

THE POTENTIAL ANALOGUE METHOD OF
SYNTHESIZING IMPEDANCE FUNCTIONS

by

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INTRODUCTION

This thesis will describe a method of synthesizing a network from a specified gain and/or phase frequency response. If the response is given in an analytic form, several standard methods exist which will obtain a network to give this response (Storer, 16). However, if the response is not given in an analytic form but in the form of a response curve, it then becomes necessary to find an analytic form so that synthesis is possible. Since to try an analytic approach to approximate any but the most elementary curves would be very difficult and time-consuming, the author suggests the use of an analogue device.

The problem, then, is to find an analogue to work with, such that the effect on the response curve by varying parameters can be predicted more easily than the effect of varying circuit parameters can be predicted. The method used is the potential analogue, where the complex frequency plane will be represented by a two-dimensional voltage plane on conducting paper. The zeros and poles of the transfer or gain function of the network will be represented by charges on this voltage plane. These charges will be placed on the conducting paper at the same places that poles and zeros of the gain function would be placed on the complex frequency plane.

The easiest way to describe the analogue is mathematically. To do this, it is best to discuss properties of networks first, properties of the two-dimensional potential surface, and then relate the two.

For a given network, if the input voltage is denoted by E and the output is denoted by V , then the ratio, V/E , will be described as the transmission ratio. This ratio will have the following form (Darlington, 6):

$$F(s) = V/E = e^{\alpha} e^{j\beta} \quad (1)$$

α = gain in nepers

β = phase shift in radians

The transmission function will be defined as the logarithm of this ratio.

$$G(s) = \ln (V/E) = \alpha + j\beta \quad (2)$$

For a finite lumped element network, the ratio V/E is a rational fraction and may be expressed as follows:

$$V/E = K \frac{\prod_{n=1}^L (s - s_n)}{\prod_{k=1}^M (s - s_k)} \quad (3)$$

When $s = s_n$, $V/E = 0$, and when $s = s_k$, V may have a value even when $E = 0$.

Therefore, the first case will be known as the zeros and the second will be known as the poles of the transmission ratio. Here s is the complex frequency variable associated with the LaPlace transforms

$$s = \sigma + j\omega$$

and K is a constant unaffected by frequency and alters only the absolute level of the gain, so it will be ignored.

The numerator and denominator of the rational fraction of Equation (3) are finite polynomials in s . If the network consists of real elements, the coefficients of these polynomials are real (Van Valkenburg, 18). Therefore the following rules apply to the

poles and zeros:

1. Poles and zeros must be real or complex conjugates.
2. Poles must have negative real parts for stable operation.
3. There must be at least as many poles as there are zeros.

($m \geq l$, Storer, 16)

$$G(s) = l \ln (V/E) = l \ln K \frac{\prod_l (s - s_n)}{\prod_m (s - s_k)} \quad (4)$$

$$= l \ln K + \sum_l l \ln (s - s_n) - \sum_m m \ln (s - s_k)$$

Separating real and imaginary parts gives:

$$\alpha = \alpha_0 + \sum_l l \ln (s - s_n) - \sum_m m \ln (s - s_k) \quad (5)$$

$$\beta = \beta_0 + \sum_l l \text{ phase } (s - s_n) - \sum_m m \text{ phase } (s - s_k)$$

As previously mentioned, $l \ln K$ will be ignored in the following discussions.

LOGARITHMIC POTENTIALS

In two-dimensional potential theory it is convenient to speak of a point charge q in a two-dimensional plane (x, y) and to regard this plane as a complex variable plane $z = x + jy$. The charge and the plane are the result of cutting infinite charged line filaments. For a point charge q at the origin of this plane, the voltage at any point is proportional to the magnitude of the charge and the logarithm of the distance from the charge, i.e.,

$$V = K_1 q \ln \rho + K_2 \quad (6)$$

Here K_1 and K_2 are constants for a coherent system of electromagnetic units. In the following statements, the constants will be omitted for ease of discussion.

If the polar coordinates $z = \rho e^{j\theta}$ are introduced, the potential may be considered in complex form.

$$W = q \ln z = q \ln \rho + jq \theta \quad (7)$$

The real part of this last expression is the potential and the imaginary part is the stream function.

If the charge is at a point z_m instead of the origin, the complex potential is:

$$W = q \ln (z - z_m) \quad (8)$$

For a set of point charges, the total potential is the sum of the individual potentials (Stratton, 17).

$$W = \sum q_m \ln (z - z_m) \quad (9)$$

In general, W is a complex number

$$W = V + j\psi$$

where V denotes the potential and ψ the stream function. The function in Equation (9) is analytic everywhere except at the points occupied by the charges.

To obtain some of the properties of the potential and the stream function, it is most convenient to resort to the theory of analytic functions of a complex variable (Goldman, 9).

