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COMPUTER-PROGRAM FOR FINDING
THE COMPLEX ROOTS OF A TRANSENDENTAL EQUATION
BY
THE NEWTON-RAPHSON METHOD

by 557

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GLOSSARY OF SYMBOLS

C	Integration variable
C_n	Mode number C-plane
$d\ell$	Dipole incremental length
$\bar{E}_n$	Electric field
F_n	Height gain function
H	Ionosphere height
$\bar{H}_n$	Magnetic field
$H_n^{(2)}$	Hankel function (second kind, order n)
I	Dipole current
$\bar{J}_n$	Current density vector
j	$= \sqrt{-1}$
k_n	Wave number in medium n
n	Subscript media n
N_n	Index of refraction for medium n
R	Distance ray travels
R_g	Fresnel reflection co-efficient, ground
R_i	Fresnel reflection co-efficient, ionosphere
r	Distance between source point and receiver point
r_o	Earth radius
S	Integration variable
S_n	Mode number S-plane
t	Time
v_n	$= \sqrt{\lambda^2 - k_n^2}$

Z	Receiver height
Z_o	Source height
γ_n	Propagation constant
δ	Dirac delta function
δ_n	Reciprocal of the derivative of the mode equation at nth mode
∂	Partial derivative operator
∇x	Curl vector operator
∇^2	Laplacian operator
ε_n	Complex permittivity
θ	Cylindrical angle measured from X-axis
λ	Integration variable
μ_n	Permeability
$\bar{\pi}_n$	Hertz vector
ρ	Horizontal distance between source and receiver
σ_n	Conductivity
ω	Radian frequency
ω_o	Center radian frequency

CHAPTER I

INTRODUCTION TO THE REPORT

1.1 INTRODUCTION

The object of this report is to set forth the most useful method for finding the complex roots of fairly complicated functions. Such functions are encountered quite often in many fields of electrical engineering. In some cases it is almost impossible to find the roots by hand-calculations. The Newton-Raphson method has been found to be a powerful tool for finding roots. This method has been demonstrated by a parallel-plate waveguide problem which is of great interest these days. A brief introduction about the problem is given in the next paragraph.

During the past few years, there has been a considerable interest in the propagation of electromagnetic pulses caused by nuclear bursts or lightning. These pulses travel large distances over the surface of the earth and into the ground. The propagation of electromagnetic pulse has engineering significance in many areas such as nuclear test detection systems, damage to electronic equipment and interruption of communication systems. A number of papers have been published in the recent literature. Johler [1] found that the pulses caused by nuclear burst contain components as low as 10 Hz. More recently, good work has been done by Bernotski [2] in this field. He attacked this problem with a particular emphasis on the near zone problem (20 to 300 Km) and on very low frequency pulses (50 Hz to 10 kHz). A mathematical model, studied by Bernotski, is reviewed in Chapter III of this report. In the model the earth and the ionosphere form the parallel planes of a parallel-plate waveguide. An integral solution is obtained for the vertical electric dipole source by solving Maxwell's equations subject to the appropriate boundary conditions at the walls of the guide. The integral solution is then approximated by a mode series. The evaluation of the mode series involves the solution of a transendental mode equation as a function of frequency.

1.2 OBJECTIVE OF THE REPORT

The objective of this report was to write a general computer program which could find the roots of the transendental mode equation for any number of modes over any range of frequency. The Newton-Raphson method was used to find the roots. Although there are many numerical methods, this was found to be very rapid. The Newton-Raphson method, its pitfalls and its speed of convergence are discussed in detail in Chapter II. The Chapter III discusses the mathematical model of the waveguide problem and the approximation of its integral solution by mode series.

In Chapter IV the numerical solution to the transendental mode equation is presented. Computer programming and the problems encountered are discussed in this chapter. Results are presented for two specific cases. The last chapter concludes the report.

CHAPTER II

NUMERICAL METHOD OF FINDING ROOTS

2.1 INTRODUCTION

Determining the roots of equations is encountered frequently in modern computing since it is required in a great variety of applications. Generally, a function of x, F(x), is given and it is required to find the values of x for which

$$F(x) = 0 \tag{2.1}$$

The function F may be algebraic or transcendental and it is generally differentiable.

In practice, generally the functions are quite complicated and have no simple closed formula for their roots. In such cases the roots can only be found by methods of approximating the roots. These methods involve two steps:

(1) Finding an approximate root.

(2) Refining the approximation to some prescribed degree of accuracy.

The first approximation is generally known from physical considerations. Sometimes graphical methods can be used as discussed by Kaiser [8]. Special methods exist for the important case in which F(x) is a polynomial. (See Anthony [9])

Refining an initial approximation or "guess" is done by a numerical method in which a succession of approximations is made and this is known as an <u>iterative</u> technique. Each step, or approximation, is called <u>iteration</u>. If the iterations produce approximations that approach the solution more and more closely, it is said that the iteration method <u>converges</u>.

Although there are several iterative techniques for the solution of equations of this problem only the Newton-Raphson method is discussed in this chapter.

2.2 THE NEWTON-RAPHSON METHOD

The Newton-Raphson method is a very rapid method for computing the real roots of $F(x) = 0$ if the derivative of $F(x)$ can be found easily.

To derive a formula for computing roots by this method, let 'a' denote an approximate value of the desired root. Suppose 'h' denotes the correction which must be applied to 'a' to get exact value of the root. Then

$$x = a + h \tag{2.2}$$

is a root of equation (2.1). Replacing x by (a + h), equation (2.1) becomes

$$F(a + h) = 0 \tag{2.3}$$

Expanding by Taylor's theorem, equation (2.3) becomes

$$F(a + h) = F(a) + hF'(a) + \frac{1}{2!} h^2F''(a) + \ldots = 0 \tag{2.4}$$

Assuming that the terms involving h^2 and higher powers of h are small enough to be neglected, equation (2.4) reduces to

$$F(a) + hF'(a) \simeq 0 \tag{2.5}$$

that is, h is approximately equal to $- [F(a)/F'(a)]$. Therefore, from equation (2.2) it follows, in general, that

$$x = a - [F(a)/F'(a)] \tag{2.6}$$

is a better approximation than $x = a$.

Equation (2.6) can be used again with each corrected estimate of the root. Now starting with x_0 instead of 'a' as the first approximation, the successive approximations will be

$$x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}$$

then

$$x_2 = x_1 - \frac{F(x_1)}{F'(x_1)}$$

leading to

$$x_{r+1} = x_r - \frac{F(x_r)}{F'(x_r)}$$

The procedure for Newton-Raphson iteration will be more clear from the flow diagram, Fig. 2.1, as described by Watson [10]. In all straight forward cases the successive corrections quickly become smaller and smaller indicating that the sequence x_0, x_1, x_2 . . . converges rapidly towards an accurate value of the root. Cases of failure are discussed in section 2.5.

2.3 GEOMETRIC SIGNIFICANCE OF THE NEWTON-RAPHSON METHOD

Geometrically speaking, the Newton-Raphson method is based on the fact that the tangent at any point of any curve is a close approximation to the curve for a short distance on each side of the point of contact.

A graphical representation of the Newton-Raphson method is shown in Fig. 2.2. The curve PS represents the graph of $y = F(x)$ near the root. Draw a tangent from the point P whose abscissa is x_0. This tangent intersects the x-axis in some point T. Then draw another tangent from P_1 whose abscissa is OT. This tangent meets the x-axis in some point T_1 between T and S. A third tangent can be drawn from P_2 whose abscissa is OT_1, this tangent cutting x-axis at a point T_2 between T_1 and S, and so on. It is evident that if the curvature of the graph does not change sign between P and S the points T, T_1, T_2 . . . will approach the point S as a limit. In other words, the intercepts OT, OT_1, OT_2 . . . will approach the intercept OS as a limit. Since OS represents the real root of the equation, it follows that OT, OT_1, OT_2 . . . are successive

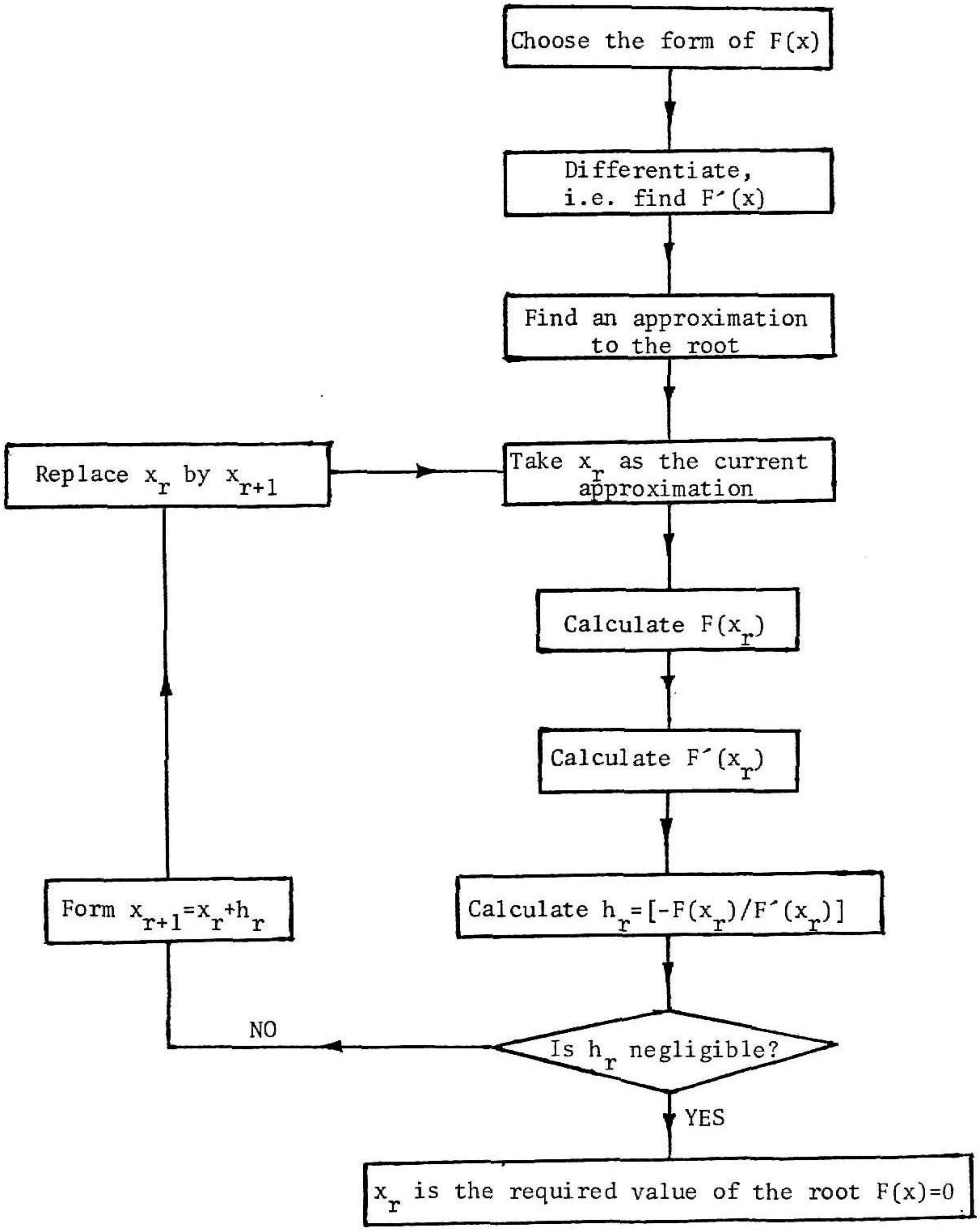


FIG. 2.1 FLOW DIAGRAM FOR NEWTON-RAPHSON METHOD

approximations to the desired root. This is the geometric significance of the Newton-Raphson method.

