ANALYSIS OF MICROBIAL GROWTH IN A TOWER SYSTEM

by

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Chapter 1

INTRODUCTION

Many chemical processes such as food processing operations, pharmaceutical processes, biological waste treatment, and brewing operations involve the use of microorganisms. The mechanism of biological enzymatic reactions was investigated by Michaelis and Menten (1)* who proposed the commonly used kinetic expression for enzymatic reactions. In 1942, Monod (2) proposed a kinetic model of biological growth based on the Michaelis-Menten equation. Although Monod's model depends only on the cell and substrate concentrations, it is widely used to describe the kinetics of growth.

Batch processes have played an important role in the fermentation industry. However, continuous processes are advantageous in some applications and since 1950 researchers have begun to enhance their effort to study and employ continuous processes. The tower fermentor considered in this work may be operated either batchwise or continuously; however, the emphasis of this work is on continuou operation.

Recently, several researchers investigated a tower fermentor for conducting growth processes (3, 4, 5, and 6). Prokop et al. (3) investigated an eight stage tower fermentor with cocurrent flow of air and medium from the bottom to the top of the column. Their work consisted of an experimental study of the residence time distribution of liquid media and cells as well as a steady state

* Equations, figures, tables, appendices, and references cited will all be found within the chapter in which they are cited except where specific reference is made to another chapter.

investigation of the substrate and cell concentrations throughout the fermentor. They found that the residence time characteristics of the continuous phase and the dispersed (microorganisms) phase are distinct. A mathematical model was developed and efforts were made to evaluate the model parameters. Falch and Gaden (4) studied continuous culture in a 4 stage tower fermentor, in which air and medium were introduced into the column countercurrently. The effects of plate and agitator design on backflow, oxygen transfer rate and gas hold up have been reported by Kitai et al.(5). These investigators found that excellent oxygen transfer rates (K_La values as high as 400 hr^{-1} were reported) could be obtained in a tower fermentor. They also found that the cell concentrations in the system were influenced by foaming and they introduced a depletion factor to account for differences in cell concentrations between the flow from a stage and the bulk concentration in that stage.

A tower type fermentor which may be operated continuously has several potential advantages; (1) high productivity, (2) reduced operating costs, and (3) flexibility of operation and control. The important factors which influence the operating performance in a tower type fermentor are feed geometry, system dilution rate, backflow, and sedimentation. The degree of mixing between stages is influenced by the backflow rate, while the sedimentation greatly influences the cell concentration in each stage. The feed geometry of a tower type system is also an important property because it allows for a wide range of growth rates and cell physiological states.

The tower system investigated in this study is assumed to be constructed such that compartments are separated by perforated plates. Air and influent enter the column and flow cocurrently up the column as shown in Fig. 1. Air bubbles, which are introduced at the bottom, circulate the media. This circulation causes each compartment to behave as an approximately completely mixed stage. The oxygen for biological growth is supplied from air bubbles and the transfer rate depends on the distribution of bubbles, bubble size, and air velocity. Since the oxygen transfer and mass transfer rate have been investigated by other researchers (5, 7, 8, and 9), this factor will not be considered in this study.

Continuous processes can be classified into three different flow models with respect to their macroscopic degree of mixing; (1) plug flow (piston flow), (2) complete mixing, and (3) partial mixing. Fortunately, flow behavior corresponding to each of these three different flow models can be approached by properly designing and operating the tower system. This flexibility of operation which is due to a combination of geometrical and hydraulic flow properties is one of the advantages of the tower type system. Plug flow is achieved if the number of stages is large and fluid backflow is prevented. When the backflow rate is increased the system approaches that of complete mixing. Partial mixing is the most general case found in the tower system, as it is the intermediate case between the above two extreme cases.

The present study includes three main topics which are; (1) modeling of microbial growth in tower systems, (2) analysis of steady state performance and the stability of growth processes in

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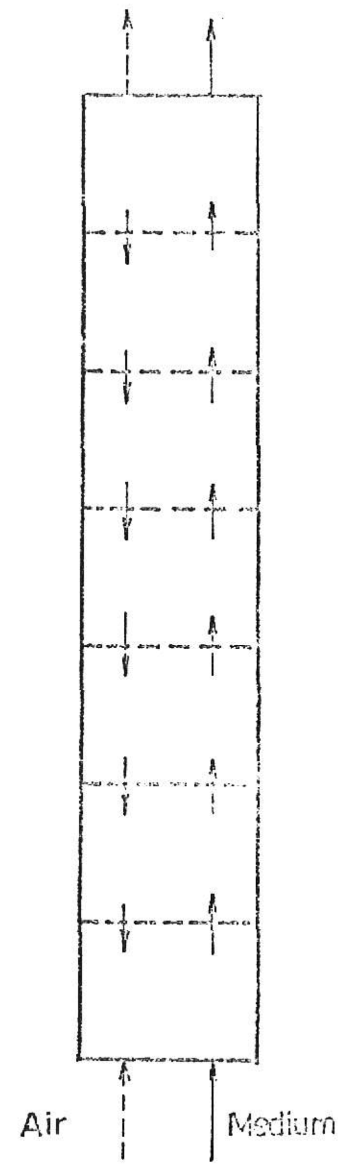

Fig. 1. Schematic diagram of flow characteristics of a tower fermentation system.

tower systems, and (3) optimization of biological waste treatment processes using the tower system. Modeling the microbial growth in a tower system which is operated continuously is an important problem. The growth rate and cell distribution, which are both of interest, are strongly affected by the backflow rate, cell sedimentation, and flow geometry. The backflow rate depends on the gravitational force, air flow rate and the design of the perforated plates. Falch and Gaden (4) noted that backflow between stages is also influenced by the degree of agitation. In this study, two different models to represent backflow are used; in the one the backflow rate is assumed to be constant from stage to stage, and in the other there is a constant ratio of flow rates. Cell sedimentation is assumed to occur in each compartment of the tower system. Previous research (3), (5) shows that the cell concentration in the upward stream is more dilute than that in the bulk liquid. This is because of the heterogeneity of the system and surface behavior. The distribution of cells between the foam and liquid phases and the tendency for the cells to sediment can all be considered in modeling cell sedimentation; however, only a simple empirical model is used in this work (the details are presented in Chapter 2). Increasing the backflow rate and the sedimentation increase the cell retention time, i.e., a higher concentration of cells exists in each compartment.

To understand the performance of a tower system at steady state, it is desirable to determine the cell and substrate concentration distributions by solving a set of 2N nonlinear

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mass balance equations simultaneously for an N-stage tower system. To solve these nonlinear simultaneous algebraic mass balance equations, two computational schemes which have been used; one is a linearization scheme in which the nonlinear equations are linearized to be linear equations, and the other is a method in which the independent variables are substituted sequentially to obtain a one dimensional problem which can be solved using the regula falsi method (10). The cell concentration profiles which are illustrated in Chapter 2 show that the cell concentration distributions are influenced by hydraulic properties, feed geometry and number of stages. In general, the cell concentration changes rapidly at dilution rates near the washout point. Because of this, operating instabilities may be experienced at dilution rates near washout. Operation near the washout dilution is often avoided because of these factors. Biological growth is the result of biological synthesis and metabolism. Although many biochemical reactions are very rapid, the growth process is much slower and a moderately long retention time is required to maintain growth in a continuous fermentor because the cells may be washed out if the dilution rate increases beyond a certain value. Since the tower system is very flexible in its operation and control, a wide range of washout dilution rates can be realized. The maximum dilution rate (washout dilution rate) depends on the characteristics of the system, especially the feed geometry, number of stages, backflow rate, and sedimentation parameter. A detailed investigation of the washout criteria for several different cases is presented in Chapter 3.

In Chapter 4, the optimal design of a tower type system for use in the activated sludge process is considered. The process assumes the waste influent is fed to a tower aeration reactor and the activated sludge or cells are recycled back. Since the hydraulic behavior (fluid backflow and cell sedimentation, for example) may greatly influence the rate of biological growth, it is fundamentally important to investigate its effect on the performance of a biological waste treatment process. In Chapter 4, the process is analyzed by employing mathematical modeling and optimization procedures to determine the optimal policy. The objective function to be minimized is the total holding time. To solve the optimum design problem, a search technique is employed. Since the objective function and the process model are only approximations, the obtained optimal policy will probably deviate from the true optimum. In spite of this, studies of this type of problem can give us a better understanding of the biological waste treatment process.

A tower type biological system has certain advantages over the conventional stirred tank system. Redispersion and circulation of air bubbles make the individual stage almost equivalent to a completely mixed tank. Appearance of backflow and sedimentation sometimes are desirable in biological processes and these may be realized in the tower system. The principal disadvantage of a tower system may be its relative instability and foaming which may make control difficult. However, the great benefit of a tower type system is its wide range of operation and the possibility of varying feed position and substrate concentration. In addition, a tower type system may require less space and land requirements and be easily integrated into an industrial complex.

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Chapter 2

MODELING AND ANALYSIS OF TOWER FERMENTATION PROCESSES: I. STEADY STATE PERFORMANCE

2-1 INTRODUCTION

Continuous culture processes are being investigated and employed on an increasing scale industrially. Many types of continuous culture schemes that might be used in biological processes are classified by Herbert (1). A classification based on continuous flow pattern could divide these schemes into three different models; (a) plug flow model, (b) completely mixed model, and (c) partial mixing model. These three different flow models depend on the degree of macroscopic mixing of media. A plug flow fermentor is difficult to construct because of aeration requirements. The flow behavior of many reaction systems lies between plug flow and completely mixed flow. The dispersion model or the multistage model with backflow is an example of the partial mixing model.

In this chapter, a tower system similar to that investigated by others (2, 3, 4, 5, and 6) is studied. The tower system is separated into several compartments by sieve plates. Varying degree of flow non-ideality, which have been experimentally observed, are considered in modeling continuous culture microbial growth processes in the tower system. The mathematical representation of a tower fermentation system is considered by Prokop et al. (2) and the concepts of backflow, bypassing and cell sedimentation are introduced. In this work, the mathematical representation of growth processes in the tower system is investigated further and the effect of the number of stages, backflow parameter and sedimentation parameter on system performance is examined.

2-2 KINETIC MODEL

Many chemical processes, such as fermentation, biological waste treatment and brewing processes, involve the use of microorganisms. Biological growth involves the biosynthesis and metabolism of bateria and consists of a complex sequence of biochemical reactions. In 1942, Monod (7) presented a model for the kinetics of biological growth. The equation proposed by Monod (7) to describe the relationship between growth rate and concentration of limiting nutrient has the same form as the Michaelis-Menten equation, which describes the kinetics of enzymatic reactions. This is a gross oversimplification of the very complex phenomena that occur and Gaudy (8) and Tsuchiya, Fredrickson, and Aris (9) have presented the results of their effort for a more complete treatment. Since much more work and information have to be gained before such a representation process will be well developed, Monod's equation is employed in this study. Although the constants in Monod's equation are strongly dependent on the heterogeneity of the population and operating condition, no attempt will be made to include the effects of types of microorganisms involved in this mathematical model of the growth process.

The growth of microorganisms will be expressed in terms of a single growth rate equation which is at all times a function of the concentrations of the growth limiting substrate (organic nutrients) and organisms. If oxygen and trace nutrients are available in sufficient quantities, the growth rate of organisms may be expressed as follows:

$$\frac{dX}{dt} = \mu X \tag{1}$$

where μ is the specific growth rate and X is the concentration of bacterial cells. In 1942, Monod (7) showed that the value of μ is not constant, but depends on the concentration of growth limiting substrate, S, according to the equation

$$\mu = \mu_{max} \frac{S}{K_S + S} \tag{2}$$

where

μ_{max} = maximum specific growth rate when the organic concentration is not limiting the rate of growth, hr^{-1}.

K_S = concentration of organics at which the specific growth rate observed is one half the maximum value.

Substituting Equation (2) into Equation (1) gives

$$\frac{dX}{dt} = \frac{\mu_{max} S X}{K_S + S} \tag{3}$$

Sanitary engineers have modified Equation (3) by introducing an endogeneous metabolism term. Equation (3) with this term added becomes

$$\frac{dX}{dt} = \frac{\mu_{max} S X}{K_S + S} - k_D X \tag{4}$$

where

k_D = specific endogeneous microbial attrition rate, hr^{-1}.

Monod (7) also showed that the relationship between the growth of bacteria and utilization of substrate is given by

$$-\frac{dX}{dS} = \frac{\text{weight of bacteria formed}}{\text{weight of substrate used}} = Y \tag{5}$$

where the ratio Y is the yield factor. From Equations (3) and (5) we have the following equation for the rate of substrate utilization.

$$-\frac{dS}{dt} = \frac{\mu_{max} S X}{Y(K_S + S)} \tag{6}$$

Simplified forms of Equations (4) and (6) result if K_S is much larger than S or K_S is much smaller than S. When $K_S \gg S$, Equation (4) reduces to

$$\frac{dX}{dt} = \frac{\mu_{max} S X}{K_S} - k_D X \tag{7}$$

while when $K_S \ll S$, Equation (4) reduces to

$$\begin{aligned} \frac{dX}{dt} &= \mu_{max} X - k_D X \\ &= (\mu_{max} - k_D) X \end{aligned} \tag{8}$$

Equations (7) and (8) show that the growth rate may be zero or first order with respect to the concentration of organic nutrients Similar simplifications can be written for Equation (6).

Biological growth is not a simple ordinary chemical reaction. In fact, the growth rate is not only dependent on the concentration of substrate and organisms, but also dependent on the history of the organisms. An advanced study of kinetics of microbial growth and cell physiological state had been reported by Kono (16, 17). The concept of growth activity is introduced and employed to relate the theoretical growth curve to experimental work.

2-3 FLOW AND CONTINUOUS CULTURE CHARACTERISTICS

Several, researchers (2,3,4,5) have investigated the flow and continuous culture characteristics of a multi-stage continuous tower system. Because of the vertical construction and hydraulic properties, the performance of these tower systems is complicated and additional research is needed to provide an adequate mathematical representation. The behavior of a continuous tower fermentor ranges between that of plug flow and complete mixing. Kitai (6) reported that the degree of liquid phase mixing in each compartment of a sieve tray tower fermentor is approximately that of complete mixing. Prokop et al. (2) found that the residence time distribution of fluid in an eight stage fermentation process can be approximately represented by a CSTR's-in-series system.

The tower fermentor considered in this work is assumed to be separated into compartments by perforated plates. Each compartment is assumed to be a completely mixed stage. The media and air flow cocurrently from the bottom to the top stage through sieve holes in the perforated plates. Several experimental studies have reporte that backflow and cell sedimentation appear in this type of tower system. Before going into a detailed investigation of the tower system operating performance, backflow and cell sedimentation will be described.

(2) Backflow rate

The presence of backflow in the tower system is expected. A portion of media in the upper stage will return to the lower stages due to gravitational forces. The amount and variation of backflow is difficult to determine quantitatively; however, the existence of backflow in the tower system has been shown by tracer study (2). The multistage system with backflow has been investigated by Haddad and Wolf (10), Miyauchi and Vermeulen (11) and others (12, 13). For the schematic flow diagram shown in Fig. 1. The over-all volumetric balance for each stage is

$$F^{i-1} + f_{i+1} = F^{i} + f_{i}, \quad i = 2, 3, \ldots, N-1$$

$$F^{0} + f_{2} = F^{1} \qquad \text{for stage 1} \tag{9}$$

and

$$F^{N-1} = F^{N} + f_{N} \qquad \text{for stage N}$$

If the backflow rate is assumed to be constant, the following relationships are obtained.

$$f = f_1 = f_2 = \ldots = f_N \tag{10}$$

$$F^1 = F^2 = \ldots = F^{N-1} \tag{11}$$

and

$$F^0 = F^N \tag{12}$$

In this case, a backflow parameter may be defined as the ratio of backflow rate to the inlet flow rate, which is

$$G' = \frac{f}{F^0} \tag{13}$$

A flow rate profile for an eight stage system with recycle is illustrated in Fig. 2. In this system a feed flow rate of unity is assumed and a recycle flow rate of $r = 0.25$ is considered; therefore $F^0 = 1.25$. Figure 2 shows that when the backflow is assumed to be constant from stage to stage, the forward flow is greater

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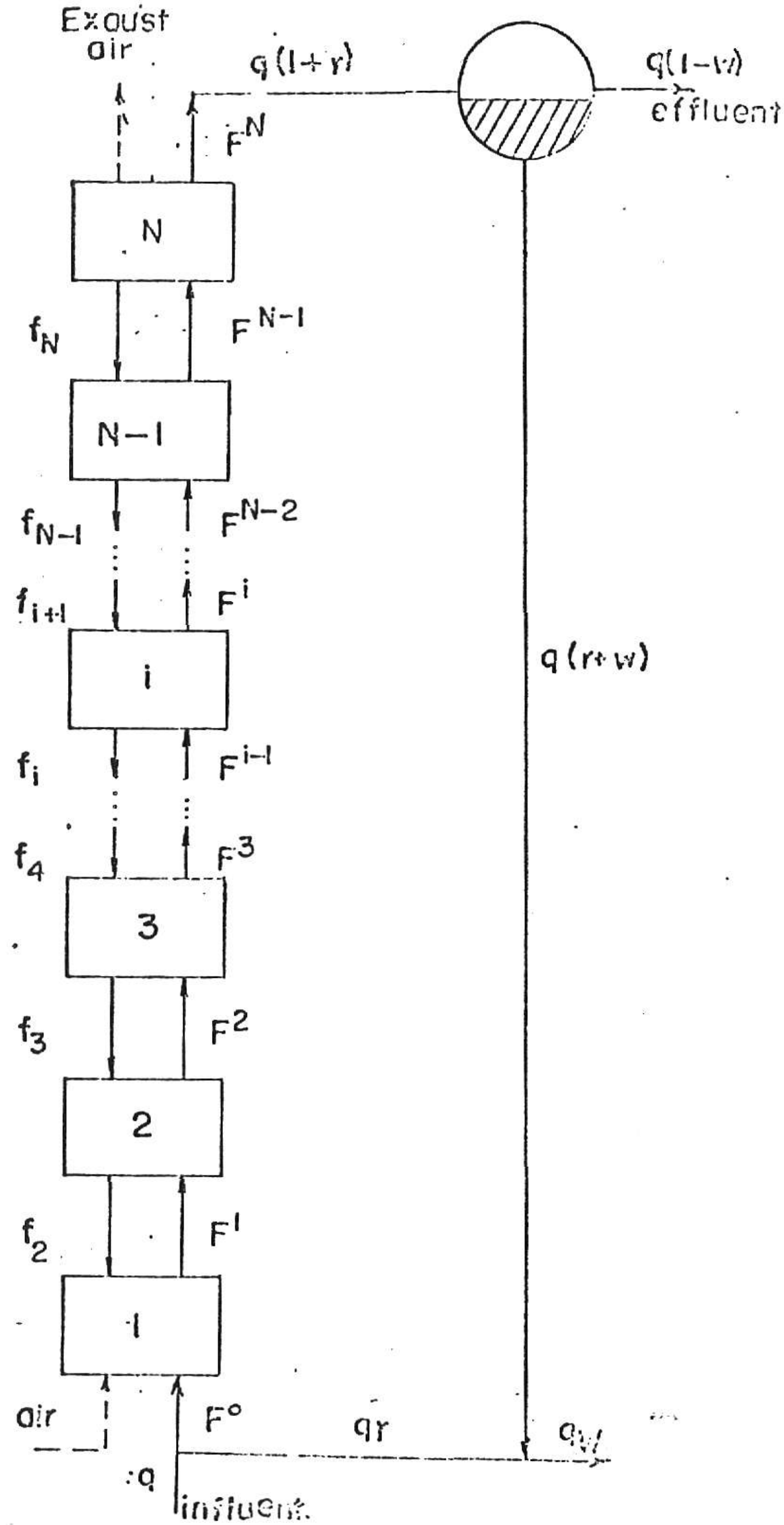

Fig.1. Backflow cell model of a tower type fermenter

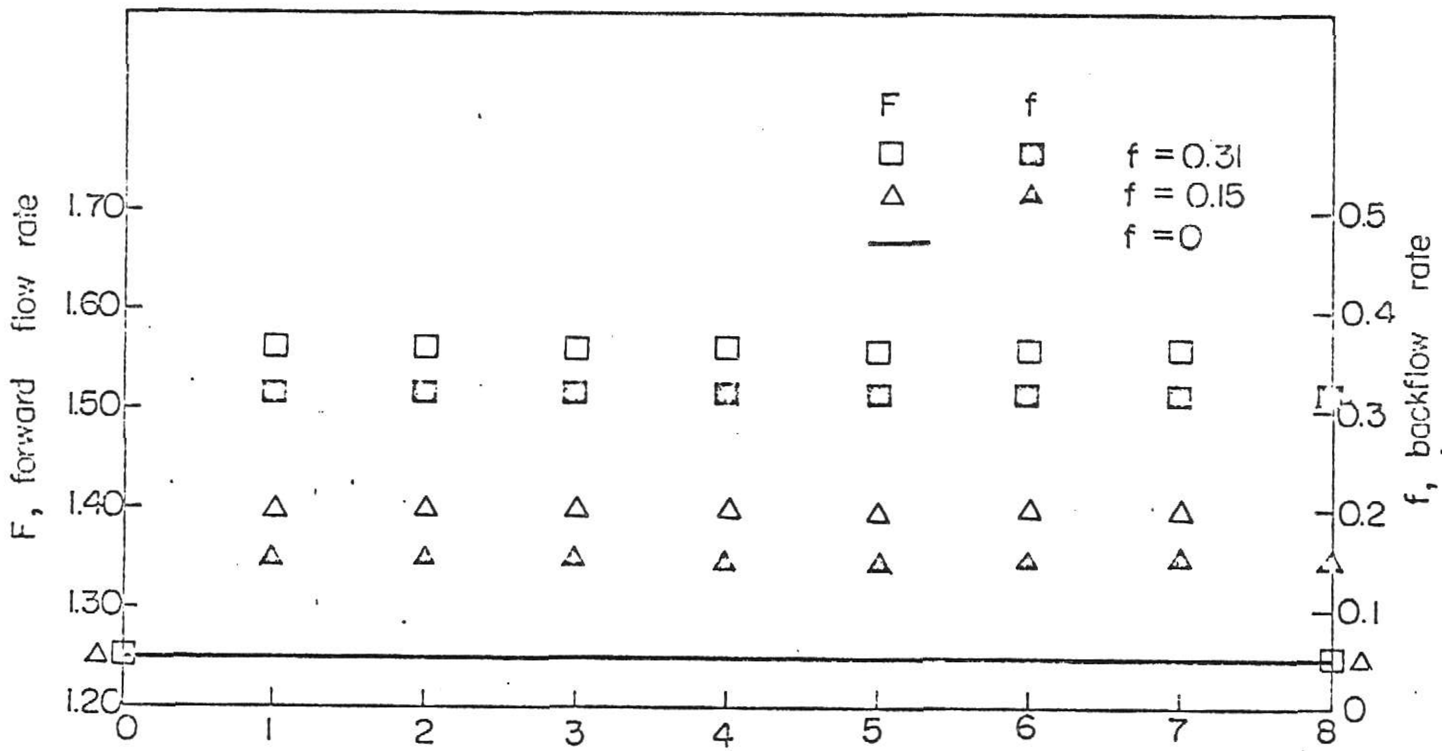

Fig. 2. Flow rate profile for 8 stage system with constant backflow, $F^{\circ} = 1.25$.

than F^0 and F^N for flow between stages.

Instead of assuming the backflow rate to be the same at every stage, we may assume a constant backflow ratio G where

$$G = \frac{f_i}{F_i} = \frac{f_i}{F^i + f_i} \tag{14}$$

and F_i is the total flow from stage i. According to the definition of the backflow ratio, the expression of F^i and f_i can be written as follows;

$$F^i = F_i\ (1-G) \tag{15}$$

$$f_i = F_i G \tag{16}$$

Substituting Equations (15) and (16) into Equation (9), a general expression for F^i and f_i in terms of F^0 can be obtained for systems which are fed at the first stage as

$$F^i = \sum_{j=0}^{N-i} \frac{F^0\ G^j}{(1-G)^j} \tag{17}$$

and

$$f_i = \sum_{j=1}^{N-i+1} \frac{F^0\ G^j}{(1-G)^j} \tag{18}$$

The detailed derivation is shown in Appendix 1.

A flow rate profile of an 8 stage system of constant backflow ratio is illustrated in Fig.3. It can be seen that the forward and backflow rates decrease in the last few stages and

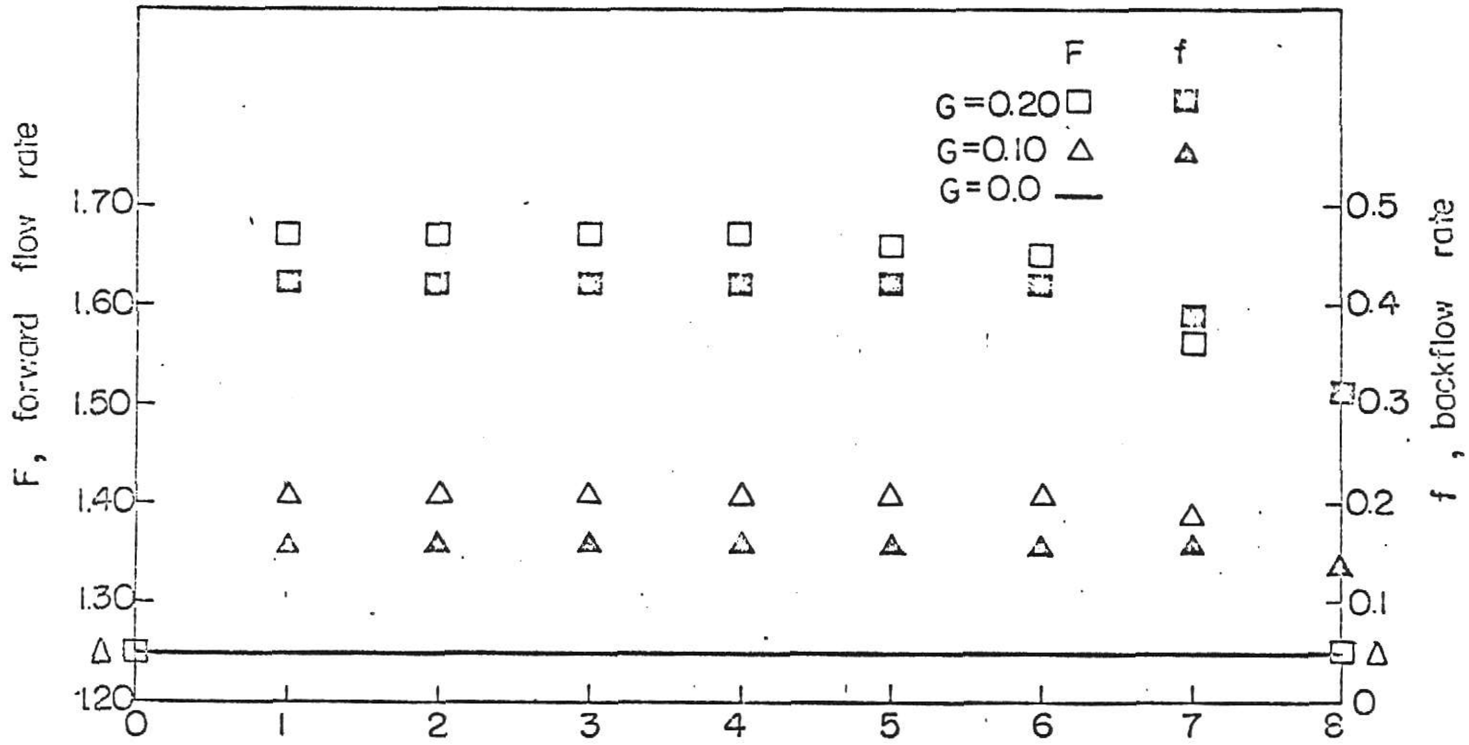

Fig. 3. Flow rate profile for 8 stage system with constant backflow ratio G for $F° = 1.25$.

almost remain constant in the bottom stages. The phenomenon is indicated by observation of Equations (17) and (18) in that the effect of the geometric terms in these equations diminishes when j is increased.

(b) Cell sedimentation

Cell sedimentation is often observed in tower systems as has been reported by Falch (5), Kitai (4), and Prokop et al. (2) Sedimentation occurs because the density of the microbial cells exceeds that of the liquid media. Flocculation of cells may also increase sedimentation rates by decreasing the surface area to mass ratio. One knows that each bacteria contains a net electric charge on its own surface which will keep the bacteria dispersed as long as bacteria remain in the log phase of growth cycle. Flocculation will occur when bacteria are under the declining growth phase.

Consideration of cell sedimentation in the tower system is important, but it is difficult to model because of the many factors which influence cell movement. Kitai (4) reported that the cell distribution is also influenced by foaming; he defined a depletion factor to describe the difference in cell concentration between the forward stream and bulk liquid. Moreover, cell sedimentation is also influenced by the operating conditions and types of cells. The definition of sedimentation factor, δ, used in this study is

$$X^i = \delta \bar{X}^i \tag{19}$$

where X^i is the cell concentration in the ith stage and $\bar{X}^i$ is the cell concentration in the upward stream which is the stream leaving the ith stage. The cell concentration in the backflow stream is assumed to be the same as that in the stage.

The parameter δ is assumed to be constant through out the whole system. This assumption may not be valid; however, the assumption of constant δ introduces only one additional parameter to describe the system. This simplified model will be used to consider the cell sedimentation in tower systems.

(c) Modeling the microbial growth in tower systems

For a tower fermentor in which liquid media is fed to the first stage, the material balances can be written around each stage as follows:

Stage 1:

$$V_1 \frac{dS^1}{dt} = F^0 S^0 - F^1 S^1 + f_2 S^2 - \frac{\mu_1}{Y} X^1 V_1 \qquad (20)$$

$$V_1 \frac{dX^1}{dt} = F^0 X^0 - F^1 \bar{X}^1 + f_2 X^2 + (\mu_1 - k_D) X^1 V_1$$

Stage i, i = 2, 3, ..., N-1

$$V_i \frac{dS^i}{dt} = F^{i-1} S^{i-1} - F^i S^i - f_i S^i + f_{i+1} S^{i+1} - \frac{\mu_i}{Y} X^i V_i \qquad (21)$$

$$V_i \frac{dX^i}{dt} = F^{i-1} \bar{X}^{i-1} - F^i \bar{X}^i - f_i X^i + f_{i+1} X^{i+1} + (\mu_i - k_D) X^i V_i$$

and stage N

$$V_N \frac{dS^N}{dt} = F^{N-1} S^{N-1} - F^N S^N - f_N S^N - \frac{\mu_N}{Y} X^N V_N \qquad (22)$$

$$V_N \frac{dX^N}{dt} = F^{N-1} \bar{X}^{N-1} - F^N \bar{X}^N - f_N X^N + (\mu_N - k_D) X^N V_N$$

In order to make the equations as general as possible dimensionless variables are introduced. These are defined as follows:

$y_1^i = \frac{S^i}{S^f}$, dimensionless substrate concentration.

$y_2^i = \frac{X^i}{YS^f}$, dimensionless organism concentration.

$d_i = \frac{F^i}{\mu_{max} V}$, dimensionless dilution rate.

$b_i = \frac{f_i}{\mu_{max} V}$, dimensionless backflow rate.

$v_i = \frac{V_i}{V}$, volume fraction.

In addition, the saturation constant K_S can be made dimensionless by dividing by S^f; that is

$$K_1 = \frac{K_S}{S^f}$$

Equations (20), (21), and (22) may be rewritten using these dimensionless group; this yields for

Stage 1

$$v_1 \frac{dy_1^1}{dt} = d_0 y_1^0 - d_1 y_1^1 + b_2 y_1^2 - r_1 y_2^1 v_1$$

$$v_1 \frac{dy_2^1}{dt} = d_0 y_2^0 - d_1 \bar{y}_2^1 + b_2 y_2^2 + (r_1 - k_D)\, y_2^1\, v_1 \qquad (23)$$

Stage i, i = 2, 3, ..., N-1

$$v_i \frac{dy_1^i}{dt} = d_{i-1}\, y_1^{i-1} - d_i y_1^i - b_i y_1^i + b_{i+1}\, y_1^{i+1} - r_i y_2^i\, v_i \qquad (24)$$

$$v_i \frac{dy_2^i}{dt} = d_{i-1}\bar{y}_2^{i-1} - d_i\bar{y}_2^i - b_i y_2^i + b_{i+1} y_2^{i+1}$$

$$+ (r_i - k_D) y_2^i v_i$$

and Stage N

$$v_N \frac{dy_1^N}{dt} = d_{N-1}y_1^N - d_N y_1^N - b_N y_1^N - r_N y_2^N v_N$$

$$v_N \frac{dy_2^N}{dt} = d_{N-1}\bar{y}_2^{N-1} - d_N \bar{y}_2^N - b_N y_2^N + (r_N - k_D) y_2^N v_N \tag{25}$$

where $r_i = \dfrac{y_1^i}{K_1 + y_1^i}$

The performance equations at steady state can be obtained by letting the left hand sides of Equations (23), (24), and (25) equal zero. For an N compartment tower system there are 2xN simultaneous nonlinear algebraic equations which must be solved to determine the steady state performance. These are of interest, but solving these 2xN simultaneous nonlinear equations may be tedious and cumbersome work. Computation schemes are presented in a following section.

(d) Limiting cases

(1) CSTR's-in-series model

This basic or simplified flow model can be obtained from the complex flow model of a tower system. If there is no backflow between stages, the tower system may be represented by a CSTR's-in-series model. The material balances for this model can be obtained from equations (23), (24), and (25) by ignoring the backflow terms.

This yields for

Stage 1

$$v_1 \frac{dy_1^1}{dt} = d_0 y_1^0 - d_1 y_1^1 - r_1 y_2^1 v_1$$

$$v_1 \frac{dy_2^1}{dt} = d_0 y_2^0 - d_1 \bar{y}_2^1 + (r_1 - k_D) y_2^1 v_1 \qquad (26)$$

Stage i, i = 2, 3, ..., N-1

$$v_i \frac{dy_1^i}{dt} = d_{i-1} y_1^{i-1} - d_i y_1^i - r_i y_2^i v_i$$

$$v_i \frac{dy_2^i}{dt} = d_{i-1} \bar{y}_2^{i-1} - d_i \bar{y}_2^i + (r_i - k_D) y_2^i v_i \qquad (27)$$

and stage N

$$v_N \frac{dy_1^N}{dt} = d_{N-1} y_1^{N-1} - d_N y_1^N - r_N y_2^N v_N$$

$$v_N \frac{dy_2^1}{dt} = d_{N-1} \bar{y}_2^{N-1} - d_N \bar{y}_2^N + (r_N - k_D) y_2^N v_N \qquad (28)$$

If the system is under steady operation, the left hand sides of Equations (26), (27), and (28), which represent the accummulation of cells and substrate inside each stage, are equal to zero.

If the system is composed of equal volume stages, then

$$v = v_1 = v_2 = \dots = v_N$$

(2) Plug flow model

The plug flow model assumes that the media passes through successive portions of the reactor without mixing. Each element of media stays in the reactor for the same period of detention time.

A plug flow reactor is impossible to build practically, but a theoretical investigation of it can be useful in understanding the characteristics of practical systems which approach plug flow. Increasing the number of stages in a tower system and decreasing the backflow rate to zero allows plug flow conditions to be approached.

The growth of a microbial population in a plug flow reactor is identical to growth in a batch reactor; however, the plug flow system is continuous. The realization of plug flow conditions in the tower system has important implications for fundamental research on growth processes and also for the continuous production of enzymes and other cellular products.

(3) System with recycle

In biological growth processes, the system is composed of several phases as the microorganisms form a separate dispersed phase. In order to retain cells within the system to increase the amount of microbial growth, either a semi-permeable membrane or a sedimentation tank and recycle pump may be used to return the concentrate cells back to the system.

In biological waste treatment, for example, biological metabolism is neglected in modeling the secondary clarifier, where cells are settling down to form a concentrated sludge. The concentration of substrate doesn't change significantly as the media passes through the secondary clarifier. A cell separation parameter, β, is assumed and used in the cell balance

$$q(1 + r) X^N = q(r + w) \beta X^N + q(1 - w) X^e$$

around the clarifier which is shown in the schematic flow diagram of Fig. 1. The sludge is concentrated to a bottoms concentration βX^N and a portion qw of the bottoms flow is wasted.

The cell balance around the mixing point where the influent enters is

$$q(1 + r) X^0 = \beta rqX^N \tag{29}$$

and for substrate it is

$$q(1 + r) S^0 = rqS^N + q\bar{S}^f \tag{30}$$

where r is the recycle ratio and a sterile feed is assumed in this continuous feed-back system.

2-4 COMPUTATIONAL SCHEMES

The steady state concentration of cells and substrate in each compartment of the tower system is important. These concentrations can be obtained by solving the steady state simultaneous nonlinear algebraic equations, Equations (23), (24), and (25), for an N stage tower system. In this chapter, two computational schemes are proposed that can be used.

(1) Linearization scheme (14)

The difficulty of solving these nonlinear simultaneous equations is due to the presence of nonlinear terms. The scheme proposed here is to linearize these nonlinear terms to obtain linear simultaneous algebraic equations. To solve these linear simultaneous algebraic equations, an IBM scientific subroutine SIMQ is available. An iteration procedure is employed in this proposed scheme in order to obtain an accurate solution. The procedure is as follows:

(1) Assume a set of initial solutions which is used as an expansion point in a Taylor series expansion, $(y_1^i, \bar{y}_2^i)_I$, $i = 1, 2, \ldots, N$.

(2) Expand the nonlinear terms around the initial solutions $(y_1^i, \bar{y}_2^i)_I$ using a Taylor series expansion and ignore the second order and higher order terms in the expansion.

(3) Employ IBM SIMQ subroutine to solve these linear simultaneous equations.

(4) Compare the new solutions with the initial solutions. If the value of

$$\sum_{i=1}^{N} \left\{ \left|(y_1^i)_I - y_1^i\right| + \left|(\bar{y}_2^i)_I - \bar{y}_2^i\right| \right\}$$

is less than a satisfactory error value the computation is terminated. Otherwise, iteration has to be employed as follows:

(5) Adapt the new solutions as the new initial solutions. Then return to step (2). This process is repeated until step (4) is satisfied.

The procedure presented here works very well to solve this type of nonlinear simultaneous algebraic equations provided a reasonably good initial guess can be made.

(2) Regula Falsi (15)

The regula falsi one dimensional search technique may be used to find numerical values of the concentrations if the endogenous phase is neglected and there is no recycle. An over-all material balance at steady state for a system with sterile feed gives

$$\bar{y}_2^N = 1 - y_1^N \tag{31}$$

This equation together with Equation (19) in the form

$$y_2^i = \delta \bar{y}_2^i, \qquad i = 1, 2, \ldots, N \tag{32}$$

may be used with Equations (23), (24), and (25) in their steady state form as follows:

(1) Assume substrate concentration of stage N, y_1^N.

(2) Obtain cell concentration in exit, $\bar{y}_2^N$, from Equation (31) and cell concentrations in stage N from Equation (32).

(3) Compute y_1^{N-1} and $\bar{y}_2^{N-1}$ using Equation (25).

(4) Using Equations (24) and (32), compute y_2^{N-1}, y_1^{N-2}, $\bar{y}_2^{N-2}$, $\bar{y}_2^{N-2}$, ..., y_1^1, $\bar{y}_2^1$, and y_2^1.

(5) Calculate y_1^0 from Equation (23) in the from

$$d_0 y_1^0 - d_1 y_1^1 + b_2 y_1^2 - r_1 y_2^1 v_1 = 0 \tag{33}$$

(6) Check to see if $y_1^0 - 1.0 = 0$

(7) Use the regula falsi method to assume new values of y_1^N until a value of y_1^N which gives a value of y_1^0 sufficiently close to unity is found.

2-5 STEADY STATE PERFORMANCE

Employing these two computational schemes, the steady state behavior of the tower system is investigated for several combinations of the feed geometry, number of stages, and hydraulic parameters. The conventional tower system, which is fed at the bottom (Case 1) and the system proposed by Prokop et al. (2), which is fed at the second stage (Case 2), are the two feed geometries that are considered. The effects of cell sedimentation and backflow rate on the cell concentration profile for 2, 4, systems are investigated for various values of b, δ, and $K_1 = 0.1$ in all cases. Some of the results are illustrated in Figs. 4 through 11 where the cell concentration distribution in the tower system is plotted as a function of dilution rate.

The effects of backflow on the system performance are shown in Figs. 4, 5, and 8 for 2 and 4 stage systems, respectively. The results show that increasing the backflow rate increases the operating range. This is because increasing the backflow rate increases the degree of mixing which will produce a wider operating range.

Washout occurs when the cell concentration decreases to zero. Since the operating dilution rate must be less than the washout dilution rate, increasing the washout dilution rate usually widens the operating range and enables one to increase productivity. The presence of cell sedimentation increases the washout dilution rate and thus, allows a wider operating range and a higher output of cells. A higher concentration of cells exits in the bulk liquid when sedimentation is present, but it is the increased dilution rate which allows productivity to be increased. Increasing the concentration of cells in the bulk liquid allows the microbial growth in each compartment to be increased by increasing the system dilution rate.

For the extreme case of a very small dilution rate with a relatively large backflow rate the cell concentration distribution obeys a geometric relationship of the form

$$\bar{y}_2^{N-1} = \delta\, \bar{y}_2^{N}$$

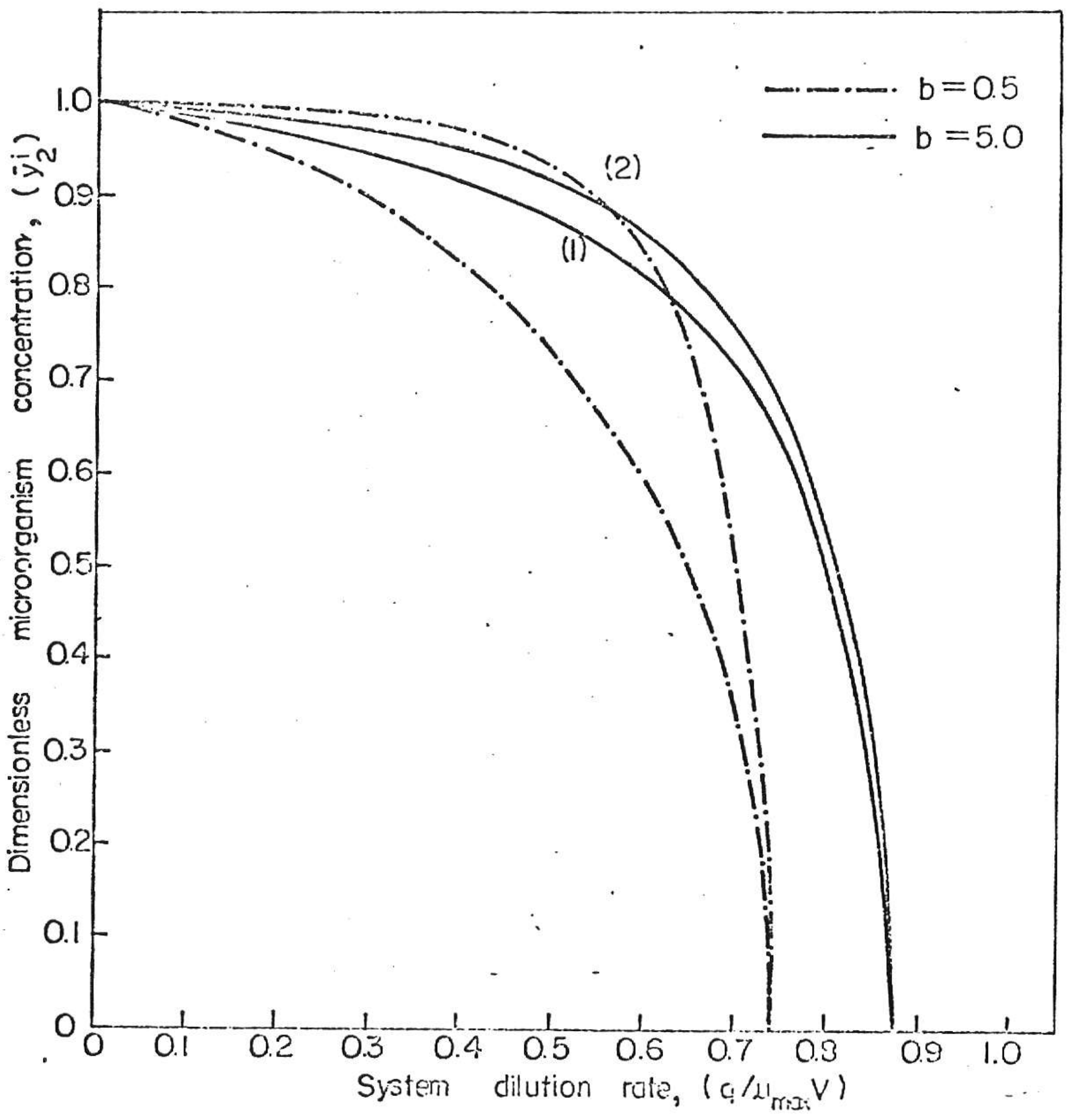

Fig. 4. Effect of backflow rate on the steady state microorganism concentration for a 2 stage system with $\delta = 1.0$ (Case 1).

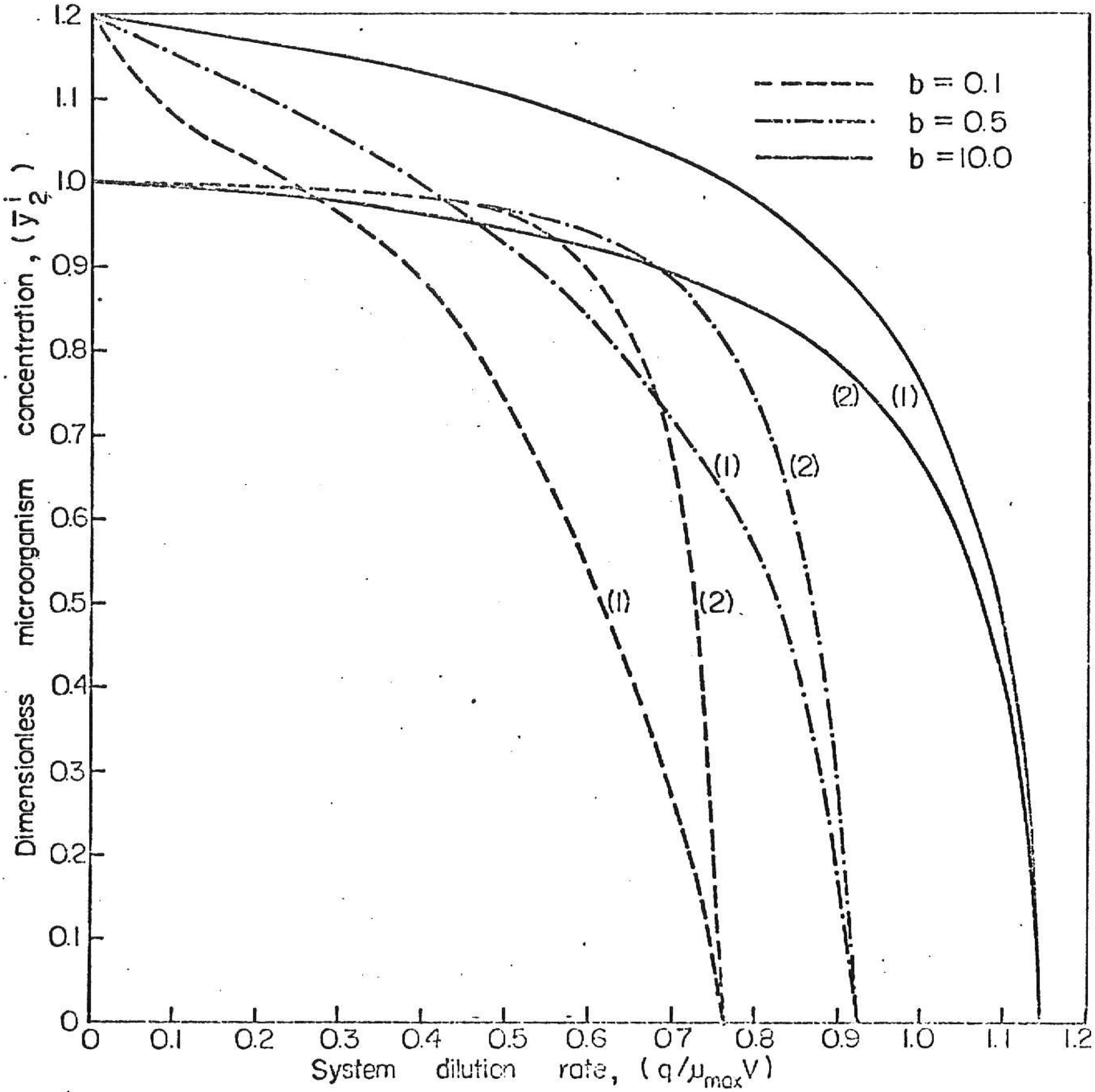

Fig. 5. Effect of backflow rate on the steady state dimensionless microorganism concentration for a 2 stage system with $S = 1.2$ (Case I).

