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MATRIX GAME THEORY

by

LI-CHING TINA KUNG

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Ray A. Waller
Major Professor

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1. INTRODUCTION

1.1 Terminology

The mathematical theory of games of strategy deals with situations involving two or more participants with conflicting interests. The outcome of such games is usually controlled partly by one side and partly by the opposing side or sides; it depends to some extent on chance (not controlled by man power), but primarily on the intelligence and skill employed by the participants. Most people are familiar with games like poker, bridge, and chess where there are many conflicting situations and where chance as well as skill is involved. The theory of games is also appliable in certain areas of economics, operation research, politics, and military science.

A game of strategy is described by its set of rules. These rules specify that each participant (either one person or more) is called a "player". The rules define the amount of information, if any, each player receives. If the game requires the use of chance devices, the rules describe how the chance events shall be interpreted. They also define the terms for playing such as when the game ends, the amount each player pays or receives (game payoff), and the objective of each player.

We use the word "move" to mean a point in a game at which one of the player (or chance, in some cases) picks out an alternative from some set of alternatives, and use the word "choice" to express the alternative picked out. For example, "John won by a clever choice in his fifth move".

From the rules one can obtain such general properties of the game as the number of moves, the number of players, and the payoff. The game is "finite" if each player has a finite number of moves and a finite number of choices available at each move. Other games are called "infinite". We distinguish a game according to the number of players, i.e. as one-person games, two-person games, and so on.

An important and fundamental concept in game theory is that of a "strategy". In the actual play of a game, instead of making his decision at each move each player may formulate in advance of the play a plan for playing the game from beginning to end. Such a plan must be complete and cover all possible contingencies that may arise in the play. Such a complete prescription for the play of a game by the player is called a "strategy" of that player. A player using a strategy does not lose any freedom of action since the strategy specifies the player's actions in terms of the information that might become available.

Consider an n-person game with players $P_1, P_2, \ldots, P_n$ and let π_i (for $i=1,2,\ldots,n$) be the payoff made to P_i at the end of the play. Then if

$$\sum_{i=1}^{n} \pi_i = 0$$

the game is called an n-person zero-sum game. Otherwise it is called an n-person non-zero-sum game.

This report, however, considers two-person, zero-sum, finite game only since most parlor games and many military games are of this type. Sometimes, they are also called "rectangular games" or "matrix games" because the set of payoffs may be displayed as a rectangular matrix. The following example, which is taken from Owen [10], is shown in a game tree, then we put the payoffs as a rectangular matrix.

Example 1.1.1 A game is played by giving each of two players an entire suit of cards (thirteen cards). A third suit is shuffled, and the cards of this third suit are then turned up, one by one. Each time one has been turned up, each player turns up one of his cards at will: the one who turns up the larger card "wins" the third card. If both turn up a card of the same denomination, neither wins. This continues until the three suits are exhausted. At this point, each player totals the number of spots on the cards he has "won"; the "score" (i.e. payoff) is the difference between what the two players have.

Since the game tree is too large to contain thirteen card suits, we will give part of the tree of an analogous game using three-card suits which is shown in Fig. 1.1.1.

There is a single chance move, the shuffle, which orders the cards in one of the six possible ways, each having a probability of 1/6. After this the moves correspond to the two players, I and II, including the initial point, several branches are similar to those we have already drawn.

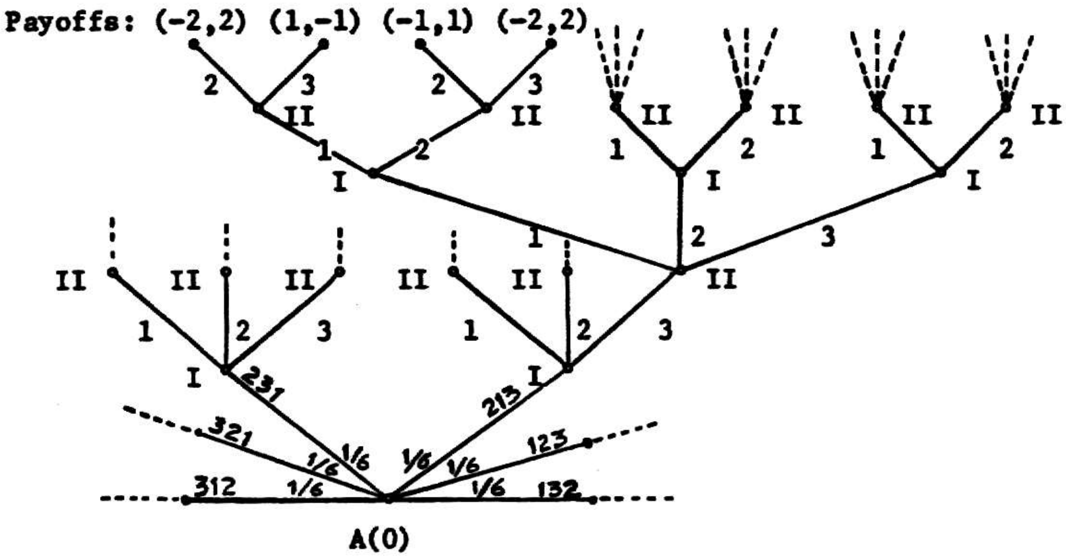

Figure 1.1.1

The payoff (-2,2) shown in Fig. 1.1.1 means that player I lost 2 to player II and player II won 2 from player I. Accordingly the sum of the payoffs of the two players is zero (i.e. $-2 + 2 = 0$), this is, obviously, a two-person zero-sum game. Next, let us put the payoffs in a rectangular matrix. We will give a matrix which only contains the payoffs shown in Fig. 1.1.1. That is, when the third card suit is ordered in 213, and player I used the strategy 312 and 321, player II used the strategy 123 and 132, the game is determined by player I's payoff matrix as follows:

$$\begin{array}{cc} & \text{Player II's strategies} \\ & \begin{array}{cc} 123 & \qquad 132 \end{array} \\ \text{Player I's strategies} \quad \begin{array}{c} 312 \\ 321 \end{array} & \begin{bmatrix} -2 & \qquad 1 \\ -1 & \qquad -2 \end{bmatrix} . \end{array}$$

The whole payoff matrix for Fig. 1.1.1 is a 6×36 matrix.

1.2 Historical development

The theory of games of strategy was first proposed in 1921 by a French mathematician, Emile Borel. The first successful analysis and the accompanying proofs were offered by John von Neumann in 1928. In 1944 his significant work in the field of game theory appeared, The Theory of Games and Economic Behavior. It was authored by von Neumann [16] and a collaborating economist, Oskar Morganstern. The real significance of this book was that it represented one of those rare occasions in scientific publication where a new field was rather thoroughly explored by the first major work to be published in that field. In this sense von Neumann and Morganstern published their work about the same time that linear programming appeared on the scene. It was then recognized that game-theory problems could be formulated as special cases of linear programming; the elements of the simplex method of linear programming as proposed by George Dantzig (see Koopmans [4], pp. 330, 339, 359) were later used to prove the minimax theorem in game theory, and to provide solutions to games of large size. Since that time a significant library of books and articles on the subject of game theory has appeared in scientific literature. A representative sampling of these works appears in the references at the end of this report. Besides, the n-person zero-sum games, n-person non-zero-sum games, and infinite games can be found in Burger [1], Karlin [3], Luce [6], Maschler [7], Owen [10], Rapoport [11], Tucker [13], and von Neumann [16].

2. MATRIX GAMES

2.1 Definition of matrix games

A matrix game Γ is played by two players I and II, usually denoted as P_1 and P_2, respectively. Suppose P_1 has m strategies, which may be denoted by the numbers

$$\alpha = 1, 2, \ldots, m.$$

Suppose P_2 has n strategies, which we may designate by

$$\beta = 1, 2, \ldots, n.$$

The two players begin play by choosing their own strategy. Neither has prior knowledge of the other's choice. There is no cheating or collusion. Both reveal their selections simultaneously. If P_1 chose strategy α and P_2 chose strategy β, then the pair of strategies, (α,β), determines a play of the game and a payoff to the two players. Let $\pi_1(\alpha,\beta)$ be the payoff to P_1 and let $\pi_2(\alpha,\beta)$ be the payoff to P_2. Since the game is zero-sum, we have

$$\pi_1(\alpha,\beta) + \pi_2(\alpha,\beta) = 0.$$

But we prefer to express this by writing

$$\pi_1(\alpha,\beta) = \pi(\alpha,\beta)$$

and

$$\pi_2(\alpha,\beta) = -\pi(\alpha,\beta).$$

The game Γ is thus described by P_1's payoff matrix,

$$R = \begin{bmatrix} \pi(1,1) & \pi(1,2) & \cdots & \pi(1,n) \\ \pi(2,1) & \pi(2,2) & \cdots & \pi(2,n) \\ \vdots & & & \vdots \\ \pi(m,1) & \pi(m,2) & \cdots & \pi(m,n) \end{bmatrix}.$$

In this matrix each row represents a strategy for P_1 and each column represents a strategy for P_2. If P_1 chooses the strategy α (or row α) and P_2 chooses the strategy β (or column β), then P_2 should pay P_1 the amount $\pi(\alpha,\beta)$. P_1 wants $\pi(\alpha,\beta)$ to be as large as possible, but he controls only the choice of his strategy α. P_2 wants $\pi(\alpha,\beta)$ to be as small as possible but he controls only the choice of his strategy β. In terms of this payoff to P_1, we may refer to P_1 as the maximizing player and P_2 as the minimizing player.

2.2 Relations among expectations

From Dresher [2] we can get the following relations. For any strategy α which P_1 may choose, he can be sure of getting at least

$$\min_{\beta \leq n} \pi(\alpha,\beta)$$

where the minimum is taken over all of P_2's strategies. P_1 is at liberty to choose α; therefore, he can make his choice in such a way as to insure that he gets at least

$$\max_{\alpha \leq m} \min_{\beta \leq n} \pi(\alpha,\beta).$$

Similarly, for any strategy β which P_2 may choose, he can be sure of getting at least

$$\min_{\alpha \leq m} (-\pi(\alpha,\beta)) = -\max_{\alpha \leq m} \pi(\alpha,\beta).$$

That is, for any strategy β which P_2 may choose, he can be sure that P_1 gets no more than

$$\max_{\alpha \leq m} \pi(\alpha,\beta).$$

Since P_2 is at liberty to choose β, he can choose it in such a way that P_1 will get at most

$$\min_{\beta \leq n} \max_{\alpha \leq m} \pi(\alpha,\beta).$$

Therefore, there exists a way for P_1 to play so that P_1 gets at least

$$\max_{\alpha \leq m} \min_{\beta \leq n} \pi(\alpha,\beta)$$

and there exists a way for P_2 to play so that P_1 gets no more than

$$\min_{\beta \leq n} \max_{\alpha \leq m} \pi(\alpha,\beta).$$

In general, those two quantities are different, but they satisfy the dominance relation contained in the following theorem as presented by Dresher [2].

Theorem 2.2.1 Let A and B be two sets, let f be a function of two variables such that $f(x,y)$ is a real number whenever $x \varepsilon A$ and $y \varepsilon B$, and suppose that

$$\max_{x \varepsilon A} \min_{y \varepsilon B} f(x,y)$$

and

$$\min_{y \varepsilon B} \max_{x \varepsilon A} f(x,y)$$

both exist. Then

$$\max_{x \varepsilon A} \min_{y \varepsilon B} f(x,y) \leq \min_{y \varepsilon B} \max_{x \varepsilon A} f(x,y). \tag{2.2.1}$$

Proof: For any fixed x and y, we have, by definition of a minimum,

$$\min_{y \varepsilon B} f(x,y) \leq f(x,y)$$

and, by the definition of a maximum,

$$f(x,y) \leq \max_{x \varepsilon A} f(x,y),$$

hence

$$\min_{y \in B} f(x,y) \leq \max_{x \in A} f(x,y). \tag{2.2.2}$$

Since the left-hand side of (2.2.2) is independent of y, we have, by taking the minimum of both sides,

$$\min_{y \in B} f(x,y) \leq \min_{y \in B} \max_{x \in A} f(x,y). \tag{2.2.3}$$

Since the right-hand side of (2.2.3) is independent of x and by taking the maximum of both sides, we have

$$\max_{x \in A} \min_{y \in B} f(x,y) \leq \min_{y \in B} \max_{x \in A} f(x,y)$$

which completes the proof.

