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Representation of solutions to Helmholtz's equation

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It is proved that any potential of a single layer v is identically equal to a potential of a double layer w in the bounded domain, $\mathscr{D}$, and a necessary and sufficient condition for $v \equiv w$ in $\Omega = \mathbb{R}^3 \setminus \mathscr{D}$, the exterior domain, is given.

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I. INTRODUCTION

Let $\mathscr{D} \subset \mathbb{R}^3$ be a bounded domain with a smooth closed boundary Γ (an obstacle) and $\Omega = \mathbb{R}^3 \setminus \mathscr{D}$ be the exterior domain. Let

$$v(\sigma) = \int_\Gamma g(x,t)\sigma\, dt, \quad w(\mu) = \int_\Gamma \frac{\partial g(x,t)}{\partial n_t} \mu(t) dt,$$
$$g = \frac{\exp(ik|x-y|)}{4\pi|x-y|}, k > 0; \tag{1}$$

n_t is the exterior unit normal to Γ at point t. The following questions are discussed in this paper: given a potential $v(\sigma)$, can one find a potential $w(\mu)$ such that $v \equiv w$ in $\mathscr{D}$? $v \equiv w$ in Ω?

Since the potentials solve Helmholtz's equation in $\mathscr{D}$ and in Ω, the above question is connected with the representation of solutions to Helmholtz's equation. Some basic properties of the potentials and some results on the representation of solutions to Helmholtz's equation are given in Ref. 1. These questions are also of interest in the singularity and eigenmode expansions methods.[2] Below, $N(A) = \{f: Af = 0\}$ denotes the nullity of a linear operator A, $(u,v) = \int_\Gamma u\bar{v}\, ds, Q\sigma = \int_\Gamma g(s,t)\sigma(t)dt, H^q = W^{2,q}(\Gamma)$ is the Sobolev space.

II. RESULTS

Theorem 1: For any $w(\mu)$ $[v(\sigma)]$, there exists a $v(\sigma)$ $[w(\mu)]$ such that $v(\sigma) \equiv w(\mu)$ in $\mathscr{D}$. The $v(\sigma)$ $[w(\mu)]$ is uniquely defined.

Theorem 2: A necessary and sufficient condition for a $v(\sigma)$ to be identically equal to a $w(\mu)$ in Ω is $(v, \nu_j) = 0$, $1 \leqslant j \leqslant r'$, where $\{\nu_j\}$ forms a basis of $N(I + A)$, $Af \equiv \int_\Gamma (\partial g(s,t)/\partial n_s) f(t) dt$. A necessary and sufficient condition for a $w(\mu)$ to be identically equal to a $v(\sigma)$ in Ω is $(\mu, \sigma_j) = 0, 1 \leqslant j \leqslant r$, where $\{\sigma_j\}$ forms a basis of $N(I - A)$.

Corollary 1: If a problem $(\nabla^2 + k^2)u = 0$ in $\mathscr{D}$, $k \geqslant 0$, $u^+ = f$ is solvable, then the solution can be represented as $u = v(\sigma)$ and also as $u = w(\mu)$.

III. PROOFS

A. Auxiliary results

Let us denote by $v^\pm$, $\partial v^\pm/\partial n$ the limit values on Γ from the interior $(+)$ and exterior $(-)$ of a function and its normal derivative. It is well known[1] that

$$\frac{\partial v^\pm}{\partial n} = \frac{A\sigma \pm \sigma}{2}, \qquad w^\pm = \frac{A'\mu \mp \mu}{2}, \tag{2}$$

where the operator A is defined in Theorem 2, and $A'\mu = \int_\Gamma (\partial g(s,t)/\partial n_t)\mu(t)dt$. Note that $A^* = \overline{A}'$, where the star denotes the adjoint in operator $H = L^2(\Gamma)$ and the bar denotes complex conjugation.

Lemma 1: If $(\nabla^2 + k^2)u = 0$ in Ω, $u = 0$, or $\partial u^-/\partial n = 0$ on Γ, $k > 0$, and $|x|(\partial u/\partial|x| - iku) \to 0$ as $|x| \to \infty$, then $u \equiv 0$.

