BENDING AND BUCKLING OF A PRISMATIC BAR
SUBJECTED TO AXIAL AND LATERAL LOAD SIMULTANEOUSLY

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by

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SYNOPSIS

A method for determining the capacity of restrained bar of a constant cross section with different system of loading is described. The type of failure considered is that due to bending or buckling by lateral and longitudinal load acting simultaneously, and also due to restraining moments applied at the ends of bar. The restraining moments are created by applying equal or unequal eccentric load at the ends of bar. The results of the standard case of lateral loading are modified for asymmetrical lateral loading on a compressed bar, with the help of the principle of superposition. In general, the effect of the axial load on laterally loaded bar is discussed in limiting case of failure for each type of loading. The expression for deflection is derived for each case in multiple of two quantities of which, the first represents the deflection due to only lateral load and the second quantity, which is a trigono-

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metric expression, is the effect of axial load. This will afford enough information to investigate the critical value of axial load for which the bar remains in elastic stability before it buckles to failure.

INTRODUCTION

With the advancement in the use of high strength alloys or steel in engineering structures, especially in bridges, ships, and aircrafts, it becomes necessary to discuss the problems of elastic stability. Urgent practical requirements have given rise in recent years to extensive investigations, both theoretical and experimental, of the conditions governing the stability of such structural elements as bars, plates, and shells.(1)[2]

The first problems of elastic instability concerning lateral buckling of compressed members were solved by L. Euler. However, at that time, the principal structural materials necessitated large sections of structural members for which the question of elastic stability does not arise. Only with the beginning of extensive construction of steel railway bridges in the last five decades did the question of buckling of compression members become of practical importance. Thus, Euler's solution for slender bars became useful in such conditions. The use of steel led naturally to types of structures embodying slender compression members, thin plates, and thin shells. Experience

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(2) Number in parentheses refers to corresponding items in Appendix I.

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showed that such structures may fail in some cases not on account of high stresses, surpassing the strength of materials, but owing to insufficient elastic stability of slender or thin walled members.

The lateral buckling and bending of compressed members is a practical case of elastic instability. In the modern design of bridges, ships, and aircraft, we are confronted by a variety of stability problems. We have solid struts; built-up or "lattice work" columns; and tubular members, where there is a possibility of local buckling, as well as buckling as a whole. In the use of thin sheet material, as in plate girders, and airplane structures, we have to keep in mind that thin plates may prove unstable under the action of forces in their own planes, and fail by buckling sideways. Thin cylinders or shells, such as vacuum vessels, which have to withstand uniform external pressure, may exhibit instability and collapse at a relatively low stress if the thickness of the shell is too small in comparison with the diameter. The thin cylindrical shell may buckle also under axial compression, bending, torsion, or a combination of these.

If a beam is submitted to the action of lateral loads alone, the deflection of the beam and the stress produced are proportional to the magnitudes of the loads. Little change in the positions of the loads have only a small effect on deflections and stresses. Such small deflections do not affect the value of bending moment and shear forces in the beam. In actual calculation of these quantities the deflections are entirely neglected, and all necessary distances from the initial straight form of the beam are determined. However, conditions are entirely different when axial and lateral loads act simultaneously. The stresses and deflections are not proportional to the magnitude of the longitudinal force. A slightest eccentricity in the application of the axial load or a little initial deflection in the beam may have a substantial effect on the final deflection of the beam and on the stresses produced in this condition. Here the effect of small deflections of the beam should be considered in the calculation of bending moments and shear forces.

When the magnitude of the axial compressive force approaches a certain limiting value, usually called the critical load, the deflections become very sensitive to the slightest change in the position of the point of application or in magnitude of the axial load, and the bending of the beam becomes a characteristic sudden lateral buckling. To study this, the derivations of concrete examples are produced here.

ILLUSTRATIVE PROBLEMS

The first example chosen is a single load acting on a compressed bar. (1)[3] (2)[4]

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[4]Page No. 26.

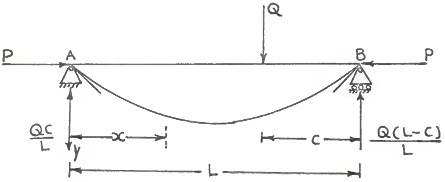

Fig. 1. Single load on a compressed bar.

If an axial compressive force acts on an ideal prismatic bar, it produces only uniform compression; however, if there is also a lateral load acting on the bar, the axial force produces some effect on the bending of the bar. This effect can be investigated by considering the deflection curve of the bar. (See Fig. 1.)

Denoting the corresponding flexual rigidity by EI and axial compressive force by P, the differential equations of the deflection curve for the left and right portions of the bar are

$$EI\,\frac{d^2y}{dx^2} = -\frac{Qcx}{L} - Py \tag{1}$$

$$EI\,\frac{d^2y}{dx^2} = -\frac{Q\,(L-C)\,(L-x)}{L} - Py \tag{2}$$

Knowing $M = -EI\,\dfrac{d^2y}{dx^2}$

in which L is the span and C is the distance of the load Q from the right support B.

For simplification denoting $\frac{P}{EI} = K^2$ the equation (1) becomes

$$\frac{d^2y}{dx^2} + K^2y = -\frac{QC}{L}\,\frac{x}{EI}$$

The corresponding homogeneous equation is $\frac{d^2y}{dx^2} + K^2y = 0$

The auxiliary equation is $D^2 + K^2 = 0$

The roots of this equation are $\pm iK$

Hence, the complementary solution is $y = A \cos Kx + B \sin Kx$

And, particular solution is $Y_p = -\frac{QC}{PL}x$

Therefore, the general solution is

$$Y = A \cos Kx + B \sin Kx - \frac{QC}{PL}x \qquad (3)$$

Similarly, the general solution of the equation (2) is

$$Y = C \cos Kx + D \sin Kx - \frac{Q(L-C)(L-x)}{PL} \qquad (4)$$

The constants of integration A, B, C, D can be determined from the conditions at the ends of the bar and at the point of application of the load Q.