The application of the above result to matrices rests on the fact that a matrix, $R = [\pi(\alpha,\beta)]$, where $\alpha = 1, \ldots, m$ and $\beta = 1, \ldots, n$, can be regarded as a real-valued function f of two variables, such that $f(x,y)$, for $x = 1, \ldots, m$ and $y = 1, \ldots, n$, is defined by the equation

$$f(x,y) = \pi(\alpha,\beta).$$

Corollary 2.2.2 Let $R = [\pi(\alpha,\beta)]$ be an arbitrary $m \times n$ payoff matrix of a game Γ. Then

$$\max_{\alpha} \min_{\beta} \pi(\alpha,\beta) \leq \min_{\beta} \max_{\alpha} \pi(\alpha,\beta).$$

Proof: This follows from Theorem 2.2.1, by taking A to be the set of the first m positive integers and B to be the set of the first n positive integers.

Example 2.2.1 Suppose the payoff matrix of a game is given by

$$
\begin{array}{l c c}
 & & \text{Row min.} \\
R = & \begin{bmatrix} 3 & 5 & -2 & 2 & 1 \\ 3 & 6 & -1 & 2 & 4 \\ 4 & 3 & 6 & 7 & 8 \end{bmatrix} & \begin{array}{c} -2 \\ -1 \\ 3^* \end{array} . \\
\text{Col. max.} & \begin{array}{ccccc} 4^* & 6 & 6 & 7 & 8 \end{array} &
\end{array}
$$

Then

$$\max_{\alpha} \min_{\beta} \pi(\alpha,\beta) = 3$$

and

$$\min_{\beta} \max_{\alpha} \pi(\alpha,\beta) = 4.$$

In this game, P_1 can receive at least 3. P_1 can guarantee this amount by playing his third strategy. The most P_2 needs to pay or the most that P_1 can get is 4. P_2 can assure this upper bound by playing his first strategy.

2.3 Games with a pure strategy

In this section we introduce a special case for playing two-person zero-sum games with only two choices to each player. These are denoted as 2×2 games. When a player plays one row all of the time (or one column all of the time in the case of player II), he is said to be playing a "pure strategy". When one of the players elects to play a pure strategy, the other player will always logically counter with a pure strategy himself. Let us use some examples to explain it.

Example 2.3.1 Suppose there is a game Γ with payoff matrix

$$R = \begin{bmatrix} -2 & 1 \\ 3 & 5 \end{bmatrix}.$$

Here P_1 would play his second row all the time, since to do so guarantees that his opponent cannot win. An intelligent opponent P_2 will obviously see that his best response in this case is to play his first column, thereby minimizing his losses (3 points per play loss instead of 5). So in this game P_1 and P_2 are playing pure strategies.

The same reasoning is used when the game is intentionally biased against P_1. Let us see another example.

Example 2.3.2 Suppose there is a payoff matrix

$$R = \begin{bmatrix} -2 & 5 \\ -4 & -2 \end{bmatrix}.$$

In this case P_2 will choose the first column on each play, since this strategy guarantees that he cannot lose. P_1 must counter on each play by choosing his first row, thereby limiting his losses per play to 2 points instead of 4. So P_1 and P_2 are playing the pure strategies row 1 and column 1, respectively.

We see then that in a 2×2 game when one of the players elects to play a pure strategy, this automatically insures that his opponent will counter with a pure strategy (if his opponent wants to behave rationally), since one of the two choices open to his opponent will always be preferable to the other, unless of course they have identical values.

2.4 Saddle points

Saddle points are defined by Mckinsey [9] as follows.

Definition 2.4.1 Suppose f is a real-valued function such that $f(x,y)$ is defined whenever $x \varepsilon A$ and $y \varepsilon B$; then a point (x^*,y^*), where $x^* \varepsilon A$ and $y^* \varepsilon B$ is called a "saddle point" of f if (x^*,y^*) satisfies the following two conditions:

$$(1) \quad f(x,y^*) \leq f(x^*,y^*) \qquad \text{for all } x \varepsilon A,$$

$$(2) \quad f(x^*,y^*) \leq f(x^*,y) \qquad \text{for all } y \varepsilon B.$$

The following theorem by Mckinsey [9] establishes the necessary and sufficient conditions for a game to have a saddle point.

Theorem 2.4.2 Let f be a real-valued function such that $f(x,y)$ is defined whenever $x\varepsilon A$ and $y\varepsilon B$. Moreover, suppose that

$$\max_{x\varepsilon A}\min_{y\varepsilon B} f(x,y)$$

and

$$\min_{y\varepsilon B}\max_{x\varepsilon A} f(x,y)$$

both exist. Then a necessary and sufficient condition for

$$\max_{x\varepsilon A}\min_{y\varepsilon B} f(x,y) = \min_{y\varepsilon B}\max_{x\varepsilon A} f(x,y)$$

is that f has a saddle point (x^*,y^*).

Proof: To see the sufficient condition first, suppose that (x^*,y^*) is a saddle point of f. Then, by definition, we have, for all $x\varepsilon A$ and $y\varepsilon B$,

$$f(x,y^*) \leq f(x^*,y^*), \qquad (2.4.1)$$

$$f(x^*,y^*) \leq f(x^*,y). \qquad (2.4.2)$$

From (2.4.1) we have

$$\max_{x\varepsilon A} f(x,y^*) \leq f(x^*,y^*) \qquad (2.4.3)$$

and from (2.4.2) we have

$$f(x^*,y^*) \leq \min_{y\varepsilon B} f(x^*,y). \qquad (2.4.4)$$

From (2.4.3) and (2.4.4), we have

$$\max_{x\varepsilon A} f(x,y^*) \leq f(x^*,y^*) \leq \min_{y\varepsilon B} f(x^*,y). \qquad (2.4.5)$$

Since

$$\min_{y\varepsilon B}\max_{x\varepsilon A} f(x,y) \leq \max_{x\varepsilon A} f(x,y^*)$$

and

$$\min_{y \in B} f(x^*,y) \leq \max_{x \in A} \min_{y \in B} f(x,y).$$

we conclude from (2.4.5) that

$$\min_{y \in B} \max_{x \in A} f(x,y) \leq f(x^*,y^*) \leq \max_{x \in A} \min_{y \in B} f(x,y). \qquad (2.4.6)$$

But by Theorem 2.2.1, the first term of (2.4.6) is not less than the third, hence we conclude that all three members are equal, i.e.

$$f(x^*,y^*) = \max_{x \in A} \min_{y \in B} f(x,y) = \min_{y \in B} \max_{x \in A} f(x,y).$$

Next, to prove the necessary condition, let x^* be a member of A which makes

$$\min_{y \in B} f(x,y)$$

a maximum, and y^* be a member of B which makes

$$\max_{x \in A} f(x,y)$$

a minimum; i.e. let x^* and y^* be members of A and B, respectively, which satisfy the conditions

$$\begin{aligned} \min_{y \in B} f(x^*,y) &= \max_{x \in A} \min_{y \in B} f(x,y), \\ \max_{x \in A} f(x,y^*) &= \min_{y \in B} \max_{x \in A} f(x,y). \end{aligned} \qquad (2.4.7)$$

We shall show that (x^*,y^*) is a saddle point of f.

Since we are supposing that

$$\max_{x \in A} \min_{y \in B} f(x,y) = \min_{y \in B} \max_{x \in A} f(x,y),$$

we see from (2.4.7) that

$$\min_{y \in B} f(x^*,y) = \max_{x \in A} f(x,y^*). \qquad (2.4.8)$$

From the definition of a minimum, we have

$$\min_{y \in B} f(x^*,y) \leq f(x^*,y^*)$$

and hence from (2.4.8) we have

$$\max_{x \in A} f(x,y^*) \leq f(x^*,y^*),$$

then for all x in A,

$$f(x,y^*) \leq f(x^*,y^*)$$

which is condition (1) of Definition 2.4.1. In a similar way, we can show that condition (2) of Definition 2.4.1 is also satisfied, which completes the proof.

Corollary 2.4.3 Let $R = [\pi(\alpha,\beta)]$ be any $m \times n$ payoff matrix of a game. Then

$$\max_{\alpha \leq m} \min_{\beta \leq n} \pi(\alpha,\beta) = \min_{\beta \leq n} \max_{\alpha \leq m} \pi(\alpha,\beta)$$

holds if and only if the game has a saddle point (α^*,β^*).

Proof: This follows from Theorem 2.4.2, by taking A to be the set of the first m positive integers, B to be the set of the first n positive integers, and $\pi(\alpha,\beta)$ instead of $f(x,y)$.

Corollary 2.4.4 For every saddle point (α^*,β^*)

$$\pi(\alpha^*,\beta^*) = \max_{\alpha} \min_{\beta} \pi(\alpha,\beta) = \min_{\beta} \max_{\alpha} \pi(\alpha,\beta).$$

Proof: This coincides with the last equation of the sufficiency proof of Theorem 2.4.2.

Therefore, by Corollary 2.4.3, P_1 can choose a pure strategy α^* so as to get at least the common value $\pi(\alpha^*,\beta^*)$ and P_2 can choose a pure strategy β^*, so as to keep P_1 from getting more than $\pi(\alpha^*,\beta^*)$. In this case there are pure strategies α^* and β^* for the two players such that, for all α and β

$$\pi(\alpha,\beta^*) < \pi(\alpha^*,\beta^*) < \pi(\alpha^*,\beta). \tag{2.4.9}$$

By (2.4.9), P_1 can not do better than to choose α^*; similarly, P_2 can not do better than to choose β^*. We refer to α^*, β^* as "optimal strategies" of P_1 and P_2, respectively. The matrix $R = [\pi(\alpha,\beta)]$ is said to have a saddle point at α^*, β^* and its value is $\pi(\alpha^*,\beta^*)$, we call it the "value" of the game and designate it by v.

Example 2.4.1 Consider a game Γ with payoff matrix

$$\begin{array}{ccc} & & \text{Row min.} \\ R = & \begin{bmatrix} 0 & 7 & 1 & 2 \\ -5 & 4 & 8 & -3 \\ 1 & 7 & 2 & 4 \end{bmatrix} & \begin{array}{c} 0 \\ -5 \\ 1^* \end{array} \\ \text{Col. max.} & \begin{array}{cccc} 1^* & 7 & 8 & 4 \end{array} & \end{array}.$$

In this example

$$\max_{\alpha} \min_{\beta} \pi(\alpha,\beta) = \min_{\beta} \max_{\alpha} \pi(\alpha,\beta) = 1.$$

So, this game has a saddle point at the third row and the first column. The value of the game is 1 which, by observation, is the minimum of the third row and the maximum of the first column. Therefore, in playing this matrix game, the optimal strategy for P_1 is to choose row 3 which makes P_1 sure that he will get at least 1 and the optimal strategy for P_2 is to choose column 1 which can keep P_1 from getting more than 1.

A game Γ may have several saddle points depending on the payoff matrix. In such a case all the saddle points have the same value. Each location of a saddle point provides another solution or pair of optimal strategies.

Example 2.4.2 Consider a game Γ with payoff matrix

$$R = \begin{bmatrix} 6 & 5 & 6 & 5 \\ 1 & 4 & 2 & -1 \\ 8 & 5 & 7 & 5 \\ 0 & 2 & 6 & 2 \end{bmatrix} \begin{array}{c} \text{Row min.} \\ 5^* \\ -1 \\ 5^* \\ 0 \end{array}.$$

$$\begin{array}{ccccc} \text{Col. max.} & 8 & 5^* & 7 & 5^* \end{array}$$

Then

$$\max_{\alpha} \min_{\beta} \pi(\alpha,\beta) = \min_{\beta} \max_{\alpha} \pi(\alpha,\beta) = 5.$$

So there is a saddle point. Actually, there are four of them. In this case, P_1 may play the pure strategy row 1, or he may play row 3. P_2 may play either column 2 or column 4; he may mix these two, if he wishes, in any way. The value of the game is 5.

3. MIXED STRATEGIES AND THE SOLUTION FOR ALL GAMES

3.1 Concept of mixed strategies

We have discussed a matrix game with saddle points in the front part, and we can find the value and the optimal strategies for two players of the game directly, so we call it a "strictly determined game". But not all matrix games have saddle points. When a matrix game does not have a saddle point, of course, we can not find the optimal strategies for P_1 and P_2 and the value of the game directly, so we call it a "non-strictly determined game". When a saddle point exists, every player must chooses the strategy which corresponds to the saddle point to assure optimal results, i.e. there exists a pure optimal strategy for each player to play a game which has a saddle point. However, when a game has no saddle point, at least one of the players can not find his pure optimal strategy. So the players should choose their strategies by combining their pure strategies, i.e. they must use "mixed strategies". Before we explain mixed strategies in detail, let us see a game without a saddle point which is taken from Williams [17].

Example 3.1.1 Stone-Water-Scissors-Glass-Paper Game.
The relations among stone, water, scissors, glass, and paper are shown in Fig. 3.1.1 which represents that stone is thicker than glass and paper, water wets stone and paper, scissors cost more than water and stone, glass is more brittle than water and scissors, and paper is more flexible than scissors and glass.