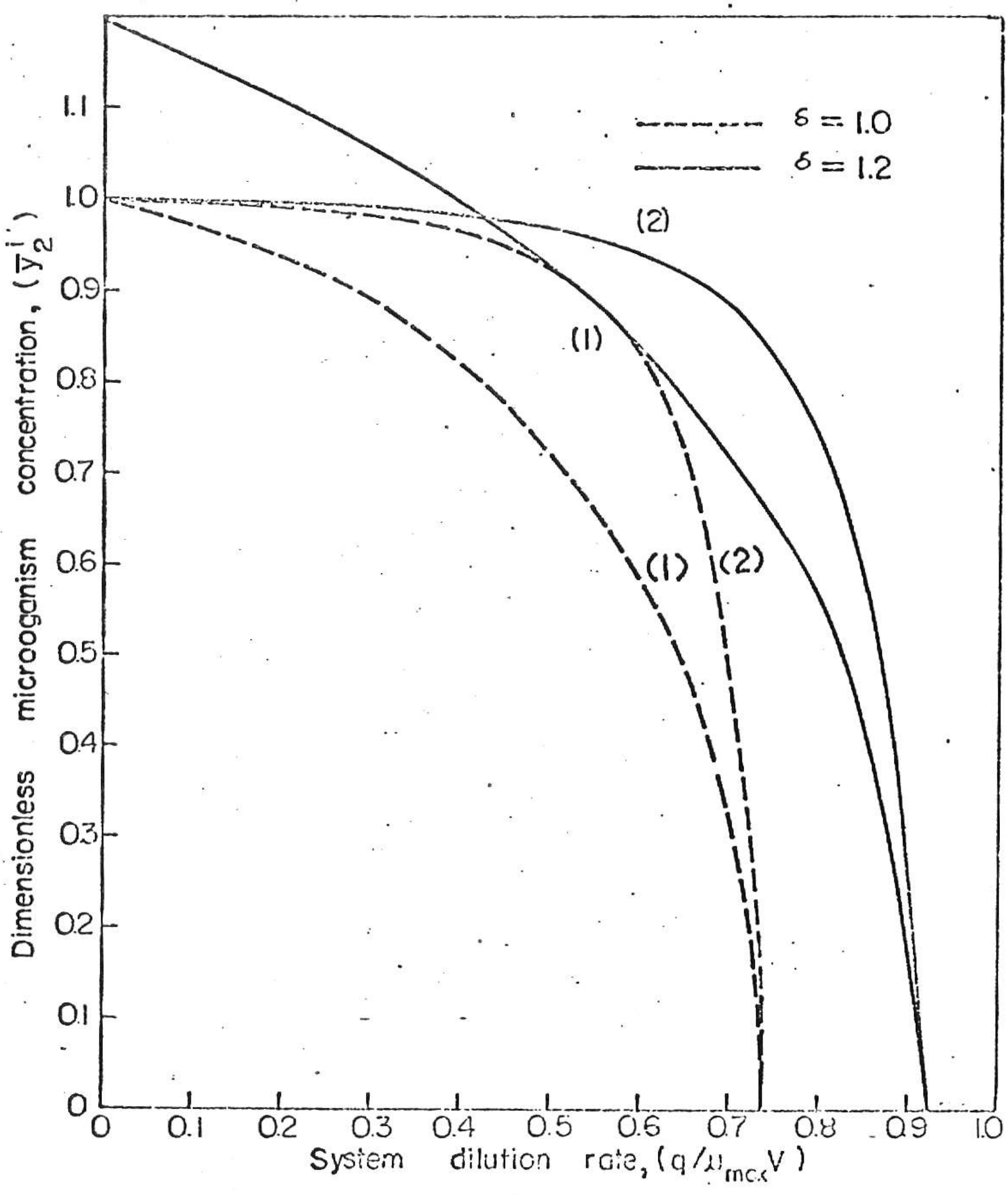

Fig. 6. Effect of sedimentation on dimensionless microorganism concentration for a 2 stage system with b = 0.5 (Case I).

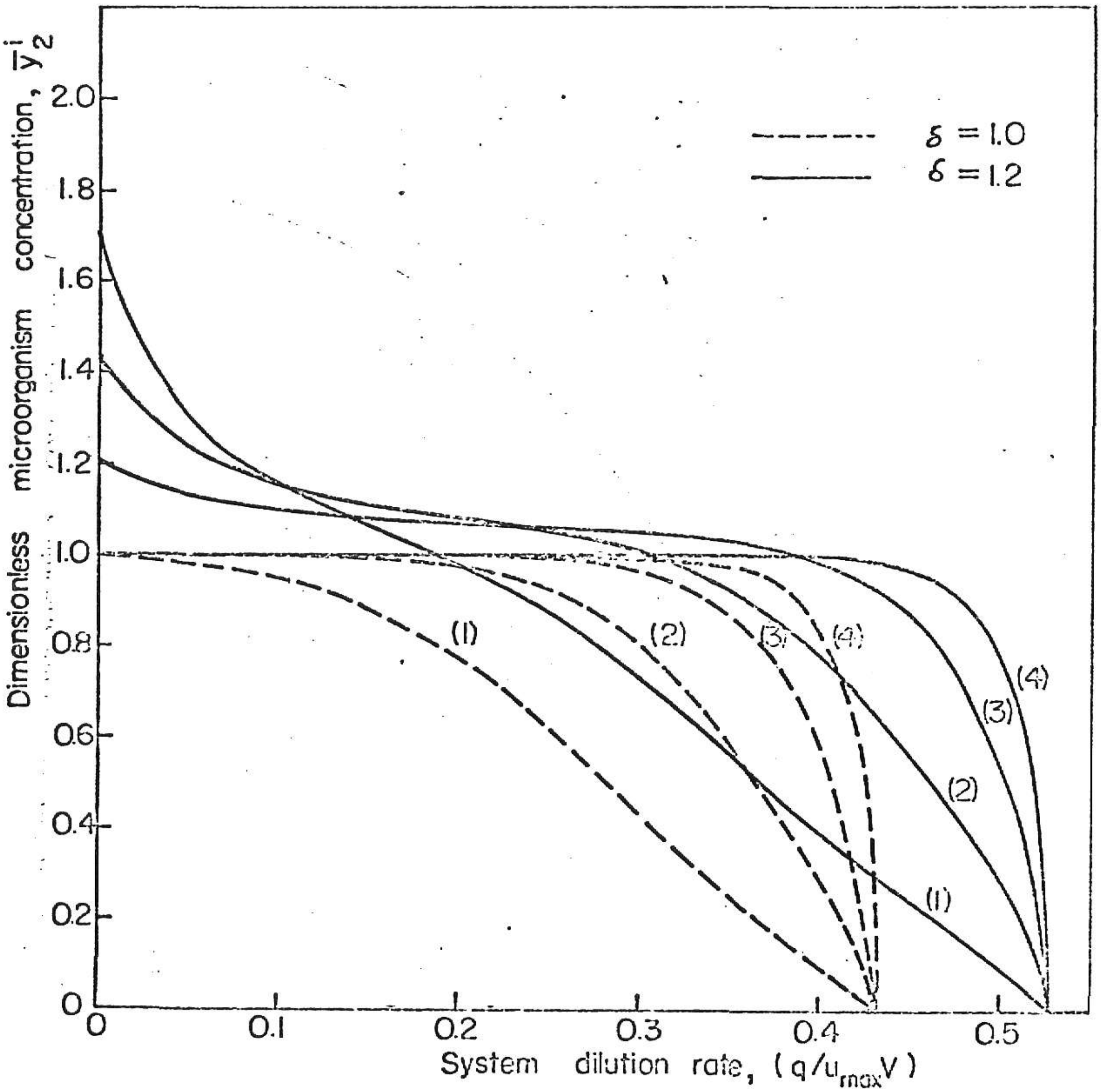

Fig. 7. Effect of sedimentation on the dimensionless microorganism concentration for a 4 stage system with b = 0.1 (Case 1).

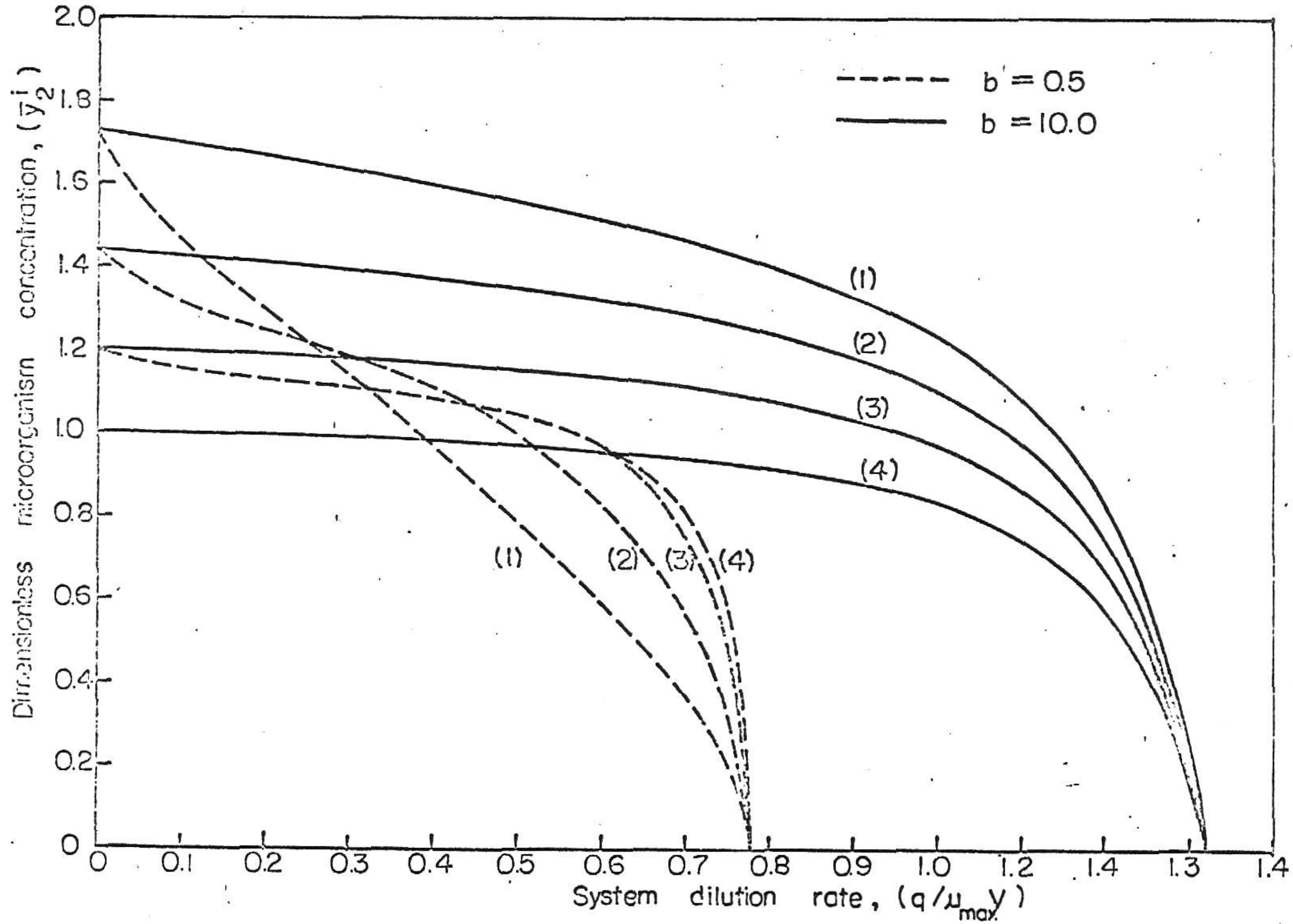

Fig. 8. Effect of backflow rate on the dimensionless microorganism concentration for a 4 stage system with $\delta = 1.2$ (Case 1).

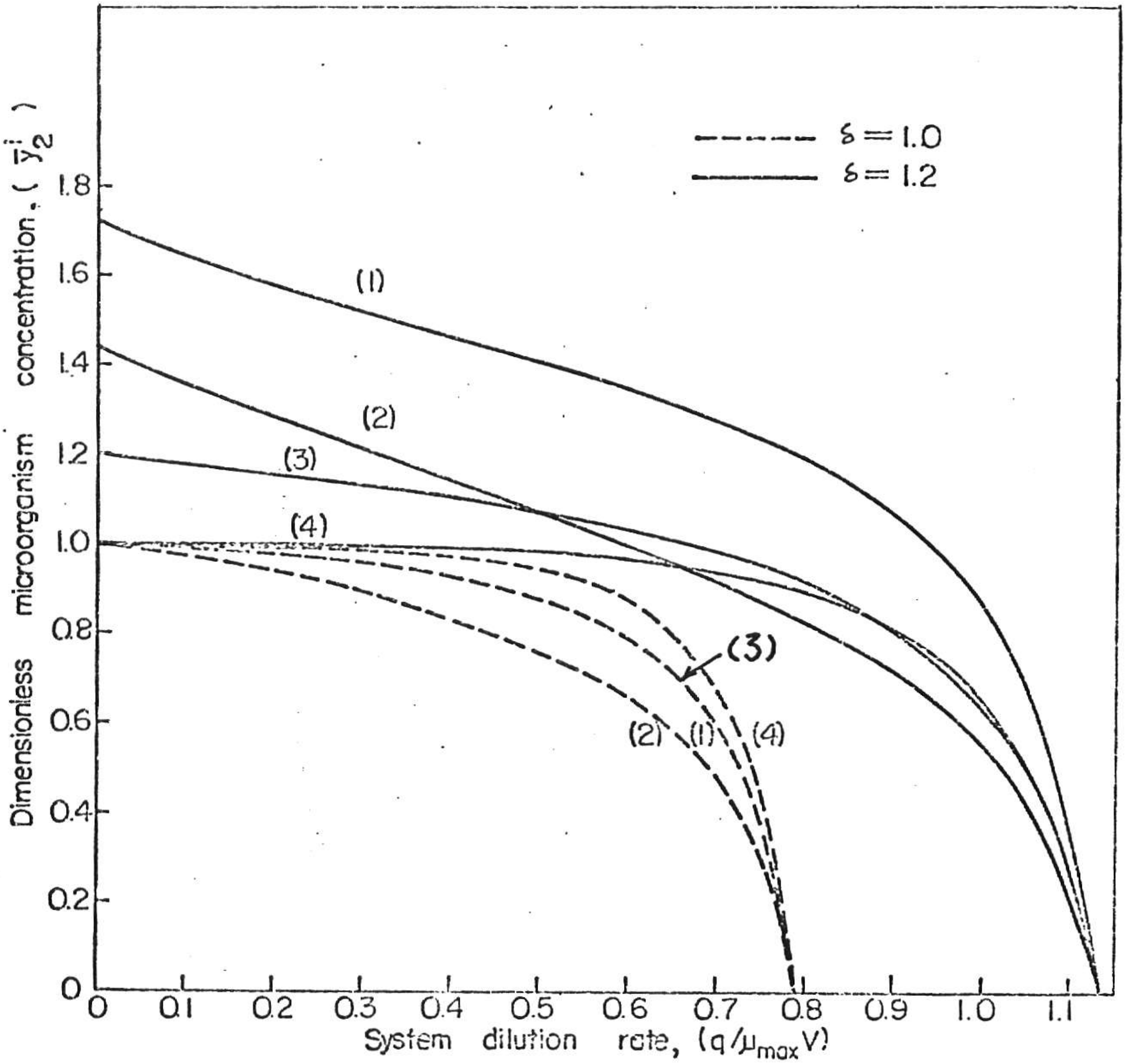

Fig. 9. Effect of sedimentation on the dimensionless microorganism concentration for a 4 stage system with b = 1.0 (Case 2).

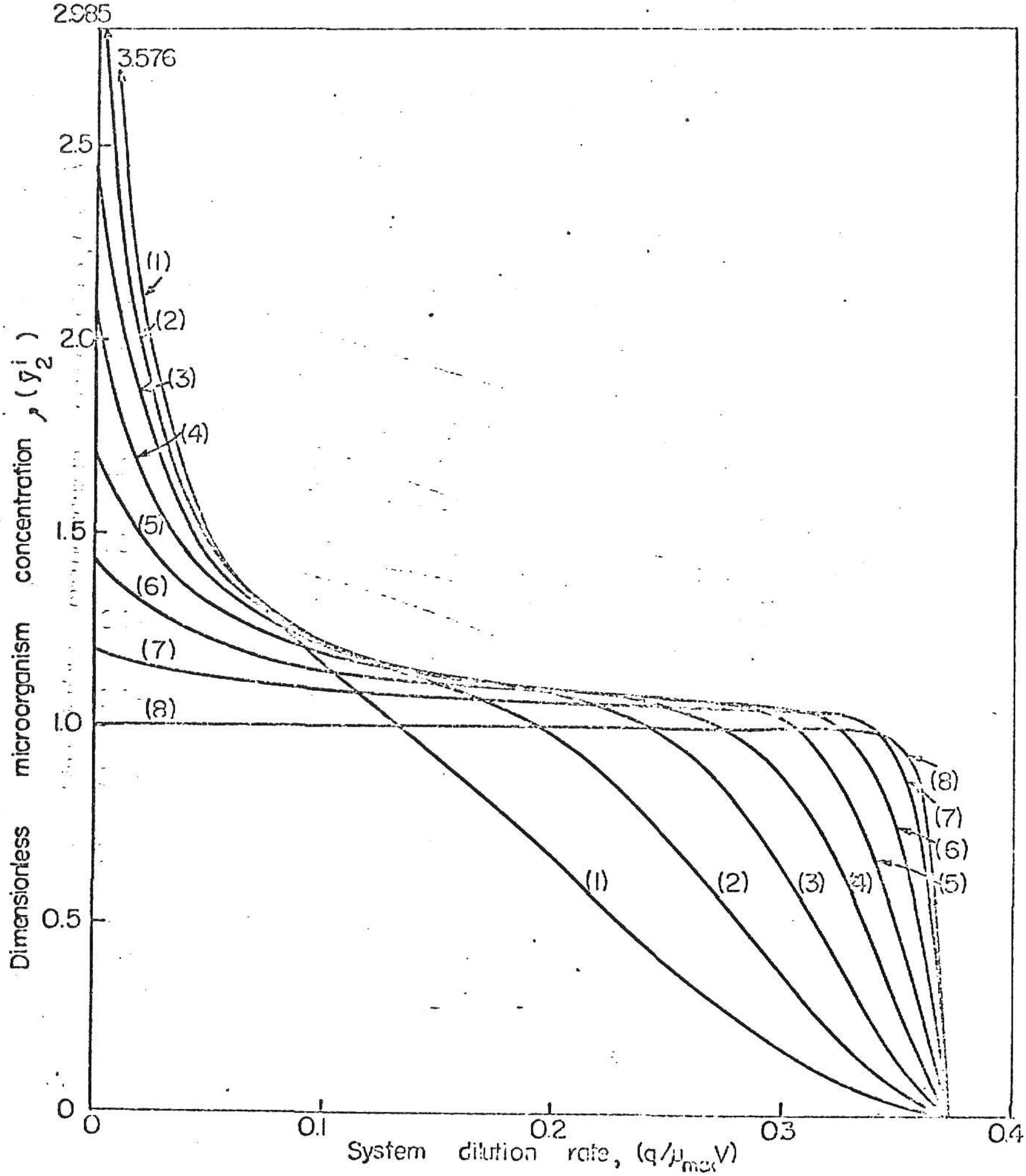

Fig. 10. Dimensionless microorganism concentration for an 8 stage system with b = 1.0 (Case I).

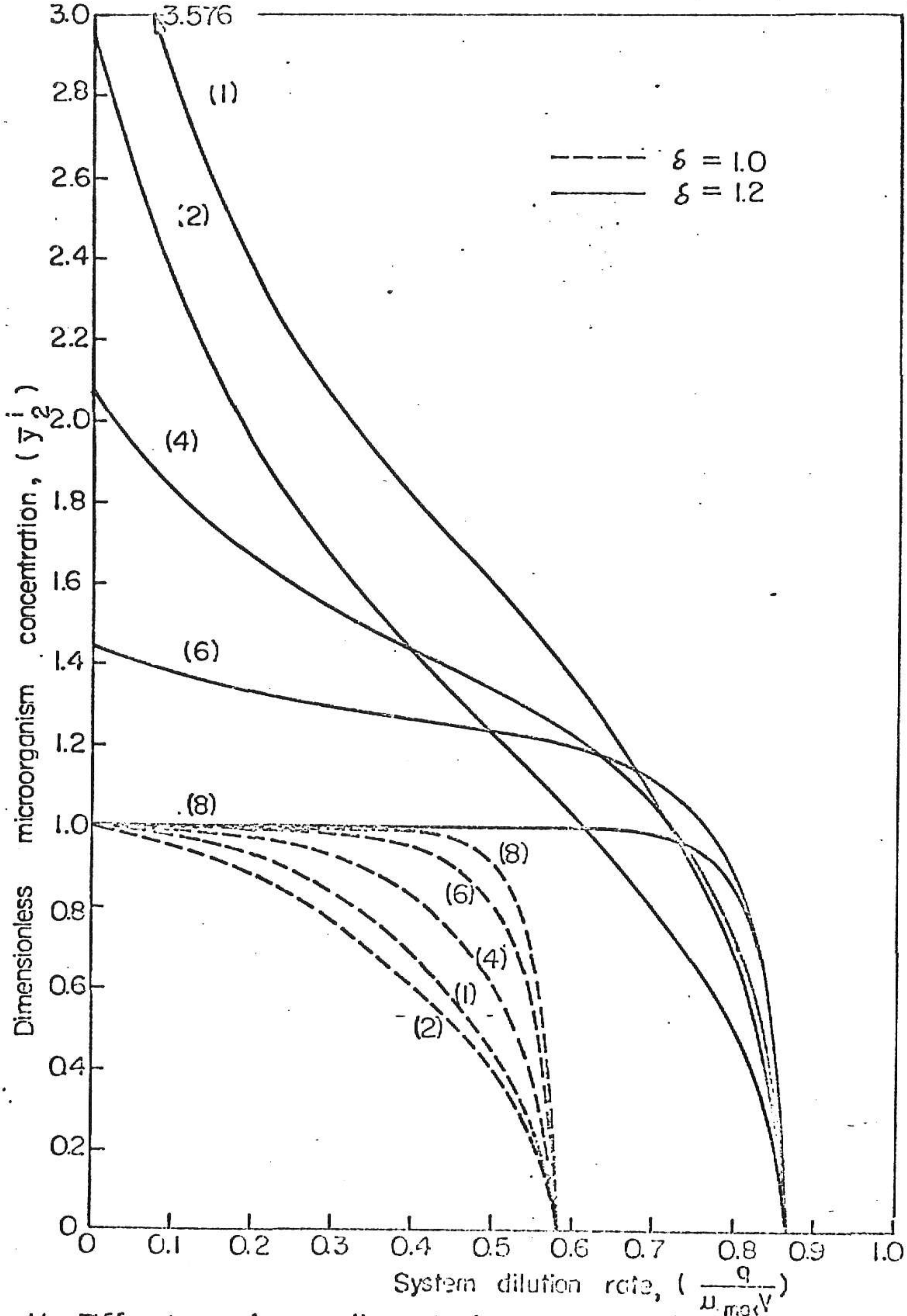

Fig. 11. Effect of sedimentation on the dimensionless microorganism concentration for an 8 stage system with b = 1.0 (Case 2).

$$\bar{y}_2^{N-2} = (\delta)^2\ \bar{y}_2^N$$
$$\vdots \qquad (34)$$
$$\bar{y}_2^1 = (\delta)^{N-1}\ \bar{y}_2^N$$

This relationship can be derived from the cell mass balance equations (the detailed analysis is given in Appendix II). Since the backflow rate is assumed to be constant, as the dilution rate approaches zero in Figs. 5 through 11, the dimensionless backflow rate becomes much larger than the dilution rate and the values given in Equation (34) as shown in Figs. 5 through 11. When the dimensionless backflow rate b is much greater than the system dilution rate the largest cell concentrations are in the bottom stage as shown by the solid lines of Fig. 5, for example.

When the backflow rate is small relative to the dilution rate, the largest cell concentration will occur in the top stage as in the system with no sedimentation as shown by the dashed lines in Fig. 5, for example.

By comparison of the behavior of Cases 1 and 2 (see Figs. 10 and 11), it can be seen that a system with feed at the second stage allows the system to be operated at a much wider range of dilution rates. The cell concentration at the same dilution rate is larger for Case 2 than for Case 1 for the lower stages. The first stage of Case 2 which is fed by backflow continuously inoculates the second stage. The results shown in Figs. 9 and 11 for Case 2 indicate that the cell concentration in the first stage is always greater than that in the second stage even when there is no cell sedimentation. For Case 2 there is a critical backflow rate at which the cells will never be washed out if the backflow rate is less than this critical value. A detailed analysis of this and washout phenomena in general will be presented in Chapter 3.

Increasing the number of stages affects the range of operating conditions that can be attained in the tower system. When the backflow rate is small, increasing the number of stages reduces the range of dilution rates which give stable growth conditions, especially for Case 1. This disadvantage can be overcome by increasing backflow and cell sedimentation parameters. Moreover, when the system is composed of a large number of stages a range of residence time distributions from complete mixing to that of the N-CSTR's-in-series system (for a system with N stages) is attainable by proper regulation of the backflow parameter. By feeding media to the second stage of a tower system with a large number of stages, suitable control of both backflow rate and system dilution rate enables one to continuously obtain almost any desired physiological growth condition.

2-6 SUMMARY

A mathematical model that can be used to describe cell growth in a tower fermentation system with backflow and cell sedimentation is presented. Two computational procedures to determine steady state performance are described and some results are obtained for 2, 4 and 8 stage tower systems. The effects of feed geometry, backflow rate and cell sedimentation on steady state behavior are investigated. The results show that control of backflow, and cell sedimentation within the tower together with control of feed flow rate and feed geometry will allow a wide range of operating conditions to be realized in a tower system.

NOMENCLATURE

b_i = Dimensionless backflow rate.

d_i = Dimensionless dilution rate, $(\frac{F^i}{\mu_{max} V})$.

G = Backflow ratio, $(\frac{f_i}{F_i})$.

F^i = The upward flow stream leaving stage i.

F_i = The total flow rate leaving stage i.

f_i = The backflow rate coming from stage i.

K_S = The concentration of organic at which the specific growth rate observed is one half the maximum value, mg/liter.

K_1 = Dimensionless saturation constant, $(\frac{K_S}{S^f})$.

k_D = Specific endogenous microbial attrition rate, hr^{-1}.

S^i = The concentration of organics in stage i, mg/liter.

S^0 = The concentration of organics in the stream coming into stage 1, mg/liter.

S^f = The concentration of organics in the influent, mg/liter.

X^i = The concentration of cell in stage i, mg/liter.

$\bar{X}^i$ = The concentration of cell in the stream leaving stage i, mg/liter.

y_1^i = The dimensionless concentration of substrate in stage i, $(\frac{S^i}{S^f})$.

y_2^i = The dimensionless cell concentration in stage i, $(\frac{X^i}{Y S^f})$.

$\bar{y}_2^i$ = The dimensionless cell concentration in the stream leaving stage i, $(\frac{\bar{X}^i}{Y S^f})$.

V_i = The liquid volume of stage i.

v_i = Volume fraction, $(\frac{V_i}{V})$.

β = Secondary clarifier parameter.

δ = Sedimentation parameter.

μ = Specific growth rate.

μ_{max} = Maximum specific growth rate when the organic concentration is not limiting the rate of growth, hr^{-1}.

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APPENDIX I

DETERMINATION OF FLOW RATE IN EACH STREAM FOR THE SYSTEM WITH CONSTANT BACKFLOW RATIO

The schematic diagram of the backflow cell model is shown in Fig. 1. In the case of a constant backflow ratio, the defining relation between the flows leaving a stage is given in Equation (14). To determine quantitatively the flow rate in each stream for this system, the volumetric flow rate balances around each stage are needed. These are

$$F^0 + f_2 = F^1 \tag{A-1}$$

$$F^{i-1} + f_{i+1} = F^i + f_i \ , \ i = 2, 3, \ldots, N-1 \tag{A-2}$$

and

$$F^{N-1} = F^N + f_N \tag{A-3}$$

Equation (14) can be used to obtain

$$F^i = F_i(1-G), \ i = 2, 3, \ldots, N \tag{A-4}$$

and

$$f_i = F_i G = \frac{F^i G}{1-G} \ , \ i = 2, 3, \ldots, N \tag{A-5}$$

An overall balance gives

$$F^N = F^0 \tag{A-6}$$

The total flow from stage N is obtained by rearranging Equation (A-4) to obtain

$$F_N = \frac{F^N}{1-G} = \frac{F^0}{1-G} \tag{A-7}$$

Substituting Equation (A-7) into Equation (A-7) yields

$$f_N = \frac{F^N G}{1-G} = \frac{F^0 G}{1-G} \tag{A-8}$$

Substituting Equations (A-6) and (A-8) into Equation (A-3) gives

$$F^{N-1} = F^0 + \frac{F^0 G}{1-G} \tag{A-9}$$

Equations (A-5) and (A-9) can be used to obtain

$$f_{N-1} = \frac{F^0 G}{1-G} + F^0\left(\frac{G}{1-G}\right)^2 \tag{A-10}$$

Equation (A-2) for i = N-1 is

$$F^{N-2} = F^{N-1} + f_{N-1} - f_N \tag{A-11}$$

Substituting Equations (A-8), (A-9), and (A-10) into Equation (A-11) gives

$$F^{N-2} = F^0\left[1 + \frac{G}{1-G} + \left(\frac{G}{1-G}\right)^2\right] \tag{A-12}$$

Equations (A-5) and (A-12) yield

$$f_{N-2} = F^0\left[\frac{G}{1-G} + \left(\frac{G}{1-G}\right)^2 + \left(\frac{G}{1-G}\right)^3\right] \tag{A-13}$$

The general form

$$F^{N-i} = \sum_{j=0}^{i} \frac{F^0 G^j}{(1-G)^j} \tag{A-14}$$

and

$$f_{N-i} = \sum_{j=1}^{i+1} \frac{F^0 G^j}{(1-G)^j} \tag{A-15}$$

can be verified by induction. Equation (A-2) may be written in the form

$$F^{N-i-1} = F^{N-i} + f_{N-i} - f_{N-i+1}$$

$$= \sum_{j=0}^{i} \frac{F^0 G^j}{(1-G)^j} + \sum_{j=1}^{i+1} \frac{F^0 G^j}{(1-G)^j} - \sum_{j=1}^{i} \frac{F^0 G^j}{(1-G)^j}$$

$$= \sum_{j=0}^{i+1} \frac{F^0 G^j}{(1-G)^j} \qquad \text{(A-16)}$$

From Equations (A-5) and (A-16), one obtains

$$f_{N-i-1} = \sum_{j=1}^{i+2} \frac{F^0 G^j}{(1-G)^j} \qquad \text{(A-17)}$$

These results show that Equations (A-14) and (A-15) are valid for i = 2, 3, ..., N-1. It is easy to show that Equations (A-14) and (A-15) may also be written in the form used to express Equations (17) and (18).

APPENDIX II

DETERMINATION OF CELL DISTRIBUTION IN THE CASE OF VERY SMALL DILUTION RATE

The cell mass balances at steady state around each stage can be obtained from Equations (23), (24) and (25) by letting the accumulation term equal zero. In the particular case of an equal volume system with constant backflow rate

$$d_i = d + b \;, \; i = 1, 2, \ldots, N \tag{A-1}$$

where d is the dimensionless system dilution rate, which is defined as

$$d = \frac{F^0}{\mu_{max} V}$$

If the endogenous metabolism rate is neglected, Equations (23), (24), and (25) can be rewritten as follows:

$$N\delta b\bar{y}_2^{\,2} - N(d + b)\,\bar{y}_2^{\,1} + r_1 y_2^1 = 0 \tag{A-2}$$

$$N(d + b)\bar{y}_2^{\,i-1} + Nb\delta\bar{y}_2^{\,i+1} - N(d + b)\bar{y}_2^{\,i} - N\delta b\bar{y}_2^{\,i} + r_i y_2^i = 0 \;,$$

$$i = 2, 3, \ldots, N-1 \tag{A-3}$$

and

$$N(d + b)\bar{y}_2^{\,N-1} - N(d + \delta b)\bar{y}_2^{\,N} + r_N y_2^N = 0 \tag{A-4}$$

If $d \to 0$, Equation (A-4) becomes

$$\bar{y}_2^{\,N-1} = \delta\,\bar{y}_2^{\,N} - \frac{r_N y_2^N}{N\;b} \tag{A-5}$$

The term $\frac{r_N y_2^N}{Nb}$ is small if b is large compared to d, because

$y_1^N \rightarrow 0$ as $d \rightarrow 0$. Therefore, one can obtain Equation (A-6).

$$\bar{y}_2^{N-1} = \delta \bar{y}_2^{N} \tag{A-6}$$

Following a procedure similar to that used to obtain Equation (A-5), Equation (A-3) reduces to

$$\bar{y}_2^{i-1} + \delta \bar{y}_2^{i+1} - \bar{y}_2^{i} - \delta \bar{y}_2^{i} = 0 , \qquad i = 2, 3, \ldots, N-1 \tag{A-7}$$

If i = N-1, substituting Equation (A-6) into Equation (A-7) gives

$$\bar{y}_2^{N-2} = \delta \bar{y}_2^{N-1} \tag{A-8}$$

Successive substitution of Equation (A-8) into Equation (A-7) provides the desired result

$$\bar{y}^{\,i-1} = \delta \bar{y}^{\,i} , \qquad i = 2, 3, \ldots, N \tag{A-9}$$

Chapter 3

MODELING AND ANALYSIS OF TOWER FERMENTATION PROCESSES: II. WASHOUT BEHAVIOR

3-1 INTRODUCTION

Recently, several workers (1, 2, 3, 4, and 5) have investigated the performance of tower fermentation processes. Considerable research has been reported; however, very little has been published on prediction of fermentor performance. In this work a steady state model incorporating backflow and cell sedimentation is investigated to determine the critical conditions which lead to washout of the microbial population.

The tower fermentation system investigated in this work is assumed to be composed of a number of stages or compartments separated by sieve trays with the compartments arranged vertically above one another. Experimental research has shown that backflow of culture media often is present (1, 3, and 4) and that the extent of backflow is influenced by the design of the sieve trays (3, 4). Cell sedimentation or differences in mean residence time of cells and liquid have also been observed (1, 3). It is, therefore, desirable that flow models which consider these nonidealities be used to investigate system performance.

A fermentation process is which a sterile feed is employed is subject to washout; that is, if the steady state feed flow rate is too large all cells will be washed from the system because the system dilution rate exceeds the maximum specific growth rate of the cells. Mathematically, at washout the cell concentration is zero and the substrate concentration is equal to that of the feed. A detailed discussion of washout is presented elsewhere (6).

Herbert (7, 8, 9) discussed the washout dilution rate of several different flow models such as the single completely mixed continuous fermentor (chemostat), the multistage completely mixed tanks in series system, and the plug flow continuous system. Mathematical analysis and detailed comments are given in his papers. Powell (10) investigated the washout of systems with feedback and presented the comparisons of the multistage system with the plug flow system. The effect of mixing on the washout and steady state performance of continuous cultures is reported by Fan et al. (11). A detailed analysis of washout for flow models which can be used to model the tower system with backflow and sedimentation of cells, has not been reported. The present study is concerned with the prediction of washout conditions for a number of tower fermentor feed geometries under various conditions of backflow and cell sedimentation.

The flow system of the present study and kinetic model will be briefly described in Section 3-2, where a complex flow model of a tower system is proposed and the kinetic model of Monod's equation is assumed. The washout criteria will be presented in Section 3-3, in which the concept of washout will be given. The characterisic equation of washout is derived in Section 3-4 and the effect of sedimentation and feed geometry on the characteristic equation are also presented.

Finally, the results are presented and the effects of kinetic parameters, backflow, sedimentation, and feed geometry on the critical washout dilution rate are examined.

3-2 SYSTEM DESCRIPTION

(a) Flow model

The tower system under consideration is assumed to be separated into several compartments by perforated plates (sieve trays) whcih have many holes. Although most of the experimental work has been with equal volume compartments, the models developed here are not restricted to be of equal volume. A schematic diagram of a general flow model of the tower fermentation system is shown in Fig. 1. The generality of this flow diagrm is that feed and withdrawal streams may appear anywhere in the system. Here, f_i, is the backflow, from stage i to stage i-1. q_i^0 represents the feed stream to stage i and q_i^e represents the exit stream from stage i. Feed may be introduced at any stage and similarly, products may be withdrawn from any stage. The backflow, f_i, depends on the area of sieve hole, the number of sieve holes, hydrodynamic pressure differences, and foam formation. The upward flow is denoted by $(q_i + f_{i+1})$ where q_i is the net upward flow from stage i. For the general case (Fig. 1), the flow model for stage 2 to N-1 has the same general form and each compartment may be considered as a homogeneous stage. The volumetric balance for the ith homogeneous stage can be written as follows

$$q_i^0 + (q_{i-1} + f_i) + f_{i+1} = q_i^e + (q_i + f_{i+1}) + f_i \ , \qquad (1)$$

$$i = 2, 3, \ldots, N-1$$

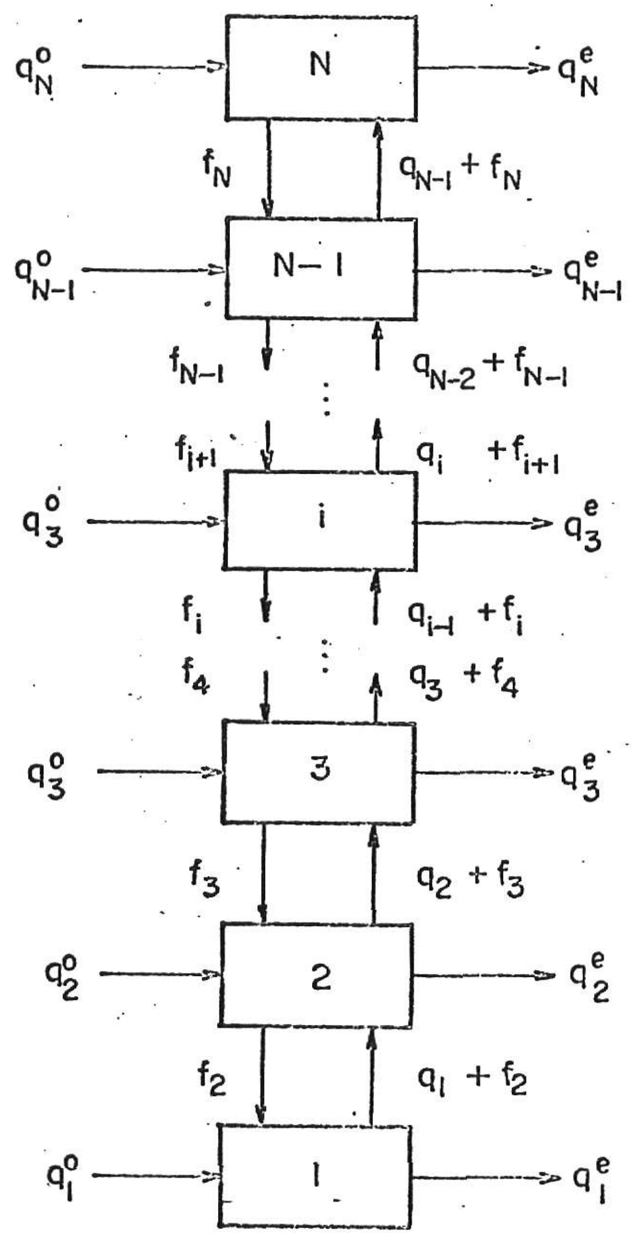

Fig. 1. Backflow cell model. (General case)

For the first and last compartments (heterogeneous stages) we may write, respectively,

$$q_1^0 + f_2 = q_1^e + (q_1 + f_2) \tag{2}$$

and

$$q_N^0 + (q_{N-1} + f_N) = q_N^e + f_N \tag{3}$$

The subscript indicates the stage number, while superscripts 0 and e denote feed and withdrawal streams, respectively. These equations may be employed regardless of whether there is feed and/or withdrawal at a particular stage.

(b) Kinetic model

The kinetic model has been described in Chapter 2. Although many workers investigated the microbial growth rate by considering environmental factors, the physiological state of cells, and other factors, the basic kinetic model proposed by Monod (15) in 1942 is still very widely used. The growth rate of microorganisms is expressed as follows

$$\frac{dX}{dt} = \frac{\mu_{max} \; S \; X}{K_S + S}$$

as shown previously in Equation (3) in Chapter 2. The microbial growth rate is between zero and first order respect to the concentration of substrate. The rate of substrate utilization is

$$- \frac{dS}{dt} = \frac{\mu_{max} \; S \; X}{Y(K_S + S)}$$

which is also given in Equation (6) in Chapter 2. It is assumed that these two equations describe the kinetics of microbial growth in the tower system considered here.

(c) Cell and substrate mass balances

Material balance equations to describe the concentration of cell mass and that of organic nutrients in each compartment at steady state can be developed using the complete mixing concept and the kinetic model of Monod.

The material balances for substrate and cells around the first stage are, respectively,

$$q_1^0\, S^0 + f_2\, S^2 - (q_1^e + q_1 + f_2)S^1 - \frac{1}{Y}\mu_1\, X^1\, V_1 = 0 \tag{4}$$

and

$$q_1^0\, X^0 + f_2\, X^2 - (q_1^e + q_1 + f_2)\, X^1 + \mu_1\, X^1\, V_1 = 0 \tag{5}$$

where μ_1 denotes the specific microbial growth rate, which is equal to $\left(\frac{\mu_{max}\, S^1}{K_S + S^1}\right)$.

Similarly, for any one of stages 2 through N-1, say the ith stage, the substrate and cell material balances are, respectively

$$q_i^0 S^0 + (q_{i-1} + f_i)S^{i-1} + f_{i+1}\, S^{i+1} - (q_i^e + q_i + f_{i+1} + f_i)S^i$$
$$- \frac{1}{Y}\mu_i\, X^i\, V_i = 0\ , \quad i = 2, 3, \ldots, N\text{-}1 \tag{6}$$

and

$$q_i^0 X^0 + (q_{i-1} + f_i)X^{i-1} + f_{i+1}\, X^{i+1} - (q_i^e + q_i + f_{i+1} + f_i)X^i$$
$$+ \mu_i\, X^i\, V_i = 0\ , \quad i = 2, 3, \ldots, N\text{-}1 \tag{7}$$

For the last stage, the substrate and cell material balances are respectively

$$q_N^0 S^0 + (q_{N-1} + f_N)S^{N-1} - (q_N^e + f_N)S^N - \frac{1}{Y}\mu_N X^N V_N = 0 \tag{8}$$

and

$$q_N^0 X^0 + (q_{N-1} + f_N)X^{N-1} - (q_N^e + f_N)X^N + \mu_N X^N V_N = 0 \tag{9}$$

In order to make the results as general as possible, dimensionless variables are introduced as follows:

$y_1^i = \frac{S^i}{S^0}$ dimensionless substrate concentration

$y_2^i = \frac{X^i}{YS^0}$ dimensionless cell concentration

$d_i = \frac{q_i}{\mu_{max} V}$ dimensionless dilution rate

$b_i = \frac{f_i}{\mu_{max} V}$ dimensionless backflow rate

$v_i = \frac{V_i}{V}$ volume fraction

In addition, the saturation constant K_S can be made dimensionless by dividing by S^0 ; that is,

$$K_1 = \frac{K_S}{S^0}$$

Thus, K_1 is the dimensionless saturation constant which is equal to the dimensionless substrate concentration at which the specific growth rate observed is one half the maximum value.

Equations (4), (6), and (8) may be divided by $(V\mu_{max}S^0)$, which yields,

$$d_1^0 y_1^0 + b_2 y_1^2 - (d_1^e + d_1 + b_2) y_1^1 - r_1 y_2^1 v_1 = 0 \tag{10}$$

$$d_i^0 y_1^0 + (d_{i-1} + b_i) y_1^{i-1} + b_{i+1} y_1^{i+1} - (d_i^e + d_i + b_{i+1} + b_i) y_1^i$$
$$- r_i y_2^i v_i = 0 \ , \ i = 2, 3, \ldots, N-1 \tag{11}$$

and

$$d_N^0 y_1^0 + (d_{N-1} + b_N) y_1^{N-1} - (d_N^e + b_N) y_1^N - r_N y_2^N v_N = 0 \tag{12}$$

Similarly, Equations (5), (7), and (9) may be divided by $(Y V\mu_{max}S^0)$ to obtain

$$d_1^0 y_2^0 + b_2 y_2^2 - (d_1^e + d_1 + b_2) y_2^1 + r_1 y_2^1 v_1 = 0 \tag{13}$$

$$d_i^0 y_2^0 + (d_{i-1} + b_i) y_2^{i-1} + b_{i+1} y_2^{i+1} - (d_i^e + d_i + b_{i+1} + b_i) y_2^i$$
$$+ r_i y_2^i v_i = 0 \ , \ i = 2, 3, \ldots, N-1 \tag{14}$$

and

$$d_N^0 y_2^0 + (d_{N-1} + b_N) y_2^{N-1} - (d_N^e + b_N) y_2^N + r_N y_2^N v_N = 0 \tag{15}$$

where

$$r_i = \frac{S^i}{K_S + S^i} = \frac{y_1^i}{K_1 + y_1^i}$$

Equations (10) through (15) describe the steady state behavior of substrates and cells in a tower type continuous culture system. One can obtain the substrate and cell concentration at a certain steady state flow condition by solving these equations simultaneously.

3-3 WASHOUT CRITERIA

Herbert (7, 8, 9), Powell (10), and Fan et. al. (11) have presented a detailed discussion of washout behavior for several different kinds of flow models. Basically, when washout occurs, there are no cells in the system, and furthermore washout can only occur for systems with a sterile feed stream. The mathematical solution of the system of equations at the washout condition is as follows:

$$S^i = S^e = S^f, \qquad i = 1, 2, \ldots, N \tag{16}$$

and

$$X^i = X^e = 0 , \qquad i = 1, 2, \ldots, N \tag{17}$$

A limiting process in which $S^i \rightarrow S^0$ and X^i approaches zero may be used to derive the washout dilution rate (11). Consider, for example, a fermentor which is assumed to be a completely mixed tank with liquid volume V with a flow rate, q, passing through the system. A cell balance around the fermentor can be written as

$$V \frac{dX}{dt} = -qX + V X \tag{18}$$

at steady state, $\frac{dX}{dt} = 0$, that is, no cells are accumulated in the fermentor, and Equation (18) reduces to

$$\frac{q}{\mu_{max} V} = d = \frac{\mu}{\mu_{max}} = \frac{S}{K_S + S} \tag{19}$$

Here, d is the dimensionless dilution rate and it is equal to $\frac{\mu}{\mu_{max}}$ which is characteristic of continuous culture at steady state in a completely mixed fermentor. The specific growth rate at washout, μ_W, is

$$\frac{\mu_W}{\mu_{max}} = \frac{S^0}{K + S^0} \tag{20}$$

since the concentration of substrate in the fermentor approaches that in the influent at conditions near the wasgout point. No cells will grow in the system if the dimensionless dilution rate, d, exceeds $\frac{\mu_W}{\mu_{max}}$, the critical value of the dimensionless growth [Equation (20)] . As shown in Fig. 2, d_W is the maximum dimensionless dilution rate which will support growth at steady state; however, d_W is a critical condition and operation at this dilution rate is often unstable.