Geometrically, the fundamental formula for the Newton-Raphson method can be derived as follows. Suppose in Fig. 2.2, $QT = h_1$, $TT_1 = h_2$ . . . . The slope of the graph at P is obviously $F'(x_0)$. Also from Fig. 2.2

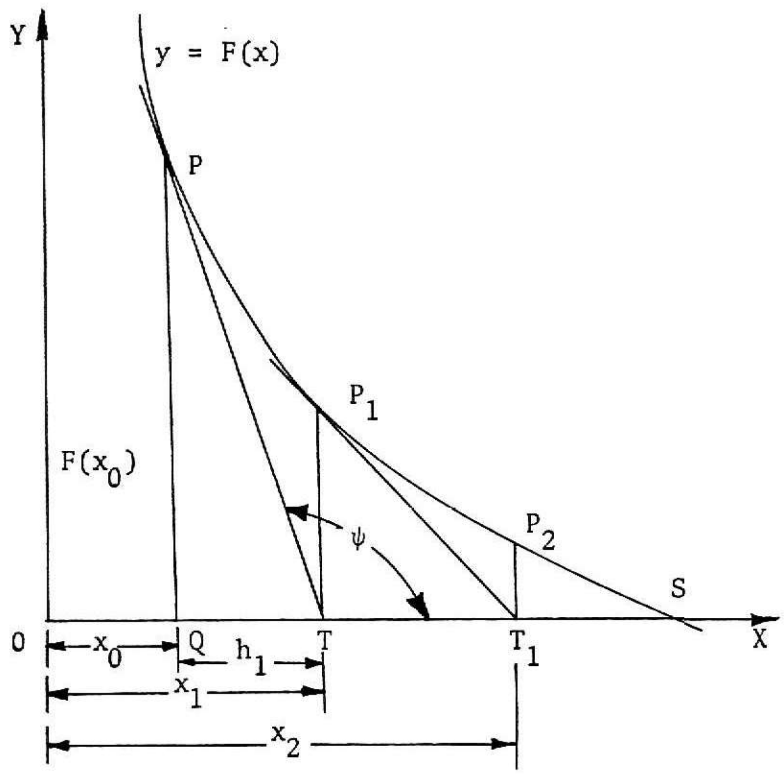


FIG. 2.2 GRAPHICAL REPRESENTATION OF NEWTON-RAPHSON METHOD

$$\text{Slope of graph at P} = \tan \angle \psi$$

$$= - PQ/QT$$

$$= - F(x_0)/h_1$$

Therefore, it follows that

$$F'(x_0) = - F(x_0)/h_1$$

or $\qquad h_1 = - F(x_0)/F'(x_0)$

leading to $h_{r+1} = - F(x_r)/F'(x_r)$

Hence, the real root after the rth iteration will be

$$x_{r+1} = x_r - F(x_r)/F'(x_r)$$

which is the fundamental formula for the Newton-Raphson method.

From the preceding discussion it is evident that in the Newton-Raphson method the graph of the given function is replaced by a tangent at each successive step in the approximation process.

2.4 SPEED OF CONVERGENCE

In the Newton-Raphson method the speed of convergence tends to increase markedly as the error becomes small. This can be seen as follows.

For convenience the basic convergence scheme is rewritten as

$$x_{r+1} = x_r - \frac{F(x_r)}{F'(x_r)} \tag{2.7}$$

If x is the desired root, then the error at step r, denoted by e_r, will be

$$e_r = x - x_r \tag{2.8}$$

and at step r + 1, the error will be

$$e_{r+1} = x - x_{r+1} \tag{2.9}$$

Subtracting x from both of equation (2.7) and then by rearranging, it can be seen that

$$x - x_{r+1} = x - x_r + \frac{F(x_r)}{F'(x_r)} \tag{2.10}$$

Using equations (2.8) and (2.9), equation (2.10) becomes

$$e_{r+1} = e_r + \frac{F(x - e_r)}{F'(x - e_r)} \tag{2.11}$$

Expanding (2.11) by Taylor series,

$$e_{r+1} = e_r + \frac{F(x) - e_r F'(x) + (e_r^{2}/2)\ F''(x) + \ldots\ .}{F'(x) - e_r F''(x) + (e_r^{2}/2)\ F'''(x) + \ldots} \tag{2.12}$$

Making use of the fact that $F(x) = 0$, and dividing the numerator by the denominator, equation (2.12) reduces to

$$e_{r+1} = e_r - e_r - \frac{e_r^{2}}{2} \frac{F''(x)}{F'(x)} + \text{higher-order terms}$$

or
$$e_{r+1} \simeq - \frac{e_r^{2}}{2} \frac{F''(x)}{F'(x)} \tag{2.13}$$

It is obvious from equation (2.13) that the absolute error at step <u>r + 1</u> is proportional to the square of the absolute error at step <u>r</u>. In other words, if an answer is correct to one decimal place at one step, it should be accurate to two decimal places at the next step, four at the next, eight at the next, and so forth. This rapid convergence, where the error at one step is proportional to the square of the previous error, is called "second-order" convergence.

2.5 PITFALLS OF THE NEWTON-RAPHSON METHOD

It was seen in the previous section that the Newton-Raphson method produces fast results when it works. Sometimes, this method does not converge but instead oscillates back and forth. Fig. 2.3 shows a case where the initial guess is such that the iteration oscillates between P and Q and never converges to the desired root x. Another case of oscillation is shown in Fig. 2.4 where the function is symmetrical about the desired root.

This method will also fail when the initial approximation x_0 is such that the value of $F(x_0)/F'(x_0)$ is not small enough, as shown in Fig. 2.5.

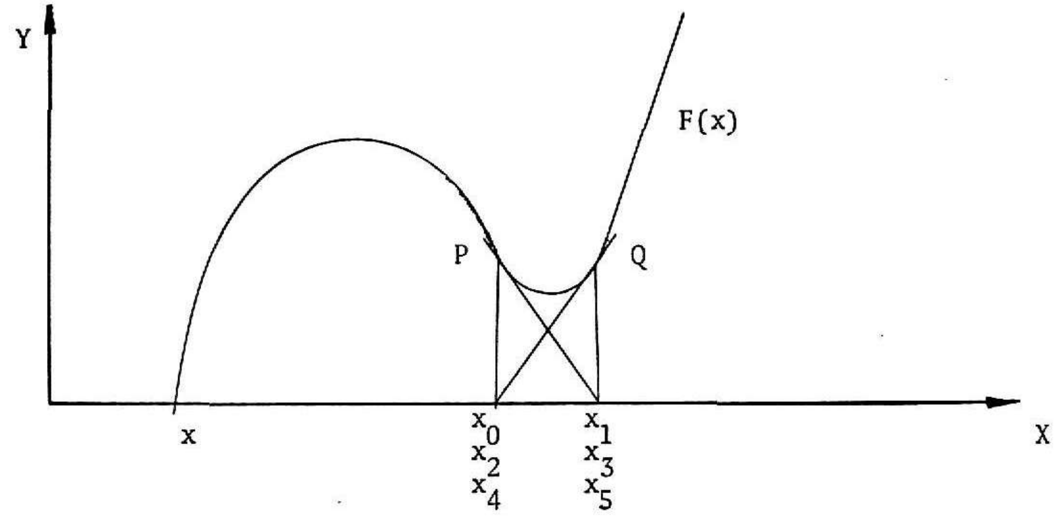


FIG. 2.3 x_0 NOT CLOSE ENOUGH

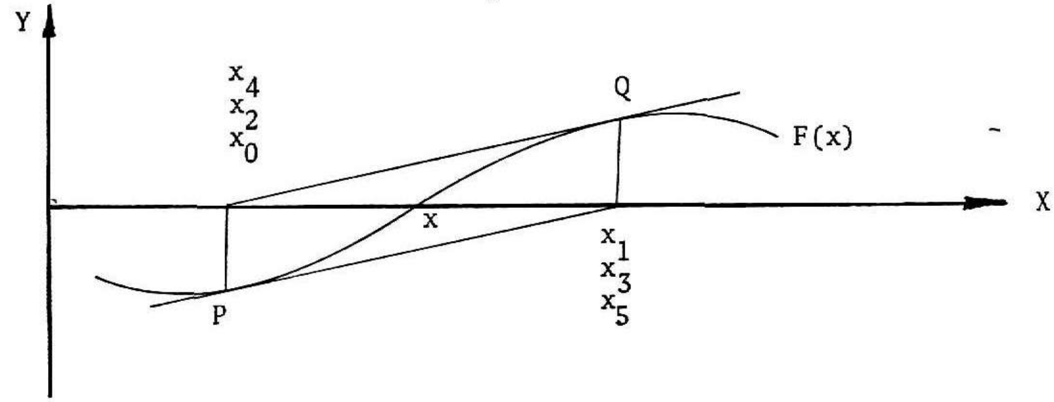


FIG. 2.4 SECOND DERIVATIVE $F''(x) = 0$

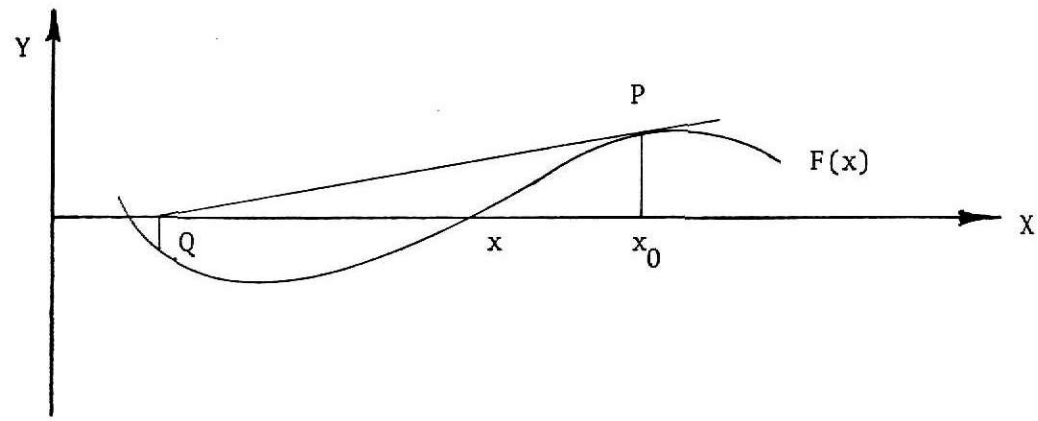


FIG. 2.5 $F(x_0)/F'(x_0)$ TOO LARGE

The slope of the tangent at P (the first approximation) is small and it would cross the x-axis at Q thus making the second estimate of the root worse than the first. In this case the initial guess should be close to the desired root x.

This method may also fail if a pole and a zero of equation (2.1) lie close to each other. In such a case the left side of equation (2.1) will never approach to the value zero and hence the iterations will never converge. So there is a possibility that we may lose the root. In order to avoid such situations, the poles of the equation should be removed.

2.6 COMPLEX ROOTS OF EQUATIONS BY THE NEWTON-RAPHSON METHOD

So far in this chapter it was discussed how to find the real roots of a equation by the Newton-Raphson method. If the initial guess x_0 is real, then

$$x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}$$

will also be real and all the x_r will be real. If, on the other hand, the initial guess is complex, then all x_r will also be complex, and so on. The pitfalls of this method, discussed in section 2.5, for real roots will also hold good in the case of complex roots.

In general, the Newton-Raphson method can find the complex as well as the real roots of complex equations, assuming the iteration process converges.

CHAPTER III

STUDY OF A PRACTICAL EXAMPLE TO DEMONSTRATE THE NEWTON-RAPHSON METHOD

3.1 INTRODUCTION

To demonstrate the usefulness of the Newton-Raphson method, a practical example—the waveguide problem—is discussed in this chapter. The mathematical formulation of the waveguide problem, originally done by Bernotski [2], has been reproduced here. Section 3.2 deals with the mathematical model for the problem developed on certain assumptions. In the next section the integral equation solutions for the vertical electric dipole are found. Their conversion into one of the well-known series solutions—the mode series solution—is presented in section 3.4. The last section discusses the significance of the roots of the transendental mode equation.

3.2 THE MATHEMATICAL MODEL

The mathematical model for the earth-ionosphere waveguide is shown in Fig. 3.1. In this model a semi-infinite ionosphere is assumed to be situated at a distance H from the semi-infinite earth; thus forming a parallel plate waveguide. The point S represents the location of a source (vertical electric dipole) distant Z_o above the earth and R represents the location of the receiver at height Z from earth. The horizontal distance between the source and the receiver is ρ in the cylindrical co-ordinate system.