First the derivative of W is unique and may be written:

$$\frac{dW}{dz} = \frac{\partial V}{\partial x} + j \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} - j \frac{\partial V}{\partial y} \quad (10)$$

where V and ψ must satisfy the Cauchy-Riemann relations

$$\frac{\partial \psi}{\partial x} = - \frac{\partial V}{\partial y} \quad \text{and} \quad \frac{\partial \psi}{\partial y} = \frac{\partial V}{\partial x} \quad (11)$$

The stream function (ψ) and the potential (V) are therefore not independent. The components of the electric intensity are obtained

from V by the relation $E = -\text{grad } V$.

$$E_x = -\frac{\partial V}{\partial x} = -\frac{\partial \psi}{\partial y} = -R_e \left(\frac{dW}{dz} \right)$$

$$E_y = -\frac{\partial V}{\partial y} = \frac{\partial \psi}{\partial x} = I_m \left(\frac{dW}{dz} \right)$$

Compare (4) and (9):

$$G(s) = \sum^l \ln(s - s_n) - \sum^m \ln(s - s_k) \quad (4)$$

$$W = \sum_{m=1}^{l+m} q_m [\ln(z - z_m)]$$

If $G(s)$ is identified with the complex potential W , then unit negative charges will be located at the zeros (s_n) and unit positive charges at the poles (s_k) of the transmission function $F(s)$. The complex potential in the s -plane, then, is

$$W = \sum \ln(s - s_n) - \sum \ln(s - s_k) \quad (12)$$

The real part of this function is the potential and the imaginary part is the stream function. Then, by definition of gain and phase in Equation (2), the gain of the associated network is given by the potential on the imaginary axis (real frequency axis) and the phase by the corresponding stream function.

The poles and zeros of $F(s)$ locate the charges producing the complex potential W and they form a discrete set of points. When $F(s)$ corresponds to practical problems, the set of points are usually arranged along well defined lines instead of being distributed at random in the plane. Recognizing this, it is usually easier to work in the potential plane using a continuous large distribution over a convenient contour. However, this contour has to be approximated by lumped charges since the poles and zeros

of $G(s)$ are discrete. This approach is only of value when an analytic expression is available for $F(s)$ and is of not much use for an arbitrary $F(s)$ that is not given analytically.

It has been assumed that the gain (α) corresponds to the voltage (V) and that the phase (β) corresponds to the stream function (ψ) of the complex potential W , but it would have been just as valid to reverse these two and interpret (α) as the stream function of another complex potential jW ; then (ψ) would be the negative of the potential (V). Since it is generally easier to equate (α) to (V) and (β) to (ψ), this discussion will interpret them in this fashion.

Since the electric intensity (E) is the gradient of the potential, $\frac{\partial \alpha}{\partial \omega}$ is analogous to E in the direction of the negative frequency axis ($j\omega$). Also the variation of (α) with frequency is analogous to the electric intensity (E) in the direction of the negative real s -axis (σ).

The analogies most commonly used may be summarized as in the following table.

Table 1. Comparison of the complex-frequency plane and the potential plane.

Complex-frequency plane	Potential plane
Transmission function $G(s)$	Complex potential W
Gain (α)	Potential (V)
Phase (β)	Stream function (ψ)
$-\frac{\partial \alpha}{\partial \omega}$	Field along real frequency axis
$-\frac{\partial \beta}{\partial \omega}$	Field across real frequency axis

Restating the pole and zero limitations in the complex potential:

1. The charge distribution must be symmetrical about the real axis.
2. The positive charges must be in the negative half of the plane.
3. The net charge must be non-negative.

FILTER NETWORKS

Now that the analogue has been justified mathematically, it is necessary to restrict the discussion so that analytic and experimental results may be compared. This is necessary since a general discussion and experimental data for general networks is nebulous and out of the scope of this paper. Filters are members of an important class of networks. To discuss filters, it is convenient to discuss the steady-state conditions of filters since they are primarily frequency sensitive devices. When this restriction is made

$$s = j\omega$$

$$F(s) = F(j\omega)$$

To facilitate the discussion, the low-pass filter case only will be mentioned. The transmission ratio $F(j\omega)$ and the cutoff frequency (ω_c) will be normalized to unity. These restrictions do not affect the generality of the discussion of filters since by utilizing frequency transformations and multiplying constants, it is possible to discuss any filter type in any steady-state situation.

The "ideal" filter gain characteristic is shown in Plate I. It is desirable to approximate this characteristic closely since it is impossible to duplicate it (Goldman, 9). Two very important classes of functions are used to perform this approximation. These two functions are the Butterworth and Chebyshev functions.

Butterworth response is a "flat - flat" or maximally flat response. That is, the transmission ratio will be one that has the most possible derivatives with respect to ω of its absolute value equal to zero (Van Valkenburg, 18), i.e.,

$$\frac{d^n F(j\omega)}{d\omega^n} = 0 \quad n = 1, 2, \dots \quad (13)$$

The result will be a transmission ratio that will appear as

$$\left| F(j\omega_p) \right| = \frac{1}{\sqrt{1 + \omega_p^{2n}}} \quad (14)$$

where ω_p is normalized to one, i.e., at the cutoff frequency $\omega_p = 1$ and $F(j\omega_p) = \frac{1}{\sqrt{2}} = .707$ and $n =$ order of filter.

Notice that as n increases, the approximation to the ideal curve becomes better (Plate I). To find the positions of the poles of transmission ratio that will give an absolute magnitude function of the form of Equation (14), let $s = j\omega_p$ or $\omega_p = s/j$. The square root factor of Equation (14) then becomes:

$$\begin{aligned} 1 - s^{2n} & \text{ if } n \text{ is odd} \\ 1 + s^{2n} & \text{ if } n \text{ is even} \end{aligned} \quad (15)$$

Then the corresponding functions are:

$$Q(s) = \frac{1}{1 + s^{2n}} \quad n \text{ is even} \quad (16)$$

$$H(s) = \frac{1}{1 - s^{2n}} \quad n \text{ is odd} \quad (17)$$

The poles of $Q(s)$ will occur at $s^{2n} = -1$. The poles of $H(s)$ will occur at $s^{2n} = 1$. Taking the $2n^{\text{th}}$ root of these poles, one obtains for $Q(s)$:

$$s_m = e^{\pm j \left(\frac{2m-1}{2n} \right) \pi} \quad m = 1, 2, 3, \dots, n \quad (18)$$

for $H(s)$:

$$s_k = e^{\pm j \frac{k\pi}{n}} \quad k = 0, 1, 2, \dots, n \quad (19)$$

These equations locate the poles for $Q(s)$ when n is even and $H(s)$ when n is odd. Both $Q(s)$ and $H(s)$ have symmetry about both axes and no roots occur on the imaginary axis regardless of whether n is odd or even. These poles lie on the locus of a circle of unity radius.

