma) \leq 1/M \, \|\varphi(y) - \varphi(x)\|$.

(*) Therefore $\rho(x,y) \geq 1/M \, \|\varphi(y) - \varphi(x)\| > 0$.

Therefore ρ is a metric.

If x and y are not in the same chart, we just consider each of the open charts which cover the path from x to y and argue as above.

On the other hand, let $x,y \in D(\varphi)$. Define $g(t) = \frac{b-t}{b-a}\varphi(x) + \frac{t-a}{b-a}\varphi(y)$ $: [a,b] \to H$. Then obviously g is C^1.

Let $\sigma = \varphi^{-1} \circ g$, then σ is a C^1-path joining x and y.

Also $d\sigma = d\varphi^{-1} \circ dg$.

$$\rho(x,y) \le \int_a^b \|\sigma'(t)\|\,dt \le \int_a^b \|d\varphi^{-1}_{g(t)}\|\,\|g'(t)\|\,dt$$

$$\le M' \int_a^b \|g'(t)\|\,dt$$

where $M' = \sup_t \|d\varphi^{-1}_{g(t)}\| < \infty$.

$$\|g'(t)\| = \left\|-\frac{\varphi(x)}{b-a} + \frac{\varphi(y)}{b-a}\right\| = \frac{1}{b-a}\|\varphi(y) - \varphi(x)\|.$$

So

$$(**) \qquad \rho(x,y) \le \frac{M'}{b-a}\|\varphi(y) - \varphi(x)\|.$$

Combining (*) and (**) we get that ρ is a compatible metric. Q.E.D.

3.6. Definition. If M is a C^{k+1}-Riemannian manifold then the metric ρ defined above on each component of M is called the Riemannian metric of M. If each component of M is a complete metric space in this metric then M is called a complete C^{k+1}-Riemannian manifold.

3.7. Definition. If σ is a C^1-map of an open interval (a,b) into a Riemannian manifold M we define the length of σ, $L(\sigma)$ to be

$$\lim_{\substack{\alpha\to a\\ \beta\to b}} \int_a^b \|\sigma'(t)\|\,dt.$$

Note. $L(\sigma)$ may be infinite.

3.8. Proposition. If M is a C^{k+1}-Riemannian manifold and $\sigma : (a,b) \to M$

is a C^1-curve of finite length, then the range of σ is a totally bounded subset of M, hence has compact closure if M is complete.

Proof. If $L(\sigma) < \infty$, then given $\epsilon > 0$ there exist a and b such that

$$\int_a^b \|\sigma'(t)\| \, dt + \epsilon < L(\sigma).$$

For this ϵ and a,b choose $t_0 = a < t_1 < t_2 < \ldots < t_n < b = t_{n+1}$ so that

$$\int_{t_i}^{t_{i+1}} \|\sigma'(t)\| \, dt < \epsilon.$$

Hence $\sigma((a,b))$ is contained in the finite union of ϵ-balls about the $\sigma(t_i)$ $i = 1, 2, \ldots, n$. Q.E.D.

3.9. <u>Proposition</u>. Let x be a C^k-vector field on a complete C^{k+1}-Riemannian manifold M ($k \geq 1$) and $\sigma : (a,b) \to M$ be a maximum solution curve of X. If $b < \infty$ then

$$\int_0^b \|X(\sigma(t))\| \, dt = \infty,$$

hence in particular $\|X(\sigma(t))\|$ is unbounded on $[0,b)$. Similarly, if $a > -\infty$, then

$$\int_a^0 \|X(\sigma(t))\| \, dt = \infty,$$

hence $\|X(\sigma(t))\|$ is unbounded on $(a,0]$.

Proof. If $\int_0^b \|X(\sigma(t))\|$ were finite, then by 3.8 we have

$$\int_0^b \|\sigma'(t)\| \, dt = \int_0^b \|X(\sigma(t))\| \, dt < \infty.$$

Hence $\sigma(t)$ would have a limit point as $t \to b$ contradicting [L1, Theorem 4, p. 65]. Q.E.D.

3.10. <u>Definition</u>. Let $f : M \to R$ be a C^{k+1}-real valued function on a C^{k+1}-Riemannian manifold M. Given $p \in M$, df_p is a continuous linear functional on M_p, hence there is a unique vector $\nabla f_p \in M_p$ such that $df_p(v) = \langle v, \nabla f_p \rangle_p$ for all $v \in M_p$. ∇f_p is called the <u>gradient</u> of f at p and $\nabla f : p \to \nabla f_p$ is called the <u>gradient</u> of f.

3.11. <u>Proposition</u>. ∇f is a C^k-vector field in M.

Proof. Let $\varphi : D(\varphi) \to H$ be a chart and H be a Hilbert space with inner product $\langle \; , \; \rangle$. Let T be the canonical identification of H^* with H, i.e. if $\ell \in M^*$, $v \in H$, then $\ell(v) = \langle v, T\ell \rangle$. Since T is a linear isomorphism it is C^∞. Define $g = f \circ \varphi^{-1}$. Then $g \in C^{k+1}$ and $dg : U \to H^* \in C^k$. Thus $T \circ dg = \lambda$ is C^k. Now by definition of G^φ

$$\langle G^\varphi(x) d\varphi_x(\nabla f_x), v \rangle = \langle \nabla f_x, d\varphi_x^{-1}(v) \rangle_x$$

$$= df_x \circ d\varphi_x^{-1}(v)$$

$$= dt_{\varphi(x)}(v) = \langle Tdg_{\varphi(x)}, v \rangle.$$

So $d\varphi_x(\nabla f_x) = (G^\varphi(x))^{-1}\lambda(\varphi(x))$. Thus $x \to d\varphi_x(\nabla f_x)$ is a C^k-map of $D(\varphi)$ into H. By the definition of C^k-structure on $T(M)$, ∇f is a C^k-vector field on H.

Q.E.D.

IV. CONDITION (C)

In this section, we assume that M is a C^{k+2}-Riemannian manifold ($k \geq 1$) without boundary.

4.1. Definition. A C^{k+2}-function $f : M \to R$ is called a C^{k+2}-Morse function if all the critical points are non-degenerate and it also satisfies the following condition (C):

(C) If S is any subset of M on which f is bounded but on which $\|\nabla f\|$ is not bounded away from zero then there is a critical point of f adherent to S, that is, belongs to $\bar{S}$ the closure of S.

4.2. Remarks.

(1) If M is compact, then condition (C) is always satisfied. In fact (C) is satisfied if f is proper.

(2) Condition (C) only gives a critical point in $\bar{S}$, there may not exist a sequence in S converging to that point. For example, let $M = R$, f a constant function, S the set of integers. Then obviously, S has no limit point.

(3) But if S is such that for every x in S, $\|\nabla f_x\| \neq 0$, then by condition (C) we can find a critical point $y \in \bar{S}$ and a sequence in S converging to y.

4.3. Proposition. If a and b are two real numbers then there are at most a finite number of critical points of a C^2-Morse function f satisfying $a < f(p) < b$. Hence the critical values of f are isolated and there at most a finite number of critical points of f on any critical level.

Proof. Suppose $\{p_n\}$ is a sequence of distinct critical points of f satis-

fying $a < f(p_n) < b$. *Since critical points are isolated we can choose for each* n a regular point q_n such that $\rho(p_n,q_n) < 1/n$. Since ∇f is continuous, we may assume that $\|\nabla f q_n - \nabla f p_n\| < 1/n$ and $a < f(q_n) < b$. Since $\nabla f p_n = 0$, so we have $0 < \|\nabla f q_n\| < 1/n$ and $a < f(q_n) < b$. By condition (C), there is a subsequence of $\{q_n\}$ converging to a critical point q of f. Hence the corresponding subsequence of $\{p_n\}$ will also converge to p. But this contradicts the fact that the critical points of f are isolated. Q.E.D.

4.4. Lemma. Let M be a C^2-complete Riemannian manifold and let $\sigma : (\alpha,\beta) \to M$ be a maximum solution curve of ∇f. Then either $\lim_{t\to\beta} f(\sigma(t)) = \infty$ or else $\beta = \infty$ and $\sigma(t)$ has no critical point of f as a limit point as $t \to \beta$. Similarly either $\lim_{t\to\alpha} f(\sigma(t)) = -\infty$ or else $\alpha = -\infty$ and $\sigma(t)$ has a critical point of f as a limit point as $t \to \alpha$.

Proof. Let $g(t) = f(\sigma(t))$. Then $g'(t) = df_{\sigma(t)}(\sigma'(t)) = df_{\sigma(t)}(\nabla f_{\sigma(t)}) = \|\nabla f_{\sigma(t)}\|^2 \geq 0$. So g is monotone increasing, hence has a limit point B as $t \to \beta$.

Suppose $B < \infty$. Then since $B \geq g(t) = g(0) + \int_0^t g'(s)ds = g(0) + \int_0^t \|\nabla f_{\sigma(s)}\|^2 ds$, it follows that $\int_0^\beta \|\nabla f_{\sigma(s)}\|^2 ds < \infty$. If $\beta < \infty$, then by Schwartz inequality $\int_0^\beta \|\nabla f_{\sigma(s)}\| ds \leq \beta^{1/2} (\int_0^\beta \|\nabla f_{\sigma(s)}\|^2 ds)^{1/2} < \infty$ which would contradict proposition 3.9. Hence $\beta = \infty$ and the fact that $\int_0^\infty \|\nabla f_{\sigma(s)}\|^2 ds < \infty$ will imply that $\|\nabla f_{\sigma(s)}\|$ cannot be bounded away from zero for $0 \leq s < \infty$. If $\|\nabla f\| = 0$ for all except a finite set, then the lemma is obvious. So we assume S is an infinite set on which $\|\nabla f\| \neq 0$ and $\|\nabla f\|$ is not bounded away from zero, then since $f(\sigma(0)) \leq f(\sigma(s)) \leq B$ for $0 \leq s < \infty$ so by condition (C) $\sigma(t)$ has a critical point of f as limit point as $t \to \beta$. Q.E.D.

4.5. Proposition. If M is complete and f has no critical values in the closed interval $[a,b]$ then ∇f is C^{k+1}-strongly transverse to f on $[a,b]$, hence by corollary 2.7 (4), $M_a = \{x \in M \mid f(x) \leq a\}$ and $M_b = \{x \in M \mid f(x) \leq b\}$ are C^{k+1}-isomorphic.

Proof. Since critical points of f are isolated, there is a $\delta > 0$ such that f has no critical values in $[a - \delta, b + \delta]$. Let $V = f^{-1}(a - \delta, b + \delta)$. Then $(\nabla f)f = \|\nabla f\|^2 > 0$ and C^{k+1} in V. Let $p \in V$ and let $\sigma : (\alpha,\beta) \to M$ be the maximal integral curve of ∇f with initial condition p. We want to show for some t_1, t_2 such that $\alpha < t_2 < 0 < t_1 < \beta$ that $\sigma(t_1)$ and $\sigma(t_2)$ are not in V, i.e. $f(\sigma(t_1)) \leq a - \delta$, and $f(\sigma(t_2)) \geq b + \delta$. Suppose for example that $f(\sigma(t)) < b + \delta$ for $0 < t < \beta$. Then by lemma 4.4 $\sigma(t)$ would have a critical point p_0 as limit point as $t \to \beta$.

Since f is continuous and $f(\sigma(t))$ is monotone we have $a - \delta < f(p) = f(\sigma(0)) \leq f(p_0) = \lim_{t\to\beta} f(\sigma(t)) \leq b + \delta$. So we get $f(p_0)$ is a critical value in $[a - \delta, b + \delta]$, which is a contradiction. Q.E.D.

V. HANDLES

Let H be a separable Hilbert space, D^k the closed unit ball of dimension k $(0 \leq k \leq \infty)$. By the smoothness theorem for regular levels, D^k is a closed C^∞-submanifold of H. Also the boundary ∂D^k of D^k is S^{k-1} the unit sphere in H. We call $D^k \times D^\ell$ a handle of index k and coindex ℓ.

5.1. Definition. Let M be a C^r-Hilbert manifold and N a closed submanifold of M. Let f be a homeomorphism of $D^k \times D^\ell$ onto a closed subset h of M. We say that M arises from N by a C^r-attachment of a handle of type (k,ℓ) if

(1) $M = N \cup h$.

(2) $f \mid S^{k-1} \times D^\ell$ is a C^r-isomorphism onto $h \cap \partial N$.

(3) $f \mid D^k \times D^\ell$ is a C^r-isomorphism onto $M \setminus N$.

5.2. Remark. We actually have $N \cap h = \partial N \cap h$. For if $x \in N \cap h$, then since $h = f(D^k \times D^\ell)$ we have $f^{-1}(x) \in D^k \times D^\ell$. Now if $f^{-1}(x) \in D^k \times D^\ell$, then by (3) of definition 5.1 $x = f(f^{-1}(x))$ would be in $M \setminus N$ i.e. $x \notin N$. This cannot happen since $x \in N \cap h$. Thus $f^{-1}(x) \times S^{k-1} \times D^\ell$, so $f(f^{-1}(x)) = x \in f(S^{k-1} \times D^\ell)$, i.e. $x \in h \cap \partial N$. Q.E.D.

5.3. Definition. Suppose $N = N_0, N_1, \ldots, N_s = M$ is a sequence of C^r-manifolds such that N_{i+1} arises from N_i by a C^r-attachment f_i of a handle of type (k_i,ℓ_i). If the images of the f_i are disjoint, then we shall say that M arises from N by disjoint C^r-attachments $(f_1, \ldots, f_s)$ of handles of type $((k_1,\ell_1), \ldots, (k_s,\ell_s))$.

5.4. Lemma. Let $\lambda : R \to R$ be a C^∞-function which is monotone non-increasing and satisfying $\lambda(x) = 1$ if $x \leq 1/2$; $\lambda(x) > 0$ if $x < 1$ and $\lambda(x) = 0$ if $x \geq 1$. For $0 \leq s \leq 1$ let $\sigma(s)$ be the unique solution of $\lambda(\sigma)/(1 + \sigma) = \frac{2}{3}(1 - s)$ in the interval $[0,1]$. Then σ is strictly monotone increasing, continuous C^∞ in $[0,1)$ and $\sigma(0) = 1/2$, $\sigma(1) = 1$. Moreover if $\epsilon > 0$ and $u^2 - v^2 \geq -\epsilon$ and u^2

$- v^2 - 3\epsilon/2\ \lambda(u^2/\epsilon) \leq -\epsilon$ then $u^2 \leq \epsilon\sigma(\frac{v^2}{\epsilon + u^2})$.

Proof. Clearly $\lambda(\sigma)/(1 + \sigma)$ is strictly monotonically decreasing if $0 \leq \sigma \leq 1$. By definition of λ, $\lambda(\sigma)/(1 + \sigma) = 1$ if $\sigma = 0$ and $\lambda(\sigma)/(1 + \sigma) = 0$ if $\sigma = 1$. Thus σ exists and is continuous and monotone. By inspection we see that if $s = 0$, then $\sigma(0) = 1/2$ is a solution. Thus by the uniqueness of the solution, $\sigma(0) = 1/2$. Similarly $\sigma(1) = 1$.

The derivative of $\lambda(\sigma)/(1 + \sigma)$ is

$$\left[\frac{\lambda(\sigma)}{1 + \sigma}\right]' = \frac{(1 + \sigma)\lambda'(\sigma) - \lambda(\sigma)}{(1 + \sigma)^2}$$

so $(\lambda(\sigma)/(1 + \sigma))' = 0$ iff $\lambda'(\sigma) = \lambda(\sigma)/(1 + \sigma)$. But $\lambda(\sigma)/(1 + \sigma) = \frac{2}{3}(1 - s) \geq 0\ \forall\ 0 \leq s \leq 1$ whereas $\lambda'(\sigma) \leq 0\ \forall\ \sigma$. So $\lambda(\sigma)/(1 + \sigma) = \lambda'(\sigma)$ is only possible at $s = 1$. Thus $\lambda(\sigma)/(1 + \sigma)$ has a non-vanishing derivative in $[0,1)$. It follows from the inverse function theorem that σ is C^∞ in $[0,1)$.