Since the deflections at the end of the bar, are zero, then at A, $x = 0$, $y = 0$

Substituting these values in equation (3) we have

$$0 = A \cos 0 + B \sin 0 - \frac{QC}{PL}0$$

$$A = 0$$

At B, $x = L$ and $y = 0$

Substituting this in equation (4) we have

$$0 = C \cos KL + D \sin KL - \frac{Q(L-C)(L-L)}{PL}$$

$$C = -D \tan KL \qquad (5)$$

At the point of application of the load Q the two portions of the deflection curve as given by the equations (3) and (4), have the same deflection at Q and same slope at Q. So rewriting and simplifying the equations (3) and (4) we have

$$B \sin K(L-C) - \frac{QC}{PL}(L-C) = D \sin K(L-C)$$
$$- \tan KL \cos K(L-C) - \frac{QC}{PL}(L-C)$$

and

$$BK \cos K(L-C) - \frac{QC}{PL} = DK \cos K(L-C) - \tan KL \sin K(L-C)$$
$$+ \frac{Q(L-c)}{PL}$$

from which

$$A = 0 \qquad B = \frac{Q \sin KC}{PK \sin KL}$$
$$C = -\frac{Q \sin (L-C)}{PK} \qquad D = -\frac{Q \sin K(L-C)}{PK \tan KL} \qquad (6)$$

Substituting these values of constant in the equations (3) and (4) we have

$$y = \frac{Q \sin KC}{PK \sin KL} \sin Kx - \frac{QC}{PL}x \text{ for } x < (L-C) \qquad (7)$$
$$y = \frac{Q \sin K(L-C)}{PK} \cos Kx - \frac{Q \sin K(L-C)}{PK \tan KL} \sin Kx - \frac{Q(L-C)(L-x)}{PL}$$

which simplifies to

$$y = \frac{Q \sin K(L-C)}{PK \sin KL} \sin K(L-x) - \frac{Q(L-C)(L-x)}{PL} \text{ for } x > (L-C) \quad (8)$$

It will be necessary to find the deflection at the center and compare it with the standard value of deflection on a simply supported beam when load is at center. This value is $\frac{QL^3}{48EI}$

From equation (7) when $x = C = L/2$

$$y_{L/2} = \frac{Q \sin \frac{KL}{2}}{PK \sin KL} \sin \frac{KL}{2} - \frac{QL}{4P}$$

$$\text{Sin } KL = 2 \sin \frac{KL}{2} \cos \frac{KL}{2}$$

Therefore

$$y_{L/2} = \frac{Q \tan \frac{KL}{2}}{2PK} - \frac{QL}{4P} \qquad \text{(A)}$$

But it is already known that $K^2 = \frac{P}{EI}$

$$\frac{KL}{2} = \frac{L}{2} \sqrt{\frac{P}{EI}} = u \text{ say for convenience} \qquad (9)$$

So that $K = \frac{2u}{L}$ and $P = \frac{4u^2 EI}{L^2}$

Substituting values of K and P in (A) we have

$$y = \frac{Q(\tan u)L^3}{16u^3 EI} - \frac{QL^3}{16u^2 EI}$$

$$= \frac{QL^3}{16EI} \left[\frac{\tan u - u}{u^3}\right]$$

$$= \frac{QL^3}{48EI} \quad \frac{3}{u^3} (\tan u - u) \qquad (10)$$

From expression (10) it is obvious that with a load Q only at the center of the beam the deflection is $\frac{QL^3}{48EI}$ while the second factor $\frac{3}{u^3}(\tan u - u)$ is the effect of the axial load P.

In the equation $u = \frac{L}{2}\sqrt{\frac{P}{EI}}$, when P is small, the quantity u is also small and the factor $\frac{3}{u^3}(\tan u - u)$ approaches unity.

This can be shown by selecting first two terms of series for tan u and substituting it in the factor.

$$\tan = u + \frac{u^3}{3} + \frac{2u^5}{3\cdot5} \quad - - - - -$$

As u increases the factor also increases. The factor approaches to infinity as u increases to $\pi/_2$

The value of P when $u = \pi/_2$ is found from the relation

$$\frac{L}{2}\sqrt{\frac{P}{EI}} = u \qquad \text{But } u = \pi/_2$$

$$P = \frac{\pi^2 EI}{L^2}$$

This allows us to conclude that when the axial load increases to limiting value, the smallest lateral load on the beam may produce large deflection. This limiting value of compressive force is denoted by critical load Pcr.

It can be observed that when lateral load and axial load act simultaneously, the maximum bending moment can be obtained by multiplying the bending moment due to lateral load with the trigonometric factor such as $\frac{3}{u^3}$ (tan u-u). The value of this factor approaches to unity when the compressive force decreases but the same factor increases to infinity when the value of u approaches to $\pi/_2$, i.e., when compressive force has the value $P_{cr} = \frac{\pi^2 EI}{L^2}$ as derived above, which is known as critical value Pcr.