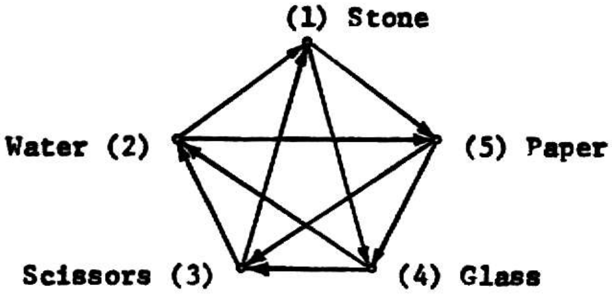

Figure 3.1.1

Now, the two players name one of the five objects simultaneously. If both name the same object, the game is a draw. If we denote the five strategies by the number in Fig. 3.1.1 and let 1, -1, and 0 represent the payoff of win, loss, and draw for P_1, then the payoff matrix for P_1 is

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ -1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 0 \end{bmatrix} \end{array} .$$

Since

$$\max_{\alpha} \min_{\beta} \pi(\alpha,\beta) = -1$$

and

$$\min_{\beta} \max_{\alpha} \pi(\alpha,\beta) = 1 ,$$

these two quantities are not equal, there is no saddle point in this game.

Since the result of a strategy chosen by a player will depend on what his opponent chooses. It is very important to discover his opponent's choice of strategy. But if a player, say P_1, who chose his strategies so steadily that his opponent, P_2, discovered which strategy P_1 will use in the next play, then his opponent P_2 can choose the optimal strategy to get as much as possible from each play by knowing P_1's strategy. Of course, it is a disadvantage for P_1. Hence, every player will concentrate on keeping his own intentions secret. The best way to do this is by using a random device for choosing a strategy. So, a player, instead of choosing a single strategy, may leave the choice of the strategy to chance. That is, he may choose a probability distribution over his set of strategies and then the associated random device selects the particular strategy for the play of the game. Such a probability distribution over the whole set of the pure strategies of a player is a "mixed strategy".

The game now requires each player to select independently a mixed strategy. We shall denote mixed strategies by vectors. Let x_α be the probability of selecting strategy α, (α=1,2,...,m). Then a mixed strategy for P_1 can be denoted as a row vector

$$\underline{x}' = [x_1,\ldots,x_m] \text{ where } \sum_{\alpha=1}^{m} x_\alpha = 1 \text{ and } x_\alpha \geq 0, \quad \alpha=1,\ldots,m. \qquad (3.1.1)$$

Similarly, let y_β be the probability of selecting strategy β, (β=1,2,...,n). Then a mixed strategy for P_2 is a column vector

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix} \quad \text{where} \quad \sum_{\beta=1}^{n} y_\beta = 1 \quad \text{and} \quad y_\beta \geq 0, \quad \beta=1,\ldots,n. \tag{3.1.2}$$

We notice that if $x_\alpha = 1$ for some α, then $\underline{x}$ is called a pure strategy. Similarly, if $y_\beta = 1$ for some β, then $\underline{y}$ is a pure strategy.

The set of all mixed strategies for P_1 is denoted by S_m which is a subset of m-dimensional vectors which satisfies (3.1.1). And the set of all mixed strategies for P_2 is denoted by S_n which is a subset of n-dimensional vectors which satisfies (3.1.2).

Having defined mixed strategies as probability distributions, we need to compute the payoffs which will be measured in terms of expectation. Suppose P_1 chooses strategy α and P_2 chooses mixed strategy $\underline{y}$; the expected payoff to P_1 is

$$s_\alpha = \sum_{\beta=1}^{n} \pi(\alpha,\beta) y_\beta \tag{3.1.3}$$

which is given by the component α of the column vector

$$\underline{s} = R\underline{y} = \begin{bmatrix} s_1 \\ s_2 \\ \cdot \\ \cdot \\ \cdot \\ s_m \end{bmatrix}. \tag{3.1.4}$$

If P_2 chooses strategy β and P_1 chooses mixed strategy $\underline{x}$, the expected payoff to P_1 is

$$t_\beta = \sum_{\alpha=1}^{m} \pi(\alpha,\beta) x_\alpha \tag{3.1.5}$$

which is the component β of the row vector

$$\underline{t}' = \underline{x}'R = [t_1, t_2, \ldots, t_n]. \tag{3.1.6}$$

When P_1 and P_2 use mixed strategies, $\underline{x}$, $\underline{y}$, respectively, since their choices are independent, the expected payoff to P_1 is

$$E(\underline{x},\underline{y}) = \underline{x}'R\underline{y} = \sum_{\beta=1}^{n} \sum_{\alpha=1}^{m} \pi(\alpha,\beta) x_\alpha y_\beta \tag{3.1.7}$$

$$= \underline{t}'\underline{y} = \underline{x}'\underline{s}.$$

Suppose P_1 chooses his strategy by using a mixed strategy $\underline{x}$. Then he can expect to receive at least

$$\min_{\underline{y}} \underline{x}'R\underline{y}$$

where the minimum is taken over all possible mixed strategies available to P_2. Since P_1 has the choice of $\underline{x}$, he will choose $\underline{x}$ so that this minimum is as large as possible. Hence P_1 can select a mixed strategy, call it $\underline{x}^*$, which will assure him an expectation of at least

$$\max_{\underline{x}} \min_{\underline{y}} \underline{x}'R\underline{y}$$

irrespective of what P_2 does. Similarly, for each mixed strategy, $\underline{y}$, chosen by P_2, the most he will have to pay to P_1 is

$$\max_{\underline{x}} \underline{x}'R\underline{y}$$

where the maximum is taken over all mixed strategies available to P_1. Since P_2 has the choice of $\underline{y}$, he will choose $\underline{y}$ so that this maximum is as small as possible. Hence P_2 can select a mixed strategy, call it $\underline{y}^*$, which will make the expectation of P_1 at most

$$\min_{\underline{y}} \max_{\underline{x}} \underline{x}'R\underline{y}$$

irrespective of what P_1 does. Then from the above remarks, and from Theorem 2.2.1, we get

$$\max_{\underline{x}} \min_{\underline{y}} \underline{x}'R\underline{y} \leq \min_{\underline{y}} \max_{\underline{x}} \underline{x}'R\underline{y}$$

or (3.1.8)

$$\max_{\underline{x}} \min_{\underline{y}} E(\underline{x},\underline{y}) \leq \min_{\underline{y}} \max_{\underline{x}} E(\underline{x},\underline{y}).$$

"The minimax theorem" states that these quantities always have a common value, v, or that

$$\max_{\underline{x}} \min_{\underline{y}} \underline{x}'R\underline{y} = \min_{\underline{y}} \max_{\underline{x}} \underline{x}'R\underline{y} = v. \tag{3.1.9}$$

This remarkable result is the fundamental theorem of game theory. We shall prove this theorem in the next section.

If for some $\underline{x}^*$ in S_m and $\underline{y}^*$ in S_n, we have

$$\underline{x}'R\underline{y}^* \leq \underline{x}^{*\prime}R\underline{y}^* \leq \underline{x}^{*\prime}R\underline{y}$$

or (3.1.10)

$$E(\underline{x},\underline{y}^*) \leq E(\underline{x}^*,\underline{y}^*) \leq E(\underline{x}^*,\underline{y})$$

for all $\underline{x}$ in S_m and all $\underline{y}$ in S_n, then we call $E(\underline{x}^*,\underline{y}^*)$ the "value" of the game (to P_1), also denoted by v, and call the pair $(\underline{x}^*,\underline{y}^*)$ a "solution" of the game. Hence we can also write (3.1.10) as

$$E(\underline{x},\underline{y}^*) \leq v \leq E(\underline{x}^*,\underline{y}). \tag{3.1.10'}$$

If $\underline{x}^*$ and $\underline{y}^*$ are mixed strategies which satisfy condition (3.1.10), then, by making use of $\underline{x}^*$, P_1 can make sure that he will get at least $E(\underline{x}^*,\underline{y}^*) = v$, regardless of what P_2 does; and, similarly, by making use of $\underline{y}^*$, P_2 can keep P_1 from getting more than v, regardless of what P_1 does. Therefore we refer to $\underline{x}^*$, $\underline{y}^*$ as "optimal (mixed) strategies".

3.2 Proof of the minimax theorem

The following theorem, the most important of game theory, has been proved in many ways. Here we shall give the proof that was given by von Neumann [16].

Theorem 3.2.1 (The minimax theorem)

For any matrix $R = [\pi(\alpha,\beta)]$ where $\alpha=1,2,\ldots,m$ and $\beta=1,2,\ldots,n$, we have that

$$\max_{\underline{x}} \min_{\underline{y}} \underline{x}'R\underline{y} = \min_{\underline{y}} \max_{\underline{x}} \underline{x}'R\underline{y} = v$$

where $x\varepsilon S_m$ and $y\varepsilon S_n$. And S_m, S_n are the sets of probability distributions over P_1's and P_2's strategies, respectively.

From this theorem it follows that every finite two-person zero-sum game has optimal mixed strategies. Before proving the minimax theorem, we start with some definitions.

Definition 3.2.2 Let $a_1,\ldots,a_n$ be n real numbers, not all of them are zero, and let b be any real number, then all points (vectors) $[x_1,\ldots,x_n]$ of Euclidean n-dimensional space, E_n, such that

$$\sum_{i=1}^{n} a_i x_i = b$$

form a "hyperplane" of E_n.

Definition 3.2.3 If $\sum_{i=1}^{n} a_i x_i = b$ is the equation (defined by Definition 3.2.2) of a hyperplane of E_n. Then it cuts E_n into two parts:

(1) The set of all points $[x_1,\ldots,x_n]$ in E_n such that

$$\sum_{i=1}^{n} a_i x_i > b.$$

(2) The set of all points $[x\ ,\ldots,x\ ]$ in E_n such that

$$\sum_{i=1}^{n} a_i x_i < b.$$

We call them the two "half-spaces" produced by the hyperplane.

Definition 3.2.4 A subset C of E_n is said to be "convex", if and only if for any $\underline{x}$, $\underline{y}\varepsilon C$ and $0 \leq t \leq 1$, we have $t\underline{x} + (1-t)\underline{y}\ \varepsilon\ C$.

Definition 3.2.5 Let K be any set. Then its "convex hull" is the smallest convex set which contain K.

Definition 3.2.6 Let $\underline{x}_1,\dots,\underline{x}_r$ be r points in E_n. Then the point $\underline{y}$ is said to be a "convex linear combination" of these r points, $\underline{x}_1,\dots,\underline{x}_r$ if there exists a vector $[c_1,\dots,c_r]\varepsilon S_r$ such that

$$\underline{y} = \sum_{i=1}^{r} c_i \underline{x}_i .$$

Definition 3.2.7 The "length" of a point (vector) $\underline{x} = [x_1,\dots,x_n]$ of E_n is defined by

$$|\underline{x}| = \sqrt{\sum_{i=1}^{n} x_i^2}\ .$$

Definition 3.2.8 The "distance" of two points $\underline{x} = [x_1,\dots,x_n]$ and $\underline{y} = [y_1,\dots,y_n]$ of E_n is the length of their difference, i.e.

$$|\underline{x} - \underline{y}| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

When proving the minimax theorem, we will use the following two lemmas. (This method of proof follows the work of Owen [10]).

Lemma 3.2.9 (Theorem of supporting hyperplane)

Let B be a closed convex set of points in n-dimensional Euclidean space, and let $\underline{x} = [x_1,\dots,x_n]$ be a point not in B. Then there exist numbers $P_1,\dots,P_n$, P_{n+1} such that

$$\sum_{i=1}^{n} p_i x_i = p_{n+1} \qquad (3.2.1)$$

and

$$\sum_{i=1}^{n} p_i y_i > p_{n+1}, \qquad \text{for all } \underline{y}\varepsilon B. \qquad (3.2.2)$$

By Definition 3.2.2, all points $\underline{x} = [x_1,\ldots,x_n]$ which fulfill (3.2.1), form a hyperplane. By Definition 3.2.3, (3.2.2) is a half-space produced by the hyperplane (3.2.1). Hence, geometrically, this lemma means that we can pass a hyperplane through $\underline{x}$ such that B lies entirely "above" the hyperplane. This fact is illustrated in Fig. 3.2.1 for the case n = 2 (plane).

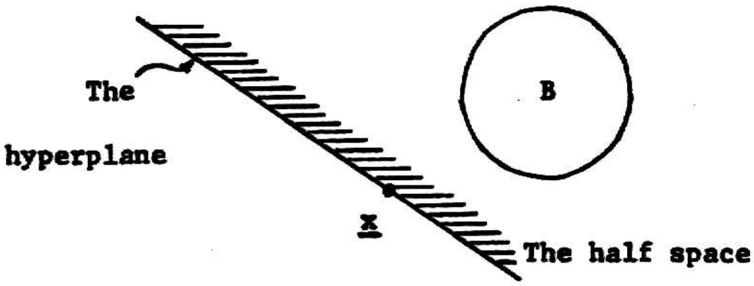

Figure 3.2.1

We observe that (3.2.2) clearly excludes (3.2.1), since $\underline{x}$ belongs to the hyperplane, $\underline{x}$ does not belong to the half space. We now prove this lemma.