Lemma 2: Let the problem

$$(\nabla^2 + k^2)u = 0 \quad \text{in} \quad \mathscr{D}, \quad \frac{\partial u^+}{\partial n} = 0 \tag{3}$$

have r' linearly independent solutions u_j, $1 \leqslant j \leqslant r'$. Then the equation

$$A'\mu + \mu = 0 \tag{4}$$

has precisely r' linearly independent solutions $\mu_j = u_j^+$, $1 \leqslant j \leqslant r'$, $u_j = w(\mu_j)$. Let the problem

$$(\nabla^2 + k^2)u = 0 \text{ in } \mathscr{D}, \quad u^+ = 0 \tag{5}$$

have r linearly independent solutions ϕ_j, $1 \leqslant j \leqslant r$. Then the equation

$$A\sigma - \sigma = 0 \tag{6}$$

has precisely r linearly independent solutions $\sigma_j = \partial\phi_j^+/\partial n$, $\phi_j = v(\sigma_j)$.

Lemma 3: If (5) has only the trivial solution, then $Q: H^q \to H^{q+1}$ is an isomorphism.

Lemma 4: If (5) has r linearly independent solutions, then equation $Q\sigma = f$ is solvable iff $(f, \bar{\sigma}_j) = 0$, $1 \leqslant j \leqslant r$.

Lemmas 1 and 2 are proved, e.g., in Ref. 1. Lemmas 3 and 4 are proved in the Appendix.

B. Proof of Theorem 1

(i) Assume first that (*) problem (5) has only the trivial (zero) solution. If $v(\sigma) = w(\mu)$ in $\mathscr{D}$, then (2) implies that (**) $A'\mu - \mu = 2v(\sigma)$. If $w(\mu)$ is given, then the above equation is an equation for σ. This equation is uniquely solvable (by Lemma 3) because of (*). If σ is its solution, then the corresponding $v(\sigma)$ satisfies (***) $v^+(\sigma) = w^+(\mu)$ and, again by (*), $v(\sigma) \equiv w(\mu)$ in $\mathscr{D}$. If $v(\sigma)$ is given, then (**) is an equation for μ. If this equation is solvable, then the corresponding $w(\mu)$ satisfies (***), and, by (*), $v(\sigma) \equiv w(\mu)$ in $\mathscr{D}$. It remains to be proved that (**) is solvable for μ. By Fredholm's alternative, it is so if $(v(\sigma), \nu_j) = 0$, where ν_j are all linearly independent solutions to the equation $\overline{A}\nu - \nu = 0$. [Notice that $\overline{A} = (A')^*$]. By Lemma 2, the functions $\nu_j = \bar{\sigma}_j = \overline{\partial\phi_j^+/\partial n}$,

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$v^{\pm}(\sigma_j) = 0 = v^{\pm}(\bar{v}_j)$. Therefore $(v^+(\sigma),v_j) = (\sigma,v^+(\bar{v}_j)) = 0$. (Or $\int_\Gamma v^+(\sigma)(\partial\phi_j^+/\partial n)ds = \int_\Gamma (\partial v^+/\partial n)\phi_j^+ \, ds = 0$). Thus (**) is solvable. In the proof of the solvability of (**), the assumption (*) was not used.

(ii) Assume now that (5) has $r>0$ linearly independent solutions. If $v(\sigma)$ is given then $w(\mu)$, where μ solves (**), satisfies (***). Therefore $w(\mu) = v(\sigma) + \Sigma_{j=1}^r c_j\phi_j(x)$ in $\mathscr{D}$, where $c_j = \text{const}$. By Lemma 2, Eq. (6) has precisely r linearly independent solutions. Thus the equations $A^*\bar{\eta} - \bar{\eta} = 0$,

$$A'\eta - \eta = 0 \tag{7}$$

have precisely r linearly independent solutions. Let η_j, $1 \leqslant j \leqslant r$, be r such solutions of (7), $w(\eta_j)$ be the corresponding potentials. Then $w(\eta_j) = \phi_j$ are r linearly independent solutions to (5). Therefore $v(\sigma) = w(\mu - \Sigma_{j=1}^r c_j\eta_j)$. Thus if $v(\sigma)$ is given, one can find a $\mu' = \mu - \Sigma_{j=1}^r c_j\eta_j$ such that $w(\mu') \equiv v(\sigma)$ in $\mathscr{D}$. This μ is uniquely defined by the requirement $w(\mu') \equiv v(\sigma)$ in $\mathscr{D}$.