The complexity of the washout analysis increases greatly as the flow models become more complex. Although many of the multistage fermentation systems have superficial similarities in construction, they may differ greatly in operating principles and their washout conditions may also be quite different. In a chemostat, the specific growth rate is equal to the dilution rate as shown by Equation (19); however, for more complex flow systems, the specific growth rate varies with position. In these complex

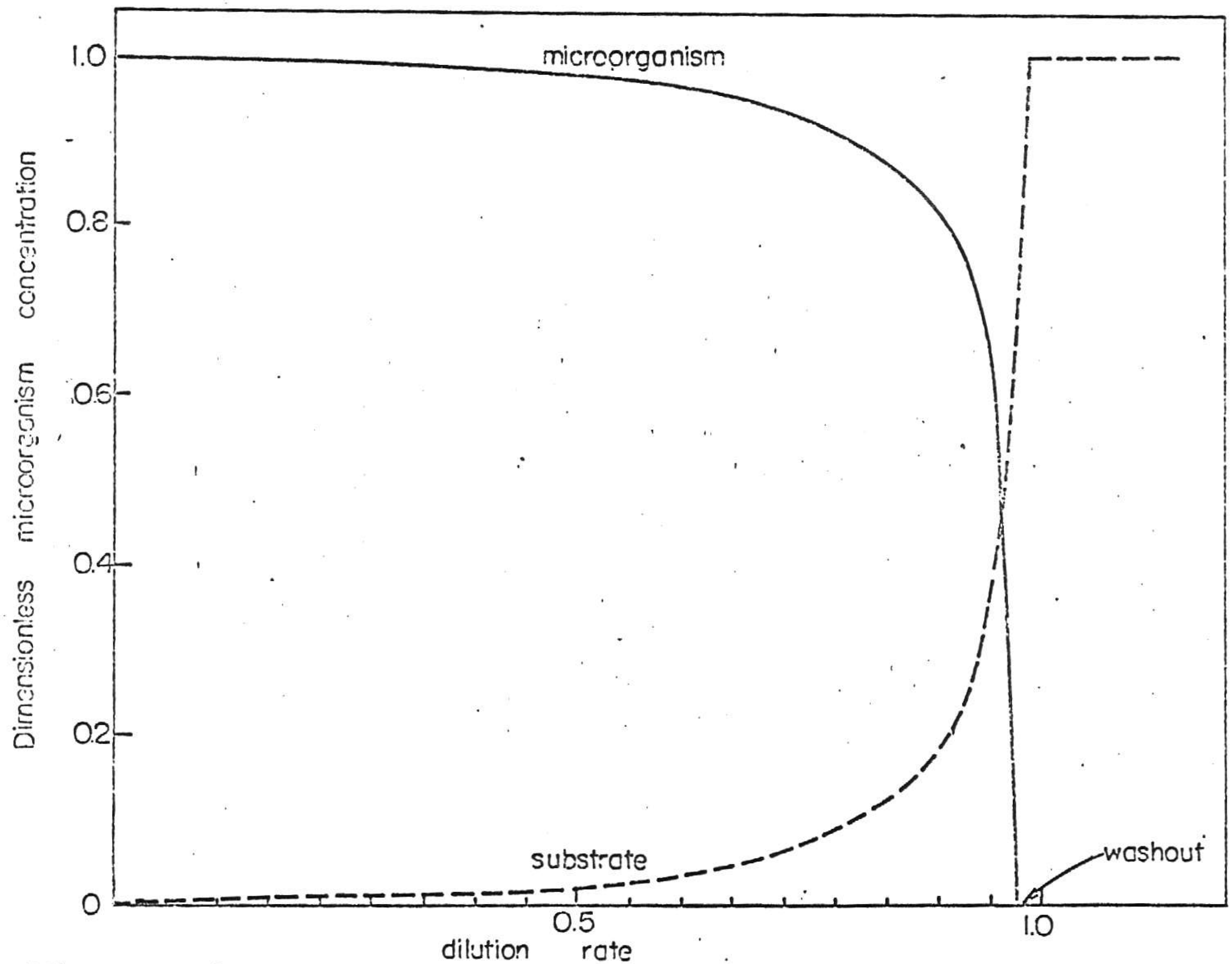

Fig. 2. Microorganism and substrate concentration profile in one completely mixed continuous tank with $K_1 = 0.02$.

flow systems, the system dilution rate can not generally be equated to a meaningful specific growth rate. The system dilution rate may be defined as the total flow through the system divided by the total volume of all stages. In dimensionless form, it is

$$d = \frac{q}{\mu_{max}(V_1 + V_2 + \ldots + V_N)} = \frac{q}{\mu_{max} V}$$

In the tower system the concentration of substrate varies from stage to stage resulting in a different specific growth rate in each stage. In the case of a system with backflow, the adjacent stages directly influence each other and the entire system must be analyzed to determine the washout conditions. Equations (16) and (17) which are satisfied at washout may be used for any type of flow system.

3-4 THE CHARACTERISTIC EQUATION OF WASHOUT DILUTION RATE

(a) Development of characteristic equation

At the critical washout dilution rate for the multistage continuous culture, the conditions, $y_1^i = 1$ and $y_2^i = 0$, $i = 1, 2, \ldots, N$, will be satisfied. To determine the critical washout dilution rate, Equations (13) to (15) need to be rearranged and simplified. Since washout only occurs when a sterile feed is employed, y_2^0 is assumed to equal zero. y_2^N can be expressed in terms of y_2^{N-1} by rearranging Equation (15)

$$y_2^N = \frac{(d_{N-1} + b_N)\, y_2^{N-1}}{(d_N^e + b_N) - r_N v_N} \tag{22}$$

Substituting for y_2^N in Equation (14) for $i = N-1$ enables

y_2^{N-1} to be expressed in terms of y_2^{N-2} as

$$y_2^{N-1} = \frac{(d_{N-2} + b_{N-1})\, y_2^{N-2}}{(d_{N-1} + d_{N-1} + b_N + b_{N-1}) - \dfrac{b_N(d_{N-1} + b_N)}{(d_N^e + b_N) - r_N v_N} - r_{N-1} v_{N-1}} \tag{23}$$

Repeating this successive substitution, y_2^i can be expressed in terms of y_2^{i-1} (see Table 1). Finally, the characteristic equation for the washout dilution rate is obtained by substituting y_2^2 into Equation (13) to obtain Equation (25) (see Table 2). Note that y_2^1 is a common factor in the left hand side of Equation (25). By eliminating y_2^1, the characteristic equation for the washout dilution rate takes the form shown in Equation (26) which is also contained in Table 2.

Equation (26) is a useful general equation, in which b_i, d_i, and d_i^e are variables depending on system design parameters and operational conditions. Note that at washout y_1^i, $i = 1,2,\ldots,N$, which are included in r_i are all equal to 1.0.

The volumetric flow rate balances are given by Equations (1), (2), and (3). Generally, the values of b_i and all except one of the values of d_i^e are controlled or fixed by the design of the system. The volumetric flow rate balances allow the remaining value of d_i^e to be calculated once the feed flow rates are fixed. Thus, in this work the feed flow rates are assumed to be the independent variables which are to be analyzed for their critical washout values.

The first term of Equation (26) is a compound fraction.

Table 1. Relation between y_2^i and y_2^{i-1}.

$$y_2^i = \cfrac{(d_{i-1} + b_i)\, y_2^{i-1}}{(d_i^e + d_i + b_{i+1} + b_i) - \cfrac{b_{i+1}\,(d_i + b_{i+1})}{\begin{array}{c}\vdots \\ (d_{N-1}^e + d_{N-1} + b_N + b_{N-1}) - \cfrac{b_N\,(d_{N-1} + b_N)}{(d_N^e + b_N) - r_N v_N} - r_{N-1} v_{N-1}\end{array}} - r_i v_i} \tag{24}$$

Table 2. Characteristic equations for washout for flow geometry of general case (Fig. 1).

$$(d_1^e+d_1+b_2)y_2^1-\cfrac{b_2(d_1+b_2)y_2^1}{(d_2^e+d_2+b_3+b_2)-\cfrac{b_3(d_2+b_3)}{(d_3^e+d_3+b_4+b_3)-\cfrac{b_4(d_3+b_4)}{\begin{matrix}\vdots\\(d_{N-2}^e+d_{N-2}+b_{N-1}+b_{N-2})-\cfrac{}{(d_{N-1}^e+d_{N-1}+b_N+b_{N-1})-\cfrac{b_N(d_{N-1}+b_N)}{(d_N^e+b_N)-r_Nv_N}-r_{N-1}v_{N-1}}-r_{N-2}v_{N-2}\end{matrix}}-r_3v_3}-r_2v_2}-r_1v_1y_2^1=0 \qquad (25)$$

$$(d_1^e+d_1+b_2)-\cfrac{b_2(d_1+b_2)}{(d_2^e+d_2+b_3+b_2)-\cfrac{b_3(d_2+b_3)}{(d_3^e+d_3+b_4+b_3)-\cfrac{b_4(d_3+b_4)}{\begin{matrix}\vdots\\(d_{N-2}^e+d_{N-2}+b_{N-1}+b_{N-2})-\cfrac{}{(d_{N-1}^e+d_{N-1}+b_N+b_{N-1})-\cfrac{b_N(d_{N-1}+b_N)}{(d_N^e+b_N)-r_Nv_N}-r_{N-1}v_{N-1}}-r_{N-2}v_{N-2}\end{matrix}}-r_3v_3}-r_2v_2}-r_1v_1=0 \qquad (26)$$

Careful observation shows that the compound fraction is a fraction repeating from stage to stage, in order to account for the multi-stage system behavior. The general term is of the form

$$\frac{\vdots}{(d^e_j+d_j+b_{j+1}+b_j)-\cfrac{b_{j+1}(d_j+b_{j+1})}{(d^e_{j+1}+d_{j+1}+b_{j+2}+b_{j+1})-\cfrac{b_{j+2}(d_{j+1}+b_{j+2})}{\vdots}-r_{j+1}v_{j+1}}-r_jv_j}$$

Every component of the compound fraction has some significance. The denominator of the compound fraction is composed of three terms; $(d^e_j+d_j+b_{j+1}+b_j)$ are the streams leaving the jth stage, the sub-compound fraction is the stream coming into the jth stage, and the term r_jv_j represents the reaction term at the jth stage.

This characteristic equation can be used to examine the washout dilution rate of most feed geometries. The specific equation for each case can be obtained by making several simplifying assumptions. Several specific cases will be discussed in the following sections.

(b) Effect of sedimentation on washout

Sedimentation of cells often occurs in biological growth processes as observed by several workers (1, 3, 4). It is often an important factor which needs to be considered in modeling biological processes. The mechanism of cell sedimentation in continuous fermentations is a very complex phenomenon. It is known that the extent of flocculation, cell physiological state, and many other factors influence cell sedimentation; however, a very simple model, Equation (3) in Chapter 2, is employed in this study. The dimensionless value of δ is used to indicate the ratio of the concentration of cells in the bulk liquid in a compartment, X^i, to that in the stream flowing upward through the sieve plate to the next stage, $\bar{X}^i$, that is,

$$\delta = \frac{X^i}{\bar{X}^i} = \frac{y_2^i}{\bar{y}_2^i}$$

where the value of δ is always greater than 1.0. In this analysis, the exit stream from any stage is assumed to have the same concentration $\bar{X}^i$ as the stream flowing upward through the sieve plate.

Including the effect of sedimentation in the analysis of washout in multistage continuous culture requires that the material balance equations for microbial cells be modified. The mass balance equations for the system with a sedimentation factor, δ, included in Equations (13) to (15) are as follows:

$$\delta b_2\bar{y}_2^2 - (d_1^e + d_1 + b_2)\bar{y}_2^1 + r_1\delta\bar{y}_2^1 v_1 = 0 \tag{27}$$

$$(d_{i-1}+b_i)y_2^{i-1} + \delta b_{i+1}\bar{y}_2^{i+1} - (d_i^e+d_i+ b_{i+1}+\delta b_i)\bar{y}_2^i$$

$$+r_i\delta y_2^i v_i = 0\ ,\ i = 2,3,\ldots,N-1 \tag{28}$$

$$(d_{N-1}+b_N)\bar{y}_2^{N-1} - (d_N^e+\delta b_N)\bar{y}_2^N + r_N\delta\bar{y}_2^N v_N = 0 \tag{29}$$

These equations can be combined following the procedure used to arrive at Equation (26) to obtain Equation (30) which may be found in Table 3.

Equation (30) is a general characteristic equation for predicting the washout dilution rate in multistage tower system. If $\delta=1$, Equation (30) reduces to Equation (26). The following section will discuss several specific cases which are all simplifications of

Table 3. Characteristic equation for washout with sedimentation for flow geometry of general case (Fig. 1).

$$(d_1^e+d_1+b_2)-\cfrac{\delta b_2(d_1+b_2)}{(d_2^e+d_2+b_3+\delta b_2)-\cfrac{\delta b_3(d_2+b_3)}{(d_3^e+d_3+b_4+\delta b_3)-\cfrac{\delta b_4(d_3+b_4)}{\ddots\; (d_{N-2}^e+d_{N-2}+b_{N-1}+\delta b_{N-2})-\cfrac{}{(d_{N-1}^e+d_{N-1}+b_N+\delta b_{N-1})-\cfrac{\delta b_N(d_{N-1}+b_N)}{(d_N^e+\delta b_N)-r_N\delta v_N}-r_{N-1}\delta v_{N-1}}-r_{N-2}\delta v_{N-2}}-r_3\delta v_3}-r_2\delta v_2}-r_1\delta v_1=0 \quad (30)$$

Equation (30).

(c) Special cases

In order to illustrate the washout behavior of multistage tower fermentation processes, several specific cases which are of practical importance are considered. The feed and effluent geometry for these cases is greatly simplified from the general case considered previously. Only one feed location and one exit location is considered for each special case. These special cases are employed to obtain the numerical results which are presented in this work.

Case 1

The schematic diagram for Case 1 is shown in Fig. 3. It is the most basic case in that feed is introduced at the first stage, while the outlet stream is withdrawn from the last stage. In investigating the washout dilution rate for this system, it is assumed that there is equal volume in each individual stage ($v_i = \frac{1}{N}$) and a fixed value of backflow. The flow rate can be determined by solving equations (1), (2), and (3) in dimensionless form. If the dimensionless influent flow rate is assumed to be d_i^0, and if

$$d_j^0 = 0 \quad \text{for } j \neq 1 \tag{31}$$

and

$$d_k^e = 0 \quad \text{for } k \neq N \tag{32}$$

it can be shown that

$$d_1^0 = d = d_1 = \ldots = d_i = \ldots = d_N = d_N^e \tag{33}$$

In this case, the backflow rate is assumed to be a constant

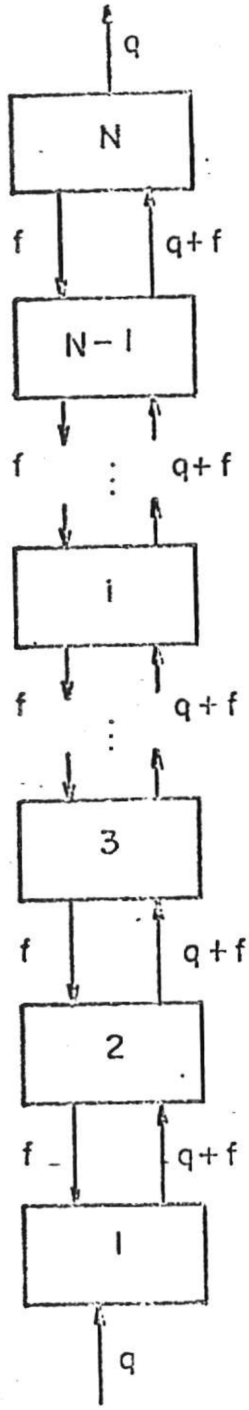

Fig. 3. Backflow cell model, Case I.

value, which might be dependent on the design of the system. The characteristic equation for the washout dilution rate for this case can be obtained by substituting Equations (32) and (33) and $b_i = b$ into Equation (30). At the washout condition,

$$r = r_1 = r_2 = \dots = r_N$$

where

$$r = \frac{1.0}{K_1 + 1.0} \tag{34}$$

These simplifications allow the characteristic equation to be written in the form shown in Equation (35) (see Table 4). For fixed values of b, δ, N, K_1, and v_i, $i = 1, 2, \dots, N$, the critical washout dilution rate can be determined using this equation.

Case 2

The schematic diagram for this case in which the feed is introduced at the second stage is shown in Fig. 4. Prokop et al. (1) constructed such a system with 8 stages. They found that this particular feed geometry provides a wide operating range, because it allows the first stage to produce microorganisms which can be continuously used to inoculate the second stage.

At the washout condition, $r = r_1 = r_2 = \dots = r_N$, where r is defined in Equation (34). The characteristic equation for the washout dilution rate is obtained from Equation (30) by noting that feed is introduced only at stage 2 and that all effluent leaves from stage N. It is shown in Equation (36) in Table 4. This feed geometry modification greatly changes the washout characteristics of the system, especially when b is small.

Case 3

In this case, the feed stream may be introduced at any stage

Table 4. Characteristic equations for washout for Cases 1 and 2, respectively.

$$(d+b)-\cfrac{\delta b(d+b)}{(d+\delta b+b)-\cfrac{\delta b(d+b)}{(d+\delta b+b)-\cfrac{\delta b(d+b)}{\begin{matrix}\vdots\\ (d+\delta b+b)-\cfrac{}{(d+\delta b+b)-\cfrac{\delta b(d+b)}{(d+\delta b)-r\delta v_N}-r\delta v_{N-1}}-r\delta v_{N-2}\end{matrix}}-r\delta v_3}-r\delta v_2}-r\delta v_1=0 \qquad (35)$$

$$(0+b)-\cfrac{\delta b(0+b)}{(d+\delta b+b)-\cfrac{\delta b(d+b)}{(d+\delta b+b)-\cfrac{\delta b(d+b)}{\begin{matrix}\vdots\\ (d+\delta b+b)-\cfrac{}{(d+\delta b+b)-\cfrac{\delta b(d+b)}{(d+\delta b)-r\delta v_N}-r\delta v_{N-1}}-r\delta v_{N-2}\end{matrix}}-r\delta v_3}-r\delta v_2}-r\delta v_1=0 \qquad (36)$$

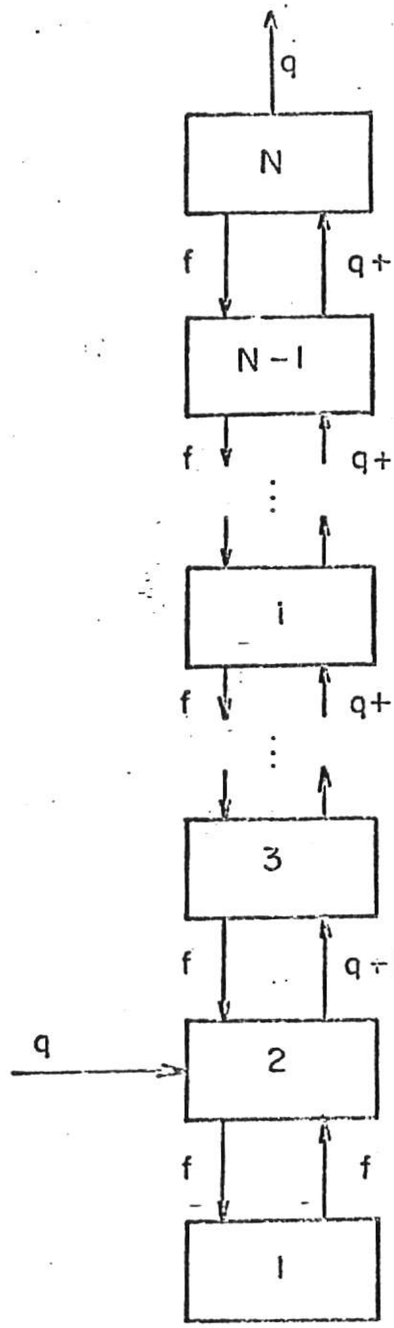

Fig. 4. Backflow cell model with feed to the second stage. (Case 2)

and the exit stream is located at stage N. It is assumed that the internal backflow rates below the feed stage are controlled of fixed at the value b' and those above the feed stage at the value b by the design of the system. Substituting these backflow rate relationships into Equation (30), and noting the feed and exit geometry, the characteristic equation of the washout dilution rate becomes that shown in Equation (37) (see Table 5) The internal flow rate b' below the stage (the ith stage) might be equal to b. The effects of differences between these two internal flow rates will be examined later by comparing results from this case with i = 2, which is shown in Fig. 5, with those from Case 2.

Case 4

The schematic diagram for this case is shown in Fig. 6. It is a general case for tower fermentors with one feed stream and one exit stream. In this case two different backflow rates are assumed with b' being the backflow above the exit stage and below the feed stage and b being the backflow for those stages between the feed and exit locations. The characteristic equation for the washout dilution rate which is obtained for this case is shown in Equation (38) in Table 6.

Equations (35) through (38) are the characteristic equations which can be used to determine the washout dilution rate; however, these equations are in implicit form. For fixed values of b, b', δ, K_1, N, and v_i, i=1,2,...,N, the appropriate characteristic equation can be used to determine the washout dilution rate d. Examination of Equations (35) through (38) shows that one must solve an Nth order polynomial to find the washout dilution rate

Table 5. Characteristic equation for washout for Case 3.

$$(0+b') - \cfrac{\delta b'(0+b')}{(0+\delta b'+b') - \cfrac{\delta b'(0+b')}{\begin{array}{c}\vdots\\ (0+\delta b'+b') - \cfrac{\delta b'(0+b')}{(d+\delta b'+b) - \cfrac{\delta b(d+b)}{(d+\delta b+b) - \cfrac{\delta b(d+b)}{\begin{array}{c}\vdots\\ (d+\delta b+b) - \cfrac{}{(d+\delta b+b) - \cfrac{\delta b(d+b)}{(d+\delta b)-r\delta v_N} - r\delta v_{N-1}} - r\delta v_{N-2}\end{array}} - r\delta v_{i+1}} - r\delta v_i} - r\delta v_{i-1}\end{array}} - r\delta v_2} - r\delta v_1 = 0 \quad (37)$$

Table 6. Characteristic equation for washout for Case 4.

$$(0+b') - \cfrac{\delta b'(0+b')}{(0+\delta b'+b') - \cfrac{\delta b'(0+b')}{(0+\delta b'+b') - \cfrac{\delta b'(0+b')}{\vdots \; (0+\delta b'+b') - \cfrac{\delta b'(0+b')}{(d+\delta b'+b) - \cfrac{\delta b(d+b)}{(d+\delta b+b) - \cfrac{\delta b(d+b)}{\vdots \; (d+\delta b+b) - \cfrac{\delta b(d+b)}{(d+\delta b+b') - \cfrac{\delta b'(0+b')}{(0+\delta b'+b') - \cfrac{\delta b'(0+b')}{\vdots \; (0+\delta b'+b') - \cfrac{}{(0+\delta b'+b') - \cfrac{\delta b'(0+b')}{(0+\delta b')-r\delta v_N} - r\delta v_{N-1}} - r\delta v_{N-2}} - r\delta v_{j+1}} - r\delta v_j} - r\delta v_{j-1}} - r\delta v_{i+1}} - r\delta v_i} - r\delta v_{i-1}} - r\delta v_3} - r\delta v_2} - r\delta v_1 = 0 \qquad (38)$$

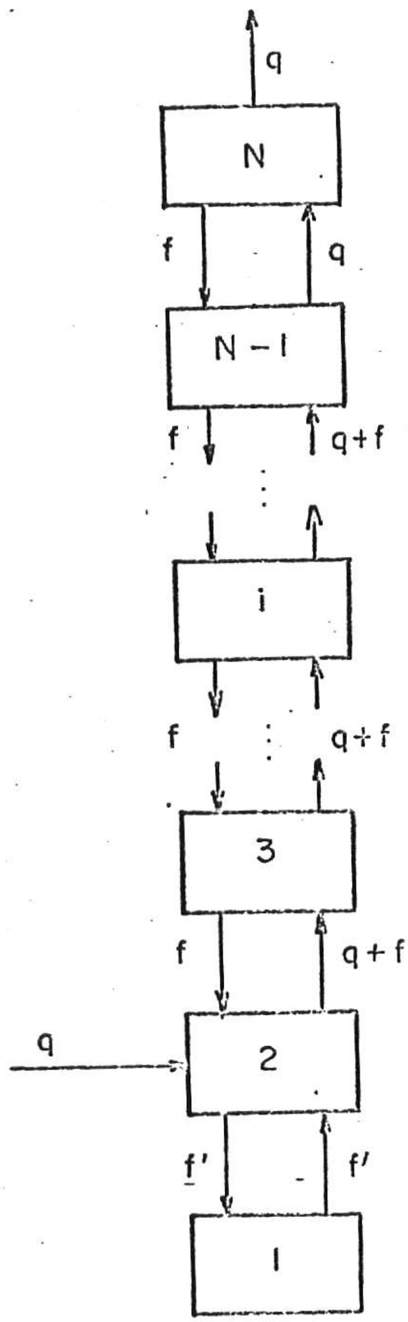

Fig. 5. Backflow cell model with feed to the second stage and controlled backflow to first stage. (Case 3)

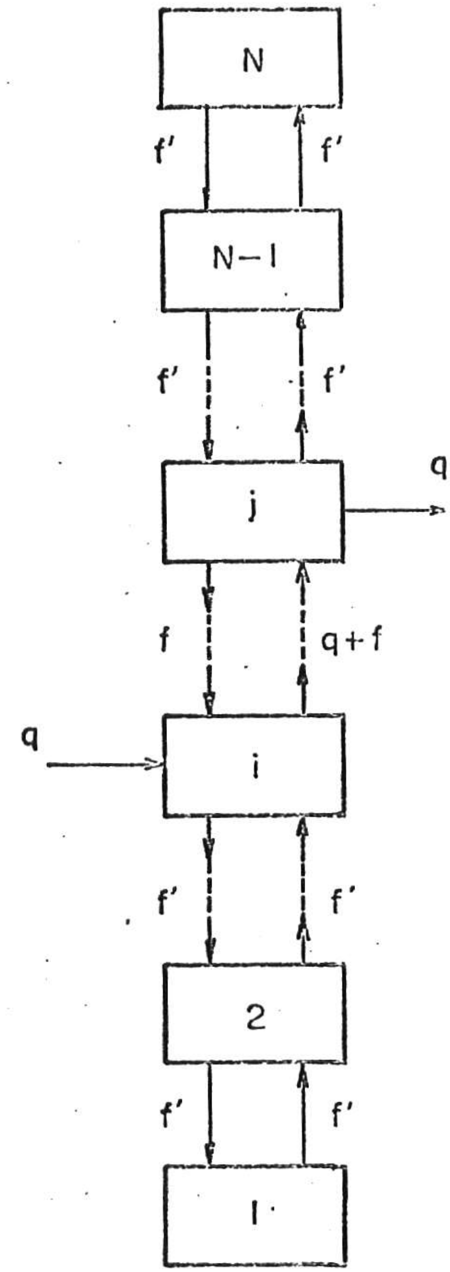

Fig. 6. Backflow cell model with feed to the i^{th} stage and withdrawal from j^{th} stage (Case 4).

for an N stage system. The regula falsi method (12) is used in this study to solve this polynomial equation.

3-5 RESULTS

The characteristic washout equation for several different cases have been presented in the preceeding section. Numerical solutions of these equations have been carried out using the regula falsi iteration method to obtain numerical results for various values of b, δ, N, and K_1. Stages of equal volume are assumed in obtainning these results which are plotted in Figs. 7 through 18.

Three flow models are considered in this numerical study, namely, Case 1, Case 2, and Case 3 with i=2. The results obtained show that the washout dilution rate definitely depends upon the flow geometry. The effects of the various physical factors on the washout dilution rate will be discussed separately in the following sections.

(a) Effect of backflow

The effect of backflow for Case 1 on the dimensionless washout dilution rate is illustrated in Figs. 7, 8, and 9 for systems with 2, 4, and 8 compartments, respectively. The results show that the appearance of backflow in the system increases the critical dilution rate.

The system with backflow can be analyzed by considering the two extreme cases; one is that of no backflow and the other is that of very large backflow. When the backflow rate in the system is large, the system acts as one completely mixed tank. The resulting washout dilution rate obtained from Equation (35) shows that the value converges to

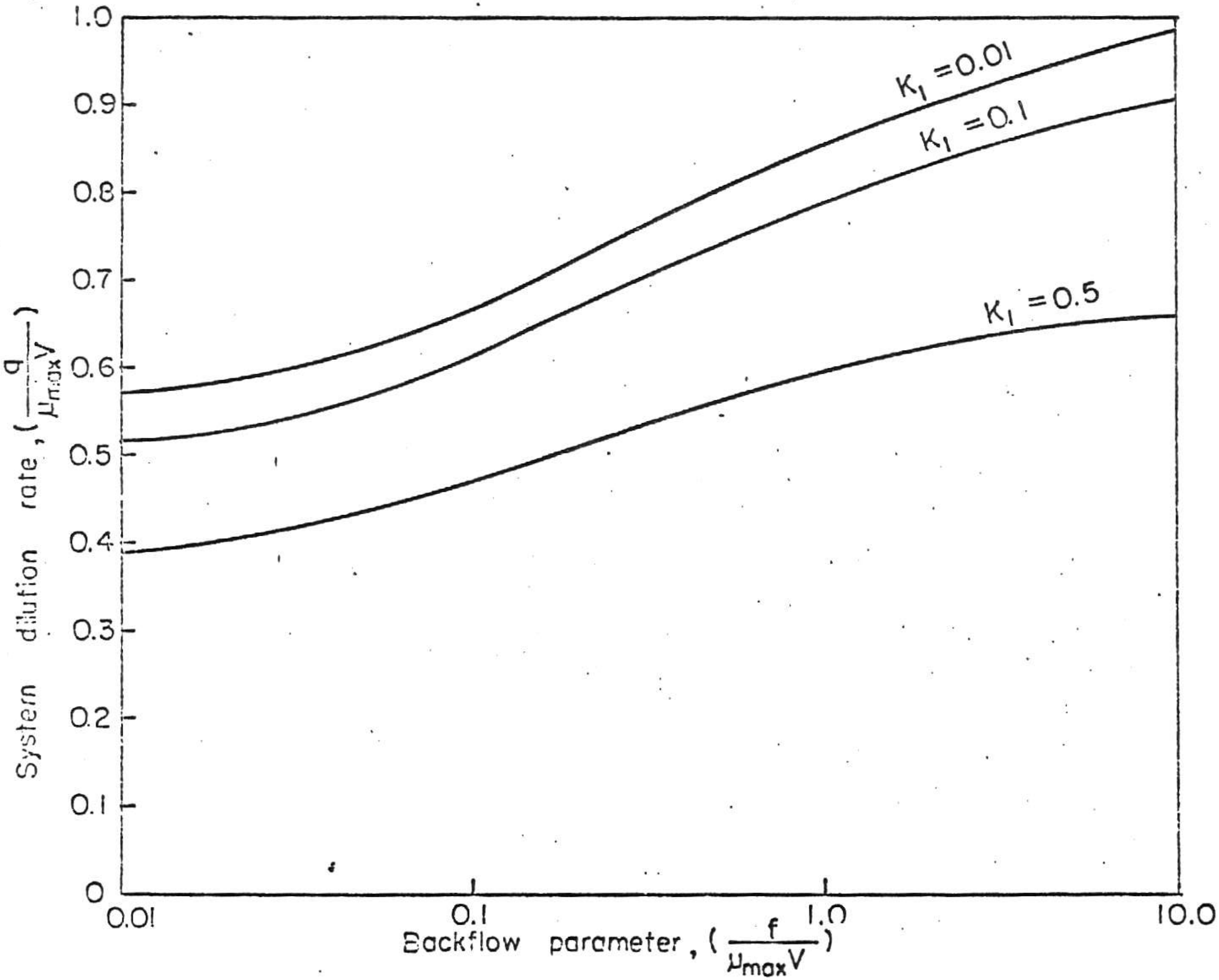

Fig. 7. Effect of backflow on washout dilution rate for 2 stage system. (Case 1)

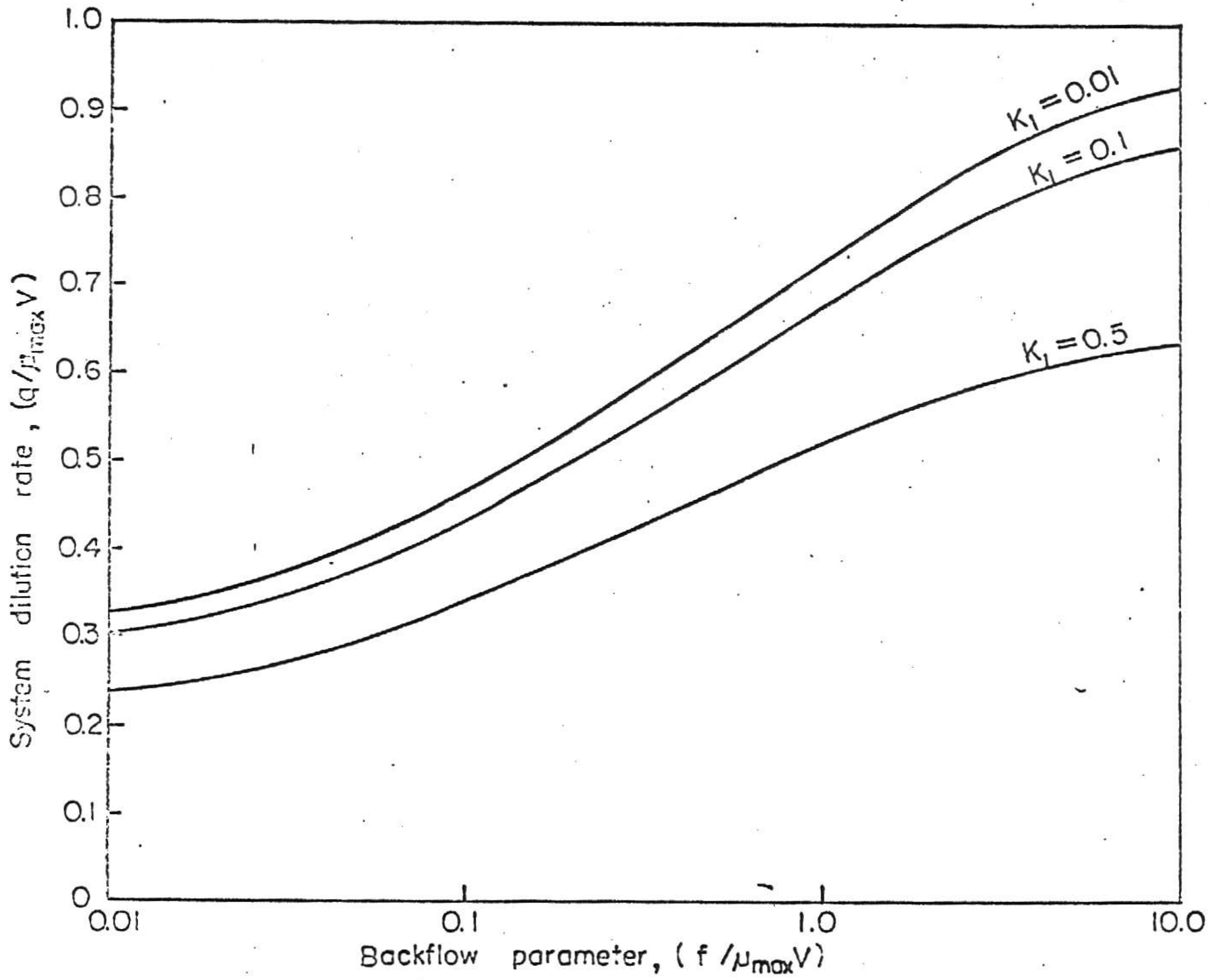

Fig. 8 Effect of backflow on washout dilution rate for 4 stage system (Case 1).

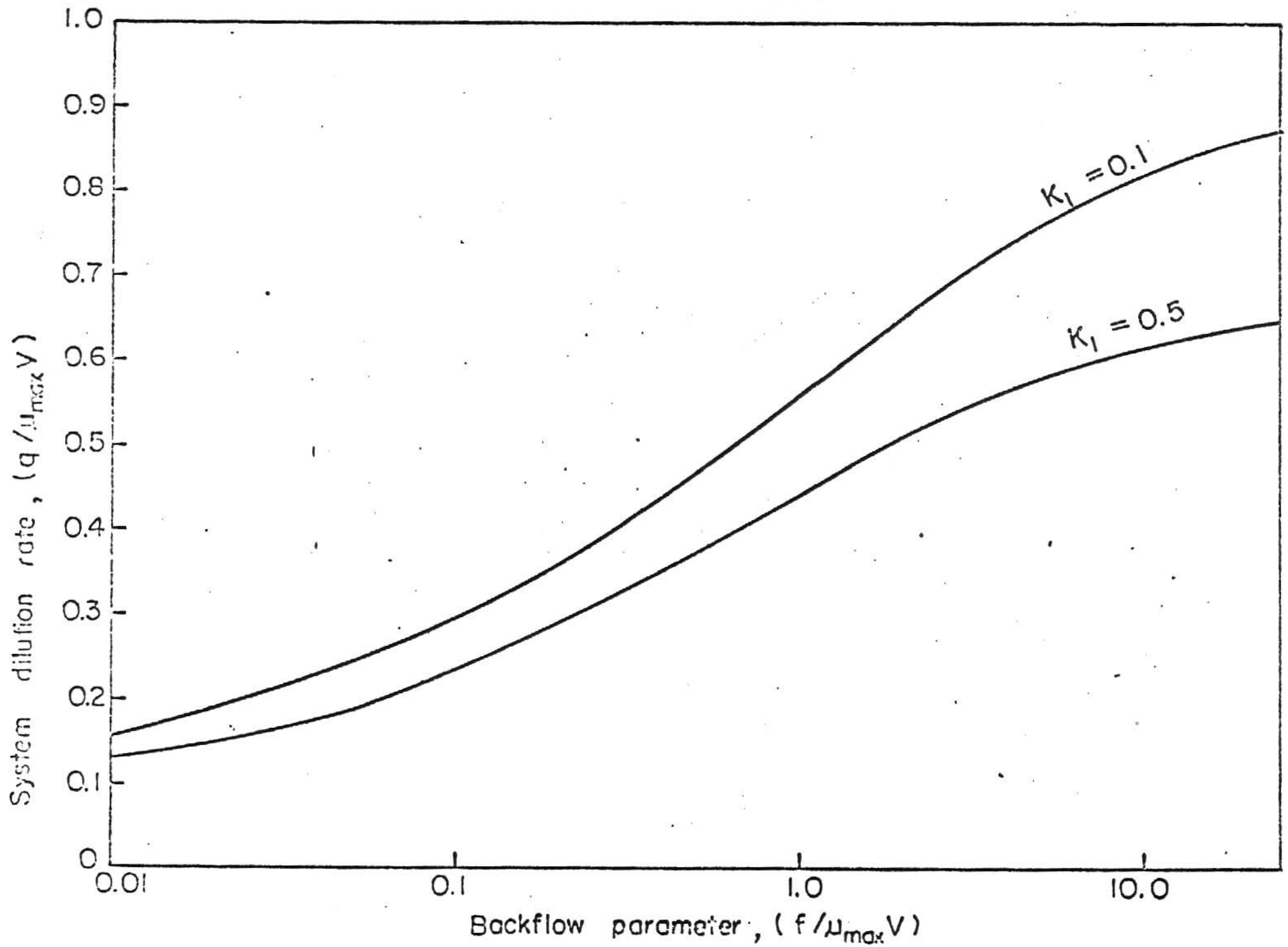

Fig. 9. Effect of backflow on washout dilution rate for 8 stage system. (Case I)

$$d_w = \frac{1.0}{K_1 + 1.0} \tag{39}$$

for one completely mixed tank when b is large. Figures 7, 8, 9, and 10 show that for different values of K_1, and N, the washout dilution rates converge to the values obtained from Equation (39).

When the system is without backflow, the model becomes the N CSTR's-in-series model. Since the effluent from the first stage becomes the feed for the second stage, organisms will be completely washed out only if organisms are washed out from the first stage. Otherwise, the organisms will never be washed out from the system. The results show that they are in agreement with the value obtained from the equation for washout from the first stage of an N stage equal volume system, namely,

$$d_w = \frac{1.0}{N(K_1 + 1.0)} \tag{40}$$

Since this equation also applies to the subsequent stages when the inflo to those stages is sterile, Equation (40) provides the lower limit for d_w as b approaches zero. The left hand side Figs. 7 through 10 converges to the value obtained from Equation (40), which indicates that the washout dilution rate for the tower system with no backflow rate converges to this limiting case.

The effect of backflow for Cases 2 and 3 on the dimensionless dilution rate is illustrated in Figs. 11 to 14. A detailed discussion of the effect of feed geometry will be presented in part (b). In general, except at low backflow rates the existence of backflow in the system increases the critical dilution rate.

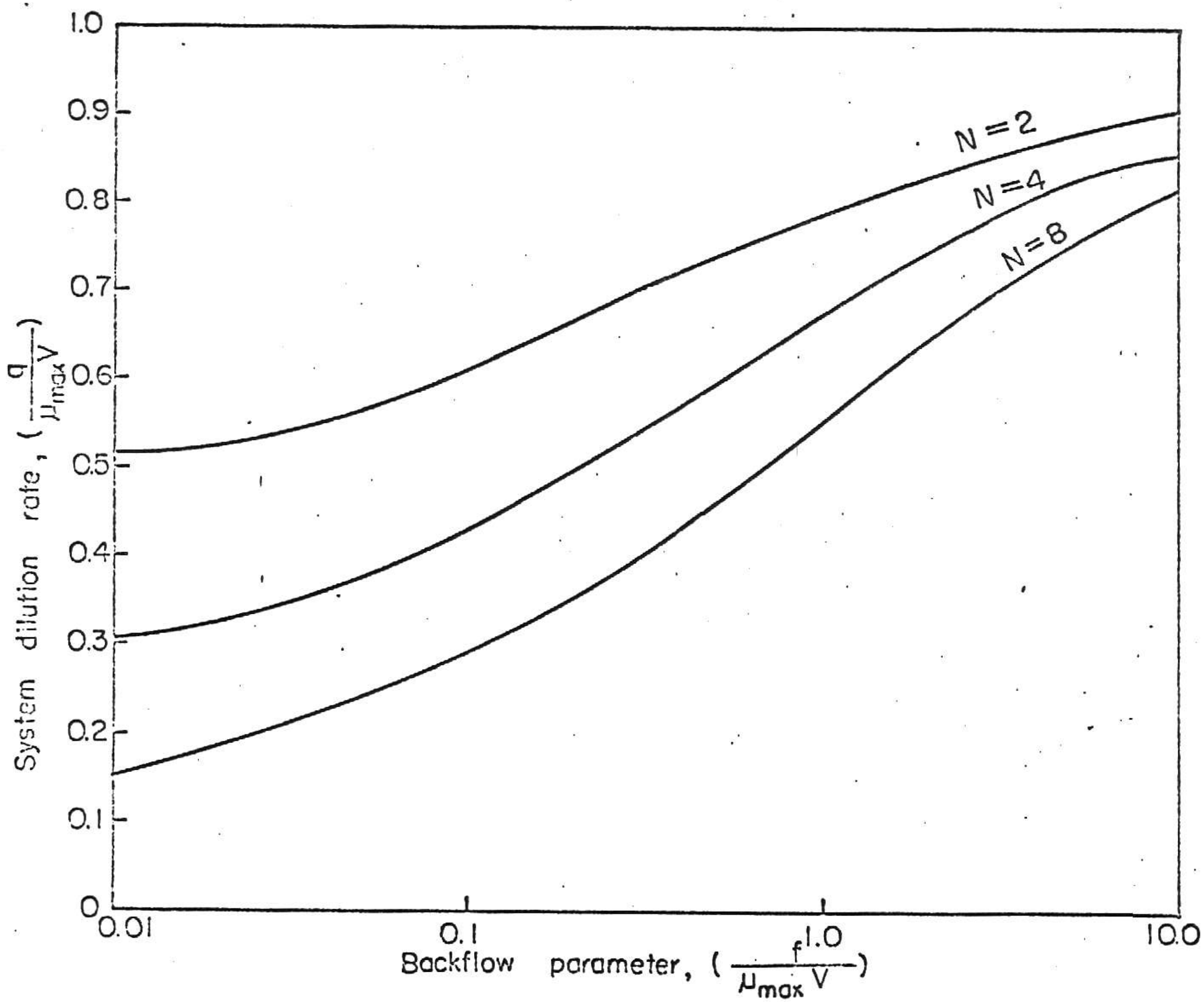

Fig. 10. Effect of backflow and numbers of stages on washout dilution rate, $K_1 = 0.1$ (Case I).