Now consider the function $f(u,v) = u^2 - \epsilon\sigma(\frac{v^2}{\epsilon + u^2})$ defined in the region $u^2 - v^2 \geq -\epsilon$, $u^2 - v^2 - 3\epsilon/2\ \lambda(u^2/\epsilon) \leq -\epsilon$. Take partial derivative of f with respect to u:

$$\begin{aligned} f_u &= 2u + 2u\epsilon\sigma'(\frac{v^2}{\epsilon + u^2})\ \frac{v^2}{(\epsilon + u^2)^2} \\ &= 2u[1 + \epsilon\sigma'(\frac{v^2}{\epsilon + u^2})\ \frac{v^2}{(u^2 + \epsilon)^2}\ ] \end{aligned}$$

Since σ is monotonic increasing, $\sigma' \geq 0$. So $f_u = 0$ iff $u = 0$. For v fixed, $u = 0$ is the only critical point of f. Also it is easy to see that f has a minimum at $u = 0$. Since f is monotonic increasing with v fixed, f must assume its maximum on the boundary. On the boundary curve $u^2 - v^2 = -\epsilon$ we have $\frac{v^2}{\epsilon + u^2} = 1$ so $f(u,v) = u^2 - \epsilon$. If (u,v) is not also on the other boundary curve, i.e. $u^2 - v^2 - 3\epsilon/2\ \lambda(u^2/\epsilon) < -\epsilon$ or $-3\epsilon/2\ \lambda(u^2/\epsilon) < -u^2 + v^2 - \epsilon = 0$, so $\lambda(u^2/\epsilon) > 0$. Hence $u^2 < \epsilon$ so $f(u,v) < 0$. On the other hand if (u,v) is on the boundary $u^2 - v^2 - 3\epsilon/2\ \lambda(u^2/\epsilon) = -\epsilon$ we have

$$\frac{v^2}{\epsilon + u^2} = 1 - \frac{3}{2(1 + u^2/\epsilon)}\lambda(u^2/\epsilon).$$

Now on this boundary $u^2/\epsilon \geq 1/2$ for otherwise $u^2/\epsilon < 1/2$ implies $\lambda(u^2/\epsilon) = 1$. Hence then $\dfrac{v^2}{\epsilon + u^2} < 1 - 3/2(1 + 1/2) = 1 - 1 = 0$. Then (u,v) cannot be on the boundary curve $u^2 - v^2 = -\epsilon$. Clearly, $u^2/\epsilon \leq 1$ so $u^2/\epsilon = \sigma(\rho)$ for some $\rho = \rho$ in $[0,1]$.

By definition of $\sigma(\rho)$

$$\frac{v^2}{\epsilon + u^2} = 1 - \frac{3}{2}\frac{\lambda(\sigma(\rho))}{1 + \sigma(\rho)} = 1 - (1 - \rho) = \rho$$

hence $f(u,v) = u^2 - \epsilon\sigma\left(\dfrac{v^2}{\epsilon + u^2}\right) = \epsilon\sigma(\rho) - \epsilon\sigma(\rho) = 0$, i.e. f vanishes on this boundary.

Thus $f \leq 0$ everywhere on the boundary of the region and hence also is in the interior. Q.E.D.

5.5. <u>Theorem</u>. Let B be the ball of radius 2ϵ about the origin in a Hilbert space H. Define $f : B \to R$ by $f(v) = \|Pv\|^2 - \|Qv\|^2$ where P is an orthogonal projection on a subspace H^ℓ of dimension ℓ and $Q = (1 - P)$ is a projection on a subspace H^k of dimension k. Let $g(v) = f(v) - 3\epsilon/2\ \lambda(\|Pv\|^2/\epsilon)$ where $\lambda : R \to R$ is as in the above lemma. Then $M = \{x \in B \mid g(x) \leq -\epsilon\}$ arises from $N = \{x \in B \mid f(x) \leq -\epsilon\}$ by a C^∞-attachment F of a handle h of type (k,ℓ).

Proof. Let D^k, D^ℓ be the unit discs in H^k and H^ℓ respectively. Let h be the set $\{x \in B \mid f(x) \geq -\epsilon$ and $g(x) \leq -\epsilon\}$. So $M = N \cup h$ and $N \cap h \subseteq \partial N$. Define $F : D^k \times D^\ell \to H$ by

$$F(x,y) = (\epsilon\sigma(\|x\|^2)\|y\|^2 + \epsilon)^{1/2}x + (\epsilon\sigma(\|x\|^2))^{1/2}y$$

where σ is as in the lemma. Then

$$\begin{aligned} f(F(x,y)) &= \epsilon[\sigma(\|x\|^2)\|y\|^2 - (1 + \sigma(\|x\|^2)\|y\|^2)\|x\|^2] \\ &= \epsilon[\sigma(\|x\|^2)\|y\|^2(1 - \|x\|^2)] \\ &\geq -\epsilon\|x\|^2 \geq -\epsilon. \end{aligned}$$

$$\begin{aligned} g(F(x,y)) = \epsilon[\sigma(\|x\|^2)\|y\|^2(1 - \|x\|^2) - \|x\|^2 \\ - \tfrac{3}{2}\lambda(\sigma(\|x\|^2)\|y\|^2]. \end{aligned}$$

Since λ is monotonically decreasing

$$g(F(x,y)) \leq \epsilon[\sigma(\|x\|^2)(1 - \|x\|^2) - \|x\|^2 - \tfrac{3}{2}\lambda(\sigma(\|x\|^2))]$$

but by definition of σ we have

$$\lambda(\sigma(\|x\|^2)) = \tfrac{2}{3}(1 + \sigma(\|x\|^2))(1 - \|x\|^2).$$

Substituting, we have $g(F(x,y)) \leq -\epsilon$. Hence F maps $D^k \times D^\ell$ into h. Conversely, suppose $w \in h$ and let $u = Pw$, $v = Qw$. So $\|u\|^2 - \|v\|^2 \geq -\epsilon$ and $\|u\|^2 - \|v\|^2 - 3\epsilon/2\ \lambda(\|u\|^2/\epsilon) \leq -\epsilon$. Thus $(\|v\|^2)/(\epsilon + \|u\|^2) \leq 1$ so $x = (\epsilon + \|u\|^2)^{1/2}v \in D^k$. Also $\sigma(\|v\|^2/(\epsilon + \|u\|^2))$ is well-defined and by the lemma

$$\frac{\|u\|^2}{\epsilon\sigma(\|v\|^2/(\epsilon + \|u\|^2))} \leq 1$$

so $y = \epsilon\sigma(\|v\|^2/(\epsilon + \|u\|^2))^{-1/2}u \in D^\ell$. Thus $G(w) = ((\epsilon + \|Pw\|^2)^{-1/2}Qw,$ $\epsilon\sigma(\|Qw\|^2/(\epsilon + \|Pw\|^2))^{-1/2}Pw)$ *defines a map of h into* $D^k \times D^\ell$. *Then* $GF(x,y) = (x,y)$. Since

$$F(w,y) = (\epsilon\sigma(\|x\|^2)\|y\|^2 + \epsilon)^{1/2}x + (\epsilon\sigma(\|x\|^2))^{1/2}y$$

with $x \in D^k$, $y \in D^\ell$. Write $w = ax + by$ where

$$a = (\epsilon\sigma(\|x\|^2)\|y\|^2 + \epsilon)^{1/2}$$

$$b = (\epsilon\sigma(\|x\|^2))^{1/2} \text{ so } a = (b^2\|y\|^2 + \epsilon)^{1/2}$$

Then

$$Pw = by \qquad \|Pw\|^2 = b^2\|y\|^2$$

$$Qw = ax \qquad \|Qw\|^2 = a^2\|x\|^2.$$

So

$$G(w) = (\epsilon + b^2\|y\|^2)^{-1/2}ax,\ (\epsilon\sigma(\frac{a^2\|x\|^2}{\epsilon + b^2\|y\|^2}))^{-1/2}by)$$

$$= (x,y).$$

Therefore $GF(x,y) = (x,y)$.

On the other hand,

$$FG(w) = F((\epsilon + \|Pw\|^2)^{-1/2}Q(w),\ (\epsilon\sigma(\frac{\|Qw\|^2}{\epsilon + \|Pw\|^2}))^{-1/2}Pw)$$

Put

$$x = (\epsilon + \|Pw\|^2)^{-1/2}Qw \in D^k \text{ and}$$

$$y = \epsilon\sigma(\frac{\|Qw\|^2}{\epsilon + \|Pw\|^2})^{1/2}Pw \in D^\ell.$$

Then

$$\|x\|^2 = (\epsilon + \|Pw\|^2)^{-1}\|Qw\|^2 \in D^\ell$$

$$\|y\|^2 = [\epsilon\sigma(\|Qw\|^2/(\epsilon + \|Pw\|^2))]^{-1}\|Pw\|^2.$$

Thus

$$(\epsilon\sigma(\|x\|^2)\|y\|^2 + \epsilon)^{1/2}x$$

$$= (\epsilon\sigma(\frac{\|Qw\|^2}{\epsilon + \|Pw\|^2}) \cdot \frac{\|Pw\|^2}{\epsilon\sigma(\frac{\|Qw\|^2}{\epsilon + \|Pw\|^2})} + \epsilon)^{1/2}$$

$$\cdot (\epsilon + \|Pw\|^2)^{-1/2}Qw$$

$$= (\|Pw\|^2 + \epsilon)^{1/2}(\epsilon + \|Pw\|^2)^{-1/2}Qw = Qw.$$

Similarly $(\epsilon\sigma(\|x\|^2))^{1/2}y = Pw$. Therefore $FG(w) = Pw + Qw = w$.

Thus F is a homeomorphism of $D^k \times D^\ell$ onto h. But since σ is C^∞ with non-vanishing derivative in $[0,1)$, it follows that F is a C^∞-isomorphism on $D^k \times D^\ell$.

On $S^{k-1} \times D^\ell$, F reduces to $F(x,y) = (\epsilon(\|y\|^2 + 1))^{1/2}x + \epsilon^{1/2}y$ (so $fF(x,y) = -\epsilon$) which is clearly a C^∞-isomorphism onto $N \cap h$, the set where $f = -\epsilon$ and $\|Pw\|^2 \leq \epsilon$. Q.E.D.

VI. MAIN THEOREM

6.1. Lemma. Let Q and f be bounded, symmetric, non-degenerate bilinear forms of a Hilbert space H; Q positive definite. Then there exists an admissible inner product $< , >$ in H such that $Q(v,v) = <Gv,v>$ and $f(v,v) = \|Pv\|^2 - \|(1 - P)_v\|^2$ where P is an orthogonal projection which commutes with G.

Proof. Since $Q(u,v)$ is an admissible inner product in H, $f(v,v) = Q(Av,v)$ where A is self adjoint invertible operator with respect to this inner product. Let $G = |A|^{-1}$ and $P = h|A|$ where h is the characteristic function of $[0,\infty)$ and define $<u,v> = Q(|A|u,v)$.

Then $Q(u,v) = Q(|A|\ |A|^{-1}u,v) = Q(|A|Gu,v) = <Gu,v>$. Since any function of A is self adjoint relative to $< , >$, P is an orthogonal projection, G a positive operator in this inner product, and both being functions of A; they commute. Now $|\lambda|^{-1}\lambda = h(\lambda) - (1 - h(\lambda))$ so $GA = P - (I - P)$ so $f(v,v) = Q(Av,v) = <GAv,v> = \|Pv\|^2 - \|(1 - P)v\|^2$. Q.E.D.

6.2. Remark. Let M be a complete C^{k+2}-Riemannian manifold $(k \geq 1)$ and f a C^{k+2}-Morse function on M. Let c be a critical value of f. Without loss of generality we assume that $c = 0$. Let $p_1, p_2, \ldots, p_r$ be the critical points of f with $f(p_i) = 0$.

Let k_i and ℓ_i be respectively the index and coindex of f at p_i. By the Morse lemma we can find for some $\delta < 1$ a C^k-chart φ_i at p_i whose image is the ball of radius 2δ in a Hilbert space H_i such that $\varphi_i(p_i) = 0$ and $f\varphi_i^{-1}(v) = \|P_iv\|^2 - \|1 - P_i)v\|^2$ where P_i is an orthogonal projection in H_i of rank ℓ_i and $(1 - P_i)$ has rank k_i. Moreover, if G^i is the positive operator in H_i uniquely defined by $<d\varphi_{p_i}^{-1}(u),d\varphi_{p_i}^{-1}(v)> = <G^iu,v>$, then by 6.1 there is a positive operator G_i' which commutes with P_i and $<d\varphi_{p_i}^{-1}(v),d\varphi_{p_i}^{-1}(v)>_{p_i} = <G^iv,v> = <G_i'v,v>$ for all v. Therefore $G^i = G_i'$ commutes with P_i.

6.3. <u>Proposition.</u> Let V be a neighborhood of zero in a Hilbert space H with inner product $< , >$ made into a C^{k+1}-Riemannian manifold ($k \geq 0$) by defining $<u,v>_w = <G(w)u,v>$ where G is a C^k-map of v into the invertible positive operators on H. Let P be an orthogonal projection in H which commutes with $G(0)$ and define $f(v) = \|Pv\|^2 - \|(1 - P)v\|^2$. Then for $\epsilon > 0$ sufficiently small, if we define $g(v) = f(v) - 3\epsilon/2\ \lambda(\|Pv\|^2/\epsilon)$ then $(\nabla f)g$ is C^k and does not vanish on the 2ϵ ball about the origin except at the origin.

Proof. Let $\Omega(x) = G(x)^{-1}$. Then $\Omega(0)$ commutes with P. Let $T(x) = P\Omega(x) - \Omega(x)P$ so $P\Omega(x) = T(x) + \Omega(x)P$. Note that

$$\begin{aligned}\|(2P - I)x\|^2 &= <(2P - I)x,(2P - I)x> \\ &= <2Px,2Px> - 2<2Px,x> + <x,x> \\ &= \|x\|^2.\end{aligned}$$

Therefore $\|(2P - I)x\| = \|x\|$. Hence

$$\begin{aligned}<Px,\Omega(x)(2P - I)x> &= <Px,P\Omega(x)(2P - I)x> \\ &= <Px,T(x)(2P - I)x> + <Px,\Omega(x)P(x)> \\ &\geq <Px,T(x)(2P - I)x> \text{ since } \Omega(x) \text{ is positive.}\end{aligned}$$

So by Schwartz inequality we have

$$\begin{aligned}|<Px,T(x)(2P - I)x>|^2 &\leq \|Px\|^2\|Tx\|^2\|(2P - I)x\|^2 \\ &\leq \|Tx\|^2\|x\|^2\end{aligned}$$

or $|<Px,T(x)(2P - I)x>| \leq \|Tx\|\ \|x\|^2$, hence $<Px,T(x)(2P - I)x> \geq -\|Tx\|\ \|x\|^2$. Thus

(1) $\langle Px, \Omega(x)(2P - I)\rangle \geq - \|Tx\| \; \|x\|^2.$

Now since $\|u\|^2 = \langle u,u\rangle = \langle u, G(x)^{1/2}\Omega(x)^{1/2}u\rangle = \langle G(x)^{1/2}u, \Omega(x)^{1/2}u\rangle$ $= \langle G(x)\Omega(x)^{1/2}, \Omega(x)^{1/2}u\rangle \leq \|G(x)\| \; \|\Omega(x)^{1/2}u\|^2 = \|G(x)\| \; \langle \Omega(x)^{1/2}u, \Omega(x)^{1/2}u\rangle$ $= \|G(x)\| \; \langle u, \Omega(x)u\rangle$. Thus $\langle (2P - I)x, \Omega(x)(2P - I)x\rangle \geq \|G(x)\|^{-1} \; \|(2P - 1)x\|^2$ or

(2) $\langle (2P - 1)x, \Omega(x)(2P - I)x\rangle \geq \|G(x)\|^{-1} \; \|x\|^2.$

