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QUASILINEARIZATION APPLIED TO OPTIMAL IDENTIFICATION OF AQUIFER DIFFUSIVITY IN STREAM INTERACTION SYSTEM

by

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NOTATION

D,	aquifer diffusivity, equal to k/s'
h,	height of water table above the impermeable layer
$h_0(t)$,	boundary condition at $x = 0$
h(x),	initial condition for water table
H,	maximum height of water table above the impervious layer
k,	hydraulic conductivity
L,	distance from river to the water divide
p,	first particular equation
q,	second particular equation
Q,	flow rate per unit width
s',	specific storage
t,	time
T,	total number of observations
W,	weighting factor
x,	horizontal distance
y =	x/L
Δy =	increment of y
θ =	h/H
θ^*,	observations on θ
τ =	$(H/L^2)\ t$

CHAPTER 1

INTRODUCTION

A large portion of the problems encountered in physical and engineering science can be represented by mathematical models which are governed by a series of differential equations with two-point or multipoint boundary conditions [22]. Usually, these problems appear in nonlinear forms. Most nonlenear differential equations cannot be solved analytically [32]. In addition, the problem becomes very difficult when the governing equation contains some unknown parameters which cannot be measured directly and the measurable variables are the dependent variables of the differential equation. One of the most frequently used methods to obtain solutions for these problems is the trial-and-error method which is very tedious and inefficient. The quasilinearization technique which was first developed by Bellman and Kalaba [1] is a powerful tool for solving nonlinear boundary value problems. It involves decoupling the system of differential equations by means of linearization into a series of initial value problems that may be repetitively solved in such a way that their solutions will converge to the solutions of the original problems [15].

The problem to be solved is that of identifying aquifer diffusivity in an unconfined aquifer and stream interaction system. The system is represented by a second order non-linear differential equation with initial and boundary conditions.

Boussinesq (1904) obtained the analytical solution of this type of second order differential equation by the use of simplifying assumptions. What he did is to derive an equation that applies to the flow of ground water into a ditch. Hornberger etc. (1970) [3] obtianed numerical solutions using finite difference approximation and the predictor-corrector. Yeh (1970) [37] solved it for a semi-infinite system and presented dimensionless curves for a variety of boundary conditions. Yeh and Singh (1970) [6] solved the transient flow problem of subsurface drainage between parallel drains. The assumption of an initial profile is removed in this method. The governing nonlinear partial differential equation is reduced to ordinary differential equations which are easily solvable on the computer. In addition to the above approaches, the identification problem has been studied by Rowe (1960) [5] and Pinder (1969) [4]. In Rowe's work, an equation was derived from which transmissibility and the coefficient of storage of an aquifer can be estimated. A linear change with the time of the depth of a nearby surface body of water was assumed in the derivation. The equation appears to be accurate when the boundary conditions are satisfied. In Pinder's work, the diffusivity of a homogeneous isotropic aquifer can be determined from the response in the aquifer to fluctuations in river stage. Observed changes in the head in the aquifer are compared with theoretic head values computes assuming a series of diffusivities. The values are generated by approximating the stage hydrograph as a series of discrete steps and assuming the influence of each increment. The diffusivity of the aquifer is calculated from the best fitting theoretic response curve. Both the work of Rowe and Pinder are essentially based on linearization and trail and error graphical manipulations. Yeh and Tauxe (1971) [7] presented a systematic technique based on pumping tests for converting field observations into the desired parameters for an unconfined aquifer system. Most existing techniques have required graphical matching in which the governing equation is solved for all possible boundary conditions and is plotted as type curves. A data curve from pumping tests is then plotted. By superimposing the data curve on type curves, it may be possible to find a match point. From this match point, common values for different variables are found so that the aquifer parameters are computed. In this present work, a fairly new technique, quasilinearization [1,15], is presented. The following is a brief review in this area.

Lee (1967) [19] combined the use of the variational equations and the quasilinearization technique in order to obtain the optimum temperature profile in a tubular reactor. He considers these parameters as additional state variables instead of following previous procedure for designing a system which is to choose several parameters and select the most promising combination. Lee (1968) [20] used the invariant imbedding concept [35,15] to derive useful estimator equations for nonlinear dynamic system. Lee and Hwang (1970) [17] and Lee (1968) [36] estimated parameters or coefficients in differential equations arising in stream quality modeling. The parameter estimation problem is treated as a multipoint boundary-value problem by the quasilinearization technique [15,36]. Lee (1971) used the invariant imbedding concept [15] to obtain useful estimation. equations of dynamic stream pollution models instead of the mehtod which was employed in Lee and Hwang's 1970 paper. In the present work, the governing nonlinear partial differential equation is replaced by a system of nonlinear ordinary differential equations for which the technique of quasilinearization is applied. In this system, we use a parameter as a new variable. This estimated parameter will be improved in each iteration under the criterion that the derivation is minimized between the actual value of dependent variable adn the observation data. The procedure is straightforward and converges quadratically. The technique of quasilinearization requires neither graphical matching nor trail and error manipulations.

CHAPTER 2

QUASILINEARIZATION AND PARAMETER ESTIMATION

2-1 INTRODUCTION

The quasilinearization technique is a generalized Newton-Raphson method for functional equations [1,15]. In addition to linearization of the nonlinear equations, it provides a sequence of functions which converge to the solutions of the original equations. In this chapter, a breif description of the initial-value problem will be given in section 2. The technique of quasilinearization is introduced in section 3. The charactersitics are discussed in section 4. In the fifth section, the least square method is introduced to find the new estimation of parameter.

2-2 INITIAL-VALUE PROBLEM

Initial-value problems are those in which all conditions are given at one point. This particular point can be the initial or final point of the entire interval. Any initial value problem can be represented as a set of one or more coupled first-order ordinary differential equations, each with an initial condition. Because any initial value problem can be expressed as a set of first-order ordinary differential equations, the primary concern will be to develop numerical methods for the solution of first-order differential equations.

There are various numerical integration methods available for obtaining the solution of ordinary differential equations, such as the Euler Method, Runge-Kutta method, Adams multistep method, and Predictor-Corrector method, etc. [12]. These methods have different advantages and disadvantages for each one. The efficiency of these and other modern methods have been compared in a recent paper [13] which gives an excellent picture of the current state of the art. For a detailed description of these methods, readers are referred to Ralston [25] and Tompkin [26]. The Rung-Kutta method will be used in this work.

2-3 QUASILINEARIZATION

The generalized Newton-Raphson method for differential equations will be discussed first before explaining how the quasilinearization technique words. Consider the nonlinear differential equation

$$\frac{dx}{dt} = f(x(t),\ t) \qquad \text{and} \qquad x(t_0) = c \tag{2-1}$$

The function f can be expanded around the function $x_0(t)$ by the use of the Taylor series [12].

$$f(x(t),\ t) = f(x_0(t),\ t) + (x(t) - x_0(t))f_{x_0}(x_0(t),\ t) \tag{2-2}$$

With the second and higher order terms omitted. The expression f_{x_0} represents partial differentiation of the function f with respect to x_0 which is a known value. Combining equations (2-1) and (2-2), the following equation is obtained after rearrangement.

$$\frac{dx}{dt} = f_{x_0}(x_0(t),\ t)\ x(t) + f(x_0(t),\ t) - f_{x_0}(x_0(t),\ t)\ x_0(t) \tag{2-3}$$

$x_0(t)$, $f_{x_0}(x_0(t),\ t)$, $f(x_0(t),\ t)$ are known functions of t. The only unknown variable is x(t) which appears as the first degree term. Hence, equation (2-3) is a linear differential equation with variable coefficients. This is the algoirthm which quasilinearization employs to linearize non-linear equations.

The above discussion concerns only the single differential equation.

Now let us consider a general nonlinear system [15].

$$\frac{d\bar{X}}{dt} = \bar{f}(\bar{X}, t) \tag{2-4}$$

where $\bar{X}$ and $\bar{f}$ are m-dimensional vectors with components $x_1, x_2, \ldots, x_n$ and $f_1, f_2, \ldots, f_m$ respectively.