To support the method of finding bending moment as described above, an expression for the same is derived in a following manner. (1)[5]

From the equation (7)

$$y = \frac{Q \sin KC}{PK \sin KL} \sin Kx - \frac{QC}{PL}x$$

$$\frac{dy}{dx} = \frac{Q \sin KC}{P \sin KL} \cos Kx - \frac{QC}{PL}$$

$$\frac{d^2y}{dx^2} = - \frac{Q \sin KC}{P \sin KL} \cdot K \sin Kx$$

at $x = L/_2$ and $C = L/_2$

$$\frac{d^2y}{dx^2} = - \frac{Q \sin \frac{KL}{2}}{P \sin KL} \cdot K \sin \frac{KL}{2}$$

$$= - \frac{QK}{2P} \; \frac{\sin^2 \frac{KL}{2}}{\cos KL/_2 \; \sin KL/_2}$$

$$\frac{d^2y}{dx^2} = - \frac{QK}{2P} \tan \frac{KL}{2}$$

$$\text{B.M. Max.} = - EI \left(\frac{d^2y}{dx^2}\right)_{x = L/_2}$$

$$= EI \frac{QK}{2P} \tan \frac{KL}{2} \qquad \frac{KL}{2} = u$$

and $\quad K = \sqrt{\frac{P}{EI}}$

$$\text{B.M. Max.} = \frac{QL}{4} \frac{\tan u}{u}$$

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[5]Page No. 5.

Again we see that the first factor in above expression represents the bending moment produced by the load Q alone, while the second factor is the magnification factor, representing the effect of the axial force P on the maximum bending moment.

EXAMPLE PROBLEM 2 (1)[6]

The result of the single acting lateral load on a compressed bar can be generalized for several lateral loads. (See Figure 2.) It is seen from the equations (7) and (8) that when a lateral load Q_2 is added to the lateral load Q_1, the resultant deflection can be obtained by the principle of superposition. It can be shown that the same method of superposition can also be used if several lateral loads are acting. For instance, the case of two lateral loads as shown in Fig. 2:

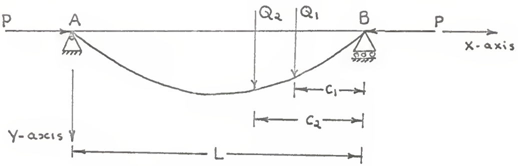

Fig. 2. Two lateral loads acting on a compressed bar.

The differential equation of the deflection curve of the left portion of the beam x $(L-C_2)$ is

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[6]Page No. 6.

$$EIy'' = -\frac{Q_1C_1}{L}x - \frac{Q_2C_2}{L}x - Py \qquad (10)$$

If we consider now the load Q_1 and the load Q_2 separately acting on the compressed bar and denote the deflection as Y_1 when the load Q_1 is acting and as Y_2 when the load Q_2 is acting, for the left portion of the beam the following will be the equations for the deflection curve.

$$EIY_1'' = -\frac{Q_1C_1}{L}x - PY_1$$

$$EIY_2'' = -\frac{Q_2C_2}{L}x - PY_2$$

Adding these two equations together, the following expression is obtained

$$EI(Y_1'' + Y_2'') = -\frac{Q_1C_1}{L}x - \frac{Q_2C_2}{L}x - P(Y_1 + Y_2)$$

When the deflections Y_1 and Y_2 are added, this equation is the same as equation (10) when it was obtained considering the loads Q_1 and Q_2 acting simultaneously. The same conclusion can be made also for the middle and for the right portions of the bar. From this it can be concluded that in the case of several loads acting on a compressed bar, the resultant deflections can be obtained by superposition of the deflections produced by each lateral load acting together with the longitudinal force P. The solution for n loads can be directly generalized from the equations (7) and (8). (See Fig. 3.)

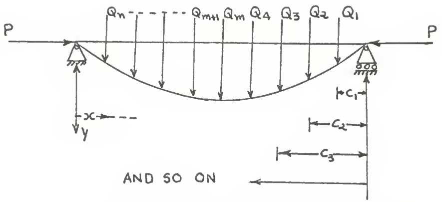

Fig. 3. n Lateral loads acting on a compressed bar.

$$Y = \frac{\sin Kx}{PK \sin KL} \sum_{i=1}^{m} Q_i \sin KC_i - \frac{x}{PL} \sum_{i=1}^{m} Q_i C_i$$

$$+ \frac{\sin K(L-x)}{PK \sin KL} \sum_{i=m+1}^{n} Q_i C_i \sin K(L-C_i)$$

$$- \frac{L-x}{PL} \sum_{i=m+1}^{n} Q_i (L-C_i) \qquad (11)$$

This is achieved by taking the summation of "n" loads.

EXAMPLE PROBLEM 3

The problem of a uniformly distributed load may be approached by use of equation (11). (See Fig. 4.)

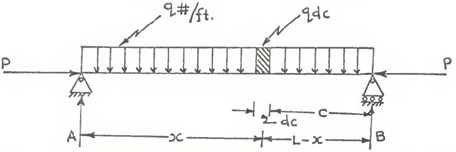

Fig. 4. Uniformly distributed lateral load acting on a compressed bar.

Consider a small length dc at 'c' from B so that the load acting at that point is q(dc). Substituting the value of Q = qdc and integrating to required limits we have:

$$y = \frac{\sin Kx}{PK \sin KL} \int_0^{L-x} qdc \sin KC - \frac{x}{PL} \int_0^{L-x} qc\, dc$$

$$+ \frac{\sin K(L-x)}{PK \sin KL} \int_{L-x}^{L} q \sin K(L-C)dc - \frac{L-x}{PL} \int_{L-x}^{L} q(L-C)dc \tag{12}$$

$$= \frac{\sin Kx}{PK \sin KL} \left[- \frac{q}{K} \cos KC \right]_0^{L-x} - \frac{xq}{PL} \left[\frac{C^2}{2}\right]_0^{L-x}$$

$$+ \frac{\sin K(L-x)}{PK \sin KL}\, q \left[\cos K(L-C) \frac{1}{K}\right]_{L-x}^{L} - \frac{(L-x)\, q}{PL} - \left[\frac{(L-C)^2}{2}\right]_{L-x}^{L}$$

$$y = \frac{\sin Kx}{PK^2 \sin KL}\, q \left[1 - \cos (L-x) K\right] - \frac{xq}{2\, PL} (L-x)^2$$