Since $T(0) = P\Omega(0) - \Omega(0)P = 0$ while $\|G(0)\|^{-1} > 0$ we can find a neighborhood U of the origin such that for $x \in U$, $\|G(x)\|^{-1} > \frac{3}{2} \|T(x)\| \sup |\lambda'|$. Since $\lambda' \leq 0$ it follows that for $x \in U$ using (1), (2) $4[\langle (2P - I)x, \Omega(x)(2P - I)x\rangle - \frac{3}{2}\lambda' (\|Px\|^2/\epsilon)\langle Px, \Omega(x)(2P - I)x\rangle] \geq 4[\|G(x)\|^{-1} \; \|x\| - \frac{3}{2}\lambda'(\|Px\|^2/\epsilon)(-\|Tx\| \cdot \|x\|^2)] = 4[\|G(x)\|^{-1} - \frac{3}{2}|\lambda'(\|Px\|^2/\epsilon)| \; \|Tx\|] \; \|x\| \geq 0$. The above is always positive unless $x = 0$. Since $f(v) = \langle Pv, Pv\rangle - \langle (1 - P)v, v\rangle$ $= \langle (2P - I)v, v\rangle$ hence $df_x(y) = 2\langle (2P - I)x, y\rangle = 2\langle \Omega(x)(2P - I)x, y\rangle_x$. Thus ∇f_x $= 2\Omega(x)(2P - I)x$. Since $g(v) = f(v) - \frac{3\epsilon}{2}\lambda(\|Pv\|^2/\epsilon)$

$$dg_x(y) = df_x(y) - \frac{3\epsilon}{2}\lambda'(\|Px\|^2/\epsilon)\,\frac{2}{\epsilon}\langle Px, y\rangle$$

$$= df_x(y) - 3\lambda'(\|Px\|^2/\epsilon)\langle Px, y\rangle.$$

Hence,

$$\nabla f_x(g) = dg_x(\nabla f_x)$$

$$= df_x(\nabla f_x) - 3\lambda'(\|Px\|^2/\epsilon)\langle Px, \nabla f_x\rangle$$

$$= 2[\langle (2P - I)x, \nabla f_x\rangle - 3\lambda'(\|Px\|^2/\epsilon)\langle Px, 2\Omega(x)(2P - I)x\rangle]$$

$$= 2[\langle (2P - I)x, 2\Omega(x)(2P - I)x\rangle$$

$$- 2[3\lambda'(\|Px\|^2/\epsilon)\langle Px, 2\Omega(x)(2P - I)x]$$

$$= 4\Gamma[\langle (2P - I)x, \Omega(x)(2P - I)x\rangle$$

$$- \frac{3}{2}\lambda'(\|Px\|^2/\epsilon)\langle Px, \Omega(x)(2P - I)x\rangle]$$

$$\geq 0.$$

The equality holds only if $x = 0$. This shows that $(\nabla f)g$ is non-vanishing on U except at the origin. Q.E.D.

Without loss of generality, we may assume that U contains an open ball around the origin of radius 2δ, where δ was chosen before. Since critical values are isolated, we can choose $\epsilon < \delta^2$ so small that 0 is the only critical value of f in $(-3\epsilon, 3\epsilon)$. Let $W = f^{-1}(-2\epsilon, \infty)$. Define $g : W \to R$ by $g(\varphi_i^{-1}(v)) = f(\varphi_i^{-1}(v)) - \frac{3\epsilon}{2}\lambda(\|P_i v\|^2/\epsilon)$ and equal to $f(w)$ if $w \notin \bigcup_{i=1}^r D(\varphi_i)$.

Note that if $w = \varphi_i^{-1}(v) \in W$ and $f(w) \neq g(w)$ then by the definition of g, $\lambda(\|P_i v\|^2/\epsilon) \neq 0$ so $\|P_i v\|^2 < \epsilon$ hence $f(w) = \|P_i v\|^2 - \|(1 - P_i)v\|^2 < \epsilon$ and since $w \in W$, $f(w) > -2\epsilon$ or $\|P_i v\|^2 - \|(1 - P_i)v\|^2 > -2\epsilon$. So $\|(1 - P_i)v\|^2 \leq 2\epsilon + \|P_i v\|^2 < 3\epsilon$. Hence $\|v\|^2 = \|(1 - P_i)v\|^2 + \|P_i v\|^2 < 3\epsilon + \epsilon = 4\epsilon < (2\delta)^2$. Thus $A = \{w \in W \cap D(\varphi_i) \mid f(w) \neq g(w)\}^- \subset D(\varphi_i)^0$. So if $w \in A$ then $w \in D(\varphi_i)^0$ and $g(w) = g(\varphi_i^{-1}(v)) = f(\varphi_i^{-1}(v)) - \frac{3\epsilon}{2}\lambda(\|P_i v\|^2/\epsilon)$ for some $v \in D(\varphi_i)$ is C^k. If $w \notin A$, then $f(w) = g(w)$ hence is C^k also.

Claim. $\{w \in W \mid f(w) \leq \epsilon\} = \{w \in W \mid g(w) \leq \epsilon\}$. For if $w \in W$ and $f(w) > \epsilon$ then $f(w) = g(w)$, otherwise by the above argument we must have $f(w) < \epsilon$. Moreover $(\nabla f)g$ is non-vanishing on $D(\varphi_i)$ except at p_i. Let $\bar{f} = f \circ \varphi^{-1}$, then $\bar{f}(v) = f \circ \varphi(v) = \|P_i v\|^2 + \|(1 - P_i)v\|^2$ satisfies the above proposition. So if $\bar{g} = g \circ \varphi_i^{-1}$ then $\nabla\bar{f}(\bar{g})$ is non-vanishing in a ball of radius 2δ except the origin. Since $df_x(v) = d\bar{f}_{\varphi_i(x)} \circ d\varphi_{ix}(v)$, $dg_x = d\bar{g}_{\varphi_i(x)} \circ d\varphi_{ix}$, $df_x(v) = d\bar{f}_{\varphi_i(x)}(d\varphi_{ix}(v)) = \langle d\varphi_{ix}(v), \nabla\bar{f}_{\varphi_i(x)}\rangle_{\varphi_i(x)} = \langle d\varphi_{ix}(v), \nabla\bar{f}_{\varphi_i(x)}\rangle = \langle v, \nabla f_x\rangle_x$

$= \langle d\varphi_{ix}^{-1} \circ d\varphi_{ix}(v), d\varphi_{ix}^{-1} \circ d\varphi_{ix}(\nabla f_x)\rangle_x = \langle G^{\varphi_i}(x) d\varphi_{ix}(\nabla f_x)\rangle = \langle d\varphi_{ix}(v),$ $G^{\varphi_i}(x)^* d\varphi_{ix}(\nabla f_x)\rangle$. Thus $\nabla \bar{f}_{\varphi_i(x)} = G^{\varphi_i}(x)^* d\varphi_{ix}(\nabla f_x)$. Hence $(\nabla f)g(x) = dg_x(\nabla f_x)$ $= d\bar{g}_{\varphi_i(x)} \circ d\varphi_{ix}(\nabla f_x) = d\bar{g}_{\varphi_i(x)} \circ G^{\varphi_i}(x)^{*-1}(\nabla \bar{f}_{\varphi_i(x)})$. Then by the same proof as in the Hilbert space case, we can show that $(\nabla f)g$ is non-vanishing except at p_i. Thus we proved:

6.4. Proposition. $(\nabla f)g$ is C^k and does not vanish on $D(\varphi_i)$ except at p_i.

6.5. Proposition. If σ is the maximal integral curve of ∇f with initial condition p, then ∇f is defined in $V = \{w \in W \mid -\frac{5\epsilon}{4} < g(w) < \frac{5\epsilon}{4}\}$.

Proof. Since $f - \frac{3\epsilon}{2} \le g \le f$ we have for $w \in V$, $f - \frac{3\epsilon}{2} \le g < \frac{5\epsilon}{4}$ implies $f(w) < \frac{5\epsilon}{4} + \frac{3\epsilon}{2} = \frac{11}{4}\epsilon < 3\epsilon$, also $-3\epsilon < -\frac{5\epsilon}{4} < g(w) \le f(w)$. Thus if $w \in V$, -3ϵ $< f(w) < 3\epsilon$. But by our choice, 0 is the only critical value of f in $(-3\epsilon,\epsilon)$. Then the only possible critical points of f in V could be $p_1, \ldots, p_r$. But $g(p_i) = -\frac{3\epsilon}{2} < -\frac{5\epsilon}{4}$ so $p_i \notin V^-$. Thus f has no critical point in V^-. Now let $p \in V$ and let $\sigma : (\alpha,\beta) \to M$ be the maximal integral curve of ∇f with initial condition r. Then by 4.4 either $f(\sigma(t)) \to \infty$ as $t \to \beta$ so $\sigma(t)$ gets outside V as $t \to \beta$ or else σ has a critical point of f as limit point as $t \to \beta$, but since V^- has no critical point, $\sigma(t)$ must get outside V. Q.E.D.

6.6. Corollary. ∇f is C^k-strongly transverse to g on $[-\epsilon,\epsilon]$.

Proof. This follows from 6.4 and 6.5 and noting that f has no critical point in V.

6.7. Theorem. Let f be a C^{k+2}-real valued Morse function on a complete C^{k+2}-Riemannian manifold M $(k \ge 1)$. Let $p_1, p_2, \ldots, p_r$ be the distinct critical points of f on $f^{-1}(c)$ and let k_i and ℓ_i be the index and coindex of f at p_i respectively. If $a < c < b$ and c is the only critical value of f in $[a,b]$ then $\{x \in M \mid f(x) \le b\}$ is C^k-isomorphic to $\{x \in M \mid f(x) \le a\}$, with r-handles of type $(k_1,\ell_1, \ldots, (k_r,\ell_r)$ disjointly C^k-attached.

Proof. By the strong transversality theorem there is a C^k-isomorphism h of W onto itself such that $h(w) = w$ if $|g(w)| \geq \frac{3\epsilon}{2}$ and h maps $\{w \in W \mid g(w) \leq -\epsilon\}$ C^k-isomorphically onto $\{w \in W \mid g(w) \leq \epsilon\} = \{w \in W \mid f(w) \leq \epsilon\}$. (Put a $= -\epsilon$, $b = \epsilon$, $\delta = \epsilon$ in that theorem.) We extend h to a C^k-isomorphism of M by defining $h(x) = x$ if $x \notin W$. By 5.5 $\{x \in M \mid g(x) \leq \epsilon\}$ arises from $\{x \in M \mid f(x) \leq -\epsilon\}$ by disjoint C^k-attachment of γ-handles of type (k_1, ℓ_1), ..., (k_r, ℓ_r). But $\{x \in M \mid g(x) \leq \epsilon\} = \{x \in M \mid f(x) \leq \epsilon\}$. Write $[a,b] = [a,-\epsilon] \cup [-\epsilon,\epsilon] \cup [\epsilon,b]$ and note that f has no critical point in $[a,-\epsilon] \cup [e,b]$. Hence M_a is C^k-isomorphic to $M_{-\epsilon}$ and M_ϵ is C^k-isomorphic to M_b where $M_d = \{x \in M \mid f(x) \leq d\}$, $d = a,b,\epsilon$, or $-\epsilon$. By the above argument, we know that M_ϵ arises from $M_{-\epsilon}$ by attaching handles of type (k_i, ℓ_i), $i = 1, 2, \ldots, r$. Q.E.D.

VII. TOPOLOGICAL IMPLICATIONS

7.1. Theorem. Let M be a connected C^1-manifold, $f : M \to R$ a non-constant C^1-function and K the set of critical points of f. Then if we denote $\dot{K}$ as the the boundary of K, then $f(\dot{K}) = f(K)$.

Proof. Since K is closed, $\dot{K} \subset K$ so $f(\dot{K}) \subset f(K)$. Let $p \in K$. We will find $x \in \dot{K}$ such that $f(x) = f(p)$. This will show that $f(K) \subset f(\dot{K})$. Choose $q \in M$ with $f(q) \neq f(p)$ and $\sigma : I \to M$ a C^1-path such that $\sigma(0) = p$ and $\sigma(1) = q$ by 3.2. Put $g(t) = f(\sigma(t))$. Then $g'(t) = df_{\sigma(t)}(\sigma'(t))$. Since g is not constant, $g' \not\equiv 0$. So $\sigma(I) \not\subset K$. (For if $\sigma(I) \subset K$, then $df_{\sigma(t)} = 0$ for all t hence so is $g'(t)$.) Let $t_0 = \inf \{t \in I \mid \sigma(t) \notin K\}$. Then $x = \sigma(t_0) \in \dot{K}$ and $g'(t) = 0$ for $0 \leq t \leq t_0$. So $f(x) = g(t_0) = g(0) = f(p)$, since g is constant in $[0,t_0]$. Q.E.D.

7.2. Remark. Note that the connectedness of M is essential. For let M be the disjoint union of two open sets with no boundary and f is constant on each component, then $f(K) \neq f(\dot{K})$.

7.3. Theorem. Let M be a C^1-Riemannian manifold, $f : M \to R$ a C^1-function satisfying condition (C) and K the set of critical points of f. Then $f \mid \dot{K}$ is proper. By $f \mid \dot{K}$ is proper we mean that given $-\infty < a < b < \infty$, $\dot{K} \cap f^{-1}[a,b]$ is compact.

Proof. Let $\{p_n\}$ be a sequence in $\dot{K}$ with $a \leq f(p_n) \leq b$. Since $\dot{K}$ is closed, it suffices to show that $\{p_n\}$ has a convergent subsequence.

Since $p_n \in \dot{K}$, by the definition of boundary point we can choose $q_n \notin K$ arbitrarily close to p_n. In particular, since $\|\nabla f\|$ is continuous and $\|\nabla f p_n\| = 0$ we can choose q_n such that $\|\nabla f q_n\| < 1/n$, $a - 1 < f(q_n) < b + 1$ and $\rho(q_n,p_n) < 1/n$ where ρ is the Riemannian metric for M. By condition (C) we can find a subsequence $\{q_n\}$ converging to a critical point p of f. Since $\rho(q_n,p_n)$

$< 1/n$, the corresponding subsequence of $\{p_n\}$ will converge to p. Q.E.D.

7.4. Remark. $f \mid K$ need not be proper. For example if M is not compact f = constant, then f satisfies (C) and K = M is not compact.

7.5. Theorem. Let M be a complete C^2-Riemannian manifold and $f : M \to R$ a C^2-function satisfying condition (C). If f is bounded below on a component M_0 of M then $f \mid M_0$ assumes its greatest lower bound.

Proof. Without loss of generality, assume M is connected, and f is not constant. Let $B = \inf \{f(x) \mid x \in M\}$. Given $\epsilon > 0$, choose $p \in M$ such that $f(p) < B + \epsilon$. If $\sigma : (\alpha,\beta) \to M$ is a maximum integral curve of ∇f with initial condition p then by theorem 4.4 $\alpha = -\infty$ and $\sigma(t)$ has a critical point q as limit point as $t \to -\infty$. Since $f(\sigma(t))$ is monotonic increasing $f(q) < B + \epsilon$.

By 7.1 we can find x in $\dot{K}$ such that $f(x) < B + \epsilon$. Let $\epsilon = 1/n$, n = 1, 2, ... Choose x_n such that $B \leq f(x_n) < B + 1/n$. Then by 7.2, a subsequence of $\{x_n\}$ will converge to a point x and clearly we must have $f(x) = B$. Q.E.D.

7.6. Corollary. If the set of critical points K of f has no interior and if f is bounded below on M then f assumes its greatest lower bound.

Proof. Let $B = \inf_{x \in M} f(x)$. For each n, we can choose $x_n \in K$ (a minimum of f on some component of M) such that $B \leq f(x_n) \leq B + 1/n$. Since K has no interior and is closed $K = \dot{K}$. So by 7.2 a subsequence of $\{x_n\}$ will converge to a point x where $f(x) = B$. Q.E.D.

7.7. Remark. Note in 7.6 the hypothesis "f is bounded below" is replaced by "f is bounded above", then the conclusion becomes "f assumes its least upper bound". To see this, we just consider -f.

7.8. First Morse Inequality. Let M be a complete C^2-Riemannian manifold and $f : M \to R$ be a C^2-*Morse function*. *Also let* f *be bounded below.* Then there are at least as many critical points of index zero as there are components of M.