Proof: Let $\underline{z}$ be that point in B whose distance from $\underline{x}$ is a minimum. (Such a point exists because B is closed.) Now, let

$$p_i = z_i - x_i, \qquad i=1,\ldots,n,$$

$$p_{n+1} = \sum_{i=1}^{n} z_i x_i - \sum_{i=1}^{n} x_i^2 .$$

Therefore,

$$\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} z_i x_i - \sum_{i=1}^{n} x_i^2 = p_{n+1},$$

i.e. (3.2.1) holds. We must show that (3.2.2) also holds.

Now

$$\sum_{i=1}^{n} p_i z_i = \sum_{i=1}^{n} z_i^2 - \sum_{i=1}^{n} z_i x_i,$$

hence

$$\sum_{i=1}^{n} p_i z_i - p_{n+1} = \sum_{i=1}^{n} z_i^2 - 2\sum_{i=1}^{n} z_i x_i + \sum_{i=1}^{n} x_i^2$$

$$= \sum_{i=1}^{n} (z_i - x_i)^2 > 0.$$

Therefore

$$\sum_{i=1}^{n} p_i z_i > p_{n+1}. \tag{3.2.3}$$

i.e. the point $\underline{z} \varepsilon B$ satisfies (3.2.2). That is not enough as we have to show that all the points in B satisfy (3.2.2). That can be proved by contradiction. Suppose that there exists $\underline{y} \varepsilon B$ such that

$$\sum_{i=1}^{n} p_i y_i \leq p_{n+1}. \tag{3.2.4}$$

Because B is convex, the line joining $\underline{y}$ to $\underline{z}$ must be entirely contained in B, i.e. for all $0 \leq r \leq 1$,

$$\underline{w} = r\underline{y} + (1-r)\underline{z} \ \varepsilon \ B.$$

Now the square of the distance from $\underline{x}$ to $\underline{w}$ is given by

$$d^2(\underline{x},\underline{w}) = \sum_{i=1}^{n} (x_i - ry_i - (1-r)z_i)^2.$$

Therefore

$$\frac{\partial d^2}{\partial r} = 2\sum_{i=1}^{n} (z_i - y_i)(x_i - ry_i - (1-r)z_i)$$

$$= 2\sum_{i=1}^{n} p_i y_i - 2\sum_{i=1}^{n} p_i z_i + 2r\sum_{i=1}^{n} (z_i - y_i)^2.$$

If we evaluate this at $r = 0$, (i.e. $\underline{w} = \underline{z}$),

$$\left.\frac{\partial d^2}{\partial r}\right|_{r=0} = 2\sum_{i=1}^{n} p_i y_i - 2\sum_{i=1}^{n} p_i z_i.$$

But recall (3.2.4)

$$\sum_{i=1}^{n} p_i y_i \leq p_{n+1}$$

and recall (3.2.3)

$$\sum_{i=1}^{n} p_i z_i > p_{n+1}.$$

Thus

$$\left.\frac{\partial d^2}{\partial r}\right|_{r=0} < 0.$$

It follows that, for r close enough to zero,

$$d(\underline{x},\underline{w}) < d(\underline{x},\underline{z}).$$

But this contradicts that the point $\underline{z}\varepsilon B$ whose distance from $\underline{x}$ is a minimum. Therefore, for all $\underline{y}\varepsilon B$, (3.2.2) must hold.

The above lemma is used to prove the next lemma.

Lemma 3.2.10 Let any $m \times n$ matrix $A = [a_{ij}]$. Then either (i) there exists an element $[x_1,\ldots,x_m]$ of S_m such that

$$a_{1j}x_1+a_{2j}x_2+\ldots+a_{mj}x_m \geq 0, \qquad j=1,\ldots,n,$$

or (ii) there exists an element $[y_1,\ldots,y_n]$ of S_n such that

$$a_{i1}y_1+a_{i2}y_2+\ldots+a_{in}y_n \leq 0, \qquad i=1,\ldots,m.$$

Proof: In this proof we shall use the delta symbols of Kronecker, which are defined as

$$\begin{aligned} \delta_{ij} &= 0 \qquad \text{if } i\neq j, \\ &= 1 \qquad \text{if } i=j. \end{aligned}$$

Let

$$\begin{aligned} \delta^{(1)} &= [\delta_{11}, \delta_{21},\ldots,\delta_{m1}], \\ \delta^{(2)} &= [\delta_{12}, \delta_{22},\ldots,\delta_{m2}], \\ &\vdots \\ \delta^{(m)} &= [\delta_{1m}, \delta_{2m},\ldots,\delta_{mm}]. \end{aligned}$$

Thus $\delta^{(j)}$, for $j=1,\ldots,m$, is the point of E_m with 1 in its jth coordinate and 0 in all other coordinates.

Also let

$$\begin{aligned} a^{(1)} &= [a_{11}, a_{21}, \ldots, a_{m1}], \\ a^{(2)} &= [a_{12}, a_{22}, \ldots, a_{m2}], \\ &\vdots \\ a^{(n)} &= [a_{1n}, a_{2n}, \ldots, a_{mn}]. \end{aligned}$$

Thus $a^{(j)}$, for $j=1,\ldots,n$, is the point of E_m whose coordinates are the components of the jth column of matrix A.

Let C be the convex hull of the set of $m + n$ points

$$\delta^{(1)},\ldots,\delta^{(m)},a^{(1)},\ldots,a^{(n)}.$$

Let $g = [0,\ldots,0]$ be the origin of E_m. We can consider two cases, $g\varepsilon C$ and $g\notin C$.

(1) If $g\varepsilon C$, then g is a convex linear combination of the points $\delta^{(1)},\ldots,$ $\delta^{(m)}$, $a^{(1)},\ldots, a^{(n)}$. Hence there is a vector $[u_1,\ldots,u_m,v_1,\ldots,v_n]\varepsilon S_{m+n}$ such that

$$u_1\delta^{(1)}+\ldots+u_m\delta^{(m)}+v_1a^{(1)}+\ldots+v_na^{(n)} = g,$$

it can also be expressed as

$$u_1\delta_{i1}+\ldots+u_m\delta_{im}+v_1a_{i1}+\ldots+v_na_{in} = 0, \qquad i=1,\ldots,m,$$

from the definition of the delta symbols, we have

$$u_i+v_1a_{i1}+\ldots+v_na_{in} = 0, \qquad i=1,\ldots,m. \qquad (3.2.5)$$

Since $[u_1,\ldots,u_m,v_1,\ldots,v_n]\varepsilon S_{m+n}$, u_i is non-negative and hence, from (3.2.5), we have

$$v_1a_{i1}+\ldots+v_na_{in} \leq 0, \qquad i=1,\ldots,m. \qquad (3.2.6)$$

We also know that $v_j \geq 0$ for $j=1,\ldots,n$. If $v_1=v_2=\ldots=v_n=0$, then by (3.2.5), we have

$$u_i = 0, \qquad i=1,\ldots,m.$$

Hence

$$\sum_{i=1}^{m} u_i + \sum_{j=1}^{n} v_j = 0 \neq 1,$$

which contradicts the fact that $[u_1,\ldots,u_m,v_1,\ldots,v_n]\varepsilon S_{m+n}$. Hence at least one of v_j, for $j=1,\ldots,n$, is greater than zero. This implies that

$$v_1+\ldots+v_n > 0. \tag{3.2.7}$$

So we can let

$$\begin{aligned} y_1 &= v_1/(v_1 + \ldots + v_n), \\ y_2 &= v_2/(v_1 + \ldots + v_n), \\ &\cdots\cdots\cdots\cdots \\ y_n &= v_n/(v_1 + \ldots + v_n), \end{aligned} \tag{3.2.8}$$

and we see that $[y_1,\ldots,y_n]\varepsilon S_n$.

From (3.2.6), (3.2.7), and (3.2.8) we can conclude that

$$a_{i1}y_1+\ldots+a_{in}y_n \leq 0, \qquad i=1,\ldots,m,$$

which is the condition (ii) of this lemma.

(2) Now, consider the case $g\notin C$. By Lemma 3.2.9, there exists a hyperplane which contains g and C lies entirely above the hyperplane. Let the equation of this hyperplane be

$$\sum_{i=1}^{m} h_i t_i = h_{m+1}.$$

Since g lies on the hyperplane, we have

$$\sum_{i=1}^{m} h_i \cdot 0 = h_{m+1},$$

hence $$h_{m+1} = 0.$$

Thus the equation of the hyperplane is

$$\sum_{i=1}^{m} h_i t_i = 0, \tag{3.2.9}$$

and from Lemma 3.2.9, we also have that for every point $[t_1,\ldots,t_m]\varepsilon C$ satisfies

$$\sum_{i=1}^{m} h_i t_i > 0. \tag{3.2.10}$$

In particular, the inequality (3.2.10) must hold for $\delta^{(1)},\ldots,\delta^{(m)}$ of C; thus

$$h_i\delta_{1i} +\ldots+h_m\delta_{mi} > 0, \qquad i=1,\ldots,m,$$

from the definition of the delta symbols, we have

$$h_i > 0, \qquad i=1,\ldots,m. \tag{3.2.11}$$

Moreover, (3.2.10) must hold for the points $a^{(1)},\ldots,a^{(n)}$; thus

$$h_1a_{1j}+\ldots+h_ma_{mj}> 0, \qquad j=1,\ldots,n. \tag{3.2.12}$$

From (3.2.11), we have

$$h_1+\ldots+h_m > 0. \tag{3.2.13}$$

Hence we can let

$$\begin{aligned} x_1 &= h_1/(h_1 + \ldots + h_m), \\ x_2 &= h_2/(h_1 + \ldots + h_m), \\ &\ldots\ldots\ldots\ldots \\ x_m &= h_m/(h_1 + \ldots + h_m). \end{aligned} \tag{3.2.14}$$

and we see that $[x_1,\ldots,x_m]\varepsilon S_m$.

From (3.2.12), (3.2.13), and (3.2.14), we conclude that

$$x_1a_{1j} +\ldots+x_ma_{mj}> 0, \qquad j=1,\ldots,n,$$

and hence

$$x_1a_{1j} +\ldots+x_ma_{mj}\geq 0, \qquad j=1,\ldots,n,$$

which is the condition (i) of this Lemma.

With the above two lemmas, we are able to prove the minimax theorem.

Proof of the minimax theorem: If condition (i) of Lemma 3.2.10 holds, then there is an element $[x_1,\ldots,x_m]\varepsilon S_m$ such that

$$\sum_{\alpha=1}^{m} \pi(\alpha,\beta)x_\alpha \geq 0, \qquad \beta=1,\ldots,n.$$

Hence for every $y\varepsilon S_n$

$$E(\underline{x},\underline{y}) = \sum_{\beta=1}^{n} [\sum_{\alpha=1}^{m} \pi(\alpha,\beta)x_\alpha]y_\beta \geq 0. \tag{3.2.15}$$

Since (3.2.15) holds for every $y\varepsilon S_n$, we have

$$\min_{\underline{y}} E(\underline{x},\underline{y},) \geq 0,$$

and hence

$$\max_{\underline{x}} \min_{\underline{y}} E(\underline{x},\underline{y}) \geq 0. \tag{3.2.16}$$

If condition (ii) of Lemma 3.2.10 holds, then there is an element $[y_1,\ldots,y_n]\varepsilon S_n$ such that

$$\sum_{\beta=1}^{n} \pi(\alpha,\beta)y_\beta \leq 0, \qquad \alpha=1,\ldots,m.$$

Hence for every $\underline{x}\varepsilon S_m$

$$E(\underline{x},\underline{y}) = \sum_{\alpha=1}^{m} [\sum_{\beta=1}^{n} \pi(\alpha,\beta)y_\beta]x_\alpha \leq 0. \tag{3.2.17}$$

Since (3.2.17) holds for every $x\varepsilon S_m$, we have

$$\max_{\underline{x}} E(\underline{x},\underline{y}) \leq 0,$$

hence

$$\min_{\underline{y}} \max_{\underline{x}} E(\underline{x},\underline{y}) \leq 0. \tag{3.2.18}$$

Since either condition (i) or (ii) of Lemma 3.2.10 holds, then at least one of the inequalities (3.2.16) or (3.2.18) must hold, and hence the following can not be true

$$\max_{\underline{x}} \min_{\underline{y}} E(\underline{x},\underline{y}) < 0 < \min_{\underline{y}} \max_{\underline{x}} E(\underline{x},\underline{y}). \tag{3.2.19}$$