Because of the interest of near-zone problem, earth has been assumed to be flat. For simplicity a homogeneous ionosphere has been chosen as the upper bound of the waveguide. For the same reason a ground with constant conductivity has been selected although Wait [3] indicates that the ground

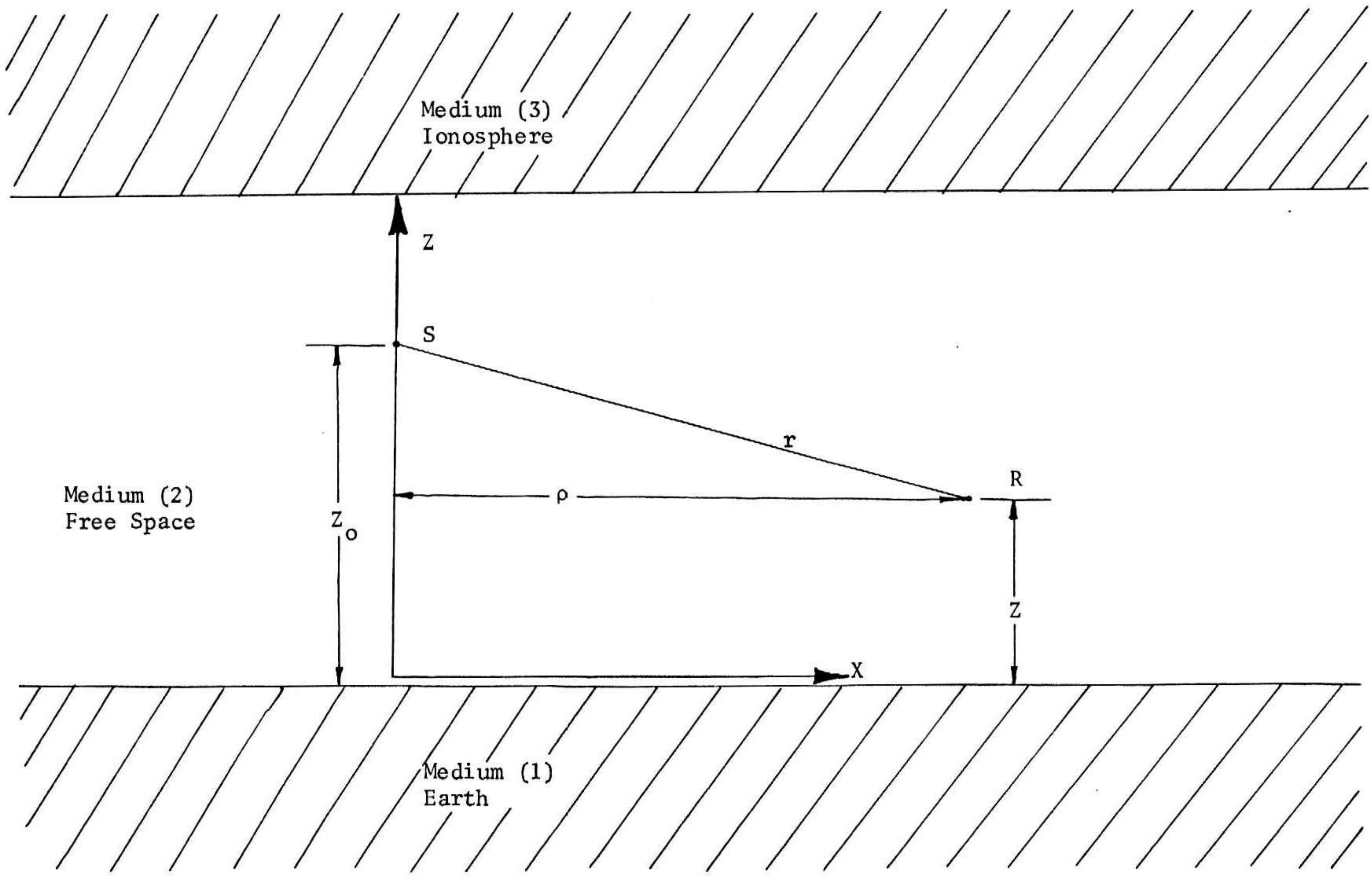


FIG. 3.1 EARTH IONOSPHERE WAVEGUIDE MODEL

conductivity may be a function of frequency.

3.3 INTEGRAL EQUATION SOLUTIONS

The model of section 3.2 is used to derive the integral equation solutions. Let the earth be represented as medium (1), the space between earth and ionosphere as medium (2), and the ionosphere as medium (3). For sinusoidal time variations Maxwell's curl equation in phasor form for any medium n can be written as:

$$\nabla \times \bar{E}_n = -j\omega\varepsilon_n \bar{H}_n \tag{3.1}$$

$$\nabla \times \bar{H}_n = (\sigma_n + j\omega\varepsilon_n)\, \bar{E}_n + \bar{J}_n \tag{3.2}$$

The electric and magnetic fields for any medium, in terms of the electric Hertz vector $\bar{\pi}_n$, can be expressed as:

$$\bar{E}_n = -\gamma_n^2\, \bar{\pi}_n + \nabla(\nabla\cdot\bar{\pi}_n) \tag{3.3}$$

$$\bar{H}_n = (\sigma_n + j\omega\varepsilon_n)\, \nabla \times \bar{\pi}_n \tag{3.4}$$

where $\gamma_n^2 = -\varepsilon_n\mu_n\omega^2 + j\sigma_n\mu_n\omega$

Substituting (3.3) and (3.4) into equations (3.1) and (3.2) and using the vector identity

$$\nabla \times \nabla \times \bar{\pi}_n = \nabla(\nabla\cdot\bar{\pi}_n) - \nabla^2\, \bar{\pi}_n$$

we get:

$$(\nabla^2 - \gamma_n^2)\, \bar{\pi}_n = -\frac{\bar{J}_n}{(\sigma_n + j\omega\varepsilon_n)} \tag{3.5}$$

Since the source—an electric dipole oriented in the Z direction—is present only in the space between earth and ionosphere, equation (3.5) for each medium becomes

$$(\nabla^2 - \gamma_1^{\ 2})\ \pi_{z_1} = (\nabla^2 - \gamma_3^{\ 2})\ \pi_{z_3} = 0 \tag{3.6}$$

$$(\nabla^2 - \gamma_2^{\ 2})\ \pi_{z_2} = -\ \frac{I d\ell}{j\omega\varepsilon_2}\ \frac{\delta(Z - Z_o)\ \delta(\rho)}{2\pi\rho} \tag{3.7}$$

where $Id\ell$ is the dipole moment and δ is the Dirac delta function. It is also assumed here that the conductivity of medium (2) is zero.

Using notations similar to Sommerfeld [4], the solution to equations (3.6) and (3.7) in integral form can be written as:

For medium (1),

$$\pi_{z_1} = \frac{I d\ell}{8\pi(j\omega\varepsilon_2)} \int_{-\infty}^{+\infty} \frac{G_1\ e^{+v_1 Z}}{v_1}\ H_o^{(2)}\ (\lambda\rho)\ \lambda d\lambda \tag{3.8}$$

For medium (2),

$$\pi_{z_2} = \frac{I d\ell}{8\pi(j\omega\varepsilon_2)} \left[\int_{-\infty}^{+\infty} \frac{B_1\ e^{v_2 Z} + B_2\ e^{-v_2 Z}}{v_2}\ H_o^{(2)}\ (\lambda\rho)\ \lambda d\lambda + \frac{2\ e^{-\gamma_2 r}}{r} \right] \tag{3.9}$$

where $\dfrac{I d\ell}{4\pi(j\omega\varepsilon_2)}\ \dfrac{e^{-\gamma_2 r}}{r}$ is the dipole field in free space.

For medium (3),

$$\pi_{z_3} = \frac{I d\ell}{8\pi(j\omega\varepsilon_2)} \int_{-\infty}^{+\infty} \frac{I_1\ e^{-v_3 Z}}{v_3}\ H_o^{(2)}\ (\lambda\rho)\ \lambda d\lambda \tag{3.10}$$

where $v_n = \sqrt{\lambda^2 - k_n^{\ 2}} = \sqrt{\lambda^2 + \gamma_n^{\ 2}}$

$\lambda = k_2 S$

$k_n = j\gamma_n$

$r = \sqrt{\rho^2 + (Z - Z_o)^2}$

and G_1, B_1, B_2 and I_1 are unknown co-efficients and can be determined by boundary conditions. The boundary conditions for the earth and the ionosphere are:

At $Z = 0$ $$E_{1\rho} = E_{2\rho} \tag{3.11a}$$

and $$H_{1\theta} = H_{2\theta} \tag{3.11b}$$

At $Z = H$ $$E_{2\rho} = E_{3\rho} \tag{3.12a}$$

and $$H_{2\theta} = H_{3\theta} \tag{3.12b}$$

Expanding either side of equations (3.3) and (3.4) in cylindrical co-ordinates, it can be seen that

$$E_{\rho n} \propto \frac{\partial^2 \pi_{z_n}}{\partial\rho\,\partial Z} \tag{3.13}$$

and $$H_{\theta n} = (\sigma_n + j\omega\varepsilon_n)\,\frac{\partial \pi_{z_n}}{\partial\rho} \tag{3.14}$$

At the boundaries Z = 0 and Z = H, equations (3.13) and (3.14) yield the following conditions:

$$\left.\frac{\partial \pi_{z_1}}{\partial Z}\right|_{Z=0} = \left.\frac{\partial \pi_{z_2}}{\partial Z}\right|_{Z=0} \tag{3.15}$$

$$(\sigma_1 + j\omega\varepsilon_1)\,\pi_{z_1}\Big|_{Z=0} = j\omega\varepsilon_2\,\pi_{z_2}\Big|_{Z=0} \tag{3.16}$$

$$\left.\frac{\partial \pi_{z_2}}{\partial Z}\right|_{Z=H} = \left.\frac{\partial \pi_{z_3}}{\partial Z}\right|_{Z=H} \tag{3.17}$$

$$(j\omega\varepsilon_2)\,\pi_{z_2}\Big|_{Z=H} = (\sigma_3 + j\omega\varepsilon_3)\,\pi_{z_3}\Big|_{Z=H} \tag{3.18}$$

Now substituting the value of $\frac{e^{-\gamma_2 r}}{r}$ given by Sommerfeld's formula as:

$$\frac{e^{-\gamma_2 r}}{r} = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{e^{-v_2|Z - Z_o|}}{v_2} H_o^{(2)} (\lambda\rho) \lambda d\lambda \tag{3.19}$$

equation (3.9) becomes:

$$\pi_{z_2} = \frac{Id\ell}{8\pi(j\omega\varepsilon_2)} \left[\int_{-\infty}^{+\infty} \left[\frac{e^{-v_2|Z - Z_o|}}{v_2} + \frac{B_1 e^{v_2 Z}}{v_2} + \frac{B_2 e^{-v_2 Z}}{v_2} \right] H_o^{(2)} (\lambda\rho) \lambda d\lambda \right] \tag{3.20}$$

where $0 < Z < H$

Since equations (3.8), (3.20) and (3.10) have the same limits of integration and each integral must converge to represent the physical fields, so taking the partial derivatives inside the integral sign will not affect the convergence. Thus the conditions given by equations (3.15) - (3.18) give rise to the following equations

$$G_1 = e^{-v_2 Z_o} + B_1 - B_2 \tag{3.21}$$

$$\frac{\sigma_1 + j\omega\varepsilon_1}{v_1} G_1 = \frac{j\omega\varepsilon_2}{v_2} (e^{-v_2 Z_o} + B_1 + B_2) \tag{3.22}$$

$$- e^{-v_2(H - Z_o)} + B_1 e^{v_2 H} - B_2 e^{-v_2 H} = -I_1 e^{-v_3 H} \tag{3.23}$$

$$\frac{j\omega\varepsilon_2}{v_2} (e^{-v_2(H - Z_o)} + B_1 e^{v_2 H} + B_2 e^{-v_2 H}) = \frac{\sigma_3 + j\omega\varepsilon_3}{v_3} I_1 e^{-v_3 H} \tag{3.24}$$

Solving the equations (3.21) - (3.24) for B_1 and B_2 we get:

$$B_1 = R_i e^{-2v_2 H} \frac{e^{v_2 Z_o} + R_g e^{-v_2 Z_o}}{1 - R_i R_g e^{-2v_2 H}} \tag{3.25}$$

$$B_2 = R_g \frac{e^{-v_2 Z_o} + R_i\, e^{-v_2(2H - Z_o)}}{1 - R_i R_g\, e^{-2v_2 H}} \tag{3.26}$$

where R_i and R_g are the Fresnel reflection co-efficients given by

$$R_g = \frac{\dfrac{v_2}{j\omega\varepsilon_2} - \dfrac{v_1}{\sigma_1 + j\omega\varepsilon_1}}{\dfrac{v_2}{j\omega\varepsilon_2} + \dfrac{v_1}{\sigma_1 + j\omega\varepsilon_1}} \tag{3.27}$$

$$R_i = \frac{\dfrac{v_2}{j\omega\varepsilon_2} - \dfrac{v_3}{\sigma_3 + j\omega\varepsilon_3}}{\dfrac{v_2}{j\omega\varepsilon_2} + \dfrac{v_3}{\sigma_3 + j\omega\varepsilon_3}} \tag{3.28}$$

Substituting the values of B_1 and B_2 in equation (3.20) and simplifying, equation (3.20) becomes

For $Z < Z_o$

$$\pi_{z_2} = \frac{Id\ell}{8\pi(j\omega\varepsilon_2)} \int_{-\infty}^{+\infty} \frac{H_o^{(2)}(\lambda\rho)}{v_2} \frac{(e^{v_2 Z} + R_g e^{-v_2 Z})(e^{-v_2 Z_o} + R_i e^{v_2(Z_o - 2H)})}{(1 - R_i R_g e^{-2v_2 H})} \lambda d\lambda \tag{3.29}$$

For $Z > Z_o$

$$\pi_{z_2} = \frac{Id\ell}{8\pi(j\omega\varepsilon_2)} \int_{-\infty}^{+\infty} \frac{H_o^{(2)}(\lambda\rho)}{v_2} \frac{(e^{v_2 Z_o} + R_g e^{-v_2 Z_o})(e^{-v_2 Z} + R_i e^{v_2(Z - 2H)})}{(1 - R_i R_g e^{-2v_2 H})} \lambda d\lambda \tag{3.30}$$

The above equations (3.29) and (3.30) are the integral equation solutions. In the next section these solutions are approximated by a series representation known as the mode series solution.