Proof. By 7.5 we know that f assumes its minimum on each component. But each point where f assumes its minimum is a critical point since f is a Morse function is of index 0. Thus there are at least as many critical points of index zero as there are components of M. Q.E.D.

<u>Note</u>. The first Morse inequality will be proved in 7.22 as $R_0^* \leq C_0^*$ where C_0^* is the number of critical points of index zero and $R_0^* = \dim H_0(M)$. Since R_0^* is equal to the number of path components thus we shall prove 7.8. But in 7.22 we shall use the fact that f is at least C^3. This is because we shall need Morse lemma in which f is at least C^3.

7.9. <u>Lemma</u>. Let X be a convex subset of a Banach space E, and let A be a subset of X. If A is a retract of X, then A is a strong deformation retract of X.

Proof. Let $r : X \to A$ be a retract. So $r \mid A = id_A$. Define $h : X \times I \to x$ by $h(x,t) = (1 - t)x + tr(x)$. Then h is a strong deformation retract. Q.E.D.

7.10. <u>Theorem</u>. If H is an infinite-dimensional Hilbert space, D the unit ball of H, and S the unit sphere of H, then S is a strong deformation retract. Q.E.D.

Proof. We first construct a fixed point free map of D into D.

<u>Claim</u>. We can embed R as a closed subset F of S.

Choose $\{x_n\}_{n=1}^{\infty}$ an orthonormal set in H. Define $f(t) = [\cos \frac{\pi(t - n)}{2}]x_n + [\sin \frac{\pi(t - n)}{2}]x_{n+1}$ for $n \leq t \leq n + 1$. Then f is a homeomorphism onto F $= f(R)$. Then the map $h : F \to F$ given by $h(f(t)) = f(t + 1)$ is continuous and fixed point free. Since F is homeomorphic to R which is solid, by the Tietze extension theorem we can extend h to $\tilde{h} : D \to F$. Also $\tilde{h} : D \to D$ is fixed point free. Define $r : D \to S$ by rx = point where the directed line segment from h(x) to x meets S.

The directed segment S_x is defined by $S_x(t) = (1 - t)h(x) + tx$. Solving

$t(x)$ in $\|S_x(t)\|^2 = 1$ we have

$$t(x) = \frac{-(h(x),\ x - h(x)) + \sqrt{(h(x),\ x - h(x))^2 - \|x - h(x)\|^2\ \|h(x)\|^2}}{\|x - h(x)\|^2}$$

So $t(x)$ is continuous in x. Thus $r(x) = S_x(t(x))$ is continuous. Obviously r is a retraction : $D \to S$. This theorem follows from 7.9. Q.E.D.

7.11. Theorem. Let H_i be a Hilbert space of dimension d_i, $i = 1, 2, \ldots, n$; D_i the closed unit disc in H_i, and S_i the unit sphere in H_i. Let $g_i : S_i \to x$ be continuous with disjoint images in a topological space X. Suppose moreover that $d_i < \infty$ for $i = 1, 2, \ldots, m$ and $d_i = \infty$ for $i > m$. Then $X \cup_{g1} D_1 \cup \cdots \cup_{gm} D_m$ is a strong deformation retract of $X \cup_{g1} D_1 \cup \cdots \cup_{gn} D_n$.

Proof. This follows from the above theorem by induction.

7.12. Corollary. If H_* denotes the singular homology functor with any coefficient G then

$$H_*(X \cup_{g1} D_1 \cup \cdots \cup_{gn} D_n,\ X) \cong \Sigma_{i=1}^{m} H_*(D^{di}, S^{di-1})$$

In particular,

$$H_r(X \cup_{g1} D_1 \cup \cdots \cup_{gn} D_n) \approx G^{c(r)}$$

where $c(r)$ is the number of indices $i = 1, 2, \ldots, n$ such that $d_i = r$.

Let N be a Hilbert manifold with boundary and suppose M arises from N by disjoint C^r-attachments $(f_1, f_2, \ldots, f_n)$ of handles of type $(d_i, e_1), \ldots, (d_n, e_n)$. Define attaching maps $g_i : S^{di-1} \to \partial N$ by $g_i(y) = f_i(y,0)$. Then $N \cup_{fi} (D^{d1} \times 0) \cup \cdots \cup_{fn} (D^{dn} \times 0)$ can be identified with $N \cup_{g1} D^{d1} \cup \cdots \cup_{gn} D^{dn}$.

7.13. Lemma. Let D_n be the unit ball in H_i and $S_i = \partial D_i$, $i = 1, 2$. Then $(D_1 \times \{0\}) \cup (S_1 \times D_2)$ is a strong deformation retract of $D_1 \times D_2$.

Proof. Since $D_1 \times D_2$ is a convex subset of $H_1 \times H_2$ it suffices to define a retraction $r : D_1 \times D_2 \to D_1 \times 0 \cup S_1 \times D_2$ as follows:

$$r(x,y) = \left(\frac{2x}{2 - \|y\|}, 0\right) \text{ if } \|x\| \geq \frac{\|y\|}{2}, \ y \neq 0$$

$$= \left(\frac{x}{\|x\|}, (2\|x\| + \|y\| - 2) \cdot \frac{y}{\|y\|} \right) \text{ if } \|x\| \geq 1 - \frac{\|y\|}{2}$$

and $y \neq 0$

$$= (x,0) \text{ if } y = 0.$$

Q.E.D.

By a finite induction, we can prove

7.14. Theorem. $N \cup \bigcup_{i=1}^{n} f_i(D^{di} \times 0)$ is a strong deformation retract of M. Hence if $di < \infty$, $i = 1, 2, \ldots, m$, $d_i = \infty$, $i > m$, then $N \cup_{g1} D^{d1} \cup \cdots \cup_{gm} D^{dm}$ is a strong deformation retract of M.

Then we can restate the Morse theory as:

7.15. Theorem. Let M be a complete C^3-Riemannian manifold, $f : M \to R$ a C^3-Morse function. Let c be a critical value of f, $p_1, \ldots, p_n$ be the critical points of f at level c, and let d_i be the index of p_i. Assume $d_i < \infty$ for all i. If c is the only critical value of f in a closed interval $[a,b]$, then M_b has as a deformation retract M_a with cells of dimension $d_1, \ldots, d_n$ disjointly attached to ∂M_a by homeomorphisms of the boundary spheres. Hence if H_* is the singular homology functor with coefficient group G then $H_k(M_b, M_a) \approx G^{c(k)}$ where c(k) is the number of critical points of index k on the level c.

7.16. Definition. Let F be a fixed field and H_* the singular homology functor with coefficient F. We call the pair of spaces (X,Y) admissible if $H_*(X,Y)$ is of finite type, i.e. each $H_k(X,Y)$ is finite dimensional and $H_k(X,Y) = 0$ except for finitely many k.

From the exact homology sequence of a triple (X,Y,Z) it follows that if (X,Y) and (Y,Z) are admissible then so is (X,Z).

Let S be an integer valued function on admissible pairs. Then S is said to be <u>subadditive</u> if $S(X,Z) \leq S(X,Y) + X(Y,Z)$ for all triples (X,Y,Z) such that (X,Y) and (Y,Z) are admissible. S is said to be <u>additive</u> if the above inequality becomes an equality.

By induction we have if $X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_0$ and each (X_{i+1},X_i) is admissible it follows that (X_n,X_0) is admissible and

$$S(X_n,X_0) \leq \Sigma_{i=0}^{n-1} S(X_{i+1},X_i)$$

if S is subadditive, equality holding if S is additive.

7.17. <u>Definition</u>. For each non-negative integer k we define integer valued functions R_k and S_k on admissible pairs by

$$R_k(X,Y) = \dim H_k(X,Y)$$

$$S_k(X,Y) = \sum_{m\leq k} (-1)^{k-m} R_m(X,Y)$$

and

$$\chi(X,Y) = \sum_{m=0}^{\infty} (-1)^m R_m(X,Y).$$

7.18. <u>Lemma</u>. R_k and S_k are subadditive and χ is additive.

Proof. Let (X,Y,Z) be a triple of spaces such that (X,Y) and (Y,Z) are admissible. Then from the exact homology sequence of (X,Y,Z)

$$\cdots \to H_m(Y,Z) \xrightarrow{im} H_m(X,Z) \xrightarrow{jm} H_m(X,Y) \xrightarrow{\partial m} H_{m-1}(Y,Z) \to \cdots$$

we get three short exact sequences:

$$0 \to \mathrm{Im}(\partial_{m+1}) \to H_m(Y,Z) \to \mathrm{Im}(im) \to 0$$

$$0 \to \mathrm{Im}(im) \to H_m(X,Z) \to \mathrm{Im}(jm) \to 0$$

$$0 \to \mathrm{Im}(jm) \to H_m(X,Y) \to \mathrm{Im}(\partial m) \to 0$$

from which we obtain

$$Rm(Y,Z) = \dim Hm(Y,Z) = \dim \mathrm{Im}(\partial_{m+1}) + \dim \mathrm{Im}(im)$$

$$Rm(X,Z) = \dim Hm(X,Z) = \dim \mathrm{Im}(im) + \dim \mathrm{Im}(jm)$$

$$Rm(X,Y) = \dim Hm(X,Y) = \dim \mathrm{Im}(jm) + \dim \mathrm{Im}(\partial m).$$

Hence

$$(*) \qquad Rm(X,Z) - Rm(X,Y) - Rm(Y,Z) = -(\dim \mathrm{Im}(\partial m) + \dim \mathrm{Im}\ (\partial_{m+1})).$$

Thus $Rm(X,Z) \leq Rm(X,Y) + Rm(Y,Z)$. Also,

$$\Sigma_{m=0}^{k} (-1)^{k-m} [Rm(X,Z) - Rm(X,Y) - Rm(Y,Z)]$$

$$= \Sigma_{m=0}^{k} (-1)^{k-m+1} [\dim \mathrm{Im}(\partial m) + \dim \mathrm{Im}(\partial_{m+1})].$$

That is,

$$S_k(X,Z) - S_k(X,Y) - S_k(Y,Z) = (-1)^{k+1} \dim \mathrm{Im}\ \partial_0 - \dim \mathrm{Im}(\partial k + 1).$$

So $S_k(X,Z) \leq S_k(X,Y) + X_k(Y,Z)$.

Similarly,

$$\Sigma_{m=0}^{\infty} (-1)^{m} [Rm(X,Z) - Rm(X,Y) - Rm(Y,Z)]$$

$$= \Sigma_{m=0}^{\infty} (-1)^{m+1} (\dim \mathrm{Im}(\partial m) + \dim (\mathrm{Im}\ \partial_{m+1}))$$

$$= \lim_{k\to\infty} (-1)^{k+1} \dim (\mathrm{Im}\ \partial k) - \dim (\mathrm{Im}\ \partial_0) = 0 - 0 = 0.$$

So χ is additive. Q.E.D.

7.19. <u>Theorem</u>. Let M be a complete C^3-*Riemannian manifold*, $f : M \to R$ a C^3-Morse function. Let a and b be regular values of f, $a > b$. For each non-negative integer m, let R_m denote the mth betti-number of (M_b, M_a) relative to some fixed field F, and let C_m denote the number of critical points of f of index m in $f^{-1}([a,b])$. Then

(1) $R_0 \leq C_0$.

(2) $R_1 - R_0 \leq C_1 - C_0$.

(3) $\Sigma_{m=0}^{k} (-1)^{k-m} Rm \leq \Sigma_{m=0}^{k} (-1)^{k-m} C_m$.

(4) $\chi(M_b, M_a) = \Sigma_{m=0}^{\infty} (-1)^m Rm = \Sigma_{m=0}^{\infty} (-1)^m C_m$.

Proof. Let $c_1 < c_2 < \ldots < c_n$ be critical values of f in $[a,b]$. Choose a_i, $i = 0, 1, 2, \ldots, n$ such that $a = a_0 < c_1 < a_1 < c_2 < \ldots < a_{n-1} < c_n < a_n = b$. Put $X_i = M_{a_i} = \{x \in M \mid f(x) \leq a_i\}$. Then by theorem 7.14 (X_{i+1}, X_i) is admissible and $R_k(X_{i+1}, X_i) = \dim H_k(X_{i+1}, X_i) = C_i(k) =$ the number of critical points of index k on the level c_i. Then by definition

$$S_k(X_{i+1}, X_i) = \Sigma_{m=0}^{k} (-1)^{k-m} R_m(X_{i+1}, X_i)$$

$$= \Sigma_{m=0}^{k} (-1)^{k-m} C_i(m),$$

and

$$\Sigma_{i=0}^{n-1} S_k(X_{i+1}, X_i) = \Sigma_{i=0}^{n-1} \Sigma_{m=0}^{k} (-1)^{k-m} C_i(m)$$

$$= \Sigma_{m=0}^{k} (-1)^{k-m} \Sigma_{i=0}^{n-1} C_i(m)$$

$$= \Sigma_{m=0}^{k} (-1)^{k-m} C_m.$$

By subadditivity

$$\Sigma_{m=0}^{k} (-1)^{k-m} R_m = S_k(M_b, M_a) \le \Sigma_{i=0}^{n-1} S_k(X_{i+1}, X_i) = \Sigma_{m=0}^{k} (-1)^{k-m} C_m.$$

This proves (3).

(1) and (2) follow from (3) immediately.

Now

$$\chi(X_{i+1}, X_i) = \Sigma_{m=0}^{\infty} (-1)^m R_m(X_{i+1}, X_i)$$

$$= \Sigma_{m=0}^{\infty} (-1)^m C_i(m)$$

while

$$\Sigma_{i=0}^{n-1} \chi(X_{i+1}, X_i) = \Sigma_{i=0}^{n-1} \Sigma_{m=0}^{\infty} (-1)^m C_i(m)$$

$$= \Sigma_{m=0}^{\infty} (-1)^m \Sigma_{i=0}^{n-1} C_i(m)$$

$$= \Sigma_{m=0}^{\infty} (-1)^m C_m.$$

Then (4) follows from the additivity of χ. Q.E.D.

7.20. <u>Corollary</u>. $R_k \le C_k$ for all k. Consequently $\Sigma R_k \le \Sigma C_k$ = the total number of critical points.

Proof. By (3) of the above theorem, we have

$$\sum_{m=0}^{k} (-1)^{k-m} R_m \le \Sigma_{m=0}^{k} (-1)^{k-m} C_m$$

and

$$\sum_{m=0}^{k-1} (-1)^{k-1-m} R_m \le \Sigma_{m=0}^{k-1} (-1)^{k-1-m} C_m.$$

Adding we get $R_k \le C_k$.

7.21. <u>Corollary</u>. *If f is bounded below then the conclusions of 7.18 and 7.19 remain valid if we interpret Rm = the mth betti-number of M_b and C_m = the number of critical points of f having index m in M_b.*

Proof. Choose $a <$ g.l.b. f. Then M_a is empty.

7.22. <u>Corollary</u>. If f is bounded below then for each non-negative integer m, $R_m^* \leq C_m^*$ where R_m^* is the mth betti-number of M and C_m^* is the total number of critical points of f having index m.

Proof. By 7.20 we have $C_m^* \geq R_m(M_b)$ for any regular value b of f. Hence it will suffice to show that if $R_m^* = \dim H_m(M,F) \geq k$ for some integer k, then $R_m(M_b) \geq k$ for some regular value b of f. (For if we put $k = R_m^*$, then we have $C_m^* \geq R_m(M_b) \geq R_m^*$). Let $h_1, h_2, \ldots, h_k$ be linearly independent elements in $H_m(M;F)$, $z_1, z_2, \ldots, z_k$ be singular cycles of M which represent them, and C a compact set containing the support of $z_1, z_2, \ldots, z_m$. Then as $b \to \infty$ through regular values of f then the interiors of M_b form an increasing sequence of open sets and cover M, hence $C \subset M_b$ for some regular value b of f. Then z_1, $z_2, \ldots, z_k$ are singular cycles of M_b and no non-trivial linear combination could be homologous to zero in M_b. Hence $R_m(M_b) \geq k$.