Let R_c be the matrix which arises from R by subtracting c from each element of R,

$$R_c = \begin{bmatrix} \pi(1,1)-c,\ldots,\pi(1,n)-c \\ \cdot \\ \cdot \\ \cdot \\ \pi(m,1)-c,\ldots,\pi(m,n)-c \end{bmatrix}$$

and let E_c be the expectation function for R_c, so that for any $\underline{x}$ and any $\underline{y}$ that are members of S_m and S_n, respectively,

$$\begin{aligned} E_c(\underline{x},\underline{y}) &= \sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} [\pi(\alpha,\beta)-c]x_\alpha y_\beta \\ &= \sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} \pi(\alpha,\beta)x_\alpha y_\beta - c \\ &= E(\underline{x},\underline{y}) - c. \end{aligned} \tag{3.2.20}$$

Since the inequality (3.2.19) does not hold for the matrix R, the following conditions cannot hold for R_c

$$\max_{\underline{x}} \min_{\underline{y}} E_c(\underline{x},\underline{y}) < 0 < \min_{\underline{y}} \max_{\underline{x}} E_c(\underline{x},\underline{y}),$$

and from (3.2.20), we conclude that the following condition does not hold:

$$\max_{\underline{x}} \min_{\underline{y}} E(\underline{x},\underline{y})-c < 0 < \min_{\underline{y}} \max_{\underline{x}} E(\underline{x},\underline{y})-c.$$

Hence the following condition does not hold:

$$\max_{\underline{x}} \min_{\underline{y}} E(\underline{x},\underline{y}) < c < \min_{\underline{y}} \max_{\underline{x}} E(\underline{x},\underline{y}). \tag{3.2.21}$$

Since the inequality (3.2.21) is false for every c, we conclude that the following is false:

$$\max_{\underline{x}} \min_{\underline{y}} E(\underline{x},\underline{y}) < \min_{\underline{y}} \max_{\underline{x}} E(\underline{x},\underline{y}),$$

hence the following relation is true:

$$\max_{\underline{x}} \min_{\underline{y}} E(\underline{x},\underline{y}) \geq \min_{\underline{y}} \max_{\underline{x}} E(\underline{x},\underline{y}). \tag{3.2.22}$$

But (3.1.8) states that

$$\max_{\underline{x}} \min_{\underline{y}} E(\underline{x},\underline{y}) \leq \min_{\underline{y}} \max_{\underline{x}} E(\underline{x},\underline{y}). \tag{3.2.23}$$

By (3.2.22) and (3.2.23), it follows that

$$\max_{\underline{x}} \min_{\underline{y}} E(\underline{x},\underline{y}) = \min_{\underline{y}} \max_{\underline{x}} E(\underline{x},\underline{y}),$$

or, by (3.1.7)

$$\max_{\underline{x}} \min_{\underline{y}} \underline{x}'R\underline{y} = \min_{\underline{y}} \max_{\underline{x}} \underline{x}'R\underline{y}.$$

3.3 Solutions for 2 x 2 matrix games

A 2 x 2 matrix game is the simplest type of matrix game. Therefore, we first determine solutions for them. The following theorem is proved by Owen [10].

Theorem 3.3.1 Let R be a 2 x 2 game matrix. Then if R does not have a saddle point, its unique optimal strategies and value will be given by

$$\begin{aligned} \underline{x}' &= \frac{\underline{j}' \text{ (adj. R)}}{\underline{j}' \text{ (adj. R)}\underline{j}}, \\ \underline{y} &= \frac{\text{(adj. R)}\underline{j}}{\underline{j}' \text{ (adj. R)}\underline{j}}, \\ v &= \frac{\text{(det. R)}}{\underline{j}' \text{ (adj. R)}\underline{j}} \end{aligned} \tag{3.3.1}$$

where (adj. R) is the adjoint of R, (det. R) is the determinant of R, and $\underline{j}' = [1,1]$.

Proof: Let the 2 x 2 payoff matrix of a game is given by

$$R = \begin{bmatrix} \pi(1,1), & \pi(1,2) \\ \pi(2,1), & \pi(2,2) \end{bmatrix}.$$

If there is a saddle point, then we can get the solution of the game immediately, if not, we can get the solution of a game by some formulas which are shown as follows.

Since there is no saddle point, mixed strategies must be used. Let $\underline{x}'=[x_1,x_2]$ and $\underline{y}'=[y_1,y_2]$ be the optimal strategies of P_1 and P_2, respectively. And the components of $\underline{x}'$ and $\underline{y}'$ are positive. Let v be the value of the game, we have

$$\underline{x}'R\underline{y} = v,$$

or
$$x_1y_1\pi(1,1)+x_1y_2\pi(1,2)+x_2y_1\pi(2,1)+x_2y_2\pi(2,2) = v,$$

or
$$x_1[\pi(1,1)y_1+\pi(1,2)y_2]+x_2[\pi(2,1)y_1+\pi(2,2)y_2] = v. \tag{3.3.2}$$

Since y is by hypothesis an optimal strategy, the two terms in parentheses on the left-hand side of (3.3.2) are both less than or equal to v. Suppose one of them were less than v; i.e., suppose

$$\pi(1,1)y_1+\pi(1,2)y_2 < v,$$
$$\pi(2,1)y_1+\pi(2,2)y_2 \leq v.$$

Then, since $x_1>0$ and $x_1+x_2=1$, the left-hand side in (3.3.2) would be strictly smaller than v. It follows that both the terms in parentheses in (3.3.2) must be equal to v. Hence,

$$\pi(1,1)y_1+\pi(1,2)y_2 = v,$$
$$\pi(2,1)y_1+\pi(2,2)y_2 = v,$$

or, in matrix form,

$$R\underline{y} = \begin{bmatrix} v \\ v \end{bmatrix} = v\underline{i} \qquad \text{where } \underline{i} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} . \tag{3.3.3}$$

Similarly, it can be seen that

$$\pi(1,1)x_1+\pi(2,1)x_2 = v,$$
$$\pi(1,2)x_1+\pi(2,2)x_2 = v,$$

or, in matrix form,

$$\underline{x}'R = \begin{bmatrix} v,v \end{bmatrix} = v\underline{i}'. \tag{3.3.4}$$

We also know that

$$x_1+x_2=1,$$
$$y_1+y_2=1,$$

or, in matrix form,

$$\underline{x}'\underline{j} = 1, \tag{3.3.5}$$

$$\underline{j}'\underline{y} = 1. \tag{3.3.6}$$

The four equations, (3.3.3), (3.3.4), (3.3.5), and (3.3.6), allow us to solve for $\underline{x}$, $\underline{y}$, and v. If R is non-singular, from (3.3.4), we have

$$\underline{x}' = v\underline{j}'R^{-1},$$

then by (3.3.5), we have

$$v\underline{j}'R^{-1}\underline{j} = 1,$$

or

$$v = \frac{1}{\underline{j}'R^{-1}\underline{j}} \; ,$$

and

$$\underline{x}' = \frac{\underline{j}'R^{-1}}{\underline{j}'R^{-1}\underline{j}} \; .$$

Similarly, we find

$$\underline{y} = \frac{R^{-1}\underline{j}}{\underline{j}'R^{-1}\underline{j}} \; .$$

If R is singular, the above is of course meaningless; it can be written in the following form.

$$\underline{x}' = \frac{\underline{j}'\,(\text{adj}.R)}{\underline{j}'\,(\text{adj}.R)\underline{j}} \; ,$$

$$\underline{y} = \frac{(\text{adj}.R)\underline{j}}{\underline{j}'\,(\text{adj}.R)\underline{j}} \; , \tag{3.3.7}$$

$$v = \frac{(\text{det}.R)}{\underline{j}'\,(\text{adj}.R)\underline{j}} \; ,$$

where (adj.R) is the adjoint of R and (det.R) is the determinant of R. We see that (3.3.7) gives the value of the 2 x 2 game, whether R is singular or not.

Example 3.3.1 Solve the game matrix

$$R = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}.$$

First of all, we check whether the game has saddle point or not. Since $\max_{\alpha} \min_{\beta} \pi(\alpha,\beta) = 0$ and $\min_{\beta} \max_{\alpha} \pi(\alpha,\beta) = 1$, the two quantities are not equal and this 2 x 2 matrix game has no saddle point. Therefore, we can apply Theorem 3.3.1. Now, the adjoint of R is

$$\text{adj.R} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}.$$

And

$$\text{det. R} = 2,$$

$$\underline{i}' \,(\text{adj.R}) = [1,1] \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = [3,1],$$

$$(\text{adj.R})\underline{i} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix},$$

$$\underline{i}' \,(\text{adj.R})\underline{i} = [1,1] \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4.$$

Thus by (3.3.1), we have

$$x' = \frac{1}{4}\,[3 \quad 1] = [\tfrac{3}{4} \quad \tfrac{1}{4}].$$

$$\underline{y} = \frac{1}{4} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

$$v = \frac{2}{4} = \frac{1}{2}.$$

3.4 A graphical method of solution

Whenever the size of the payoff matrix of a game is $2 \times n$ or $m \times 2$ ($n>2$, $m>2$), i.e., one of the two players only has two pure strategies, we can not apply Theorem 3.3.1 to solve it, but we can use a graphical method to find the solutions and the value of the game. We shall illustrate the method by some examples of $2 \times n$ matrix games which are given by Mckinsey [9].

Example 3.4.1 Suppose the payoff matrix of a game Γ is

$$\begin{array}{cc} & P_2\text{'s strategies} \\ P_1\text{'s strategies} & \begin{bmatrix} 2 & 3 & 11 \\ 7 & 5 & 2 \end{bmatrix}. \end{array}$$

Since there is no saddle point in this payoff matrix, mixed strategies must be used. Let $[x, 1-x]$ be the mixed strategy of P_1 where x is between zero and one. If P_2 uses his first strategy (pure strategy), then the expected payoff to P_1 will be

$$2x+7(1-x) = 7-5x.$$

Similarly, if P_2 uses his second (pure) strategy, then the expected payoff to P_1 is

$$3x+5(1-x) = 5-2x,$$

and if P_2 uses his third (pure) strategy, then the expected payoff to P_1 is

$$11x+2(1-x) = 2+9x.$$

We now plot, over the interval $[0,1]$, the three lines $y=7-5x$, $y=5-2x$, and $y=2+9x$ in Fig. 3.4.1.

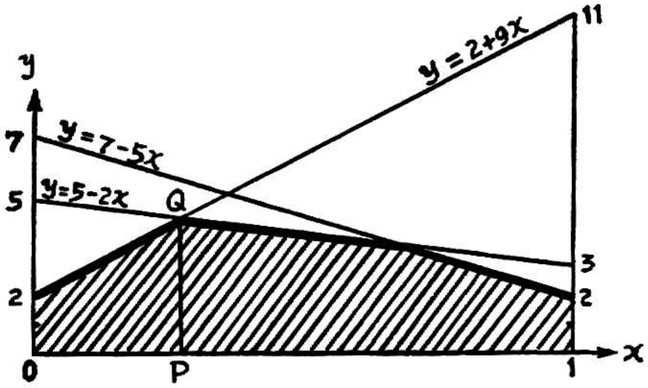

Figure 3.4.1

For each choice of x by P_1, he can be certain of getting at least the minimum of the ordinates of the three lines at x. Thus if P_1 wants to choose an optimal x, then he must choose an x which will make the minimum of the three ordinates as large as possible; hence, from Fig. 3.4.1 it is apparent that the optimal x will be the segment OP and that the value of the game is PQ. Therefore we can find an optimal strategy for P_1 (in this game, moreover, we see from the figure that there is only one optimal strategy for P_1) and the value of the game by solving the two equations

$$y = 5-2x,$$

$$y = 2+9x,$$

simultaneously. And we can find that $x = 3/11$, $y = 49/11$. Hence, $\underline{x}' = [3/11 \quad 8/11]$ is the optimal strategy for P_1 and the value of the game is 49/11.

Moreover, from Fig. 3.4.1, it is clear that no optimal mixed strategy for P_2 will contain his first strategy, hence we can determine an optimal mixed strategy for P_2 by using the matrix

$$\begin{bmatrix} 3 & 11 \\ 5 & 2 \end{bmatrix}.$$

We solve this 2 x 2 matrix game, and find an optimal strategy for P_2 is the column vector

$$\begin{bmatrix} 9/11 \\ 2/11 \end{bmatrix}.$$

Since in the original payoff matrix of the game Γ, P_2 has three strategies, we say that P_2 has an optimal strategy as $\underline{y}'=[0,\ 9/11,\ 2/11]$.

From the minimax theorem it follows that every finite matrix game has a mixed strategies solution. And the above example has only one optimal strategy for P_1 and P_2. In some cases, depending on the payoff matrix, the game may have many optimal mixed strategies. Now we turn to an example where P_1 has many optimal strategies.