3.4 THE MODE SERIES SOLUTION

The integral solutions given by equations (3.29) and (3.30) can be approximated by a series representing $2\pi j$ times the sum of the residues at the poles of the integrand which lie in the lower half of the S-plane or λ-plane. The location of the poles of the integrand are given by the roots of the transendental equation

$$1 - R_i \; R_g \; e^{-2jk_2CH} = 0 \tag{3.31}$$

where $v_2 = jk_2C = jk_2\sqrt{1 - S^2}$

Using the notations of Wait [3], the mode series for equations (3.29) and (3.30) becomes

$$\pi_{z_2} \simeq \frac{Id\ell}{2\omega\varepsilon_2 H} \sum_{n=-\infty}^{\infty} H_o^{(2)}(k_2\rho \; S_n) \; F_n(Z_o) \; F_n(Z) \; \delta_n(C_n) \tag{3.32}$$

where F_n is the height gain function given by

$$F_n(Z) = \frac{e^{jk_2C_nZ} + R_g \; e^{-jk_2C_nZ}}{2\sqrt{R_g(C_n)}}$$

and $$\delta_n(C_n) = \left[1 + \frac{\left. \left| \dfrac{\partial R_i(C) \; R_g(C)}{\partial C} \right| \right|_{C=C_n}}{2k_2H \; R_i(C_n) \; R_g(C_n)} \right]^{-1}$$

3.5 SIGNIFICANCE OF THE ROOTS OF THE TRANSENDENTAL EQUATION

In order to study the significance of the roots of the transendental equation (3.31) it is essential to find out what the Hankel function of argument $k_2S_n\rho$ describes. Referring to Jordan [5], when the Hankel function of zero order and second kind is appropriately combined with the time factor

$e^{j\omega t}$, then it represents the outward-traveling wave. In other words, it represents the behavior of each mode with distance from the source. That this is so, is evident since this function can be replaced by an exponential function for large values of the argument. For example, for large values of $k_2S_n\rho$ the Hankel function of zero order and second kind can be expressed as

$$H_o^{(2)}(k_2S_n\rho) \quad \sqrt{\frac{2}{\pi k_2S_n\rho}}\, e^{-j(k_2S_n\rho\ -\ \pi/4)}$$

Now it should be observed that the argument k_2S_n can be associated with the propagation constant, $\gamma = \alpha + j\beta$, in the usual rectangular waveguides. Thus we see that the phase velocity and the attenuation of each mode can be predicted from the S-plane plots. The phase velocity is given by the reciprocal of the real part of S and the attenuation per wave length by 2π times the imaginary part of S.

A numerical solution to the transendental equation (3.31) is presented in the next chapter.

CHAPTER IV

NUMERICAL SOLUTION TO THE TRANSENDENTAL EQUATION

4.1 INTRODUCTION

In this chapter a demonstration of finding the roots of a transendental equation by the Newton-Raphson method on the computer is given. The computer program, written in Fortran IV language, is described in section 4.2. Problems encountered in programming are also discussed in this section. This program takes about 3 minutes of computer time to find the roots of the equation for four modes over the frequency range 0.1 kHz - 50 kHz.

Parameters were chosen for two specific cases. The results for these cases are presented in graphical form and their physical interpretations are discussed in the last section of this chapter.

4.2 ROOTS OF THE TRANSENDENTAL EQUATION

It was mentioned in the last section of the previous chapter that in order to approximate the integral solutions by mode series one of the essentialities is to find the roots of the transendental equation (3.31). For the sake of convenience, equation (3.31) and all information about it is repeated here:

$$1 - R_i R_g e^{-2jk_2 CH} = 0 \qquad (4.1)$$

where

$$R_i = \frac{\dfrac{v_2}{\sigma_2 + j\omega\varepsilon_2} - \dfrac{v_3}{\sigma_3 + j\omega\varepsilon_3}}{\dfrac{v_2}{\sigma_2 + j\omega\varepsilon_2} + \dfrac{v_3}{\sigma_3 + j\omega\varepsilon_3}} \qquad (4.2)$$

$$R_g = \frac{\dfrac{v_2}{\sigma_2 + j\omega\varepsilon_2} - \dfrac{v_1}{\sigma_1 + j\omega\varepsilon_1}}{\dfrac{v_2}{\sigma_2 + j\omega\varepsilon_2} + \dfrac{v_1}{\sigma_1 + j\omega\varepsilon_1}} \tag{4.3}$$

$$v_n = j\, k_n \sqrt{1 - \left(\frac{k_2}{k_n}\right)^2 S^2} \tag{4.4}$$

$$k_n = \omega\sqrt{\mu_n \varepsilon_n} \tag{4.5}$$

$$S = \sqrt{1 - C^2} \tag{4.6}$$

$$\omega = 2\pi f \tag{4.7}$$

Looking at equation (4.1) it is quite obvious that this is of the form:

$$F(C) = 0 \tag{4.8}$$

which has already been discussed in Chapter I. So it follows that this can be solved by the Newton-Raphson method. Now it may be recalled that to start with Newton-Raphson method it is essential to find a "good" initial guess. The initial guess must be good because otherwise the iterations procedure may fail as discussed in section 2.5 of Chapter II. In our case a good initial guess which worked quite satisfactorily was found as follows:

Defining the index of refractions N_1 and N_3 for earth and ionosphere respectively as

$$N_1 = \frac{k_1}{k_2} \tag{4.9}$$

$$N_3 = \frac{k_3}{k_2} \tag{4.10}$$

the equations (4.2) and (4.3) become

$$R_g = \frac{N_1^{\,2}C - \sqrt{N_1^{\,2} - 1 + C^2}}{N_1^{\,2}C + \sqrt{N_1^{\,2} - 1 + C^2}} \tag{4.11}$$

$$R_i = \frac{N_3^{\,2}C - \sqrt{N_3^{\,2} - 1 + C^2}}{N_3^{\,2}C + \sqrt{N_3^{\,2} - 1 + C^2}} \tag{4.12}$$

At extremely low frequencies and for reasonable conductivities for the earth and the ionosphere it can be seen from equations (4.9) and (4.10) that

$$N_1^{\,2} \text{ and } N_3^{\,2} >> 1, \text{ since } N_1^{\,2} \text{ and } N_3^{\,2} \propto \frac{1}{f}$$

Assuming $C^2 << N_1^{\,2}$ or $N_3^{\,2}$, equations (4.11) and (4.12) reduce to

$$R_g \simeq \frac{1 - \frac{1}{N_1C}}{1 + \frac{1}{N_1C}}$$

$$\simeq 1 - \frac{2}{N_1C}$$

$$\simeq e^{-\frac{2}{N_1C}}$$

Similarly $R_i \simeq e^{-\frac{2}{N_3C}}$

Substituting the approximate values of R_g and R_i in equation (4.1), it can be seen that

$$e^{\pm 2n\pi j} - e^{-\frac{2}{N_1C}}\, e^{-\frac{2}{N_3C}}\, e^{-2jk_2CH} = 0$$

or
$$\frac{2}{N_1C} + \frac{2}{N_3C} + 2jk_2CH = \pm 2n\pi j \tag{4.13}$$

Solving equation (4.13) for C gives

$$C \simeq \pm \left\{\frac{n\pi}{2k_2H} + \sqrt{\left(\frac{n\pi}{2k_2H}\right)^2 + \frac{j}{k_2H}\left(\frac{1}{N_1} + \frac{1}{N_3}\right)}\right\} \tag{4.14}$$

For n = 0,

$$C = \pm \sqrt{\frac{j}{k_2H}\left(\frac{1}{N_1} + \frac{1}{N_3}\right)} \tag{4.15}$$

Since C^2 is assumed to be much less than N_1 or N_3, this means that the second term under the radical sign in equation (4.14) will be very small. So, for $n \neq 0$, C can be approximated as

$$C \simeq \pm \frac{n\pi}{k_2H} \tag{4.16}$$

Starting with the initial guess as given by equations (4.15) and (4.16), the sequence of successive approximations will follow as given by the following equation

$$C_r + 1 = C_r - \frac{F(C_r)}{F'(C_r)} \tag{4.17}$$

Since the initial approximations for n = 0 and $n \neq 0$ are complex, so all C_r will also be complex, and so on.

4.3 COMPUTER PROGRAMMING

A computer program (Appendix I) was written to find the roots of the transendental equation (4.1) by the Newton-Raphson method. Equations (4.15) and (4.16) were used as initial approximations. The Table 4.1 on the next page gives the list of computer variables used in the computer program for different problem variables.

TABLE 4.1 LIST OF COMPUTER VARIABLES

S. No.	Problem Variable	Computer Variable
1.	Permeability and permittivity of free space; μ_o, ε_o	UO, EO
2.	Relative permeability and relative permittivity of medium; μ_r, ε_r	UR, ER
3.	Permeability and permittivity of medium; μ, ε	U, E
4.	Conductivity of medium, σ	G
5.	Ionosphere height, H	H
6.	Complex j	J
7.	Value of π	PHI
8.	Radian frequency, ω	W
9.	Mode number, n	MN
10.	Frequency in K.Hz	FK
11.	Numerators of R_i and R_g	NRI, NRG
12.	Denominators of R_i and R_g	DRI, DRG
13.	Exponential term, e^{-2jk_2CH}	EX
14.	Function of C, F(C)	F
15.	Derivatives of NRI, NRG, DRI, DRG, EX and F	DNRI, DNRG, DDRI, DDRG, DEX, DERF
16.	Ratio $F(C)/F'(C)$	DELTA
17.	Absolute value of DELTA	AV

The transendental equation (4.1) was not taken as F(C) but it was simplified first. It is quite obvious that the poles of this equation are the zeros of the denominators of R_i and R_g.

In order to avoid the pitfall due to poles, discussed in section 2.5 of Chapter II, the poles of the transendental equation were removed by multiplying both sides of it by the denominators of R_i and R_g. Thus, in terms of computer variables, F(C) takes the form as

$$F = DRI * DRG - NRI * NRG * EX \tag{4.18}$$

and its derivative will be

$$\begin{aligned} DERF = {} & DDRI * DRG + DRI * DDRG - DNRI * NRG * EX - NRI * DNRG * EX \\ & - NRI * NRG * DEX \end{aligned} \tag{4.19}$$

Then, the ratio $F(C)/F'(C)$ will be given by

$$DELTA = \frac{F}{DERF} \tag{4.20}$$

Since a transendental equation has infinite set of roots, so it is very essential to find a logical numbering system to order the roots. A system found by Bernotski [2] was adopted in which the roots are traced as the frequency varies in steps over a certain range for each mode. A flow chart describing the operations in the computer program is shown in Fig. 4.1 on the next page.

Before considering the problem of convergence or lack of convergence, let us point out the situation in which errors in the data or the program could cause the machine to run in the loop indefinitely, thus wasting time and money. In order to save such situations a counter, named N, has been established in the program which allows the iteration to be performed a maximum of 25 times and then skips to statement 70, which prints out "DOES NOT CONVERGE." If the answer had been obtained in less than 25 iterations, it