These roots are the poles of $Q(s)$ and $H(s)$ as defined by Equations (16) and (17). They are not, however, necessarily network functions. Transmission ratios for networks cannot have poles in their right half plane as $Q(s)$ and $H(s)$ have.

Because of the symmetry of $Q(s)$ and $H(s)$ about both axes, there are as many poles in their right half plane as there are in the left half plane. If the poles in the right half plane were grouped together and called $f_r(s)$ and the poles in the left half plane called $f_\ell(s)$, it is possible to write

$Q(s) = f_r(s) f_\ell(s)$. In steady-state analysis ($s = j\omega_p$),

$$|f_r(j\omega_p)| = |f_\ell(j\omega_p)| \quad (19)$$

$$\text{Therefore } |Q(j\omega_p)| = |f_\ell(j\omega_p)|^2 \quad (20)$$

$$\text{or } |f_\ell(j\omega_p)| = \sqrt{|Q(j\omega_p)|}$$

EXPLANATION OF PLATE I

Fig. 1. Gain of Butterworth filters plotted versus ω .

Fig. 2. Gain of Chebyshev filters plotted versus ω .

PLATE I

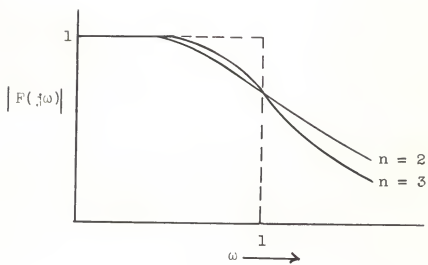


Fig. 1.

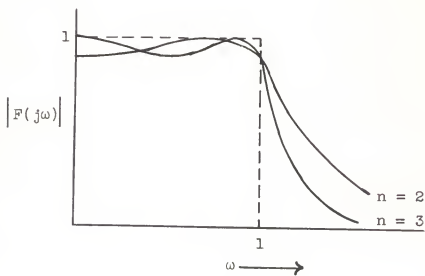


Fig. 2.

where
$$|Q(j\omega_p)| = \left(\frac{1}{1 + \omega_p^{2n}} \right) \quad (21)$$

Substituting into Equation (21), one obtains

$$f_\ell(j\omega_p) = \frac{1}{\sqrt{1 + \omega_p^{2n}}} \quad (22)$$

The same technique and the same form would result for the poles of $H(s)$. Therefore only the poles in the left-hand plane are necessary to give the Butterworth or "flat-flat" responses.

This shows, then, that the poles of this Butterworth transmission ratio lie on the locus of a semicircle in the left half portion of the s -plane. It is also apparent, as in the case of ordinary transmission ratios, that this has symmetry about the real axis.

The Chebyshev or "equal ripple" filter improves one of the disadvantages of the Butterworth filter. The disadvantage of the Butterworth filter is that as ω increases the approximation of the ideal becomes poor. The Chebyshev filter spreads this error over the whole pass band. The transmission ratio is (18):

$$F(j\omega) = \frac{1}{\sqrt{1 + \epsilon^2 C_n^2(\omega)}} \quad (23)$$

where ϵ is a constant ($\epsilon < 1$), C_n is an n th order Chebyshev polynomial.

The n above will be related to the number of poles, and hence their locations. To find the location of the poles and their function in Equation (23), start with the definition of the Chebyshev polynomial.

$$\begin{aligned} C_n(\omega) &= \cos(n \cos^{-1} \omega) \text{ for } \omega \leq 1 \\ C_n(\omega) &= \cosh(n \cosh^{-1} \omega) \text{ for } \omega > 1 \end{aligned} \quad (24)$$

When $\omega \leq 1$, then $|C_n(\omega)| \leq 1$ and the magnitude of $|F(j\omega)|$ varies between 1 and $\frac{1}{\sqrt{1+\epsilon}}$. When $\omega > 1$, then $|C_n(\omega)|$ increases without bound and $|F(j\omega)|$ approaches zero.

To find the poles of Equation (23), set the denominator equal to zero.

$$1 + C_n^2(\omega_p) = 0 \quad \text{Let } s = j\omega_p \text{ or } \omega_p = s/j \quad (25)$$

$$1 + C_n^2(s/j) = 0 \text{ or } C_n^2(s/j) = -1/\epsilon$$

$$C_n(s/j) = \pm j/\sqrt{\epsilon}$$

$$\text{where} \quad C_n(s) = \cos(n \cos^{-1} x) \quad (26)$$

Since the inverse cosine of a complex number is, in general, complex, let

$$\cos^{-1}(s/j) = (\alpha - jd) \quad (27)$$

such that (from Equations (25) and (26))

$$\cos(n\alpha - jnd) = \pm j/\sqrt{\epsilon} \quad (28)$$

Expand Equation (27) to obtain:

$$\cos n\alpha \cosh nd + j \sin n\alpha \sinh nd = \pm j/\sqrt{\epsilon}$$

The following identities may be written.