Example 3.4.2. Consider a payoff matrix of a game Γ is

$$\begin{array}{cc} & P_2\text{'s strategies} \\ P_1\text{'s strategies} & \begin{bmatrix} 2 & 4 & 11 \\ 7 & 4 & 2 \end{bmatrix} \end{array}.$$

In this payoff matrix, again, there is no saddle point, mixed strategy must be used. Let $[x,1-x]$ be the mixed strategy of P_1 where x is between zero and one. If P_2 uses his pure strategy step by step, we can get the following three equations:

$$y = 7-5x,$$

$$y = 4,$$

$$y = 2+9x.$$

Then we plot this three lines in Fig. 3.4.2. Again, discuss as in Example 3.4.1 and we can find that the value of the game is 4 and that any x will be optimal for P_1, so long as it satisfies $OP_1 \leq x \leq OP_2$. We can find OP_1 by solving the following two equations: $y=4$, $y=2+9x$ simultaneously. And we get $x=2/9$, i.e., $OP_1=2/9$.

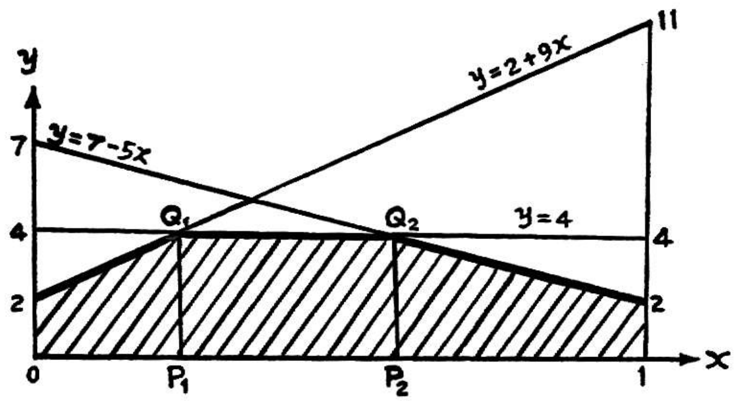

Figure 3.4.2

Also, we can find OP_2 by solving the following two equations: $y=4$, $y=7-5x$ simultaneously. And we get $x=3/5$, i.e., $OP_2=3/5$. Thus an optimal strategy for P_1 is any vector $[x, 1-x]$ where $2/9 \leq x \leq 3/5$. And an optimal strategy for P_2 in this game is $\underline{y}'=[0, 1, 0]$.

If an m x n matrix where both m and n are greater than 2, then it becomes impracticable to use this method for solving this game. So if the matrix size is large, we will transform the game into a linear programming problem as discussed later.

3.5 Dominance

Before introducing other methods for solving a game, we are going to explain a very important technique which can reduce the size of a matrix to smaller size. Then we can follow the smaller matrix to solve the original matrix game. Of course, it makes the problem easier to be solved.

Definition 3.5.1 Given any matrix $R = [\pi(\alpha,\beta)]$, where $\alpha=1,\ldots,m$ and $\beta=1,\ldots,n$. (i) For any two rows i and j, if

$$\pi(i,\beta) \geq \pi(j,\beta), \qquad \text{for all } \beta,$$

then we say the i<u>th</u> row "dominates" the j<u>th</u> row, or the j<u>th</u> row is dominated by the i<u>th</u> row.

(ii) Similarly for the case of columns, for any two columns k and ℓ, if

$\pi(\alpha,k) \geq \pi(\alpha,\ell)$, for all α,

then we say the $k\underline{th}$ column "dominates" the $\ell\underline{th}$ column, or the $\ell\underline{th}$ column is dominated by the $k\underline{th}$ column.

But how do we apply this concept to a matrix game? The best way is to give an example, which is taken from May [7], to explain it.

Example 3.5.1 Suppose the payoff matrix for P_1 of a game Γ is

$$R = \begin{bmatrix} 2 & 0 & 1 & 4 \\ 1 & 2 & 5 & 3 \\ 4 & 1 & 3 & 2 \end{bmatrix}.$$

It is easily checked that there is no saddle point. So we want to see whether we can reduce the size of the matrix or not. It is seen that the fourth column dominates the second column. We notice that the columns are the strategies of P_2, and P_2 wants to minimize P_1's receipt, hence P_2 would like to cross out the larger one, i.e., the fourth column (so P_2 will cross out the one which dominates others) and leave the matrix

$$R = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 5 \\ 4 & 1 & 3 \end{bmatrix}.$$

Again, in this matrix, we find that the third row dominates the first row. The rows are the strategies of P_1 and P_1 wants to get as much as possible, therefore, P_1 would like to cross out the smaller one, i.e., the first row (so P_1 will cross out the one which is dominated by others) and leave the matrix

$$R = \begin{bmatrix} 1 & 2 & 5 \\ 4 & 1 & 3 \end{bmatrix}.$$

In this matrix, we see that the third column dominates the second column. Thus as before, P_2 will cross out the one which dominates the others, and leave a 2 x 2 matrix

$$R_3 = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}.$$

The above 2 x 2 matrix R_3 does not have a saddle point. So we can apply Theorem 3.3.1 to solve it, and determine the optimal strategies for P_1 and P_2, respectively to be $\underline{x}'=[3/4, 1/4]$ and $\underline{y}'=[1/4, 3/4]$. The value of the game matrix R_3 is $v=7/4$.

Dresher [2] extended this concept to a more general case. Given any m x n matrix $R = [\pi(\alpha,\beta)]$, without any loss of generality we can consider the elements of the mth row which are not smaller than or equal to another row for every corresponding element, but are all smaller than or equal to some convex linear combinations of the corresponding elements of other rows. That is, there is a member $\underline{t}'=[t_1,\ldots,t_{m-1}]$ of S_{m-1} such that

$$\pi(m,\beta) \leq \sum_{\alpha=1}^{m-1} \pi(\alpha,\beta)t_\alpha , \qquad \text{for } \beta=1,\ldots,n.$$

In this case P_1 can cross out the mth row since P_1 could always get more by applying mixed strategies to the first m-1 rows. If one of the elements of $\underline{t}\epsilon S_{m-1}$ is one, then the others are all zeros, and this reduces to the case of Definition 3.5.1.

Example 3.5.2 If the game Γ has a payoff matrix

$$R = \begin{bmatrix} 12 & 0 \\ 3 & 1 \\ 0 & 3 \end{bmatrix}.$$

We can check that there is no saddle point in this matrix. Then we follow Definition 3.5.1 to see whether there is any dominated strategy in order to reduce the size of the matrix R. The result is that no strategy is dominated by any other pure strategies. But we notice that the elements of the second row are all smaller than the following convex linear combinations of the corresponding elements of the first and third rows, i.e.

$$3 < 1/3 \cdot 12 + 2/3 \cdot 0,$$

$$1 < 1/3 \cdot 0 + 2/3 \cdot 3.$$

This means that if P_1 chooses the first row and the third row with ratio 1 : 2, P_1 can always get more than choosing the second row. Therefore, certainly, P_1 will not need the second row anymore and crosses it out to leave the matrix

$$R_1 = \begin{bmatrix} 12 & 0 \\ 0 & 3 \end{bmatrix}.$$

Since there is no saddle point in R_1, we can use Theorem 3.3.1 to solve R_1 and we find that the optimal strategies for P_1 and P_2, respectively, are $\underline{x}'=[1/5, 4/5]$ and $\underline{y}'=[1/5, 4/5]$. The value of the game matrix R_1 is $v=12/5$.

So far, we have determined the solution and the value of the reduced matrix game. But, in fact, we need the solution and the value of the original matrix game. What are the relations between the solutions and the values of those matrices? The relations are stated in the following theorems which are presented and proved in Mckinsey [9].

Theorem 3.5.2 Consider a matrix game Γ having payoff matrix $R = [\pi(\alpha,\beta)]$ where $\alpha=1,\dots,m$ and $\beta=1,\dots,n$. Suppose that, for some α, the $\alpha\underline{th}$ row of R is dominated by convex linear combination of the other rows of R; let R' be the matrix obtained from R by crossing out the $\alpha\underline{th}$ row; and let Γ' be the matrix game whose payoff matrix is R'. Then the value of Γ' is the same as the value of Γ, every optimal strategy for P_2 in Γ' is also an optimal strategy for P_2 in Γ, and if $\underline{x}_1'$ is any optimal strategy for P_1 in

Γ' and $\underline{x}'$ is <u>the α-place extension</u>[#] of $\underline{x}'_1$, then $\underline{x}'$ is an optimal strategy for P_1 in Γ.

Proof: Let

$$R = \begin{bmatrix} \pi(1,1), & \dots, & \pi(1,n) \\ \vdots & & \vdots \\ \pi(m,1), & \dots, & \pi(m,n) \end{bmatrix} .$$

We can suppose, without loss of generality, that the last row of R is dominated by a convex linear combination of other rows. Thus there exists a member $\underline{t}'=[t_1,\dots,t_{m-1}]$ of S_{m-1} such that

$$\pi(m,\beta) \leq \sum_{\alpha=1}^{m-1} \pi(\alpha,\beta)t_\alpha, \qquad \beta=1,\dots,n. \tag{3.5.1}$$

Let v be the value of Γ', let $\underline{x}'_1=[x_1,\dots,x_{m-1}]$ be an optimal strategy for P_1 in Γ' and let $\underline{y}'=[y_1,\dots,y_n]$ be an optimal strategy for P_2 in Γ'. Then, from the definition of the optimal strategy, we have

$$\sum_{\beta=1}^{n} \pi(\alpha,\beta)y_\beta \leq v, \qquad \alpha=1,\dots,m-1 \tag{3.5.2}$$

and

$$v \leq \sum_{\alpha=1}^{m-1} \pi(\alpha,\beta)x_\alpha, \qquad \beta=1,\dots,n. \tag{3.5.3}$$

To prove this theorem, we must show that v is also the value of Γ, that $\underline{y}$ is an optimal strategy for P_2 in Γ, and that $[x_1,\dots,x_{m-1},0]$ is an optimal strategy for P_1 in Γ. By the definition of the optimal strategy, we must show that

$$\sum_{\beta=1}^{n} \pi(\alpha,\beta)y_\beta \leq v, \qquad \alpha=1,\dots,m \tag{3.5.2'}$$

[#] Consider a mixed strategy $\underline{h}'=[h_1,\dots,h_n]$ of S_n and $1 \leq i \leq n+1$, then the i-place extension of the mixed strategy $\underline{h}'$ is the vector
$[h_1,\dots,h_{i-1},0,h_i,\dots,h_n]$.

and

$$v \leq \sum_{\beta=1}^{m-1} \pi(\alpha,\beta)x_\alpha + \pi(m,\beta) \cdot 0, \qquad \beta=1,\ldots,n. \qquad (3.5.3')$$

Since (3.5.3') is obviously the same as (3.5.3), we need only to prove that (3.5.2') holds. By (3.5.2), we need only to prove that

$$\sum_{\beta=1}^{n} \pi(m,\beta)y_\beta \leq v.$$

By using (3.5.1) and (3.5.2), we have

$$\begin{aligned}\sum_{\beta=1}^{n} \pi(m,\beta)y_\beta &\leq \sum_{\beta=1}^{n} \sum_{\alpha=1}^{m-1} \pi(\alpha,\beta)t_\alpha y_\beta \\ &= \sum_{\alpha=1}^{m-1} \sum_{\beta=1}^{n} \pi(\alpha,\beta)y_\beta t_\alpha \\ &\leq \sum_{\alpha=1}^{m-1} vt_\alpha = v,\end{aligned}$$

which completes the proof.

The next theorem is concerned with the case of columns and the proof is omitted since it is similar to the last one.

Theorem 3.5.3 Consider a matrix game Γ has payoff matrix $R = [\pi(\alpha,\beta)]$ where $\alpha=1,\ldots,m$ and $\beta=1,\ldots,n$. Suppose that, for some β, the βth column of R dominates some convex linear combination of the other columns of R; let R' be the matrix obtained from R by crossing out the βth column; and let Γ' be the matrix game whose payoff matrix is R'. Then the value of Γ' is the same as the value of Γ; every optimal strategy for P_1 in Γ' is also an optimal strategy for P_1 in Γ, and if $\underline{y}_1$ is an optimal strategy for P_2 in Γ', and $\underline{y}$ is the β-place extension of $\underline{y}_1$, then $\underline{y}$ is an optimal strategy for P_2 in Γ.

So by Theorem 3.5.2 and Theorem 3.5.3 we can get the solutions and the values for the original matrices R in Example 3.5.1 and Example 3.5.2.