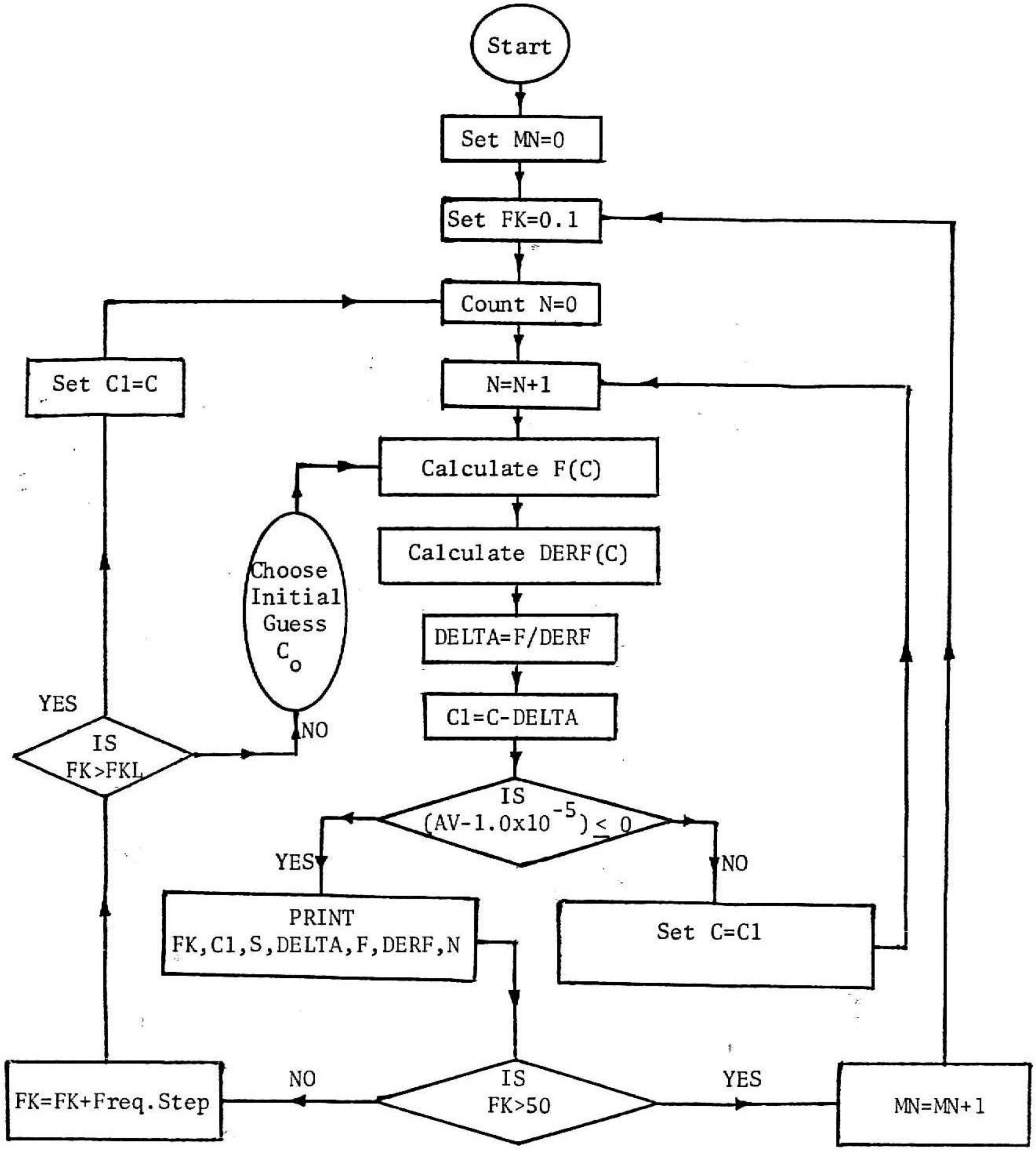


FIG. 4.1 FLOW CHART OF COMPUTER PROGRAM

would be printed out in the normal fashion.

Once DELTA has been calculated, the obvious next step is to test the convergence. For this we ask the computer whether the absolute value of DELTA is sufficiently close to zero i.e. less than, say m, depending upon the accuracy required. In our case the value of m is 10^{-5}. This is generally a valid test, since the difference between two successive iteration must go to zero if the process is to converge. If the convergence does not occur to the desired accuracy, the iteration process continues. If the limit of iterations has not been reached but the process has converged to desired accuracy, the computer prints out the values of C, S, DELTA, F, DERF and N in a tabular form as shown in Appendix II. Next the computer takes a frequency step and assumes the previous value of C as the initial guess and goes through the same process again as discussed above.

Stepping the frequency was found to be the most troublesome problem. Frequency step should not be too large because the root at the last frequency may not be a good initial guess at the new frequency. In the range of 0.1 kHz to FKL, which is defined as

$$FKL = 2 * MN$$

the frequency step was kept as 0.1 kHz and at each step the initial guess was calculated from the equations (4.14) and (4.15) since in this range the roots for a particular mode were found to be so far apart from each other that the previous root did not prove to be a good initial guess at the new frequency. During the period FKL to 20 kHz, the step size was kept 0.2 kHz. From 20 kHz to 50 kHz, the frequency step was boosted to 2.0 kHz because it was found that the roots for each mode lie very close to each other at higher frequencies.

4.4 RESULTS AND DISCUSSION

Two cases have been studied, namely, (a) Day Over Poor Land (b) Day Over Normal Land. In each case the height of the ionosphere was 70 kilometers. Other parameters used for each media are listed in Tables 4.2 and 4.3.

TABLE 4.2 DAY OVER POOR LAND

Medium	Relative Permeability	Relative Permittivity	Conductivity in mhos
Earth	1.000	15.000	0.236×10^{-3}
Free space	1.000	1.000	0.000
Ionosphere	1.000	1.000	0.118×10^{-5}

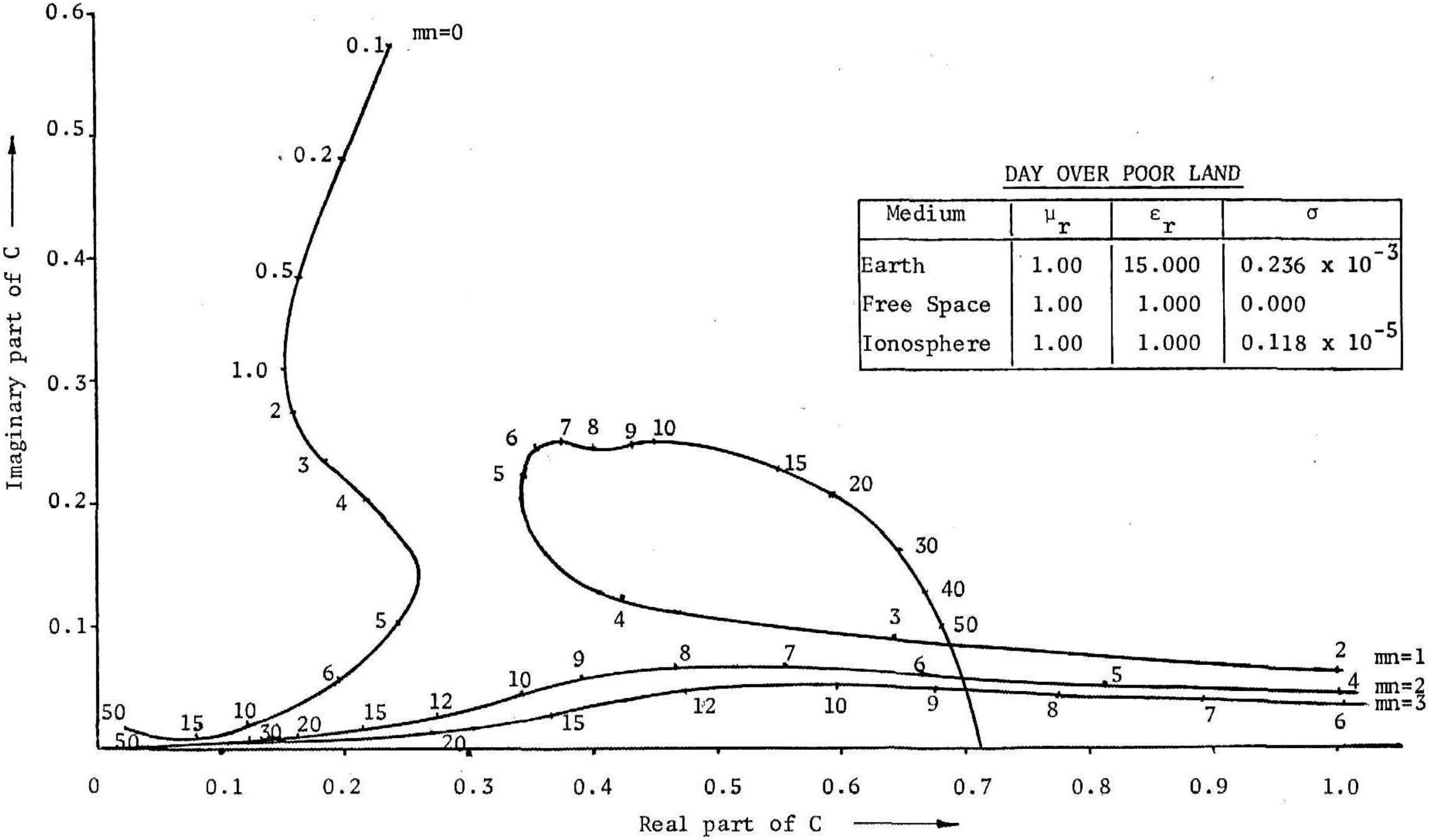
TABLE 4.3 DAY OVER NORMAL LAND

Medium	Relative Permeability	Relative Permittivity	Conductivity in mhos
Earth	1.000	15.000	0.500×10^{-2}
Free space	1.000	1.000	0.000
Ionosphere	1.000	1.000	0.177×10^{-5}

The roots for the first four modes were plotted in the complex planes, namely, C-plane and S-plane. These plots are shown in Figs. 4.2 - 4.5 for each case.

Recalling section 3.5 of the previous chapter, the phase velocity and the attenuation of each mode can be interpreted from the S-plane plots. The Figs. 4.4 and 4.5 show that the frequencies less than 3 kHz for $n = 0$ mode have phase velocities less than the speed of light since the value of S is greater than one. But the phase velocities of the higher frequencies of this

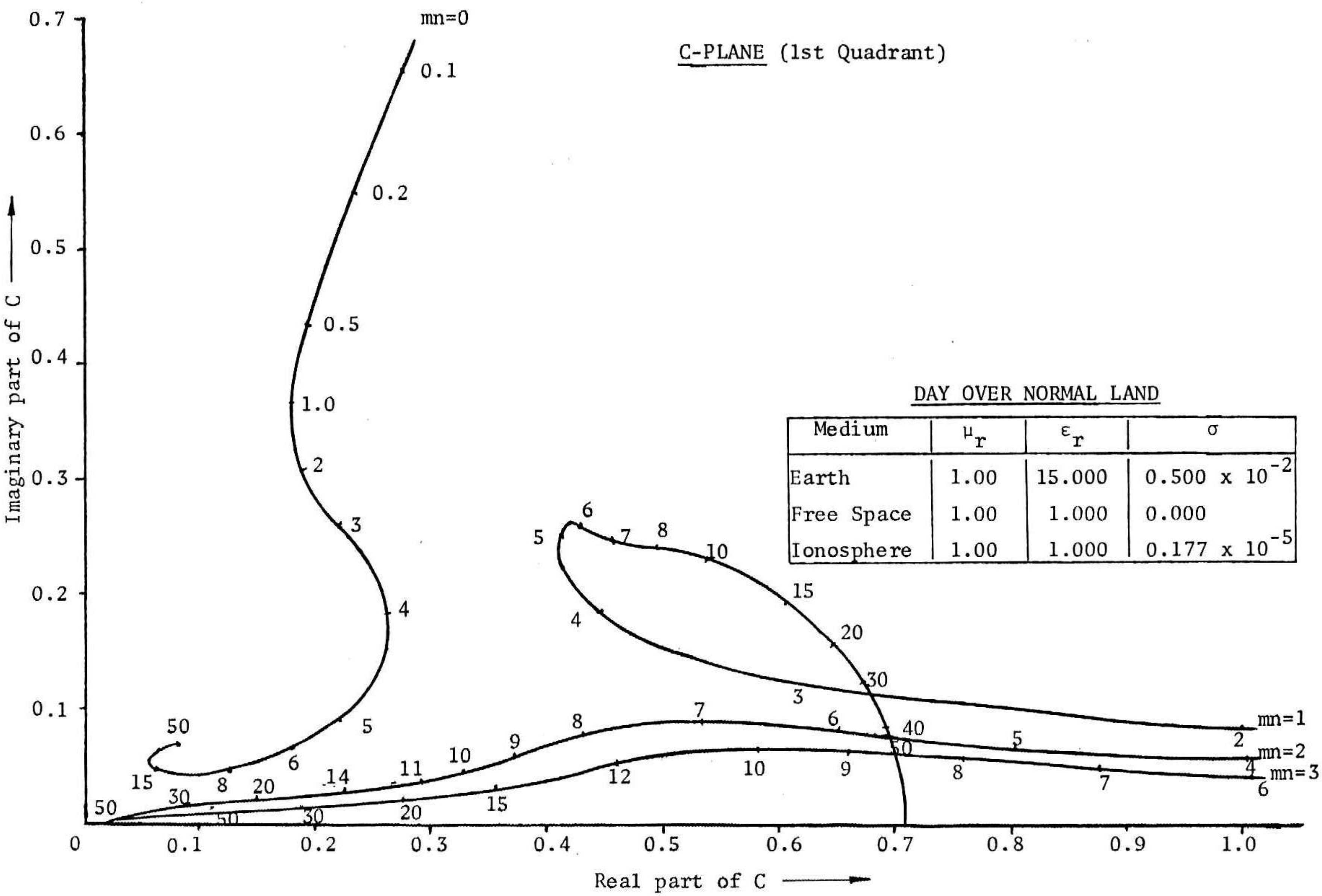
C-PLANE (1st Quadrant)