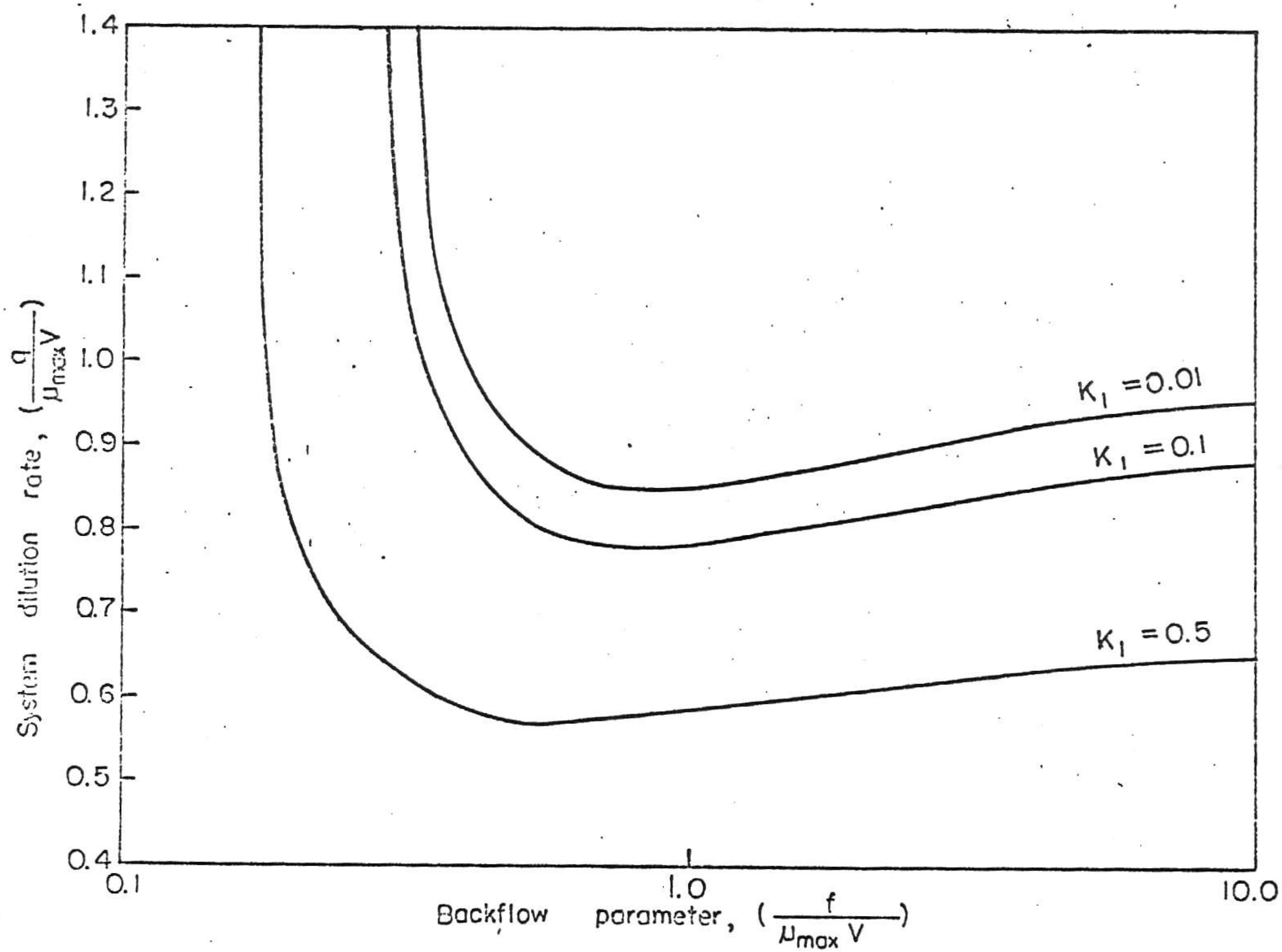

Fig.11. Effect of backflow on washout dilution rate for 4 stage system. (Case 2)

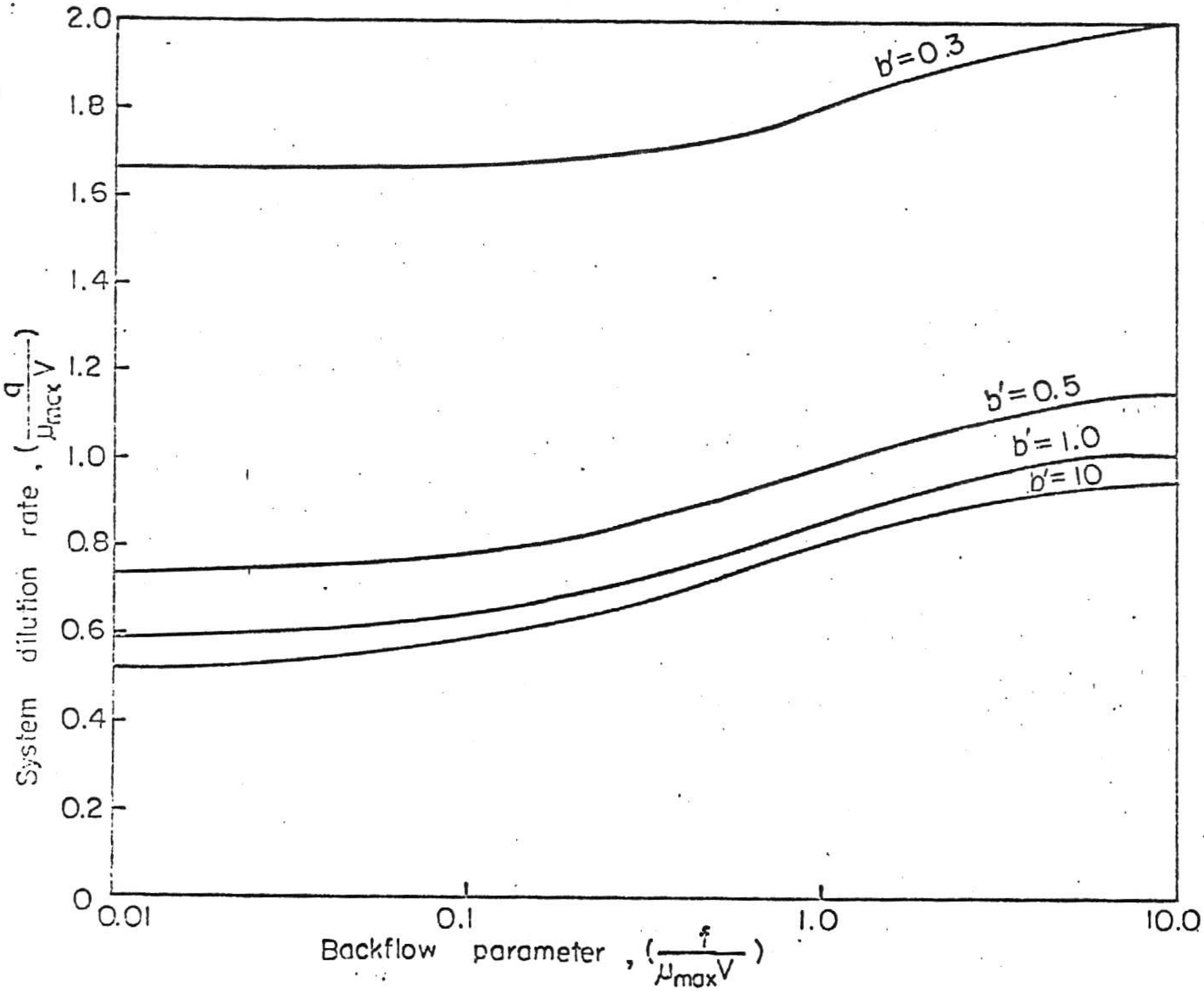

Fig.12. Effect of backflow on washout dilution rate for 4 stage system , K_1 = 0.01 . (Case 3)

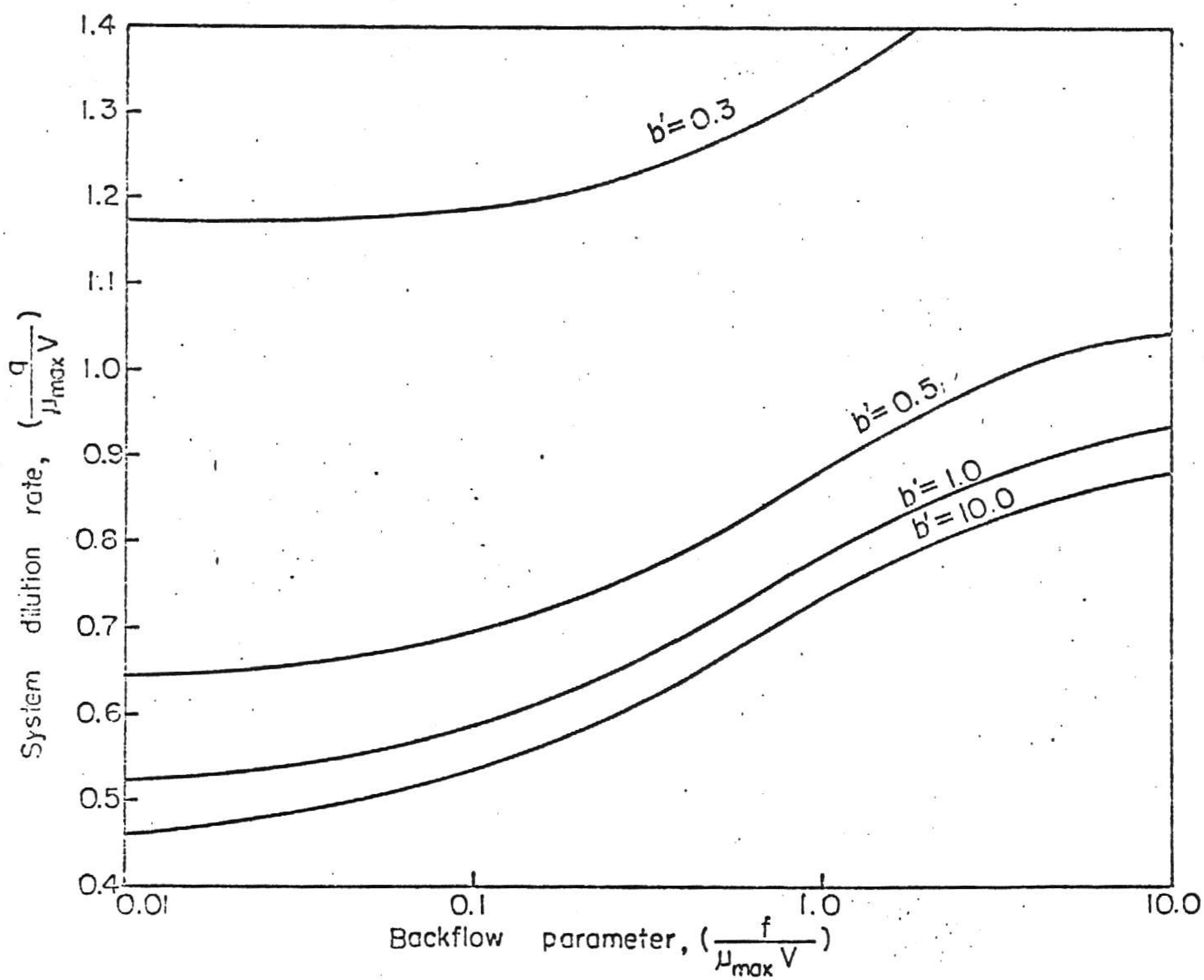

Fig. 13. Effect of backflow on washout dilution rate for 4 stage system, $K_1 = 0.1$. (Case 3)

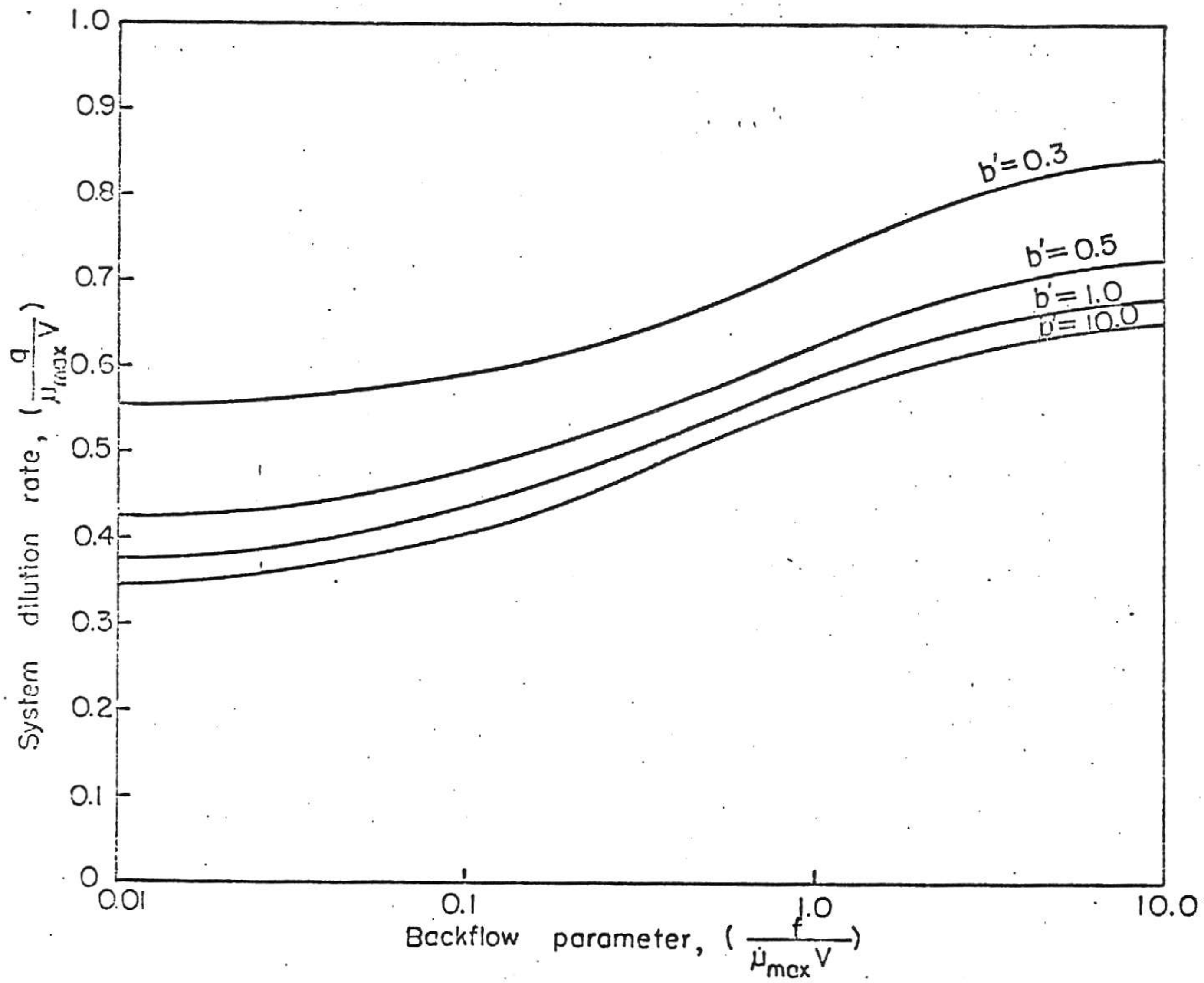

Fig.14. Effect of backflow on washout dilution rate for 4 stage system, K_I = 0.5. (Case 3)

(b) Effect of feed geometry

In the numerical results for Cases 2 and 3, the feed stream is introduced at the second stage of the tower system. The advantage of this feed geometry is that it allows the first stage to produce microorganisms which can be used to continuously inoculate the second stage. This growth of microorganisms in the bottom stage is beneficial in preventing complete washout of microoganisms from the system. When the feed is introduced to the second stage, the range and flexibility of operation are greatly increased, especially when flow rates to the first stage are controlled. In this study, two different cases of feed to the second stage are considered; Case 2 (see Fig. 4) assumes that the backflow between the first and second stage is the backflow rate in the other stages; Case 3 (see Fig. 5) assumes that the backflow between the first and second stages can be fixed independently of the backflow rate in the other stages. Some results for Case 2 appear in Fig. 11 while results for Case 3 are presented in Figs. 12 to 14. The Case 2 results in Fig. 11 show that at low backflow rates to the first stage, the system will not wash out for any feed flow rate. This same result also occurs for Case 3 when the backflow between the first and second stages is sufficiently small.

The critical value of backflow between stage 1 and 2 is able to be examined by investigating the material balance around the first stage. For backflow rates below this value the system will never be washed out. This critical value of backflow can be determined as follows:

A cell material balance around the first stage gives

$$b_2 y_2^2 - b_2 y_2^1 + \frac{1}{N} r_1 y_2^1 = 0 \tag{41}$$

where b_2 is the backflow coming from the second stage, which becomes b for Case 2 or b' for Case 3. The most dilute cell concentration coming from the second stage that can be assumed at washout is $y_2^2 = 0$; therefore Equation (41) becomes

$$- b_2 y_2^1 + \frac{1}{N} r_1 y_2^1 = 0 \tag{42}$$

where the critical backflow parameter b_2 is defined as

$$b_2 = \frac{r_1}{N} \tag{43}$$

If the backflow parameter, b_2 (or b'), is less than this critical backflow parameter is plotted against the number of stages in Fig. 15. The critical backflow parameter decreases as the number of stages increases because of the existence of N in Equation (43).

Because of the effect of feed geometry, the critical washout dilution rate of Case 2 behaves a convex curve with respect to the backflow parameter as shown in Fig. 11. The critical washout dilution rate increases rapidly as the backflow decreases to the critical backflow value below which the cells always reproduce in the first stage and continuously inoculate the second stage. As this critical backflow rate is approached, the washout dilution rate approaches infinity.

In both Cases 2 and 3, at large backflow rates the washout dilution rate increases with increasing backflow rate. The degree

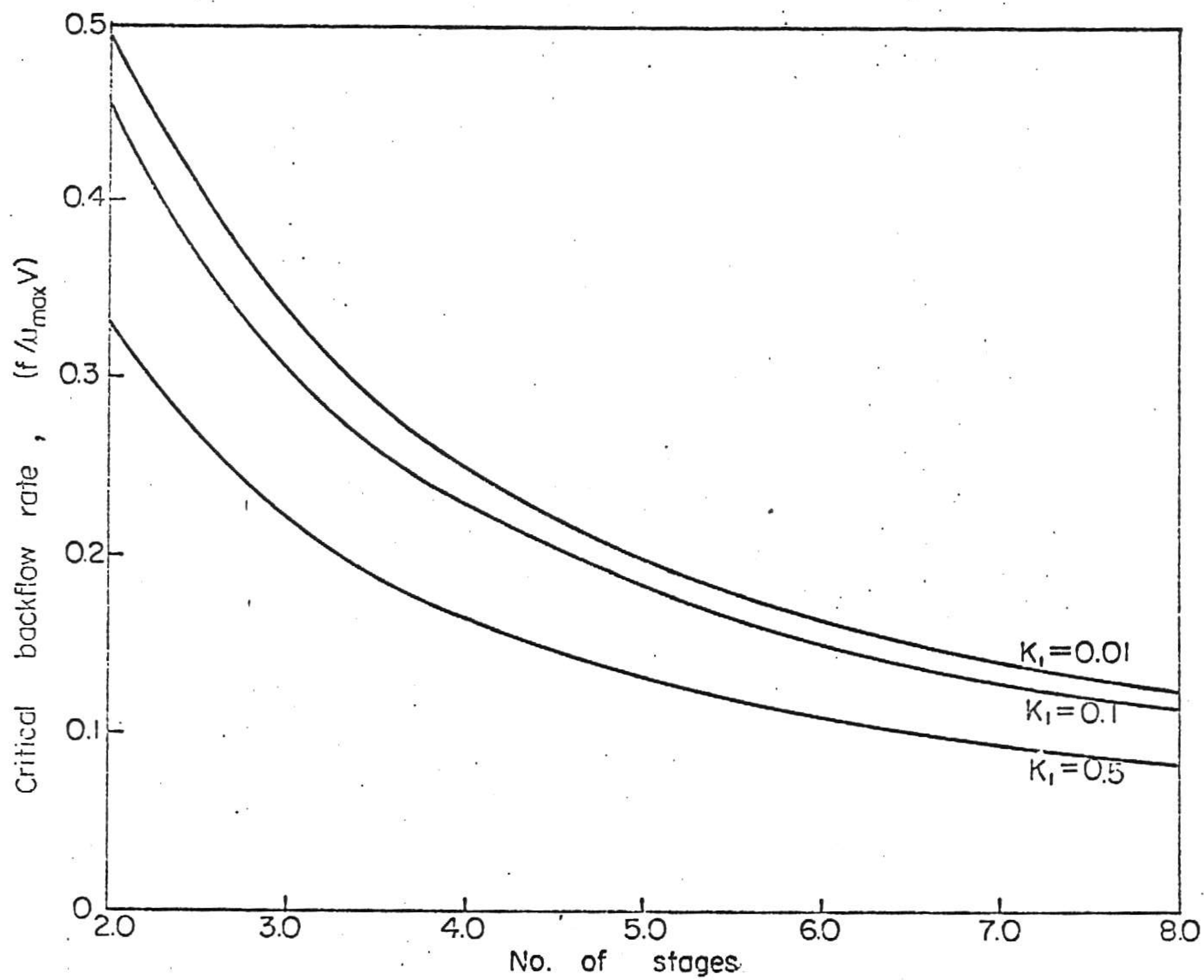

Fig. 15. Critical backflow parameter for backflow model. (Case 2)

of mixing in the system is increased as the backflow rate increases, and at large backflow rates, the system may be assumed to be one completely mixed tank. Equation (39) can, therefore, be used to predict the critical washout dilution rate for these systems with large backflow rates. As shown at the right hand side of Figs. 11 through 14, the results are in agreement with the value obtained from Equation (39).

The critical washout dilution rate for Case 3 depends on the values of both b and b'. The results shown in Figs. 12 to 14 are for the values of b' of 0.3, 0.5, 1.0, and 10. These constant backflow rates, b', are all greater than the critical backflow parameter of 0.247 ($K_1 = 0.01$, $N = 4$) at which there is no washout because of continuous inoculation from the first stage [see Equation (43)]. Because of this, these results show a monitonically increasing relationship between the washout dilution rate and the backflow parameter b. Figures 12 to 14 also show that the critical washour dilution rate increases as b' decreases.

(c) Effect of sedimentation

The appearance of cell sedimentation is a natural phenomena of multiphase processes in which there are density defferences between the solid and liquid phases. In tower fermentation processes, there are three phases (solid, liquid, and gas). A continuous liquid phase contains the liquid media of disslved organic and inorganic nutrients. Oxygen is supplied through the gas phase and cells are suspended in the liquid phase. The cells may sediment due to density differences or pass up the column with air bubbles due to surface forces. A higher washout dilution rate is predicted for a system with sedimentation. while accumulation of cells in a foaming effluent may reduce the washout dilution rate. The numerical values used in this investigation assume that sedimentation is the primary factor influencing the deviation of the cell flow behavior from the liquid flow.

The characteristic equations to predict the critical washout dilution rate for several defferent cases are given in Equations (35) to (38). The sedimentation parameter, δ, which appears in these equations accounts for the cell sedimentation which is considered. The results in Figs. 16 to 18, which assume that is equal to 1.2, show that the critical dilution rate is larger when sedimentation is present. Increases in the critical washout dilution rate are expected for systems with sedimentation; however, the specific effects of the sedimentation on the system are of considerable interest, especially, the extreme cases of the system at large and small backflow rates.

Mathematical analysis of the effect of sedimentation on a multistage tower fermentation process for Case 1 is based on the cell mass balances around each stage in the system. These dimensionless cell mass balance equations are described by Equation (27), (28), and (29), in which the parameter, δ, is employed to account for the sedimentation of cells in each stage. A good understanding of the interaction of sedimentation and backflow on the critical washout dilution rate can be obtained through analysis of the extreme values of no backflow and very large backflow for the system with sedimentation.

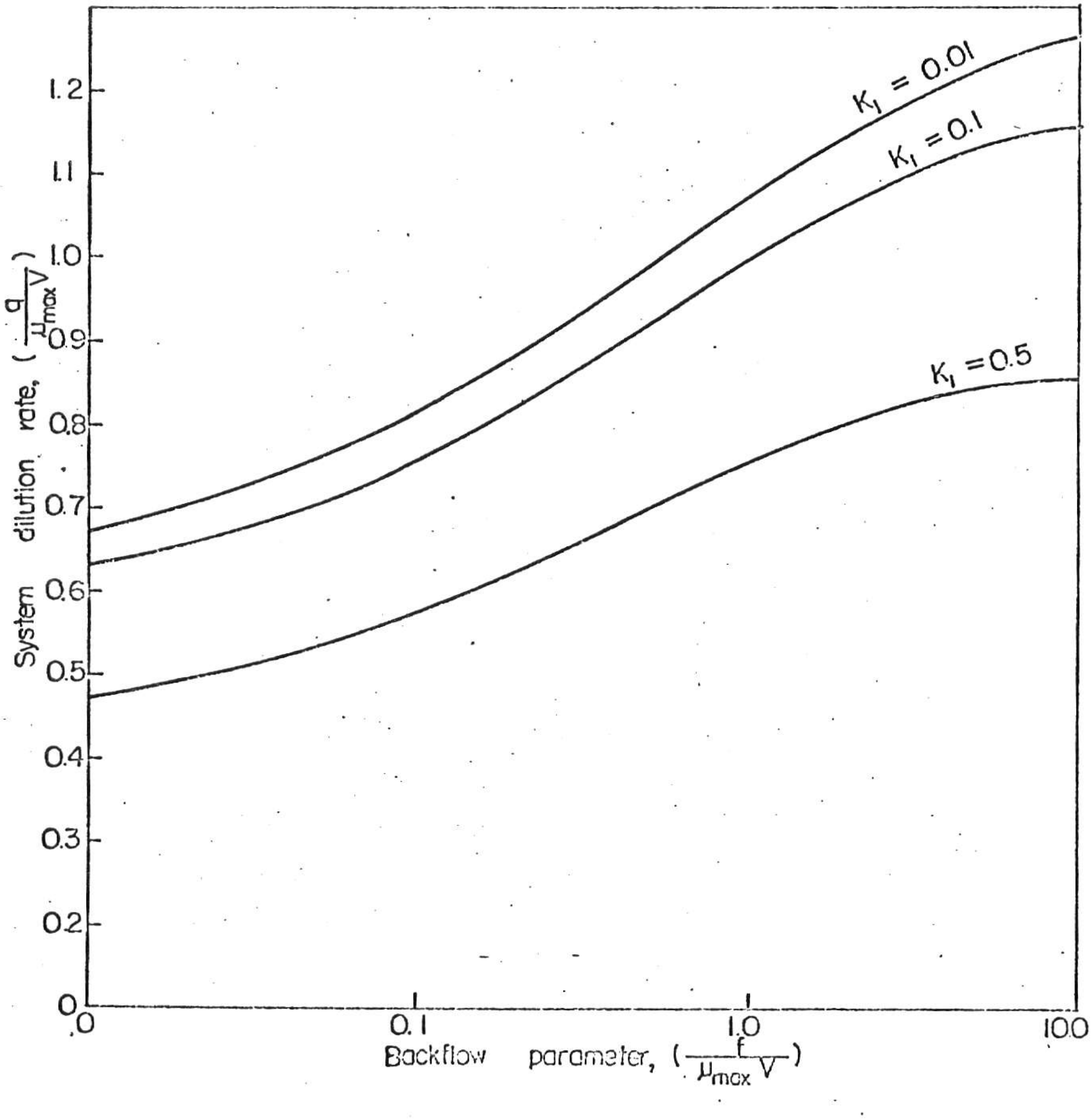

Fig. 16. Effect of backflow on washout dilution for a 2 stage system with $\delta = 1.2$. (Case I)

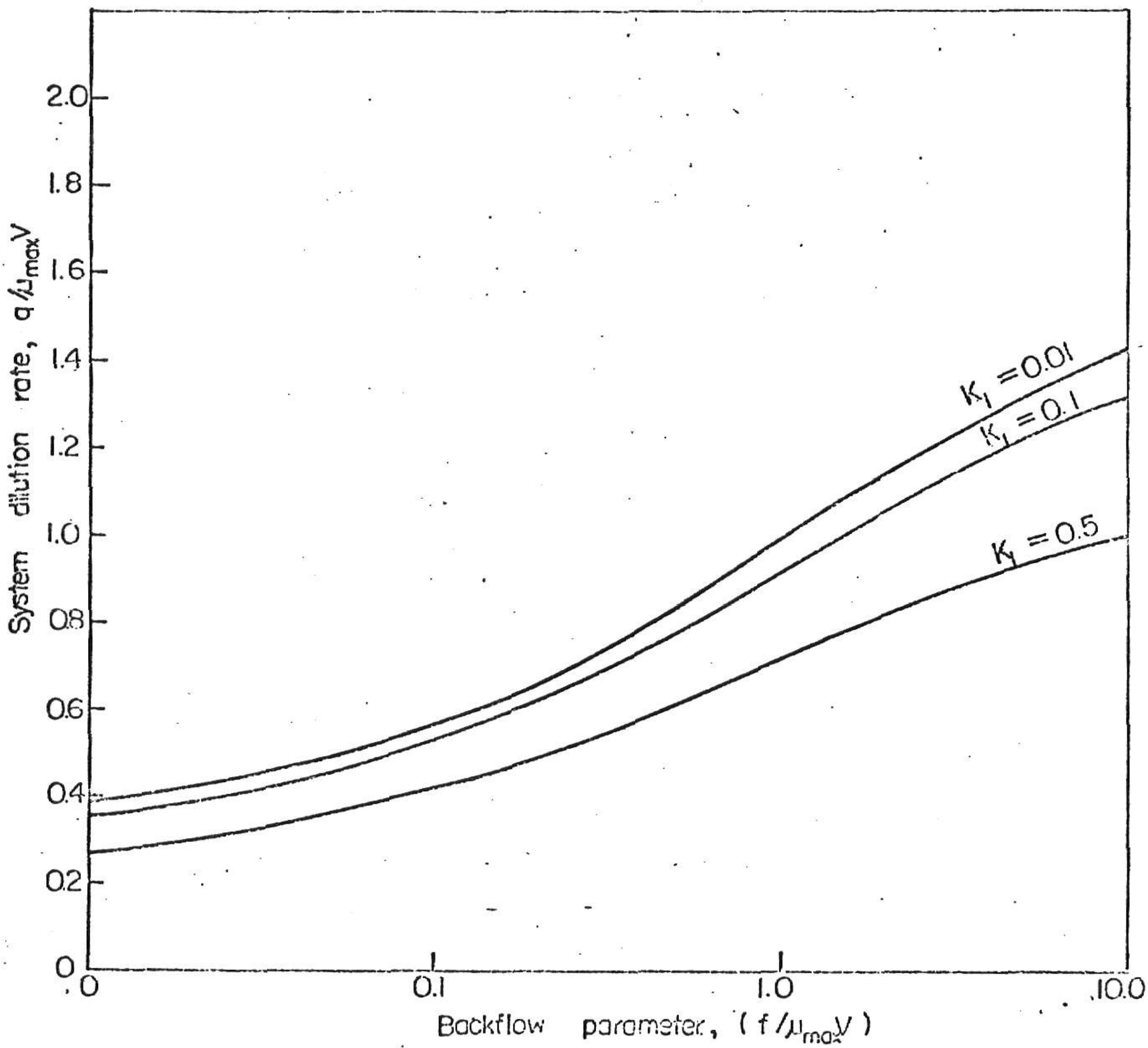

Fig.17. Effect of backflow on washout dilution rate for a 4 stage system with $\delta = 1.2$. (Case 1)

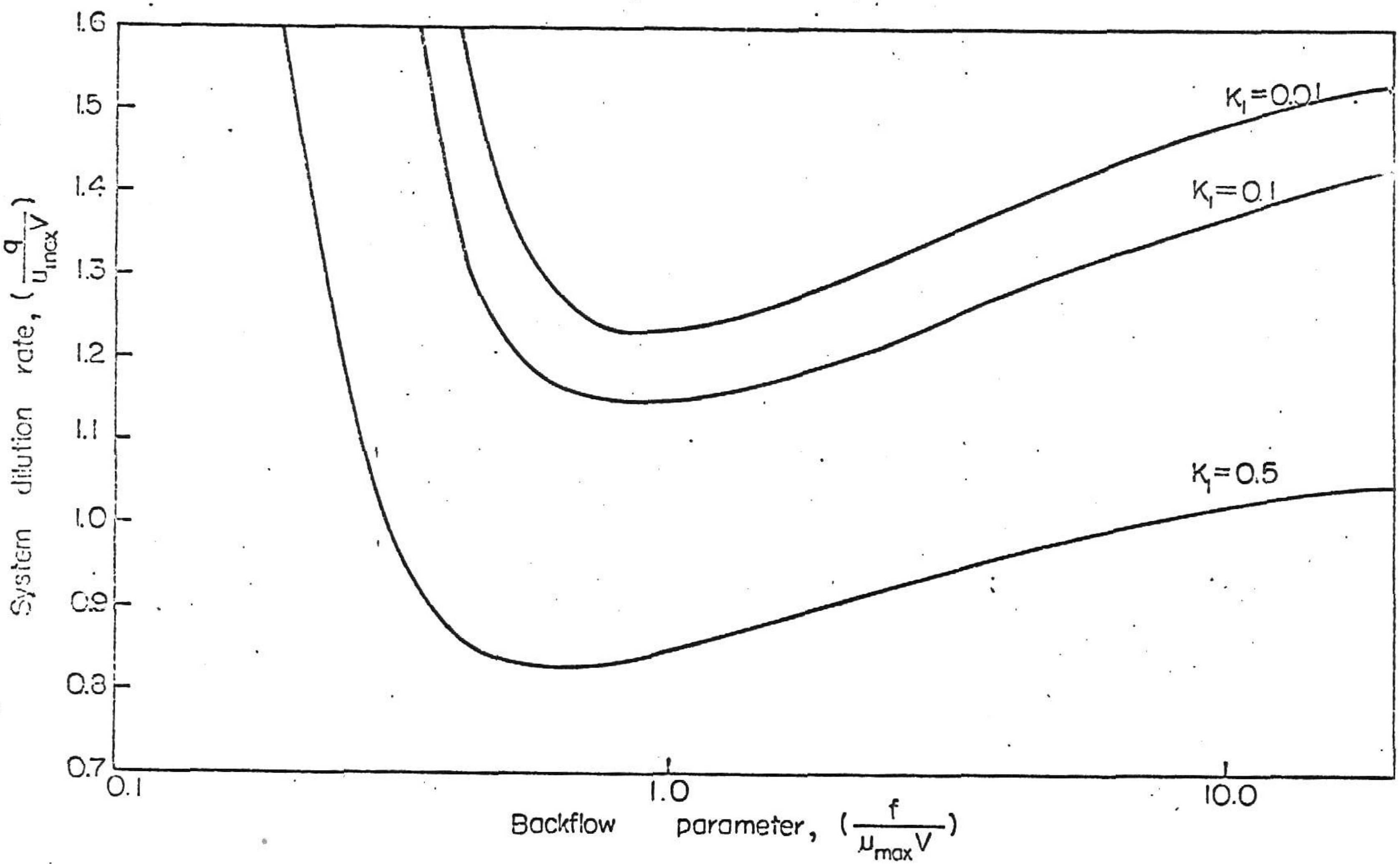

Fig. 18 . Effect of backflow on washout dilution rate for a 4 stage system with =1.2. (Case 2)

Small backflow rate, $b \rightarrow 0$

Essentially, the system becomes an N CSTR's-in-series system when backflow approaches zero. The critical washout dilution rate can be determined by substituting the limiting value of backflow rate b = 0 into Equation (27) which reduces to

$$- dy_2^1 + \frac{y_1^1 \delta y_2^1}{N(K_1 + y_1^1)} = 0 \qquad (44)$$

At washout, $y_1^1 = 1.0$ and Equation (44) reduces to

$$d_w' = \frac{\delta}{N(K_1 + 1.0)} \qquad (45)$$

where d_w', is the critical washout dilution rate when sedimentation is present. It is related to the system washout dilution rate, d_w, for an N stage system with no sedimentation as follows;

$$d_w' = \delta d_w \qquad (46)$$

Obviously, $d_w' > d_w$ if $\delta > 1.0$, i.e., the critical washout dilution rate for N-stage systems with sedimentation at small backflow ($b \rightarrow 0$) is δ times that for the system with no sedimentation.

Large backflow rate, $b \rightarrow \infty$

The degree of mixing between stages is increased as the backflow parameter increases. but according to the model there must be a difference in cell concentrations between stages when sedimentation is present. The cell concentration distribution in the system when $b \rightarrow \infty$ may be expressed by a set of relations

$$\begin{aligned}
\bar{y}_2^{N-1} &= \delta\,\bar{y}_2^{N} \\
\bar{y}_2^{N-2} &= \delta\,\bar{y}_2^{N-1} = (\delta)^2\bar{y}_2^{N} \\
&\vdots \\
\bar{y}_2^{1} &= \delta\bar{y}_2^{2} = \;.\;.\;.\; = (\delta)^{N-1}\bar{y}_2^{N}
\end{aligned} \tag{47}$$

The deailed derivation of these relations is presented in the Appendix to this chapter. The Appendix also gives a detailed derivation of the critical washout dilution rate when $b \rightarrow \infty$; that is,

$$d_w' = \frac{1.0}{N(K_1 + 1.0)}\left[(\delta)^N + (\delta)^{N-1} + \;....\; + (\delta)^2 + \delta\right]$$

$$= \frac{1.0}{N(K_1 + 1.0)} \sum_{i=1}^{N} (\delta)^i \tag{48}$$

This critical dilution rate is related to d_w for the same system with no sedimentation by the relation

$$d_w' = \frac{\sum_{i=1}^{N} (\delta)^i d_w}{N} \tag{49}$$

Equation (45) and (48) represent the critical washout dilution rates for N-stage systems with sedimentation at large and small backflow rates respectively. Tables 7 gives the critical washout dilution rates for limiting backflow rates for several values of K_1, N, and δ. Figures 16 and 17 show that the extreme values of the critical washout dilution rate for large and small backflow rates in agreement with those in Table 7 ($\delta = 1.2$). For Case 1, the critical washout dilution rates for other intermediate backflow

Table 7. Critical washout dilution rates for limiting backflow rates for several values of K_1, N, and δ, for Case 1.

N	K_1	b=0		b→∞	
		δ=1.0	δ=1.2	δ=1.0	δ=1.2
1	0.01	0.990	1.188		
	0.1	0.909	1.090		
	0.5	0.666	0.800		
2	0.01	0.495	0.594	0.990	1.306
	0.1	0.454	0.545	0.909	1.200
	0.5	0.333	0.400	0.666	0.800
4	0.01	0.247	0.297	0.990	1.594
	0.1	0.227	0.272	0.909	1.463
	0.5	0.165	0.200	0.666	1.073
8	0.01	0.123	0.148	0.990	2.443
	0.1	0.113	0.136	0.909	2.243
	0.5	0.083	0.100	0.666	1.645

rates lie between these extreme results. For the case of very large backflow and no sedimentation ($b \to \infty$, $\delta = 1.0$), increasing the stage number doesn't affect the critical washout dilution rate, because the system acts as one completely mixed tank. However, when sedimentation is present ($b \to \infty$, $\delta > 1.0$), the critical washout dilution rate increases with increasing stage number, because the distribution of cell concentration is not uniform from stage to stage in these cases, as is discussed in Equation (47). The sedimentation parameter appears in the form of a geometric series in the system with sedimentation and large backflow rates [see Equation (48)], and because of this a change in the number of stages results in a change of performance as predicted by this sedimentation model.

For Case 2, the critical backflow rate can be derived using the same procedure as that discussed previously, for Case 2, except that the sedimentation parameter has to be taken into account. Upon introducing the sedimentation parameter, Equation (43) which predicts the critical backflow rate for washout of the first stage reduces to

$$b_2 = \frac{\delta}{N(K_1 + 1.0)} \qquad (50)$$

Because of the behavior of the first stage, for nonzero backflow rates the system will never be washed out when the backflow, b_2, is less than that obtained from Equation (50). Thus, as shown in Fig. 18 the critical washout dilution rate approaches infinity as b_2 approaches from above the critical value given by Equation

(50). A convex curve, as shown in Fig. 18, results for the same reasons as those presented for the case of no sedimentation; however, because of the effects of sedimentation this convexity is more pronounced in Fig. 18. When the backflow rate approaches infinity, the critical washout dilution rates for Case 2 are in agreement with those shown in Table 7. Equation (48) for $b \to \infty$ can also be used for Case 2.

In summary, the general effect of cell sedimentation is to increase the critical washout dilution rate. In Case 2, the critical backflow rate governing washout of the first stage is also increased by sedimentation.

(d) Effect of kinetic constants

The nonlinear kinetic model used in this work indicates that the microbial growth rate depends not only on the cell and substrate concentrations, but also on the kinetic constants, μ_{max} and K_S. Determination of these kinetic constants μ_{max} and K_S, which is often based on batch kinetic data, is important as these constants affect system performance. The maximum specific growth rate, μ_{max}, affects the dimensional critical washout dilution rate, because the dimensionless dilution rate is defined as

$$d = \frac{q}{V\mu_{max}} \tag{51}$$

Therefore, the dimensionless critical washout dilution rate is independent of μ_{max} because μ_{max} does not appear in Equation (19). But μ_{max} linearly affects the dimensional critical washout dilution rate in that

$$\frac{q}{V} = d\mu_{max} \tag{52}$$

The constant K_1 ($K_1 = K_S/S^0$) also plays an important part, but it does not have as great an effect on washout as μ_{max}. In general, the dimensionless critical washout dilution rate decreases as K_1 increases, because a large K_1 indicates a slow dimensionless specific growth rate. In contrast, a higher dimensionless critical washout dilution rate is obtained when K_1 is small. The value of K_1 is determined by the value of K_S and the initial substrate concentration (S^0). The various values of K_1 used in Figs. 7 through 18 show that the dimensionless critical washout dilution rate is increases as K_1 decreases.

3-6 CONCLUSIONS

In a chemostat, the critical washout dilution rate is directly related to the maximum specific growth rate because the growth rate is assumed to be everywhere the same within the system. In the tower system the specific growth rate varies from stage to stage and the liquid and cell mean residence times also may vary from stage to stage because of choice of flow geometry, backflow rate, and sedimentation rate. The results show that feed geometry, cell sedimentation, backflow rate, and the growth kinetic constants all influence the critical washout dilution rate.

The feed geometry can greatly influence the critical washout dilution rate, especially at small and moderate values of backflow. When the backflow rate is small, changing the feed geometry can prevent washout from occuring as is shown by the analysis of Cases 2 and 3. By feeding to the second stage, the first stage may be used to continuously inoculate the second stage. This allows for great operating flexibility in that large feed flow rates can be used to obtain growth rates near the maximum specific growth rate in the stages above the feed point, for example.

The influence of backflow rate on the critical washout dilution rate depends on the feed geometry at low backflow rates, but it becomes independent of feed geometry at very large backflow rates. For Case 1 the critical washout dilution rate increases as backflow and sedimentation increase; however, for Case 2 the backflow parameter has an opposite effect at low backflow rates because of the stable operation of the first stage that is created under these conditions.

NOMENCLATURE

b_i = Dimensionless backflow rate, $(\frac{f^i}{_{max}V})$.

d_i = Dimensionless dilution rate, $(\frac{F^i}{_{max}V})$.

d_w = Washout dimensionless dilution rate.

f_i = The backflow rate coming from stage i, liter/hr.

K_S = The concentration of organic at which the specific growth rate observed is one half the maximum value, mg/liter.

K_1 = The dimensionless organic concentraion at which the specific growth rate observed is one half the maximum value.

q = Influent flow rate.

q_i^0 = Influent flow rate to stage i.

q_i^e = Effluent flow rate from stage i.

r_i = Dimensionless specific growth rate.

S^0 = The concentration of organics in a feed stream, mg/liter.

S^i = The concentration of organics in stage i, mg/liter.

V_i = The liquid volume of stage i.

V = The total liquid volume of the system.

v_i = Volume fraction, $(\frac{V_i}{V})$.

X^i = The cell concentration in stage i, mg/liter.

$\bar{X}^i$ = The cell concentration in the stream leaving stage i, mg/liter.

y_1^i = The dimensionless concentration of substrate in stage i, $(\frac{S^i}{S^f})$.

y_2^i = The dimensionless cell concentration in stage i, $(\frac{X^i}{YS^f})$.

$\bar{y}_2^i$ = The dimensionless cell concentration in the stream leaving stage i, $(\frac{\bar{X}^i}{YS^f})$.

μ_{max} = Maximum specific growth rate when the organic concentration is not limiting the rate of growth, hr^{-1}.

μ = Specific growth rate.

β = Secondary clarifier parameter.

δ = Sedimentation parameter.

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APPENDIX

The degree of mixing between stages is increased as the backflow rate increases. A multistage system acts as one completely mixed tank when the backflow rate is large, and there is no sedimentation of cells. When sedimentation is present, the cells concentrations are defferent from stage to stage, but the substrate concentration is the same in each stage. The cell concentration distribution predicted for systems with sedimentation at large backflow rate can be obtained from a mathematical analysis of the following equations which are the cell mass balance equations for Case 1

$$\delta b\bar{y}_2^2 - (d + b)\bar{y}_2^1 + \frac{y_1^1 \delta \bar{y}_2^1}{N(K_1 + y_1^1)} = 0 \tag{A-1}$$

$$(d + b)\bar{y}_2^{i-1} + \delta b\bar{y}_2^{i+1} - (d + b + \delta b)\bar{y}_2^i + \frac{y_1^i \delta \bar{y}_2^i}{N(K_1 + y_1^i)} = 0 ,$$

$$i = 2, 3, \ldots, N-1 \tag{A-2}$$

and

$$(d + b)\bar{y}_2^{N-1} - (d + \delta b)\bar{y}_2^N + \frac{y_1^N \delta \bar{y}_2^N}{N(K_1 + y_1^N)} = 0 \tag{A-3}$$

If Equation (A-1) is divided by b

$$\bar{y}_2^2 - (\frac{d}{b} + 1)\bar{y}_2^1 + \frac{y_1^1 \delta \bar{y}_2^1}{Nb(K_1 + y_1^1)} = 0 \tag{A-4}$$

As b approaches infinite, Equation (A-4) reduces to

$$\bar{y}_2^1 = \delta \bar{y}_2^2 \tag{A-5}$$

Similarly, Equation (A-2) can be divided by b to obtain

$$(\frac{d}{b} + 1)\bar{y}_2^{i-1} + \delta\bar{y}_2^{i+1} - (\frac{d}{b} + 1 + \delta)\bar{y}_2^i + \frac{y_1^i\delta\bar{y}_2^i}{Nb(K_1 + y_1^1)} = 0 \quad \text{(A-6)}$$

which reduces to

$$\bar{y}_2^{i-1} + \delta\,\bar{y}_2^{i+1} - (1 + \delta)\bar{y}_2^i = 0, \qquad i = 2, 3, \ldots, N-1 \quad \text{(A-7)}$$

as b . When i = 2, Equations (A-5) and (A-7) can be combined to yield

$$\bar{y}_2^2 = \delta\,\bar{y}_2^3 \quad \text{(A-8)}$$

Similarly, a general relatiinship of the cell concentration between stages can be obtained by repeating this process for i = 3, 4, ..., N-2, and then employing Equation (A-3) when i = N-1. we can write

$$\bar{y}_2^i = \delta\,\bar{y}_2^{i+1}, \qquad i = 1, 2, \ldots, N-1 \quad \text{(A-9)}$$

The cell concentration at each each stage can be expressed in terms of $\bar{y}_2^N$. These linear relations with respect to $\bar{y}_2^N$ are given in Equation (47).

The critical washout dilution rate for N-stage equal volume system(Case 1) with sedimentation and large backflow rate can be obtained by adding up Equations (A-1), (A-2), and (A-3) to obtain

$$-d\bar{y}^N + \frac{y_1^1\delta\bar{y}_2^1}{N(K_1 + y_1^1)} + \frac{y_1^2\delta\bar{y}_2^2}{N(K_1 + y_1^2)} + \ldots + \frac{y_1^{N-1}\delta\bar{y}_2^{N-1}}{N(K_1 + y_1^{N-1})}$$

$$+ \frac{y_1^N\delta\,\bar{y}_2^N}{N(K_1 + y_1^N)} = 0 \quad \text{(A-11)}$$

At washout, since the substrate is not consumed at all in the system, we have

$$y_1^1 = y_1^2 = \dots = y_1^{N-1} = y_1^N = 1 \tag{A-12}$$

When the backflow rate is large, Equations (A-12) and (47) may be substituted into Equation (A-11) to obtain the critical washout dilution rate as

$$d_w' = \frac{1.0}{N(K_1 + 1.0)} \left[(\delta)^N + (\delta)^{N-1} + \dots + (\delta)^2 + \delta \right]$$

$$= \frac{1.0}{N(K_1 + 1.0)} \sum_{i=1}^{N} (\delta)^i \tag{A-16}$$

Chapter 4

MODELING AND OPTIMIZATION OF A TOWER ACTIVATED SLUDGE SYSTEM

4-1 INTRODUCTION

As efforts are made to reduce the quantity of pollutants discharged into receiving streams and to improve the performance of waste water treatment systems, an increasing emphasis will be placed on the design, control, and performance of waste treatment systems. New or modified processes which reduce capital or operating costs, give better control, or provide improved performance will be introduced. Cocurrently aerated multistage tower type activated sludge waste treatment processes such as those shown in Figs. 1 and 2 may give improved performance at reduced cost and at the same time provide opportunities for good process control.