First, we see that, in Example 3.5.1, P_1 crossed out the first row and P_2 crossed out the third column and the fourth column. So, for the original payoff matrix R, the optimal strategies for P_1 and P_2, respectively, are $\underline{x}'=[0, 3/4, 1/4]$ and $\underline{y}'=[1/4, 3/4, 0, 0]$. The value of the original game is also 7/4.

Next, in Example 3.5.2, P_1 crossed out the second row only. So, for the original payoff matrix R, the optimal strategies for P_1 and P_2, respectively, are $\underline{x}'=[1/5, 0, 4/5]$ and $\underline{y}'=[1/5, 4/5]$. The value of the game remains 12/5.

3.6 Method of approximating the value of a game

In this section we shall introduce an approximate method of solving matrix games which will enable us to find the value of such games to any desired degree of accuracy and also to approximate optimal strategies. Suppose that two players play a long sequence of plays of a given game where neither knows an optimal strategy because they are ignorant of game theory, perhaps, or because the matrix of the game is too large for them to be able to make the required computations. In the long sequence of plays of a given game, one can keep track of his opponent's past plays and choose at each play the optimal pure strategy against the accumulated mixed strategy of the opponent's past plays. At each play of the long sequence we can calculate the upper and lower bounds for the value of the game and an approximation to an optimal strategy for each player.

This method can be illustrated by the following example which is taken from Dresher [2].

Example 3.6.1 Consider a game Γ with payoff matrix

$$\begin{array}{cc} & \begin{array}{ccc} C1 & C2 & C3 \end{array} \\ \begin{array}{c} R1 \\ R2 \\ R3 \end{array} & \begin{bmatrix} 2 & 1 & 0 \\ 2 & 0 & 3 \\ -1 & 3 & -3 \end{bmatrix} \end{array},$$

where R_i, i=1,2,3, are the strategies of P_1, and C_j, j=1,2,3, are the strategies of P_2. Suppose the series of plays is begun by P_1 and he chooses strategy R_1 in his first play. The successive method for getting an approximate solution and the upper and lower bounds of the value of the game is shown by Table 3.6.1.

Table 3.6.1

N	$S_1(N)$	$R_1(N)$	$R_2(N)$	$R_3(N)$	$\underline{V}(N)$	$S_2(N)$	$C_1(N)$	$C_2(N)$	$C_3(N)$	$\overline{V}(N)$
1	R_1	2	1	$\underline{0}$	.000	C_3	0	$\underline{3}$	-3	3.000
2	R_2	4	$\underline{1}$	3	.500	C_2	1	$\underline{3}$	0	1.500
3	R_2	6	$\underline{1}$	6	.333	C_2	2	$\underline{3}$	3	1.000
4	R_2	8	$\underline{1}$	9	.250	C_2	3	3	$\underline{6}$	1.500
5	R_3	7	$\underline{4}$	6	.800	C_2	4	3	$\underline{9}$	1.800
6	R_3	6	7	$\underline{3}$	.500	C_3	4	$\underline{6}$	6	1.000
7	R_2	8	7	$\underline{6}$	.857	C_3	4	$\underline{9}$	3	1.286
8	R_2	10	$\underline{7}$	9	.875	C_2	5	$\underline{9}$	6	1.125
9	R_2	12	$\underline{7}$	12	.778	C_2	6	$\underline{9}$	9	1.000
10	R_2	14	$\underline{7}$	15	.700	C_2	7	9	12	1.200
11	R_3	13	10	12	.909	C_2	8	9	$\underline{15}$	1.364
12	R_3	12	13	$\underline{9}$	.750	C_3	8	$\underline{12}$	12	1.000

In Table 3.6.1, the notations of the column headings are explained as follows:

(i) $S_1(N)$ is the pure strategy chosen by P_1 on the Nth play. For instance, in Table 3.6.1, $S_1(1)=R_1$, $S_1(2)=R_2$, and $S_1(6)=R_3$.

(ii) $S_2(N)$ is the pure strategy chosen by P_2 on the Nth play. For instance, in Table 3.6.1, $S_2(1)=C_3$, $S_2(2)=C_2$, and $S_2(6)=C_3$.

(iii) $R_j(N)$, j=1,2,3, are the total receipts of P_1 after N of his play if P_2 uses his pure strategy C_j, j=1,2,3, respectively, constantly.

(iv) $C_i(N)$, i=1,2,3, are the total receipts of P_1 after N plays of P_2 if P_1 chooses his pure strategy R_i, i=1,2,3, respectively, constantly.

(v) $\underline{v}(N)$ is the minimum that P_1 can expect to receive on the average after N of his plays, or we can express it as

$$\underline{v}(N) = \frac{1}{N} \min_j R_j(N), \qquad j=1,2,3. \tag{3.6.1}$$

(vi) $\overline{v}(N)$ is the maximum that P_1 can expect to receive on the average after N plays of P_2, or we can express it as

$$\overline{v}(N) = \frac{1}{N} \max_i C_i(N), \qquad i=1,2,3. \tag{3.6.2}$$

Table 3.6.1 has been completed as following steps:

Step 1. For the first play of the game, assume that P_1 chooses strategy R_1, i.e., $S_1(1)=R_1$. Then P_1 will receive 2, 1, or 0 depending on what P_2 chooses (C_1, C_2, or C_3), therefore the total receipts of P_1 are $R_1(1)=2$, $R_2(1)=1$, and $R_3(1)=0$. The minimum of $R_j(1)$, j=1,2,3, is 0, so by (3.6.1), $\underline{v}(1)=0$.

Step 2. Since P_2 wants to minimize P_1's receipt, P_2 will, of course, choose C_3 for his first play, i.e., $S_2(1)=C_3$. Then P_1 will get 0, 3, or -3 depending on what strategy P_1 uses, so the total receipts of P_1 after first play of P_2 are $C_1(1)=0$, $C_2(1)=3$, and $C_3(1)=-3$. The maximum of $C_i(1)$ is 3, so by (3.6.2), $\overline{v}(1)=3$.

Step 3. For the second play, P_1 will choose R_2 since that maximized his receipts against P_2's first play, i.e., $S_1(2)=R_2$. Then P_1 will receive 2, 0, or 3 depending on what P_2 chooses (C_1,C_2, or C_3), therefore after two plays, the total receipts of P_1 are $R_1(2)=2+2=4$, $R_2(2)=1+0=1$, and $R_3(2)=0+3=3$. Also by (3.6.1), we have $\underline{v}(2)=1/2$.

Step 4. Again P_2 will minimize P_1's receipt, and choose C_2 for N=2, i.e., $S_2(2)=C_2$. Then P_1 will get 1, 0, or 3 depending on what strategy P_1 uses, so the total receipts of P_1 after second play of P_2 are $C_1(2)=0+1=1$, $C_2(2)=3+0=3$, and $C_3(2)=-3+3=0$. By (3.6.2), we have $\overline{v}(2)=3/2=1.5$.

The procedures for the successive N are all the same. If the minimum and maximum of $R_j(N)$ and $C_i(N)$, respectively, are not unique, the player may choose any one of the possible pure strategies which satisfy the requirement.

After N steps, an approximation to an optimal strategy will be obtained from the relative frequencies of each of the pure strategies in Table 3.6.1. Thus at N=12, P_1 has chosen R_1 for one time, R_2 for seven times, R_3 for four times, so we have the approximate optimal strategy for P_1 at N=12 as

$$\underline{x}'(12) = [1/12,\ 7/12,\ 4/12].$$

Similarly, we have the approximate optimal strategy for P_2 at N=12 as

$$\underline{y}'(12) = [0/12,\ 8/12,\ 4/12].$$

The value of the game, v, is approximated by $\underline{v}(N)$ and $\overline{v}(N)$. We have for all N,

$$\underline{v}(N) \leq v \leq \overline{v}(N).$$

Thus at N=12, the value of the game v is between 0.75 and 1.00.

Robinson [11] has shown that if

$$v = \lim_{N\to\infty} \underline{x}(N) \qquad \text{and} \lim_{N\to\infty} \underline{y}(N)$$

exist, then these limits are a solution of the game, and the value is

$$v = \lim_{N\to\infty} \overline{v}(N) = \lim_{N\to\infty} \underline{v}(N). \tag{3.6.3}$$

It can be proved that in the previous example we have

$$\lim_{N\to\infty} \underline{x}'(N) = [0,\ 2/3,\ 1/3],$$

$$\lim_{N\to\infty} \underline{y}'(N) = [0,\ 2/3,\ 1/3],$$

and

$$v = 1.00.$$

This successive method of solving a game was proposed as a means for actually computing the value of a game. However, the convergence of (3.6.1) and (3.6.2) to (3.6.3) is extremely slow, so this method is impractical to solve a game. Therefore we will introduce an efficient method to solve the game in the next section.

3.7 Solution of matrix games by linear programming

In this section, we will introduce the most popular method for solving a matrix game, especially when the size of the payoff matrix of a given game is large. Since the principle of linear programming is a technique for maximizing or minimizing some objective function subject to certain constraints, we can use it to solve matrix games.

In order to get the optimal strategies for P_1 and P_2 and the value of the game by linear programming methods, we need to transform the matrix game into a linear programming problem where both the objective function and the constraints are stated in the form of linear equations. Consider the payoff matrix of a given game Γ to be $R = [\pi(\alpha,\beta)]$ where $\alpha=1,2,\dots,m$ and $\beta=1,2,\dots,n$. Let the value of the game be denoted by v. If the optimal strategy for P_1 is $\underline{x}'=[x_1,\dots,x_m]$ of S_m then, by the definition of the optimal strategy, we have

$$\sum_{\alpha=1}^{m} \pi(\alpha,\beta)x_{\alpha} \geq v, \qquad \beta=1,\ldots,n,$$

$$x_{\alpha} \geq 0, \qquad \alpha=1,\ldots,m, \tag{3.7.1}$$

$$\sum_{\alpha=1}^{m} x_{\alpha} = 1,$$

and P_1 wants to make v as large as possible, i.e.

$$\text{maximize } v. \tag{3.7.2}$$

By writing (3.7.1) and (3.7.2), we have not yet reached a linear programming formulation, because v may be negative. If all the elements of the given payoff matrix for P_1 are positive, then, of course, the value of the game (for P_1), v, will be positive, and there is no problem. If some of the elements of the payoff matrix for P_1 are negative, then the value of the game may not be non-negative. In this case, we may add an amount large enough to all entries in the payoff matrix in order to make sure that the value of the game is positive. This increases the value of the game by the same amount but does not change the solution. Therefore, we can assume the value of the game, v, to be positive, then we can define a new variable

$$X_{\alpha} = \frac{x_{\alpha}}{v}, \qquad \alpha=1,\ldots,m. \tag{3.7.3}$$

If we divide the inequalities of (3.7.1) by v, and use the notation expressed in (3.7.3), we have

$$\sum_{\alpha=1}^{m} \pi(\alpha,\beta)X_{\alpha} \geq 1, \qquad \beta=1,\ldots,n,$$

$$X_{\alpha} = \frac{x_{\alpha}}{v} \geq 0, \qquad \alpha=1,\ldots,m \tag{3.7.1'}$$

$$\sum_{\alpha=1}^{m} X_\alpha = \frac{1}{v} ,$$

and further maximizing v in (3.7.2) is equivalent to minimizing 1/v, so we can state (3.7.1') as

$$\text{Minimize} \quad \sum_{\alpha=1}^{m} X_\alpha = \frac{1}{v} ,$$

$$\text{Subject to} \quad \sum_{\alpha=1}^{m} \pi(\alpha,\beta) X_\alpha \geq 1, \qquad \beta=1,\ldots,n \tag{3.7.4}$$

$$\text{where } X_\alpha = \frac{x_\alpha}{v} \geq 0, \qquad \alpha=1,\ldots,m.$$

Thus, the matrix game as stated in (3.7.4) has been reduced to a linear programming problem in the usual form.

Similarly, we can get a set of inequalities for P_2. If $\underline{y}'=[y_1,\ldots,y_n]$ of S_n is an optimal strategy for P_2, then we have a form which is similar to (3.7.4) as follows:

$$\text{Maximize} \quad \sum_{\beta=1}^{n} Y_\beta = \frac{1}{v} ,$$

$$\text{subject to} \quad \sum_{\beta=1}^{n} \pi(\alpha,\beta) Y_\beta \leq 1, \qquad \alpha=1,\ldots,m \tag{3.7.5}$$

$$\text{where } Y_\beta = \frac{y_\beta}{v} \geq 0, \qquad \beta=1,\ldots,n.$$

The two sets of inequalities are dual to each other; by solving one of them, the other is solved implicitly. If we have found X_α, Y_β (α =1,...,m and β=1,...,n) and the minimum of $\sum_{\alpha=1}^{m} X_\alpha$, which equals the maximum of $\sum_{\beta=1}^{n} Y_\beta$ then we have the value of the game v, and

$$x_\alpha = X_\alpha \cdot v, \qquad \alpha=1,\ldots,m,$$

$$y_\beta = Y_\beta \cdot v, \qquad \beta=1,\ldots,n,$$

which is the solution we need.