DAY OVER POOR LAND

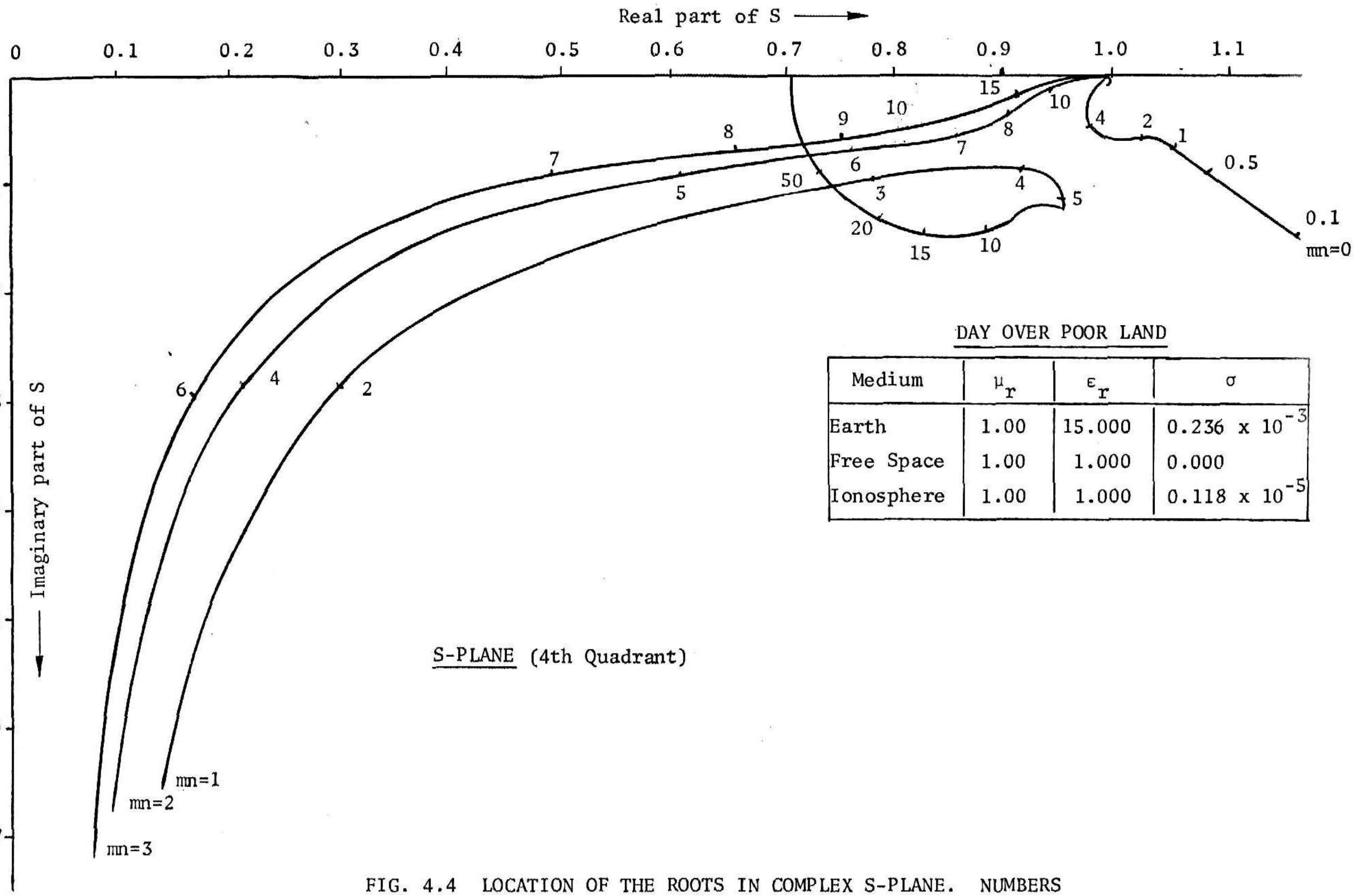
Medium	μ_r	ε_r	σ
Earth	1.00	15.000	0.236×10^{-3}
Free Space	1.00	1.000	0.000
Ionosphere	1.00	1.000	0.118×10^{-5}

FIG. 4.2 LOCATION OF THE ROOTS IN THE COMPLEX C-PLANE. NUMBERS ALONG EACH CURVE INDICATE f IN kHz



Medium	μ_r	ε_r	σ
Earth	1.00	15.000	0.500×10^{-2}
Free Space	1.00	1.000	0.000
Ionosphere	1.00	1.000	0.177×10^{-5}

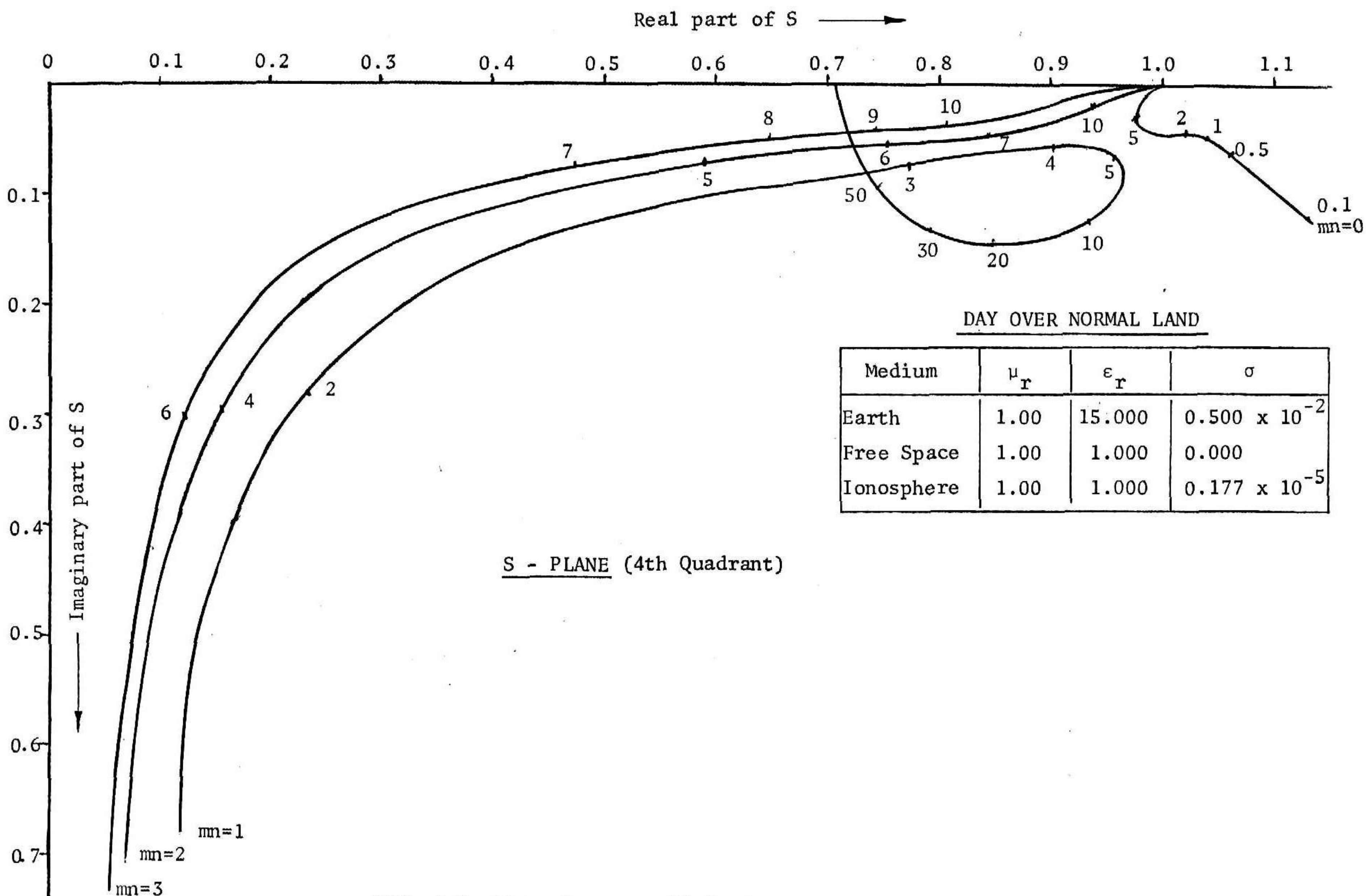
FIG. 4.3 LOCATION OF THE ROOTS IN THE COMPLEX C-PLANE. NUMBERS ALONG EACH CURVE INDICATE f IN kHz



DAY OVER POOR LAND

Medium	μ_r	ε_r	σ
Earth	1.00	15.000	0.236×10^{-3}
Free Space	1.00	1.000	0.000
Ionosphere	1.00	1.000	0.118×10^{-5}

FIG. 4.4 LOCATION OF THE ROOTS IN COMPLEX S-PLANE. NUMBERS ALONG EACH CURVE INDICATE f IN kHz



DAY OVER NORMAL LAND

Medium	μ_r	ε_r	σ
Earth	1.00	15.000	0.500×10^{-2}
Free Space	1.00	1.000	0.000
Ionosphere	1.00	1.000	0.177×10^{-5}

FIG. 4.5 LOCATION OF ROOTS IN COMPLEX S-PLANE. NUMBERS ALONG EACH CURVE INDICATE f IN kHz

mode are about the same as the speed of light. For other modes the real part of S is always less than one and hence their phase velocities are faster than the speed of light.

At low frequencies, the imaginary part of S is quite large for each mode, except for the $n = 0$ mode. This shows that the lower frequencies are highly attenuated.

CHAPTER V

CONCLUSIONS

The Newton-Raphson method was found to be an efficient method for finding the roots of a transendental equation on the computer. Prior to the advent of computers, the solutions to transendental equations have been considered relatively inaccessible, and obtainable only by means of a considerable effort. But, as can be seen from the sample output (Appendix II), the convergence is so fast that the computer takes less than 3 minutes to find the roots for four modes over the frequency range 0.1 kHz - 50 kHz. The only drawback of this method is that to start with we must have a good initial guess.

The computer program is quite general for the waveguide problem—general in the sense that it can be used for predicting the velocity and attenuation of electromagnetic pulses for any case such as Day land, Night land, Sea land etc., just by changing the data cards. The ionospheric height (in meters) is specified on the first data card and the next three cards carry the values of UR, ER and G in order for each media according to the format statements 2 and 3 respectively. This program can be used in conjunction with a computer subroutine for the Hankel Functions to calculate the sum of the modes.

A simple model was used in the study of the waveguide problem. The study can be extended to more complicated model where layered ionospheres may be used to make it a real world problem.

The Newton-Raphson method can be applied to many complicated problems in electrical engineering where it is required to find the roots of an equation in which one parameter is varied over a certain range. One such application could be in the field of Network Synthesis where it may be necessary to find the stability of a network when a parameter is changed over a certain range. Stability of a network can be checked from the location of the poles and zeros of the network function in the complex-plane.

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APPENDIX I

Computer Program

```
$JOB            SCS,TIME=10,PAGES=25
C       FINDING COMPLEX ROOTS OF COMPLEX FUNCTIONS BY
C       NEWTON-RAPHSON METHOD
C       EXAMPLE. SOLVING FOR ROOTS OF 1-RI(C).RG(C).EXP(C)=0
        COMPLEX CMPLX,CEXP,CSQRT
        COMPLEX C,J,NRI,NRG,DRI,DRG,EX,F,DERF,E(3),K(3),
       1V(3),VE(3),DVE(3),DNRI,DNRG,DDRI,DDRG,DEX,DELTA,
       1S,C1,CM(3),NJ(3),NI(3)
        DIMENSION UR(3),ER(3),G(3),U(3),REL(3)
        REAL IMG(3)
        READ(1,2)H
      2 FORMAT(F10.0)
        WRITE(3,200)H
    200 FORMAT(/,55X,' H=',F10.3,' METERS')
        WRITE(3,4)
      4 FORMAT(/,48X,'UR',10X,'ER',12X,'G')
        DO 10 I=1,3
        READ(1,3)UR(I),ER(I),G(I)
      3 FORMAT(3F10.0)
        WRITE(3,5)UR(I),ER(I),G(I)
      5 FORMAT(44X,F7.3,5X,F7.3,5X,E10.3)
     10 CONTINUE
        J=CMPLX(0.0,1.0)
        PHI=3.14159
        U0=PHI*4.0 E-07
        E0=8.854 E-12
C       BEGIN MODE NUMBER LOOP
        DO 110 L=1,4
        MN=L-1
        FKL=2.0*MN
        WRITE(3,300)MN
    300 FORMAT(/,'    MODE NO.=',I3)
        WRITE(3,400)
    400 FORMAT(///,3X,'FK',12X,'C1',20X,'S',19X,'DELTA'
       1,22X,'F',24X,'DERF',16X,'N')
        FK=0.1
C       BEGIN FREQUENCY LOOP
     90 W=2.0*PHI*FK*1000.0
        DO 20 I=1,3
        U(I)=U0*UR(I)
        REL(I)=E0*ER(I)
        IMG(I)=-G(I)/W
        E(I)=CMPLX(REL(I),IMG(I))
        K(I)=W*CSQRT(U(I)*E(I))
     20 CONTINUE
        DO 120 I=1,3
        NJ(I)=(K(2)/K(I))
        NI(I)=(NJ(I))**2
    120 CONTINUE
C       CHOOSE INITIAL GUESS AT EACH FREQ. FOR EACH MODE
        IF(MN.EQ.0)GO TO 140
        IF(FK.GT.FKL)GO TO 130
        C=(MN*PHI)/(K(2)*H)
        GO TO 130
    140 IF(FK.GT.1.0)GO TO 130
        C=CSQRT(J*(NJ(1)+NJ(3))/(K(2)*H))
    130 N=0
C       BEGIN ITERATION LOOP
     30 S=CSQRT(1.0-C**2)
        N=N+1
```

```
      IF((N-25).EQ.0)GO TO 70
      DO 60 I=1,3
C     BEGIN DEFINITION OF F AND DERF
      CM(I)=CSQRT(1.0-NI(I)*(S**2))
      V(I)=J*K(I)*CM(I)
      VE(I)=V(I)*E0/E(I)
      DVE(I)=J*K(I)*(1.0/CM(I))*NI(I)*C*E0/E(I)
   60 CONTINUE
      NRI=VE(2)-VE(3)
      NRG=VE(2)-VE(1)
      DRI=VE(2)+VE(3)
      DRG=VE(2)+VE(1)
      EX=CEXP(-2.0*V(2)*H)
      DNRI=DVE(2)-DVE(3)
      DNRG=DVE(2)-DVE(1)
      DDRI=DVE(2)+DVE(3)
      DDRG=DVE(2)+DVE(1)
      DEX=(-2.0*J*H*EX*K(2))
      F=DRI*DRG-NRI*NRG*EX
      DERF=DDRI*DRG+DRI*DDRG-DNRI*NRG*EX-NRI*DNRG*EX-NRI*NRG*DEX
C     END DEFINITION OF F AND DERF
      DELTA=F/DERF
C     ITERATION STEP
      C1=C-DELTA
C     TESTING CONVERGENCE
      AV=CABS(DELTA)
      IF(AV-1.0 E-05)50,50,40
   40 C=C1
      GO TO 30
C     END ITERATION LOOP
   70 WRITE(3,500)
  500 FORMAT(' DOES NOT CONVERGE')
   50 WRITE(3,600)FK,C1,S,DELTA,F,DERF,N
  600 FORMAT(F5.1,05X,F7.3,'+J',F7.3,05X,F7.3,'+J',F7.3,04X,E9.2
     1,'+J',E9.2,04X,E10.3,'+J',E10.3,005X,E10.3,'+J',E10.3,005X
     1,I3)
      IF(MN.EQ.0)GO TO 150
      IF(FK.GE.FKL)GO TO 160
      FK=FK+0.10001
      GO TO 170
  150 IF(FK.GT.0.8)GO TO 160
      FK=FK+0.10001
      GO TO 170
  160 C1=C
      IF(FK.GE.19.99)GO TO 180
      FK=FK+0.2
      GO TO 170
  180 FK=FK+2.0
  170 IF(FK-52.0)90,90,110
C     END FREQUENCY LOOP
  110 CONTINUE
      STOP
C     END MODE NUMBER LOOP
      END
```