Although not much has been published on experimental aspects of microbial growth in tower type systems (1, 2, 3, 4, and 5), the tower type systems have been employed industrially (12) and appear sufficiently promising to warrant further investigation. In most of the experimental work (1, 2, and 4) sieve plates have been used in the tower to induce dispersion of the air bubbles which pass up through the tower. This together with the jet action of the fluid provides agitation of the fluid. This tower system which is divided into several compartments by sieve plates may be considered to be a multistage system. The multistage system with backflow model which was proposed by Miyauchi (6) and Haddad (7) may be used to describe the complicated flow patterns which occur

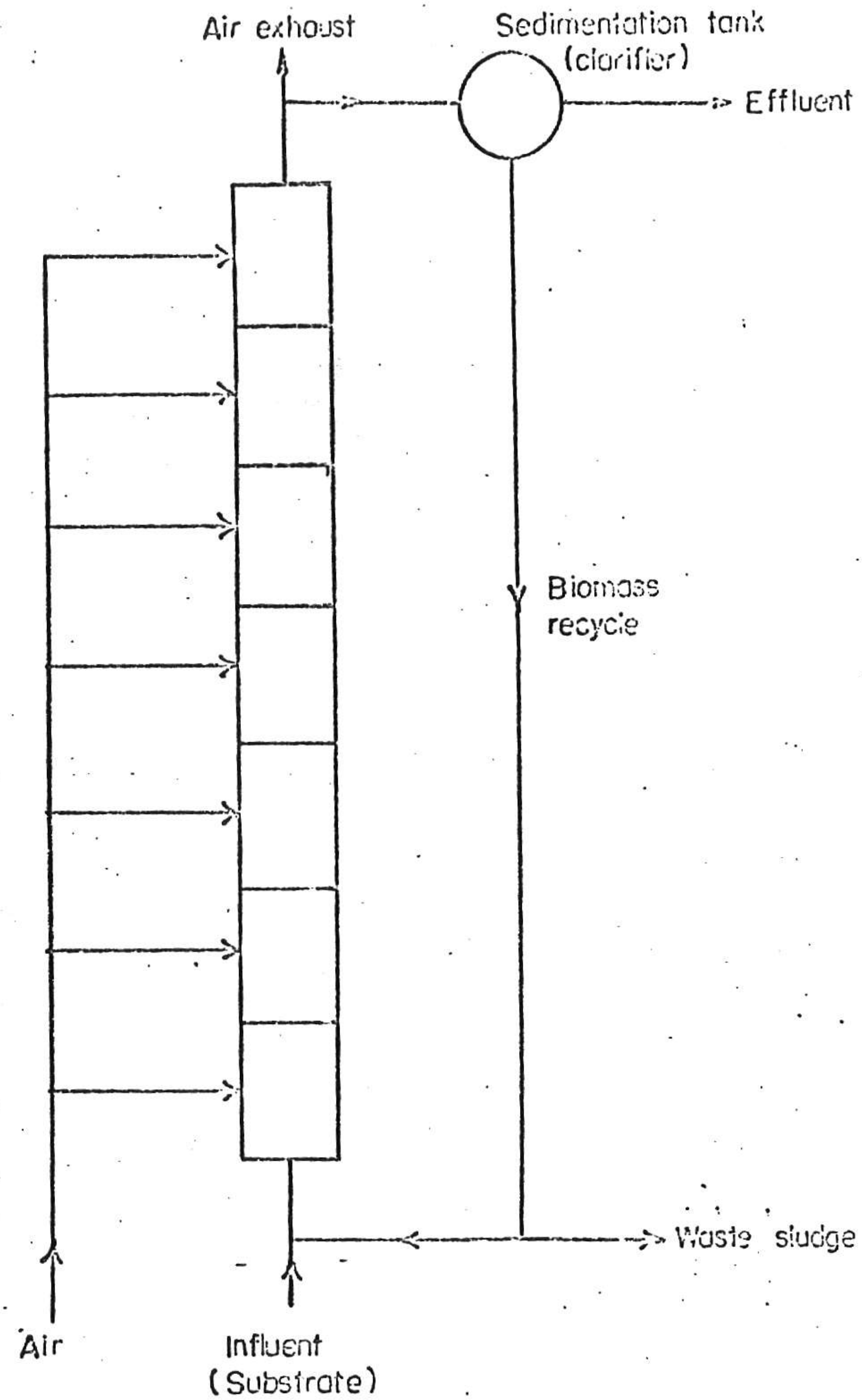

Fig. 1. Schematic diagram of a tower type biological waste treatment system with influent fed at the bottom stage

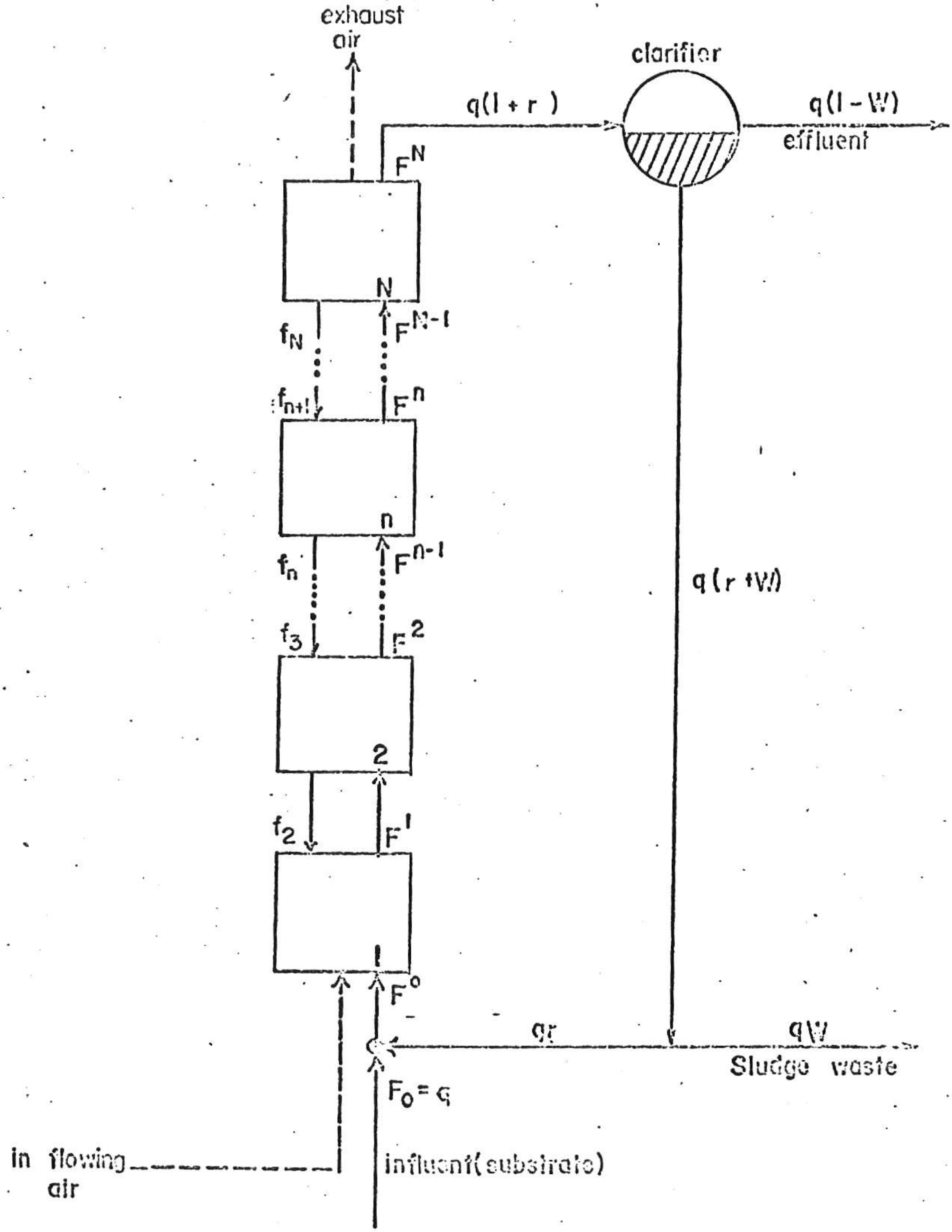

Fig.2. Schematic diagram of tower type biological treatment process with flow notation of liquid streams.

in the tower.

Sedimentation of organisms within the tower has been reported (1, 2, 4) and excellent oxygen transfer rates have been noted (4). The advantages of a tower type biological waste treatment system which can be visualized include reduction in aeration tank volume because of some sludge sedimentation within the aeration tank and better control of desired mixing patterns within the aeration tank. Good oxygen transfer and improved oxygen utilization can be obtained because of the sieve plates. The repeated redispersion of air by the sieve plates enable volumetric mass transfer coefficients of up to 400 $hours^{-1}$ and yet the method of providing for these high oxygen transfer rates also allows the solids in the system to settle. Since experimental studies have shown the highest cell concentrations to be in the lower part the tower, improved oxygen utilization can be obtained besause oxygen concentration profiles in the gas phase are similar to those of the active mass.

In addition to the primary advantages cited above, a tower type biological waste treatment system may require less space and land requirements and be easily integrated into an industrial complex. The flexibility of operation may enable the process to be controlled at a variety of operating conditions with the best operating condition depending on the feed condition.

The purpose of this work is to present mathematical models which can be used to characterize a tower type activated sludge system and then to investigate some of the design implications of these models. Optimization procedures are used to find optimum values of certain design variables.

A tower type biological waste treatment process in which compartments are separated by sieve plates is shown in Fig. 2. Substrate and air flow are cocurrent from the bottom to the top of the column. In this study, varying degrees of flow nonideality, i.e., fluid backflow and cell sedimentation are examined. The former seems to be due to interaction of the cocurrent gas and liquid flow (1). It is particularly dependent upon sieve plate hole void area and fluid flow rates. Sedimentation is probably a function of plate design as well as cell size and density.

In this work, the tower type biological waste treatment process with feeding patterns as shown in Fig. 2 is considered. In all of the cases considered here, all of the influent is fed to the first stage. Cases with backflow and sedimentation between each stage are compared with cases without backflow and sedimentation under optimal conditions. The effects of recycle of organisms and endogeneous respiration are included in the model used in this investigation; however, sufficient oxygen is assumed to be present for the biological oxidation and the oxygen concentration is not considered in the mathematical model.

The process is analyzed by employing mathematical modeling and optimization procedures to determine the optimal policy of several of the design variables. A mathematical model which describes the growth kinetics of the biological waste treatment and a mathematical model which represents the hydrodynamic behavior of the flow system are included in the systems model.

In the system approach, simulation and mathematical optimization procedures are used to find the optimum of the mathematical design problem. However, since the model is only an approximation of the system, the optimum of the mathematical problem probably will deviate from the true optimum. Because of this and shortage of experimental information, engineering judgement and experience must be used in evaluating the results which are presented.

4-2 KINETIC MODEL

The growth kinetic model used here is the same as that used previously (8, 9, 10). It is a modified form of Monod's equation (8) which was based on Michaelis-Menten enzyme reaction. The additional assumptions are that oxygen and other trace nutrients are available in sufficient quantites and that growth is limited only by the organic which provides the carbon substrate for cell growth. The kinetic model for cell growth is

$$\frac{dX}{dt} = \frac{\mu_{max}\, SX}{K_S + S} - k_D X \tag{1}$$

where

$\frac{dX}{dt}$ = growth rate, mg/liter-hr.

S = concentration of organic nutrients, mg/liter

X = concentration of active microoganism, mg/liter

μ_{max} = maximum specific growth rate, hr^{-1}.

K_S = saturation constant, mg/liter.

k_D = specific endogeneous microbial attrition rate, hr^{-1}.

When growth occurs according to Equation (1), the organic nutrients are being consumed at a rate

$$\frac{dS}{dt} = \frac{-\mu_{max} S X}{Y(K_S + S)} \qquad (2)$$

where

$-\frac{dS}{dt}$ = rate with which organic nutrients are consumed, mg/liter-hr.

Y = yield factor, dimensionless.

4-3 MATHEMATICAL REPRESENTATION OF THE PROCESS

(a) Flow parameters

All feed is assumed to be allocated to the first tank where it is mixed with a recycle stream with high concentration of organisms. The hydraulic model that is considered in this work is shown schematically in Fig. 2. The model for a system with N compartments is composed of a sequence of N completely mixed stages connected in series followed by a secondary clarifier. A portion of the sludge from the bottom of the clarifier is removed and sent to the sludge disposal system, and the remainder is recycled. In Fig. 2, q is the volumetric flow rate, qr is the recycle flow rate and qw is the flow rate to the sludge digestion system. The concentration of the food or organic waste in the nth stage is represented by S^n. X^n is the concentration of the active microorganisms in the nth stage, and β is the separation concentration efficiency. f_n is the backflow rate from the nth stage to the (n-1)th stage, F_n is the volumetric flow rate leaving the nth stage, i,e., the flow rate F^n plus the flow rate f_n due to backflow to the (n-1)th stage. The backflow parameter G which is a measure of the flow leaving each stage which flows down the column is defined in terms of f_n, F^n and F_n as follows

$$G = \frac{f_n}{F_n} = \frac{f_n}{F^n + f_n} \tag{6}$$

When G = 0, there is no backflow and the system reduces to the simple case of the CSTR's-in-series model, while as G becomes large the system approaches one complete mixing stage.

Delta (δ) is the sedimentation factor in each stage. For simplicity, a uniform cell concentration, X^n, is assumed throughout the nth stage; however, the cell concentration in the upward flow leaving that stage, $\bar{X}^n$, is assumed to be less than X^n because of sedimentation. The sedimentation factor, δ which is defined by the equation

$$\delta = \frac{X^n}{\bar{X}^n} \tag{7}$$

is always greater than or equal to one. For simplicity, the backflow leaving each stage is assumed to have a cell concentration equal to that in that stage.

(b) Analysis at the inlet

As shown in Fig. 2, the influent is mixed with the sludge recycle stream at the inlet and introduced into the first stage. A balance of the volumetric flow gives

$$F^0 = q + q\,r \tag{8}$$

where F^0 is the volumetric flow rate coming into the first stage. The organic balance at this point is

$$q\,S^f + q\,r\,S^N = q(1 + r)\,S^0 \tag{9}$$

where S^f is the concentration of the organic waste in the influent fed to the system. If the cell concentration in the influent is assumed to be equal to zero, the organism balance is

$$q(1 + r)\,X^0 = q r \beta X^N \tag{10}$$

or

$$X^0 = \frac{r \beta X_2^N}{1 + r} \tag{11}$$

(c) Analysis at each stage

A substrate balance around the first stage as shown in Fig. 2 gives

$$F^0 S^0 + f_2 S^2 - F^1 S^1 - V_1 \left(\frac{\mu_{max}\, S^1 X^1}{Y(K_S + S^1)}\right) = 0 \tag{12}$$

where the growth kinetics are those described before in Equation (2). Similarly, an organism balance around the first stage gives

$$F^0 X^0 + f_2 X^2 - F^1 \bar{X}^1 + V_1 \left(\frac{\mu_{max}\, S^1 X^1}{K_S + S^1} - k_D X^1\right) = 0 \tag{13}$$

A volumetric flow rate balance around the first stage yields

$$F^1 = F^0 + f_2 \tag{14}$$

Similarly, organic substrate, organism, and volumetric flow rate balances around the ith stage can be obtained as follows:

$$F^{i-1}S^{i-1} + f_{i+1}S^{i+1} - F^iS^i - f_iS^i - V_i\left(\frac{\mu_{max}\, S^i\, X^i}{Y(K_S + S^i)}\right) = 0 \quad (15)$$

$$F^{i-1}\bar{X}^{i-1} + f_{i+1}X^{i+1} - F^i\bar{X}^i - f_iX^i + V_i\left(\frac{\mu_{max}S^iX^i}{K_S + S^i} - k_DX^i\right)$$

$$= 0 \quad (16)$$

and

$$F^{i-1} + f_{i+1} = F^i + f_i \quad , \quad i = 2, 3, \ldots, N-1 \quad (17)$$

Balances on the organics, organisms and volumetric flow rate around the Nth stage are of the form

$$F^{N-1}S^{N-1} - F^NS^N - f_NS^N - V_N\left(\frac{\mu_{max}\, S^N\, X^N}{Y(K_S + S^N)}\right) = 0 \quad (18)$$

$$F^{N-1}\bar{X}^{N-1} - F^N\bar{X}^N - f_NX^N + V_N\left(\frac{\mu_{max}S^NX^N}{K_S + S^N} - k_DX^N\right) = 0 \quad (19)$$

and

$$F^{N-1} = F^N + f_N \quad (20)$$

(d) Analysis of secondary clarifier

The flow entering the secondary clarifier is denoted by $q(1 + r)$; the effluent flow from the clarifier is $q(1 - w)$ and the bottom flow is $q(r + w)$. This is schematically represented in Fig. 2. Since the organic waste is assumed to pass through the clarifier unchanged, the concentration in the effluent and bottoms is given by S^N. If the sludge is concentrated in the clarifier to a bottoms concentration of βX^N, the organism balance around the clarifier is

$$q(1+r)X^N = q(1 - w)X^e + q(r + w)\beta X^N \tag{21}$$

where X^e is the organism concentration in the effluent stream and β is the separator concentration efficiency. For any given set of influent flow rate and concentration, recycle flow rate, and waste sludge flow rate, the value of β is assumed constant in this investigation.

(e) Dimensionless variables

In order to make the results as general as possible, the concentrations are put into a dimensionless form. The organic waste concentrations are made dimensionless by dividing them by the concentration of the organic waste in the influent, S^f, and the organism concentrations by the product, YS^f. This gives rise to the following dimensionless variables.

$$y_1^i = \frac{S^i}{S^f}, \qquad y_2^i = \frac{X^i}{YS^f} \tag{22}$$

where y_1^i and y_2^i are the dimensionless concentrations of the organic waste and organism respectively in the ith stage.

The saturation constant K_S may also be made dimensionless by dividing it by S^f. that is

$$K_1 = \frac{K_S}{S^f} \tag{23}$$

Thus, K_1 is the dimensionless organic concentration at which the specific growth rate observed is one half the maximum value.

(f) Dimensionless material balance equations

Equations (12) through (19) can be rewritten in terms of dimensionless variables by using the dimensionless groups introduced in the previous section. The dimensionless organic material balance equations are

$$F^0 y_1^0 + f_2 y_1^2 - F^1 y_1^1 - V_1\left(\frac{\mu_{max} y_1^1 y_2^1}{K_1 + y_1^1}\right) = 0 \tag{24}$$

$$F^{i-1} y_1^{i-1} + f_{i+1} y_1^{i+1} - F^i y_1^i - f_i y_1^i - V_1\left(\frac{\mu_{max} y_1^i y_2^i}{K_1 + y_1^i}\right) = 0 \tag{25}$$

where i = 2, 3,..., N-1 and

$$F^{N-1} y_1^{N-1} - F^N y_1^N - f_N y_1^N - V_N\left(\frac{\mu_{max} y_1^N y_2^N}{K_1 + y_1^N}\right) = 0 \tag{26}$$

Similarly, the organism dimensionless balance equations are as follows:

$$F^0 y_2^0 + f_2 y_2^2 - F^1 \bar{y}_2^1 + V_1\left(\frac{\mu_{max} y_1^1 y_2^1}{K_1 + y_1^1} - k_D y_2^1\right) = 0 \tag{27}$$

$$F^{i-1} \bar{y}_2^{i-1} + f_{i+1} y_2^{i+1} - F^i \bar{y}_2^i - f_i y_2^i + V_i\left(\frac{\mu_{max} y_1^i y_2^i}{K_1 + y_1^i} - k_D y_2^i\right) = 0 \tag{28}$$

where i = 2, 3, ..., N-1 and

$$F^{N-1} \bar{y}_2^{N-1} - F^N \bar{y}_2^N - f_N y_2^N + V_N\left(\frac{\mu_{max} y_1^N y_2^N}{K_1 + y_1^N} - k_D y_2^N\right) = 0 \tag{29}$$

(g) The mathematical objective function

The mathematical objective function for this problem which is to be minimized is the total holding time of the biological growth chamber.

$$J = \sum_{i=1}^{N} \frac{V_i}{q(1 + r)} = \sum_{i=1}^{N} \bar{t}_i \tag{30}$$

for a fixed degree of treatment (fixed value of y_1^N) where V_i is the liquid volume of the ith stage and $\bar{t}_i$ is the mean residence time of the fluid in that stage. Although this mathematical objective function is rather simple, it can be used to obtain useful information about the effect of hydraulic flow pattern on the performance of the system.

4-4 COMPUTATION

Analytical expressions for the organic substrates and organisms for each stage are represented by Equations (24) through (29). The flow rate relationships are given by Equations (14), (17), and (20). In order to minimize the total holding time, the number of degrees of freedom in this system must by known. The result of this analysis, which follows that of Erickson and Fan (9), shows that (N-1) decision variables need to be considered in optimizing this N stage system. For example, in practice the relative volumes of individual stages could be selected so as to implement the optimal design. The optimal policy can be determined by solving 2 x N simultaneous equations with search techniques. The techniques employed in this study are the Golden section search and Simplex pattern search techniques. Optimal results are obtained for one, two, and three stage systems.

Additional details on the computation scheme used to obtain the optimal policy of the tower type biological treatment system are given in the Appendix I.

4-5 RESULTS AND DISCUSSION

Optimal results are obtained for 90, 95, 98, and 99% treatment for several values of the backflow and sedimentation parameters and K_1. The following values for the constants and parameters are used in this investigation.

$y_2^f = 0$

$\mu_{max} = 0.1\ hr^{-1}$

$k_D = 0.002\ hr^{-1}$

$r = 0.25$ dimensionless

$\beta = 4.0$ dimensionless

$K_1 = 0.01,\ 0.02,\ 0.05,\ 0.1,\ 0.2,\ 0.5$ dimensionless

$y_1^N = 0.01,\ 0.02,\ 0.05,\ 0.1$ dimensionless

$C = 0.0,\ 0.05,\ 0.10,\ 0.15,\ 0.20,\ 0.25$ dimensionless

$\delta = 1.0,\ 1.1,\ 1.2$ dimensionless

The results of this investigation are presented in tables in Appendix II and Figs. 3 through 16. For all cases investigated, the required holding time (the optimal total holding time) increases as the percentage of treatment increases; however, this increase in required holding time becomes much more rapid as the percentage of treatment approaches 100. As the number of aeration stages in the system is increased, the total required holding time decreases for a fixed percentage of treatment. This reduction of the required total holding time is partially the result of an increase in the number of degrees of freedom.

The variation of total holding time with percent treatment,

K_1 (saturation constant), G (backflow parameter), and δ (sedimentation parameter) as parameters is shown in Figs. 3 through 5 for the two stage system, and Figs. 6 through 8 for the three stage system. The results show that the total holding time increases with increasing values of K_1. This is due to the fact that the specific growth rate decreases as K_1 increases. In Fig. 3 and 6 where there is no sedimentation, the total holding time increases when backflow appears in the system. Because of the appearance of backflow in the system, a higher concentration of organisms is created in the lowest stage (see Fig. 13), along with a more dilute concentration of substrate (see Fig. 14). If one carefully examines the growth model of Monod's equation, increasing the backflow in the system with no sedimentation decreases the specific growth rate in the earlier stages. Thus, although the organism concentration in the earlier stages increases, the required total holding time increases because of the change in substrate concentration. The effect of backflow, therefore, is undesirable in the case of the system with no sedimentation.

As shown in Figs. 4 and 7, the effect of backflow may reduce the total required holding time when sedimentation is present in the system. Backflow still reduces the specific growth rate; however, the increase in the concentration of organisms through the system is more enhanced than when no sedimentation is present because of the combined effects of sedimentation and backflow (see Fig. 13).

Comparison of the effects of the sedimentation parameter and backflow on the total holding time of the two and three stage systems with certain values of saturation constant are shown

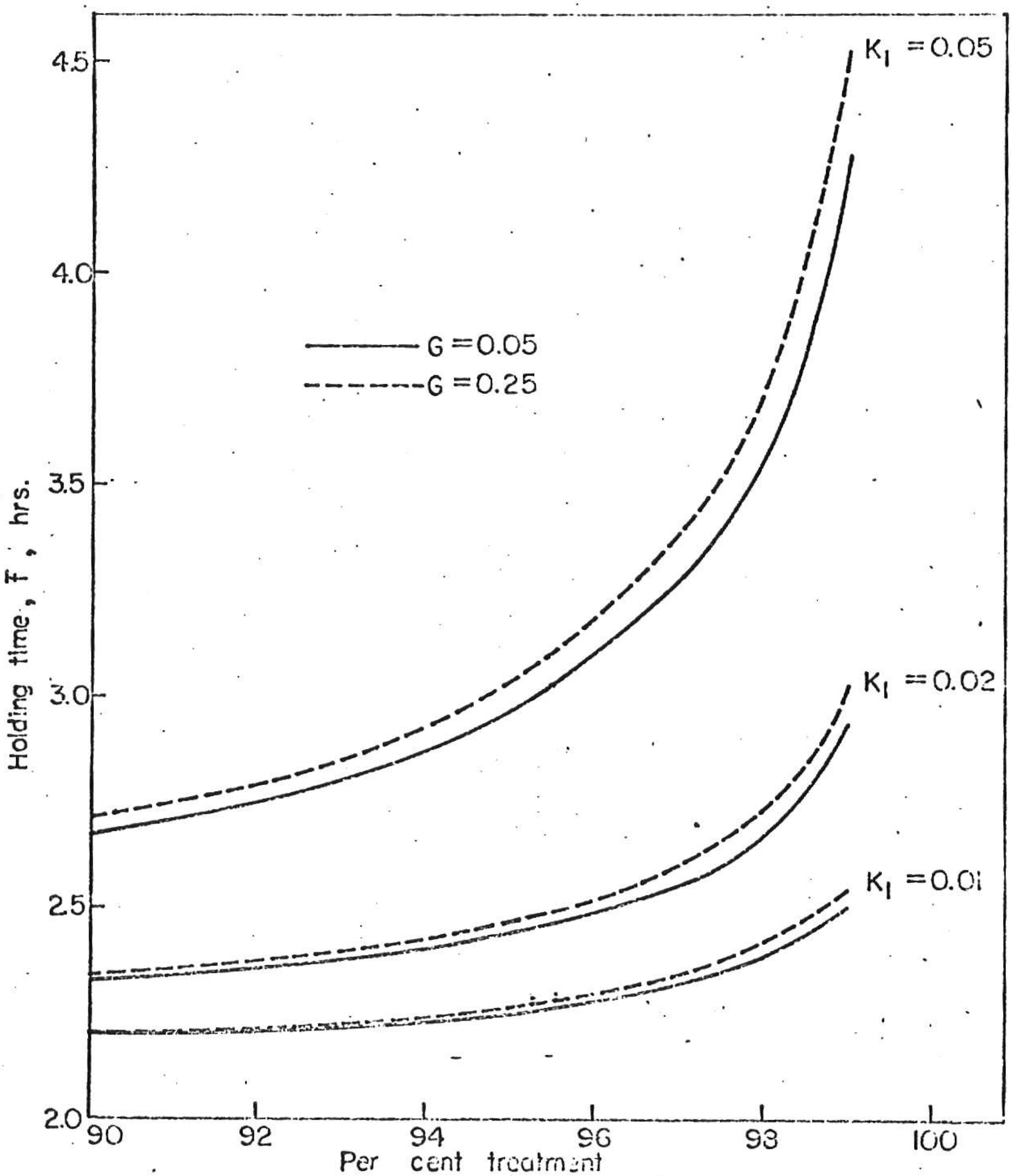

Fig. 3. Effect of backflow parameter G on the variation of optimal holding time with per cent treatment for several values of K_1 ($\delta = 1.0$, two stages).

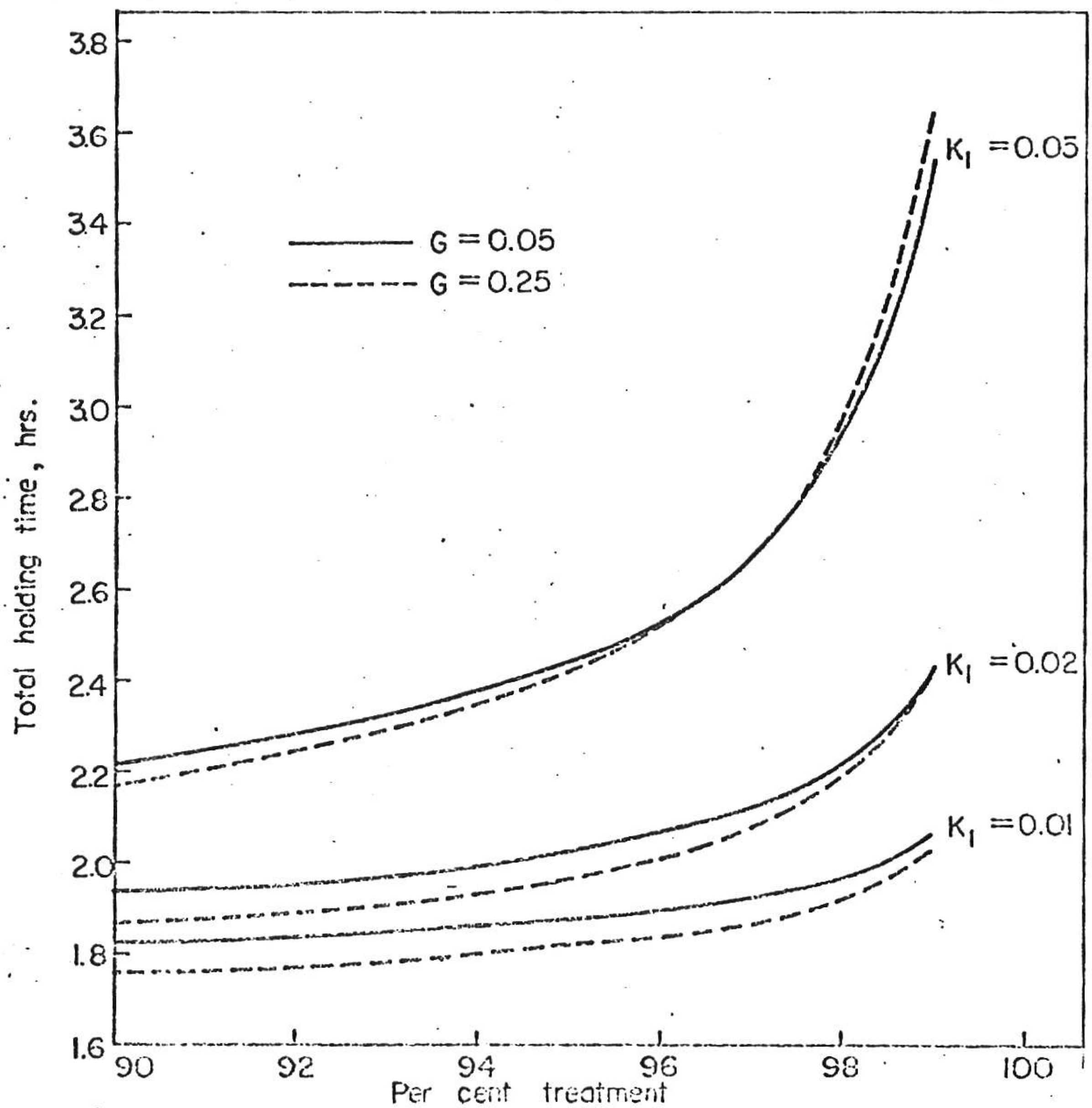

Fig. 4. Effect of back flow parameter G on the variation of optimal holding time with per cent treatment for several values of K_1 ($\delta = 1.2$, 2 stages).

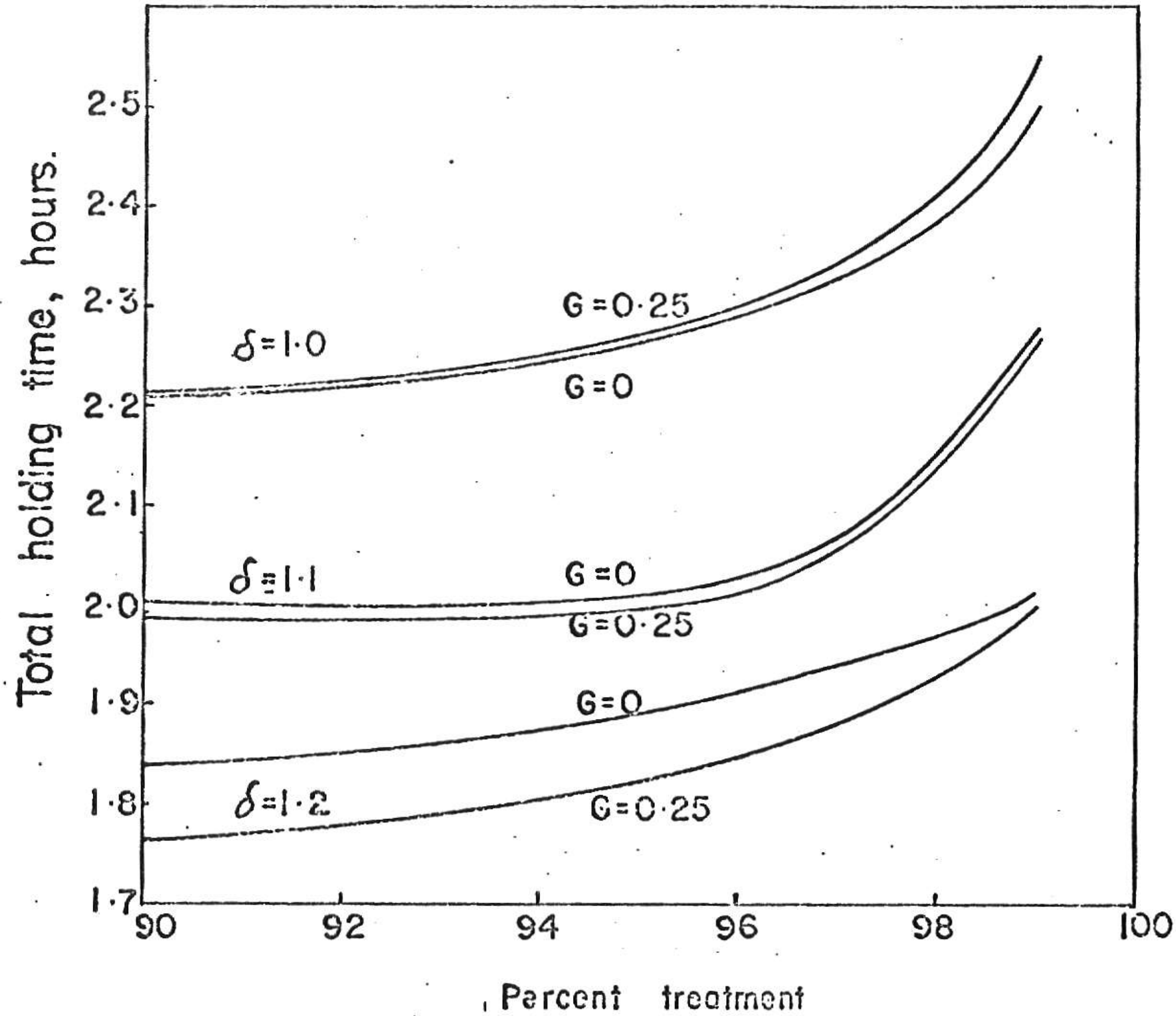

Fig. 5 Effect of sedimentation and backflow on optimal total holding time for a two stage system, $K_1 = 0{\cdot}01$ and $\beta = 4{\cdot}0$.

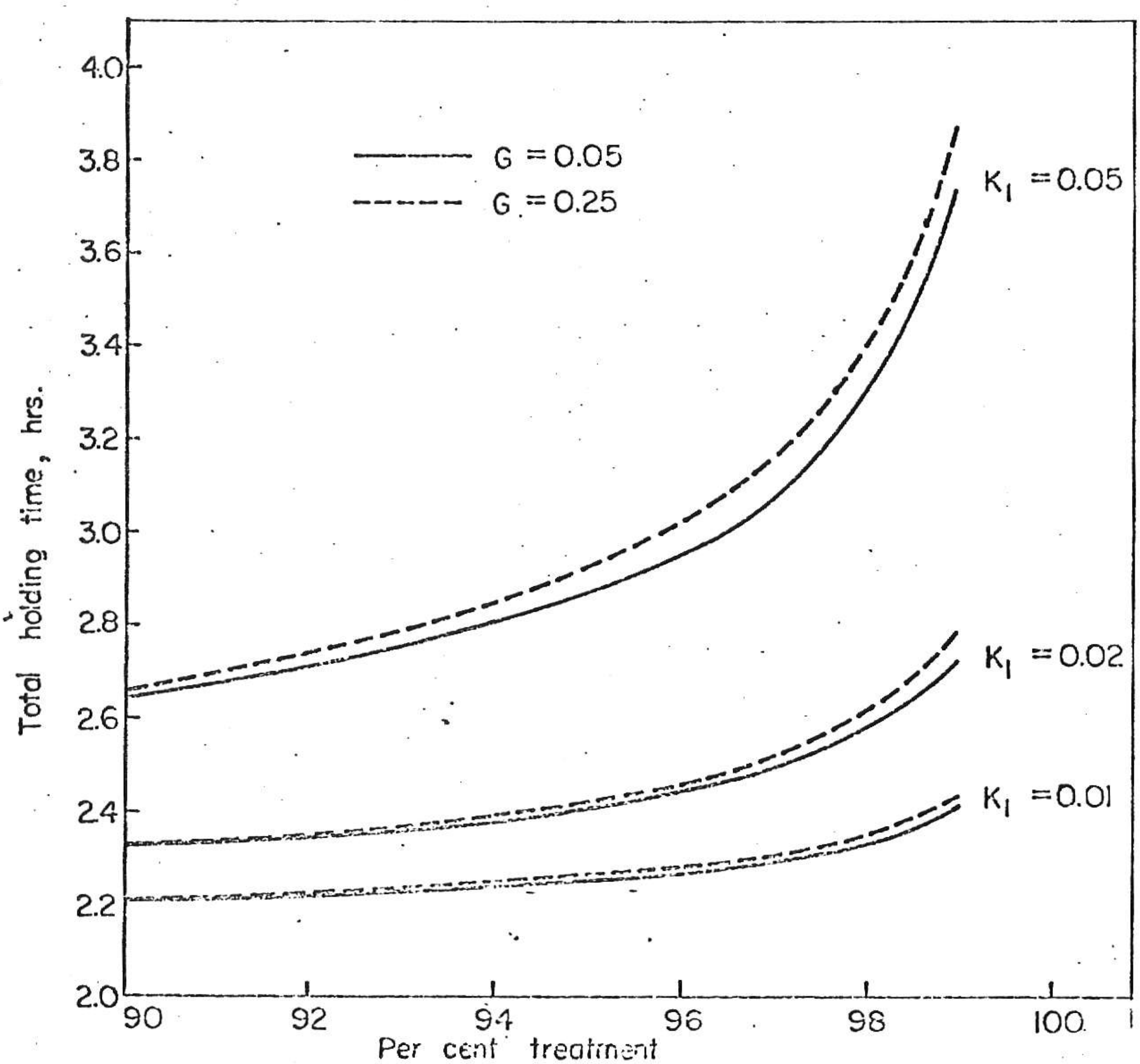

Fig. 6. Variation of total holding time with per cent treatment for a three stage system with K_1 and G as parameters, $\delta = 1.0$.

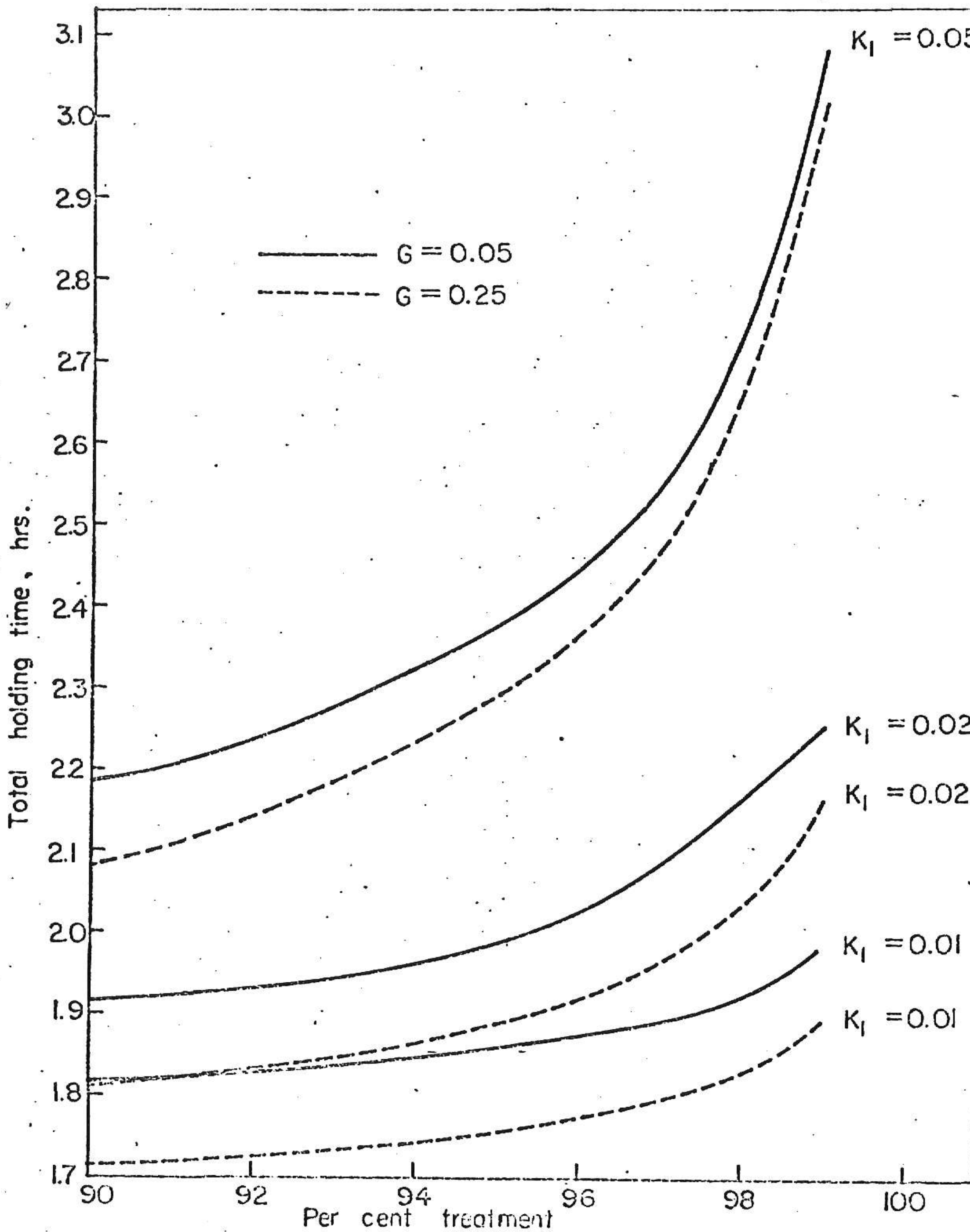

Fig. 7. Effect of backflow and K_1 on total holding time for a three stage system with sedimentation ($\theta = 1.2$).

respectively in Figs. 5 and 8. The results show that the sedimentation parameter definitely reduces the total holding time. On the other hand, the backflow may be either desirable or undesirable as the effects of backflow on the total holding time of the system with sedimentation become favorable only when the sedimentation parameter is greater than a certain value.

In Figs. 9 and 10, the optimal total holding time is plotted as a function of the sedimentation parameter, δ. The results in Fig. 9 for the two stage system and those in Fig. 10 for the three stage system show that the total holding time decreases as the sedimentation parameter increases for all values of K_1 and G. The results also show that the effect of backflow depends on both K_1 and δ with each set of curves showing a point of intersection where equal performance is obtained with either amount of backflow. On the left hand side of the intersection increasing backflow increases the total required holding time, while on the right hand side increasing backflow reduces the required total holding time. These two figures show that the point of intersection shifts with changes in the saturation constant. These figures also show which combination of values of the sedimentation parameter and saturation constant is favorable to improved operation by increasing backflow.