$$+ \frac{\sin K(L-x)}{PK^2 \sin KL}\, q \left[1 - \cos Kx\right] + \frac{L-x}{PL}\, q(-x^2/_2)$$

$$y = \frac{q}{PK^2} \frac{1}{\sin KL}\, 2 \left[\sin Kx \sin^2 \frac{K(L-x)}{2} + \sin K(L-x) \sin^2 \frac{Kx}{2}\right]$$

$$- \frac{q}{2PL} \left[x(L-x)^2 + x^2(L-x)\right]$$

$$y = \frac{2q}{PK^2}\frac{1}{\sin KL}\,2\left[\sin\frac{Kx}{2}\cos\frac{Kx}{2}\sin^2\frac{K(L-x)}{2}\right.$$

$$\left. + 2\ \text{xin}\,\frac{K(L-x)}{2}\cos\frac{(L-x)K}{2}\sin^2\frac{Kx}{2}\right]$$

$$-\frac{q}{2PL}\left[x(L-x)\ \ \{L-x+x\}\right]$$

$$= \frac{4q}{PK^2\sin KL}\sin\frac{Kx}{2}\sin\frac{K(L-x)}{2}\left[\cos\frac{Kx}{2}\sin\frac{K(L-x)}{2}\right.$$

$$\left. + \cos\frac{K(L-x)}{2}\sin\frac{Kx}{2}\right] - \frac{q}{2PL}Lx(L-x)$$

$$= \frac{4q}{PK^2\sin KL}\sin\frac{Kx}{2}\sin\frac{K(L-x)}{2}\sin\left(\frac{Kx}{2}+\frac{KL}{2}-\frac{Kx}{2}\right)$$

$$-\frac{q}{2P}x\,(L-x)$$

$$= \frac{4q\sin\frac{Kx}{2}\sin\frac{K(L-x)}{2}}{2PK^2\sin\frac{KL}{2}\cos\frac{KL}{2}}\sin\frac{KL}{2} - \frac{q}{2P}x(L-x)$$

$$= \frac{2q}{PK^2\cos\frac{KL}{2}}\ \frac{1}{2}\left[\cos\left(\frac{Kx}{2}-\frac{KL}{2}+\frac{Kx}{2}\right)\right.$$

$$\left. - \cos\left(\frac{KL}{2}-\frac{Kx}{2}+\frac{Kx}{2}\right)\right] - \frac{q}{2P}x\,(L-x)$$

$$y = \frac{q}{PK^2\cos\frac{KL}{2}}\left[\cos\left(Kx-\frac{KL}{2}\right) - \cos\frac{KL}{2}\right] - \frac{q}{2P}x\,(L-x)$$

$$y = \frac{q}{PK^2}\ \frac{\cos\frac{KL}{2}\left(1-\frac{2x}{L}\right)}{\cos\frac{KL}{2}} - 1\quad - \frac{q}{2P}x\,(L-x)$$

If $\quad u = \frac{KL}{2} = \frac{L}{2}\sqrt{\frac{P}{EI}} \qquad u^2 = \frac{L\,P}{4EI}$

$$P = \frac{4EIu^2}{L^2} \qquad \text{also} \qquad K^2 = \frac{4u^2}{L^2}$$

$$y = \frac{qL^2\ L^2}{4EIu^2\ 4u^2}\left[\frac{\cos u(1 - \frac{2x}{L}}{\cos u} - 1\right] - \frac{qL^2\ x(L-x)}{8\ EIu^2}$$

$$y = \frac{qL^4}{16EIu^4}\left[\frac{\cos u(1 - \frac{2x}{L}}{\cos u} - 1\right] - \frac{qL^2}{8EIu^2}\,x\,(L-x) \tag{13}$$

As P increases u also increases. (Maximum value of $u = \pi/2$ (say) so that $\cos u = 0$ and hence, the deflection becomes ∞.) Therefore, we can conclude that axial force P has considerable effect on the bending of the beam when the beam is loaded laterally.

EXAMPLE PROBLEM 4

A triangular load is an interesting situation to be considered. The results will be similar to the preceding example of a uniformly distributed load. (See Fig. 5.)

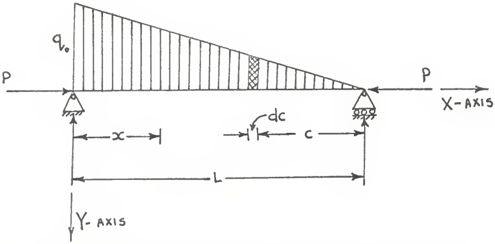

Fig. 5. Triangular distributed lateral load acting on compressed bar.

Considering a section at a distance c from B having width dc.

q_o is the maximum load at left hand side.

The value of load at distance c from B $\quad q = \frac{q_o C}{L}$

Substituting $\frac{q_o C}{L}$ for q in equation (12) we have

$$y = \frac{\sin Kx}{PK \sin KL} \int_0^{L-x} \frac{q_o C}{L}\, dc \sin KC - \frac{x}{PL} \int_0^{L-x} \frac{q_o C^2}{L}\, dc$$

$$+ \frac{\sin K(L-x)}{PK \sin KL} \int_{L-x}^{L} \frac{q_o C}{L} \sin K(L-C) dc - \frac{L-x}{PL} \int_{L-x}^{L} \frac{q_o C}{L} (L-C) dc \qquad (14)$$

$$y = \frac{q_o \sin Kx}{PK\, L \sin KL} \left[- \frac{C}{K} \cos KC + \frac{1}{K^2} \sin KC \right]_0^{L-x} - \frac{q_o}{PL^2}\, x \left[\frac{C^3}{3} \right]_0^{L-x}$$

$$+ \frac{q_o \sin K(L-x)}{PK\, L \sin KL} \left[\frac{C}{K} \cos K(L-C) + \frac{1}{K^2} \sin (L-C) K \right]_{L-x}^{L}$$