Consider now the case when $w(\mu)$ is given and $v(\sigma)$ is to be found such that $w(\mu) \equiv v(\sigma)$ in $\mathscr{D}$. In this case (**) is an equation for σ of the type $Q\sigma = f \equiv (A'\mu - \mu)/2$. By Lemma 4, this equation is solvable iff $(f,\bar{\sigma}_j) = 0$, where σ_j, $1 \leqslant j \leqslant r$ solve (6). One has $(A'\mu - \mu,\bar{\sigma}_j) = (\mu,\overline{(A - I)\sigma_j}) = 0$. Thus (**) is solvable for σ. As above, the requirement $w(\mu) \equiv v(\sigma)$ in $\mathscr{D}$ defines σ uniquely. Theorem 1 is proved.

C. Proof of Theorem 2

(i) Assume that (5) has only the trivial solution. Given $w(\mu)$, one can find the unique σ from the equation

$$w^-(\mu) = (A'\mu + \mu)/2 = v^-(\sigma). \tag{8}$$

Because of Lemma 1, $w(\mu) \equiv v(\sigma)$ in Ω. Suppose now that $v(\sigma)$ is given. Then μ is to be found from (8). If this equation is solvable, then as above, $w(\mu) \equiv v(\sigma)$ in Ω. Equation (8) is solvable for μ iff $(v^-(\sigma),v_j) = 0, 1 \leqslant j \leqslant r$, where $\bar{A}v_j + v_j = 0$. Notice that $A\bar{v}_j + \bar{v}_j = 0$, $v(\bar{v}_j)$ solve (3). If (3) has only the trivial solution, then (8) is uniquely solvable for μ, and $v(\sigma) \equiv w(\mu)$ in Ω. If (3) has r' linearly independent solutions, then (4) has r' linearly independent solutions, and, by Fredholm's alternative, the equation $\bar{A}v + v = 0$ has r' linearly independent solutions: v_j, $v(v_j)$ solve (3), $v(v_j) = w(\mu_j)$ in $\mathscr{D}$. Therefore

$$\begin{aligned}(v^-(\sigma),v_j) &= (\sigma,v^-(\bar{v}_j)) = (\sigma,w^+(\mu_j)) \\ &= (\sigma,w^+(\mu_j)) = (\sigma, \overline{\tfrac{1}{2}(A' - I)\mu_j}) \\ &= -(\sigma,\bar{\mu}_j).\end{aligned}$$

Thus $(\sigma,\bar{\mu}_j) = 0$, $1 \leqslant j \leqslant r'$ is a necessary and sufficient condition for a potential $v(\sigma)$ to be identically equal to $w(\mu)$ in Ω in the case when (3) has r' linearly independent solutions.

(ii) Assume that (5) has r linearly independent solutions. Given $w(\mu)$ one has to find σ from (8). By Lemma 4, (8) is solvable for σ iff $(A'\mu + \mu,\bar{\sigma}_j) = 0$, $1 \leqslant j \leqslant r$, i.e. $(\mu, \overline{(A + I)\sigma_j}) = 0$, or $(\mu,\bar{\sigma}_j) = 0$, $1 \leqslant j \leqslant r$, because $\sigma_j = A\sigma_j$. If $v(\sigma)$ is given, the analysis does not depend on the assumption about the number of the linearly independent solutions of (5) and is given above in part (i). In the case when problem (5) [(3)] has a nontrivial solution and $w(\mu)$ [$v(\sigma)$] is given, the density $\sigma(\mu)$ is not uniquely defined. But the difference between two densities σ and $\tilde{\sigma}$ (μ and $\tilde{\mu}$) generates $v \equiv 0$ in Ω ($\mu \equiv 0$ in Ω) because of Lemma 1.

Proof of Corollary 1: If the Dirichlet problem in Corollary 1 is solvable, then its solution by Green's formula is of the form $u(x) = v(\sigma) + w(\mu)$. By Theorem 1, $v(\sigma)$ [$w(\mu)$] can be substituted by $w(\tilde{\mu})$ [$v(\tilde{\sigma})$]. Thus $u = w(\mu + \tilde{\mu}) = v(\sigma + \tilde{\sigma})$.