The influences of backflow on the total holding time of the system are shown in Figs. 11 and 12 in which sedimentation parameter and saturation constant appear as parameters. Again, the effect of sedimentation is greater than that of backflow on the total required holding time. As shown previously, there are some cases in which the total holding time is decreased by backflow and others

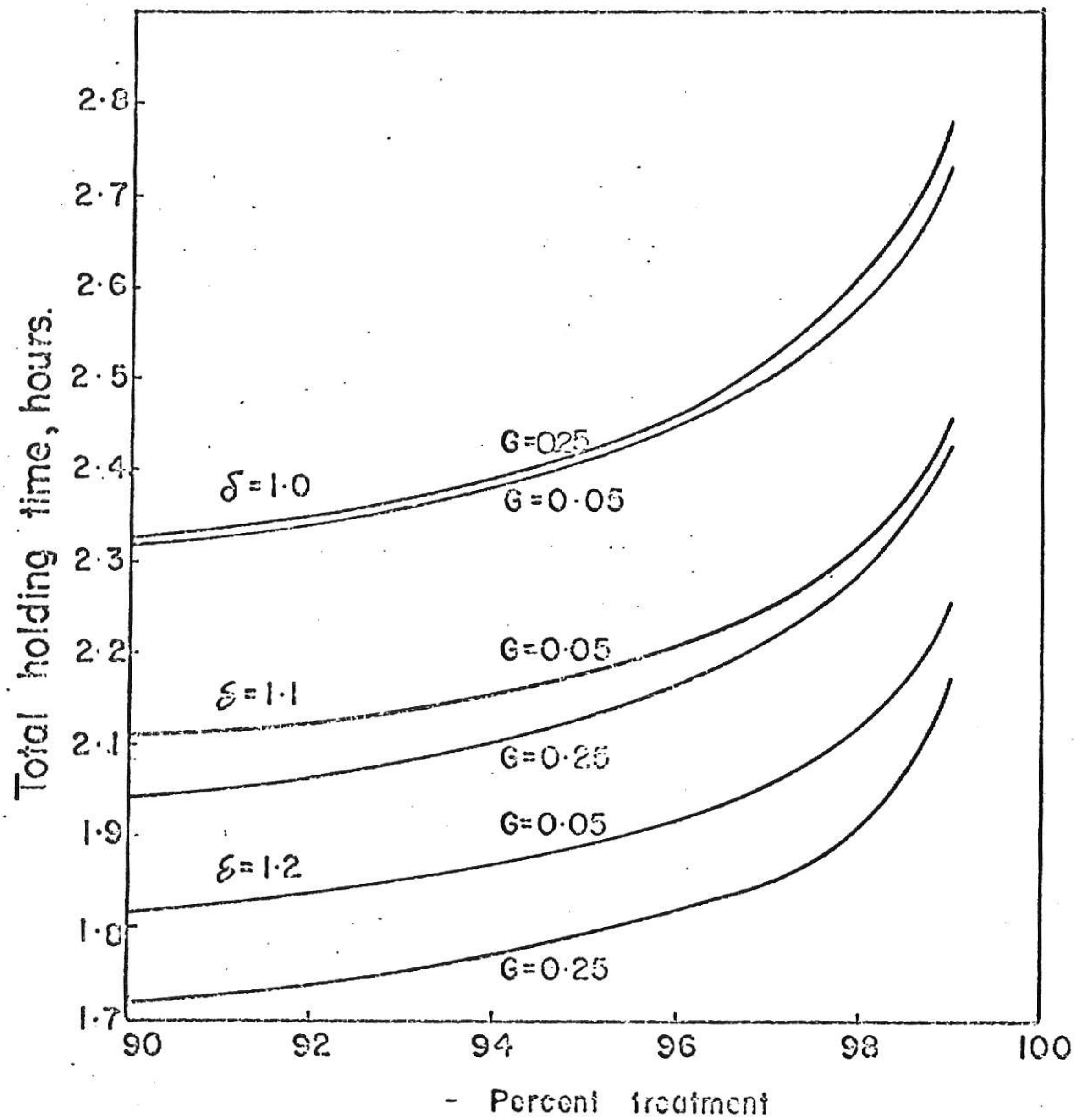

Fig. 8 . Effect of sedimentation and backflow on optimal total holding time for a three stage system $K_1 = 0{\cdot}02$ and $\beta = 4{\cdot}0$

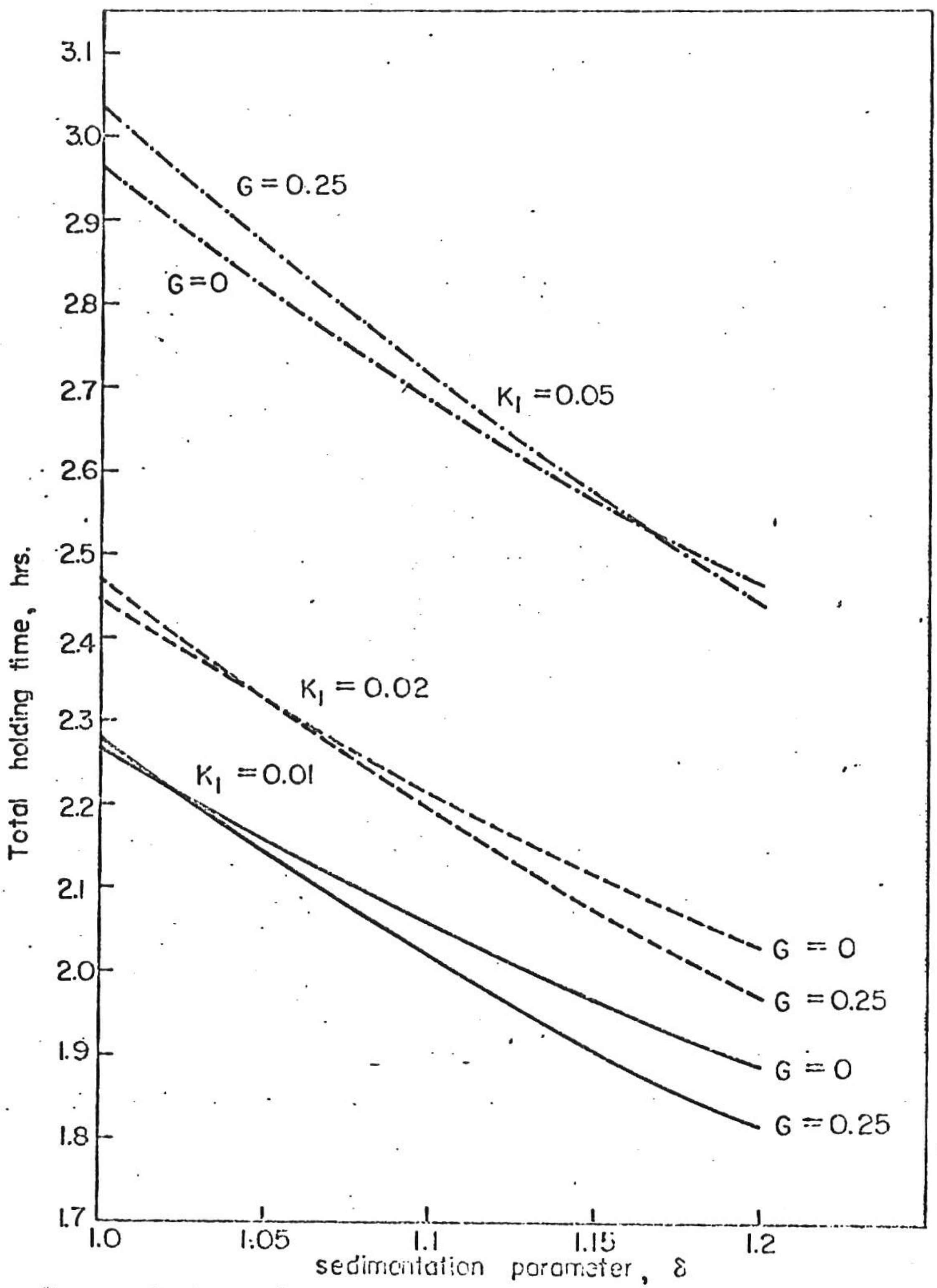

Fig. 9. Variation of total holding time with δ for a two stage system with 95 % treatment. ($\beta = 4.0$)

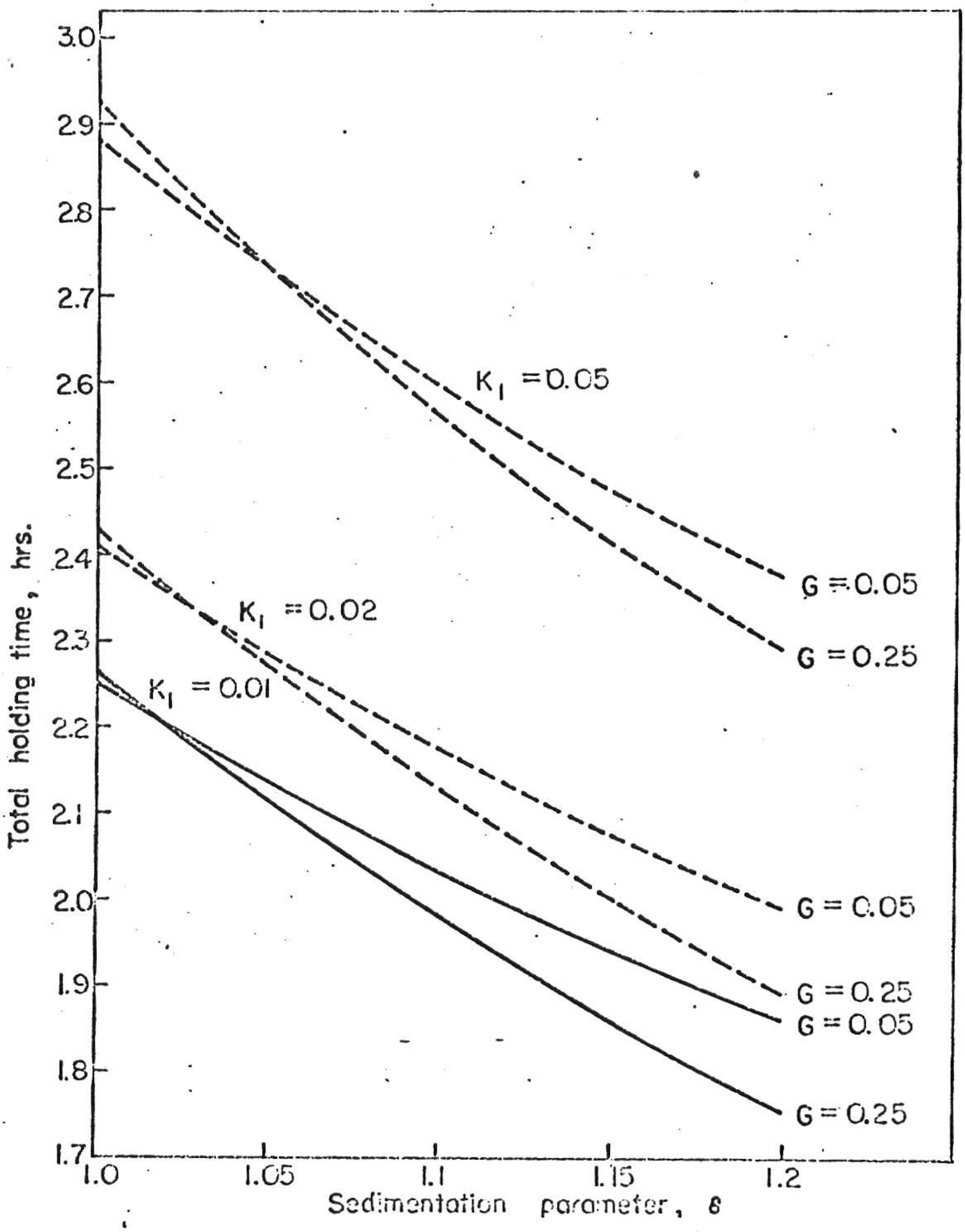

Fig. 10. Variation of total holding time with sedimentation parameter for a three stage system with 95 % treatment. (β = 4.0).

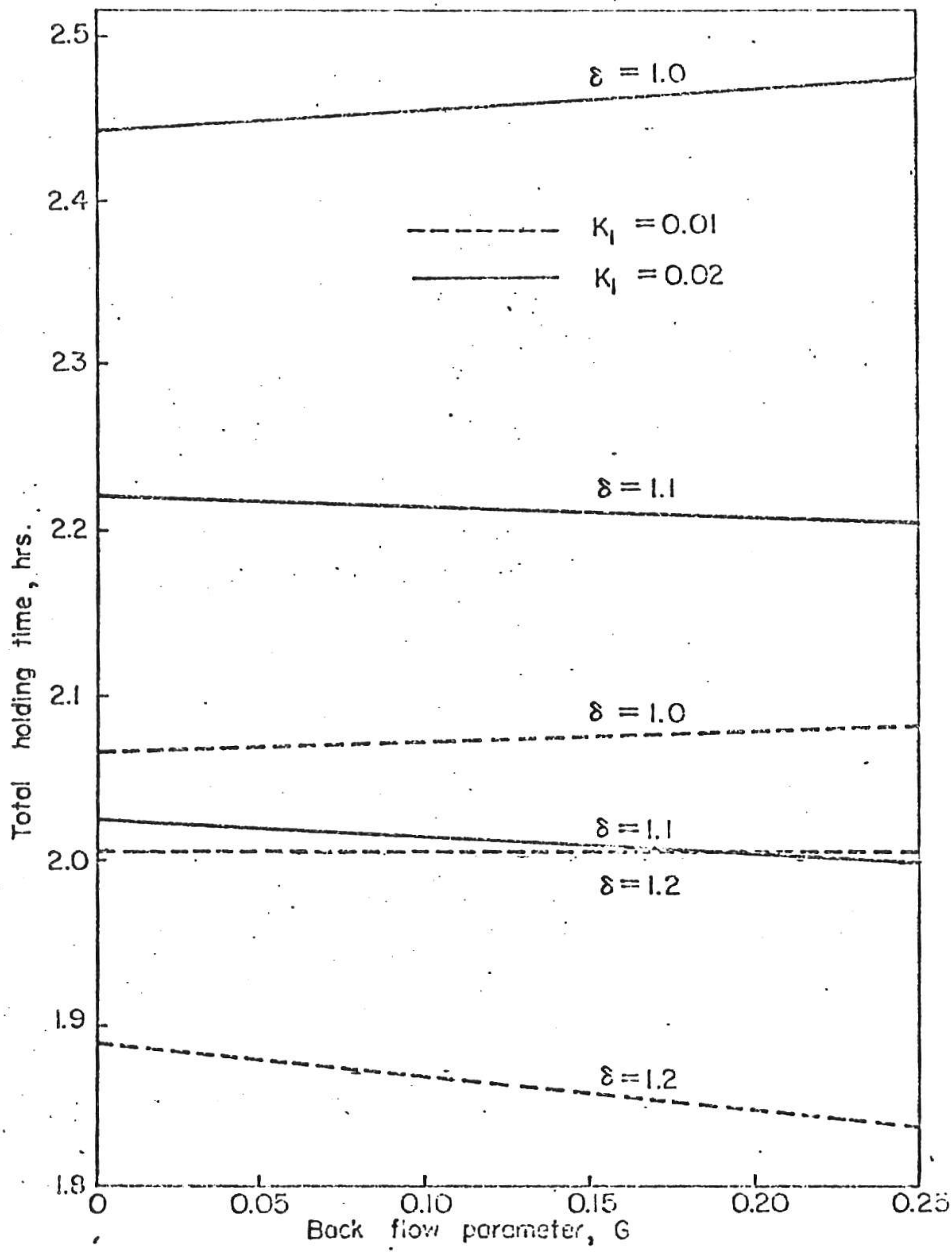

Fig. 11. Variation of total required holding time with back flow for a two stage system. (95 % treatment, β = 4.0)

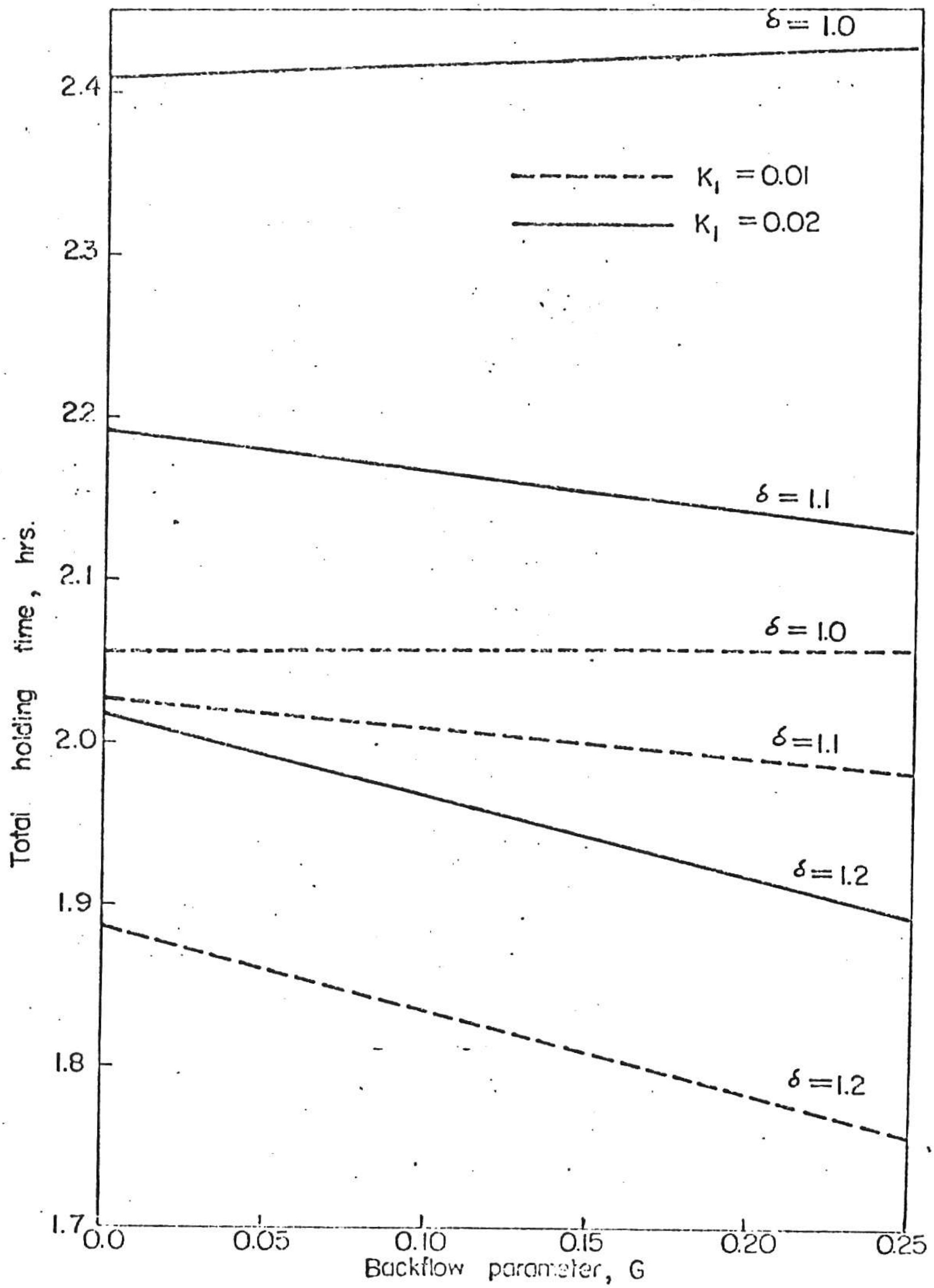

Fig. 12. Variation of total required holding time with backflow for a three stage system (95 % treatment, $\beta = 4.0$).

where it is increased by backflow.

The change in the concentrations of organisms and substrate in each stage of the system with backflow and sedimentation are shown respectively in Figs. 13 and 14. At high values of the backflow and sedimentation parameters, the highest concentration of organisms is present in the first stage and the concentration decreases from the bottom to the top (see the case of $G = 0.25$, $\delta = 1.2$). For some combinations of sedimentation and backflow, a higher concentration of organisms appears in the middle stage (see the case of $G = 0.25$, $\delta = 1.1$). With no sedimentation, the concentration increases from the bottom to the top. Figures 13 and 14 show not only the distribution of the concentration of organisms and substrate in the system, but also the effects of backflow and sedimentation on the distribution of the concentration of organisms.

The effect of number of stages, sedimentation, and backflow on required volume is illustrated in Figs. 15 and 16 by examining relative volume requirements. The effect of the number of stages on the required volume relative to that required for a one stage system is shown in Fig. 15. Increasing the number of stages in the activated sludge system reduces the total required volume. The effect of sedimentation and backflow is also important. As shown in Fig. 15, a two stage system with backflow ($G = 0.25$) and sedimentation ($\delta = 1.2$) requires less volume than a three stage system with no backflow and no sedimentation. Figure 16 illustrates the effect of backflow and sedimentation on the volume requirement relative to that of a three stage system with no backflow and no sedimentation. It shows that sedimentation is desirable for

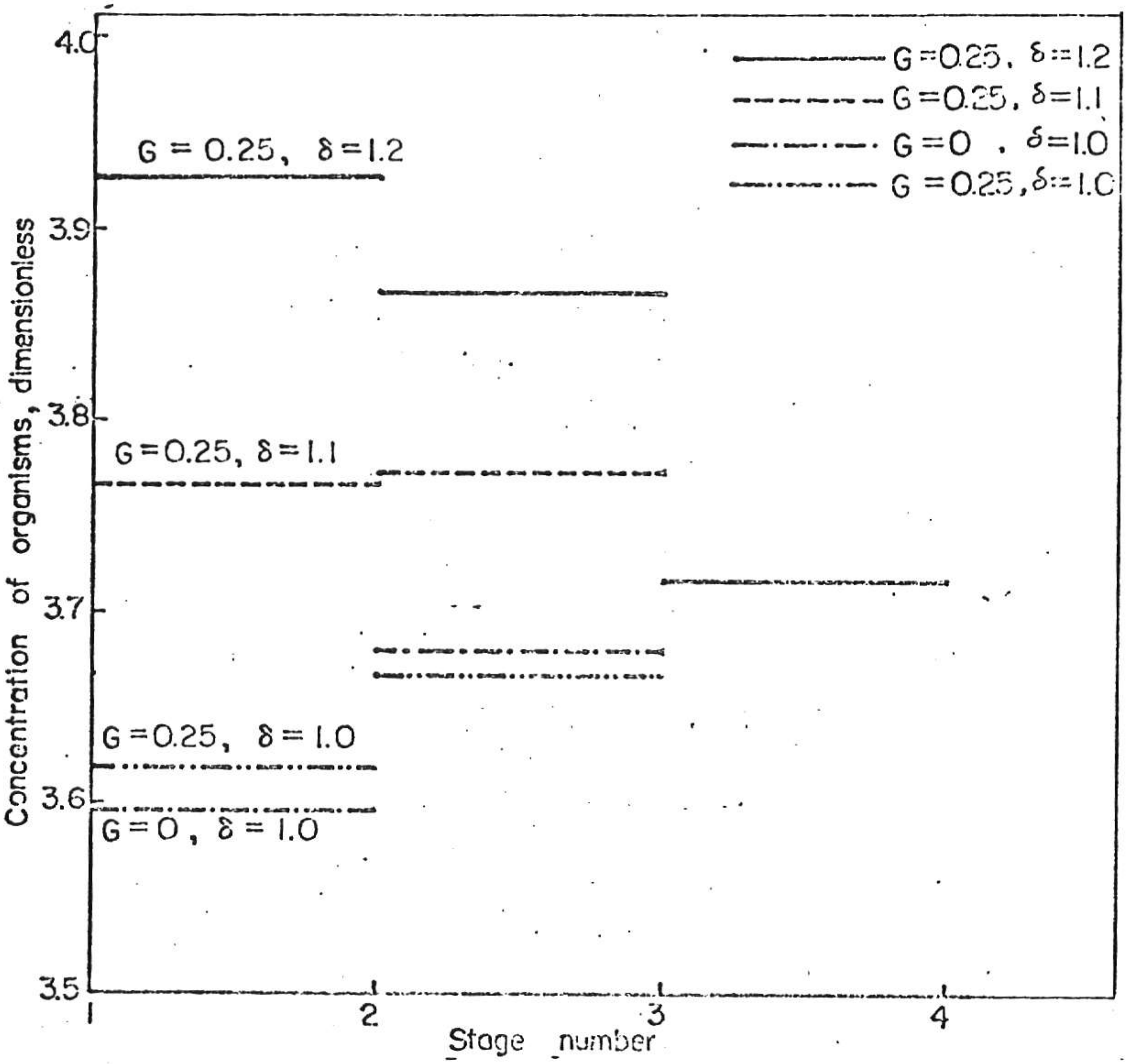

Fig.13. Organism concentration vs stage number in a three stage system with backflow and sedimentation for 95% treatment, $K_1 = 0.01$, $\beta = 4.0$.

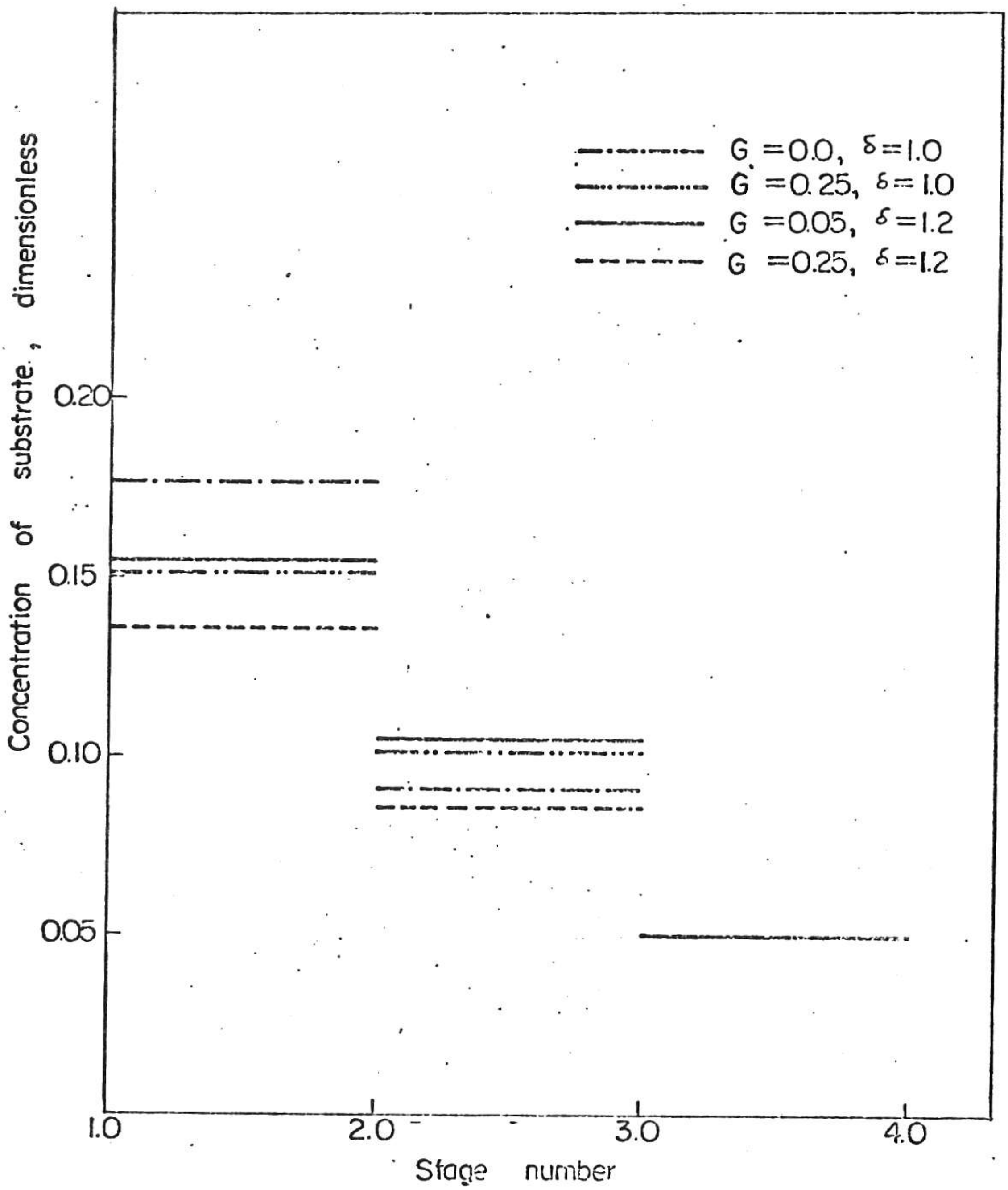

Fig.14. Substrate concentration vs stage number in a three stage system with backflow and sedimentation for 95 % treatment, $K_1 = 0.01$, $\beta = 4.0$.

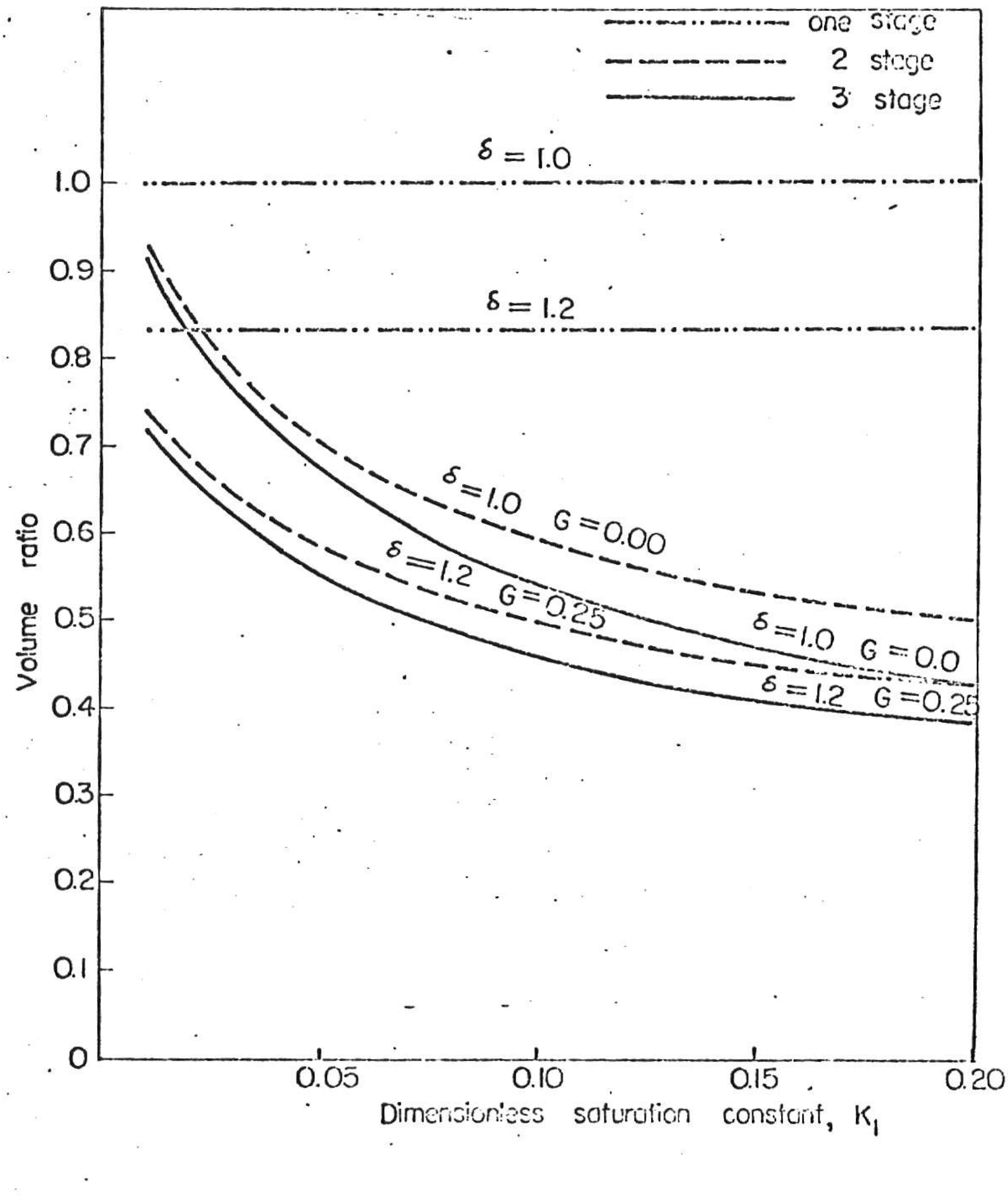

Fig. 15. Ratio of optimal volume requirements for multistage systems based on the one stage system with no sedimentation. (95 % treatment, $\beta = 4.0$)

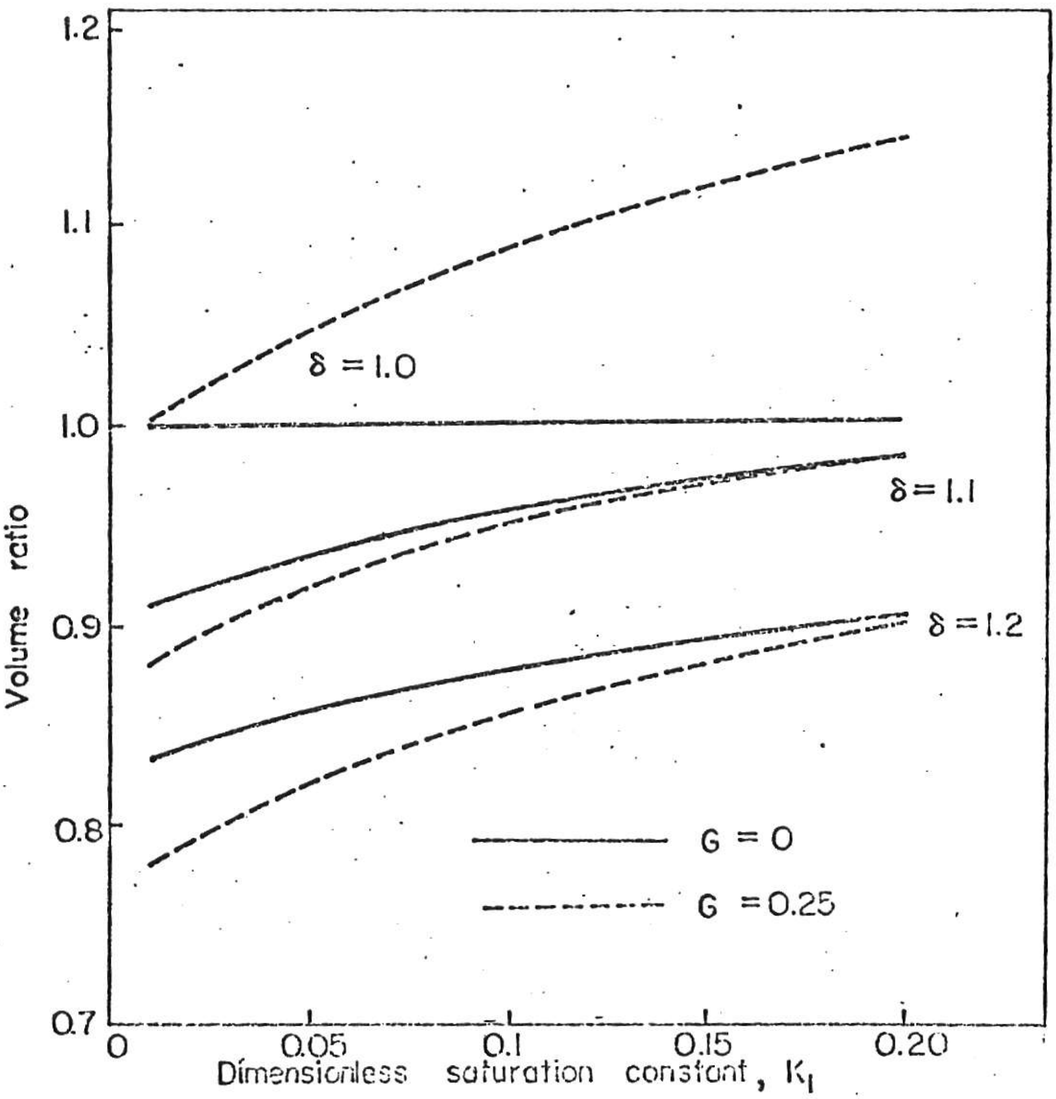

Fig. 16. Ratio of optimal volume requirements for three stage systems based on the system with no sedimentation and no backflow (95 % treatment, β = 4.0)

optimal performance and that it reduces the total required volume. However, the effect of backflow depends on the value of the sedimentation parameter and the saturation constant. For the case of no sedimentation, more volume is required for optimal performance as the backflow and saturation constant increase; however, for the cases with sedimentation, backflow becomes desirable as it reduces the total required volume. The effect of backflow on reducing the volume ratio is small.

Since it may be possible to achieve the optimal pattern of flow and mixing within an activated sludge system without greatly affecting the costs of separation and sludge disposal, it is desirable to increase one's knowledge about the effect of the hydraulic regime on the performance of the system. Although the results presented here indicate that the backflow and sedimentation can be used to reduce the volume requirements under some conditions, one should remember that the mathematical model used here to describe the biological waste treatment process is only an approximation of the actual process. For example, the model does not consider either the physiological state of the organisms or the heterogeneity of the population. If, for example, a lag in growth occurs when the sludge from the third tank is recycled back to the first tank, the required volume will be greater than that predicted and the advantage of multiple stages will be less than that predicted. The backflow parameter may be adjusted by increasing or decreasing the number of sieve holes, the sieve plate hole void area, and the geometry of the holes. The sedimentation parameter may depend on organism physiological state, hydraulic position, and surface behavior; however, a uniform sedimentation parameter throughout the entire system is assumed in this work. The results presented here can be employed for achieving optimal performance by adjusting backflow and sedimentation parameters; however, additional work is needed to determine specific values for these parameters.

4-6 CONCLUSIONS

The present work presents a mathematical model which can be used to describe a tower type biological waste treatment system. Fluid backflow and cell sedimentation are included in the model which is used to investigate system optimal performance for various values of percentage treatment, saturation constant, fluid backflow, and cell sedimentation. The results show that the percentage treatment, saturation constant, and cell sedimentation significantly affect the optimal volume requirements.

Some of the advantages of a tower type activated sludge process that can be visualized are the reduction in aeration tank volume because of sedimentation and optimally controlled backflow, better control of desired mixing patterns within the aeration tank, and improved oxygen transfer and utilization. The tower system will also require less land space and be easily integrated into an industrial complex.

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NOMENCLATURE

F^n = The upward flow stream leaving stage n.

F_n = The total flow rate leaving stage n, $(F^n + f_n)$.

f_n = The backflow rate coming from stage n.

G = Backflow ratio, $(\frac{f_i}{F_i})$.

J = Objective function.

K_S = The concentration of organic at which the specific growth rate observed is one half the maximum value, mg/liter.

K_1 = The dimensionless organic concentration at which the specific growth rate observed is one half the maximum value.

k_D = Specific endogeneous microbial attrition rate, hr^{-1}.

q = Volumetric flow rate of feed to the overall system, liter/hr.

r = Recycle ratio.

S^n = The concentration of organics in stage n, mg/liter.

S^f = The concentration of organics in the feed, mg/liter.

$\bar{t}_n$ = The holding time of stage n.

V_i = The liquid volume of stage n.

V = Total liquid volume of the system.

X^f = The concentration of organisms in the feed, mg/liter.

X^n = The concentration of organisms in stage n, mg/liter.

$\bar{X}^n$ = The concentration of organisms in the upward stream leaving stage n, mg/liter.

y_1^n = The dimensionless concentration of the organic waste in the nth tank, (S^i/S^f).

y_2^n = The dimensionless concentraion of organisms in the nth stage, $(\frac{X^i}{YS^f})$.

$\bar{y}_2^n$ = Dimensionless concentration of organisms in the upward stream leaving stage n.

μ_{max} = Maximum specific growth rate when the organic concentration is not limiting the rate of growth, hr^{-1}.

β = Secondary clarifier parameter.

δ = Sedimentation parameter.

APPENDIX

The schematic diagram of a tower system with recycle is shown in Fig. 2. Influent, q, mixed with recycle of activated sludge, qr, comes into the first stage. The volumetric flow rate at the inlet mixing point can be written as

$$F^0 = q + qr$$
$$= q(1 + r) \qquad (A-1)$$

At steady state, the flow rate leaving the top stage is

$$F^N = F^0 = q(1 + r) \qquad (A-2)$$

The volumetric flow rate balances around each stage are

$$F^0 + f_2 = F^1 \qquad (A-3)$$

$$F^{n-1} + f_{n+1} = F^n + f_n \,, \quad n = 2, 3, \ldots, N-1 \qquad (A-4)$$

and

$$F^{N-1} = F^N + f_N \qquad (A-5)$$

Let

$$F_n = F^n + f_n \qquad (A-6)$$

The backflow ratio, G, which is assumed to be constant is defined as

$$G = \frac{f_n}{F_n} \qquad (A-7)$$

Therefore,

$$F^n = F_n(1 - G) \qquad (A-8)$$

The flow rates at the last stage may be written in the form

$$F^N = F_N(1 - G) \tag{A-9}$$

and

$$f_N = F_N G \tag{A-10}$$

Substituting Equation (A-2) into Equation (A-9) yields

$$F_N = \frac{q(1 + r)}{1 - G} \tag{A-11}$$

The backflow rate, which can be obtained by substituting Equation (A-11) into (A-10), is

$$f_N = \frac{q(1 + r)G}{1 - G} \tag{A-12}$$

Substituting Equations (A-11) and (A-12) into Equation (A-5) gives

$$F^{N-1} = q(1 + r) + \frac{q(1 + r)G}{1 - G} \tag{A-13}$$

Using the relations given in Equations (A-7) and (A-8), one obtains

$$F_{N-1} = \frac{q(1 + r)}{1 - G} + \frac{q(1 + r)G}{(1 - G)^2} \tag{A-14}$$

and

$$f_{N-1} = \frac{q(1 + r)G}{1 - G} + \frac{q(1 + r)G^2}{(1 - G)^2} \tag{A-15}$$

Substituting Equations (A-12), (A-13), and (A-15) into the volumetric balance equation of the (N-1)th stage yields

$$F^{N-2} = \sum_{i=0}^{2} \frac{q(1 + r)}{(1 - G)^i} G^i \tag{A-16}$$

and

$$f_{N-2} = \sum_{j=1}^{3} \frac{q(1 + r)}{(1 - G)^j} G^j \qquad \text{(A-17)}$$

By successive substitution into the volumetric balance equations, the flow rate relation for the nth stage is obtained as

$$F^n = \sum_{i=0}^{N-n} \frac{q(1 + r)}{(1 - G)^i} G^i \, , \qquad n = 2, 3, \ldots, N-1 \quad \text{(A-18)}$$

and

$$f_n = \sum_{j=1}^{N-n+1} \frac{q(1 + r)}{(1 - G)^j} G^j \, , \qquad n = 2, 3, \ldots, N-1 \quad \text{(A-19)}$$

Substituting Equations (A-1) and (A-20) into Equation (A-3) gives

$$F^1 = q(1 + r) + \sum_{j=1}^{N-1} \frac{q(1 + r)}{(1 - G)^j} G^j \qquad \text{(A-20)}$$

which can also be written as

$$F^1 = \sum_{i=0}^{N-1} \frac{q(1 + r)}{(1 - G)^i} G^i \qquad \text{(A-21)}$$

Equations (A-1), (A-2), (A-12), (A-13), (A-18), (A-19), (A-20) and (A-21) allow the flows to be written in terms of q, r, and G.

The objective function which is to be minimized is given by Equation (30). The equality constraints may be written in dimensionless form as follows:

$$y_1^0 = \frac{1}{1 + r} (r \, y_1^N + 1) \qquad \text{(A-22)}$$

$$y_2^0 = \frac{r\beta}{1 + r} y_2^N \qquad \text{(A-23)}$$

$$y_1^0 + \bar{f}_2 y_1^2 - \bar{F}^1 y_1^1 - \bar{t}_1 \left(\frac{\mu_{max}\, y_1^1 y_2^1}{K_1 + y_1^1}\right) = 0 \qquad \text{(A-24)}$$

$$\bar{F}^{i-1} y^{i-1} + \bar{f}_{i+1} y^{i+1} - \bar{F}_i y_1^i - t_i \left(\frac{\mu_{max}\, y_1^i y_2^i}{K_1 + y_1^1}\right) = 0 \qquad \text{(A-25)}$$

$$\bar{F}^{N-1} y_1^{N-1} - \bar{F}_N y_1^N - \bar{t}_N \left(\frac{\mu_{max}\, y_1^N y_2^N}{K_1 + y_1^1}\right) = 0 \qquad \text{(A-26)}$$

$$y_2^0 + f_2 y_2^2 - \bar{F}^1 \bar{y}_2^1 + \bar{t}_1 \left(\frac{\mu_{max}\, y_1^1 y_2^1}{K_1 + y_1^1} - k_D y_2^1\right) = 0 \qquad \text{(A-27)}$$

$$\bar{F}^{i-1} \bar{y}_2^{i-1} + \bar{f}_{i+1} y_2^{i+1} - \bar{F}^i \bar{y}_2^i - \bar{f}_i y_2^i$$

$$+ \bar{t}_i \left(\frac{\mu_{max}\, y_1^i y_2^i}{K_1 + y_1^i} - k_D y_2^i\right) = 0 \qquad \text{(A-28)}$$

and

$$\bar{F}^{N-1} \bar{y}^{N-1} - \bar{F}^N \bar{y}_2^N - \bar{f}_N y_2^N + t_N \left(\frac{\mu_{max}\, y_1^N y_2^N}{K_1 + y_1^N} - k_D y_2^N\right) = 0 \qquad \text{(A-29)}$$

In these equations,

$$\bar{t}_i = \frac{V_i}{q(1 + r)} \qquad \text{(A-30)}$$

is the mean holding time in each stage and

$$\bar{F}^i = \frac{F^i}{q(1 + r)}, \text{ and } \bar{f}_i = \frac{f_i}{q(1 + r)} \qquad \text{(A-31)}$$

are the dimensionless flow rates.

The mathematical problem of minimizing Equation (30) subject to Equations (A-2), (A-9), (A-12), (A-18), (A-19), (A-20), (A-21) and (A-22) through (A-29) for fixed values of y_1^N, β, r, G, and δ, can be accomplished by using direct search optimization procedures. If the concentraion of organics in each stage, y_1^i, are taken to be the independent decision variables of the optimization problem, the optimization problem can be put in a form in which the objective function, J, can be computed for various values of the design variables. Equations(A-24) through (A-26) may be written in the form

$$t_i = (\bar{F}^{i-1} y_1^{i-1} + \bar{f}_{i+1} y_1^{i+1} - \bar{F}_i y_1^i) / \left(\frac{\mu_{max} \; y_1^i y_2^i}{K_1 + y_1^i} \right) ,$$

$$i = 1, 2, \ldots, N \tag{A-32}$$

where

$$\bar{F}^0 = 1$$

$$\bar{f}_{N+1} = 0$$

Equation (A-32) can be written as

$$\bar{t}_i = \frac{A_i}{B_i \; y_2^i} \tag{A-33}$$

and

$$A_i = \bar{F}_{i-1} y_1^{i-1} + \bar{f}_{i+1} y_1^{i+1} - \bar{F}_i y_1^i$$

and

$$B_i = \frac{\mu_{max} \; y_1^i}{K_1 + y_1^i}$$

Substituting Equation (A-33) into Equations (A-27), (A-28) and (A-29), one obtains

$$\bar{F}^{i-1}\bar{y}_2^{i+1} + \bar{f}_{i+1}y_2^{i+1} - \bar{F}^{i}\bar{y}_2^{i} - f_i y_2^i + \frac{A_i}{B_i y_2^i}(B_i y_2^i - k_D y_2^i) = 0$$

$$i = 1, 2, \ldots, N \qquad \text{(A-34)}$$

Rearranging Equation (A-34) yields

$$y_2^i = \frac{(\bar{F}^{i-1}/\delta)y_2^{i-1} + \bar{f}_{i+1}y_2^{i+1} + A_i(1 - \frac{k_D}{B_i})}{(\bar{F}_i/\delta) + \bar{f}_i}$$

$$= R_i y_2^{i-1} + T_i y_2^{i+1} + P_i, \qquad i = 1, 2, \ldots, N \qquad \text{(A-35)}$$

where

$$R_i = \frac{\bar{F}^{i-1}}{\bar{F}^i + \delta \bar{f}_i}$$

$$T_i = \frac{\bar{f}_{i+1}}{(\bar{F}^i/\delta) + \bar{f}_i}$$

and

$$P_i = \frac{A_i(1 - \frac{k_D}{B_i})}{(\bar{F}^i/\delta) + \bar{f}_i}$$

There are N simultaneous equations of the form of Equation (A-35) for an N stage tower system. Eliminating y_2^1 from the first two equations of Equation (A-35), which represent the organism balances around the first and the second stages, followed by successive elimination of y_2^i by using Equation (A-35) for the third and following stages leads to an equation for y_2^n in terms of y_2^0. In order to simplify the resulting expression, the following new notation is introduced

$$
\begin{aligned}
W_0 &= 1 \\
W_1 &= 1 \\
W_2 &= W_1 - W_0 T_1 R_2 \\
W_3 &= W_2 - W_1 T_2 R_3 \\
&\vdots \\
W_n &= W_{n-1} - W_{n-2} T_{n-1} R_n \ , \quad n = 2,\ 3,\ \ldots,\ N
\end{aligned}
\tag{A-36}
$$

$$
\begin{aligned}
Q_1 &= W_0 P_1 R_2 \\
Q_2 &= R_3(Q_1 + W_1 P_2) \\
Q_3 &= R_4(Q_2 + W_2 P_3) \\
&\vdots \\
Q_n &= R_{n+1}(Q_{n-1} + W_{n-1} P_n) \ , \quad n = 2,\ 3,\ \ldots, N
\end{aligned}
\tag{A-37}
$$

and

$$
\begin{aligned}
Z_1 &= R_1 y_2^0 \\
Z_2 &= R_1 R_2 y_2^0 + Q_1 \\
Z_3 &= R_1 R_2 R_3 y^0 + Q_2 \\
&\vdots \\
Z_n &= \mathop{}_{i=1}^{n} R_i y_2^0 + Q_{n-1} \ , \quad n = 2,\ 3,\ \ldots,\ N
\end{aligned}
\tag{A-38}
$$

By using these redefined parameters and Equation (A-35), the general expression y_2^n is obtained as

$$
y_2^n = \frac{Z_n}{W_n} + \frac{W_{n-1}}{W_n}(T_n y_2^{n+1} + P_n) \ , \quad n = 1,\ 2,\ \ldots,\ N \tag{A-39}
$$

Since $T_N = 0$, y_2^N is obtained in terms of y_2^0 and the substrate concentration of each stage as

$$y_2^N = \frac{Z_N}{W_N} + \frac{W_{N-1}}{W_N} P_N \tag{A-40}$$

Substituting Equations (A-23) and (A-38) into Equation (A-40) yields

$$y_2^N = \frac{Q_{N-1} + W_{N-1}P_N}{W_N - \frac{r\beta}{1 + r} \sum_{i=1}^{N} R_i} \tag{A-41}$$

Now, y_2^N becomes a function of substrate concentrations only.

For a desired degree of treatment, y_1^N is fixed. Selecting values for the variables y_1^n, n = 2, 3, .., N-1 is sufficient to specify the values of the dependent variables and the objective function, J.

The suggested computational procedure to compute J is as follows;

1. Assume y_1^n, n = 1, 2, ..., N-1.
2. Compute y_2^N using Equation (A-41).
3. Compute y_2^0 using Equation (A-23)
4. Compute y_2^n using Equation (A-35), n = N-1, N-2, ..., 1.
5. Compute $\bar{t}_i$ using Equation (A-32).
6. Compute J using Equation (31).

To obtain the minimum value of J, the values of y_1^n, n = 2, 3, ..., N must be the optimum values; that is values which allow J to take on its minimum value must be selected. A direct search optimization procedure may be used to systematically assume sets of values y_1^n , $n = 2, 3, \ldots, N$ until the optimum values of these design variables have been found. The simplex pattern search technique (10) has been used as an optimization subroutine to determine the optimum values of these design variables.