7.23. <u>Remark</u>. The assumption that f is bounded below is necessary in 7.22 as can be seen by considering the identity map of R which has no critical points whereas $R_0^*(R) = 1$.

7.24. If M is a torus, then $R_0^* = R_2^* = 1$, $R_1^* = 2$, and $R_k^* = 0$ for $k \geq 3$. Hence $\Sigma R_k^* = 4$, i.e., if f has only non-degenerate critical point then f must have at least 4 critical points. This is not true for f admitting degenerate critical points. See Pittcher [1]. Schwartz [1] generalized the Lusternik-Schnirelman theory of the lower bound of the number of critical points to a pair (M,f) where f satisfies condition (C).

VIII. THE MANIFOLDS $H_1(I,V)$ AND $\Omega(V;P,Q)$

8.1. Definition. Let $H_0(I,R^n) = \{\sigma : I \to R^n$ measurable and $\int_0^1 \|\sigma(t)\|^2\, dt < \infty\}$. Then $H_0(I,R^n)$ becomes a Hilbert space if we define the inner product $< , >_0$ by $<\sigma,\rho>_0 = \int_0^1 <\sigma(t),\rho(t)>\, dt$ where $< , >$ is the inner product in R^n.

We define $H_1(I,R^n) = \{\sigma : I \to R^n \mid \sigma$ absolutely continuous and $\sigma' \in H_0(I,R^n)\}$. Then $H_1(I,R^n)$ is a Hilbert space under the inner product $< , >_1$ defined by $<\sigma,\rho>_1 = <\sigma(0),\rho(0)> + <\sigma',\rho'>_0$. In fact, the map : $(p,g) \to \sigma : R^n \oplus H_0(I,R^n) \to H_1(I,R^n)$ where $\sigma(t) = p + \int_0^t g(s)ds$ is an isometry onto. Clearly $p = \sigma(0)$. Define $L : H_1(I,R^n) \to H_0(I,R^n)$ by $L\sigma = \sigma'$ and $H_1^*(I,R^n) = \{\sigma \in H_1(I,R^n) \mid \sigma(0) = \sigma(1) = 0\}$. Let $H_0^*(I,R^n) = \{g \in H_0(I,R^n) \mid \int_0^1 g(t)dt = 0\}$.

8.2. Theorem. L is a bounded linear transformation of norm one. $H_1^*(I,R^n)$ is a closed linear subspace of codimension 2n in $H_1(I,R^n)$ and L maps $H_1^*(I,R^n)$ isometrically onto $H_0^*(I,R^n)$.

Proof. Since $\|L\sigma\|_0^2 = \|\sigma'\|_0^2 \le \|\sigma(0)\|^2 + \|\sigma'\|_0^2 = \|\sigma\|_1^2$ therefore $\|L\| \le 1$. But if σ is such that $\sigma(0) = 0$, then we see that $\|L\| = 1$.

It is easy to see that $H_1^*(I,R^n)$ is a linear subspace of $H_1(I,R^n)$. To see that it is closed, let $\sigma_k \in H_1^*(I,R^n)$ and let $\sigma_k \to \sigma$ in $H_1(I,R^n)$ where $\sigma_k(t) = \int_0^t \sigma_k'(s)ds$, $\sigma_k(1) = 0$ and $\sigma(t) = \sigma(0) + \int_0^t \sigma'(s)ds$. Then as $\|\sigma_k - \sigma\|_1^2 = <\sigma(0),\sigma(0)> + \int_0^1 \|\sigma_k'(s) - \sigma'(s)\|^2\, ds$ tends to zero, we must have $\sigma(0) = 0$ and $\sigma_k' \to \sigma'$ almost everywhere. So $\sigma(1) = \int_0^1 \sigma'(s)ds = \lim_k \int_0^1 \sigma'(s)ds = \lim_k \sigma_k(1) = 0$. So $\sigma \in H_1^*(I,R^n)$. Clearly, H_0^* is closed and L maps $H_1^*(I,R^n)$ isometrically onto $H_0^*(I,R^n)$.

Claim. $H_0^*(I,R^n)$ is the orthogonal complement in $H_0(I,R^n)$ of the set of constant maps of I into R^n.

If $g \in H_0^*(I,R^n)$ and h is a constant function on I then $\langle g,h\rangle_0 = \int_0^1 \langle g(t), h(t)\rangle dt = h \int_0^1 g(t)dt = 0$. On the other hand if $\langle g,h\rangle_0 = 0$ for all constant maps, then in particular let $h \equiv 1$, then we have $\int_0^1 g(t)dt = \langle g,1\rangle_0 = 0$ so $g \in H_0^*(I,R^n)$. But $H_0^*(I,R^n)$ has codimension n in $H_0(I,R^n)$ and $H_1(I,R^n) = H_0(I,R^n) \oplus R^n$ so $H_0^*(I,R^n)$ has codimension 2n in $H_1(I,R^n)$. Q.E.D.

8.3. <u>Theorem</u>. If $\rho \in H_1^*(I,R^n)$ and λ is an absolutely continuous map of I into R^n then $\int_0^1 \langle\lambda'(t),\rho(t)\rangle dt = \langle\lambda,-L\rho\rangle_0$.

Proof. Since the composition map $t \to \rho(t) \to \langle\lambda(t),\rho(t)\rangle$ is absolutely continuous, $\langle\lambda'(t),\rho(t)\rangle'$ exists and is equal to $\langle\lambda'(t),\rho(t)\rangle + \langle\lambda(t),\rho'(t)\rangle$. So $0 = \langle\lambda(1),\rho(1)\rangle - \langle\lambda(0),\rho(0)\rangle = \int_0^1 \langle\lambda(t),\rho(t)\rangle' dt = \int_0^1 \langle\lambda'(t),\rho(t)\rangle dt + \int_0^1 \langle\lambda(t),\rho'(t)\rangle dt$. Therefore, $\int_0^1 \langle\lambda'(t),\rho(t)\rangle dt = -\int_0^1 \langle\lambda(t),\rho'(t)\rangle\, dt = \int_0^1 \langle\lambda(t),-L\rho\rangle dt = \langle\lambda,-L\rho\rangle_0$. Q.E.D.

8.4. <u>Remark</u>. Let $C^0(I,R^n)$ be the Banach space of all continuous maps of I into R^n with norm $\|\ \|_\infty$ defined by $\|\sigma\|_\infty = \sup\{\|\sigma(t)\| : t \in I\}$. Let i be the inclusion map : $C^0(I,R^n) \to H_0(I,R^n)$ and let $\sigma,\rho \in C^0(I,R^n)$, then $\|i(\sigma) - i(\rho)\|_0^2 = \|\sigma - \rho\|_0^2 = \int_0^1 \|\sigma(t) - \rho(t)\|^2\, dt \le \|\sigma - \rho\|_\infty^2$. So is is uniformly continuous.

We recall that by the Ascoli-Arzela theorem a subset S of $C^0(I,R^n)$ is totally bounded iff it is bounded and equicontinuous. Thus S is also totally bounded in $H_0(I,R^n)$.

8.5. <u>Theorem</u>. If $\sigma \in H_1(I,R^n)$ then $\|\sigma\|_\infty \le 2\|\sigma\|_1$.

Proof. Since

$$\|\sigma\|_1^2 = \|\sigma(0)\|^2 + \|\sigma'\|_0^2$$

$$\|\sigma\|_1 \ge \|\sigma(0)\|$$

$$\|\sigma\|_1 \geq \|\sigma'\|_0.$$

Therefore

$$\begin{aligned}
\|\sigma(t)\| &\leq \|\sigma(0)\| + \|\sigma(t) - \sigma(0)\| \\
&\leq \|\sigma\|_1 + \left\| \int_0^t \sigma'(s)ds \right\| \\
&\leq \|\sigma\|_1 + \int_0^t \|\sigma'(s)\| \, ds \\
&\leq \|\sigma\|_1 + \int_0^1 \|\sigma'(s)\| \, ds \\
&= \|\sigma\|_1 + \|\sigma'\|_0 \leq 2\|\sigma\|_1.
\end{aligned}$$

Thus $\|\sigma\|_\infty \leq 2\|\sigma\|_1$. Q.E.D.

8.6. <u>Corollary</u>. The inclusion maps of $H_1(I,R^n)$ into $C^0(I,R^n)$ and $H_0(I,R^n)$ is completely continuous. f is said to be completely continuous if f maps every bounded subset onto a totally bounded set.

Proof. Suppose S is bounded in $H_1(I,R^n)$, then by 8.5 S is bounded in $C^0(I,R^n)$. For any $\sigma \in S$, $\|\sigma(S) - \sigma(t)\| \leq \int_s^t \|\sigma'(x)\| \, dx \leq |t - s|^{1/2} \|\sigma'\|_0$ $\leq |t - s|^{1/2} \|\sigma\|_1 \leq |t - s|^{1/2} K$ (where K is a bound for S in $H_1(I,R^n)$.

This says that S is equicontinuous in $C^0(I,R^n)$. So S is totally bounded in $C_0(I,R^n)$ by Ascoli-Arzela theorem.

We have shown that every totally bounded set in $C_0(I,R^n)$ is totally bounded in $H_0(I,R^n)$. Thus the composition map $H_1(I,R^n) \to C_0(I,R^n) \to H_0(I,R^n)$ is completely continuous. Q.E.D.

8.7. <u>Lemma</u>. Let F be a C^1-map of R^n into $L^s(R^n,R^p)$. Then the map $\overline{F}$ of $H_1(I,R^n)$ into $L^s(H_1(I,R^n),H_1(I,R^p))$ defined by $\overline{F}(\sigma)(\lambda_1, \ldots, \lambda_s)(t)$ $= F(\sigma(t))(\lambda_1(t), \ldots, \lambda_s(t))$ is continuous. Moreover if F is C^3 then $\overline{F}$ is C^1

and $d\bar{F} = \overline{dF}$.

Proof. First we note that

$$(\bar{F}(\sigma)(\lambda_1, \ldots, \lambda_s))'(t) = dF_{\sigma(t)}(\sigma'(t)(\lambda_1(t), \ldots, \lambda_s(t))$$

$$+ \Sigma_{i=1}^{s} F(\sigma(t))(\lambda_1(t), \ldots, \lambda_1'(t), \ldots, \lambda_s(t))$$

This can be seen from the following consideration:

$$\bar{F}(\sigma)(\lambda_1, \ldots, \lambda_s)(t + \Delta t) - \bar{F}(\sigma)(\lambda_1, \ldots, \lambda_s)(t)$$

$$= F(\sigma(t + \Delta t)(\lambda_1(t + \Delta t), \ldots, \lambda_s(t + \Delta t))$$

$$- F(\sigma(t))(\lambda_1(t + \Delta t), \ldots, \lambda_s(t + \Delta t))$$

$$+ F(\sigma(t))(\lambda_1(t + \Delta t), \ldots, \lambda_s(t + \Delta t))$$

$$- F(\sigma(t))(\lambda_1(t), \lambda_2(t + \Delta t), \ldots, \lambda_s(t + \Delta t)) + \ldots$$

$$+ F(\sigma(t))(\lambda_1(t), \ldots, \lambda_{s-1}(t), \lambda_s(t + \Delta t))$$

$$- F(\sigma(t))(\lambda_1(t), \ldots, \lambda_s(t)).$$

Hence

$$\|(\bar{F}(\sigma)(\lambda_1, \ldots, \lambda_s))'(t)\|$$

$$\leq \|dF_{\sigma(t)}\| \; \|\sigma'(t)\| \; \|\lambda_1(t)\| \ldots \|\lambda_s(t)\|$$

$$+ \Sigma_{i=1}^{s} \|F(\sigma(t))\| \; \|\lambda_1(t)\| \ldots \|\lambda_1'(t)\| \ldots \|\lambda_s(t)\|$$

$$\leq (\sup_t \|dF_{\sigma(t)}\|) \; \|\sigma'(t)\| \; \|\lambda_1\|_\infty \cdots \|\lambda_s\|_\infty$$

$$+ \Sigma_{i=1}^{s} (\sup_t \|F(\sigma(t))\|) \; \|\lambda_1\|_\infty \cdots \|\lambda_1'(t)\| \ldots \|\lambda_s\|_\infty.$$

But $\|\lambda_i\|_\infty \le 2\|\lambda_i\|_1$, we have

$$\le (\sup \|dF_{\sigma(t)}\|) \cdot \|\sigma'(t)\| \prod_{i=1}^{s} 2^s \|\lambda_i\|_1$$

$$+ 2^{s-1} \Sigma_{i=1}^{s} (\sup_t \|F(\sigma(t))\|) \|\lambda_i'(t)\|$$

$$\cdot \prod_{i=1}^{n} \|\lambda_1\|_1 \cdots \|\hat{\lambda}_i\|_1 \cdots \|\lambda_s\|_1.$$

Take $\| \ \|_0$ and use the triangle inequality:

$$\|(\bar{F}(\sigma)(\lambda_1, \ldots, \lambda_s))'\|_0$$

$$\le 2^s \prod_{i=1}^{s} \|\lambda_i\|_1 \sup \|dF_{\sigma(t)}\| \|\sigma'\|_0$$

$$+ 2^{s-1} \Sigma_{i=1}^{s} \sup_t \|F(\sigma(t))\| \Pi \|\lambda_1\|_1 \cdots \|\lambda_i'\|_0 \cdots \|\lambda_s\|_1.$$

Again use the fact that $\|\lambda_i'\|_0 \le \|\lambda_i\|_1$. So we have

$$\|(\bar{F}(\sigma)(\lambda_1, \ldots, \lambda_s))'\|_0$$

$$\le 2^s \prod_{i=1}^{s} \|\lambda_i\|_1 \sup_t \|dF_{\sigma(t)}\| \|\sigma'\|_0$$

$$+ 2^{s-1} s \sup \|F(\sigma(t))\| \prod_{i=1}^{s} \|\lambda_i\|_1$$

$$\le 2^s \Pi_{i=1}^{s} \|\lambda_i\|_1 (\sup_t \|dF_{\sigma(t)}\| \|\sigma'\|_0 + s \sup \|F(\sigma(t))\|)$$

$$= 2^s \Pi_{i=1}^{s} \|\lambda_i\|_1 L(\sigma)$$

where $L(\sigma) = \sup_t \|dF_{\sigma(t)}\| \|\sigma'\|_0 + s \sup \|F(\sigma(t))\|$. Also

$$\|\bar{F}(\sigma)(\lambda_1, \ldots, \lambda_s)\|_\infty$$

$$= \sup_t \|F(\sigma(t))(\lambda_1(t), \ldots, \lambda_s(t))\|$$

$$\le \sup_t \left[\|F(\sigma(t))\| \prod_{i=1}^{s} \|\lambda_i(t)\| \right] \text{ (since } F(\sigma(t)) \text{ is multilinear)}$$

$$\le \prod_{i=1}^{s} \|\lambda_i\|_\infty \sup_t \|F(\sigma(t))\|$$

$$\le 2^s \, \Pi_{i=1}^{s} \|\lambda_i\|_1 \sup_t \|F(\sigma(t))\|.$$

Since $\|\rho\|_1^2 = \|\rho(0)\|^2 + \|\rho'\|_0^2 \le \|\rho\|_\infty^2 + \|\rho'\|_0^2$. So

$$\|\overline{F}(\sigma)(\lambda_1, \ldots, \lambda_s)\|_1^2$$

$$\le \|\overline{F}(\sigma)(\lambda_1, \ldots, \lambda_s)\|_\infty^2 + \|(\overline{F}(\sigma)(\lambda_1, \ldots, \lambda_s))'\|_0^2$$

$$\le [2^s \, \Pi_{i=1}^{s} \|\lambda_i\|_1]^2 [\sup_t \|F(\sigma(t))\|]^2 + [2^s \, \Pi_{i=1}^{s} \|\lambda_i\|_1 \, L(\sigma)]^2$$

$$= [2^s \, \Pi_{i=1}^{s} \|\lambda_i\|_1]^2 [(\sup_t \|F(\sigma(t))\|)^2 + L(\sigma)^2]$$

$$= [2^s \, \Pi_{i=1}^{s} \|\lambda_i\|_1^2]^2 \, K_1(\sigma).$$

Therefore

$$\|\overline{F}(\sigma)(\lambda_1, \ldots, \lambda_s)\|_1 \le 2^s \, \Pi_{i=1}^{s} \|\lambda_i\|_1 \, K_1(\sigma) = K(\sigma) \prod_{i=1}^{s} \|\lambda_i\|.$$