If we choose a set of initial approximations for $x_1, x_2, \ldots, x_m$ and denote them as $x_{1,0}, x_{2,0}, \ldots, x_{m,0}$, Equation (2-4) can be linearized by the use of the following vector equation.

$$\frac{d\bar{X}}{dt} = \bar{f}(\bar{X}, t) = \bar{f}(\bar{X}_0, t) + J(\bar{X}_0)(\bar{X} - \bar{X}_0) \tag{2-5}$$

where $\bar{X}_0$ is an m-dimensional vector with components $x_{1,0}, x_{2,0}, \ldots, x_{m,0}$. The Jacobi matrix $J(\bar{X}_0)$ is defined by

$$J(\bar{X}_0) = \begin{bmatrix} \frac{\partial f_1}{\partial x_{1,0}} & \frac{\partial f_1}{\partial x_{2,0}} & \cdots & \frac{\partial f_1}{\partial x_{m,0}} \\ \frac{\partial f_2}{\partial x_{1,0}} & \frac{\partial f_2}{\partial x_{2,0}} & \cdots & \frac{\partial f_2}{\partial x_{m,0}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_{1,0}} & \frac{\partial f_m}{\partial x_{2,0}} & \cdots & \frac{\partial f}{\partial x_{m,0}} \end{bmatrix} \tag{2-6}$$

The solution we get is an improved set of solutions. Let this improved solution be $\bar{X}_1$ which can be used as a new initial approximation. A new improved solution $\bar{X}_2$ can now be obtained. If this procedure is continued, the following recurrence relation is obtained:

$$\frac{d\bar{X}_{n+1}}{dt} = \bar{f}(\bar{X}_n, t) + J(\bar{X}_{n+1} - \bar{X}_n)(\bar{X}_n) \tag{2-7}$$

where $J(\bar{X}_n)$ is the Jacobi matrix defined as

$$J(\bar{X}_n) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1,n} & \frac{\partial f_1}{\partial x_2,n} & \cdots & \frac{\partial f_1}{\partial x_m,n} \\ \frac{\partial f_2}{x_1,n} & \frac{\partial f_2}{x_2,n} & & \frac{\partial f_2}{x_m,n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1,n} & \frac{\partial f_m}{\partial x_2,n} & & \frac{\partial f_m}{\partial x_m,n} \end{bmatrix} \tag{2-8}$$

2-4 LINEAR DIFFERENTIAL EQUATIONS

The general linear ordinary differential equation of first order which we will consider is

$$L(y) = y' + a(t)\, y = h(t) \tag{2-9}$$

Here $a(t)$ and $h(t)$ are given functions continuous on an open interval $J = (\alpha,\beta)$.

A solution of the nonhomogeneous Equation (2-9) y_p is usually called a particular solution. In many problems the nonhomogeneous term $h(t)$ may be very complicated; however, if $h(t)$ can be expressed as the sum of a finite number of functions, we can make use of the linearity of the differential equation to replace the original problem by several simpler ones [11,38]. Suppose it is possible to write $h(t)$ as $h_1(t) + h_2(t) + \ldots + h_m(t)$; then

Equation (2-9) becomes

$$L(y) = y' + a(t)Y = h(t) = h_1(t) + h_2(t) \ldots + h_m(t) \tag{2-10}$$

If we can find particular solutions y_{pi} of the differential equations

$$L(y) = h_i(t), \qquad i = 1,2, \ldots, m \tag{2-11}$$

then it follows by direct substitution that

$$y_p(t) = y_{p_1}(t) + y_{p_2}(t) + \ldots + y_{p_m}(t) \tag{2-12}$$

is a particular solution of Equation (2-10).

In general it is easier to find solutions of Equation (2-11) and to add the results than to try to solve Equation (2-9) as it stands. This method of constructing the solution of a complicated problem by adding solutions of a simpler problem is known as the method of superposition [38].

2-5 LEAST SQUARES

The method of least squares is the oldest and most widely used parameter estimation procedure. Some of its popularity is due to the fact that it can be applied directly to the deterministic model, without any cognizance being taken of the probability distribution of the observations. Especially, in case of pure curve fitting, where the coefficients have no physical significance, the least squares method is usually adequate [29]. The least squares procedure in its simplest form consists of finding the values of U which minimize the function

$$\phi(U) = \sum_{k=1}^{n} E_k^2(U) \tag{2-12}$$

$$E_k(U) = y_k - f(X_k, U) \tag{2-13}$$

The objective function of Equation (2-12) is often unsatisfactory for the reasons that the various quantities y_k may represent entities having different physical dimensions, or measured on different scales. Besides, some observations may be known to be less reliable than others, and we want to make sure that our parameter estimates will be less influenced by these than by the more accurate ones [12,34,38]. The solution to both of these problem is the same; assign a non-negative weight factor W_k to each $E_k(U)$, and minimize

$$\phi(U) = \sum_{k=1}^{n} W_k \cdot E_k^2(U) \tag{2-14}$$

We choose small W_k for y_k which are either measured on a large scale, or which are highly unreliable, and conversely for large W_k.

We derive the normal equation easily

$$\frac{\partial\phi}{\partial U} = 2 \sum_{k=1}^{n} W_k \cdot E_k(U) \cdot \frac{\partial E_k(U)}{\partial U} = 0 \; . \tag{2-15}$$

Substitute Equation (2-13) into Equation (2-15)

$$\frac{\partial\phi}{\partial U} = -2 \cdot \sum_{k=1}^{n} W_k \cdot (y_k - f(x_k,U)) \cdot \frac{\partial f(x_k, U)}{\partial U} = 0 \tag{2-16}$$

Assume the function of $f(x_k,U)$ which is dependent on U to be continuous.

CHAPTER 3

PROBLEM FORMULATION

3-1 PROBLEM DEFINITION

Figure 1 shows schematically the configuration of an unconfined aquifer and stream interaction system [40]. For a homogeneous and isotropic medium, if the curvature of the free surface is small, the Dupuit-Forchheimer assumptions [16] may be assumed to be valid. The frow θ through a unit width at a distance x and with a head h is [40]

$$\theta = k\ h\ \partial h/\partial x \tag{3-1}$$

in which k is the hydraulic conductivity of the aquifer.

The continuity equation may be expressed as

$$(\partial\theta/\partial x)\ dxdt = S'\ (\partial h/\partial t) \cdot dt \cdot dx \tag{3-2}$$

in which S' is the specific storage of the aquifer and t is the time.

Substituting Equation (3-1) into (3-2), one gets

$$k(\partial/\partial x)\ (h\ \partial h/\partial x) = S'\ \partial h/\partial t \tag{3-3}$$

subject to the following initial and boundary conditions.

$$\begin{aligned} h &= h(x) & 0 \le x \le L \quad & t = 0 \\ h &= h_0(t) & x = 0 \quad & t > 0 \\ \partial h/\partial x &= 0 & x = L \quad & t > 0 \end{aligned} \tag{3-4}$$

3-2 ANALYTICAL FORMULATION

Equation (3-3) can be rewritten in a more convenient form:

$$\partial h/\partial t = (k/S')(\partial/\partial x)\ (h\ \partial h/\partial x) \tag{3-5}$$

To make the head and distance dimensionless the following changes in variable are used:

$$\theta = h/H$$

$$y = x/L \tag{3-6}$$

$$\tau = (H/L^2)\ t$$

where H is the maximum height of the water table above the impervious layer, and is a known constant, and L also a known constant, is the distance from the river to the water divide.

Substituting these variables into Equation (3-5) yields

$$\partial\theta/\partial\tau = D\ (\partial/\partial y) \cdot (\theta\ \ \partial\theta/\partial y) \tag{3-7}$$

subject to

$$\begin{array}{lll} \theta = h(x)/H & 0 \leq y \leq 1 & \tau = 0 \\ \theta = h_0(t)/H & y = 0 & \tau > 0 \\ \partial\theta/\partial y = 0 & y = 1 & \tau > 0\ . \end{array} \tag{3-8}$$

where diffusivity $D = K/S'$

The dependent variable θ in Equation (3-7) is a function of two independent variable, y and τ. The governing equation can, however, be integrated by using a finite difference approximation. If one of the independent variable is discretized while the other is kept continuous, the derivative with respect to the discretized variable becomes readily available

[12]. The space variable y is discretized to replace Equation (3-7) by a system of nonlinear ordinary differential equations while the time variable τ is kept continuous. The distance y between 0 and 1 is divided into n equal intervals, where i = 1, 2, ..., n. The value of i = 0 and i = n correspond to the boundary conditions. To minimize the truncation error, the central difference method is used [12, 28]. The finite difference approximation of the Equation (3-7) is derived as follows:

For the subscripts of i + 1 and i, $\theta \frac{d\theta}{dy}$ can be expressed as $[\frac{\theta_{i+1} + \theta_i}{2}] \cdot \frac{\theta_{i+1} - \theta_i}{\Delta y}$; for the subscripts of i and i - 1, $\frac{\theta d\theta}{dy}$ can be expressed as $[\frac{\theta_{i-1} + \theta_i}{2}] \cdot \frac{\theta_i - \theta_{i-1}}{\Delta y}$. After these treatments, the expression of $\theta \cdot \frac{d\theta}{dy}$ is derivative with respect to y again as shown in Equation (3-7). That is, the difference between $[\frac{\theta_{i+1} + \theta_i}{2}] \cdot \frac{\theta_{i+1} - \theta_i}{\Delta y}$ and $[\frac{\theta_{i-1} + \theta_i}{2}] \cdot \frac{\theta_i - \theta_{i-1}}{\Delta y}$ is divided by Δy.