Here, we present an example which is taken from Levin [5].

Example 3.7.1 Let the payoff matrix of a matrix game be

$$R_0 = \begin{bmatrix} 1 & 2 & -1 \\ -2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} .$$

Since neither a saddle point exists nor the size of the matrix can be reduced to a smaller matrix by dominance, we use the linear programming method to solve it.

Since two elements of the matrix R_o are negative, the value of the game may not be non-negative. If we add two to every element of the matrix R_o, we have

$$R = \begin{bmatrix} 3 & 4 & 1 \\ 0 & 3 & 3 \\ 4 & 2 & 3 \end{bmatrix} .$$

The value of the game matrix R_o is increased by two but the optimal strategies for P_1 and P_2 remain the same. If v_o represents the value of the game matrix R_o and v represents the value of the game matrix R, then $v=v_o+2$, or $v_o=v-2$.

Let the optimal strategy for P_1 be denoted by the row vector $\underline{x}'=[x_1,x_2,x_3]$ of S_3 then, by the definition of the optimal strategy, we have

$$3x_1+0x_2+4x_3 \geq v,$$

$$4x_1+3x_2+2x_3 \geq v,$$

$$1x_1+3x_2+3x_3 \geq v$$

where

$$x_1+x_2+x_3 = 1,$$

$$x_1,\ x_2,\ x_3 \geq 0,$$

and P_1 wants to make v as large as possible, so P_1 wants to maximize v.

Since v is positive, we can define

$$X_\alpha = \frac{x_\alpha}{v}, \qquad \alpha=1,2,3.$$

Also

$$X_1, X_2, X_3 \geq 0$$

and

$$X_1+X_2+X_3 = \frac{x_1}{v} + \frac{x_2}{v} + \frac{x_3}{v} = \frac{1}{v}.$$

So we can get the form as in (3.7.4)

$$\text{Minimize } X_1+X_2+X_3 = \frac{1}{v},$$

$$\begin{aligned} \text{subject to} \quad & 3X_1+0X_2+4X_3 \geq 1, \\ & 4X_1+3X_2+2X_3 \geq 1, \\ & 1X_1+3X_2+3X_3 \geq 1 \\ \text{where} \quad & X_1, X_2, X_3 \geq 0. \end{aligned}$$

Now, we have reduced P_1's problem to a linear programming problem. Similarly, we can also reduce P_2's problem to a linear programming problem as follows. Let $\underline{y}'=[y_1,y_2,y_3]$ of S_3 be an optimal strategy for P_2 then, we have

$$\begin{aligned} & 3y_1+4y_2+1y_3 \leq v, \\ & 0y_1+3y_2+3y_3 \leq v, \\ & 4y_1+2y_2+3y_3 \leq v \\ \text{where} \quad & y_1+y_2+y_3 = 1, \\ & y_1, y_2, y_3 \leq 0. \end{aligned}$$

and P_2 wants to make P_1's receipt as small as possible, so P_2 likes to minimize v. Again v is positive, we can define

$$Y_\beta = \frac{y_\beta}{v}, \qquad \beta=1,2,3.$$

Also

$$Y_1, Y_2, Y_3 \geq 0.$$

So we can get the form as in (3.7.5)

$$\text{Maximize } Y_1+Y_2+Y_3 = \frac{1}{v},$$
$$\text{subject to } 3Y_1+4Y_2+1Y_3 \le 1,$$
$$0Y_1+3Y_2+3Y_3 \le 1,$$
$$4Y_1+2Y_2+3Y_3 \le 1$$
$$\text{where} \quad Y_1, Y_2, Y_3 \ge 0.$$

(3.7.6) and (3.7.7) are dual to each other; by solving one of them, the other is solved implicitly. So we can choose either one. Here, we choose to solve the set of inequalities (3.7.7). Three iterations are required for the solution to this problem. They are shown in Table 3.7.1. The simplex procedure and notation used are standard. Details of the calculations are omitted.

Table 3.7.1 Simplex solution for P_2's strategies

Table 1

C_j			1	1	1	0	0	0	
	Basis	P_0	Y_1	Y_2	Y_3	S_1	S_2	S_3	
0	S_1	1	3	4	1	1	0	0	Θ=1/3.
0	S_2	1	0	3	3	0	1	0	Θ=undefined.
0	S_3	1	4	2	3	0	0	1	Θ=1/4 (this row is replaced).
	Z_j	0	0	0	0	0	0	0	
	C_j-Z_j		1	1	1	0	0	0	

Table 2

0	S_1	1/4	0	5/2	-5/4	1	0	-3/4	$\Theta = \frac{1}{4} \div \frac{5}{2} = \frac{1}{10}$ (this row is replaced).
0	S_2	1	0	3	3	0	1	0	Θ=1/3.
1	Y_1	1/4	1	1/2	3/4	0	0	1/4	$\Theta = \frac{1}{4} \div \frac{1}{2} = \frac{1}{2}$.
	Z_j	1/4	1	1/2	3/4	0	0	1/4	
	C_j-Z_j		0	1/2	1/4	0	0	-1/4	

Table 3

1	Y_2	1/10	0	1	-1/2	2/5	0	-3/10
0	S_2	7/10	0	0	9/2	-6/5	1	9/10
1	Y_1	1/5	1	0	1	-1/5	0	2/5
	Z_j	3/10	1	1	1/2	1/5	0	1/10
	C_j-Z_j		0	0	1/2	-1/5	0	1/10

$\theta=\frac{1}{10}\div(-\frac{1}{2})=-\frac{1}{5}$.

$\theta=\frac{7}{10}\div\frac{9}{2}=\frac{7}{45}$ (this row is replaced).

$\theta=1/5$.

Table 4

1	Y_2	8/45	0	1	0	4/15	1/9	-1/5
1	Y_3	7/45	0	0	1	-4/15	2/9	1/5
1	Y_1	2/45	1	0	0	1/15	-2/9	1/5
	Z_j	17/45	1	1	1	1/15	1/9	1/5
	C_j-Z_j		0	0	0	-1/15	-1/9	-1/5

stop when all the elements of this row are non-positive.

From Table 4 of Table 3.7.1, the value of the objective function is

$$\text{Maximize } Y_1+Y_2+Y_3 = \frac{1}{v} = 17/45 \tag{3.7.8}$$

and

$$Y_1 = 2/45, \quad Y_2 = 8/45, \text{ and } Y_3 = 7/45. \tag{3.7.9}$$

From (3.7.8), we have $v = 45/17$.

Then we can get

$$y_1 = Y_1 \cdot v = (2/45)\cdot(45/17) = 2/17,$$

$$y_2 = Y_2 \cdot v = (8/45)\cdot(45/17) = 8/17,$$

$$y_3 = Y_3 \cdot v = (7/45)\cdot(45/17) = 7/17.$$

Hence, $\underline{y}'$ =[2/17, 8/17, 7/17] is an optimal strategy for P_2.

Since (3.7.6) and (3.7.7) are dual to each other, we can find the optimal strategy for P_1 directly from Table 4 of Table 3.7.1. They appear in the row C_j-Z_j under the columns S_1, S_2, and S_3, i.e., -1/15, -1/9, -1/5. We disregard the minus sign, since negative values for strategies would have no meaning to the players. But these values are X_1, X_2, and X_3, and we need

$$x_1 = X_1 \cdot v = (1/15)\cdot(45/17) = 3/17,$$
$$x_2 = X_2 \cdot v = (1/9)\cdot(45/17) = 5/17,$$
$$x_3 = X_3 \cdot v = (1/5)\cdot(45/17) = 9/17.$$

Hence, $\underline{x}' = [3/17, 5/17, 9/17]$ is an optimal strategy for P_1. The value of the original game matrix R_o is

$$v_o = v-2$$
$$= (45/17)-2 = 11/17.$$

Also we can solve this problem by using the set of inequalities (3.7.6), and we will get the same solution and value of the game.

When the payoff matrix of a given game can not be reduced below 3 x 3, linear programming offers an efficient method for finding the optimal strategies for P_1 and P_2 and the value of the game. But, sometimes, the size of the matrix is quite large and the simplex table will be too much to hold. In this case, the most efficient method for solving these large linear programming problems is to use computer programs.

4. SUMMARY AND CONCLUSION

In this report, we discuss matrix games, which are also called finite two-person zero-sum games. There are two participants (two persons) in the game. Each one has a finite set of strategies. And the gain of one player is the loss of the other. The payoffs between the two players for a given game form a payoff matrix (or game matrix). According to the saddle point of a payoff matrix, which is explained in Section 2.4, the matrix games can be distinguished into two kinds: (1) The first kind is strictly determined games which contain one or more saddle points in the payoff matrix. In this case, both players use pure strategies. (2) The second is non-strictly determined games which have no saddle points in the payoff matrix, and mixed strategies must be used. Both players want to get the optimal result under a given game. An optimal play will imply a strategy that will either maximize a player's gain or minimize his loss. The main theorem, the minimax theorem, which was proved by von Neumann [16], assures that every matrix game has optimal mixed strategies for both players. Therefore, given any matrix game, we can find the optimal strategies for both players and the value of the game.

The solutions for a matrix game can be obtained by a variety of methods. Some of those methods are discussed in this report. When a payoff matrix has a saddle point, of course, there is no problem. But when there is no saddle points, the most efficient method to solve it depends on the size of the payoff matrix. If the matrix is 2×2, we can use the results presented in Theorem 3.3.1. If the game matrix is $2 \times n$ or $m \times 2$ $(n>2, m>2)$, a graphical method can be used. When the size of the game matrix is greater than or equal to 3×3, the technique of dominance is used to check whether or not the payoff matrix can be reduced to a smaller matrix. If it can be reduced so that one dimension is 2, then previous methods can be applied to solve it. If not, the most general method of solution is the simplex algorithm as presented in Section 3.7. That is, the matrix game problem is restated as a linear programming problem and solved by a method of solution for linear programming problems using the simplex algorithm. When the payoff matrix is too large, programs for solving the simplex algorithm are available for most electronic computers.

Besides those already discussed there are several other types of games. A brief statement about some of those games follows.

If the sum of the payoffs due to each player in a given game is not zero, we say it is a "non-zero-sum game". When a game involves more than two persons (participants), we call it n-person game.

In a two-person zero-sum game, one player's receipt is always the other player's loss. Thus there is no reason to consider the possibility of cooperation or negotiation between the players. However, the existence of more than two players and/or payoffs that do not add to zero introduces the possibility of cooperation and bargaining. For example, in an n-person game two or more players may decide to cooperate in the hope that by acting together they can more easily beat the opposition. Similarly, when the sum of the payoffs is not zero the players may be able to cooperate in such a way that they will maximize the total payoff rather than maximizing the payoff to a single player. The theorems of non-zero-sum games and n-person games can be found in Burger [1], Maschler [7], Rapoport [11], Tucker [13] and von Neumann [16].

In a finite game, each player selects a strategy from a finite set of strategies. The number of such strategies may be large, as in chess, but finite. A natural generalization is to consider games in which a player chooses a strategy from an infinite set of strategies. Such a game is called an "infinite game". There are several reasons for developing a theory of infinite games. Many military and economic problems, when viewed as games, involve an infinite number of strategies. For example, a military budget can be thought of as being divisible in an infinite number of ways between offense and defense. In economics a commodity may have an infinite number of price possibilities. The solution of infinite games is not discussed in this report, but the reader can refer to Karlin [3], Luce [6], Owen [10], and Tucker [13] for the details of those games.

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MATRIX GAME THEORY

by

LI-CHING TINA KUNG

B. S., National Changchi University, 1970

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KANSAS STATE UNIVERSITY
Manhattan, Kansas

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A matrix game is also called a two-person zero-sum game. The game is a conflict of interest which involves two persons. Whenever one of the two players wins an amount which is lost by another player, that is, the sum of the payoffs of the two players is zero.

The strictly determined games always possess one or more saddle points, so both of the players use the pure strategies. In Section 2.3, we also explained that if one of the players uses a pure strategy all the time in a 2 x 2 matrix game, then another player will also use a pure strategy all the time that will assures him to get the optimal result.

The non-strictly determined game is the case of a game without saddle points, then mixed strategies must be used. The main theorem, the minimax theorem, which was originally proved by von Neumann in 1928, insures that any matrix game has the optimal strategies for both players, at the same time, the value of the game maximizes one's receipt and minimizes another's loss.

We have discussed several methods for solving a matrix game in Section 3. The most general method for solving a game, whose payoff matrix is greater than or equal to 3 x 3, is to use linear programming method, that is, a game can be written as a linear programming problem (see Section 3.7), and the solution of the latter gives also that of the former.

Also some further topics of game theory are briefly stated in the last section of the report.