Since $\overline{F}$ is clearly multilinear, it follows from the above that $\overline{F}(\sigma) \in L^s(H_1(I,R^n), H_1(I,R^p))$. If $\rho \in H_1(I,R^n)$ then

$$((\overline{F}(\sigma) - \overline{F}(\rho))(\lambda_1, \ldots, \lambda_s))'(t)$$

$$= (\overline{F}(\sigma)(\lambda_1, \ldots, \lambda_s))'(t) - (\overline{F}(\rho)(\lambda_1, \ldots, \lambda_s))'(t)$$

$$= dF_{\sigma(t)}(\sigma'(t))(\lambda_1(t), \ldots, \lambda_s(t))$$

$$+ \Sigma_{i=1}^{s} F(\sigma(t))(\lambda_1(t), \ldots, \lambda_i'(t), \ldots, \lambda_s(t))$$

$$- [dF_{\rho(t)}(\rho'(t))(\lambda_1(t), \ldots, \lambda_s(t))$$

$$+ \Sigma_{i=1}^{s} F(\rho(t))(\lambda_1(t), \ldots, \lambda_i'(t), \ldots, \lambda_s(t))]$$

$$= [dF_{\sigma(t)}(\sigma'(t)) - dF_{\sigma(t)}(\rho'(t))](\lambda_1(t), \ldots, \lambda_s(t))$$

$$+ [dF_{\sigma(t)}(\rho'(t)) - dF_{\rho(t)}(\rho'(t))](\lambda_1(t), \ldots, \lambda_s(t))$$

$$+ \Sigma_1^s (F(\sigma(t)) - F(\rho(t)))(\lambda_1(t), \ldots, \lambda_i'(t), \ldots, \lambda_s(t))$$

Thus

$$\|(\overline{F}(\sigma) - \overline{F}(\rho))(\lambda_1, \ldots, \lambda_s)'(t)\|$$

$$\leq \|dF_{\sigma(t)}\| \ \|\sigma'(t) - \rho'(t)\| \ \|\lambda_1(t)\| \cdots \|\lambda_s(t)\|$$

$$+ \|dF_{\sigma(t)} - dF_{\rho(t)}\| \ \|\rho'(t)\| \ \|\lambda_1(t)\| \cdots \|\lambda_s(t)\|$$

$$+ \Sigma_{i=1}^{s} \|F(\sigma(t)) - F(\rho(t))\| \ \|\lambda_1(t)\| \cdots \|\lambda_i'(t)\| \cdots \|\lambda_s(t)\|$$

$$\leq (\sup_t \|dF_{\sigma(t)}\|) \ \|\sigma'(t) - \rho'(t)\|$$

$$+ (\sup_t \|dF_{\sigma(t)} - dF_{\rho(t)}\|) \ \|\rho'(t)\|$$

$$+ \Sigma (\sup \|F(\sigma(t)) - F(\rho(t))\|) \cdot \prod_{i=1}^{s} \lambda_i.$$

By the triangle inequality we have

$$\|(\overline{F}(\sigma) - \overline{F}(\rho))(\lambda_1, \ldots, \lambda_s)'\|_0$$

$$\leq [\sup_t \|dF_{\sigma(t)}\| \ \|\sigma' - \rho'\|_0 + \sup_t \|dF_{\sigma(t)} - dF_{\rho(t)}\| \ \|\rho'\|_0$$

$$+ \sup \|F(\sigma(t)) - F(\rho(t))\|] \ 2^s \cdot \prod_{i=1}^{s} \|\lambda_i\|_1$$

$$= 2^s \prod_{i=1}^{s} \|\lambda_i\|_1 \ [M(\sigma,\rho)]$$

where $M(\sigma,\rho) = \sup \|dF_{\sigma(t)}\| \cdot \|\sigma' - \rho'\|_0 + \sup \|dF_{\sigma(t)} - dF_{\rho(t)}\| \; \|\rho'\|_0$ $+ s \sup \|F(\sigma(t)) - F(\rho(t))\|$. Also we have

$$\|(\overline{F}(\sigma) - \overline{F}(\rho))(\lambda_1, \ldots, \lambda_s)\|_\infty$$

$$\leq 2^s \sup \|F(\sigma(t)) - F(\rho(t))\| \; \|\lambda_1\|_1 \cdots \|\lambda_s\|_1.$$

Hence

$$\|(\overline{F}(\sigma) - \overline{F}(\rho))(\lambda_1, \ldots, \lambda_s)\|_1$$

$$\leq \|(\overline{F}(\sigma) - \overline{F}(\rho))(\lambda_1, \ldots, \lambda_s)\|_\infty$$

$$+ \|((\overline{F}(\sigma) - \overline{F}(\rho))(\lambda_1, \ldots, \lambda_s))'\|_0$$

$$\leq [\sup \|F(\sigma(t)) - F(\rho(t))\| + M(\sigma,\rho)] \, 2^s \, \Pi_{i=1}^s \|\lambda_i\|_1$$

$$= K(\sigma,\rho) \prod_{i=1}^{s} \|\lambda_i\|_1.$$

So $|||\overline{F}(\sigma) - \overline{F}(\rho)||| \leq K(\sigma,\rho)$ where $|||\ |||$ is the norm in $L^s(H_1(I,R^n),H_1(I,R^p))$ and $K(\sigma,\rho) \to 0$ if $\sup \|F(\sigma(t)) - F(\rho(t))\|$, $\sup \|dF_{\sigma(t)} - dF_{\rho(t)}\|$ and $\|\sigma' - \rho'\|_0$ all approach zero. But if $\rho \to \sigma$ in $H_1(I,R^n)$, then since $\|\sigma - \rho\|_\infty \leq 2 \|\sigma - \rho\|_1$ so $\rho \to \sigma$ uniformly. Hence since F and dF are continuous, $F(\rho(t)) \to F(\sigma(t))$ uniformly and $dF_{\rho(t)} \to dF_{\sigma(t)}$ uniformly, so $K(\sigma,\rho) \to 0$. Thus $|||\overline{F}(\sigma) - \overline{F}(\rho)||| \to 0$ so $\overline{F}$ is continuous. This proves the first part of the lemma.

Now suppose F is C^3 so dF is C^2, then by mean value theorem, there is a C^1-map $R : R^n \to L(R^n,L^s(R^n,R^n))$ such that if $x = p + v$ then $F(x) - F(p) - dF_p(v) = R(x)(v,v)$. Then $\overline{R} : H_1(I,R^n) \to L^2(H_1(I,R^n), H_1(I,L^s(R^n,R^p)))$ is continuous by the first part of the lemma. (Here we embed $L^s(R^n,R^p)$ into some Euclidean space.) Now if $x = \rho + \sigma$ and σ are in $H_1(I,R^n)$ then

$$(\bar{F}(x) - \bar{F}(\sigma) - \overline{dF}_\sigma(\rho))(\lambda_1, \ldots, \lambda_s)(t)$$

$$= F(x(t)) - F(\sigma(t)) - dF(\sigma(t))(\lambda_1(t), \ldots, \lambda_s(t))$$

$$= R(x(t))(\rho(t),\rho(t))(\lambda_1(t), \ldots, \lambda_s(t))$$

$$= \bar{R}(x)(\rho,\rho)(\lambda_1, \ldots, \lambda_s)(t).$$

Clearly $\overline{dF}_\sigma$ is linear in ρ and $\bar{R}(x)$ is $o(\|\rho\|)$. Thus $\overline{dF}_\sigma = d\bar{F}_\sigma$. Since σ is arbitrary, we must have $\overline{dF} = d\bar{F}$. Q.E.D.

8.8. Theorem. If $\varphi : R^n \to R^p$ is a C^{k+2}-map then $\sigma \to \varphi \circ \sigma$ is a C^k-map $\bar{\varphi} : H_1(I,R^n) \to H_1(I,R^p)$. Moreover if $1 \le m \le k$ then $d^m\bar{\varphi}_\sigma(\lambda_1, \ldots, \lambda_m)(t)$ $= d^m\varphi_{\sigma(t)}(\lambda_1(t), \ldots, \lambda_m(t))$.

Proof. Let $F = d^s\varphi$ for $0 \le s \le k - 1$ then $F \in C^3$. So $F : R^n \to L^s(R^n,R^p)$ induces $\bar{F} = \overline{d^s\varphi} : H_1(I,R^n) \to L^s(H_1(I,R^n),H_1(I,L^s(R^n,R^p)))$ is C^1 and $\overline{dF} = d\bar{F}$, i.e. $\overline{d(d^s\varphi)} = d(\overline{d^s\varphi}) = \overline{d^{s+1}\varphi}$. Thus we can proceed by mathematical induction. Q.E.D.

8.9. *Theorem*. Consider R^m and R^n as complementary subspaces of R^{m+n}. Then the map $(\lambda,\sigma) \to \lambda + \sigma$ is an isometry of $H_1(I,R^m) \oplus H_1(I,R^n)$ onto $H_1(I,R^{m+n})$.

Proof. The norm of (λ,σ) in $H_1(I,R^m) \oplus H_1(I,R^n)$ is $\|(\lambda,\sigma)\| = \|\lambda\|_m + \|\sigma\|_n$ whereas $(\lambda + \sigma)(t) = (\lambda_1(t), \ldots, \lambda_m(t), \sigma_1(t), \ldots, \sigma_n(t))$. Hence it is easy to see that $(\lambda,\sigma) \to \lambda + \sigma$ is an isometry. Moreover by 8.8 the map is C^∞.

8.10. Definition. Let V be a finite dimensional C^1-manifold. We define $H_1(I,V)$ to be the set of continuous maps σ of I into V such that $\varphi \circ \sigma$ is absolutely continuous and $\|(\varphi \circ \sigma)'\|$ locally square summable for each chart φ for V.

If V is C^2 and $\sigma \in H_1(I,V)$ we define $H_1(I,V)_\sigma = \{\lambda \in H_1(I,T(V)) \mid \lambda(t) \in V_{\sigma(t)}$ for all $t \in I\}$. If $P,Q \in V$, we define $\Omega(V;P,Q) = \{\sigma \in H_1(I,V) \mid \sigma(0) = P,\ \sigma(1) = Q\}$ and if $\sigma \in \Omega(V;P,Q)$ we define $\Omega(V;P,Q)_\sigma = \{\lambda \in H_1(I,V)_\sigma \mid \lambda(0) = 0_P$ and $\lambda(1) - 0_Q\}$.

8.11. Remark. $H_1(I,V)_\sigma$ is a vector space under pointwise operations. For if $\lambda_1, \lambda_2 \in H_1(I,V)_\sigma$, then $\lambda_1(t), \lambda_2(t) \in V_{\sigma(t)}$. Thus $\lambda_1(t) + a\lambda_2(t) \in V_{\sigma(t)}$ where a is a real number, which implies $\lambda_1 + a\lambda_2 \in H_1(I,V)_\sigma$. Similarly $\Omega(V;P,Q)_\sigma$ is a subspace of $H_1(I,V)_\sigma$.

8.12. Proposition. If V is a closed C^{k+4}-submanifold of R^n $(k \geq 1)$, then $H_1(I,V)$ consists of all $\sigma \in H_1(I,R^n)$ such that $\sigma(I) \subseteq V$ and is closed in $H_1(I,R^n)$.

Proof. Let $\sigma \in H_1(I,R^n)$ and $\sigma(I) \subseteq V$. Let φ be a chart for V. Then it is easy to check that $\varphi \circ \sigma$ is absolutely continuous and $\|(\varphi \circ \sigma)'\|$ is square summable. Thus $\sigma \in H_1(I,V)$. Conversely, if $\sigma \in H_1(I,V)$ then by definition for each chart φ on V, $\varphi \circ \sigma$ is absolutely continuous and $\|(\varphi \circ \sigma)'\|$ is square summable.

Since V is a closed submanifold of R^n then i, the inclusion map of V into R^n is a chart for V. So $\sigma \in H_1(I,R^n)$.

Next, since $H_1(I,V)$ is closed in $C^0(I,R^n)$ by 8.6 is hence closed in $H_1(I,R^n)$. Q.E.D.

8.13. Theorem. If V is a closed C^{k+4}-submanifold of R^n then $H_1(I,V)$ is a closed C^k-submanifold of $H_1(I,R^n)$. Also $\Omega(V;P,Q)$ is a closed C^k-submanifold of $H_1(I,V)$. If $\sigma \in H_1(I,V)$ then the tangent space to $H_1(I,V)$ at σ is just $H_1(I,V)_\sigma$ which is equal to $\{\lambda \in H_1(I,R^n) \mid \lambda(t) \in V_{\sigma(t)}\ \forall\, t \in I\}$. If $\sigma \in \Omega(V;P,Q)$ then the tangent space to $\Omega(V;P,Q)$ at σ is just $\Omega(V;P,Q)_\sigma$ which equals $\{\lambda \in H_1(I,V)_\sigma \mid \lambda(0) = \lambda(1) = 0\}$.

Proof. Here we shall only prove that $H_1(I,V)$ is a C^k-submanifold of $H_1(I,R^n)$ and $H_1(I,V)_\sigma$ is the tangent space of $H_1(I,V)_\sigma$. The proof for $\Omega(V;P,Q)$ is similar.

Since V is a C^{k+4}-submanifold of R^n, we can find a C^{k+3}-Riemannian metric for R^n such that V is a totally geodesic submanifold. Then if $E : R^n \times R^n \to R^n$ is the corresponding exponential map (i.e. $t \to E(p,tv)$ is the geodesic starting from p with tangent vector v). E is C^{k+2}. Let $\sigma \in H_1(I,V)$ define $\varphi : H_1(I,R^n) \to H_1(I,R^n)$ by $\varphi(\lambda)(t) = E(\sigma(t),\lambda(t))$. Then by 8.8 and 8.9 φ is C^k and clearly $\varphi(0) = \sigma$. Moreover, by 8.8 $d\varphi_0(\lambda)(t) = dE_0^{\sigma(t)}(\lambda(t))$ where $E^{\sigma(t)}(v) = E(\sigma(t),v)$. Since $dE_0^{\sigma(t)}$ is the identity map of R^n, hence $d\varphi_0$ is the identity map of $H_1(I,R^n)$, so by the inverse function theorem, φ maps a neighborhood of zero in $H_1(I,R^n)$ C^k-isomorphically onto a neighborhood of σ in $H_1(I,R^n)$.

<u>Claim</u>. If λ is close enough to 0 in $H_1(I,R^n)$ then $\varphi(\lambda) \in H_1(I,V)$ iff $\lambda \in H_1(I,V)_\sigma$.

For if $\varphi(\lambda) \in H_1(I,V)$ implies $\varphi(\lambda)(t) \in V$ for all $t \in I$ or $E(\sigma(t);\lambda(t)) \in V$ for all t. That is, $\varphi(\lambda)$ is a geodesic in V, hence is a geodesic in R^n and is close to σ. Thus, $\lambda \in H_1(I,V)_\sigma$. Conversely, if $\lambda \in H_1(I,V)_\sigma$ then $\varphi(\lambda) \in H_1(I,R^n)$ and $\varphi(\lambda)$ is close to σ, which is in $H_1(I,V)$ so $\varphi(x) \in H_1(I,V)$.

Hence φ^{-1} restricted to a neighborhood of σ is a neighborhood of 0 in $H_1(I,V)_\sigma$ which is the restriction of a C^k-chart for $H_1(I,R^n)$ so is a chart in $H_1(I,V)$. Q.E.D.