$$\frac{d\theta_i}{d\tau} = D \cdot [(\frac{\theta_{i+1} + \theta_i}{2}) \cdot (\frac{\theta_{i+1} - \theta_i}{\Delta y}) - (\frac{\theta_i + \theta_{i-1}}{2}) \cdot (\frac{\theta_i - \theta_{i-1}}{\Delta y})] / \Delta y$$

$$= D \cdot \{\frac{1}{\Delta y} \cdot [(\frac{\theta_{i+1}^2 - \theta_i^2}{2\Delta y} - \frac{\theta_i^2 - \theta_{i-1}^2}{2\Delta y}]\}$$

$$= D \cdot \{\frac{1}{\Delta y} \cdot [\frac{\theta_{i+1}^2 - 2\theta_i^2 + \theta_{i+1}^2}{2\Delta y}]\}$$

$$= D \cdot \frac{1}{2(\Delta y)^2} \cdot [\theta_{i+1}^2 - 2\theta_i^2 + \theta_{i-1}^2] \qquad (3\text{-}9)$$

$$i = 1,2,3, \ldots, (n-1)$$

which is subject to the following initial and boundary conditions.

$$\theta = h(x)/H \qquad 0 \leq y \leq 1 \qquad \tau = 0$$

$$\theta = h_0(t)/H \qquad y = 0 \qquad \tau > 0$$

$$\theta_n = \theta_{n-1} \qquad y = 1 \qquad \tau > 0$$

3-3 QUASILINEARIZATION

Because Equation (3-8) is nonlinear, it is hard to solve analytically [31,32]. We know that both initial-value and linear boundary-value problems in ordinary differential equations can be solved numerically in a fairly routine fashion on modern digital computers. Hence, if we can let the Equation (3-8) be replaced by a system of linear ordinary differential equations for which the technique of quasilinearization is applies, then the problem will be more simple in comparison to the original problem. The theoretical operations to decouple the nonlinear differential equation systems by quasilinearization will be the same as that mentioned in Section (2-2).

Let us define $\bar{X}$ as a vector which has components θ_1, θ_2, θ_3, θ_{n-1}, D, and $\bar{F}$ as the vector function which has the corresponding differential equations as its components.

The n equations can be written as

$$\frac{d\bar{X}}{d\tau} = f(\bar{X},\tau) \tag{3-10}$$

$$f(\bar{X},\tau) = D \cdot \frac{1}{2(\Delta y)^2} \cdot [\theta_{i+1}^2 - 2\theta_i^2 + \theta_{i-1}^2] , \quad i = 1, 2, 3, \ldots, n-1 \tag{3-11}$$

$$f(\bar{X},\tau) = 0, \; i = n$$

The linearized form of Equation (3-10) written in the recurrence relation is

$$\frac{d\bar{X}_{n+1}}{d\tau} = \bar{f}_n + J(\bar{X}_n)[\bar{X}_{n+1} - \bar{X}_n]$$

or

$$\frac{d\bar{X}}{d\tau} = f(\bar{X}_0,\tau) + J(\bar{X}_0)(\bar{X} - \bar{X}_0) \qquad (3\text{-}12)$$

The Jacobi matrix $J(\bar{X}_0)$ is defined by

$$J(\bar{X}_0)=\begin{pmatrix} \frac{\partial f_1}{\partial\theta_1^0} & \frac{\partial f_1}{\partial\theta_2^0} & \frac{\partial f_1}{\partial\theta_3^0} & \cdots & \frac{\partial f_1}{\partial\theta_{n-1}^0} & \frac{\partial f_1}{\partial D^0} \\ \frac{\partial f_2}{\partial\theta_1^0} & \frac{\partial f_2}{\partial\theta_2^0} & \frac{\partial f_2}{\partial\theta_3^0} & \cdots & \frac{\partial f_2}{\partial\theta_{n-1}^0} & \frac{\partial f_2}{\partial D^0} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{\partial f_{n-1}}{\partial\theta_1^0} & \frac{\partial f_{n-1}}{\partial\theta_2^0} & \frac{\partial f_{n-1}}{\partial\theta_3^0} & & \frac{\partial f_{n-1}}{\partial\theta_{n-1}^0} & \frac{\partial f_{n-1}}{\partial D^0} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}_{n\text{x}n} \qquad (3\text{-}13)$$

The components of $f(\bar{X}_0, \tau)$ are obtained by substituing the known value of θ_i^0 and D^0 into Equation (3-11). One gets

$$D^0 \cdot \frac{1}{2(\Delta y)^2} \cdot [\theta_{i+1}^{0^2} - 2\theta_i^{0^2} + \theta_{i-1}^{0^2}]$$

$i = 1, 2, 3, \ldots, n-1$

AND

0 when $i = n$

The components of $\bar{X} - \bar{X}_0$ are

$$[\bar{X} - \bar{X}_0] = \begin{bmatrix} \theta_1^1 - \theta_1^0 \\ \theta_2^1 - \theta_2^0 \\ \theta_3^1 - \theta_3^0 \\ \vdots \\ \theta_{n-2}^1 - \theta_{n-2}^0 \\ \theta_{n-1}^1 - \theta_{n-1}^0 \\ D^1 - D^0 \end{bmatrix}_{n\mathrm{x}1} \tag{3-14}$$

In which the superscript 1 represents the current approximations and 0 the previous approximations.

The components of the Jacobi matrix $J(\bar{X}_0)$ are listed in the following:

For the first row

$$\frac{\partial f_1}{\partial \theta_1^0} = - 4\theta_1^0 \cdot \frac{D^0}{2(\Delta y)^2}$$

$$\frac{\partial f_1}{\partial \theta_2^0} = 2\theta_2^0 \cdot \frac{D^0}{2(\Delta y)^2}$$

$$\frac{\partial f_1}{\partial \theta_3^0} = 0$$

$$\frac{\partial f_1}{\partial \theta_4^0} = 0$$

$$\frac{\partial f_1}{\partial \theta_5^0} = 0$$

$$\vdots$$

$$\frac{\partial f_1}{\partial \theta_{n-1}^0} = 0$$

$$\frac{\partial f_1}{\partial D^0} = \frac{1}{2(\Delta y)^2} \cdot (\theta_2^{0^2} - 2\theta_1^{0^2} + \theta_2^{0^2})$$

The second row:

$$\frac{\partial f_2}{\partial \theta_1^0} = \frac{D_0}{2(\Delta y)^2} \cdot (2\theta_1^0)$$

$$\frac{\partial f_2}{\partial \theta_2^0} = \frac{D_0}{2(\Delta y)^2} \cdot (-4\theta_2^0)$$

$$\frac{\partial f_2}{\partial \theta_3^0} = \frac{D_0}{2(\Delta y)^2} \cdot (2\theta_3^0)$$

$$\frac{\partial f^2}{\partial \theta_4^0} = 0$$

$$\frac{\partial f^2}{\partial \theta_5^0} = 0$$

$$\vdots$$

$$\frac{\partial f_2}{\partial \theta^0_{n-1}} = 0$$

$$\frac{\partial f_2}{\partial D^0} = \frac{1}{2(\Delta y)^2} \cdot (\theta_3^{0^2} - 2\theta_2^{0^2} + \theta_1^{0^2})$$

The third row

$$\frac{\partial f_3}{\partial \theta^0_1} = 0$$

$$\frac{\partial f_3}{\partial \theta^0_2} = \frac{D^0}{2(\Delta y)^2} \cdot (2\theta^0_2)$$

$$\frac{\partial f_3}{\partial \theta^0_3} = \frac{D^0}{2(\Delta y)^2} \cdot (-4\theta^0_3)$$

$$\frac{\partial f_3}{\partial \theta^0_4} = \frac{D_0}{2(\Delta y)^2} \cdot (2\theta^0_4)$$

$$\vdots$$

$$\frac{\partial f_3}{\partial \theta^0_{n-1}} = 0$$

$$\frac{\partial f_3}{\partial D^0} = \frac{1}{2(\Delta y)^2} \cdot (\theta_4^{0^2} - 2\theta_3^{0^2} + \theta_2^{0^2})$$

The (n-3)'th row

$$\frac{\partial f_{n-3}}{\partial \theta_1^0} = 0$$

$$\frac{\partial f_{n-3}}{\partial \theta_2^0} = 0$$

$$\frac{\partial f_{n-3}}{\partial \theta_3^0} = 0$$

$$\vdots$$

$$\frac{\partial f_{n-3}}{\partial \theta_{n-4}^0} = \frac{D_0}{2(\Delta y)^2} \cdot (2\theta_{n-4}^0)$$

$$\frac{\partial f_{n-3}}{\partial \theta_{n-3}^0} = \frac{D_0}{2(\Delta y)^2} \cdot (-4\ \theta_{n-3}^0)$$

$$\frac{\partial f_{n-3}}{\partial \theta_{n-2}^0} = \frac{D_0}{2(\Delta y)^2} \cdot 2\theta_{n-2}^0$$

$$\frac{\partial f_{n-3}}{\partial \theta_{n-1}^0} = 0$$

$$\frac{\partial f_{n-3}}{\partial D^0} = \frac{1}{2(\Delta y)^2} \cdot (\theta_{n-2}^2 - 2\theta_{n-3}^2 + \theta_{n-4}^2)$$