$$\cos n\alpha \cosh nd = 0 \text{ and } \sin n\alpha \sinh nd = \pm 1/\sqrt{\epsilon}$$

$$\cos n\alpha = 0, \text{ since } \cosh nd \geq 1 \text{ always}$$

$$\text{Therefore} \quad \alpha = \left(\frac{2N+1}{2n} \right) \pi \quad N = 0, 1, 2, \dots, n \quad (29)$$

For these values of α , $\sin n\alpha = \pm 1$. Therefore

$$\sinh nd = \pm 1/\sqrt{\epsilon} \text{ or } d = \pm 1/n \sinh^{-1}(1/\sqrt{\epsilon}) \quad (30)$$

From Equation (27),

$$s = j \cos(\alpha - jd) \quad (31)$$

If Equation (31) is expanded, the following is obtained:

$$s = j \cos \alpha \cosh d - \sin \alpha \sinh d \quad (32)$$

or $s = \cosh d (-\tanh d \sin \alpha + j \cos \alpha)$

Equation (32) defines the roots as required.

For convenience, these roots will be related to the roots of the Butterworth or "flat-flat" roots. To do this, note that

$$\cos x = \sin (\pi/2 - x)$$

$$\sin x = \cos (\pi/2 - x)$$

Using these identities, Equation (32) becomes

$$s = \cosh d (-\tanh d \cos b + j \sin b) \quad (33)$$

where $b = \pi/2 - \alpha$. Substitute for α from Equation (29) to obtain

$$b = \left(\frac{n - 2N - 1}{2n} \right) \pi \quad N = 0, 1, 2, \dots, n \quad (34)$$

For the Butterworth response:

$$s = e^{jb'} = \cos b' + j \sin b' \quad (35)$$

where $b' = \left(\frac{2m - 1}{2n} \right) \pi$ if n is even, and $b' = \frac{k\pi}{n}$ if n is odd.

If b' is made equal to b , then

$$m = \left(\frac{n - 2N}{2} \right) \quad \text{for } n \text{ even}$$

$$k = \left(\frac{n - 1 - 2N}{2} \right) \quad \text{for } n \text{ odd}$$

If Equations (33) and (35) are compared,

$$s_1 = \cos b' + j \sin b'$$

$$s_2 = \cosh d (-\tanh d \cos b + j \sin b)$$

the similarity of the pole locations becomes obvious.

The roots for the equal ripple case (Chebyshev) may be

found from the roots of the Butterworth by the following steps.

1. Change the radius of the circle for the Butterworth from unity to $\cosh d$.

2. Multiply the real part of the poles located for the Butterworth case by $\tanh d$.

This relationship holds because b' has been set equal to b . The poles of the equal ripple (Chebyshev) response are located on an ellipse. This can be shown by noting from Equation (33) that $s = \sigma_p + j\omega_p$ with $\sigma_p = -\sinh d \cos b$, and $\omega_p = \cosh d \sin b$. Then:

$$\frac{\sigma_p^2}{\sinh^2 d} + \frac{\omega_p^2}{\cosh^2 d} = 1 \quad (36)$$

This is the equation of an ellipse with its major axis along the $j\omega$ axis and having a major semi-axis of length $\cosh d$ and a minor semi-axis of length $\sinh d$.

Another point of similarity between the Chebyshev and Butterworth responses may be noted. If a frequency ω_{p1} is chosen such that

$$\omega_{p1} = \cosh d, \quad \omega_{p1} > 1 \quad (37)$$

it is then possible to write the following:

$$C_n(\omega_{p1}) = C_n(\cosh d) = \cosh (n \cosh^{-1} \cosh d) = \cosh nd \quad (38)$$

Replace the value of d from Equation (30) into Equation (38).

$$C_n(\omega_{p1}) = \cosh (\sinh^{-1} 1/\sqrt{\epsilon}) \quad (39)$$

If $\sinh x = 1/\sqrt{\epsilon}$, then $\cosh x = \sqrt{1 + 1/\epsilon}$, (since $\cosh^2 x - \sinh^2 x = 1$). Therefore $\sinh^{-1} 1/\sqrt{\epsilon} = \cosh^{-1} \sqrt{1 + 1/\epsilon}$. Placing this in Equation (39) gives

$$C_n(\omega_{p1}) = \cosh (\cosh^{-1} \sqrt{1 + 1/\epsilon}) = \sqrt{1 + 1/\epsilon}$$

Therefore,

$$|F(j\omega_{p1})| = \frac{1}{\sqrt{1 + C_n^2(\omega_{p1})}} = \frac{1}{\sqrt{2 + \epsilon}}$$

If $\epsilon < 1$, then

$$|F(j\omega_{p1})| \doteq 1/\sqrt{2} = .707$$

This corresponds to $\omega = 1$ for the Butterworth response.

The summary of equal ripple frequency response characteristics indicates that specifications are complete if given in terms of ϵ and n .

DESCRIPTION OF EQUIPMENT

The next step is to find the most convenient form to work with for experimental work. Hanson and Lundstrom (12), in 1945, discussed the use of a large circular surface with the edge of the surface being a conductor set at the same potential as the center. Since the equations for the analogue assume an infinite surface, this introduces errors due to the finite size of the surface. This means that correction factors must be introduced to reduce the error. Since the application of these correction factors introduces complexity into the system, it is undesirable because this will defeat the whole purpose of the analogue, that is, simplicity.