8.14. <u>Theorem</u>. Let V and W be closed C^{k+4}-submanifolds of R^n and R^m respectively $(k \geq 1)$ and let $\varphi : V \to W$ be a C^{k+4}-map. Then $\bar{\varphi} : H_1(I,V) \to H_1(I,W)$ defined by $\bar{\varphi}(\sigma) = \varphi \circ \sigma$ is a C^k-map of $H_1(I,V)$ into $H_1(I,W)$. Moreover $d\bar{\varphi}_\sigma : H_1(I,V)_\sigma \to H_1(I,W)_{\bar{\varphi}(\sigma)}$ is given by $d\bar{\varphi}_\sigma(\lambda)(t) = d\varphi_{\sigma(t)}(\lambda(t))$.

Proof. Extend φ to a C^{k+4}-map of R^n into R^m then the theorem follows from 8.8 and 8.13 and then by taking proper restrictions. Q.E.D.

8.15. Definition. Let V be a C^{k+4}-manifold of infinite dimension ($k \geq 1$) and let $j : V \to R^n$ be a C^{k+4}-imbedding of V as a closed submanifold of a Euclidean space (by Whitney's theorem). Then by the above theorem, the C^k-structures induced on $H_1(I,V)$ and $\Omega(V;P,Q)$ as closed C^k-submanifolds of $H_1(I,R^n)$ are independent of j. Hence we shall regard $H_1(I,V)$ and $\Omega(V;P,Q)$ as C^k-Hilbert manifolds.

8.16. Definition. Let V be a C^{k+4}-finite dimensional Riemannian manifold ($k \geq 1$). We define a real valued function J^V on $H_1(I,V)$ called the action integral by $J^V(\sigma) = \frac{1}{2} \int_0^1 \|\sigma'(t)\|^2 \, dt$.

8.17. Theorem. Let V and W be C^{k+4}-Riemannian manifolds of finite dimension and let $\varphi : V \to W$ be a C^{k+4}-local isometry. Then $J^V = J^W \circ \bar{\varphi}$.

Proof. Since $(\bar{\varphi}(\sigma))'(t) = (\varphi \circ \sigma)'(t) = d\varphi_{\sigma(t)}(\sigma'(t))$. Since $d\varphi_{\sigma(t)}$ maps $V_{\sigma(t)}$ isometrically into $W_{\varphi(\sigma(t))}$, $\|(\bar{\varphi}(\sigma))'(t)\| = \|\sigma'(t)\|$. So $J^V(\sigma)$ $= \frac{1}{2} \int_0^1 \|\sigma'(t)\|^2 \, dt = \frac{1}{2} \int_0^1 \|(\varphi(\sigma))'(t)\|^2 \, dt = J^W \circ \bar{\varphi}(\sigma)$.

8.18. Corollary. If V is a C^{k+4}-Riemannian submanifold of the C^{k+4}-Riemannian manifold W then $J^V = J^W \mid H_1(I,V)$.

8.19. Corollary. If V is a closed C^{k+4}-submanifold of R^n then $J^V(\sigma)$ $= \frac{1}{2} \|L\sigma\|_0^2$. Consequently $J^V : H_1(I,V) \to R$ is C^k.

Proof. By definition $J^{R^n}(\sigma) = \frac{1}{2} \|L\sigma\|_0^2$. So $J^V = J^{R^n} \mid H_1(I,V)$. Since J^{R^n} is a continuous quadratic form on $H_1(I,R^n)$, J^{R^n} is C^∞, hence so the restriction to the closed C^k-submanifold $H_1(I,V)$ is C^k. Q.E.D.

8.20. Corollary. If V is a complete finite dimensional C^{k+4}-Riemannian manifold then J^V is a C^k-real valued function on $H_1(I,V)$.

Proof. By a theorem of Nash V can be C^{k+4}-imbedded isometrically in some R^n. Then the corollary follows from 8.19. Q.E.D.

8.21. Theorem. If V is a closed C^{k+4}-submanifold of R^n then $H_1(I,V)$ is a complete C^k-Riemannian manifold in the Riemannian structure induced on it as a closed C^k-submanifold of $H_1(I,R^n)$.

8.22. <u>Remark</u>. The Riemannian structure on $H_1(I,V)$ induced on it by an imbedding onto a closed submanifold of some R^n depends on the imbedding. To be more precise if V and W are closed submanifolds of Euclidean spaces and $\varphi : V \to W$ is an isometry it does not follow that $\bar{\varphi} : H_1(I,V) \to H_1(I,W)$ is an isometry.

For example, let $V = R^2$, $W = R^3$ and $\varphi : (x,y) \to (x \cos x, x \sin x, y)$. Then $\|(x,y)\|^2 = x^2 + y^2 = \|\varphi(x,y)\|^2$. But $(\varphi(x,y))'(t) = (x' \cos x - x x' \sin x, x' \sin x + x x' \cos x, y)$ where x,y are functions of t. Then if we put $\sigma(t) = (x(t),y(t))$, then as we just calculated $\|(\varphi \circ \sigma)'(t)\|^2 = x'^2(1 + x^2) + y'^2$ whereas $\|\sigma'(t)\|^2 = x'^2 + y'^2$.

8.23. <u>Theorem</u>. If V is a closed C^{k+4}-submanifold of R^n and $P,Q \in V$ then $\Omega(V;P,Q)$ is included in a translate of $H_1^*(I,R^n)$ and $\Omega(V;P,Q)_\sigma \subset H_1^*(I,R^n)$.

Proof. If σ and ρ are in $\Omega(V;P,Q)$ then $\sigma - \rho \in H_1^*(I,R^n)$, so $\sigma \in \rho + H_1^*(I,R^n)$. If $\lambda \in \Omega(V;P,Q)_\sigma$ then $\lambda + \sigma \in \Omega(V;P,Q)$. Therefore $\lambda = \lambda + \sigma - \sigma \in -\sigma + \Omega(V;P,Q) \subseteq H_1^*(I,R^n)$. Q.E.D.

8.24. <u>Corollary</u>. If we regard $\Omega(V;P,Q)$ as a Riemannian submanifold of $H_1(I,R^n)$ then the inner product $< , >_\sigma$ in $\Omega(V;P,Q)_\sigma$ is given by $<\rho,\lambda>_\sigma = <L\rho,L\lambda>_0$.

Proof. This follows from 8.23 and 8.2.

8.25. <u>Corollary</u>. If $S \subseteq \Omega(V;P,Q)$ and if J^V is bounded on S then S is totally bounded in $C^0(I,R^n)$ and $H_0(I,R^n)$.

Proof. Since $J^V(\sigma) = \frac{1}{2}\|L\sigma\|_0^2$, $\|L\sigma\|_0$ is bounded on S. Since S is included in a translate of $H_1^*(I,R^n)$ so by 8.2 L is an isometry on $H_1^*(I,R^n)$. S is bounded in $H_1(I,R^n)$ hence by 8.6 S is totally bounded in $C^0(I,R^n)$ and $H_0(I,R^n)$. Q.E.D.

8.26. If $\{\sigma_n\}$ is a sequence in $\Omega(V;P,Q)$ and $\|L(\sigma_n - \sigma_m)\|_0 \to 0$ as $n,m \to \infty$ then σ_n converges in $\Omega(V;P,Q)$.

Proof. Since $\sigma_n - \sigma_m \in H_1^*(I,R^n)$ and L is an isometry on $H_1^*(I,R^n)$; $\{\sigma_n\}$ is Cauchy in $H_1(I,R^n)$ hence convergent in $H_1(I,R^n)$ but $\Omega(V;P,Q)$ is closed in $H_1(I,R^n)$ and the corollary follows. Q.E.D.

8.27. Definition. Let V be a closed C^{k+4}-submanifold of R^n $(k \geq 1)$ and let $P,Q \in V$. If $\sigma \in \Omega(V;P,Q)$ then we define $h(\sigma)$ to be the orthogonal projection of $L\sigma$ on the orthogonal complement of $L(\Omega(V;P,Q)_\sigma)$ in $H_0(I,R^n)$.

8.28. Theorem. Let V be a closed C^{k+4}-submanifold of R^n $(k \geq 1)$, $P,Q \in V$ and let J be the restriction of J^V to $\Omega(V;P,Q)$. If we consider $\Omega(V;P,Q)$ as a Riemannian manifold in the structure induced on it as a closed submanifold of $H_1(I,R^n)$, then for each $\sigma \in \Omega(V;P,Q)$ ∇J_σ can be characterized as the unique element of $\Omega(V;P,Q)_\sigma$ mapped by L onto $L\sigma - h(\sigma)$. Moreover $\|\nabla J_\sigma\|_\sigma = \|L\sigma - h(\sigma)\|_0$.

Proof. Since $\Omega(V;P,Q)$ is a closed subspace of $H_1(I,R^n)$ and is included in $H_1^*(I,R^n)$, by 8.2 L maps $\Omega(V;P,Q)_\sigma$ isometrically onto a closed subspace of $H_0(I,R^n)$ which, therefore, is the orthogonal complement of its orthogonal complement.

Since $L\sigma - h(\sigma)$ is orthogonal to the orthogonal complement of $L(\Omega(V;P,Q)_\sigma)$ it is therefore of the form $L\lambda$ for some $\lambda \in \Omega(V;P,Q)_\sigma$ and since L is an isometry on $\Omega(V;P,Q)_\sigma$, λ is unique and $\|\lambda\|_\sigma = \|L\lambda\|_0 = \|L\sigma - h(\sigma)\|_0$. So it suffices to show that $dJ_\sigma(\rho) = \langle\lambda,\rho\rangle_\sigma$ for $\rho \in \Omega(V;P,Q)_\sigma$ or by 8.24 to show $dJ_\sigma(\rho) = \langle L\lambda, L\rho\rangle_0 = \langle L\sigma - h(\sigma), L\rho\rangle_0$ for $\rho \in \Omega(V;P,Q)_\sigma$. Since by definition $\langle h(\sigma), L\rho\rangle_0 = 0$ we need to show $dJ_\sigma(\rho) = \langle L\sigma, L\rho\rangle_0$ for $\rho \in \Omega(V;P,Q)_\sigma$. Now $J^{R^n}(\sigma) = \frac{1}{2}\|L\sigma\|_0^2$, so $dJ_\sigma^{R^n}(\rho) = \langle L\sigma, L\rho\rangle_0$ for $\rho \in H_1(I,R^n)$. Since $J = J^{R^n} \mid \Omega(V;P,Q)$, so $dJ_\sigma = dJ_\sigma^{R^n} \mid \Omega(V;P,Q)_\sigma$. Q.E.D.

IX. ACTION INTEGRAL J

Let V be a closed C^{k+4}-submanifold of R^n ($k \geq 3$), $P,Q \in V$, and $J = J^V \mid \Omega(V;P,Q)$. We proved that $\Omega(V;P,Q)$ is a complete C^k-Riemannian manifold in the Riemannian structure induced on it as a closed submanifold of $H_1(I,R^n)$ and J is a C^k-real valued function.

9.1. Definition. Define $\Omega : V \to L(R^n,R^n)$ by $\Omega(p)$ = orthogonal projection of R^n on V_p. Then Ω is C^{k+3}. If σ in $\Omega(V;P,Q)$ we define $\overline{\Omega}(V;P,Q)_\sigma$ to be the closure of $\Omega(V;P,Q)_\sigma$ in $H_0(I,R^n)$ and we define P_σ to be the orthogonal projection of $H_0(I,R^n)$ on $\overline{\Omega}(V;P,Q)_\sigma$.

9.2. Theorem. If $\sigma \in \Omega(V;P,Q)$ then $\overline{\Omega}(V;P,Q) = \{\lambda \in H_0(I,R^n) \mid \lambda(t) \in V_{\sigma(t)}$ for almost all $t \in I\}$; and if $\lambda \in H_0(I,R^n)$ then $(P_\sigma\lambda)(t) = \Omega(\sigma(t))\lambda(t)$. Also $P_\sigma(H_1^*(I,R^n)) = \Omega(V;P,Q)_\sigma$.

Proof. Define a linear transformation Π_σ on $H_0(I,R^n)$ by $(\Pi_\sigma\lambda)(t) = \Omega(\sigma(t))\lambda(t)$. Then $\Pi_\sigma^2(\lambda)(t) = \Pi_\sigma(\Pi_\sigma(\lambda))(t) = \Omega(\sigma(t))(\Pi_\sigma(\lambda)(t)) = \Omega(\sigma(t))\Omega(\sigma(t))\lambda(t) = \Omega(\sigma(t))\lambda(t) = \Pi_\sigma\lambda(t)$. Also we have

$$\begin{aligned}
\langle \Pi_\sigma(\lambda),\lambda'\rangle &= \int_0^1 \langle \Pi_\sigma(\lambda)(t),\lambda'(t)\rangle\, dt \\
&= \int_0^1 \langle \Omega(\sigma(t))\lambda(t),\lambda'(t)\rangle\, dt \\
&= \int_0^1 \langle \lambda(t),\Omega(\sigma(t))\lambda'(t)\rangle\, dt \\
&= \int_0^1 \langle \lambda(t),\Pi_\sigma(\lambda')(t)\rangle\, dt = \langle \lambda,\Pi_\sigma(\lambda')\rangle_0 .
\end{aligned}$$

So Π_σ is self-adjoint, hence is an orthogonal projection.

If $\lambda \in H_1^*(I,R^n)$ then $\lambda \in H_1(I,R^n)$ and $\lambda(0) = \lambda(1) = 0$, so $(\Pi_\sigma\lambda)(0) = \Omega(\sigma(0))\lambda(0) = 0 = (\Pi_\sigma\lambda)(1)$. Also by the definition of Ω, $(\Pi_\sigma\lambda)(t) = \Omega(\sigma(t))\lambda(t)$ is in $V_{\sigma(t)}$ for all t, so Π_σ maps $H_1^*(I,R^n)$ into $\Omega(V;P,Q)_\sigma$. Onto is also clear.

Claim. $H_1^*(I,R^n)$ is dense in $H_0(I,R^n)$.

It suffices to show that $H_1^*(I,R)$ is dense in $H_0(I,R)$. Note that $C^0(I,R)$ is dense in $H_0(I,R)$ and the polynomials on I are dense in $C^0(I,R)$.

Let $\lambda \in H_0(I,R)$. Then for any given $\epsilon > 0$ there is a polynomial p within $\epsilon/3$ neighborhood of λ. Let $M = \sup_t \{|p(t)|\}$. Choose $\delta > 0$ such that $\delta M^2 < \frac{1}{2}(\epsilon/3)^2$. Define $f \in H_1^*(I,R)$ by

$$
\begin{aligned}
f(\epsilon) &= \frac{t}{\delta}\, p(\delta) && 0 \le t \le \delta \\
&= p(t) && \delta < t < 1 - \delta \\
&= \frac{1 - t}{\delta}\, p(1 - \delta) && 1 - \delta \le t \le 1.
\end{aligned}
$$

Then

$$
\begin{aligned}
\|p - f\|_0^2 = \int_0^1 |p - f|^2\, dt &= \int_0^\delta |p - f|^2\, dt + \int_{1-\delta}^1 |p - f|^2\, dt \\
&\le \int_0^\delta M^2 dt + \int_{1-\delta}^1 M^2 dt = 2M^2\delta < (\epsilon/3)^2.
\end{aligned}
$$

So $\|p - f\|_0 \le \epsilon/3$. So $\|\lambda - f\|_0 \le \|\lambda - p\|_0 + \|p - f\|_0 \le \frac{2}{3}\epsilon < \epsilon$. Thus $H_1^*(I,R)$ is dense in $H_0(I,R)$.

$$
\begin{aligned}
\Pi_\sigma(H_1^*(I,R^n)) \subset \Pi_\sigma(H_0(I,R^n) &= \Pi_\sigma(\overline{H_1^*(I,R^n)}) \\
&\subseteq \overline{\Pi_\sigma(H_1^*(I,R^n))} = \bar{\Omega}(V;P,Q)_\sigma.
\end{aligned}
$$

So we have $\Pi_\sigma(H_0(I,R^n)) = \bar{\Omega}(V;P,Q)_\sigma$. Hence $\Pi_\sigma = P_\sigma$.

On the other hand, if $\lambda \in H_0(I,R^n)$ is fixed by Π_σ iff $\Omega(\sigma(t))\lambda(t) = \lambda(t)$ a.e. iff $\Omega(\sigma(t))\lambda(t) \in V_{\sigma(t)}$ a.e. That is, iff $\lambda(t) \in V_{\sigma(t)}$ a.e. Q.E.D.