$$- \frac{(L-x)}{PL^2}\, q_o \left[\frac{LC^2}{2} - \frac{C^3}{3} \right]_{L-x}^{L}$$

$$y = \frac{q_o \sin Kx}{PK\,L \sin KL}\left[-\frac{(L-x)}{K}\cos K(L-x) + \frac{1}{K^2}\sin K(L-x)\right]$$

$$-\frac{q_o}{PL^2}\;\frac{x(L-x)^3}{3}$$

$$+\frac{q_o \sin K(L-x)}{PK^2\,L \sin KL}\left[L - (L-x)\cos Kx - \frac{1}{K}\sin Kx\right]$$

$$-\frac{(L-x)}{PL^2}\,q_o\left[\frac{L^3}{2} - \frac{L^3}{3} - \frac{L(L-x)^2}{2} + \frac{(L-x)^3}{3}\right]$$

$$y = \frac{q_o}{PK^2\,L}\,\frac{1}{\sin KL}\left[\frac{1}{K}\sin K(L-x)\sin Kx\right.$$

$$- (L-x)\sin Kx \cos K(L-x)$$

$$+ L \sin K(L-x) - (l-x)\cos Kx \sin K(L-x)$$

$$\left.-\frac{1}{K}\sin Kx \sin K(L-x)\right]$$

$$-\frac{q_o}{PL^2}\left[\frac{x(L-x)^3}{3} + (L-x)\left\{\frac{L^3}{6} - (L-x)^2\left(L/_2 - L/_3 + x/_3\right)\right\}\right]$$

$$y = \frac{q_o}{PK^2\,l \sin KL}\left[L \sin K(L-x)\;\left\{\sin Kx \cos K(L-x)\right.\right.$$

$$\left.\left. + \cos Kx \sin K(L-x)\right\}\right]$$

$$+\frac{q_o}{PL^2}(L-x)\left[\frac{x}{3}(L-x)^2 + \frac{L^2}{6} - (L-x)^2\left(L/_6 + x/_3\right)\right]$$

$$y = \frac{q_o}{PK^2\ L \sin KL}\left[L \sin K(L-x) - (L-x) \sin KL\right]$$

$$- \frac{q_o}{PL^2}\,\frac{L}{6}\,(L-x)\left[L - (L-x)^2\right]$$

$$y = \frac{q_o}{PK^2\ L}\left[L\,\frac{\sin K(L-x)}{\sin KL} - \frac{L \sin KL}{\sin KL} + \frac{x \sin KL}{\sin KL}\right]$$

$$- \frac{q_o}{PL}\,\frac{(L-x)}{6}\,(2L-x)\ x$$

$$y = \frac{q_o}{PK^2}\left[\frac{\sin K(L-x)}{\sin KL} + \frac{x}{L} - 1\right] - \frac{q_o}{6PL}\,x(L-x)\ (2L-x)$$

Now $u = \frac{KL}{2} = L/2\ \sqrt{\frac{P}{EI}}$ $\qquad P = \frac{4EIu^2}{L^2}$

and $K^2 = \frac{4u^2}{L^2}$ (A)

We have

$$y = \frac{q_o L^4}{16EIu^4}\left[\frac{\sin 2u(1-\frac{x}{L})}{\sin 2u} - \frac{x}{L} - 1\right] - \frac{3q_o\ Lx}{8EIu^2}\,(L-x)\ (2L-x) \tag{15}$$

As P increases u increases as seen in equation A. Let u increase to $\pi/2$ such that $\sin 2u = \sin\pi = 0$; the value of the expression then becomes infinity. The same argument presented in the Problem 3 holds good; that is, the axial force P has substantial effect on a laterally loaded beam.

Using the expressions previously derived, one can solve for any system of loading on the beam. To illustrate this, two modifications are briefly discussed. (See Fig. 6.)

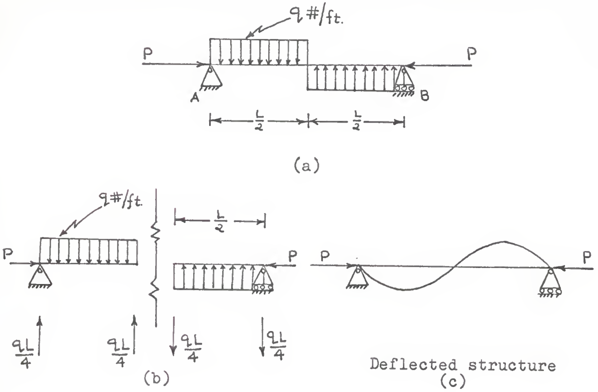

Fig. 6. Asymmetrical uniformly distributed load acting on compressed bar.

The above system of loading, as shown in Fig. 6 (a), does not occur usually in a practical problem, but theoretically it will be interesting to see the application of derived expressions. The above problem can be divided into the forms shown in Fig. 6 (b) and (c).

The two halves are solved separately taking $L/2$ as the span. At $x = L/2$ the net deflection is zero. The same problem can be solved by assuming the equation of deflected curve of the form

$$y = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L} .$$

The equation of the deflected curve is derived by substituting $^L/_2$ instead of L in expression number (13).

$$y = \frac{q\,L^4}{256EIu^4}\left[\frac{\cos u\,(1 - \frac{4x}{L})}{\cos u} - 1\right] - \frac{q\,L^2}{32EIu^2}\,x\,(^L/_2 - x)$$

This shows that at $x = {}^L/_2$ $y = 0$ which satisfies the boundary condition. This expression is true for both parts of the divided beam because of the symmetry, but in first part the direction of y is downward, while in second part the direction of y is upward. The second modification is shown in Figure 7.