APPENDIX

A. Proof of Lemma 3

If problem (5) has only the trivial solution then $N(Q) = \{0\}$ and Q is injective. The fact that $Q: H^q \to H^{q+1}$ is bounded follows from the known results about the smoothness of solutions to Helmholtz's equation. Indeed, if $\sigma \epsilon H^q$, then

$$\frac{\partial v^+}{\partial n} \epsilon H^q, \quad \nabla v \epsilon H^{q+1/2}(D), \quad v \epsilon H^{q+3/2}(D), \quad v^{\pm} \epsilon H^{q+1}.$$

Here, the well-known trace theorem is used: $u \epsilon H^q(D) \Rightarrow u^+ \epsilon H^{q-1/2}(\Gamma)$. It remains to be proved that Range $Q \equiv R(Q) = H^{q+1}$. Take any $f \epsilon H^q$. Solve the problem

$$\begin{aligned}&(\nabla^2 + k^2)v = 0 \quad \text{in} \quad \Omega, \\ &\frac{\partial u^-}{\partial n} = f, \quad |x|\left(\frac{\partial v}{\partial |x|} - ikv\right) \to 0, \quad |x| \to \infty.\end{aligned} \tag{A1}$$

The solution exists, is unique, and can be found as $v(\sigma)$ because problem (5) [and therefore Eq. (6)] has only the trivial solution, and (*)$A\sigma - \sigma = 2f$. Since A is a smoothing operator, σ has the same smoothness as f, i.e., $\sigma \epsilon H^q$. Therefore $Q\sigma = v^- \epsilon H^{q+1}$. It is now easy to show that $R(Q) = H^{q+1}$. Take any $h \epsilon H^{q+1}$ and solve the exterior Dirichlet problem with the data $u^- = h$. Calculate $\partial u^-/\partial n \equiv f \epsilon H^q$. Find (the unique) σ from (*). By the uniqueness theorem (Lemma 1), $v(\sigma) = u$ in Ω, $Q\sigma = v^- = u^- = h$. Thus $Q: H^q \to H^{q+1}$ is a linear, injective, surjective, and continuous mapping. From the Banach theorem (about inverse operator) it follows that $Q^{-1}: H^{q+1} \to H^q$ is continuous. Lemma 3 is proved.

B. Proof of Lemma 4

If $Q\sigma = f$, $Q\sigma_j = 0$, then $(f,\bar{\sigma}_j) = (Q\sigma,\bar{\sigma}_j) = (\sigma, \overline{Q\sigma_j}) = 0$. This proves the necessity. The operator $Q: H^q \to H^{q+1}$ is a Fredholm operator and ind $Q = 0$, ind $Q \equiv$ index $Q \equiv \dim N(Q) - \dim N(Q^*) = \dim N(Q) - \operatorname{codim} R(Q)$. Therefore there are $r = \dim N(Q)$ necessary and sufficient conditions of the type $(f,h_j)_{q+1} = 0$, $1 \leqslant j \leqslant r$, where $(u,v)_q$ is the inner product in H^q. Suppose that the r conditions $(f,b_j)_0 = 0$ are necessary for $f \epsilon R(Q)$. Since $H^q \subset H^0$, $q>0$, and $||\cdot||_0 \leqslant ||\cdot||_q$, one has $(f,b_j)_0 = (f,a_j)_q$. Let $L_r = \operatorname{span}(h_1,\ldots,h_r)$, $M_r = \operatorname{span}(a_1,\ldots,a_r)$. Then $L_r = M_r$. Indeed, suppose there exists $a_m \notin L_r$. Then the number of the necessary conditions for f to belong to $R(Q)$ would be at least $r+1$, namely, $(f,h_j)_q = 0$, $1 \leqslant j \leqslant r$, and $(f,a_m)_q = 0$. This contradiction proves that $L_r = M_r$. Thus the conditions $(f,\bar{\sigma}_j) = 0$, $1 \leqslant j \leqslant r$, where $\{\sigma_j\}_{j=1}^r$ form a basis of $N(Q)$, are necessary and sufficient for $f \epsilon R(Q)$.