The (n-2)'th row

$$\frac{\partial f_{n-2}}{\partial \theta_1^0} = 0$$

$$\frac{\partial f_{n-2}}{\partial \theta_2^0} = 0$$

$$\vdots$$

$$\frac{\partial f_{n-2}}{\partial \theta_{n-3}^0} = \frac{D^0}{2(\Delta y)^2} \cdot 2\theta_{n-3}^0$$

$$\frac{\partial f_{n-2}}{\partial \theta_{n-2}^0} = \frac{D^0}{2(\Delta y)^2} \cdot (-4\theta_{n-2}^0)$$

$$\frac{\partial f_{n-2}}{\partial \theta_{n-1}^0} = \frac{D_0}{2(\Delta y)^2} \cdot (2\theta_{n-1}^0)$$

$$\frac{\partial f_{n-1}}{\partial D^0} = \frac{1}{2(\Delta y)^2} \cdot (\theta_{n-1}^2 - 2\theta_{n-2}^2 + \theta_{n-3}^2)$$

The (n-1)'th row

$$\frac{\partial f_{n-1}}{\partial \theta_1} = 0$$

$$\frac{\partial f_{n-1}}{\partial \theta_2} = 0$$

$$\frac{\partial f_{n-1}}{\partial \theta_3} = 0$$

$$\vdots$$

$$\frac{\partial f_{n-1}}{\partial \theta_{n-2}} = \frac{D_0}{2(\Delta y)^2} \cdot (2\theta^0_{n-2})$$

$$\frac{\partial f_{n-1}}{\partial \theta_{n-1}} = \frac{D_0}{2(\Delta y)^2} \cdot (-4\theta^0_{n-1})$$

$$\frac{\partial f_{n-1}}{\partial D^0} = \frac{1}{2(\Delta y)^2} \cdot (\theta^2_n - 2\theta^2_{n-1} + \theta^2_{n-2})$$

The n'th row

$$\frac{\partial f_n}{\partial \theta_1} = 0$$

$$\frac{\partial f_n}{\partial \theta_2} = 0$$

$$\frac{\partial f_n}{\partial \theta_3} = 0$$

$$\vdots$$

$$\frac{\partial f_n}{\partial \theta_{n-1}} = 0$$

The linearized forms of Equation (3-11) are represented as shown in Equation (3-12). One gets,

$$\frac{d\theta_i}{d\tau} = D^0 \cdot \frac{1}{2(\Delta y)^2} \cdot (\theta^{0^2}_{i+1} - 2\theta^{0^2}_{i} + \theta^{0^2}_{i-1}) + (\theta^1_i$$

$$- \theta^0_i) \cdot \left(\frac{D_0}{2(\Delta y)^2}\right) \cdot (-4\theta^0_i) + (\theta^1_{i+1} - \theta^0_{i+1}) \cdot$$

$$\left(\frac{D_0}{2(\Delta y)^2} \cdot (2\theta^0_{i+1})\right) + (\theta^1_{i-1} - \theta^0_{i-1}) \cdot \left(\frac{D_0}{2(\Delta y)^2}\right.$$

$$\left.(2\theta^0_{i-1})\right) \cdot \delta + (D^1 - D^0) \cdot \left(\frac{1}{2(\Delta y)^2} \cdot (\theta^0_{i+1} -\right.$$

$$\left.2\theta^{0^2}_{i} + \theta^{0^2}_{i-1})\right) \tag{3-15}$$

$i = 1,2,3, \ldots, (n-2),$

$\delta = 0 \qquad \text{for } i = 1$

$\delta = 1 \qquad \text{for } i \neq 1$

and

$$\frac{d\theta^1_{n-1}}{d\tau} = \frac{D_0}{2(\Delta y)^2} \cdot (-\theta^{0^2}_{n-1} + \theta^{0^2}_{n-2}) + (\theta^1_{n-1} - \theta^0_{n-1}) \cdot$$

$$\left(\frac{D_0}{2(\Delta y)^2} \cdot (-2\theta^0_{n-1})\right) + (\theta^1_{n-2} - \theta^0_{n-2}) \cdot \left(\frac{D_0}{2(\Delta y)^2} \cdot\right.$$

$$\left.(2\theta_{n-2}^{0})\right] + (D^{1} - D^{0}) \cdot \left(\frac{-\theta_{n-1}^{0^2} + \theta_{n-2}^{0^2}}{2(\Delta y)^2}\right) \tag{3-16}$$

The Equation (3-15) and (3-16) are subject to:

$$\theta^{1} = \frac{h(x)}{H} \qquad 0 \le y \le 1 \qquad \tau = 0$$

$$\theta^{1} = \frac{h_0(t)}{H} \qquad y = 0 \qquad \tau > 0$$

$$\theta_n^{1} = \theta_{n-1}^{1} \qquad y = 1 \qquad \tau > 0 \tag{3-17}$$

In which the superscript 1 represents the current approximations and 0 the previous approximations. The method of complementary function is used to obtain the general solution and requires only the previous estimates of D^0 and the solutions of θ_i^0.

3-4 THE SOLUTION OF THE PROBLEM

The quasilinearization technique linearizes the nonlinear equation and solved it as an initial value problem. Thus, Equations (3-15) and (3-16) are both nonhomogeneous linear differential equations. As mentioned in section (2-4), when the problems of nonhomogeneous terms are present, we can let these nonhomogeneous terms be expressed as sums of a finite number of functions, which are represented by p and q in this problem. There are many combinations of p and q; they must fulfill the only requirment that each sum should be equal to each original term. But only on them satisfies the situation that parameter D appears in the solution in an explicit manner. In other words, the general solution is represented by the linear combination of p and q, in which p is multiplies by constant D to give

$$\theta = D \cdot (p + q)$$

If one rearranges the right-hand side of Equations (3-15) and (3-16), in sequence, one gets,

$$\frac{d\theta_i^1}{d\tau} = D^0 \cdot \frac{1}{2(\Delta y)^2} \cdot (\theta_{i+1}^{0^2} - 2\theta_i^{0^2} + \theta_{i-1}^{0^2}) +$$

$$(\theta_i^1 - \theta_i^0) \cdot \left[\frac{D_0}{2(\Delta y)^2} \cdot (-4\theta_i^0)\right] +$$

$$(\theta_{i+1}^1 - \theta_{i+1}^0) \cdot \left[\frac{D_0}{2(\Delta y)^2} \cdot (2\theta_i^0 + 1)\right] +$$

$$(\theta_{i-1}^1 - \theta_{i-1}^0) \cdot \left[\frac{D_0}{2(\Delta y)^2} \cdot (2\theta_{i-1}^0)\right] \cdot \delta +$$

$$D^1 \cdot \left[\frac{1}{2(\Delta y)^2} \cdot (\theta_{i+1}^{0^2} - 2\theta_i^{0^2} + \theta_{i-1}^{0^2})\right] -$$

$$D^0 \cdot \left[\frac{1}{2(\Delta y)^2} \cdot (\theta_{i+1}^{0^2} - 2\theta_i^{0^2} + \theta_{i-1}^{0^2})\right]$$

In this way, we consider D as a new variable which can be improved as iterations continue.

The iterm of $D^0 \cdot \frac{1}{2(\Delta y)^2} \cdot (\theta^{0^2}_{i+1} - 2\theta^{0^2}_i + \theta^{0^2}_{i-1})$ is eleminated.

$$\frac{d\theta^1_i}{d\tau} = \theta^1_i \cdot \left[\frac{D_0}{2(\Delta y)^2} \cdot (-4\theta^0_i)\right] - \theta^0_i \cdot \left[\frac{D_0}{2(\Delta y)^2} \cdot\right.$$

$$\left.(-4\theta^0_i)\right] + \theta^1_{i+1} \cdot \left[\frac{D_0}{2(\Delta y)^2} \cdot (2\theta^0_{i+1})\right] - \theta^0_{i+1}$$

$$\cdot \left[\frac{D_0}{2(\Delta y)^2} \cdot (2\theta^0_{i+1})\right] + \theta^1_{i-1} \cdot \left[\frac{D_0}{2(\Delta y)^2} \cdot (2\theta^0_{i-1})\right]$$

$$- \theta^0_{i-1} \cdot \left[\frac{D_0}{2(\Delta y)^2} \cdot (2\theta^0_{i-1})\right] \cdot \delta + D^1 \cdot \left[\frac{1}{2(\Delta y)^2} \cdot\right.$$

$$\left.(\theta^{0^2}_{i+1} - 2\theta^{0^2}_i + \theta^{0^2}_{i-1})\right] \tag{3-19}$$