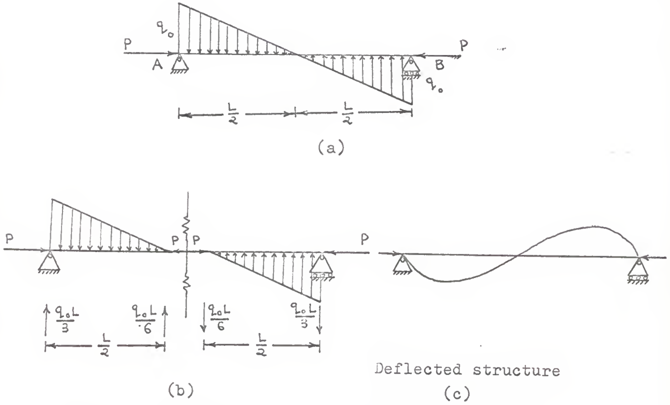

Fig. 7. Triangular distributed load acting on a compressed bar.

The system of lateral loading can be divided into two portions and each can be solved separately.

On similar lines the expression for the deflected structure can be derived by substituting $^L/_2$ instead of L in equation (15). Then we have

$$y = \frac{q_o\ L^4}{256EIu^4}\left[\frac{\sin 2u\ (1-\frac{2x}{L})}{\sin 2u} - \frac{2x}{L} - 1\right] - \frac{3q_o\ Lx}{16EIu^2}\left(^L/_2-x\right)(L-x)$$

This shows that at $x = {}^L/_2$ $y = 0$ which satisfies the boundary condition. This expression is true for both parts of the divided beam, because of the symmetry, but in first part the direction of y is downward, while in second part the direction of y is upward.

EXAMPLE PROBLEM 5 (1)[7] (2)[8] (3)[9]

In this example the bending of a compressed bar by couples is considered. Such problems are a modification of single concentrated load Q. The solution for a single concentrated force "Q" is available for us to use in this case by recalling expressions (7) and (8). The equation of the deflected curve when the couple is applied at the right hand of the beam can be derived in the following way: (See Fig. 8.)

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[7] Page No. 11.
[8] Page No. 29.
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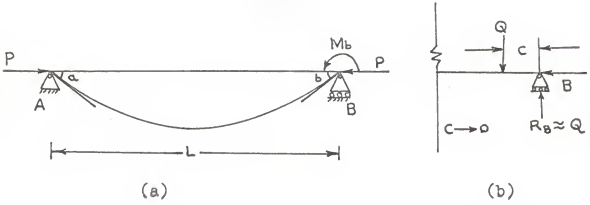

Fig. 8. Moment acting at one end of compressed bar.

Assume that the distance from one end of the beam is indefinitely decreasing and that at the same time Q is increasing so as to have the product QC finite and equal to M_b. The deflection curve is then obtained from equation (7) by substituting sin KC = KC for C is very small and QC = M_b.

$$y = \frac{Q \sin KC}{PK \sin KL} \sin Kx - \frac{QC}{PL}x \qquad \text{for } x \angle (L-C)$$

$$y = \frac{QKC}{PK} \frac{\sin Kx}{\sin KL} - \frac{QC}{PL}x \qquad \text{now } QC = M_b$$

$$y = \frac{M_b}{P} \frac{\sin Kx}{\sin KL} - \frac{M_b x}{PL}$$

$$y = \frac{M_b}{P} \left[\frac{\sin Kx}{\sin KL} - \frac{x}{L} \right] \qquad (16)$$

To find angle of rotation at the ends B and A expression (16) is differentiated.

$$\frac{dy}{dx} = \frac{M_b}{P} \left[\frac{K \cos Kx}{\sin KL} - \frac{1}{L} \right]$$

Evaluating this relation at $x = 0$ and $x = L$

$$a = y'_{x=0} = \frac{M_b}{P}\left[\frac{K}{\sin KL} - \frac{1}{L}\right]$$

$$= \frac{M_b}{P} K\left[\frac{1}{\sin 2u} - \frac{1}{2u}\right] \qquad \text{as } \frac{KL}{2} = u$$

but $\frac{L}{2} \quad \frac{P}{EI} = u$

and $\qquad P = \frac{4u\ EI}{L^2}$

'a' and 'b' are angles of rotation at A and B respectively.

Substituting these values we obtain:

$$a = \frac{M_b L}{2uEI}\left[\frac{1}{\sin 2u} - \frac{1}{2u}\right]$$

$$a = \frac{M_b L}{6EI}\frac{3}{u}\left[\frac{1}{\sin 2u} - \frac{1}{2u}\right] \qquad (17)$$

Similarly at $x = L$

$$b = \frac{M_b L}{3EI}\frac{3}{2u}\left[\frac{1}{2u} - \frac{1}{\tan u}\right] \qquad (18)$$

The expressions $\frac{M_b L}{6EI}$ and $\frac{M_b L}{3\ EI}$ obtain the values of the angles produced by the couple M_b acting alone, while the trigonometric factors represent the effect of axial load on the angles of rotation of the ends of the bar.

For calculating the effect of M_a and M_b on a compressed bar, the trigonometric factor may be abbreviated as follows to facilitate the calculation.

$$\frac{3}{u}\left(\frac{1}{\sin 2u} - \frac{1}{2u}\right) = \phi(u) \tag{19}$$

$$\frac{3}{2u}\left(\frac{1}{2u} - \tan 2u\right) = \psi(u) \tag{20}$$

If two couples M_a and M_b are acting at the ends A and B of the bar, the deflection curve is obtained by superposition. From relation (16) we obtain the deflection produced by the couple M_b. Substituting in the same equation M_a for M_b and $(L-x)$ for x, we find the deflections produced by the couple M_a. The deflection curve then is represented by

$$y = \frac{M_b}{P}\left[\frac{\sin Kx}{\sin KL} - \frac{x}{L}\right] + \frac{M_a}{P}\left[\frac{\sin K(L-x)}{\sin KL} - \frac{L-x}{L}\right] \tag{21}$$

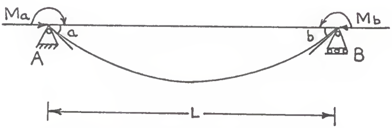

Fig. 9. Couples from either end act on compressed bar.