APPENDIX II

Table 1. Optimal results for one stage systems with 90, 95, 98, and 99 per cent treatment, β = 4.0, δ = 1.2

K_1	y_1^1	y_2^0	$\bar{y}_2^{-1}$	y_2^1	$\bar{t}$
0.01	0.01	3.034	3.792	4.550	3.472
	0.02	3.026	3.783	4.540	2.577
	0.05	2.928	3.660	4.392	2.049
	0.10	2.738	3.423	4.108	1.875
0.02	0.01	2.970	3.713	4.456	5.319
	0.02	2.995	3.764	4.693	3.472
	0.05	2.916	3.645	4.374	2.401
	0.10	2.733	3.416	4.099	2.049
0.05	0.01	2.781	3.476	3.171	11.36
	0.02	2.902	3.627	4.352	6.272
	0.05	2.880	3.600	4.320	3.472
	0.10	2.716	3.395	4.076	2.577
0.1	0.01	2.465	3.081	3.697	23.50
	0.02	2.746	3.432	4.118	11.36
	0.05	2.820	3.525	4.230	5.319
	0.10	2.688	3.360	4.032	3.472
0.2	0.01	1.833	2.291	2.769	60.34
	0.02	2.434	3.042	3.650	23.50
	0.05	2.700	3.375	4.050	9.259
	0.10	2.632	3.290	3.948	5.319
0.5					
	0.02	1.498	1.892	2.246	90.28
	0.05	2.340	2.925	3.510	23.50
	0.10	2.464	3.080	3.696	11.36

Table 2. Optimal results for two stage system for various values of K_1 and y_1^2, $\delta = 1.0$, $G = 0.0$

K_1	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	y_2^1	y_1^2	$\bar{y}_2^2$	y_2^2	$\bar{t}_1$	$\bar{t}_2$	$\bar{t}$
0.010	0.820	2.819	0.175	3.450	3.450	0.100	3.524	3.524	1.975	0.326	2.210
0.010	0.810	2.974	0.152	3.619	3.619	0.050	3.718	3.718	1.940	0.328	2.268
0.010	0.804	3.065	0.110	3.743	3.743	0.020	3.831	3.831	2.021	0.354	2.375
0.010	0.802	3.092	0.083	3.794	3.794	0.010	3.865	3.865	2.122	0.379	2.501
0.020	0.820	2.816	0.218	3.405	3.405	0.100	3.520	3.520	1.930	0.402	2.332
0.020	0.810	2.969	0.173	3.592	3.592	0.050	3.712	3.712	1.980	0.463	2.442
0.020	0.804	3.056	0.118	3.726	3.726	0.020	3.820	3.820	2.152	0.514	2.666
0.020	0.802	3.079	0.086	3.777	3.777	0.010	3.849	3.849	2.334	0.595	2.929
0.050	0.820	2.807	0.256	3.358	3.358	0.100	3.509	3.509	2.010	0.665	2.675
0.050	0.810	2.955	0.189	3.561	3.561	0.050	3.694	3.694	2.208	0.751	2.958
0.050	0.804	3.031	0.123	3.692	3.692	0.020	3.788	3.788	2.594	0.952	3.546
0.050	0.802	3.041	0.088	3.733	3.733	0.010	3.801	3.801	2.998	1.233	4.231
0.100	0.820	2.793	0.271	3.327	3.327	0.100	3.491	3.491	2.263	0.977	3.239
0.100	0.810	2.931	0.195	3.527	3.527	0.050	3.663	3.663	2.638	1.187	3.825
0.100	0.804	2.988	0.125	3.642	3.642	0.020	3.735	3.735	3.357	1.686	5.042
0.100	0.802	2.977	0.089	3.660	3.660	0.010	3.722	3.722	4.136	2.336	6.472
0.200	0.820	2.763	0.278	3.286	3.286	0.100	3.453	3.453	2.833	1.549	4.382
0.200	0.810	2.882	0.198	3.470	3.470	0.050	3.603	3.603	3.548	2.052	5.599
0.200	0.804	2.902	0.126	3.545	3.545	0.020	3.628	3.628	4.946	3.216	8.162
0.200	0.802	2.850	0.089	3.517	3.517	0.010	3.563	3.563	6.581	4.659	11.240
0.500	0.820	2.673	0.283	3.180	3.180	0.100	3.342	3.342	4.666	3.291	7.957
0.500	0.810	2.737	0.200	3.304	3.304	0.050	3.421	3.421	6.455	4.828	11.283
0.500	0.804	2.646	0.126	3.256	3.256	0.020	3.308	3.308	10.311	8.364	18.676
0.500	0.802	2.468	0.090	3.087	3.087	0.010	3.085	3.085	15.170	13.175	28.345

Table 3. Optimal results for two stage system for various values of K_1 and y_1^2, $\delta = 1.0$, $G = 0.05$

K_1	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	y_2^1	y_1^2	$\bar{y}_2^2$	y_2^2	$\bar{t}_1$	$\bar{t}_2$	$\bar{t}$
0.010	0.820	2.819	0.175	3.451	3.451	0.100	3.524	3.524	1.965	0.245	2.211
0.010	0.810	2.974	0.150	3.620	3.620	0.050	3.718	3.718	1.929	0.340	2.269
0.010	0.804	3.064	0.108	3.745	3.745	0.020	3.831	3.831	2.016	0.364	2.380
0.010	0.802	3.092	0.081	3.796	3.796	0.010	3.864	3.864	2.122	0.387	2.509
0.020	0.820	2.816	0.216	3.407	3.407	0.100	3.520	3.520	1.917	0.416	2.334
0.020	0.810	2.969	0.170	3.595	3.595	0.050	3.711	3.711	1.971	0.475	2.447
0.020	0.804	3.056	0.116	3.728	3.728	0.020	3.820	3.820	2.149	0.528	2.677
0.020	0.802	3.079	0.084	3.779	3.779	0.010	3.848	3.848	2.339	0.608	2.947
0.050	0.820	2.807	0.252	3.361	3.361	0.100	3.509	3.509	1.996	0.685	2.680
0.050	0.810	2.954	0.185	3.563	3.563	0.050	3.693	3.693	2.202	0.770	2.972
0.050	0.804	3.030	0.121	3.693	3.693	0.020	3.787	3.787	2.596	0.979	3.576
0.050	0.802	3.040	0.086	3.732	3.732	0.010	3.799	3.799	3.012	1.268	4.280
0.100	0.820	2.792	0.266	3.331	3.331	0.100	3.490	3.490	2.255	0.999	3.254
0.100	0.810	2.930	0.191	3.530	3.530	0.050	3.662	3.662	2.640	1.215	3.856
0.100	0.804	2.986	0.122	3.643	3.643	0.020	3.732	3.732	3.377	1.729	5.106
0.100	0.802	2.974	0.087	3.658	3.658	0.010	3.718	3.718	4.191	2.385	6.576
0.200	0.820	2.762	0.273	3.290	3.290	0.100	3.452	3.452	2.836	1.579	4.416
0.200	0.810	2.881	0.194	3.471	3.471	0.050	3.601	3.601	3.563	2.102	5.665
0.200	0.804	2.899	0.123	3.543	3.543	0.020	3.623	3.623	5.004	3.295	8.299
0.200	0.802	2.844	0.087	3.510	3.510	0.010	3.555	3.555	6.666	4.802	11.468
0.500	0.820	2.671	0.277	3.183	3.183	0.100	3.339	3.339	4.704	3.348	8.052
0.500	0.810	2.733	0.196	3.302	3.302	0.050	3.416	3.416	6.533	4.937	11.470
0.500	0.804	2.637	0.124	3.246	3.246	0.020	3.296	3.296	10.487	8.606	19.093
0.500	0.802	2.454	0.087	3.068	3.068	0.010	3.067	3.067	15.611	13.499	29.111

Table 4. Optimal results for two stage system for various values of K_1 and y_1^2, $\delta = 1.0$, $G = 0.10$

K_1	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	y_2^1	y_1^2	$\bar{y}_2^2$	y_2^2	$\bar{t}_1$	$\bar{t}_2$	$\bar{t}$
0.010	0.820	2.819	0.175	3.451	3.451	0.100	3.524	3.524	1.952	0.259	2.211
0.010	0.810	2.974	0.148	3.622	3.622	0.050	3.717	3.717	1.918	0.352	2.271
0.010	0.804	3.064	0.107	3.747	3.747	0.020	3.830	3.830	2.009	0.376	2.385
0.010	0.802	3.091	0.079	3.798	3.798	0.010	3.864	3.864	2.121	0.398	2.519
0.020	0.820	2.816	0.214	3.408	3.408	0.100	3.520	3.520	1.901	0.434	2.335
0.020	0.810	2.969	0.167	3.597	3.597	0.050	3.711	3.711	1.959	0.492	2.451
0.020	0.804	3.055	0.113	3.730	3.730	0.020	3.819	3.819	2.146	0.543	2.689
0.020	0.802	3.078	0.082	3.779	3.779	0.010	3.847	3.847	2.340	0.626	2.967
0.050	0.820	2.807	0.248	3.365	3.365	0.100	3.509	3.509	1.984	0.703	2.687
0.050	0.810	2.954	0.181	3.567	3.567	0.050	3.692	3.692	2.199	0.788	2.987
0.050	0.804	3.029	0.118	3.695	3.695	0.020	3.786	3.786	2.604	1.004	3.608
0.050	0.802	3.038	0.084	3.732	3.732	0.010	3.797	3.797	3.037	1.296	4.333
0.100	0.820	2.791	0.261	3.335	3.335	0.100	3.489	3.489	2.247	1.002	3.270
0.100	0.810	2.929	0.186	3.533	3.533	0.050	3.661	3.661	2.647	1.241	3.888
0.100	0.804	2.984	0.119	3.643	3.643	0.020	3.730	3.730	3.401	1.773	5.174
0.100	0.802	2.971	0.085	3.656	3.656	0.010	3.714	3.714	4.227	2.461	6.687
0.200	0.820	2.761	0.267	3.294	3.294	0.100	3.451	3.451	2.837	1.614	4.451
0.200	0.810	2.879	0.189	3.473	3.473	0.050	3.598	3.598	3.590	2.145	5.735
0.200	0.804	2.895	0.120	3.541	3.541	0.020	3.618	3.618	5.078	3.369	8.446
0.200	0.802	2.838	0.085	3.504	3.504	0.010	3.547	3.547	6.796	4.918	11.715
0.500	0.820	2.668	0.271	3.185	3.185	0.100	3.335	3.335	4.733	3.420	8.153
0.500	0.810	2.728	0.190	3.300	3.300	0.050	3.409	3.409	6.640	5.030	11.670
0.500	0.804	2.627	0.120	3.235	3.235	0.020	3.283	3.283	10.740	8.804	19.544
0.500	0.802	2.438	0.085	3.049	3.049	0.010	3.047	3.047	15.982	13.962	29.945

Table 5. Optimal results for two stage system for various values of K_1 and y_1^2, $\delta = 1.0$, $G = 0.15$

K_1	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	y_2^1	y_1^2	$\bar{y}_2^2$	y_2^2	$\bar{t}_1$	$\bar{t}_2$	$\bar{t}$
0.010	0.820	2.819	0.174	3.451	3.451	0.100	3.523	3.523	1.939	0.272	2.211
0.010	0.810	2.974	0.147	3.623	3.623	0.050	3.717	3.717	1.905	0.368	2.273
0.010	0.804	3.064	0.105	3.748	3.748	0.020	3.830	3.830	2.002	0.389	2.391
0.010	0.802	3.091	0.078	3.799	3.799	0.010	3.864	3.864	2.117	0.412	2.529
0.020	0.820	2.816	0.212	3.410	3.410	0.100	3.520	3.520	1.886	0.450	2.337
0.020	0.810	2.969	0.164	3.600	3.600	0.050	3.711	3.711	1.949	0.507	2.456
0.020	0.804	3.055	0.110	3.732	3.732	0.020	3.819	3.819	2.145	0.557	2.702
0.020	0.802	3.077	0.080	3.781	3.781	0.010	3.847	3.847	2.345	0.644	2.989
0.050	0.820	2.807	0.243	3.369	3.369	0.100	3.508	3.508	1.972	0.722	2.694
0.050	0.810	2.953	0.177	3.570	3.570	0.050	3.692	3.692	2.191	0.812	3.003
0.050	0.804	3.028	0.114	3.697	3.697	0.020	3.784	3.784	2.617	1.026	3.643
0.050	0.802	3.036	0.082	3.732	3.732	0.010	3.795	3.795	3.055	1.335	4.390
0.100	0.820	2.791	0.255	3.340	3.340	0.100	3.489	3.489	2.241	1.046	3.286
0.100	0.810	2.928	0.182	3.536	3.536	0.050	3.660	3.660	2.651	1.272	3.923
0.100	0.804	2.982	0.116	3.643	3.643	0.020	3.727	3.727	3.434	1.814	5.248
0.100	0.802	2.968	0.082	3.653	3.653	0.010	3.710	3.710	4.286	2.522	6.808
0.200	0.820	2.760	0.261	3.298	3.298	0.100	3.449	3.449	2.844	1.645	4.489
0.200	0.810	2.876	0.184	3.475	3.475	0.050	3.595	3.595	3.614	2.196	5.810
0.200	0.804	2.890	0.117	3.537	3.537	0.020	3.613	3.613	5.139	3.466	8.605
0.200	0.802	2.831	0.083	3.497	3.497	0.010	3.539	3.539	6.912	5.070	11.982
0.500	0.820	2.665	0.264	3.187	3.187	0.100	3.332	3.332	4.780	3.480	8.260
0.500	0.810	2.722	0.186	3.297	3.297	0.050	3.403	3.403	6.721	5.164	11.885
0.500	0.804	2.616	0.117	3.223	3.223	0.020	3.270	3.270	10.950	9.083	20.033
0.500	0.802	2.421	0.083	3.027	3.027	0.010	3.026	3.026	16.458	14.399	30.857

Table 6. Optimal results for two stage system for various values of K_1 and y_1^2, $\delta = 1.0, G = 0.20$

K_1	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	y_2^1	y_1^2	$\bar{y}_2^2$	y_2^2	$\bar{t}_1$	$\bar{t}_2$	$\bar{t}$
0.010	0.820	2.819	0.173	3.452	3.452	0.100	3.523	3.523	1.925	0.286	2.212
0.010	0.810	2.974	0.144	3.625	3.625	0.050	3.717	3.717	1.893	0.381	2.275
0.010	0.804	3.064	0.102	3.750	3.750	0.020	3.830	3.830	1.995	0.401	2.397
0.010	0.802	3.091	0.076	3.800	3.800	0.010	3.863	3.863	2.114	0.426	2.540
0.020	0.820	2.816	0.210	3.412	3.412	0.100	3.520	3.520	1.869	0.469	2.338
0.020	0.810	2.969	0.161	3.602	3.602	0.050	3.711	3.711	1.937	0.525	2.462
0.020	0.804	3.054	0.107	3.734	3.734	0.020	3.818	3.818	2.144	0.572	2.716
0.020	0.802	3.077	0.078	3.782	3.782	0.010	3.846	3.846	2.352	0.660	3.012
0.050	0.820	2.806	0.239	3.373	3.373	0.100	3.508	3.508	1.959	0.742	2.701
0.050	0.810	2.953	0.173	3.573	3.573	0.050	3.691	3.691	2.188	0.833	3.020
0.050	0.804	3.026	0.111	3.698	3.698	0.020	3.783	3.783	2.624	1.057	3.681
0.050	0.802	3.034	0.079	3.732	3.732	0.010	3.793	3.793	3.084	1.368	4.452
0.100	0.820	2.790	0.250	3.345	3.345	0.100	3.488	3.488	2.234	1.071	3.305
0.100	0.810	2.927	0.177	3.539	3.539	0.050	3.658	3.658	2.655	1.305	3.961
0.100	0.804	2.979	0.113	3.643	3.643	0.020	3.724	3.724	3.485	1.870	5.328
0.100	0.802	2.964	0.080	3.651	3.651	0.010	3.705	3.705	4.348	2.592	6.940
0.200	0.820	2.758	0.255	3.302	3.302	0.100	3.448	3.448	2.845	1.684	4.529
0.200	0.810	2.874	0.179	3.476	3.476	0.050	3.593	3.593	3.646	2.245	5.891
0.200	0.804	2.886	0.113	3.534	3.534	0.020	3.607	3.607	5.218	3.559	8.778
0.200	0.802	2.824	0.080	3.489	3.489	0.010	3.530	3.530	7.055	5.217	12.273
0.500	0.820	2.662	0.258	3.189	3.189	0.100	3.328	3.328	4.820	3.555	8.375
0.500	0.810	2.716	0.181	3.294	3.294	0.050	3.396	3.396	6.834	5.283	12.116
0.500	0.804	2.604	0.114	3.211	3.211	0.020	3.256	3.256	11.210	9.355	20.566
0.500	0.802	2.402	0.080	3.004	3.004	0.010	3.003	3.003	16.967	14.893	31.861

Table 7. Optimal results for two stage system for various values of K_1 and y_1^2, $\delta = 1.0$, $G = 0.25$

K_1	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	y_2^1	y_1^2	$\bar{y}_2^2$	y_2^2	$\bar{t}_1$	$\bar{t}_2$	$\bar{t}$
0.010	0.820	2.819	0.172	3.453	3.453	0.100	3.523	3.523	1.911	0.301	2.212
0.010	0.810	2.974	0.143	3.627	3.627	0.050	3.717	3.717	1.879	0.398	2.277
0.010	0.804	3.064	0.100	3.752	3.752	0.020	3.829	3.829	1.988	0.415	2.404
0.010	0.802	3.090	0.073	3.802	3.802	0.010	3.863	3.863	2.114	0.438	2.552
0.020	0.820	2.816	0.207	3.415	3.415	0.100	3.520	3.520	1.852	0.489	2.340
0.020	0.810	2.968	0.158	3.606	3.606	0.050	3.710	3.710	1.927	0.541	2.468
0.020	0.804	3.054	0.105	3.736	3.736	0.020	3.817	3.817	2.137	0.594	2.731
0.020	0.802	3.076	0.076	3.783	3.783	0.010	3.845	3.845	2.352	0.685	3.038
0.050	0.820	2.806	0.234	3.377	3.377	0.100	3.507	3.507	1.944	0.765	2.709
0.050	0.810	2.952	0.168	3.577	3.577	0.050	3.690	3.690	2.184	0.855	3.039
0.050	0.804	3.025	0.108	3.700	3.700	0.020	3.781	3.781	2.636	1.086	3.722
0.050	0.802	3.032	0.077	3.732	3.732	0.010	3.790	3.790	3.111	1.408	4.520
0.100	0.820	2.790	0.244	3.349	3.349	0.100	3.487	3.487	2.225	1.099	3.324
0.100	0.810	2.925	0.172	3.542	3.542	0.050	3.657	3.657	2.667	1.335	4.001
0.100	0.804	2.977	0.109	3.642	3.642	0.020	3.721	3.721	3.492	1.923	5.415
0.100	0.802	2.960	0.077	3.648	3.648	0.010	3.700	3.700	4.412	2.671	7.083
0.200	0.820	2.757	0.248	3.307	3.307	0.100	3.446	3.446	2.854	1.718	4.573
0.200	0.810	2.872	0.174	3.478	3.478	0.050	3.590	3.590	3.674	2.305	5.978
0.200	0.804	2.881	0.110	3.531	3.531	0.020	3.601	3.601	5.308	3.658	8.966
0.200	0.802	2.816	0.078	3.480	3.480	0.010	3.520	3.520	7.207	5.385	12.592
0.500	0.820	2.659	0.251	3.191	3.191	0.100	3.324	3.324	4.855	3.642	8.497
0.500	0.810	2.710	0.176	3.290	3.290	0.050	3.388	3.388	6.933	5.434	12.367
0.500	0.804	2.592	0.110	3.196	3.196	0.020	3.240	3.240	11.474	9.675	21.149
0.500	0.802	2.382	0.078	2.979	2.979	0.010	2.978	2.978	17.514	15.458	32.972

Table 8. Optimal results for two stage system for various values of K_1 and y_1^2, $\delta = 1.1$, $G = 0.0$

K_1	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	y_2^1	y_1^2	$\bar{y}_2^2$	y_2^2	$\bar{t}_1$	$\bar{t}_2$	$\bar{t}$
0.010	0.820	2.819	0.175	3.450	3.795	0.100	3.524	3.876	1.795	0.214	2.009
0.010	0.810	2.974	0.152	3.619	3.980	0.050	3.718	4.089	1.764	0.298	2.061
0.010	0.804	3.065	0.111	3.743	4.117	0.020	3.831	4.214	1.836	0.323	2.159
0.010	0.802	3.092	0.083	3.794	4.174	0.010	3.865	4.251	1.929	0.345	2.273
0.020	0.820	2.816	0.218	3.405	3.745	0.100	3.520	3.872	1.755	0.366	2.120
0.020	0.810	2.969	0.173	3.592	3.952	0.050	3.712	4.083	1.800	0.421	2.220
0.020	0.804	3.056	0.118	3.726	4.098	0.020	3.820	4.202	1.956	0.468	2.424
0.020	0.802	3.079	0.086	3.777	4.155	0.010	3.849	4.234	2.122	0.540	2.662
0.050	0.820	2.807	0.256	3.358	3.693	0.100	3.509	3.860	1.825	0.607	2.431
0.050	0.810	2.955	0.189	3.560	3.916	0.050	3.694	4.063	2.006	0.684	2.689
0.050	0.804	3.031	0.123	3.692	4.062	0.020	3.788	4.167	2.358	0.865	3.223
0.050	0.802	3.041	0.088	3.733	4.106	0.010	3.801	4.181	2.726	1.120	3.846
0.100	0.820	2.793	0.271	3.327	3.660	0.100	3.491	3.840	2.057	0.888	2.945
0.100	0.810	2.931	0.195	3.527	3.880	0.050	3.663	4.030	2.398	1.079	3.477
0.100	0.804	2.988	0.125	3.642	4.007	0.020	3.735	4.108	3.052	1.532	4.584
0.100	0.802	2.977	0.089	3.660	4.026	0.010	3.722	4.094	3.760	2.124	5.884
0.200	0.820	2.763	0.278	3.286	3.614	0.100	3.453	3.799	2.576	1.408	3.984
0.200	0.810	2.882	01.98	3.470	3.817	0.050	3.603	3.963	3.225	1.865	5.090
0.200	0.804	2.902	0.126	3.545	3.900	0.020	3.628	3.991	4.496	2.924	7.420
0.200	0.802	2.850	0.089	3.517	3.868	0.010	3.563	3.919	5.983	4.236	10.218
0.500	0.820	2.673	0.283	3.180	3.498	0.100	3.342	3.676	4.242	2.992	7.234
0.500	0.810	2.737	0.200	3.304	3.635	0.050	3.421	3.764	5.868	4.390	10.258
0.500	0.804	2.646	0.126	3.256	3.582	0.020	3.308	3.638	9.274	7.604	16.978
0.500	0.802	2.468	0.090	3.087	3.396	0.010	3.085	3.394	13.791	11.977	25.768

Table 9. Optimal results for two stage system for various values of K_1 and Y_1^2, $\delta = 1.1$, $G = 0.0$

K_1	Y_1^0	Y_2^0	Y_1^1	$\bar{Y}_2^1$	Y_2^1	Y_1^2	$\bar{Y}_2^2$	Y_2^2	$\bar{t}_1$	$\bar{t}_2$	t
0.010	0.820	2.819	0.174	3.369	3.816	0.100	3.524	3.876	1.781	0.220	2.001
0.010	0.810	2.974	0.149	3.640	4.004	0.050	3.718	4.089	1.749	0.304	2.054
0.010	0.804	3.064	0.108	3.764	4.141	0.020	3.831	4.214	1.824	0.330	2.154
0.010	0.802	3.092	0.081	3.816	4.197	0.010	3.864	4.251	1.919	0.352	2.272
0.020	0.820	2.816	0.214	3.426	3.768	0.100	3.520	3.872	1.739	0.374	2.113
0.020	0.810	2.969	0.169	3.614	3.976	0.050	3.711	4.083	1.786	0.429	2.215
0.020	0.804	3.056	0.115	3.747	4.122	0.020	3.820	4.202	1.946	0.477	2.424
0.020	0.802	3.079	0.084	3.798	4.178	0.010	3.848	4.233	2.118	0.551	2.668
0.050	0.820	2.807	0.250	3.381	3.719	0.100	3.509	3.860	1.813	0.615	2.427
0.050	0.810	2.954	0.184	3.583	3.941	0.050	3.693	4.062	1.996	0.696	2.692
0.050	0.804	3.030	0.120	3.713	4.084	0.020	3.787	4.166	2.354	0.885	3.239
0.050	0.802	3.040	0.086	3.752	4.127	0.010	3.799	4.179	2.734	1.143	3.877
0.100	0.820	2.792	0.264	3.350	3.685	0.100	3.490	3.839	2.047	0.901	2.947
0.100	0.810	2.930	0.190	3.549	3.904	0.050	3.662	4.029	2.395	1.098	3.493
0.100	0.804	2.986	0.122	3.662	4.028	0.020	3.732	4.106	3.064	1.562	4.626
0.100	0.802	2.974	0.087	3.677	4.045	0.010	3.718	4.090	3.790	2.168	5.958
0.200	0.820	2.762	0.272	3.308	3.639	0.100	3.452	3.797	2.571	1.430	4.001
0.200	0.810	2.881	0.193	3.490	3.839	0.050	3.601	3.961	3.229	1.904	5.133
0.200	0.804	2.899	0.123	3.561	3.918	0.020	3.623	3.986	4.544	2.978	7.521
0.200	0.802	2.844	0.087	3.528	3.881	0.010	3.555	3.911	6.029	4.366	10.395
0.500	0.820	2.671	0.276	3.200	3.520	0.100	3.338	3.672	4.268	3.029	7.298
0.500	0.810	2.733	0.195	3.320	3.652	0.050	3.416	3.757	5.928	4.468	10.397
0.500	0.804	2.637	0.123	3.263	3.589	0.020	3.296	3.626	9.532	7.777	17.309
0.500	0.802	2.454	0.087	3.084	3.392	0.010	3.067	3.374	14.103	12.291	26.394

Table 10. Optimal results for two stage system for various values of K_1 and Y_1^2, δ = 1.1, G = 0.10

K_1	Y_1^0	Y_2^0	Y_1^1	$\bar{Y}_2^1$	Y_2^1	Y_1^2	$\bar{Y}_2^2$	Y_2^2	$\bar{t}_1$	$\bar{t}_2$	$\bar{t}$
0.010	0.820	2.819	0.171	3.490	3.839	0.100	3.523	3.876	1.770	0.222	1.992
0.010	0.810	2.974	0.146	3.661	4.027	0.050	3.717	4.089	1.734	0.313	2.047
0.010	0.804	3.064	0.106	3.786	4.164	0.020	3.830	4.213	1.811	0.338	2.150
0.010	0.802	3.091	0.079	3.837	4.220	0.010	3.864	4.250	1.911	0.359	2.270
0.020	0.820	2.816	0.210	3.448	3.792	0.100	3.520	3.872	1.726	0.379	2.105
0.020	0.810	2.969	0.165	3.637	4.000	0.050	3.711	4.082	1.772	0.438	2.210
0.020	0.804	3.055	0.112	3.769	4.146	0.020	3.819	4.201	1.937	0.488	2.425
0.020	0.802	3.078	0.082	3.818	4.200	0.010	3.847	4.232	2.110	0.565	2.676
0.050	0.820	2.807	0.245	3.403	3.743	0.100	3.509	3.859	1.797	0.627	2.424
0.040	0.810	2.954	0.180	3.605	3.966	0.050	3.692	4.062	1.986	0.709	2.695
0.050	0.804	3.029	0.117	3.734	4.107	0.020	3.786	4.164	2.352	0.904	3.256
0.050	0.802	3.038	0.083	3.771	4.148	0.010	3.797	4.177	2.744	1.168	3.911
0.100	0.820	2.791	0.258	3.372	3.709	0.100	3.489	3.838	2.034	0.917	2.951
0.100	0.810	2.929	0.185	3.571	3.928	0.050	3.661	4.027	2.391	1.119	3.510
0.100	0.804	2.984	0.118	3.681	4.049	0.020	3.730	4.103	3.076	1.596	4.673
0.100	0.802	2.971	0.084	3.693	4.063	0.010	3.714	4.085	3.823	2.217	6.041
0.200	0.820	2.761	0.265	3.330	3.663	0.100	3.451	3.796	2.570	1.450	4.020
0.200	0.810	2.878	0.188	3.510	3.861	0.050	3.598	3.958	3.234	1.937	5.180
0.200	0.804	2.895	0.119	3.577	3.935	0.020	3.618	3.980	4.588	3.043	7.632
0.200	0.802	2.838	0.084	3.540	3.894	0.010	3.547	3.902	6.156	4.432	10.588
0.500	0.820	2.668	0.270	3.219	3.541	0.100	3.335	3.669	4.282	3.084	7.367
0.500	0.810	2.727	0.190	3.335	3.668	0.050	3.409	3.750	6.004	4.543	10.547
0.500	0.804	2.627	0.120	3.268	3.595	0.020	3.283	3.612	9.692	7.976	17.669
0.500	0.802	2.438	0.085	3.079	3.387	0.010	3.047	3.352	14.437	12.641	27.078

Table 11. Optimal results for two stage system for various values of K_1 and y_1^2, $\delta = 1.1$, $G = 0.15$

K_1	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	y_2^1	y_1^2	$\bar{y}_2^2$	y_2^2	$\bar{t}_1$	$\bar{t}_2$	$\bar{t}$
0.010	0.820	2.819	0.168	3.510	3.861	0.100	3.523	3.876	1.757	0.227	1.983
0.010	0.810	2.974	0.143	3.683	4.051	0.050	3.717	4.089	1.721	0.319	2.040
0.010	0.804	3.064	0.103	3.807	4.188	0.020	3.830	4.213	1.800	0.346	2.146
0.010	0.802	3.091	0.077	3.857	4.243	0.010	3.864	4.250	1.900	0.370	2.270
0.020	0.820	2.816	0.207	3.468	3.815	0.100	3.520	3.872	1.709	0.389	2.098
0.020	0.810	2.969	0.161	3.658	4.024	0.050	3.711	4.082	1.757	0.449	2.206
0.020	0.804	3.055	0.109	3.791	4.170	0.020	3.819	4.201	1.929	0.498	2.427
0.020	0.802	3.077	0.079	3.839	4.223	0.010	3.847	4.231	2.108	0.577	2.685
0.050	0.820	2.807	0.239	3.426	3.769	0.100	3.508	3.859	1.786	0.635	2.421
0.050	0.810	2.953	0.175	3.627	3.990	0.050	3.692	4.061	1.975	0.724	2.699
0.050	0.804	3.028	0.113	3.754	4.130	0.020	3.784	4.163	2.352	0.924	3.276
0.050	0.802	3.036	0.081	3.790	4.169	0.010	3.795	4.175	2.747	1.202	3.949
0.100	0.820	2.791	0.252	3.395	3.735	0.100	3.489	3.837	2.027	0.929	2.956
0.100	0.810	2.928	0.180	3.592	3.951	0.050	3.660	4.026	2.388	1.141	3.529
0.100	0.804	2.982	0.115	3.700	4.070	0.020	3.727	4.100	3.094	1.630	4.724
0.100	0.802	2.968	0.082	3.709	4.080	0.010	3.710	4.081	3.855	2.276	6.131
0.200	0.820	2.760	0.258	3.352	3.688	0.100	3.449	3.794	2.568	1.473	4.041
0.200	0.810	2.876	0.183	3.530	3.883	0.050	3.595	3.955	3.260	1.971	5.231
0.200	0.804	2.890	0.116	3.592	3.952	0.020	3.613	3.974	4.632	3.120	7.752
0.200	0.802	2.831	0.082	3.550	3.905	0.010	3.539	3.893	6.250	4.548	10.798
0.500	0.820	2.665	0.262	3.239	3.563	0.100	3.332	3.665	4.320	3.122	7.442
0.500	0.810	2.722	0.184	3.349	3.684	0.050	3.403	3.743	6.070	4.641	10.711
0.500	0.804	2.616	0.116	3.273	3.600	0.020	3.270	3.597	9.886	8.176	18.062
0.500	0.802	2.421	0.082	3.073	3.380	0.010	3.026	3.329	14.796	13.033	27.810

Table 12. Optimal results for two stage system for various values of K_1 and y_1^2, $\delta = 1.1$, $G = 0.20$

K_1	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	y_2^1	y_1^2	$\bar{y}_2^2$	y_2^2	$\bar{t}_1$	$\bar{t}_2$	$\bar{t}$
0.010	0.820	2.819	0.165	3.530	3.883	0.100	3.523	3.876	1.745	0.230	1.975
0.010	0.810	2.974	0.139	3.704	4.075	0.050	3.717	4.089	1.706	0.327	2.033
0.010	0.804	3.064	0.100	3.829	4.211	0.020	3.830	4.213	1.786	0.357	2.143
0.010	0.802	3.091	0.074	3.879	4.266	0.010	3.863	4.249	1.892	0.379	2.271
0.020	0.820	2.816	0.202	3.491	3.840	0.100	3.520	3.872	1.697	0.394	2.091
0.020	0.810	2.969	0.157	3.681	4.049	0.050	3.711	4.082	1.744	0.459	2.202
0.020	0.804	3.054	0.106	3.812	4.193	0.020	3.818	4.200	1.918	0.511	2.430
0.020	0.802	3.077	0.077	3.860	4.246	0.010	3.846	4.230	2.104	0.592	2.695
0.050	0.820	2.806	0.233	3.449	3.794	0.100	3.508	3.859	1.772	0.647	2.419
0.050	0.810	2.953	0.170	3.650	4.015	0.050	3.691	4.060	1.968	0.737	2.705
0.050	0.804	3.026	0.110	3.775	4.153	0.020	3.783	4.161	2.354	0.944	3.298
0.050	0.802	3.034	0.078	3.809	4.190	0.010	3.793	4.172	2.763	1.229	3.991
0.100	0.820	2.790	0.245	3.418	3.760	0.100	3.488	3.837	2.017	0.946	2.962
0.100	0.810	2.927	0.175	3.614	3.975	0.050	3.658	4.024	2.387	1.165	3.551
0.100	0.804	2.979	0.111	3.719	4.090	0.020	3.724	4.097	3.109	1.671	4.781
0.100	0.802	2.964	0.079	3.725	4.098	0.010	3.705	4.076	3.892	2.338	6.230
0.200	0.820	2.758	0.251	3.375	3.712	0.100	3.448	3.793	2.568	1.496	4.064
0.200	0.810	2.874	0.177	3.550	3.905	0.050	3.593	3.952	3.274	2.014	5.288
0.200	0.804	2.886	0.112	3.607	3.968	0.020	3.607	3.968	4.688	3.196	7.885
0.200	0.802	2.824	0.079	3.560	3.916	0.010	3.530	3.883	6.351	4.678	11.029
0.500	0.820	2.662	0.255	3.258	3.584	0.100	3.328	3.661	4.346	3.177	7.523
0.500	0.810	2.716	0.179	3.363	3.699	0.050	3.395	3.735	6.149	4.740	10.889
0.500	0.804	2.604	0.113	3.276	3.604	0.020	3.255	3.581	10.065	8.427	18.492
0.500	0.802	2.402	0.080	3.064	3.371	0.010	3.003	3.303	15.181	13.480	28.661

Table 13. Optimal results for two stage system for various values of K_1 and y_1^2, $\delta = 1.1$, $G = 0.25$

K_1	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	y_2^1	y_1^2	$\bar{y}_2^2$	y_2^2	$\bar{t}_1$	$\bar{t}_2$	$\bar{t}$
0.010	0.820	2.819	0.162	3.551	3.906	0.100	3.523	3.876	1.733	0.234	1.967
0.010	0.810	2.974	0.136	3.726	4.098	0.050	3.717	4.089	1.690	0.337	2.027
0.010	0.804	3.064	0.097	3.850	4.235	0.020	3.829	4.212	1.774	0.366	2.140
0.010	0.802	3.090	0.072	3.900	4.290	0.010	3.863	4.249	1.884	0.389	2.272
0.020	0.820	2.816	0.197	3.513	3.864	0.100	3.519	3.871	1.685	0.400	2.085
0.020	0.810	2.968	0.153	3.703	4.073	0.050	3.710	4.081	1.729	0.471	2.200
0.020	0.804	3.054	0.103	3.834	4.217	0.020	3.817	4.199	1.910	0.524	2.434
0.020	0.802	3.076	0.074	3.880	4.268	0.010	3.845	4.229	2.099	0.609	2.708
0.050	0.820	2.806	0.227	3.472	3.820	0.100	3.507	3.858	1.762	0.656	2.418
0.050	0.810	2.952	0.165	3.672	4.039	0.050	3.690	4.059	1.956	0.756	2.712
0.050	0.804	3.025	0.107	3.795	4.175	0.020	3.781	4.160	2.353	0.971	3.324
0.050	0.802	3.032	0.076	3.827	4.210	0.010	3.790	4.170	2.775	1.264	4.039
0.100	0.820	2.790	0.238	3.442	3.786	0.100	3.487	3.836	2.011	0.959	2.970
0.100	0.810	2.925	0.169	3.636	4.000	0.050	3.657	4.022	2.391	1.185	3.576
0.100	0.804	2.977	0.108	3.737	4.110	0.020	3.721	4.093	3.124	1.720	4.843
0.100	0.802	2.960	0.076	3.741	4.115	0.010	3.700	4.070	3.955	2.385	6.340
0.200	0.820	2.757	0.244	3.397	3.737	0.100	3.446	3.791	2.568	1.523	4.091
0.200	0.810	2.872	0.172	3.570	3.927	0.050	3.589	3.948	3.295	2.056	5.350
0.200	0.804	2.881	0.108	3.622	3.984	0.020	3.601	3.961	4.760	3.270	8.030
0.200	0.802	2.816	0.077	3.569	3.926	0.010	3.520	3.872	6.450	4.835	11.285
0.500	0.820	2.659	0.248	3.277	3.605	0.100	3.324	3.656	4.378	3.234	7.612
0.500	0.810	2.710	0.174	3.376	3.714	0.050	3.388	3.727	6.220	4.863	11.083
0.500	0.804	2.592	0.109	3.278	3.606	0.020	3.240	3.564	10.324	8.643	18.967
0.500	0.802	2.382	0.077	3.054	3.359	0.010	2.978	3.276	15.731	13.853	29.584

Table 14. Optimal results for two stage system for various values of K_1 and y_1^2, $\delta = 1.2$, $G = 0.0$

K_1	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	y_2^1	y_1^2	$\bar{y}_2^2$	y_2^2	$\bar{t}_1$	$\bar{t}_2$	$\bar{t}$
0.010	0.820	2.819	0.175	3.450	4.140	0.100	3.524	4.228	1.646	0.196	1.842
0.010	0.810	2.974	0.152	3.619	4.342	0.050	3.718	4.461	1.617	0.273	1.890
0.010	0.804	3.065	0.111	3.743	4.491	0.020	3.831	4.597	1.683	0.296	1.979
0.010	0.802	3.092	0.083	3.794	4.553	0.010	3.865	4.638	1.768	0.316	2.084
0.020	0.820	2.816	0.218	3.405	4.086	0.100	3.520	4.224	1.608	0.335	1.944
0.020	0.810	2.969	0.173	3.592	4.311	0.050	3.712	4.454	1.650	0.386	2.035
0.020	0.804	3.056	0.118	3.726	4.471	0.020	3.820	4.584	1.793	0.429	2.222
0.020	0.802	3.079	0.086	3.777	4.533	0.010	3.849	4.619	1.945	0.495	2.440
0.050	0.820	2.807	0.256	3.358	4.029	0.100	3.509	4.211	1.672	0.556	2.229
0.050	0.810	2.955	0.189	3.560	4.272	0.050	3.694	4.432	1.839	0.627	2.465
0.050	0.804	3.031	0.123	3.692	4.431	0.020	3.788	4.546	2.162	0.793	2.955
0.050	0.802	3.041	0.088	3.733	4.479	0.010	3.801	4.561	2.499	1.027	3.526
0.100	0.820	2.793	0.271	3.327	3.992	0.100	3.491	4.189	1.885	0.814	2.699
0.100	0.810	2.931	0.195	3.527	4.233	0.050	3.663	4.396	2.198	0.989	3.188
0.100	0.804	2.988	0.125	3.642	4.371	0.020	3.735	4.482	2.797	1.405	4.202
0.100	0.802	2.977	0.089	3.660	4.392	0.010	3.722	4.466	3.446	1.947	5.393
0.200	0.820	2.763	0.278	3.286	3.943	0.100	3.453	4.144	2.361	1.291	3.652
0.200	0.810	2.882	0.198	3.470	4.164	0.050	3.603	4.324	2.956	1.710	4.666
0.200	0.804	2.902	0.126	3.545	4.254	0.020	3.628	4.354	4.122	2.680	6.802
0.200	0.802	2.850	0.089	3.517	4.220	0.010	3.563	4.275	5.484	3.883	9.367
0.500	0.820	2.673	0.283	3.180	3.816	0.100	3.342	4.010	3.888	2.743	6.631
0.500	0.810	2.737	0.200	3.304	3.965	0.050	3.421	4.106	5.379	4.024	9.403
0.500	0.804	2.646	0.126	3.256	3.908	0.020	3.308	3.969	8.593	6.970	15.563
0.500	0.802	2.468	0.090	3.087	3.704	0.010	3.085	3.702	12.642	10.979	23.621

Table 15. Optimal results for two stage system for various values of K_1 and y_1^2, $\delta = 1.2$, $G = 0.05$

K_1	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	y_2^1	y_1^2	$\bar{y}_2^2$	y_2^2	$\bar{t}_1$	$\bar{t}_2$	$\bar{t}$
0.010	0.820	2.819	0.171	3.489	4.187	0.100	3.523	4.228	1.630	0.195	1.826
0.010	0.810	2.974	0.148	3.659	4.391	0.050	3.717	4.461	1.598	0.277	1.875
0.010	0.804	3.064	0.107	3.784	4.541	0.020	3.831	4.597	1.666	0.300	1.966
0.010	0.802	3.092	0.081	3.835	4.602	0.010	3.864	4.637	1.752	0.321	2.073
0.020	0.820	2.816	0.212	3.446	4.135	0.100	3.520	4.224	1.593	0.336	1.928
0.020	0.810	2.969	0.168	3.634	4.361	0.050	3.711	4.454	1.632	0.390	2.022
0.020	0.804	3.056	0.115	3.767	4.520	0.020	3.820	4.584	1.778	0.435	2.213
0.020	0.802	3.079	0.084	3.817	4.581	0.010	3.848	4.618	1.931	0.505	2.436
0.050	0.820	2.807	0.249	3.399	4.079	0.100	3.509	4.211	1.657	0.559	2.216
0.050	0.810	2.954	0.183	3.602	4.322	0.050	3.693	4.432	1.825	0.633	2.458
0.050	0.804	3.030	0.120	3.732	4.479	0.020	3.787	4.544	2.149	0.808	2.958
0.050	0.802	3.040	0.086	3.771	4.525	0.010	3.799	4.559	2.492	1.049	3.541
0.100	0.820	2.792	0.264	3.368	4.041	0.100	3.490	4.188	1.870	0.822	2.692
0.100	0.810	2.930	0.190	3.568	4.281	0.050	3.662	4.395	2.188	1.003	3.190
0.100	0.804	2.986	0.122	3.680	4.416	0.020	3.732	4.479	2.793	1.433	4.226
0.100	0.802	2.974	0.086	3.696	4.435	0.010	3.718	4.462	3.465	1.980	5.444
0.200	0.820	2.762	0.271	3.326	3.991	0.100	3.452	4.143	2.351	1.304	3.655
0.200	0.810	2.880	0.192	3.508	4.210	0.050	3.601	4.321	2.956	1.734	4.690
0.200	0.804	2.899	0.123	3.580	4.296	0.020	3.623	4.348	4.144	2.729	6.873
0.200	0.802	2.844	0.087	3.546	4.256	0.010	3.555	4.266	5.533	3.968	9.501
0.500	0.820	2.671	0.276	3.217	3.861	0.100	3.338	4.006	3.900	2.769	6.669
0.500	0.810	2.732	0.195	3.337	4.004	0.050	3.416	4.099	5.414	4.088	9.503
0.500	0.804	2.637	0.123	3.279	3.935	0.020	3.296	3.955	8.687	7.135	15.823
0.500	0.802	2.454	0.087	3.099	3.719	0.010	3.067	3.680	12.864	11.266	24.130

Table 16. Optimal results for two stage system for various values of K_1 and y_1^2, δ = 1.2, G = 0.10

K_1	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	y_2^1	y_1^2	$\bar{y}_2^2$	y_2^2	$\bar{t}_1$	$\bar{t}_2$	$\bar{t}$
0.010	0.820	2.819	0.167	3.528	4.234	0.100	3.523	4.228	1.616	0.194	1.810
0.010	0.810	2.974	0.143	3.701	4.441	0.050	3.717	4.461	1.581	0.279	1.860
0.010	0.804	3.064	0.105	3.825	4.590	0.020	3.830	4.596	1.648	0.306	1.954
0.010	0.802	3.091	0.078	3.876	4.651	0.010	3.864	4.637	1.736	0.328	2.064
0.020	0.820	2.816	0.207	3.486	4.183	0.100	3.520	4.224	1.577	0.337	1.914
0.020	0.810	2.969	0.163	3.676	4.411	0.050	3.711	4.453	1.616	0.394	2.010
0.020	0.804	3.055	0.111	3.808	4.569	0.020	3.819	4.583	1.762	0.443	2.205
0.020	0.802	3.078	0.081	3.858	4.629	0.010	3.847	4.617	1.920	0.513	2.433
0.050	0.820	2.807	0.242	3.441	4.129	0.100	3.508	4.210	1.642	0.563	2.205
0.050	0.810	2.954	0.178	3.643	4.372	0.050	3.692	4.431	1.809	0.643	2.452
0.050	0.804	3.029	0.116	3.772	4.526	0.020	3.786	4.543	2.140	0.823	2.963
0.050	0.802	3.038	0.083	3.809	4.571	0.010	3.797	4.557	2.488	1.072	3.560
0.100	0.820	2.791	0.256	3.409	4.091	0.100	3.489	4.187	1.857	0.829	2.686
0.100	0.810	2.929	0.184	3.608	4.330	0.050	3.661	4.393	2.179	1.016	3.195
0.100	0.804	2.984	0.118	3.719	4.462	0.020	3.730	4.476	2.801	1.454	4.255
0.100	0.802	2.971	0.084	3.731	4.477	0.010	3.714	4.457	3.478	2.024	5.502
0.200	0.820	2.761	0.264	3.366	4.039	0.100	3.451	4.141	2.344	1.317	3.660
0.200	0.810	2.878	0.187	3.547	4.257	0.050	3.598	4.318	2.961	1.757	4.718
0.200	0.804	2.895	0.119	3.614	4.337	0.020	3.618	4.342	4.179	2.775	6.953
0.200	0.802	2.838	0.084	3.575	4.290	0.010	3.547	4.257	5.587	4.063	9.649
0.500	0.820	2.668	0.268	3.254	3.905	0.100	3.335	4.002	3.918	2.795	6.712
0.500	0.810	2.727	0.189	3.369	4.043	0.050	3.409	4.091	5.463	4.149	9.613
0.500	0.804	2.627	0.119	3.302	3.962	0.020	3.283	3.940	8.841	7.266	16.108
0.500	0.802	2.438	0.084	3.110	3.732	0.010	3.047	3.657	13.205	11.486	24.692

Table 17. Optimal results for two stage system for various values of K_1 and y_1^2, $\delta = 1.2$, $G = 0.15$

K_1	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	y_2^1	y_1^2	$\bar{y}_2^2$	y_2^2	$\bar{t}_1$	$\bar{t}_2$	$\bar{t}$
0.010	0.820	2.819	0.162	3.568	4.282	0.100	3.523	4.228	1.604	0.190	1.794
0.010	0.810	2.974	0.139	3.742	4.490	0.050	3.717	4.461	1.563	0.283	1.846
0.010	0.804	3.064	0.101	3.866	4.640	0.020	3.830	4.596	1.631	0.311	1.942
0.010	0.802	3.091	0.076	3.916	4.700	0.010	3.864	4.636	1.721	0.334	2.055
0.020	0.820	2.816	0.200	3.527	4.233	0.100	3.520	4.224	1.564	0.336	1.900
0.020	0.810	2.969	0.158	3.717	4.461	0.050	3.711	4.453	1.598	0.400	1.998
0.020	0.804	3.055	0.108	3.849	4.618	0.020	3.819	4.582	1.746	0.452	2.198
0.020	0.802	3.077	0.079	3.897	4.677	0.010	3.847	4.616	1.907	0.525	2.432
0.050	0.820	2.806	0.235	3.483	4.179	0.100	3.508	4.210	1.631	0.564	2.195
0.050	0.810	2.953	0.173	3.685	4.422	0.050	3.692	4.430	1.796	0.651	2.447
0.050	0.804	3.027	0.112	3.812	4.575	0.020	3.784	4.541	2.134	0.836	2.971
0.050	0.802	3.036	0.080	3.847	4.617	0.010	3.795	4.554	2.495	1.087	3.583
0.100	0.820	2.791	0.248	3.451	4.141	0.100	3.489	4.186	1.849	0.832	2.681
0.100	0.810	2.928	0.178	3.649	4.379	0.050	3.660	4.392	2.173	1.030	3.202
0.100	0.804	2.982	0.114	3.757	4.508	0.020	3.727	4.473	2.809	1.479	4.288
0.100	0.802	2.968	0.081	3.766	4.519	0.010	3.710	4.452	3.500	2.067	5.567
0.200	0.820	2.759	0.256	3.407	4.088	0.100	3.449	4.139	2.342	1.326	3.668
0.200	0.810	2.876	0.181	3.585	4.302	0.050	3.595	4.314	2.965	1.786	4.750
0.200	0.804	2.890	0.115	3.647	4.377	0.020	3.613	4.335	4.217	2.826	7.043
0.200	0.802	2.831	0.081	3.604	4.324	0.010	3.539	4.247	5.676	4.138	9.814
0.500	0.820	2.665	0.261	3.290	3.948	0.100	3.332	3.998	3.927	2.834	6.761
0.500	0.810	2.722	0.183	3.401	4.081	0.050	3.403	4.083	5.526	4.209	9.735
0.500	0.804	2.616	0.116	3.322	3.987	0.020	3.270	3.924	8.954	7.469	16.423
0.500	0.802	2.421	0.082	3.118	3.742	0.010	3.026	3.631	13.453	11.859	25.313