$i = 1,2,3, \ldots, n-2$

$$\frac{d\theta^1_i}{d\tau} - \left[\frac{D_0}{2(\Delta y)^2}\right] \cdot (-4\theta^0_i) \cdot \theta^1_i -$$

$$\frac{D_0}{2(\Delta y)^2} \cdot (2\theta^0_{i+1}) \cdot \theta^1_{i+1} - \left[\frac{D_0}{2(\Delta y)^2} \cdot (2\theta^0_{i-1})\right] \cdot \delta \cdot$$

$$\theta^1_{i-1} = - \theta^0_i \cdot \left[\frac{D_0}{2(\Delta y)^2} (-4\theta^0_i)\right] - \theta^0_{i+1} \cdot \left[\frac{D_0}{2(\Delta y)^2} \cdot\right.$$

$$(2\theta^0_{i+1})\Big] - \theta^0_{i-1} \cdot \left[\frac{D_0}{2(\Delta y)^2} \cdot (2\theta^0_{i-1})\right] \cdot \delta + D^1 \cdot \left[\frac{1}{2(\Delta y)^2} \cdot\right.$$

$$\left.\theta^{0^2}_{i+1} - 2\theta^{0^2}_i + \theta^{0^2}_{i-1}\right) \quad . \tag{3-20}$$

$i = 1,2,3, \ldots, n-2$

Quasilinearization involves decoupling the system of differential equations by linearization into a series of initial value problems. Depending on the arguments which are mentioned in the beginning of this section, we can arrange the right hand side in such a manner that parameter D can appear in the general solution. Let g_1 represent the first part of the right hand side and g_2 represent the second part of the right hand side. That is

$$L(x) = g_1(t) + g_2(t) \tag{3-21}$$

Each part is corresponding the particular solution of p and q.
From Equation (3-20)

$$g_1 = D^1 \cdot \left[\frac{1}{2(\Delta y)^2} \cdot (\theta^{0^2}_{i+1} - 2\theta^{0^2}_i + \theta^{0^2}_{i-1})\right] \tag{3-22}$$

$$g_2 = -\theta^0_i \cdot \left[\frac{D^0}{2(\Delta y)^2} \cdot (-4\theta^0_i)\right] - \theta^0_{i+1} \cdot$$

$$\frac{D^0}{2(\Delta y)^2} \cdot (2\theta^0_{i+1}) - \theta^0_{i-1} \cdot \frac{D^0}{2(\Delta y)^2} \cdot$$

$$(2\theta^0_{i-1}) \cdot \delta \tag{3-23}$$

Under the assumption of linearity, the Equation (3-21) can be decoupled into

$$L(x) = g_1(t) \tag{3-24}$$

$$L(x) = g_2(t)$$

Therefore, Equation (3-20) can be written

$$L = \frac{d\theta_i^1}{d\tau} - \frac{D_0}{2(\Delta y)^2} \cdot (-4\theta_i^0) \cdot \theta_i^1 - \frac{D^0}{2(\Delta y)^2} \cdot$$

$$2\theta_{i+1}^0) \cdot \theta_{i+1}^1 - \left(\frac{D^0}{2(\Delta y)^2} \cdot (2\theta_{i-1}^0)\right) \cdot \delta \cdot \theta_{i-1}^1 =$$

$$= g_1 = D^1 \cdot \left(\frac{1}{2(\Delta y)^2} \cdot (\theta_{i+1}^{0^2} - 2\theta_i^{0^2} + \theta_{i-1}^{0^2})\right) \tag{3-25}$$

Let Equation (3-25) be divided by D^1 for both sides.

$$\frac{1}{D^1} \cdot \frac{d\theta_i^1}{d\tau} - \frac{1}{D^1} \cdot \frac{D^0}{2(\Delta y)^2} \cdot (-4\theta_i^0) \cdot \theta_i^1 - \frac{1}{D^1} \cdot$$

$$\left(\frac{D^0}{2(\Delta y)^2} \cdot (2\theta_{i+1}^0)\right) \cdot \theta_{i+1}^1 - \frac{1}{D^1} \cdot \left(\frac{D_0}{2(\Delta y)^2} \cdot\right.$$

$$(2\theta_{i-1}^0) \cdot \delta \cdot \theta_{i-1}^1 = \left(\frac{1}{2(\Delta y)^2} \cdot (\theta_{i+1}^{0^2} - 2\theta_i^{0^2} +\right.$$

$$\left.\theta_{i-1}^{0^2})\right) \tag{3-26}$$

$$\frac{d\left(\frac{\theta_i^1}{D^1}\right)}{d\tau} - \left[\frac{D^0}{2(\Delta y)^2} \cdot (-4\theta_i^0)\right] \cdot \left(\frac{\theta_i^1}{D^1}\right) - \left[\frac{D^0}{2(\Delta y)^2} \cdot\right.$$

$$\left.(2\theta_{i+1}^0)\right] \cdot \left(\frac{\theta_{i+1}^1}{D^1}\right) - \left[\frac{D_0}{2(\Delta y)^2} \cdot (2\theta_{i-1}^0)\right] \cdot \delta \cdot$$

$$\left(\frac{\theta_{i-1}^1}{D^1}\right) = \left[\frac{1}{2(\Delta y)^2} \cdot (\theta_{i+1}^{0^2} - 2\theta_i^{0^2} + \theta_{i-1}^{0^2})\right] \tag{3-27}$$

Let a new variable $SITA_i^1$ be introduced which is equal to $\frac{\theta_i^1}{D^1}$.

$i = 1,2, \ldots, n-2$

Substitute $SITA_i^1 = \frac{\theta_i^1}{D^1}$ into Equation (3-27).

$$\frac{d(SITA_i^1)}{d\tau} - \left[\frac{D^0}{2(\Delta y)^2} \cdot (-4\theta_i^0)\right] \cdot (SITA_i^1)$$

$$- \left[\frac{D^0}{2(\Delta y)^2} \cdot (2\theta_{i+1}^0)\right] \cdot (SITA_{i+1}^1) - \left[\frac{D^0}{2(\Delta y)^2} \cdot\right.$$

$$\left.(2\theta_{i-1}^0)\right] \cdot \delta \cdot (SITA_{i-1}^1) = \frac{1}{2(\Delta y)^2} \cdot (\theta_{i+1}^{0^2} -$$

$$2\theta_i^{0^2} + \theta_{i-1}^{0^2}) \tag{3-28}$$

The superscript 1 represents the current approximations and 0 the previous approximations. In addition, SITA $\frac{\theta_i^1}{D^1}$.

Suppose there exists one particular solution p which satisfies Equation (3-28). Hence, the iterative equation for p can be written in a similar form as Equation (3-28). That is

$$\frac{dp_i}{d\tau} - p_i \cdot \left[\frac{D_0}{2(\Delta y)^2} \cdot (-4\theta_i^0)\right] - p_{i+1} \cdot$$

$$\left[\frac{D^0}{2(\Delta y)^2} \cdot (2\theta_{i+1}^0)\right] - p_{i-1} \cdot \left[\frac{D^0}{2(\Delta y)^2} \cdot (2\theta_{i-1}^0)\right] \cdot \delta$$

$$= \frac{1}{2(\Delta y)^2} \cdot (\theta_{i+1}^{0^2} - 2\theta_i^{0^2} + \theta_{i-1}^{0^2}) \qquad (3\text{-}29)$$

$$\frac{dP_i}{d\tau} = p_i \cdot \left[\frac{D^0}{2(\Delta y)^2} \cdot (-4\theta_i^0)\right] + P_{i+1} \cdot \left[\frac{D^0}{2(\Delta y)^2} \cdot\right.$$

$$\left.(2\theta_{i+1}^0)\right] + p_{i-1} \cdot \left[\frac{D^0}{2(\Delta y)^2} \cdot (2\theta_{i-1}^0)\right] \cdot \delta + \frac{1}{2(\Delta y)^2} \cdot$$

$$(\theta_{i+1}^{0^2} - 2\theta_i^{0^2} + \theta_{i-1}^{0^2}) \qquad (3\text{-}30)$$