Using the relations (17), (18), (19), and (20), and the principle of superposition, we obtain the angle of rotation:

$$\left.\begin{aligned} a &= \frac{M_a L}{3EI}\psi(u) + \frac{M_b L}{6EI}\phi(u) \\ b &= \frac{M_b L}{3EI}\psi(u) + \frac{M_a L}{6EI}\phi(u) \end{aligned}\right\} \cdots \tag{22}$$

If the axial load has the eccentricity e_a and e_b at the end A and B, equation (21) can be rewritten in the following way. Substituting $M_a = Pe_a$ and $M_b = Pe_b$, we obtain:

$$y = e_b \left[\frac{\sin Kx}{\sin KL} - \frac{x}{L} \right] + e_a \left[\frac{\sin K(L-x)}{\sin KL} - \frac{L-x}{L} \right]$$

It is interesting to find the expression for deflection when $M_a = M_b = M$ at $x = L/2$.

$$y = \frac{M}{P} \left[\frac{\sin Kx}{\sin KL} - \frac{x}{L} \right] + \frac{M}{P} \left[\frac{\sin K(L-x)}{\sin KL} - \frac{L-x}{L} \right]$$

Substituting $x = L/2$

$$= \frac{M}{P} \left[\frac{\sin K^{L}/_2}{\sin KL} - 1/2 + \frac{\sin K^{L}/_2}{\sin KL} - 1/2 \right]$$

$$= \frac{M}{P} \left[\frac{2\sin K^{L}/_2}{\sin KL} - 1 \right]$$

$$y = \frac{M}{P} \left[\frac{2\sin \frac{KL}{2}}{2\sin \frac{KL}{2} \cos \frac{KL}{2}} - 1 \right]$$

$$= \frac{M}{P} \quad (\sec u - 1) \qquad \text{As } \frac{KL}{2} = u$$

$$= \frac{ML^2}{8EI} \frac{2}{u^2} \quad (\sec u - 1) \qquad P = \frac{4u^2}{L^2} EI$$

or $$y = \frac{ML^2}{8EI} \frac{2}{u^2} \left[\frac{1 - \cos u}{\cos u} \right] \qquad (23)$$

The value of the deflection at the center when only end moments are acting may be computed by using "Tafel der Warte". (See Fig. 10.)

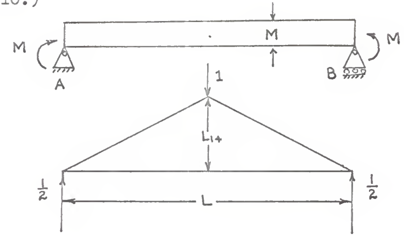

Fig. 10. Moment diagram when unit moment acts at the ends and unit load acts at the middle point of the beam.

$$y = \frac{1}{EI}\,\frac{1}{2}\,M_i\;M_K\;L'$$

$$= \frac{1}{EI}\,\frac{1}{2}\,\frac{L}{4}\,ML$$

$$= \frac{ML^2}{8EI}$$

Therefore, it can be seen that the factor $\frac{2}{u^2}$ (sec u - 1) in relation (23) is the effect of the axial load on the deflection of the beam. As U approaches $\pi/_2$ the same factor increases indefinitely because $\frac{1}{\cos u} = \frac{1}{\cos \pi/2} = \frac{1}{0} = \infty$. The same factor becomes unity when trigonometric function is expanded and only the first few terms are considered. Therefore, an axial load creates a considerable deflection and affects the bending of the beam when accompanied by lateral loads.

EXAMPLE PROBLEM 6 (1)[10] (2)[11]

In the previous discussion the critical load on a compressed bar was obtained by considering the simultaneous action of compressive and bending forces. The same result may be obtained by assuming initially that the bar is perfectly straight and simply compressed by a centrally applied load. First the case of slender prismatic bar built vertically at the bottom and loaded axially at the top is considered. (See Fig. 11.)

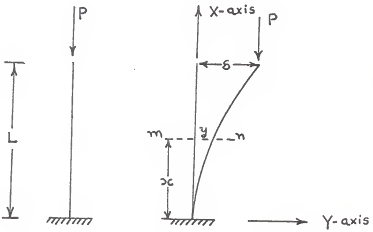

Fig. 11. Slender prismatical bar built in vertically at the bottom and loaded axially at the top.

The problem of buckling of columns was first discussed by L. Euler. If the load P is less than its critical value, the bar remains straight and undergoes only axial compression.

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This straight form of elastic equilibrium is stable. If a lateral force is applied and a small deflection produced, this deflection disappears when the lateral force is removed and the bar becomes straight again. By gradually increasing P, a condition is obtained in which the straight form of equilibrium becomes unstable and a slight lateral force may produce a lateral deflection which does not disappear when the lateral force is removed. The critical load is then defined as the axial load which is sufficient to keep the bar in a slightly bent form. This load can be calculated by using the differential equation of the deflection curve. Using coordinate axis marked in the figure, the bending moment at any cross section mn is $P(\delta - y)$ and corresponding differential equation of the curve becomes

$$EI\frac{d^2y}{dx^2} = P(\delta - y) \qquad (24)$$

$$K^2 = \frac{P}{EI}$$

We obtain

$$\frac{d^2y}{dx^2} + K^2y = K^2\delta$$

The complementary solution will be

$$y = A \cos Kx + B \sin Kx$$

while the particular solution of the equation is $y = \delta$

The general solution of the above differential equation will be

$$y = \delta + A \cos Kx + B \sin Kx \qquad (25)$$

in which A and B are constants of integration which must be adjusted to satisfy the conditions of the built-in end.