9.3. Corollary. If $\sigma \in \Omega(V;P,Q)$ then $P\sigma L\sigma = L\sigma$.

Proof. Since $(L\sigma)(t) = \sigma'(t) \in V_{\sigma(t)}$ whenever σ' is defined so $L\sigma \in \overline{\Omega}(V;P,Q)_\sigma = P\sigma(H_0(I,R^n))$ a.e. So $P\sigma L\sigma = L\sigma$ a.e. But these two functions are continuous. Q.E.D.

9.4. Theorem. Let $T \in H_0(I,L(R^n,R^p))$ and define for each $\lambda \in H_0(I,R^n)$ a measurable function $\overline{T}(\lambda) : I \to R^p$ by $\overline{T}(\lambda)(t) = T(t)\lambda(t)$. Then

(1) $\overline{T}$ is a bounded linear transformation of $H_0(I,R^n)$ into $L^1(I,R^p)$;

(2) If T and λ are absolutely continuous then so is $\overline{T}\lambda$ and $(\overline{T}\lambda)'(t) = T'(t)\lambda(t) + T(t)\lambda'(t)$;

(3) If $T \in H_1(I,L(R^n,R^p))$, $\lambda \in H_1(I,R^n)$ then $\overline{T}\lambda \in H_1(I,R^p)$.

Proof. Case 1. $p = n = 1$.

(1) Then $T : I \to L(R,R)$, $\lambda \in H_0(I,R)$ and $\overline{T}(\lambda) : I \to R$ by $\overline{T}(\lambda)(t) = T(t)\lambda(t)$. Thus

$$\begin{aligned}\int_0^1 \|T(t)\lambda(t)\|\, dt &= \int_0^1 \|(\overline{T}\lambda)(t)\|\, dt \\ &\le \int_0^1 \|T(t)\|\ \|\lambda(t)\|\, dt \\ &\le \left(\int_0^1 \|T(t)\|^2\, dt\right)^{1/2} \left(\int_0^1 \|\lambda(t)\|^2\, dt\right)^{1/2}\end{aligned}$$

by Schwartz inequality. Since $T \in H_0(I,L(R,R))$ $\|T\|^2 = \int_0^1 \|T(t)\|^2\, dt < \infty$ so $\int_0^1 \|T(t)\lambda(t)\|\, dt \le \|T\|\ \|\lambda\|_0$. This implies that $\overline{T}$ is bounded.

(2) If λ and T are absolutely continuous, then $(\overline{T}\lambda)'(t) = (T(t)\lambda(t))' = T(t)'\lambda(t) + T(t)\lambda'(t)$.

(3) If $T \in H_1(I,L(R,R))$; $\lambda \in H_1(I,R)$, then $\overline{T}\lambda \in H_1(I,R)$. Since $T \in H_1(I,L(R,R))$, $\int_0^1 \|T'(t)\|^2\, dt < \infty$. Hence

$$\int_0^1 \|(\bar{T}\lambda)'(t)\|^2\, dt = \int_0^1 \|T'(t)\lambda(t) + T(t)\lambda'(t)\|^2\, dt$$

$$\leq \int_0^1 [\|T'(t)\lambda(t)\|^2$$

$$+ 2\|T'(t)\lambda(t)\|\ \|T(t)\|\ \|\lambda'(t)\|$$

$$+ \|T(t)\lambda'(t)\|^2]\, dt$$

$$\leq \|\lambda\|_\infty^2 \int_0^1 \|T'(t)\|^2\, dt$$

$$+ 2\|\lambda\|_\infty \|T\|_1 \int_0^1 \|T'(t)\|\ \|\lambda'(t)\|\, dt$$

$$+ \|T\|_1^2 \int_0^1 \|\lambda'(t)\|^2\, dt.$$

By Schwartz inequality we have $\int_0^1 \|T'(t)\|\ \|\lambda'(t)\|\, dt$
$\leq (\int_0^1 \|T'(t)\|^2\, dt)^{1/2} (\int_0^1 \|\lambda'(t)\|^2)^{1/2} < \infty$ so
$\int_0^1 \|(\bar{T}\lambda)'(t)\|^2\, dt < \infty$, i.e. $\bar{T}\lambda \in H_1(I,R)$.

<u>Case 2</u>. General case.

Let $\lambda \in H_1(I,R^n)$ and $\lambda(t) = (\lambda_1(t), \ldots, \lambda_n(t))$. Let $T \in H_0(I,L(R^n,R^p))$ be $T(t) = (T_{ij}(t))_{ij}$ where $T_{ij}(t) \in L(R,R)$. Then $\bar{T}\lambda(t) = T(t)\lambda(t)$ $= (T_{ij}(t)_{ij}(\lambda_i(t))^t = (\Sigma_i T_{ij}(t)\lambda_i(t))_j$. If T and λ are absolutely continuous then so are the T_{ij} and λ_i and

$$(\bar{T}\lambda)'(t) = (\Sigma_i(\lambda_i(t)T_{ij}(t))')_j$$

$$= (\Sigma_i(\lambda_i' T_{ij} + \lambda_i T_{ij}'))_j$$

$$= T'(t)\lambda(t) + T(t)\lambda'(t).$$

So the theorem follows by looking at the components. Q.E.D.

9.5. Definition. Given $\sigma \in \Omega(V;P,Q)$ we define $G_\sigma \in H_1(I,L(R^n,R^p))$ by $G_\sigma = \Omega \circ \sigma$, and we define $Q_\sigma \in H_0(I,L(R^n,R^p))$ by $Q_\sigma = G'_\sigma$.

9.6. Theorem. Let $\sigma \in \Omega(V;P,Q)$. If $\rho \in H_1(I,R^n)$ then $(LP_\sigma - P_\sigma L)\rho(t) = Q_\sigma(t)\rho(t)$. Given $f \in H_0(I,R^n)$ define an absolutely continuous map $g : I \to R^n$ by $g(t) = \int_0^t Q_\sigma(s)f(s)ds$. Then if $\rho \in H_1^*(I,R^n)$ $<f,(LP_\sigma - P_\sigma L)_\rho>_0 = <g - L_\rho>_0$.

Proof. Since $P_\sigma\rho(t) = \Omega(\sigma(t))\rho(t) = G_\sigma(\rho(t)) = G_\sigma \circ \rho(t)$ and $P_\sigma(L\rho)(t) = G_\sigma(t)L\rho(t) = G_\sigma(t)\rho'(t)$, since $Q_\sigma(t)\rho(t) + P_\sigma L\rho(t) = G'_\sigma(t)\rho(t) + G_\sigma(t)\rho'(t) = (G_\sigma(t) \circ \rho(t))' = (G_\sigma \circ \rho)'(t) = ((\Omega \circ \sigma) \circ \rho)'(t) = (P_\sigma\rho)'(t) = LP_\sigma\rho(t)$, so $(LP_\sigma - P_\sigma L)\rho(t) = Q_\sigma(t)\rho(t)$. By 9.2 and the above argument, we have $s \to Q_\sigma(s)f(s)$ is summable so g is absolutely continuous.

Note that since $G_\sigma(t) = \Omega(\sigma(t))$ is self-adjoint for all t, $Q_\sigma(t) = G'_\sigma(t)$ is self-adjoint whenever it is defined, hence

$$\begin{aligned} <f,(LP_\sigma - P_\sigma L)\rho>_0 &= \int_0^1 <f(t),Q_\sigma(t)\rho(t)>dt \\ &= \int_0^1 <Q_\sigma(t)f(t),\rho(t)>dt \\ &= \int_0^1 <g'(t),\rho(t)>dt. \end{aligned}$$

Then if $\rho \in H_1^*(I,R^n)$, 8.3 gives $<f,(LP_\sigma - P_\sigma L)\rho>_0 = <g,-L\rho>_0$. Q.E.D.

9.7. Theorem. If $\sigma \in \Omega(V;P,Q)$ then $P_\sigma h(\sigma)$ is absolutely continuous and $(P_\sigma h(\sigma))'(t) = Q_\sigma(t)h(\sigma)(t)$.

Proof. If $\rho \in H_1^*(I,R^n)$ then $<P_\sigma h(\sigma),L\rho>_0 = <h(\sigma),P_\sigma L\rho>_0 = <h(\sigma), (P_\sigma L - LP_\sigma)\rho>_0$, since $<h(\sigma),LP_\sigma\rho> = 0$. Then by 9.6 $<P_\sigma h(\sigma),L\rho>_0 = <g,L\rho>_0$ if we define g to be the absolutely continuous map of $I \to R^n$. $g(t) = \int_0^t Q_\sigma(t)h(\sigma)(s)ds$. Then $<P_\sigma h(\sigma) - g, L\rho>_0 = 0$ for all $\rho \in H_1^*(I,R^n)$. Then $P_\sigma h(\sigma) - g$ is orthogonal to $L(H_1^*(I,R^n))$ so $P_\sigma h(\sigma) - g$ is constant (8.2). Since g is absolutely continuous so is $P_\sigma h(\sigma)$ and they have the same derivative. But $g'(t) = Q_\sigma(t)h(\sigma)(t)$. Q.E.D.

9.8. Lemma. Given a compact subset A of V there is a constant K such that $\int_0^1 \|Q_\sigma(t)\rho(t)\|\,dt \le K\|L\sigma\|_0\ \|\rho\|_0$ for all $\rho \in H_0(I,R^n)$ and all $\sigma \in H_1(I,R^n)$ such that $\sigma(I) \subseteq A$.

Proof. Since V is closed in R^n and A is compact subset of V so A is closed and bounded. Let $A^* = \{(p,v,x) \mid p \in A,\ v \in V_p,\ \|v\| = 1,\ x \in R^n,\ \|x\| = 1\} \subset R^n \times R^n \times R^n$. Since A^* is closed and bounded in $R^n \times R^n \times R^n$ hence it is compact. Since Ω is C^{k+3} the map $(p,v,x) \to \|d\Omega_p(v)x\|$ is continuous on A^* hence bounded by some constant K. Since $Q_\sigma(t) = \frac{d}{dt} G_\sigma(t) = \frac{d}{dt} \Omega(\sigma(t)) = d\Omega_{\sigma(t)}(\sigma'(t))$ it follows that $\|Q_{\sigma(t)}\rho(t)\| \le \|d\Omega_{\sigma(t)}\|\ \|\sigma'(t)\|\ \|\rho(t)\| \le K\|\sigma'(t)\|\ \|\rho(t)\|$. Hence $\int_0^1 \|Q_\sigma(t)\rho(t)\|\,dt \le K \int_0^1 \|\sigma'(t)\|\ \|\rho(t)\| \le K\ \|L\sigma\|_0\ \|\rho\|_0$ by Schwartz's inequality. Q.E.D.

9.9. Theorem. *Let $S \subseteq \Omega(V;P,Q)$ and suppose J is bounded on S but that $\|\nabla J\|$ is not bounded away from zero on S. Then there is a critical point of J adherent to S.*

Proof. Since $\|\nabla J\|$ is not bounded away from zero and by 8.28 we can choose a sequence $\{\sigma_n\}$ in S such that $\|\nabla J_{\sigma_n}\| = \|L\sigma_n - h(\sigma_n)\|_0 \to 0$. Since $P\sigma_n$ is a projection for each n, $\|L\sigma_n - P\sigma_n h(\sigma_n)\|_0 = \|P\sigma_n L\sigma_n - P\sigma_n h(\sigma_n)\| \le \|L\sigma_n - h(\sigma_n)\|_0 \to 0$. By 8.25, S is totally bounded so we can assume $\|\sigma_n - \sigma_m\|_\infty \to 0$ as $n,m \to \infty$. If we could prove that $\|L(\sigma_n - \sigma_m)\|_0 \to 0$ as $n,m \to \infty$, by 8.26 σ_n would converge in $\Omega(V;P,Q)$ to a point σ in the closure of S, and σ would be a critical point of J.

But $\|L(\sigma_n - \sigma_m)\|_0^2 = \langle L\sigma_n, L(\sigma_n - \sigma_m)\rangle_0 - \langle L\sigma_m, -L(\sigma_n - \sigma_m)\rangle_0$ hence it will suffice to prove that $\langle L\sigma_n, L(\sigma_n - \sigma_m)\rangle_0 \to 0$ as $m,n \to \infty$. Now $\|L\sigma_n\|_0^2 = 2J(\sigma_n)$ is bounded, hence $\|L(\sigma_n - \sigma_m)\|$ is bounded and since $L\sigma_n - P\sigma_n h(\sigma_n) \to 0$ in

$H_0(I,R^n)$, it will suffice to prove $\langle P\sigma_n h(\sigma_n), L(\sigma_n - \sigma_m)\rangle_0 \to 0$ as $n,m \to \infty$. Recalling that $\sigma_n - \sigma_m \in H_1^*(I,R^n)$ (8.23) it follows from 9.7 and 8.3 that $|\langle P\sigma_n h(\sigma_n), L(\sigma_n - \sigma_m)\rangle_0| = |\int_0^1 \langle Q\sigma_n(t)h(\sigma_n)(t), (\sigma_n - \sigma_m)(t)\rangle\, dt|$ $\leq \|\sigma_n - \sigma_m\|_\infty \int_0^1 \|Q\sigma_n(t)h(\sigma)(t)\|\, dt$ and since $\|\sigma_n - \sigma_m\|_\infty \to 0$ we only need to show $\int_0^1 \|Q\sigma_n(t)h(\sigma_n)(t)\|\, dt$ is bounded.

Since $\{\sigma_n\}$ is uniformly Cauchy, it is uniformly bounded. We can find a compact set A such that $\sigma_n(I) \subseteq A$ for all n. By the lemma there is a constant K such that $\int_0^1 \|Q\sigma_n(t)h(\sigma_n)(t)\|\, dt \leq K\|L\sigma_n\|_0 \; \|h(\sigma_n)\|_0$. Now since $\|L\sigma_n\|_0$ is bounded and $\|L\sigma_n - h(\sigma_n)\|_0 \to 0$, $\|h(\sigma_n)\|_0$ is bounded. This prove the theorem. Q.E.D.

REFERENCES

S. Lang [1] Introduction to Differentiable Manifold, Interscience Publishers, 1967.

J. Milnor [1] Morse Theory, Annals of Mathematics Studies, Number 51.

E. Pittcher [1] "Inequalities of critical point theory", Bull. Amer. Math. Soc. 64 (1958), 1-30.

R. Palais [1] "Morse theory of Hilbert manifolds", Topology 2 (1963), 299-340.

[2] "The Morse lemma for Banach spaces", Bull. Amer. Math. Soc. 75 (1969).

[3] Lecture Notes on Differential Topology, by S. Greenfield Brandies, 1964-1965.

J. Schwartz [1] Non-linear Functional Analysis, Gordon and Breach Science Publishers, 1969.

REPORT ON MORSE THEORY ON HILBERT MANIFOLDS

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ABSTRACT

The finite dimensional Morse theory has been generalized to a Hilbert manifold modelled on a separable Hilbert space in the following form:

Let M be a complete Riemannian manifold of class C^{k+2} $(k \geq 1)$ and $f : M \to R$ a C^{k+2}-function. Assume that all the critical points of f are non-degenerate and in addition

(C) If S is any subset of M on which f is bounded but on which $\|\nabla f\|$ is not bounded away from zero then there is a critical point of f adherent to S.

Then

(a) The critical values of f are isolated and there are only a finite number of critical points of f on any critical level;

(b) If there are no critical values of f in $[a,b]$ then M_b is diffeomorphic to M_a;

(c) If $a < c < b$ and c is the only critical value of f in $[a,b]$ and $p_1, \ldots, p_r$ are the critical points of f on the level c, then M_b is diffeomorphic to M_a with γ-handles of type $(k_1,\ell_1), \ldots, (k_\gamma,\ell_\gamma)$ disjointly C^k-attached, where k_i and ℓ_i are respectively the index and coindex of p_i.

The purpose of this report is to fill in more details of Palais' paper "Morse theory on Hilbert manifolds".