$i = 1,2,3, \ldots, (n-2)$

$\delta = 0, \quad \text{for } i = 1$

$\delta = 1, \quad \text{for } i \neq 1$

For the second part $L = g_2$

$$L = \frac{d\theta_i^1}{d\tau} - \left(\frac{D^0}{2(\Delta y)^2} \cdot (-4\theta_i^0)\right) \cdot \theta_i^1 - \left(\frac{D^0}{2(\ y)^2} \cdot\right.$$

$$\left.(2\theta_{i+1}^0)\right) \cdot \theta_{i+1}^1 - \left(\frac{D^0}{2(\Delta y)^2} \cdot (2\theta_{i-1}^0)\right) \cdot \delta \cdot$$

$$\theta_{i-1}^1 = - \theta_i^0 \cdot \left(\frac{D^0}{2(\Delta y)^2} \cdot (-4\theta_i^0)\right) - \theta_{i+1}^0 \cdot$$

$$\left(\frac{D^0}{2(\Delta y)^2} \cdot (2\theta_{i+1}^0)\right) - \theta_{i-1}^0 \cdot \left(\frac{D^0}{2(\Delta y)^2} \cdot\right.$$

$$\left.(2\theta_{i-1}^0)\right) \cdot \delta \tag{3-31}$$

Suppose there exists one particular solution of q which satisfies Equation (3-31). That is the term of θ in Equation (3-31) can be replaced by q. The equations are:

$$\frac{dq_i}{d\tau} - q_i \cdot \left(\frac{D^0}{2(\Delta y)^2} \cdot (-4\theta_i^0)\right) - q_{i+1} \cdot \frac{D^0}{2(\Delta y)^2}$$

$$\cdot\ (2\theta_{i+1}^0) - q_{i-1} \cdot \left(\frac{D^0}{2(\Delta y)^2} \cdot (2\theta_{i-1}^0)\right) \cdot \delta$$

$$= - \theta_i^0 \cdot \left[\frac{D^0}{2(\Delta y)^2} \cdot (-4\theta_i^0)\right] - \theta_{i+1}^0 \cdot \left[\frac{D^0}{2(\Delta y)^2} \cdot (2\theta_{i+1}^0)\right] - \theta_{i-1}^0$$

$$\cdot \left[\frac{D^0}{2(\Delta y)^2} \cdot (2\theta_{i-1}^0)\right] \cdot \delta$$

or

$$\frac{dq_i}{d\tau} = (q_i - \theta_i^0) \cdot \left[\frac{D^0}{2(\Delta y)^2} \cdot (-4\theta_i^0)\right] +$$

$$(q_{i+1} - \theta_{i+1}^0) \cdot \left[\frac{D^0}{2(\Delta y)^2} \cdot (2\theta_{i+1}^0)\right] +$$

$$(q_{i-1} - \theta_{i-1}^0) \cdot \left[\frac{D^0}{2(\Delta y)^2} \cdot (2\theta_{i-1}^0)\right] \cdot \delta \tag{3-32}$$

$i = 1,2,3, \ldots, (n-2)$

$\delta = 0$ for $i = 1$

$\delta = 1$ for $i \neq 1$

The above discussion only consider that i lies between 1 and n-2. The (n-1)'th term will be considered in the following statements. From Equation (3-16).

$$\frac{d\theta_{n-1}^1}{d\tau} - \theta_{n-1}^1 \cdot \frac{D^0}{2(\Delta y)^2} \cdot (-2\theta_{n-1}^0) - \theta_{n-2}^1 \cdot$$

$$\left[\frac{D^0}{2(\Delta y)^2} \cdot (2\theta^0_{n-2})\right] = -\,\theta^0_{n-1} \cdot \left[\frac{D^0}{2(\Delta y)^2} \cdot\right.$$

$$\left.(-2\theta^0_{n-1})\right] - \theta^0_{n-2} \cdot \left[\frac{D^0}{2(\Delta y)^2} \cdot (2\theta^0_{n-2})\right] +$$

$$D^1 \cdot \left[\frac{-\theta^{0^2}_{n-1} + \theta^{0^2}_{n-2}}{2(\Delta y)^2}\right] \tag{3-33}$$

The arguments for decoupling right hand side into two parts of g_1 and g_2 will be the same as before.

Let $g_1 = D^1 \cdot \left[\dfrac{-\theta^{0^2}_{n-1} + \theta^{0^2}_{n-2}}{2(\Delta y)^2}\right]$

$$g_2 = -\theta^0_{n-1} \cdot \left[\frac{D^0}{2(\Delta y)^2} \cdot (-2\theta^0_{n-1})\right]$$

$$-\theta^0_{n-2} \cdot \left[\frac{D^0}{2(\Delta y)^2} \cdot (2\theta^0_{n-2})\right]$$

For the first part.

$L = g_1$

or

$$\frac{d\theta^1_{n-1}}{d\tau} - \theta^1_{n-1} \cdot \frac{D^0}{2(\Delta y)^2} \cdot (-2\theta^1_{n-1}) - \theta^1_{n-2} \cdot$$

$$\left(\frac{D^0}{2(\Delta y)^2} \cdot (2\theta^0_{n-2})\right) = D^1 \cdot \left(\frac{-\theta^{0^2}_{n-1} + \theta^{0^2}_{n-2}}{2(\Delta y)^2}\right) \tag{3-34}$$

Equation (3-34) can be divided by D^1.

$$\frac{1}{D^1} \cdot \frac{d\theta^1_{n-1}}{d\tau} - \frac{\theta^1_{n-1}}{D^1} \cdot \left(\frac{D^0}{2(\Delta y)^2} \cdot (-2\theta^0_{n-1})\right) -$$

$$\frac{1}{D^1} \cdot \theta^1_{n-2} \cdot \left(\frac{D^0}{2(\Delta y)^2} \cdot (2\theta^0_{n-2})\right) =$$

$$\left(\frac{-\theta^{0^2}_{n-1} + \theta^{0^2}_{n-2}}{2(\Delta y)^2}\right) \tag{3-35}$$

Let $SITA^1_{n-1} = \theta^1_{n-1}/D^1$

$$\frac{dSITA^1_{n-1}}{d\tau} - SITA^1_{n-1} \cdot \left(\frac{D^0}{2(\Delta y)^2} \cdot (-2\theta^0_{n-1})\right)$$

$$- SITA^1_{n-2} \cdot \left(\frac{D^0}{2(\Delta y)^2} \cdot (2\theta^0_{n-2})\right) = \left(\frac{-\theta^{0^2}_{n-1} + \theta^{0^2}_{n-2}}{2(\Delta y)^2}\right)$$

Suppose there exists one particular solution of P which satisfies the above equation.

$$\frac{dP_{n-1}}{d\tau} - P_{n-1} \cdot \left(\frac{D^0}{2(\Delta y)^2} \cdot (-2\theta^0_{n-1})\right) -$$

$$P_{n-2} \cdot \left(\frac{D^0}{2(\Delta y)^2} \cdot (2\theta^0_{n-2})\right) = \left(\frac{-\theta^{0^2}_{n-1} + \theta^{0^2}_{n-2}}{2(\Delta y)^2}\right)$$

$$\frac{dP_{n-1}}{d\tau} = P_{n-1} \cdot \left(\frac{D^0}{2(\Delta y)^2} \cdot (-2\theta^0_{n-1})\right) + P_{n-2} \cdot$$

$$\left(\frac{D^0}{2(\Delta y)^2} \cdot (2\theta^0_{n-2})\right) + \left(\frac{-\theta^{0^2}_{n-1} + \theta^{0^2}_{n-2}}{2(\Delta y)^2}\right) \tag{3-36}$$

For the second part

$$L = g_2$$

$$\frac{d\theta^1_{n-1}}{d\tau} - \theta^1_{n-1} \cdot \left(\frac{D^0}{2(\Delta y)^2} \cdot (-2\theta^0_{n-1})\right) -$$

$$\theta^1_{n-2} \cdot \left(\frac{D^0}{2(\Delta y)^2} \cdot (2\theta^0_{n-2})\right) = - \theta^0_{n-1} \cdot$$

$$\left(\frac{D^0}{2(\Delta y)^2} \cdot (-2\theta^0_{n-1})\right) - \theta^0_{n-2} \cdot \left(\frac{D^0}{2(\Delta y)^2} \cdot\right.$$

$$\left.(2\theta^0_{n-2})\right) \tag{3-37}$$

Suppose there exists one particular solution of q which satisfies the Equation (3-36). Let θ_i' in Equation (3-36) be replaced by q_i. Hence,

$$\frac{dq_{n-1}}{d\tau} - q_{n-1} \cdot \left[\frac{D^0}{2(\Delta y)^2} \cdot (-2\theta^0_{n-1})\right] -$$

$$q_{n-2} \cdot \left[\frac{D^0}{2\cdot(\Delta y)^2} \cdot (2\theta^0_{n-2})\right] = -\theta^0_{n-1} \cdot$$

$$\left[\frac{D^0}{2\cdot(\Delta y)^2} \cdot (-2\theta^0_{n-1})\right] - \theta^0_{n-2} \cdot \left[\frac{D^0}{2(\Delta y)^2} \cdot (2\theta^0_{n-2})\right]$$

$$\frac{dq_{n-1}}{d\tau} = q_{n-1} \cdot \left[\frac{D^0}{2.(\Delta y)^2} \cdot (-2\theta^0_{n-1})\right] +$$

$$q_{n-2} \cdot \left[\frac{D^0}{2\cdot(\Delta y)^2} \cdot (2\cdot\theta^0_{n-2})\right] - \theta^0_{n-1} \cdot$$

$$\left[\frac{D^0}{2(\Delta y)^2} \cdot (-2\theta^0_{n-1})\right] - \theta^0_{n-2} \cdot \left[\frac{D^0}{2(\Delta y)^2} \cdot\right.$$

$$\left.(2\theta^0_{n-2})\right] \tag{3-38}$$

The boundary conditions for the particular solution of P and q will be the same as the original differential equation. Otherwise, it will be a different problem if the boundary conditions are changed. That is, the boundary conditions for P and q are the same as Equation (3-9).