Table 18. Optimal results for two stage system for various values of K_1 and y_1^2, $\delta = 1.2$, $G = 0.20$

K_1	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	y_2^1	y_1^2	$\bar{y}_2^2$	y_2^2	$\bar{t}_1$	$\bar{t}_2$	$\bar{t}$
0.010	0.820	2.819	0.158	3.607	4.329	0.100	3.523	4.228	1.591	0.188	1.779
0.010	0.810	2.974	0.135	3.783	4.539	0.050	3.717	4.460	1.547	0.285	1.833
0.010	0.804	3.064	0.098	3.908	4.689	0.020	3.830	4.596	1.615	0.317	1.932
0.010	0.802	3.091	0.073	3.957	4.748	0.010	3.863	4.636	1.705	0.342	2.048
0.020	0.820	2.816	0.194	3.568	4.282	0.100	3.519	4.223	1.552	0.334	1.886
0.020	0.810	2.968	0.153	3.759	4.510	0.050	3.711	4.453	1.582	0.406	1.987
0.020	0.804	3.054	0.104	3.890	4.668	0.020	3.818	4.582	1.734	0.459	2.193
0.020	0.802	3.076	0.076	3.938	4.725	0.010	3.846	4.615	1.898	0.535	2.433
0.050	0.820	2.806	0.228	3.524	4.229	0.100	3.508	4.209	1.614	0.570	2.185
0.050	0.810	2.953	0.167	3.726	4.471	0.050	3.691	4.429	1.781	0.663	2.444
0.050	0.804	3.026	0.108	3.852	4.622	0.020	3.783	4.539	2.130	0.851	2.981
0.050	0.802	3.034	0.078	3.885	4.662	0.010	3.793	4.552	2.494	1.115	3.609
0.100	0.820	2.790	0.241	3.492	4.191	0.100	3.488	4.185	1.838	0.840	2.678
0.100	0.810	2.927	0.173	3.689	4.427	0.050	3.658	4.390	2.164	1.048	3.212
0.100	0.804	2.979	0.110	3.794	4.552	0.020	3.724	4.469	2.809	1.517	4.326
0.100	0.802	2.964	0.078	3.800	4.560	0.010	3.705	4.446	3.527	2.114	5.641
0.200	0.820	2.758	0.248	3.447	4.136	0.100	3.448	4.137	2.338	1.340	3.678
0.200	0.810	2.874	0.176	3.623	4.348	0.050	3.593	4.311	2.968	1.820	4.787
0.200	0.804	2.886	0.111	3.681	4.417	0.020	3.607	4.328	4.260	2.883	7.143
0.200	0.802	2.824	0.079	3.631	4.357	0.010	3.530	4.236	5.742	4.255	9.997
0.500	0.820	2.662	0.253	3.327	3.992	0.100	3.328	3.993	3.949	2.867	6.816
0.500	0.810	2.716	0.177	3.432	4.118	0.050	3.395	4.075	5.572	4.297	9.870
0.500	0.804	2.604	0.112	3.342	4.010	0.020	3.255	3.907	9.125	7.645	16.771
0.500	0.802	2.402	0.079	3.124	3.749	0.010	3.003	3.604	13.742	12.262	26.003

Table 19. Optimal results for two stage system for various values of K_1 and y_1^2, $\delta = 1.2$, $G = 0.25$

K_1	y_1^0	y_2^0	y_1^1	y_2^{-1}	y_2^1	y_1^2	y_2^{-2}	y_2^2	$\bar{t}_1$	$\bar{t}_2$	$\bar{t}$
0.010	0.820	2.818	0.153	3.647	4.377	0.100	3.523	4.228	1.581	0.184	1.764
0.010	0.810	2.973	0.130	3.824	4.589	0.050	3.717	4.460	1.532	0.288	1.820
0.010	0.804	3.064	0.095	3.948	4.738	0.020	3.829	4.595	1.597	0.325	1.922
0.010	0.802	3.090	0.071	3.997	4.797	0.010	3.863	4.635	1.690	0.351	2.041
0.020	0.820	2.815	0.188	3.609	4.331	0.100	3.519	4.223	1.539	0.334	1.873
0.020	0.810	2.968	0.148	3.800	4.560	0.050	3.710	4.452	1.565	0.412	1.977
0.020	0.804	3.054	0.101	3.931	4.717	0.020	3.817	4.581	1.720	0.469	2.189
0.020	0.802	3.076	0.073	3.977	4.773	0.010	3.845	4.614	1.886	0.550	2.436
0.050	0.820	2.806	0.220	3.566	4.279	0.100	3.507	4.209	1.605	0.571	2.176
0.050	0.810	2.952	0.161	3.768	4.521	0.050	3.690	4.428	1.772	0.670	2.442
0.050	0.804	3.025	0.105	3.891	4.670	0.020	3.781	4.538	2.120	0.874	2.994
0.050	0.802	3.032	0.075	3.923	4.708	0.010	3.790	4.549	2.503	1.138	3.641
0.100	0.820	2.790	0.233	3.534	4.240	0.100	3.487	4.184	1.828	0.848	2.677
0.100	0.810	2.925	0.167	3.730	4.476	0.050	3.657	4.388	2.162	1.063	3.224
0.100	0.804	2.977	0.107	3.831	4.597	0.020	3.721	4.465	2.820	1.550	4.370
0.100	0.802	2.960	0.075	3.834	4.601	0.010	3.700	4.440	3.569	2.155	5.724
0.200	0.820	2.757	0.240	3.487	4.184	0.100	3.446	4.135	2.337	1.354	3.691
0.200	0.810	2.871	0.169	3.662	4.394	0.050	3.589	4.307	2.985	1.844	4.830
0.200	0.804	2.881	0.107	3.713	4.455	0.020	3.601	4.321	4.301	2.954	7.255
0.200	0.802	2.816	0.076	3.657	4.389	0.010	3.520	4.223	5.833	4.368	10.201
0.500	0.820	2.659	0.244	3.363	4.036	0.100	3.324	3.989	3.983	2.894	6.877
0.500	0.810	2.710	0.171	3.463	4.155	0.050	3.388	4.065	5.650	4.369	10.019
0.500	0.804	2.592	0.108	3.359	4.031	0.020	3.240	3.888	9.311	7.846	17.157
0.500	0.802	2.382	0.076	3.128	3.753	0.010	2.978	3.573	14.179	12.597	26.775

...ptimal results for three stage system for various values of K_1 and y_1, $\theta = 1.0$, $G = 0.0$.

K_1	$\bar{t}$	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	$\bar{t}_1$	y_1^2	$\bar{y}_2^2$	$\bar{t}_2$	y_1^3	$\bar{y}_2^3$	$\bar{t}_3$
0.01	2.208	0.820	2.189	0.180	3.445	1.960	0.127	3.497	0.163	0.10	3.524	0.085
0.01	2.252	0.810	2.975	0.176	3.596	1.864	0.090	3.679	0.259	0.05	3.719	0.130
0.01	2.314	0.804	3.066	0.156	3.701	1.862	0.056	3.799	0.312	0.02	3.834	0.140
0.01	2.374	0.802	3.095	0.134	3.750	1.914	0.036	3.846	0.327	0.01	3.870	0.133
0.02	2.321	0.820	2.816	0.244	3.381	1.845	0.153	3.470	0.297	0.10	3.54	0.179
0.02	2.395	0.810	2.970	0.220	3.547	1.814	0.102	3.664	0.385	0.05	3.714	0.196
0.02	2.520	0.804	3.059	0.188	3.664	1.861	0.060	3.789	0.452	0.02	3.827	0.207
0.02	2.646	0.802	3.005	0.159	3.717	1.547	0.040	3.833	0.469	0.01	3.861	0.230
0.05	2.615	0.820	2.808	0.323	3.296	1.742	0.174	3.441	0.557	0.10	3.512	0.316
0.05	2.797	0.810	2.957	0.270	3.487	1.833	0.115	3.638	0.614	0.05	3.701	0.350
0.05	3.125	0.804	3.037	0.211	3.622	2.025	0.065	3.763	0.688	0.02	3.805	0.412
0.05	3.460	0.802	3.055	0.171	3.679	2.214	0.041	3.803	0.755	0.01	3.831	0.492

Table 21. Optimal results for three stage system for various values of K_1 and y_1^3, $\delta = 1.0$, $G = 0.05$.

K_1	$\bar{t}$	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	$\bar{t}_1$	y_1^2	$\bar{y}_2^2$	$\bar{t}_2$	y_1^3	$\bar{y}_2^3$	$\bar{t}_3$
0.01	2.208	0.820	2.819	0.181	3.443	1.948	0.131	3.493	0.157	0.10	3.523	0.102
0.01	2.256	0.810	2.975	0.157	3.613	1.911	0.107	3.661	0.148	0.05	3.718	0.195
0.01	2.333	0.804	3.066	0.117	3.737	1.983	0.067	3.785	0.152	0.02	3.832	0.197
0.01	2.408	0.802	3.095	0.092	3.788	2.065	0.042	3.837	0.164	0.01	3.868	0.178
0.02	2.324	0.820	2.816	0.223	3.400	1.902	0.173	3.448	0.158	0.10	3.520	0.263
0.02	2.414	0.810	2.970	0.178	3.588	1.949	0.128	3.636	0.154	0.05	3.712	0.310
0.02	2.577	0.804	3.059	0.125	3.721	2.106	0.075	3.770	0.167	0.02	3.823	0.303
0.02	2.737	0.802	3.085	0.095	3.774	2.255	0.045	3.823	0.191	0.01	3.856	0.289
0.05	2.645	0.820	2.808	0.261	3.353	1.373	0.211	3.402	0.170	0.10	3.510	0.501
0.05	2.881	0.810	2.957	0.194	3.557	2.163	0.744	3.605	0.178	0.05	3.696	0.539
0.05	3.314	0.804	3.038	0.130	3.692	2.514	0.080	3.740	0.215	0.02	3.796	0.584
0.05	3.734	0.802	3.056	0.097	3.739	2.842	0.047	3.786	0.276	0.01	3.819	0.616

Table 22. Optimal results for three stage system for various values of K_1 and y_1^3, $\delta = 1.0$, $G = 0.10$.

K_1	$\bar{t}$	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	$\bar{t}_1$	y_1^2	$\bar{y}_2^2$	$\bar{t}_2$	y_1^3	$\bar{y}_2^3$	$\bar{t}_3$
0.01	2.208	0.820	2.819	0.181	3.444	1.938	0.131	3.493	0.162	0.10	3.523	0.107
0.01	2.257	0.810	2.975	0.155	3.614	1.906	0.105	3.663	0.149	0.05	3.718	0.201
0.01	2.336	0.804	3.066	0.116	3.738	1.978	0.065	3.787	0.154	0.02	3.832	0.202
0.01	2.413	0.802	3.095	0.091	3.789	2.061	0.041	3.838	0.170	0.01	3.868	0.181
0.02	2.325	0.820	2.816	0.221	3.401	1.896	0.171	3.450	0.155	0.10	3.520	0.272
0.02	2.417	0.810	2.970	0.176	3.589	1.945	0.126	3.638	0.151	0.05	3.712	0.320
0.02	2.584	0.804	3.059	0.123	3.723	2.106	0.073	3.772	0.169	0.02	3.823	0.307
0.02	2.747	0.802	3.085	0.094	3.775	2.253	0.044	3.823	0.199	0.01	3.855	0.294
0.05	2.650	0.820	2.808	0.257	3.353	1.977	0.207	3.405	0.160	0.10	3.510	0.512
0.05	2.891	0.810	2.957	0.191	3.560	2.170	0.140	3.608	0.172	0.05	3.696	0.548
0.05	3.333	0.804	3.037	0.127	3.694	2.525	0.077	3.742	0.218	0.02	3.796	0.589
0.05	3.762	0.802	3.055	0.098	3.739	2.849	0.045	3.787	0.288	0.01	3.818	0.624

Table 23. Optimal results for three stage system for various values of K_1 and y_1^3, $\delta = 1.0$, $G = 0.15$.

K_1	$\bar{t}$	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	$\bar{t}_1$	y_1^2	$\bar{y}_2^2$	$\bar{t}_2$	y_1^3	$\bar{y}_2^3$	$\bar{t}_3$
0.01	2.208	0.820	2.819	0.181	3.444	1.925	0.131	3.493	0.169	0.10	3.523	0.114
0.01	2.258	0.810	2.974	0.155	3.615	1.898	0.105	3.664	0.151	0.05	3.718	0.208
0.01	2.339	0.804	3.066	0.114	3.739	1.972	0.064	3.788	0.159	0.02	3.832	0.206
0.01	2.418	0.802	3.094	0.094	3.790	2.054	0.040	3.838	0.178	0.01	3.868	0.184
0.02	2.326	0.820	2.816	0.220	3.402	1.888	0.170	3.451	0.155	0.10	3.520	0.282
0.02	2.421	0.810	2.970	0.173	3.519	1.942	0.123	3.640	0.151	0.05	3.712	0.327
0.02	2.592	0.804	3.058	0.121	3.725	2.103	0.071	3.773	0.174	0.02	3.823	0.314
0.02	2.759	0.802	3.084	0.092	3.774	2.249	0.042	3.824	0.209	0.01	3.855	0.300
0.05	2.655	0.820	2.808	0.253	3.360	1.971	0.203	3.408	0.153	0.10	3.509	0.524
0.05	2.902	2.810	2.957	0.187	3.562	2.173	0.137	3.611	0.169	0.05	3.695	0.559
0.05	3.354	0.804	3.036	0.125	3.695	2.530	0.075	3.743	0.225	0.02	3.795	0.579
0.05	3.792	0.802	3.054	0.094	3.739	2.853	0.044	3.787	0.305	0.01	3.817	0.633

Table 24. Optimal results for three stage system for various values of K_1 and y_1^3, $\delta = 1.0$, $G = 0.20$.

K_1	$\bar{t}$	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	$\bar{t}_1$	y_1^2	$\bar{y}_2^2$	$\bar{t}_2$	y_1^3	$\bar{y}_2^3$	$\bar{t}_3$
0.01	2.209	0.820	3.819	0.181	3.444	1.909	0.131	3.493	0.178	0.10	3.523	0.121
0.01	2.259	0.810	2.974	0.153	3.616	1.886	0.103	3.665	0.156	0.05	3.717	0.216
0.01	2.342	0.804	3.066	0.113	3.741	1.963	0.063	3.789	0.167	0.02	3.832	0.212
0.01	2.424	0.802	3.094	0.089	3.790	2.043	0.039	3.839	0.190	0.01	3.867	0.190
0.02	2.327	0.820	2.816	0.218	3.404	1.877	0.168	3.453	0.156	0.10	3.520	0.293
0.02	2.424	0.810	2.970	0.170	3.594	1.936	0.120	3.643	0.152	0.05	3.712	0.335
0.02	2.600	0.804	3.058	0.118	3.727	2.098	0.068	3.775	0.182	0.02	3.822	0.320
0.02	2.772	0.802	3.084	0.091	3.776	2.241	0.041	3.825	0.223	0.01	3.854	0.307
0.05	2.660	0.820	2.818	0.250	3.363	1.975	0.200	3.412	0.148	0.10	3.509	0.536
0.05	2.914	0.810	2.956	0.183	3.566	2.177	0.133	3.614	0.169	0.05	3.695	0.567
0.05	3.378	0.804	3.036	0.122	3.697	2.533	0.072	3.745	0.236	0.02	3.794	0.608
0.05	3.827	0.802	3.053	0.092	3.739	2.852	0.042	3.787	0.328	0.01	3.816	0.645

Table 25. Optimal results for three stage system for various values of K_1 and y_1^3, $\delta = 1.0$, $G = 0.25$.

K_1	$\bar{t}$	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	$\bar{t}_1$	y_1^2	$\bar{y}_2^2$	$\bar{t}_2$	y_1^3	$\bar{y}_2^3$	$\bar{t}_3$
0.01	2.209	0.820	2.819	0.180	3.444	1.890	0.130	3.493	0.190	0.10	3.523	0.127
0.01	2.260	0.810	2.974	0.151	3.618	1.872	0.101	3.667	0.164	0.05	3.717	0.223
0.01	2.347	0.804	3.065	0.111	3.742	1.949	0.061	3.791	0.178	0.02	3.831	0.219
0.01	2.431	0.802	3.094	0.088	3.791	2.029	0.038	3.840	0.205	0.01	3.867	0.196
0.02	2.328	0.820	2.816	0.216	3.406	1.862	0.166	3.455	0.161	0.10	3.520	0.303
0.02	2.428	0.810	2.970	0.168	3.596	1.925	0.118	3.645	0.158	0.05	3.712	0.345
0.02	2.610	0.804	3.058	0.116	3.728	2.087	0.066	3.777	0.194	0.02	3.822	0.328
0.02	2.780	0.802	3.083	0.090	3.777	2.228	0.040	3.825	0.242	0.01	3.854	0.315
0.05	2.666	0.820	2.807	0.246	3.367	1.970	0.196	3.416	0.147	0.10	3.509	0.548
0.05	2.927	0.810	2.956	0.180	3.568	2.172	0.130	3.617	0.173	0.05	3.694	0.581
0.05	3.403	0.804	3.035	0.120	3.698	2.528	0.070	3.746	0.252	0.02	3.793	0.622
0.05	3.866	0.802	3.052	0.091	3.739	2.843	0.041	3.786	0.358	0.01	3.814	0.663

Table 26. Optimal results for three stage system for various values of K_1 and y_1^3, $\delta = 1.1$, $G = 0.0$.

K_1	$\bar{t}$	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	$\bar{t}_1$	y_1^2	$\bar{y}_2^2$	$\bar{t}_2$	y_1^3	$\bar{y}_2^3$	$\bar{t}_3$
0.01	2.007	0.820	2.819	0.181	3.444	1.778	0.131	3.493	0.140	0.10	3.523	0.088
0.01	2.050	0.810	2.975	0.158	3.612	1.743	0.108	3.661	0.135	0.05	3.718	0.171
0.01	2.118	0.804	3.066	0.119	3.736	1.805	0.069	3.784	0.137	0.02	3.832	0.155
0.01	2.185	0.802	3.095	0.093	3.787	1.880	0.043	3.836	0.145	0.01	3.868	0.159
0.02	2.112	0.820	2.817	0.225	3.398	1.732	0.175	3.446	0.146	0.10	3.520	0.232
0.02	2.192	0.810	2.970	0.180	3.585	1.771	0.130	3.634	0.144	0.05	3.713	0.277
0.02	2.337	0.804	3.059	0.127	3.720	1.914	0.077	3.769	0.151	0.02	3.823	0.271
0.02	2.479	0.802	3.085	0.096	3.773	2.049	0.046	3.822	0.169	0.01	3.856	0.260
0.05	2.401	0.820	2.808	0.265	3.349	1.788	0.215	3.398	0.164	0.10	3.510	0.448
0.05	2.611	0.810	2.958	0.198	3.554	1.960	0.148	3.602	0.168	0.05	3.697	0.482
0.05	2.996	0.804	3.038	0.132	3.691	2.277	0.082	3.739	0.195	0.02	3.797	0.523
0.05	3.372	0.802	3.057	0.098	3.738	2.575	0.048	3.786	0.243	0.01	3.820	0.553

Table 27. Optimal results for three stage system for various values of K_1 and y_1^3, $\delta = 1.1$, $G = 0.05$.

K_1	θ	y_1^0	y_2^0	y_1^1	y_2^1	θ_1	y_1^2	y_2^2	θ_2	y_1^3	y_2^3	θ_3
0.01	1.997	0.820	2.819	0.181	3.463	1.759	0.123	3.518	0.168	0.10	3.523	0.069
0.01	2.039	0.810	2.975	0.158	3.632	1.723	0.1004	3.687	0.159	0.05	3.718	0.155
0.01	2.105	0.804	3.066	0.121	3.754	1.779	0.063	3.810	0.164	0.02	3.832	0.161
0.01	2.168	0.802	3.095	0.097	3.805	1.845	0.039	3.860	0.177	0.01	3.869	0.145
0.02	2.101	0.820	2.816	0.223	3.418	1.717	0.165	3.474	0.170	0.10	3.520	0.212
0.02	2.180	0.810	2.970	0.180	3.606	1.754	0.121	3.661	0.167	0.05	3.713	0.258
0.02	2.320	0.804	3.059	0.128	3.740	1.888	0.070	3.795	0.181	0.02	3.824	0.250
0.02	2.455	0.802	3.086	0.100	3.791	2.008	0.041	3.846	0.209	0.01	3.857	0.237
0.05	2.390	0.820	2.808	0.262	3.371	1.778	0.204	3.426	0.185	0.10	3.510	0.426
0.05	2.596	0.810	2.958	0.197	3.575	1.943	0.138	3.630	0.194	0.05	3.697	0.459
0.05	2.968	0.804	3.039	0.133	3.711	2.247	0.074	3.766	0.236	0.02	3.798	0.484
0.05	3.323	0.802	3.058	0.102	3.757	2.510	0.043	3.811	0.306	0.01	3.822	0.506

Table 28. Optimal results for three stage system for various values of K_1 and y_1^3, $\delta = 1.1$, $G = 0.10$.

K_1	$\bar{t}$	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	$\bar{t}_1$	y_1^2	$\bar{y}_2^2$	$\bar{t}_2$	y_1^3	$\bar{y}_2^3$	$\bar{t}_3$
0.01	1.984	0.820	2.819	0.179	3.488	1.742	0.120	3.538	0.176	0.10	3.523	0.066
0.01	2.027	0.810	2.975	0.156	3.658	1.708	0.981	3.708	0.162	0.05	3.718	0.156
0.01	2.095	0.804	3.066	0.119	3.782	1.764	0.060	3.831	0.168	0.02	3.832	0.161
0.01	2.159	0.802	3.095	0.096	3.831	1.827	0.038	3.080	0.184	0.01	3.868	0.147
0.02	2.090	0.820	2.816	0.219	3.446	1.707	0.160	3.496	0.171	0.10	3.520	0.210
0.02	2.171	0.810	2.970	0.176	3.633	1.742	0.118	3.683	0.167	0.05	3.712	0.261
0.02	2.313	0.804	3.059	0.126	3.767	1.875	0.067	3.816	0.186	0.02	3.823	0.251
0.02	2.450	0.802	3.085	0.098	3.818	1.991	0.040	3.866	0.218	0.01	3.856	0.239
0.05	2.381	0.820	2.808	0.256	3.399	1.776	0.198	3.450	0.180	0.10	3.510	0.424
0.05	2.591	0.810	2.957	0.192	3.603	1.939	0.134	3.652	0.192	0.05	3.696	0.459
0.05	2.969	0.804	3.038	0.130	3.738	2.238	0.072	3.786	0.242	0.02	3.797	0.487
0.05	3.329	0.802	3.057	0.100	3.783	2.496	0.042	3.830	0.321	0.01	3.821	0.511

Table 29. Optimal results for three stage system for various values of K_1 and y_1^3, $\delta = 1.1$, $G = 0.15$.

K_1	$\bar{t}$	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	$\bar{t}_1$	y_1^2	$\bar{y}_2^2$	$\bar{t}_2$	y_1^3	$\bar{y}_2^3$	$\bar{t}_3$
0.01	1.970	0.820	2.819	0.172	3.522	1.738	0.122	3.553	0.155	0.10	3.523	0.076
0.01	2.016	0.810	2.974	0.150	3.694	1.705	0.100	3.725	0.138	0.05	3.717	0.172
0.01	2.088	0.804	3.066	0.112	3.818	1.765	0.062	3.848	0.144	0.02	3.832	0.178
0.01	2.158	0.802	3.094	0.089	3.868	1.835	0.039	3.897	0.161	0.01	3.868	0.162
0.02	2.078	0.820	2.816	0.212	3.480	1.707	0.162	3.512	0.143	0.10	3.520	0.226
0.02	2.163	0.810	2.970	0.168	3.669	1.745	0.118	3.700	0.138	0.05	3.712	0.280
0.02	2.316	0.804	3.058	0.119	3.802	1.882	0.069	3.833	0.157	0.02	3.823	0.275
0.02	2.464	0.802	3.084	0.091	3.853	2.009	0.041	3.883	0.189	0.01	3.855	0.265
0.05	2.375	0.820	2.808	0.247	3.434	1.785	0.197	3.467	0.141	0.10	3.509	0.448
0.05	2.596	0.810	2.957	0.184	3.638	1.952	0.134	3.669	0.154	0.05	3.695	0.489
0.05	3.000	0.804	3.036	0.123	3.772	2.265	0.073	3.802	0.203	0.02	3.795	0.531
0.05	3.392	0.802	3.054	0.093	3.816	2.551	0.043	3.845	0.276	0.01	3.817	0.563

Table 30. Optimal results for three stage system for various values of K_1 and y_1^3, $\delta = 1.1$, $G = 0.20$.

K_1	$\bar{t}$	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	$\bar{t}_1$	y_1^2	$\bar{y}_2^2$	$\bar{t}_2$	y_1^3	$\bar{y}_2^3$	$\bar{t}_3$
0.01	1.954	0.820	2.819	0.173	3.553	1.700	0.114	3.579	0.201	0.10	3.523	0.052
0.01	2.000	0.810	2.974	0.150	3.726	1.667	0.093	3.751	0.177	0.05	3.717	0.154
0.01	2.070	0.804	3.066	0.115	3.849	1.719	0.057	3.872	0.185	2.02	3.832	0.165
0.01	2.138	0.802	3.095	0.094	3.898	1.777	0.036	3.920	0.207	0.01	3.868	0.153
0.02	2.061	0.820	2.816	0.210	3.513	1.674	0.152	3.539	0.184	0.10	3.520	0.203
0.02	2.146	0.810	2.970	0.168	3.703	1.710	0.110	3.727	0.177	0.05	3.712	0.259
0.02	2.294	0.804	3.059	0.121	3.835	1.833	0.063	3.858	0.204	0.02	3.823	0.256
0.02	2.437	0.802	3.085	0.096	3.884	1.940	0.038	3.906	0.246	0.01	3.855	0.249
0.05	2.358	0.820	2.808	0.245	3.468	1.756	0.186	3.495	0.181	0.10	3.509	0.420
0.05	2.576	0.810	2.957	0.183	3.672	1.915	0.125	3.697	0.199	0.05	3.695	0.461
0.05	2.968	0.804	3.037	0.125	3.804	2.204	0.067	3.827	0.268	0.02	3.795	0.496
0.05	3.343	0.802	3.055	0.098	3.847	2.446	0.039	3.868	0.368	0.01	3.818	0.528

Table 31. Optimal results for three stage system for various values of K_1 and y_1^3, $\delta = 1.1$, $G = 0.25$.

K_1	$\bar{t}$	y_1^0	y_2^0	y_1^1	y_2^1	$\bar{t}_1$	y_1^2	y_2^2	$\bar{t}_2$	y_1^3	y_2^3	$\bar{t}_3$
0.01	1.937	0.820	2.819	0.170	3.592	1.670	0.112	3.599	0.221	0.10	3.523	0.045
0.01	1.984	0.810	2.976	0.148	3.767	1.639	0.089	3.772	0.190	0.05	3.717	0.153
0.01	2.056	0.804	3.066	0.113	3.890	1.688	0.055	3.893	0.199	0.02	3.832	0.168
0.01	2.126	0.802	3.094	0.093	3.937	1.743	0.035	3.940	0.224	0.01	3.867	0.158
0.02	2.045	0.820	2.816	0.206	3.552	1.649	0.148	3.561	0.198	0.10	3.520	0.198
0.02	2.132	0.810	2.970	0.165	3.743	1.684	0.107	3.719	0.188	0.05	3.712	0.260
0.02	2.283	0.804	3.058	0.119	3.875	1.804	0.060	3.879	0.220	0.02	3.822	0.259
0.02	2.430	0.802	3.084	0.095	3.923	1.906	0.037	3.926	0.267	0.01	3.855	0.256
0.05	2.345	0.820	2.808	0.239	3.509	1.737	0.180	3.518	0.189	0.10	3.509	0.418
0.05	2.568	0.810	2.956	0.179	3.713	1.894	0.120	3.719	0.210	0.05	3.695	0.463
0.05	2.968	0.804	3.036	0.123	3.844	2.176	0.064	3.847	0.290	0.02	3.794	0.501
0.05	3.354	0.802	3.054	0.096	3.885	2.408	0.038	3.887	0.402	0.01	3.817	0.542

Table 32. Optimal results for three stage system for various values of K_1 and y_1^3, $\delta = 1.2$, $G = 0.0$.

K_1	$\bar{t}$	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	$\bar{t}_1$	y_1^2	$\bar{y}_2^2$	$\bar{t}_2$	y_1^3	$\bar{y}_2^3$	$\bar{t}_3$
0.01	1.840	0.820	2.819	0.181	3.444	1.630	0.131	3.492	0.128	0.10	3.523	0.081
0.01	1.879	0.810	2.975	0.158	3.612	1.597	0.108	3.661	0.124	0.05	3.718	0.157
0.01	1.942	0.804	3.066	0.119	3.735	1.655	0.069	3.784	0.125	0.02	3.832	0.161
0.01	2.003	0.802	3.095	0.093	3.787	1.723	0.043	3.836	0.133	0.01	3.868	0.146
0.02	1.936	0.820	2.817	0.224	3.398	1.589	0.174	3.447	0.134	0.10	3.520	0.211
0.02	2.010	0.810	2.970	0.180	3.585	1.624	0.130	3.634	0.132	0.05	3.713	0.253
0.02	2.142	0.804	3.059	0.127	3.720	1.753	0.077	3.768	0.139	0.02	3.823	0.250
0.02	2.273	0.802	3.085	0.096	3.773	1.879	0.046	3.822	0.155	0.01	3.856	0.238
0.05	2.201	0.820	2.808	0.205	3.349	1.639	0.215	3.398	0.151	0.10	3.510	0.410
0.05	2.393	0.810	2.958	0.198	3.553	1.795	0.148	3.602	0.154	0.05	3.697	0.443
0.05	2.747	0.804	3.038	0.132	3.690	2.087	0.082	3.739	0.178	0.02	3.797	0.480
0.05	3.091	0.802	3.057	0.098	3.738	2.360	0.048	3.786	0.222	0.01	3.820	0.507

Table 33. Optimal results for three stage system for various values of K_1 and y_1^3, $\delta = 1.2$, $G = 0.05$.

K_1	$\bar{t}$	y_1^0	y_2^0	y_1^1	$\bar{y}_2^1$	$\bar{t}_1$	$\bar{y}_1^2$	y_2^2	$\bar{t}_2$	y_1^3	$\bar{y}_2^3$	$\bar{t}_3$
0.01	1.820	0.820	2.821	0.177	3.487	1.616	0.127	3.532	0.130	0.10	3.523	0.073
0.01	1.861	0.810	2.975	0.154	3.657	1.504	0.104	3.702	0.123	0.05	3.718	0.153
0.01	1.924	0.804	3.066	0.116	3.780	1.637	0.066	3.825	0.126	0.02	3.832	0.274
0.01	1.986	0.802	3.095	0.091	3.831	1.705	0.049	3.876	0.135	0.01	3.868	0.144
0.02	1.917	0.820	2.816	0.218	3.443	1.581	0.168	3.488	0.131	0.10	3.520	0.205
0.02	1.992	0.810	2.970	0.175	3.630	1.613	0.125	3.676	0.128	0.55	3.712	0.251
0.02	2.127	0.804	3.059	0.124	3.765	1.740	0.076	3.809	0.138	0.02	3.823	0.248
0.02	2.258	0.802	3.085	0.094	3.817	1.861	0.044	3.862	0.158	0.01	3.856	0.237
0.05	2.185	0.820	2.808	0.258	3.394	1.636	0.208	3.440	0.141	0.10	3.510	0.404
0.05	2.379	0.810	2.957	0.192	3.599	1.790	0.142	3.644	0.147	0.05	3.696	0.441
0.05	2.736	0.804	3.037	0.129	3.735	2.079	0.079	3.779	0.178	0.02	3.796	0.479
0.05	3.083	0.802	3.056	0.096	3.781	2.347	0.046	3.825	0.229	0.01	3.819	0.507

Table 34. Optimal results for three stage system for various values of K_1 and y_1^3, $\delta = 1.2$, $G = 0.10$.

K_1	$\bar{t}$	y_1^0	y_2^0	y_1^1	y_2^1	$\bar{t}_1$	y_1^2	y_2^2	$\bar{t}_2$	y_1^3	y_2^3	$\bar{t}_3$
0.01	1.798	0.820	2.819	0.172	3.539	1.599	0.122	3.572	0.135	0.10	3.523	0.063
0.01	1.839	0.810	2.974	0.150	3.710	1.565	0.100	3.743	0.123	0.05	3.718	0.150
0.01	1.903	0.804	3.066	0.113	3.834	1.617	0.063	3.866	0.127	0.02	3.832	0.158
0.01	1.966	0.802	3.095	0.090	3.885	1.681	0.040	3.916	0.140	0.01	3.868	0.144
0.02	1.896	0.820	2.816	0.211	3.496	1.571	0.161	3.530	0.130	0.10	3.520	0.194
0.02	1.972	0.810	2.970	0.171	3.684	1.599	0.121	3.718	0.126	0.05	3.712	0.247
0.02	2.108	0.804	3.059	0.120	3.818	1.721	0.070	3.850	0.140	0.02	3.823	0.246
0.02	2.240	0.802	3.085	0.092	3.870	1.838	0.042	3.902	0.164	0.01	3.855	0.237
0.05	2.165	0.820	2.808	0.250	3.448	1.633	0.200	3.482	0.134	0.10	3.510	0.397
0.05	2.362	0.810	2.957	0.187	3.652	1.781	0.137	3.686	0.143	0.05	3.696	0.438
0.05	2.723	0.804	3.037	0.125	3.588	2.064	0.075	3.820	0.180	0.02	3.796	0.457
0.05	3.072	0.802	3.055	0.094	3.833	2.328	0.044	3.864	0.239	0.01	3.818	0.504

Table 35. Optimal results for three stage system for various values of K_1 and y_1^3, $\delta = 1.2$, $G = 0.15$.

K_1	$\bar{t}$	y_1^0	y_2^0	y_1^1	y_2^1	$\bar{t}_1$	y_1^2	y_2^2	$\bar{t}_2$	y_1^3	y_2^3	$\bar{t}_3$
0.01	1.772	0.820	2.819	0.166	3.600	1.578	0.116	3.613	0.143	0.10	3.523	0.050
0.01	1.814	0.810	2.974	0.145	3.774	1.544	0.095	3.785	0.127	0.05	3.717	0.143
0.01	1.880	0.804	3.066	0.110	3.898	1.592	0.060	3.907	0.132	0.02	3.832	0.155
0.01	1.943	0.802	3.094	0.088	3.348	1.652	0.038	3.956	0.147	0.01	3.868	0.143
0.02	1.871	0.820	2.816	0.204	3.558	1.554	0.154	3.572	0.133	0.10	3.520	0.183
0.02	1.949	0.810	2.970	0.165	3.748	1.580	0.115	3.760	0.125	0.05	3.712	0.242
0.02	2.086	0.804	3.058	0.117	3.882	1.697	0.067	3.891	0.144	0.02	3.822	0.244
0.02	2.219	0.802	3.084	0.091	3.933	1.810	0.041	3.941	0.172	0.01	3.855	0.236
0.05	2.142	0.820	2.808	0.242	3.550	1.624	0.192	3.525	0.131	0.10	3.509	0.386
0.05	2.234	0.810	2.957	0.181	3.716	1.767	0.131	3.728	0.141	0.05	3.695	0.432
0.05	2.706	0.804	3.036	0.122	3.850	2.044	0.072	3.860	0.186	0.02	3.795	0.475
0.05	3.058	0.802	3.054	0.092	3.895	2.300	0.042	3.903	0.258	0.01	3.817	0.505

Table 36. Optimal results for three stage system for various values of K_1 and y_1^3, $\delta = 1.2$, $G = 0.20$.

K_1	$\bar{t}$	y_1^0	y_2^0	y_1^1	y_2^1	$\bar{t}_1$	y_1^2	y_2^2	$\bar{t}_2$	y_1^3	y_2^3	$\bar{t}_3$
0.01	1.7432	0.820	2.819	0.161	3.671	1.551	0.111	3.653	0.156	0.10	3.523	0.035
0.01	1.7876	0.810	2.974	0.140	3.846	1.516	0.090	3.826	0.134	0.05	3.717	0.136
0.01	1.8538	0.804	3.066	0.107	3.971	1.561	0.057	3.948	0.139	0.02	3.831	0.153
0.01	1.9176	0.802	3.094	0.086	4.020	1.617	0.036	3.996	0.150	0.01	3.867	0.143
0.02	1.843	0.820	2.816	0.197	3.630	1.533	0.147	3.614	0.140	0.10	3.519	0.169
0.02	1.923	0.810	2,970	0.160	3.821	1.556	0.110	3.802	0.131	0.05	3.712	0.235
0.02	2.061	0.804	3.058	0.114	3.955	1.668	0.064	3.932	0.151	0.02	3.822	0.242
0.02	2.169	0.802	3.084	0.089	4.005	1.775	0.039	3.981	0.184	0.01	3.854	0.237
0.05	2.116	0.820	2.807	0.234	3.582	1.610	0.184	3.568	0.132	0.10	3.509	0.374
0.05	2.319	0.810	2.956	0.175	3.788	1.747	0.125	3.770	0.144	0.05	3.695	0.427
0.05	2.687	0.804	3.035	0.119	3.922	2.017	0.069	3.900	0.196	0.02	3.794	0.473
0.05	3.043	0.802	3.053	0.090	3.965	2.265	0.040	3.941	0.272	0.01	3.816	0.505

Table 37. Optimal results of three stage system for various values of K_1 and y_1^3, $\delta = 1.2$, $G = 0.25$.

K_1	$\bar{t}$	y_1^0	y_2^0	y_1^1	y_2^1	$\bar{t}_1$	y_1^2	y_2^2	$\bar{t}_2$	y_1^3	y_2^3	$\bar{t}_3$
0.01	1.711	0.820	2.818	0.155	3.750	1.519	0.105	3.694	0.174	0.10	3.523	0.018
0.01	1.758	0.810	2.954	0.136	3.928	1.484	0.086	3.868	0.144	0.05	3.717	0.128
0.01	1.825	0.804	3.065	0.105	4.053	1.524	0.055	3.989	0.149	0.02	3.831	0.151
0.01	1.890	0.802	3.094	0.085	4.102	1.576	0.035	4.036	0.169	0.01	3.867	0.144
0.02	1.813	0.820	2.816	0.190	3.710	1.507	0.140	3.656	0.152	0.10	3.519	0.153
0.02	1.894	0.810	2.696	0.154	3.903	1.526	0.104	3.844	0.139	0.05	3.711	0.229
0.02	2.035	0.804	3.058	0.111	4.037	1.632	0.061	3.973	0.162	0.02	3.822	0.239
0.02	2.172	0.802	3.083	0.087	4.086	1.733	0.037	4.021	0.200	0.01	3.854	0.238
0.05	2.088	0.820	2.807	0.225	3.662	1.588	0.175	3.610	0.139	0.10	3.508	0.360
0.05	2.293	0.810	2.956	0.169	3.870	1.721	0.119	3.811	0.151	0.05	3.694	0.421
0.05	2.666	0.804	3.035	0.115	4.003	1.983	0.065	3.940	0.211	0.02	3.793	0.470
0.05	3.028	0.802	3.052	0.089	4.045	2.220	0.039	3.979	0.297	0.01	3.814	0.510

Chapter 5

FUTURE PROBLEMS

In large scale industrial fermentations, continuous processes are increasing in importance. Tower fermentation systems which can be operated as either continuous or batch processes have a number of unique characteristics that may lead to much greater use by biochemical industries and also in biological experiments. The most important characteristics of the tower system are related to the presence of backflow, cell sedimentation, and adjustable geometry. Essentially, a tower system has the same advantages as a multistage continuous culture system and those of a chemostat. In addition, it has several unique features which the conventional systems do not have.

Recently, a few experimental papers have been published on the tower system; however, knowledge of this system is still rather incomplete. The topics which have been studied in the present work give people a better understanding about the performance of the tower system, but in order to know how to operate and control the tower system in various applications the following topics need to be investigated.

(1) Oxygen transfer

An aerobic growth process needs oxygen to use in microbial metabolism and biosynthesis. The oxygen transfer rate may be the rate controlling factor in biological growth. The oxygen transfer rate depends upon temperature, bubble surface behavior, oxygen demand and the degree of mixing. Investigating the oxygen transfer rate has been a popular biochemical engineering topic and Kitai (1) has studied oxygen transfer in the tower system.

The bubble size distribution and bubble detention time greatly influence the oxygen transfer rate. If the controlling step is the interfacial mass transfer process the surface renewal rate may play an important role in the oxygen transfer process. From the microscopic stant point the interfacial mass transfer rate is dependent on the surface renewal rate and bubble size distribution. The correlation of the microscopic mass transfer coefficient with the macroscopic mass transfer coefficient in order to estimate the total amount of oxygen transfer in a tower system should be investigated. Also, the effects of bubble size and the degree of mixing on operating performance are of interest.

(2) Control in tower systems

Biological processes are often designed to operate on the basis of an optimal policy, but the process variables may fluctuate due to environmental conditions. The microbial growth rate depends on several process variables which may cause the operating performance to deviate from the optimal policy. Gaden (2) listed the important process variables to considered; (a) temperature, (b) pH, (c) nutrient supply, (d) oxygen supply, (e) agitation, (f) flow rate, (g) product concentration. Some of these variables are hard to adjust quantitatively, and qualitatively, only the measurable variables can be considered as control variables.

The most popular control variables are temperature, pH, and flow rate. The response function to a disturbance or forcing function in the flow to the tower system, and a study of the necessary controller to maintain a tower system at the optimal operating conditions are of interest. Productivity or precent treatment may be selected as a state variable to be optimized. The general process involves developing the system performance equations, system simulation and optimization in order to find the optimal policy, development of the dynamic equations needed for control, selection of a control system, and optimal adjustment of the controller.

(3) Graphical evaluation of the substrate and cell concentration in tower systems

Determination of the concentration of substrate and cells in the tower fermentation system by solving 2N simultaneous equations is described in Chapter 2. A graphical method might be employed to determine the concentration of substrate and cells in each compartment of the tower system. Graphical methods have been used to predict the performance of a continuous culture by Luedeking and Piret (3) who used a graphical method to predict the concentration of cells, in which the cells in continuous culture are in the same "physiological state" as those at the corresponding stage of batch cultivation.

The complexity of the flow pattern of a tower system is due to the backflow and cell sedimentation flow nonidealities. The mass balances around each stage are given in Equations (23), (24), and (25) in Chapter 2. The interaction between two adjacent stages because of backflow results in the concentrations in stage i being influenced by the concentrations in the two adjacent stages. Lelli (4) reported a graphical method to solve the backflow cell model associated with a chemical reactor. Concepts employed in solving distillation, adsorption and extraction problems are used to determine the concentration distribution in the reactor. The operating line and equilibrium line have to be identified before a stepwise procedure can be followed to predict the concentrations in each compartment. An effort should be made to develop a graphical stepwise procedure that might be employed to estimate the concentration distribution in a tower type biological system.

(4) Residence time distribution analysis in tower fermentation

The fluid dynamics play an important role in tower fermentation systems. Unfortunately, the analysis is complicated by the fact that the biological growth process is a hetergeneous system with three phases. The residence time distribution of each phase is different and each must be investigated to construct a model of the process. The microorganisms may be considered as the dispersed phase. Their aggregation into large particles, their microbial metabolism, and their sedimentation can be considered in modeling their flow behavior. The experimental study of Prokop et al. (5) indicates that the dispersed phase residence time distribution differs from that of the continuous phase in the tower system.

Many researchers (6, 7, 8) have investigated the residence time distribution of the continuous phase for the backflow cell model. In the case of a large number of stages, the laplace transform functions of the system equations can be obtained systematically; however, the inverse transform functions are difficult to obtain. Numerical methods provide an effective technique to integrate the simultaneous differential equations which describe the response of the system to an inprt of tracer. The time consumption required by the RKGS IBM scientific subroutine may be quite substantial. Other solution techniques such as that of employing collocation methods (9) should also be investigated. The residence time distribution of the dispersed phase in a multistage tower system should provide a very interesting area for further research.

(5) Consideration of the effect of the physiological state on modeling microbial growth processes.

Monod's equation to describe microbial growth assumes that all cells are in the same physiological state; however, the activity of a cell may be influenced by its age and physiological state. The microbial growth rate may depend not on the concentration of cells and substrate, but also on the history (or physiological state) of the cells. Kono (10) has reported his study which considers physiological state in modeling microbial growth by introducing a coefficient of activity which depends on environmental conditions, retention time, and competition behavior. Population balances and other stochastic models should be used to analyze the effect of cell age and cell history in modeling microbial growth processes. By properly controlling conditions in various parts of the tower system specific changes in cell physiological state may be continuously obtained to experimentally verify theoretical results.

(6) Modeling hydrocarbon fermentations in the tower system.

Hydrocarbon fermentations in which a second liquid phase is present have not been thoroughly investigated in a tower system. The introduction of an additional liquid phase and the very unique character of many growth processes when two liquid phases are present suggest a number of interesting residence time distribution and flow behavior investigations that should be conducted. The optimal feed geometry and operation of a tower system can be investigated after the flow behavior is adequately modeled.

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ANALYSIS OF MICROBIAL GROWTH IN A TOWER SYSTEM

by

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ABSTRACT

This work investigates the microbial growth process in a tower type continuous culture system. Media and air are assumed to flow cocurrently through a multistage tower type aeration system. A mathematical model that can be used to describe cell growth in tower fermentation systems with backflow and sedimentation is presented. Two computational procedures to obtain steady state simulation results are described and some results are obtained for 2, 4 and 8 stage tower systems. The effects of feed geometry, backflow rate and cell sedimentation on steady state behavior are investigated. The results show that control of backflow, and cell sedimentation within the tower together with control of feed flow rate and feed geometry will allow a wide range of operating conditions to be obtained in the tower system.

The washout behavior of tower fermentors with different hydraulic characteristics (backflow, cell sedimentation, and feed geometry) is investigated. General characteristic equations for washout are developed for a variety of feed and discharge geometries. Results show that it is possible to greatly increase the operating range by adjusting the hydraulic parameters and feed geometry; however, increasing the number of stages usually decreases the washout dilution rate. A detailed analysis of washout for several different cases is presented.

A tower type activated sludge waste treatment process is investigated. Air and influent are assumed to enter the column at the bottom and flow cocurrently up the column. A part of the

activated sludge from the sedimentation tank which clarifies the effluent leaving the top of the tower is recycled. The optimal design of tower type activated sludge waste treatment processes is investigated using mathematical models in which flow nonideality due to sedimentation of sludge and fluid backflow are considered. Mathematical optimization techniques are used to select the volume of each stage which minimizes the total volume required for aeration to a fixed degree of treatment. Optimal results have been obtained for two and three stage system for several values of the backflow parameter, the sedimentation parameter, the degree of treatment, and saturation constant.