In 1948, Huggins (13) showed a way to circumvent this drawback by using a logarithmic conformal transformation. To obtain this transformation, the circular surface is transformed into a

rectangular one by the following method.

For the circular surface

$$s = \sigma + j\omega = \rho e^{j\beta}$$

For the rectangular surface

$$M = P + jQ = \ell n s = \ell n |s| + j(\beta + 2n\pi)$$

When this transformation is made, the surface becomes an infinite series of rectangular surfaces in the Q direction to take care of the branch points. The two sides of this rectangular surface corresponding to $\rho = 0$ and $\rho = \infty$ become conducting strips. Since any point on the S plane has an infinite number of points spaced 2π distant on the M plane in the Q direction, one might suspect that a certain amount of symmetry would help reduce the number of branch points necessary. Another point to be noted is the symmetry of the charges about the axis in the S plane (Rule 1). When mapped on the M plane, these points form a symmetry with the other points already mentioned. This symmetry is around the lines $V = 2n\pi$ ($n = 0, \pm 1, \pm 2, \dots$) (Plate II). If these are charges, notice that if insulating strips are placed along the lines $Q = 0$, and $Q = \pi$, in no way will the field between these two insulators be affected since no current will cross these lines because of the symmetry of the charges about these lines. This also means that the whole surface is not necessary so that using just the portion between the two imaginary insulators is sufficient. Expanding this strip will make a plane of infinite extent in the P direction with a finite height. Notice, however, that in the P direction the distance is the logarithm of the radius of a circular tank so that by increasing the

EXPLANATION OF PLATE II

Fig. 1. Representation of complex s plane with a pole at x and its conjugate at o .

Fig. 2. Representation of complex M plane where $M = \mathcal{L}_n s = P + j Q$ with the pole (x) and its conjugate (o) shown.

Fig. 3. Representation of portion of M plane with the pole (x) shown to be used for experimental work.

PLATE II

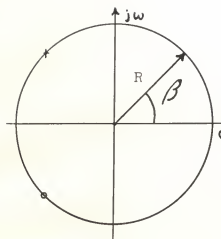


Fig. 1.

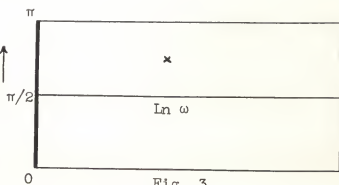


Fig. 3.

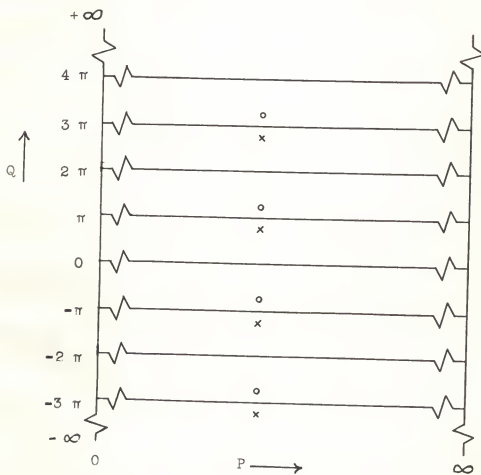


Fig. 2.

frequency scale ($j\omega$) by a decade essentially increases the diameter of the circular surface by 10.

The correction factors are fairly small for the circular surface if the distance from the center to the edge of the surface is large compared to the distance from the center to the furthest charge. If the ratio of these distances is 10, the error or correction factor is about one per cent. If this ratio is 100, the error would be about .01 per cent (Hanson and Lundstrom, 12). This, then, shows how the rectangular surface can be constructed so that the errors can be made negligible. This can be done by maintaining reasonably large spacing between charges and still using a surface of manageable size.

Even though this paper is restricted to the discussion of two special types of transmission ratios, the conformal transformation mentioned above is perfectly general. These restricted cases, however, make it quite clear why such a transformation was used. The poles that lie on a circle in the s -plane lie on a straight vertical line in the log plane. The poles that lie on an ellipse in the s -plane do not, however, lie on a straight line in the log plane. There are other transformations that will give convenient contours for the ellipse. However, all of these transformations become cumbersome in analysis, since there is no intuition about the experimental results.

The most natural transformation for the potential analogue is the logarithmic transformation. This is due to the ease with which numerical values can be obtained both mathematically and experimentally. Also, this transformation gives results that

come out directly as \ln -Db type response curves.

To experimentally determine pole and zero locations on the rectangular surface, a sheet of Teledeltos conducting paper was cut in a rectangular shape 40 inches by 36 centimeters. One decade of frequency is represented by 10 inches and 90 degrees of phase shift is represented by 18 centimeters.

To facilitate the use of a rectangular surface for the analogue, a scanning device was constructed so that one decade of frequency could be scanned and displayed on an oscilloscope. This produced quicker observation of the effect on the gain of adding or moving a charge. To do this, ten separate probes were placed on the $j\omega$ axis. The ten probes were equally placed. A mechanical commutator was inserted so that the voltages at these points could be displayed on an oscilloscope. To faithfully reproduce these voltages, an external sweep for the oscilloscope was constructed. The sweep was obtained by mechanically commutating ten equal voltages. This was done so that the distances between the traces on the oscilloscope would conform with the physical distances between the probes.

To synchronize these two commutations, the sweeping (or scanning) arms were connected to the same shaft. This shaft was then connected to a small series motor (Plate III).

To obtain the phase (β) of the transmission function ($F(s)$), an analytic approach may be used as Goldman (9), p. 244, has done.