APPENDIX II

Sample of Computer Output

H= 70000.000 METERS

UR	ER	G
1.000	15.000	0.236E-03
1.000	1.000	0.000E 00
1.000	1.000	0.118E-05

MODE NO.= 0

FK	C1	S	DELTA	F	DERF	N
0.1	0.273+J 0.653	1.173+J -0.152	-0.14E-06+J-0.65E-05	0.607E-17+J-0.694E-17	0.105E-11+J 0.959E-12	2
0.2	0.233+J 0.549	1.123+J -0.114	0.65E-07+J 0.77E-07	0.000E 00+J 0.867E-18	0.655E-11+J 0.558E-11	3
0.3	0.213+J 0.497	1.100+J -0.096	-0.65E-07+J 0.14E-06	-0.347E-17+J 0.173E-17	0.197E-10+J 0.157E-10	3
0.4	0.201+J 0.462	1.086+J -0.086	-0.97E-06+J 0.17E-05	-0.971E-16+J 0.416E-16	0.437E-10+J 0.327E-10	3
0.5	0.194+J 0.437	1.077+J -0.079	-0.94E-06+J 0.83E-06	-0.125E-15+J 0.139E-16	0.822E-10+J 0.576E-10	3
0.6	0.188+J 0.418	1.070+J -0.074	-0.21E-05+J 0.17E-05	-0.444E-15+J 0.555E-16	0.139E-09+J 0.911E-10	3
0.7	0.184+J 0.402	1.064+J -0.070	-0.44E-05+J 0.16E-05	-0.118E-14+J-0.236E-15	0.219E-09+J 0.134E-09	3
0.8	0.182+J 0.389	1.060+J -0.067	-0.93E-05+J 0.29E-05	-0.357E-14+J-0.763E-15	0.327E-09+J 0.185E-09	3
1.0	0.179+J 0.368	1.052+J -0.063	-0.34E-07+J-0.27E-07	-0.139E-16+J-0.278E-16	0.647E-09+J 0.308E-09	4
1.2	0.178+J 0.352	1.047+J -0.060	0.20E-06+J-0.46E-06	0.430E-15+J-0.444E-15	0.115E-08+J 0.444E-09	4
1.4	0.180+J 0.338	1.042+J -0.058	-0.52E-05+J 0.25E-05	-0.111E-13+J 0.179E-14	0.188E-08+J 0.557E-09	3
1.6	0.182+J 0.327	1.038+J -0.057	-0.63E-05+J 0.29E-05	-0.197E-13+J 0.466E-14	0.288E-08+J 0.567E-09	3
1.8	0.185+J 0.316	1.034+J -0.057	-0.44E-05+J-0.11E-05	-0.180E-13+J-0.644E-14	0.421E-08+J 0.443E-09	3
2.0	0.190+J 0.307	1.030+J -0.056	-0.33E-05+J-0.39E-05	-0.195E-13+J-0.226E-13	0.585E-08+J 0.202E-11	3
2.2	0.194+J 0.297	1.026+J -0.056	-0.35E-05+J-0.46E-05	-0.314E-13+J-0.326E-13	0.778E-08+J-0.884E-09	3
2.4	0.200+J 0.288	1.023+J -0.056	-0.46E-05+J-0.45E-05	-0.556E-13+J-0.333E-13	0.987E-08+J-0.237E-08	3
2.6	0.206+J 0.279	1.019+J -0.056	-0.42E-05+J-0.77E-05	-0.856E-13+J-0.728E-13	0.119E-07+J-0.460E-08	3
2.8	0.213+J 0.269	1.015+J -0.057	-0.50E-06+J 0.57E-06	-0.246E-14+J 0.115E-13	0.136E-07+J-0.761E-08	4
3.0	0.220+J 0.259	1.011+J -0.057	0.18E-07+J 0.21E-06	0.266E-14+J 0.289E-14	0.144E-07+J-0.113E-07	4
3.2	0.228+J 0.249	1.006+J -0.056	0.12E-05+J-0.14E-05	-0.333E-14+J-0.380E-13	0.141E-07+J-0.152E-07	4
3.4	0.237+J 0.236	1.002+J -0.056	-0.12E-05+J 0.99E-06	0.333E-14+J 0.355E-13	0.124E-07+J-0.188E-07	4
3.6	0.245+J 0.222	0.996+J -0.055	-0.16E-06+J-0.45E-07	-0.244E-14+J 0.289E-14	0.947E-08+J-0.210E-07	4
3.8	0.254+J 0.206	0.990+J -0.053	-0.13E-06+J-0.22E-06	-0.533E-14+J 0.133E-14	0.633E-08+J-0.210E-07	4
4.0	0.261+J 0.184	0.984+J -0.049	-0.64E-06+J-0.25E-07	-0.355E-14+J 0.120E-13	0.481E-08+J-0.189E-07	4
4.2	0.262+J 0.159	0.979+J -0.043	0.52E-06+J 0.39E-06	0.977E-14+J-0.622E-14	0.616E-08+J-0.167E-07	4
4.4	0.255+J 0.135	0.977+J -0.035	-0.61E-07+J-0.63E-07	-0.155E-14+J 0.444E-15	0.875E-08+J-0.163E-07	4
4.6	0.244+J 0.117	0.977+J -0.029	-0.33E-06+J 0.43E-07	-0.289E-14+J 0.600E-14	0.111E-07+J-0.170E-07	4
4.8	0.233+J 0.104	0.978+J -0.025	-0.11E-06+J 0.90E-07	0.222E-15+J 0.311E-14	0.132E-07+J-0.181E-07	4
5.0	0.222+J 0.093	0.980+J -0.021	-0.10E-06+J 0.13E-07	-0.133E-14+J 0.222E-14	0.152E-07+J-0.193E-07	4
5.2	0.211+J 0.085	0.981+J -0.018	-0.11E-06+J 0.19E-07	-0.155E-14+J 0.266E-14	0.171E-07+J-0.206E-07	4
5.4	0.202+J 0.078	0.983+J -0.016	-0.67E-07+J 0.39E-07	-0.444E-15+J 0.222E-14	0.191E-07+J-0.220E-07	4
5.6	0.193+J 0.072	0.984+J -0.014	0.14E-06+J-0.14E-06	-0.444E-15+J-0.622E-14	0.212E-07+J-0.234E-07	4
5.8	0.184+J 0.068	0.985+J -0.013	0.20E-06+J-0.88E-05	-0.213E-12+J-0.211E-12	0.234E-07+J-0.249E-07	3
6.0	0.177+J 0.064	0.986+J -0.011	-0.48E-06+J-0.71E-05	-0.200E-12+J-0.171E-12	0.258E-07+J-0.262E-07	3
6.2	0.170+J 0.061	0.987+J -0.010	-0.11E-05+J-0.56E-05	-0.185E-12+J-0.127E-12	0.282E-07+J-0.277E-07	3
6.4	0.163+J 0.058	0.988+J -0.010	-0.11E-05+J-0.45E-05	-0.167E-12+J-0.107E-12	0.309E-07+J-0.292E-07	3
6.6	0.157+J 0.055	0.989+J -0.009	-0.15E-05+J-0.35E-05	-0.157E-12+J-0.744E-13	0.337E-07+J-0.306E-07	3
6.8	0.151+J 0.053	0.990+J -0.008	-0.16E-05+J-0.28E-05	-0.150E-12+J-0.495E-13	0.367E-07+J-0.321E-07	3
7.0	0.146+J 0.052	0.991+J -0.008	-0.16E-05+J-0.22E-05	-0.139E-12+J-0.346E-13	0.399E-07+J-0.336E-07	3
7.2	0.140+J 0.050	0.991+J -0.007	-0.18E-05+J-0.17E-05	-0.136E-12+J-0.104E-13	0.433E-07+J-0.351E-07	3
7.4	0.136+J 0.049	0.992+J -0.007	-0.14E-05+J-0.14E-05	-0.117E-12+J-0.135E-13	0.470E-07+J-0.366E-07	3
7.6	0.131+J 0.048	0.993+J -0.006	-0.10E-05+J-0.12E-05	-0.975E-13+J-0.204E-13	0.509E-07+J-0.381E-07	3
7.8	0.127+J 0.047	0.993+J -0.006	-0.10E-05+J-0.98E-06	-0.946E-13+J-0.140E-13	0.551E-07+J-0.395E-07	3
8.0	0.123+J 0.046	0.993+J -0.006	-0.86E-06+J-0.73E-06	-0.810E-13+J-0.866E-14	0.596E-07+J-0.409E-07	3
8.2	0.119+J 0.045	0.994+J -0.005	-0.87E-06+J-0.60E-06	-0.810E-13+J-0.155E-14	0.643E-07+J-0.423E-07	3
8.4	0.116+J 0.045	0.994+J -0.005	-0.10E-05+J-0.30E-06	-0.855E-13+J 0.242E-13	0.694E-07+J-0.436E-07	3