$$y = 0 \quad \text{at} \quad x = 0$$

$$\frac{dy}{dx} = 0 \quad \text{at} \quad x = 0$$

i.e. $A = -\delta$ and $B = 0$

Hence we have

$$y = \delta - \delta \cos Kx \qquad (26)$$

The condition at the upper end requires

$$y_{x=L} = \delta$$

This will be satisfied if $\delta \cos KL = 0$ which requires either $\delta = 0$ in which case there is no deflection or $\cos KL = 0$ because $\delta \neq 0$. i.e., $KL = (2n+1) \;/_2$ when n is any integer the smallest value of KL which satisfies the condition is $\pi/_2$.

Hence $\quad KL = L\sqrt{\frac{P}{EI}} = \frac{\pi}{2}$

From which we have

$$P_{cr} = \frac{\pi^2 EI}{4L^2} \qquad (27)$$

This is the critical load for the bar in such condition. It is the smallest axial load that can keep the bar in slightly bent shape.

EXAMPLE PROBLEM 7 (2)[12]

Lateral Buckling of Prismatic Bar

As an example of a more complicated case of lateral buckling of bars, let us consider a centrally compressed strut with the lower end built in and the upper end hinged. The critical value of the compressive force is that value Pcr which can keep the strut in slightly buckled shape. During buckling a lateral reaction Q will be produced as shown in the figure.

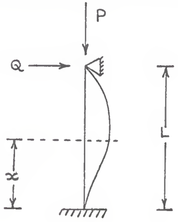

Fig. 12. Lower end built and upper end hinged compressed bar.

The differential equation of the deflection curve becomes

$$EI \frac{d^2y}{dx^2} = - Py + Q(L-x)$$

From the past experience the general solution of above differential equation is

$$y = A \cos Kx + B \sin Kx + \frac{Q}{P}(L-x)$$

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Eliminating the constants A and B by the boundary conditions

$x = 0 \qquad y = 0$

$x = 0 \qquad \frac{dy}{dx} = 0$

$x = L \qquad y = 0$

$A + \frac{Q}{P}L = 0 \qquad\qquad \text{(i)}$

$A \cos KL + B \sin KL = 0 \qquad\qquad \text{(ii)}$

$KB - \frac{Q}{P} = 0 \qquad\qquad \text{(iii)}$

Hence

$A = -\frac{Q}{P}L \qquad\qquad$ From (i)

$B = \frac{Q}{KP} \qquad\qquad$ From (ii)

$-\frac{A}{B} = \tan KL \qquad\qquad$ From (iii)

Hence, we have the following relation from the above

$\tan KL = KL$

The above relation can be solved graphically $KL = \pi/2$, $3\pi/2$ and a curve for tan KL is drawn. The root of equation is found from the intersection of line $y = KL$ with the curves. The smallest value obtained in this way is $KL = 4.493$.

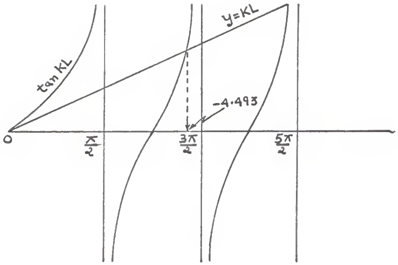

Fig. 13. Graph showing the value of K.

Then $$Pcr = K^2EI = \frac{K^2L^2}{L^2} EI$$

Hence $$Pcr = \frac{20.19EI}{L^2}$$

EXAMPLE PROBLEM 8

Finally, it will be interesting to solve some problems with the help of the results expressed previously. Let us assume such condition as expressed in problem

Fig. 14. (a) Lateral load and couple acting on a compressed bar.

This problem is divided into two problems as shown below:

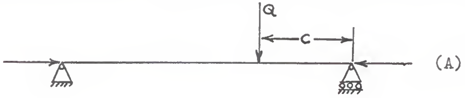

Fig. 14 (b)

Fig. 14 (c)

Solution of A and B are already discussed. Hence, by superimposing two results, we will get the required expression for the deflection of original beam.

CONCLUSION

In most of the above expressions we find the trigonometric function along with expression of the bending of bar without axial force. This trigonometric expression is the effect of the axial force. The limiting conditions of the trigonometric function shows us that the effect of the axial load is considerable and in limiting condition the bar may break due to this force.

ACKNOWLEDGMENT

I wish to thank and express my deepest appreciation and gratitude to Professor J. Mc Entyre whose encouragement, aid, and criticism rendered this report possible.

APPENDIX I

Reading References

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BENDING AND BUCKLING OF A PRISMATIC BAR
SUBJECTED TO AXIAL AND LATERAL LOAD SIMULTANEOUSLY

by

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B. S., Vallabh Vidyapith, 1960

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Approved by:

Major Professor

ABSTRACT

A method for determining the capacity of restrained bar of a constant cross section with different system of loading is described. The type of failure considered is that due to bending and buckling by lateral and longitudinal load acting simultaneously, and also due to restraining moments applied at the ends of bar. The restraining moments are created by applying equal or unequal eccentric load at the ends of bar. The results of the standard case of lateral loading are modified for asymmetrical lateral loading on a compressed bar, with the help of the principle of superposition. In general, the effect of the axial load on laterally loaded bar is discussed in limiting case of failure for each type of loading. The expression for deflection is derived for each case in multiple of two quantities of which, the first represents the deflection due to only lateral load and the second quantity , which is a trigonometric expression, if the effect of axial load. This will afford enough information to investigate the critical value of axial load for which the bar remains in elastic stability before it buckles to failure.