C. Solution of the scattering problem via an indetermined equation

Consider the problem

$$(\nabla^2 + k^2)u = 0 \quad \text{in} \quad \Omega, \quad k > 0,$$
$$u^- = f, \quad |x|(\partial u/\partial|x| - iku) \to 0, \quad |x| \to \infty. \qquad \text{(A2)}$$

The existence and uniqueness of the solution to this problem were studied extensively and a complete analysis is given in Ref. 1. It follows from the arguments in Sec. III that if (5) has only the trivial solution, then (A2) is (uniquely) solvable by a potential of the single layer $u = v(\sigma)$, while if (3) has only the trivial solution then (A2) is (uniquely) solvable by a potential of the double layer $u = w(\mu)$. Since it is known[1] that (A2) is (uniquely) solvable for any $k \geqslant 0$, one sees from Green's formula that the representation (*)$u = v(\sigma) + w(\mu)$ holds for all $k \geqslant 0$. The aim of this Section is to give a short proof of the existence of the solution to (A2) following Ref. 3, p. 98. The idea in Ref. 3 is very elegant. Let us look for a solution to (A2) of the form (*). Then

$$Q\sigma + (A'\mu + \mu)/2 = f, \quad \mu + A'\mu + 2Q\sigma = 2f. \qquad \text{(A3)}$$

This is an equation in $H = L^2(\Gamma)$ for the two unknown functions μ and σ.

Lemma A 1[3]: Let A and B be compact operators on a Hilbert space H, and $N(I + A^*) \cap N(B^*) = \{0\}$. Then the equation $(I + A)\mu + B\sigma = f$ is solvable for any $f \epsilon H$.

Let us postpone a proof of this lemma and show that Eq. (A3) is solvable. In our case, $B \to 2Q, A \to A'$. Thus one should check that $N(I + A'^*) \cap N(Q^*) = \{0\}$. Assume that $\sigma \epsilon N(I + A'^*)$, $\sigma \neq 0$, and consider $v(\sigma)$. By assumption, $\bar{v}^-(\sigma) = 0$. Thus $\bar{v}(\sigma) \equiv 0$ in Ω. On the other hand, $A'^* = \bar{A}$ and $\sigma + \bar{A}\sigma = 0$ implies that $\partial \bar{v}^+/\partial n = 0$. Since $(\nabla^2 + k^2)\bar{v} = 0$ in D, $\bar{v}^+ = \partial \bar{v}^+/\partial n = 0$, one concludes that $\bar{v} \equiv 0$ in D. Thus from (2) it follows that $\sigma = 0$. From Lemma A1 it now follows that Eq. (A3) is solvable for any $f \epsilon H$.

Proof of Lemma A 1: Let $T \equiv I + A$. Since A is compact, $R(T)$ is closed and $\dim N(T^*) = \operatorname{codim} R(T) < \infty$. It is clear that a necessary and sufficient condition for the dense solvability of the equation $T\mu + B\sigma = f$ is $N(T^*) \cap N(B^*) = \{0\}$. The dense solvability means the solvability for all f in a dense set of H. It remains to be proved that this condition is sufficient for everywhere solvability (solvability for any $f \epsilon H$). Let P be a projection onto $R(T)$ in H. Then

$$T\mu + PB\sigma = Pf \qquad \text{(A4)}$$
$$(I - P)B\sigma = (I - P)f, \qquad \text{(A5)}$$

If (A5) is solvable then (A4) and

$$T\mu + B\sigma = f \qquad \text{(A6)}$$

are solvable. Indeed, let σ solve (A5) and let $T\mu = P(f - B\sigma)$. Since the right-hand side belongs to $R(T)$, this equation has a solution μ. The pair (μ, σ) solves (A6): $T\mu + B\sigma = Pf - PB\sigma + B\sigma = Pf + (I - P)f = f$. Let us show that (A5) is everywhere solvable if $N(T^*) \cap N(B^*) = \{0\}$. This condition implies dense solvability of (A6) and therefore dense solvability of the equivalent to (A6), system (A4), (A5); in particular, dense solvability of (A5). But (A5) is an equation in the finite-dimensional space $(I - P)H$. If such an equation is densely solvable, then it is everywhere solvable. But then, as was shown above, Eqs. (A4) and (A6) are everywhere solvable. Lemma A1 is proved.

Remark. All we used in the proof are the following assumptions: (i) $R(T)$ is closed, and (ii) $\dim N(T^*) < \infty$.

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