8.6	0.112+J 0.044	0.995+J -0.005	-0.72E-06+J-0.26E-06	-0.655E-13+J 0.127E-13	0.749E-07+J-0.449E-07	3
8.8	0.109+J 0.044	0.995+J -0.005	-0.62E-06+J-0.16E-06	-0.577E-13+J 0.158E-13	0.807E-07+J-0.461E-07	3
9.0	0.106+J 0.044	0.995+J -0.005	-0.42E-06+J-0.24E-06	-0.475E-13+J-0.111E-14	0.869E-07+J-0.473E-07	3
9.2	0.103+J 0.043	0.996+J -0.004	-0.49E-06+J-0.88E-07	-0.502E-13+J 0.155E-13	0.935E-07+J-0.484E-07	3
9.4	0.100+J 0.043	0.996+J -0.004	-0.67E-06+J 0.14E-06	-0.600E-13+J 0.473E-13	0.101E-06+J-0.494E-07	3
9.6	0.098+J 0.043	0.996+J -0.004	-0.37E-06+J 0.51E-07	-0.377E-13+J 0.242E-13	0.108E-06+J-0.503E-07	3
9.8	0.095+J 0.043	0.996+J -0.004	-0.57E-06+J 0.17E-06	-0.582E-13+J 0.491E-13	0.116E-06+J-0.511E-07	3
10.0	0.093+J 0.043	0.997+J -0.004	-0.77E-07+J-0.41E-07	-0.118E-13+J-0.111E-14	0.125E-06+J-0.519E-07	3
10.2	0.091+J 0.043	0.997+J -0.004	-0.42E-06+J 0.15E-06	-0.486E-13+J 0.426E-13	0.134E-06+J-0.525E-07	3
10.4	0.089+J 0.043	0.997+J -0.004	-0.73E-07+J 0.22E-07	-0.933E-14+J 0.711E-14	0.144E-06+J-0.530E-07	3
10.6	0.087+J 0.043	0.997+J -0.004	0.93E-07+J-0.83E-07	0.999E-14+J-0.178E-13	0.155E-06+J-0.534E-07	3
10.8	0.085+J 0.043	0.997+J -0.004	0.29E-06+J-0.14E-06	0.411E-13+J-0.391E-13	0.166E-06+J-0.536E-07	3
11.0	0.083+J 0.043	0.997+J -0.004	-0.29E-06+J 0.19E-06	-0.406E-13+J 0.497E-13	0.178E-06+J-0.537E-07	3
11.2	0.081+J 0.044	0.998+J -0.004	0.27E-06+J-0.13E-06	0.457E-13+J-0.391E-13	0.191E-06+J-0.537E-07	3
11.4	0.080+J 0.044	0.998+J -0.004	0.28E-07+J 0.25E-07	0.711E-14+J 0.355E-14	0.205E-06+J-0.536E-07	3
11.6	0.078+J 0.044	0.998+J -0.003	-0.42E-06+J 0.25E-06	-0.782E-13+J 0.782E-13	0.220E-06+J-0.533E-07	3
11.8	0.077+J 0.044	0.998+J -0.003	-0.12E-06+J 0.14E-06	-0.213E-13+J 0.391E-13	0.236E-06+J-0.529E-07	3
12.0	0.076+J 0.045	0.998+J -0.003	0.13E-06+J-0.29E-07	0.320E-13+J-0.142E-13	0.253E-06+J-0.524E-07	3
12.2	0.074+J 0.045	0.998+J -0.003	-0.18E-06+J 0.18E-06	-0.391E-13+J 0.568E-13	0.271E-06+J-0.517E-07	3
12.4	0.073+J 0.045	0.998+J -0.003	-0.22E-06+J 0.16E-06	-0.568E-13+J 0.568E-13	0.291E-06+J-0.510E-07	3
12.6	0.072+J 0.045	0.998+J -0.003	-0.43E-07+J-0.18E-07	-0.142E-13+J-0.355E-14	0.311E-06+J-0.502E-07	3
12.8	0.071+J 0.046	0.999+J -0.003	0.14E-06+J-0.65E-07	0.426E-13+J-0.284E-13	0.334E-06+J-0.493E-07	3
13.0	0.070+J 0.046	0.999+J -0.003	0.14E-06+J-0.61E-07	0.462E-13+J-0.284E-13	0.358E-06+J-0.483E-07	3
13.2	0.069+J 0.046	0.999+J -0.003	0.82E-07+J-0.64E-07	0.284E-13+J-0.284E-13	0.383E-06+J-0.474E-07	3
13.4	0.068+J 0.046	0.999+J -0.003	-0.15E-06+J 0.61E-07	-0.568E-13+J 0.320E-13	0.410E-06+J-0.465E-07	3
13.6	0.067+J 0.047	0.999+J -0.003	-0.97E-07+J 0.79E-07	-0.391E-13+J 0.391E-13	0.440E-06+J-0.457E-07	3
13.8	0.066+J 0.047	0.999+J -0.003	-0.39E-06+J 0.27E-06	-0.171E-12+J 0.146E-12	0.471E-06+J-0.449E-07	3
14.0	0.065+J 0.047	0.999+J -0.003	-0.38E-06+J 0.28E-06	-0.181E-12+J 0.156E-12	0.504E-06+J-0.444E-07	3
14.2	0.064+J 0.047	0.999+J -0.003	-0.11E-06+J 0.50E-07	-0.568E-13+J 0.320E-13	0.540E-06+J-0.442E-07	3
14.4	0.064+J 0.048	0.999+J -0.003	0.12E-06+J-0.28E-07	0.675E-13+J-0.213E-13	0.578E-06+J-0.443E-07	3
14.6	0.063+J 0.048	0.999+J -0.003	0.20E-06+J-0.18E-06	0.114E-12+J-0.117E-12	0.618E-06+J-0.448E-07	3
14.8	0.062+J 0.048	0.999+J -0.003	-0.37E-06+J 0.31E-06	-0.231E-12+J 0.220E-12	0.662E-06+J-0.458E-07	3
15.0	0.062+J 0.048	0.999+J -0.003	0.23E-06+J-0.21E-06	0.153E-12+J-0.156E-12	0.708E-06+J-0.476E-07	3
15.2	0.061+J 0.049	0.999+J -0.003	-0.78E-07+J 0.37E-07	-0.568E-13+J 0.320E-13	0.757E-06+J-0.501E-07	3
15.4	0.061+J 0.049	0.999+J -0.003	-0.19E-06+J 0.19E-06	-0.142E-12+J 0.160E-12	0.809E-06+J-0.535E-07	3
15.6	0.060+J 0.049	0.999+J -0.003	-0.16E-06+J 0.96E-07	-0.131E-12+J 0.924E-13	0.865E-06+J-0.581E-07	3
15.8	0.060+J 0.049	0.999+J -0.003	-0.36E-06+J 0.33E-06	-0.316E-12+J 0.327E-12	0.924E-06+J-0.639E-07	3
16.0	0.059+J 0.050	0.999+J -0.003	-0.11E-06+J 0.10E-06	-0.103E-12+J 0.107E-12	0.987E-06+J-0.713E-07	3
16.2	0.059+J 0.050	1.000+J -0.003	-0.63E-05+J-0.72E-05	-0.716E-11+J-0.704E-11	0.105E-05+J-0.799E-07	2
16.4	0.058+J 0.050	1.000+J -0.003	-0.64E-05+J-0.57E-05	-0.770E-11+J-0.582E-11	0.112E-05+J-0.909E-07	2
16.6	0.058+J 0.050	1.000+J -0.003	-0.65E-05+J-0.45E-05	-0.821E-11+J-0.477E-11	0.120E-05+J-0.104E-06	2
16.8	0.058+J 0.050	1.000+J -0.003	-0.55E-05+J-0.42E-05	-0.752E-11+J-0.475E-11	0.128E-05+J-0.120E-06	2
17.0	0.057+J 0.051	1.000+J -0.003	-0.47E-05+J-0.41E-05	-0.693E-11+J-0.493E-11	0.136E-05+J-0.139E-06	2
17.2	0.057+J 0.051	1.000+J -0.003	-0.51E-05+J-0.28E-05	-0.781E-11+J-0.330E-11	0.145E-05+J-0.161E-06	2
17.4	0.057+J 0.051	1.000+J -0.003	-0.43E-05+J-0.26E-05	-0.718E-11+J-0.325E-11	0.154E-05+J-0.186E-06	2
17.6	0.056+J 0.051	1.000+J -0.003	-0.39E-05+J-0.23E-05	-0.690E-11+J-0.299E-11	0.164E-05+J-0.216E-06	2
17.8	0.056+J 0.051	1.000+J -0.003	-0.36E-05+J-0.19E-05	-0.678E-11+J-0.233E-11	0.174E-05+J-0.250E-06	2
18.0	0.056+J 0.052	1.000+J -0.003	-0.37E-05+J-0.12E-05	-0.712E-11+J-0.118E-11	0.185E-05+J-0.289E-06	2
18.2	0.056+J 0.052	1.000+J -0.003	-0.27E-05+J-0.14E-05	-0.585E-11+J-0.181E-11	0.197E-05+J-0.333E-06	2
18.4	0.056+J 0.052	1.000+J -0.003	-0.28E-05+J-0.88E-06	-0.620E-11+J-0.764E-12	0.209E-05+J-0.384E-06	2
18.6	0.055+J 0.052	1.000+J -0.003	-0.27E-05+J-0.46E-06	-0.613E-11+J 0.171E-12	0.221E-05+J-0.441E-06	2
18.8	0.055+J 0.052	1.000+J -0.003	-0.21E-05+J-0.54E-06	-0.507E-11+J-0.217E-12	0.234E-05+J-0.505E-06	2
19.0	0.055+J 0.052	1.000+J -0.003	-0.22E-05+J-0.59E-07	-0.536E-11+J 0.110E-11	0.247E-05+J-0.577E-06	2
19.2	0.055+J 0.052	1.000+J -0.003	-0.20E-05+J 0.13E-06	-0.522E-11+J 0.168E-11	0.261E-05+J-0.658E-06	2
19.4	0.055+J 0.053	1.000+J -0.003	-0.16E-05+J 0.63E-07	-0.429E-11+J 0.135E-11	0.276E-05+J-0.748E-06	2
19.6	0.055+J 0.053	1.000+J -0.003	-0.10E-05+J-0.17E-06	-0.307E-11+J 0.345E-12	0.291E-05+J-0.848E-06	2
19.8	0.055+J 0.053	1.000+J -0.003	-0.14E-05+J 0.52E-06	-0.392E-11+J 0.297E-11	0.306E-05+J-0.959E-06	2
20.0	0.055+J 0.053	1.000+J -0.003	-0.92E-06+J 0.23E-06	-0.272E-11+J 0.175E-11	0.322E-05+J-0.108E-05	2
22.0	0.054+J 0.054	1.000+J -0.003	-0.26E-06+J 0.16E-06	-0.771E-12+J 0.163E-11	0.492E-05+J-0.316E-05	3
24.0	0.055+J 0.054	1.000+J -0.003	0.93E-09+J-0.25E-06	-0.183E-11+J-0.149E-11	0.602E-05+J-0.745E-05	3

26.0	C.056+J 0.055	1.000+J -0.003	-0.17E-05+J 0.15E-05	0.147E-10+J 0.301E-10	0.414E-05+J-0.144E-04	3
23.0	0.059+J 0.055	1.000+J -0.C03	-0.23E-05+J 0.91E-05	0.214E-09+J 0.730E-11	-0.488E-05+J-0.223E-04	3
30.0	0.061+J 0.056	1.C00+J -0.C03	-0.57E-07+J 0.19E-06	0.390E-11+J-0.130E-11	-0.254E-04+J-0.237E-04	4
32.0	C.064+J C.057	1.000+J -0.004	-0.68E-06+J 0.29E-06	0.390E-10+J-0.147E-10	-0.559E-04+J-0.245E-05	4
34.0	0.067+J C.058	0.999+J -0.C04	-0.11E-05+J-0.20E-05	0.209E-09+J 0.841E-10	-0.759E-04+J 0.641E-04	4
36.0	0.069+J C.060	0.999+J -0.C04	0.72E-06+J-0.16E-05	0.277E-09+J 0.179E-09	-0.288E-04+J 0.186E-03	4
38.0	C.071+J 0.062	0.999+J -0.004	0.13E-05+J-0.80E-06	0.478E-09+J 0.246E-09	0.182E-03+J 0.313E-03	4
40.0	C.C73+J 0.063	0.999+J -0.C05	0.75E-06+J-0.67E-06	0.638E-09+J-0.242E-09	0.632E-03+J 0.243E-03	4
42.0	0.C75+J 0.064	0.999+J -0.C05	0.58E-06+J-0.84E-06	0.369E-09+J-0.122E-08	0.119E-02+J-0.392E-03	4
44.0	C.077+J 0.065	0.999+J -0.005	0.11E-06+J-0.64E-06	-0.113E-08+J-0.977E-09	0.119E-02+J-0.196E-02	4
46.0	C.079+J 0.067	0.999+J -0.005	-0.27E-06+J-0.60E-06	-0.262E-08+J 0.165E-08	-0.773E-03+J-0.414E-02	4
48.0	C.081+J C.068	0.999+J -0.C05	-0.53E-06+J-0.84E-06	-0.649E-09+J 0.778E-08	-0.624E-02+J-0.457E-02	4
50.0	C.083+J 0.069	0.999+J -0.C06	-0.52E-06+J-0.12E-05	0.988E-08+J 0.164E-07	-0.142E-01+J 0.200E-02	4

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COMPUTER-PROGRAM FOR FINDING
THE COMPLEX ROOTS OF A TRANSENDENTAL EQUATION
BY
THE NEWTON-RAPHSON METHOD

by

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ABSTRACT

The usefulness of the Newton-Raphson method to find the complex roots of a complex function is presented in this report. This method was found to be very efficient and rapid. It is demonstrated by a earth-ionosphere waveguide problem studied by Bernotski [2]. In the mathematical study of this problem, it is required to find the roots of a transendental equation as given below:

$$1 - R_i R_g e^{-2jk_2CH} = 0$$

where R_i and R_g are functions of C and f. A computer program is written in Fortran IV language to find the roots by the Newton-Raphson method. The computer program is quite general for this problem and it can be used to find the roots over the frequency range 0.1 kHz to 50 kHz for first four modes for any type of earth surface and ionosphere.