This, however, is not necessary since the potential analogue embodies a method for finding the phase shift directly. From

Table 1, where the analogue is outlined in detail, the following relation is used to find the phase shift.

$$-\frac{\partial \beta}{\partial \omega} \rightarrow \text{field across } j\omega \text{ axis } (E_x)$$

$$E_x = -\nabla V = -\frac{\partial V}{\partial x}$$

Therefore, $\frac{\partial \beta}{\partial \omega} \rightarrow \frac{\partial V}{\partial x}$

This means that in the approximation

$$\frac{\Delta \beta}{\Delta \omega} = \frac{\Delta V}{\Delta \sigma}$$

or $\beta_{\omega_1} = \sum \frac{\Delta V \Delta \omega}{\Delta \sigma}$

To obtain the phase at any given frequency, two rows of probes were placed along the $j\omega$ (real frequency) axis. The voltage between any two probes on opposite sides of this axis is proportional to $\frac{\Delta \beta}{\Delta \omega}$ at that value of ω . An analogue integration from zero to ω_1 will then give the phase shift at ω_1 .

CALIBRATION OF EQUIPMENT

Calibration of the gain is obtained by adjusting the voltage read from the probes to six decibels per octave when a probe is placed on the right-hand conductive strip and an opposite polarity probe is placed on the left-hand conducting strip. This corresponds to a transmission ratio with a pole at zero (Plate III).

EXPLANATION OF PLATE III

Fig. 1. Diagram of equipment used.

Fig. 2. Calibration curve of surface for six decibels per octave slope when a "pole" is placed at $\rho = 0$ and a zero at $\rho = \infty$.

PLATE III

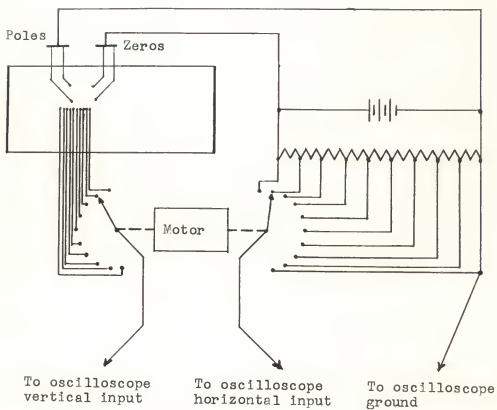


Fig. 1.

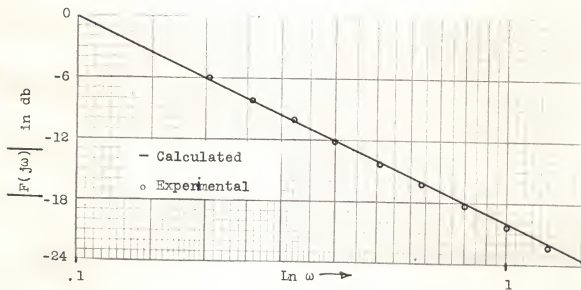


Fig. 2.

For calibrating the phase another known relationship of transmission ratios was employed. The phase of a second-order Butterworth filter has a 180-degree shift at $\omega = 1$ (Plate IV). The calibration then consists of setting the voltage across the $j\omega$ axis at $\omega = 1$ to a value that corresponds to 180 degrees. Then the voltmeter will be calibrated to read the phase shift.

EXPERIMENTAL RESULTS

The experimental results consist of graphical comparison of the calculated and experimental values for gain and phase for two Butterworth and two Chebyshev filters. (Plates IV, V, VI, and VII.) Also an arbitrary transmission function was chosen and a network was synthesized using the analogue. The results are shown on Plates VIII and IX. The synthesis was accomplished with a minimum number of probes. This was done by first observing the general shape of the curve (Plate VIII, Fig. 2), and using intuition as to where probes would likely be needed.

The intuition consists of listing the possible areas where poles and zeros can exist. The shape of the curve is such that these areas have fairly wide limits, which discounts the possibility of easily predicting the actual locations of the poles and zeros. The easiest way to handle this problem is in a tabular form (Table 2).

The curve indicates that there is either a double order pole or two single poles in the range $.4 < \omega < .7$.

The response curve shown on Plate VII was not chosen completely

EXPLANATION OF PLATE IV

Fig. 1. Actual and experimental values of gain in db plotted versus $\ln \omega$ for Butterworth ($n = 2$) filter.

Fig. 2. Actual and experimental values of phase in radians plotted versus $\ln \omega$ for Butterworth ($n = 2$) filter.

PLATE IV

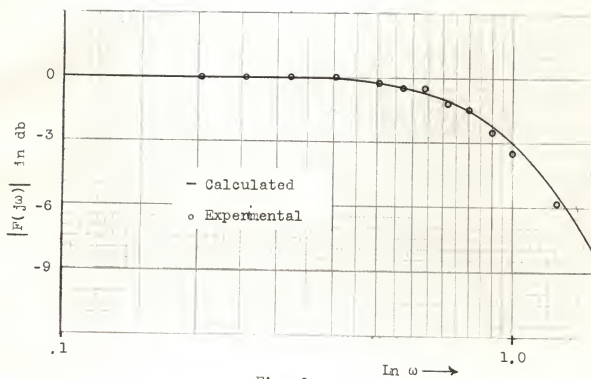


Fig. 1.