For the particular solution of P, it is subject to

$$P = h_0(t)/H \qquad y = 0 \qquad \tau > 0$$
$$P_n = P_{n-1} \qquad y = 1 \qquad \tau > 0 \tag{3-39}$$

and

For the particular solution of q, it is subject to

$$q = h_0(t)/\ H \qquad y = 0 \qquad \tau > \ 0 \tag{3-40}$$

$$q_n = q_{n-1} \qquad y = 1 \qquad \tau > \ 0$$

The initial conditions can be set to any value at our convenience. However, one factor that we should consider before setting an initial value to the particular solution of P and q is that the particular solution of P has already been divided by D' as shown in equation (3-26) and (3-35). Hence, the sum of D times the initial value of p and the initial value of q should be equal to $\frac{h(x)}{H}$. If we set the initial value of P to be zero, then the initial value of q will always be $\frac{h(x)}{H}$. In that way, it will not be necessary to calculate each time. The initial conditions for P and q are

$$P = 0 \qquad 0 \leq y \leq 1 \qquad \tau = 0 \tag{3-41}$$

and

$$q = h(x)/H \qquad 0 \leq y \leq 1 \qquad \tau = 0 \tag{3-42}$$

From the above statements, we can make a summary as:

For the particular solution of P

$$\frac{dP_i}{d\tau} = P_i \cdot \left(\frac{D^0}{2(\Delta y)^2} \cdot (-4\theta_i^0)\right) + P_{i+1} \cdot \left(\frac{D^0}{2(\Delta y)^2} \cdot\right.$$

$$\left.(2\theta_{i+1}^0)\right) + P_{i-1} \cdot \left(\frac{D^0}{2(\Delta y)^2} \cdot (2\theta_{i-1}^0)\right) \cdot \delta$$

$$+ \frac{1}{2(\Delta y)^2} (\theta_{i+1}^{0^2} - 2\theta_i^{0^2} + \theta_{i-1}^{0^2}) \qquad (3\text{-}43)$$

$i = 1,2,3, \ldots, (n-2)$

$g = 0 \qquad \text{for } i = 1$

$g = 1 \qquad \text{for } i \neq 1$

and when $i = n-1$

$$\frac{dP_{n-1}}{d\tau} = P_{n-1} \cdot \left(\frac{D^0}{2\cdot(\Delta y)^2} \cdot (-2\theta_{n-1}^0)\right) + P_{n-2} \cdot$$

$$\left(\frac{D^0}{2\cdot(\Delta r)^2} \cdot (2\theta_{n-2}^0)\right\} + \frac{-\theta_{n-1}^{0^2} + \theta_{n-2}^{0^2}}{2\cdot(\Delta y)^2} \qquad (3\text{-}44)$$

subject to

$$P = 0 \qquad 0 \leq y \leq 1 \qquad \tau = 0$$

$$P = h_0(t)/H \qquad y = 0 \qquad \tau > 0$$

$$P_n = P_{n-1} \qquad y = 1 \qquad \tau > 0 \qquad (3\text{-}45)$$

For the particular solution of q

$$\frac{dq_i}{d\tau} = (q_i - \theta_i^0) \cdot \left[\frac{D^0}{2\cdot(\Delta y)^2} \cdot (-4\theta_i^0)\right]$$

$$+ (q_{i+1} - \theta_{i+1}^0) \cdot \left[\frac{D^0}{2\cdot(\Delta y)^2} \cdot (2\theta_{i+1}^0)\right] +$$

$$(q_{i-1} - \theta_{i-1}^0) \cdot \frac{D^0}{2\cdot(\Delta y)^2} \cdot (2\theta_{i-1}^0) \cdot \delta \qquad (3\text{-}46)$$

$i = 1,2,3, \ldots, (n-2)$

$\delta = 0$ for $i = 1$

$\delta = 1$ for $i \neq 1$

and when $i = n - 1$

$$\frac{dq_{n-1}}{d\tau} = q_{n-1} \cdot \left[\frac{D^0}{2\cdot(\Delta y)^2} \cdot (-2\theta_{n-1}^0)\right] + q_{n-2} \cdot$$

$$\left[\frac{D^0}{2\cdot(\Delta y)^2} \cdot (2\cdot\theta_{n-2}^0)\right] - \theta_{n-1}^0 \cdot \left[\frac{D^0}{2(\Delta y)^2} \cdot\right.$$

$$\left.(-2\theta_{n-1}^0)\right] - \theta_{n-2}^0 \cdot \left[\frac{D^0}{2(\Delta y)^2} \cdot (2\theta_{n-2}^0)\right] \qquad (3\text{-}47)$$

Subject to

$$q = h(x)/H \qquad 0 \leq y \leq 1 \qquad \tau = 0$$

$$q = h_0(t)/H \qquad y = 0 \qquad \tau > 0 \tag{3-48}$$

$$q_n = q_{n-1} \qquad y = 1 \qquad \tau > 0$$

The general solution is represented by the linear combination of these particular equations to give

$$\theta_i^1 = D^1 \cdot P_i + q_i \tag{3-49}$$

$$i = 1,2,3, \ldots, (n-1)$$

Let us assume that observations on the change of the head in the aquifer in response to a flood wave are available at the m'th discretized point. The dimensionless head at this point is defined as $\theta_m^* (\tau_j)$ for various time τ_j.

$$j = 1,2, \ldots, T$$

The objective is to find the unknows parameter D so that the solution of θ will be in the closest agreement with the observation

$$\theta_m^* (\tau_j).$$

The general solution of the m'th discretized point, where observations are made, is

$$\theta_m^1 = D^1 \cdot P_m + q_m \tag{3-50}$$

When the least squares criterion is used, the objective function is the minimization of the weighted sum of the squares of the deviation between Equation (3-49) and $_m^*(\theta_j)$. That is

$$S_1 = \sum_{j=1}^{T} \left[\theta_m(\tau_j) - \theta_m^* (\tau_j)\right]^2 \cdot W_j \tag{3-51}$$

Where W_j is the weight of the observation point, and $\sum_{j=1}^{T} (W_j/T) = 1$

The new estimate of D^1 is found by equating the derivative of S_1 with respect to D^1 to zero.

$$\frac{\partial S_1}{\partial D^1} = 2 \cdot \sum_{j=1}^{T} \{[D^1 \cdot P_m + q_m - \theta_m^* (\tau_j)] \cdot$$

$$W_j \cdot P_m\} = 0$$

$$D^1 = \frac{\sum_{j=1}^{T} [\theta_m^*(\tau_j) \cdot P_m - q_m \cdot P_m] \cdot W_j}{\sum_{j=1}^{T} P_m^2 \cdot W_j} \tag{3-52}$$

3-5 COMPUTATIONAL PROCEDURE

1. Assume a set of reasonable initial values for $\theta_i^0(\tau)$, D^0.

   Also, the observation value of $\theta_m^*(\tau_j)$ is given.

   $i = 1,2, \ldots, n-1$

   $j = 1,2, \ldots, T$

2. Integrate Equation (3-43) and (3-45) numerically by Fourth order Runge-Kutta method.

3. Substitute the value P_m and q_m, obtained from Step 2, into Equation (3-49).

4. Substitute the new value of parameter D^1 into Equation (3-51) and check whether it satisfies the accuracy we need. If the answer is yes, then the procedure is ended. Otherwise go to Step 5.

5. Let $\theta_i^0(\tau) = \theta_i^1(\tau)$

$$D^0 = D^1$$

Then go back to Step 2.

Current value of $\theta_i^1(\tau)$ and D^1 are obtained from Step 2.

CHAPTER 4

NUMERICAL RESULTS

4-1 INTRODUCTION

In this chapter, the problem formulated in chapter 3 with an unknown parameter is solved.

4-2 NUMERICAL ASSUMPTIONS

Stiff differential equations frequently arise in physical equations due to the existence of greatly differing time constants [8,14]. The problem we are working is also a stiff differential equation system. The step size will influence the stability of our problems [14]. If we can set the step size to be less than the absolute value of the time constant, the instability problem can be avoided. Time constant is the term used by engineers and physicists to refer to the rate of decay. For example, the equation $y' = \lambda \cdot y$ has the solution $c \cdot e^{\lambda t}$. If λ is negative, then y decays by a factor of e^{-1} in time $-\frac{1}{\lambda}$. This is the time constant. Here, the step size is assumed to be 0.1.

The initial approximation of parameter and θ can affect the convergence, depending on the sensitivity of the solutions to the initial estimation. Occasionally, solutions may diverge. The best remedy is to try another estimate of the parameter. The initial estimation for θ and parameter D should be considered simultaneously. Otherwise if any one of them should fail to give a suitable initial approximation, this would cause solutions to be divergent. The reasons can be explained from Equation (3-27) and (3-18). Here, the initial value of parameter D is 0.1, the initial value of θ is shown in Table 1. Also, the following values are assumed: $H/L^2 = 1$, $h(x)/H = 1.0$, $h_0(t)/1.1 = 0.5$, $\Delta Y = 1.0$. The observation data for the 5th grid point are given in Table 2.