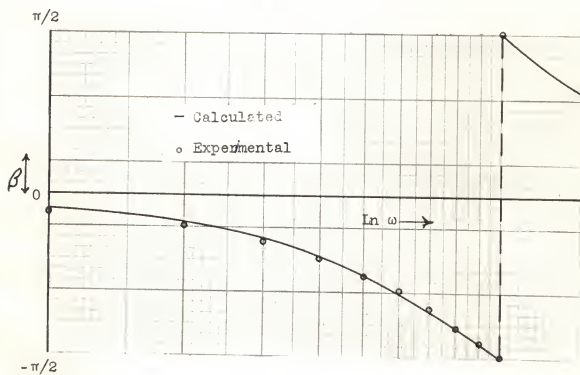


Fig. 2.

EXPLANATION OF PLATE V

Fig. 1. Actual and experimental values of gain in db plotted versus $\ln \omega$ for Butterworth ($n = 3$) filter.

Fig. 2. Actual and experimental values of phase in radians plotted versus $\ln \omega$ for Butterworth ($n = 3$) filter.

PLATE V

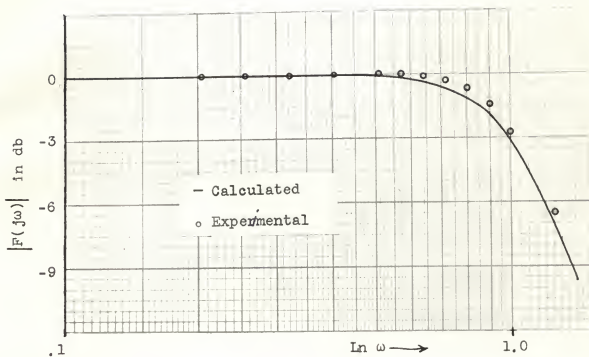


Fig. 1.

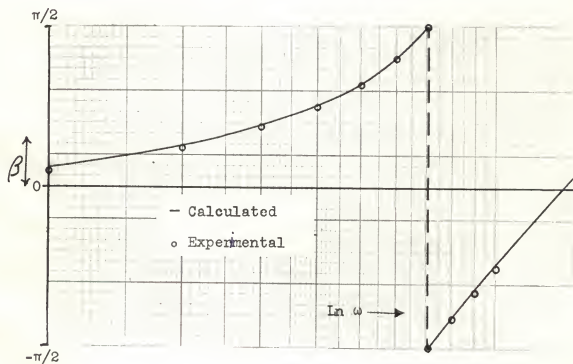


Fig. 2.

EXPLANATION OF PLATE VI

Fig. 1. Actual and experimental values of gain in db plotted versus $\ln \omega$ for Chebyshev ($n = 2$) filter.

Fig. 2. Actual and experimental values of phase in radians plotted versus $\ln \omega$ for Chebyshev ($n = 2$) filter.

PLATE VI

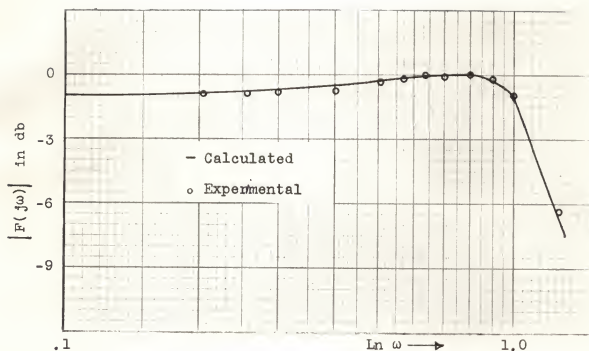


Fig. 1.

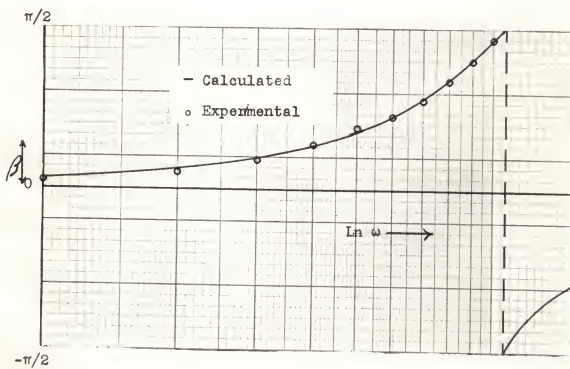


Fig. 2.

EXPLANATION OF PLATE VII

Fig. 1. Actual and experimental values of gain in db plotted versus $\ln \omega$ for Chebyshev ($n = 3$) filter.

Fig. 2. Actual and experimental values of phase in radians plotted versus $\ln \omega$ for Chebyshev ($n = 3$) filter.

PLATE VII

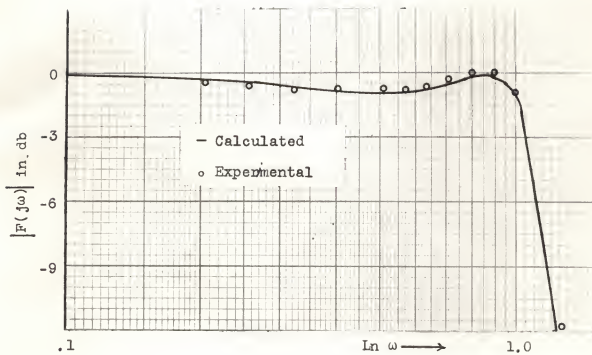


Fig. 1.

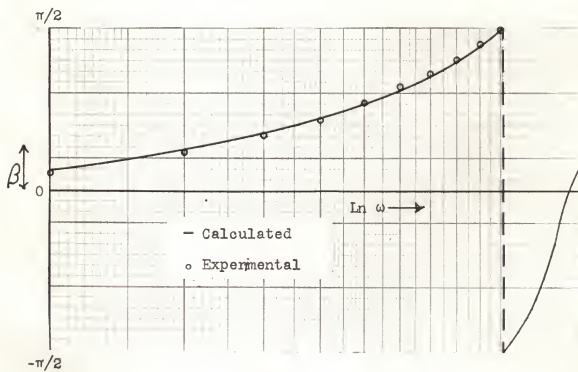


Fig. 2.

Table 2. The pole and zero locations of the transmission function shown with ranges of ω .

ω	:	Poles	:	Zeros
0		0		0
0-.4		0		1
.4-.7		2		0
.7-1.1		0		1
∞		0		0

arbitrarily. An attempt was made to make it very time consuming to try a trial-and-error method for curve fitting without using the analogue.

The results of trial-and-error adjustment using the analogue are shown on Plate VIII, Fig. 2. The complete synthesis is carried out on Plate IX.

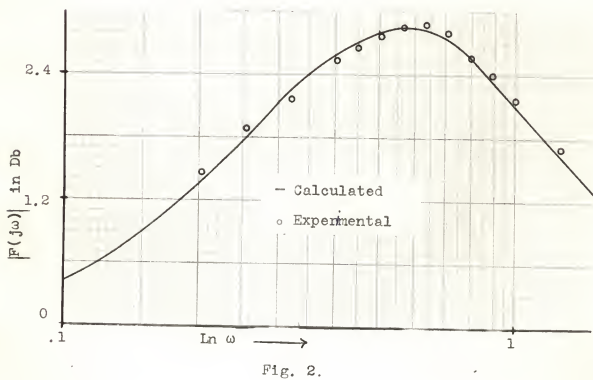
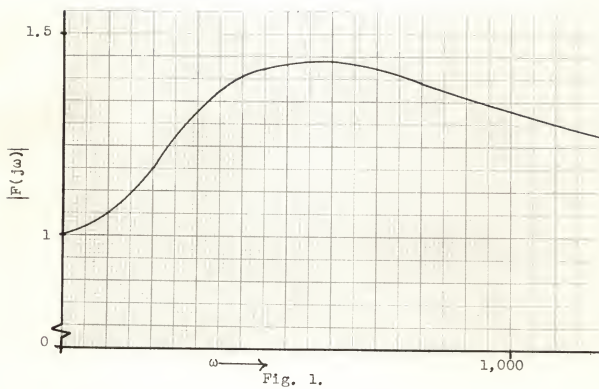
The maximum error experienced for all the transmission ratios was less than five per cent, which is well within the accuracy of circuit components used for synthesis. This indicates that if the pole and zero locations are known, the gain and phase of a transmission ratio may be found and plotted in a matter of minutes. If the pole and zero locations are not known, then with a little experimental work the gain may be determined and plotted directly, and the phase shift may be determined experimentally and plotted easily.

EXPLANATION OF PLATE VIII

Fig. 1. Arbitrary response curve showing gain plotted versus ω .

Fig. 2. Arbitrary response curve of Fig. 1 with gain in decibels plotted versus $\ln \omega$. The point on Fig. 1 where $\omega = 1,000$ has been normalized to unity for this figure. Also plotted are the experimental points, from zeros at $\omega = .32$, $\omega = .9$, and poles at $\omega = .5$, $\omega = .6$.

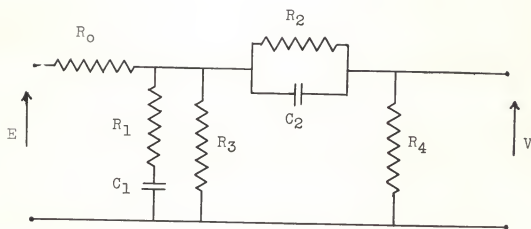
PLATE VIII



EXPLANATION OF PLATE IX

Schematic diagram synthesized from the zeros and poles discovered experimentally for the arbitrary response curve. Circuit element values are also shown.

PLATE IX



$$V/E = (S + 320)(S + 900)/(S + 500)(S + 600)$$

$$R_0 = 1,000 \text{ ohms}$$

$$R_1 = 2,450 \text{ ohms}$$

$$R_2 = 18,500 \text{ ohms}$$

$$R_3 = 1,360 \text{ ohms}$$

$$R_4 = 27,500 \text{ ohms}$$

$$C_1 = 0.45 \text{ mfd}$$

$$C_2 = 0.1625 \text{ mfd}$$

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THE POTENTIAL ANALOGUE METHOD OF
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by

KERMIT WILLIAM REISTER, JR.

B. S., University of Nevada, 1955

AN ABSTRACT OF
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For electric networks the ratio of the input to output voltages is of interest. This ratio is called the transmission ratio of a network, and it has the following form:

$$F(s) = v/E = e^{\alpha} e^{j\beta}$$

α = gain in nepers

β = phase shift in radians

The transmission function is defined as the logarithm of this ratio:

$$G(s) = \ln F(s) = \alpha + j\beta$$

For a finite lumped element network, this transmission function is of the same form as the voltage equation for a two-dimensional surface with positive and negative charges on it.

The restrictions that exist for transmission ratios must be employed with the potential plane.

To facilitate experimental work that would minimize errors due to a finite plane, logarithmic conformal transformation was used, which gave results of a " \ln -db" nature. The maximum error encountered in the experimental work was less than five per cent, which is well within the range of accuracy of ordinary circuit elements.