4-3 RESULTS

The problem converged in four iterations. The results are shown in Table 3 through Table 6. Figure 5 shows the water head corresponding to the time period under different iteration, in which only the 5th grid point is considered. Figure 6 shows the water head corresponding to the distance under different iterations, in which only the 6th time period is considered. Figure 4 and Table 8 tell us the convergence of parameter D. Figure 2 shows the final relationship between the water head and the distance during different time periods. Figure 3 shows the final relationship between the water heads and the time periods under different distances. Table 7 shows the comparison between the numerical solutions of the fifth grid point and the experimental results published by Ibrahim and Brutsacrt (1965). Table 9 shows the comparison between the present work and Yeh's result [40].

4-4 THE PROCEDURE FOR FINDING THE INITIAL ESTIMATION OF D and θ.

1. Set the initial estimation of parameter D^0 as 1 which is already known as an actual value.
2. Assume initial estimation of θ^0 as reasonable as possible.
3. Let these data run through computer programming.
4. Check whether the numerical results of parameter D are around 1 and the value of θ^1 is convergent stably or not.
5. If the answer is no, the procedure is to go back to step 1 and assume a new estimation of θ^0. The parameter D^1 is still the value of 1. If the answer is yes, the procedure is to go on to

the following steps.

6. Use the data which was obtained from the final interation of step 3 as initial estimation for θ.

   And, parameter D^0 is any value which lies between 0 and 1.

7. Let these data go through computer program for one iteration.

8. Whether D^1 lies between 0 and 1 or not.

9. If yes, a good initial estimation of θ^0 is found.

   The initial estimation of D^0 is the smallest vaule we know in the range of 0 and 1 which makes the problem of convergence.

   Here, the value of 0.1 is assumed.

   If no, go back to step 1.

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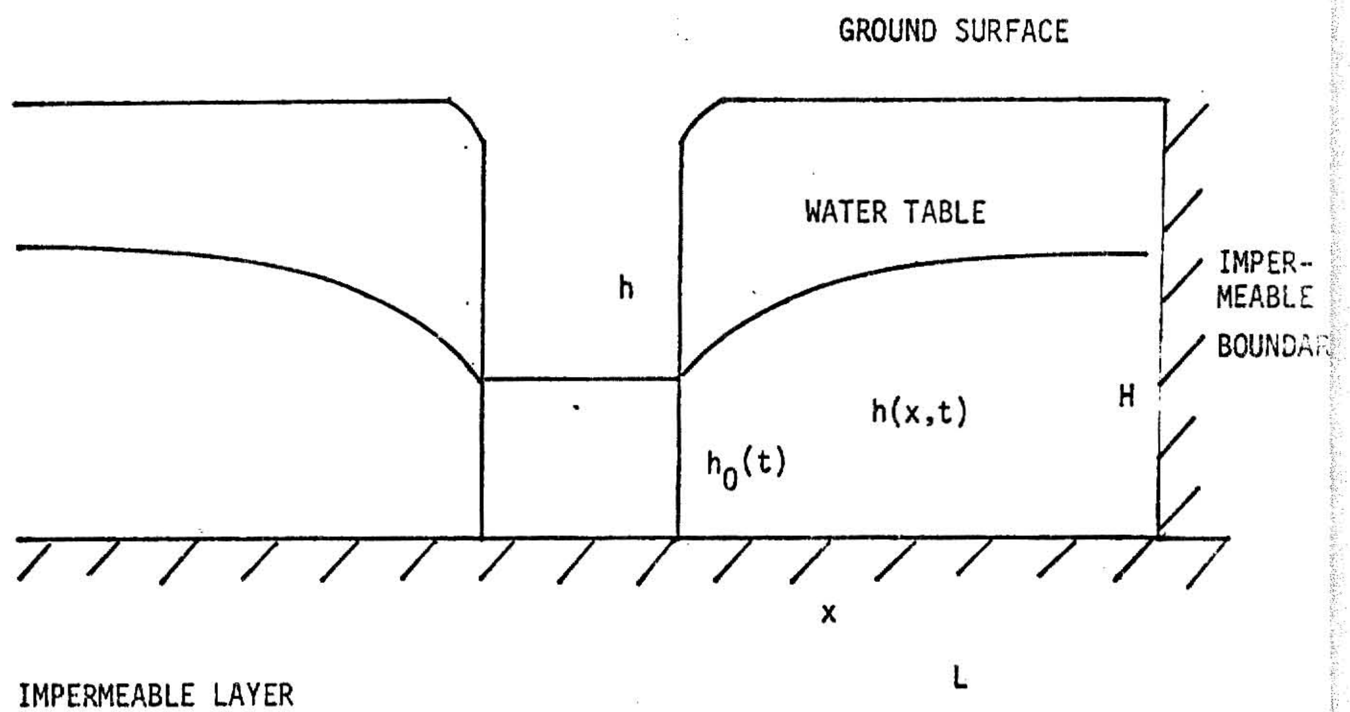

Fig. 1. The configuration of an unconfined aquifer and stream interaction system.

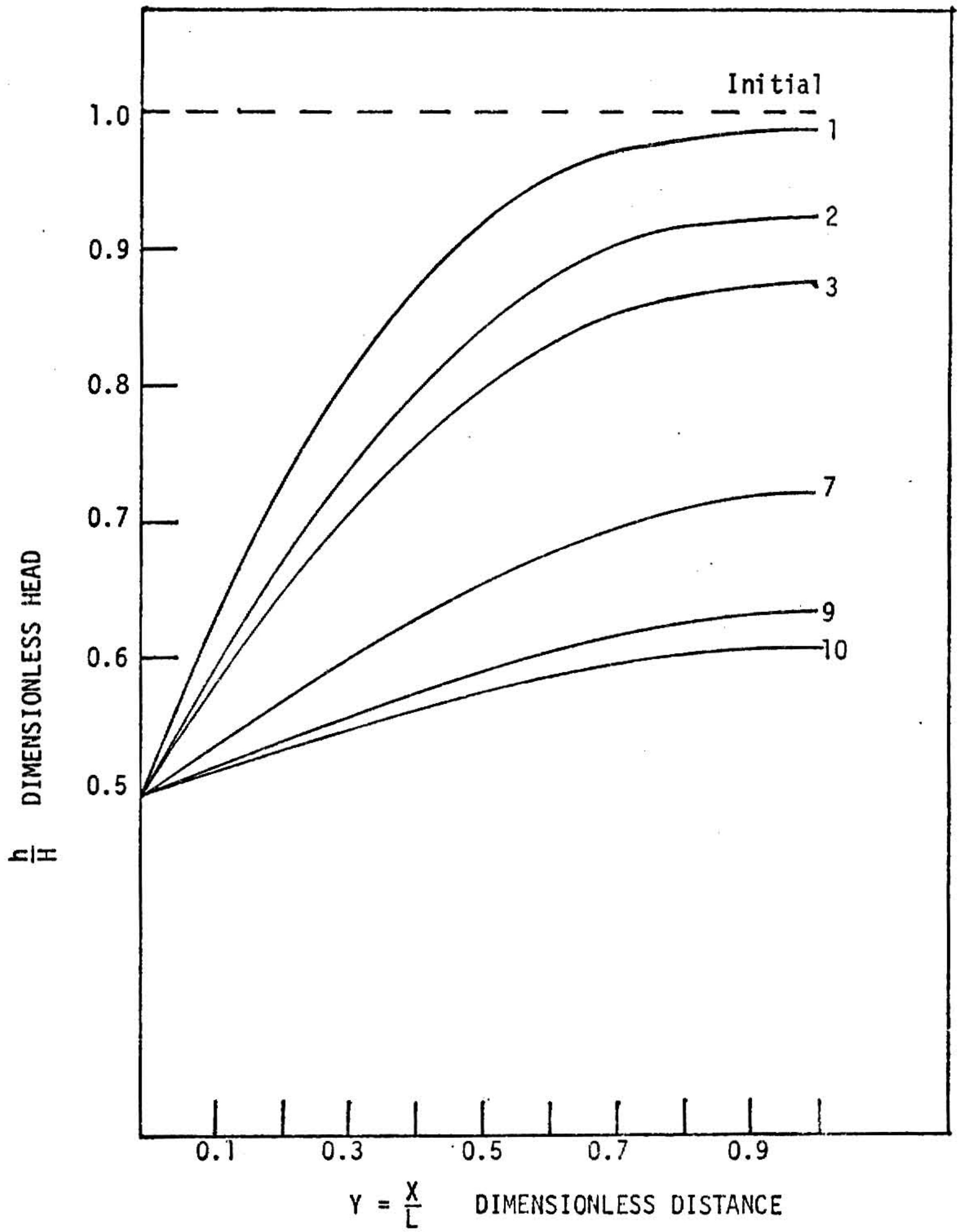

Fig. 2. The final relationship between water head and distance under different time periods.

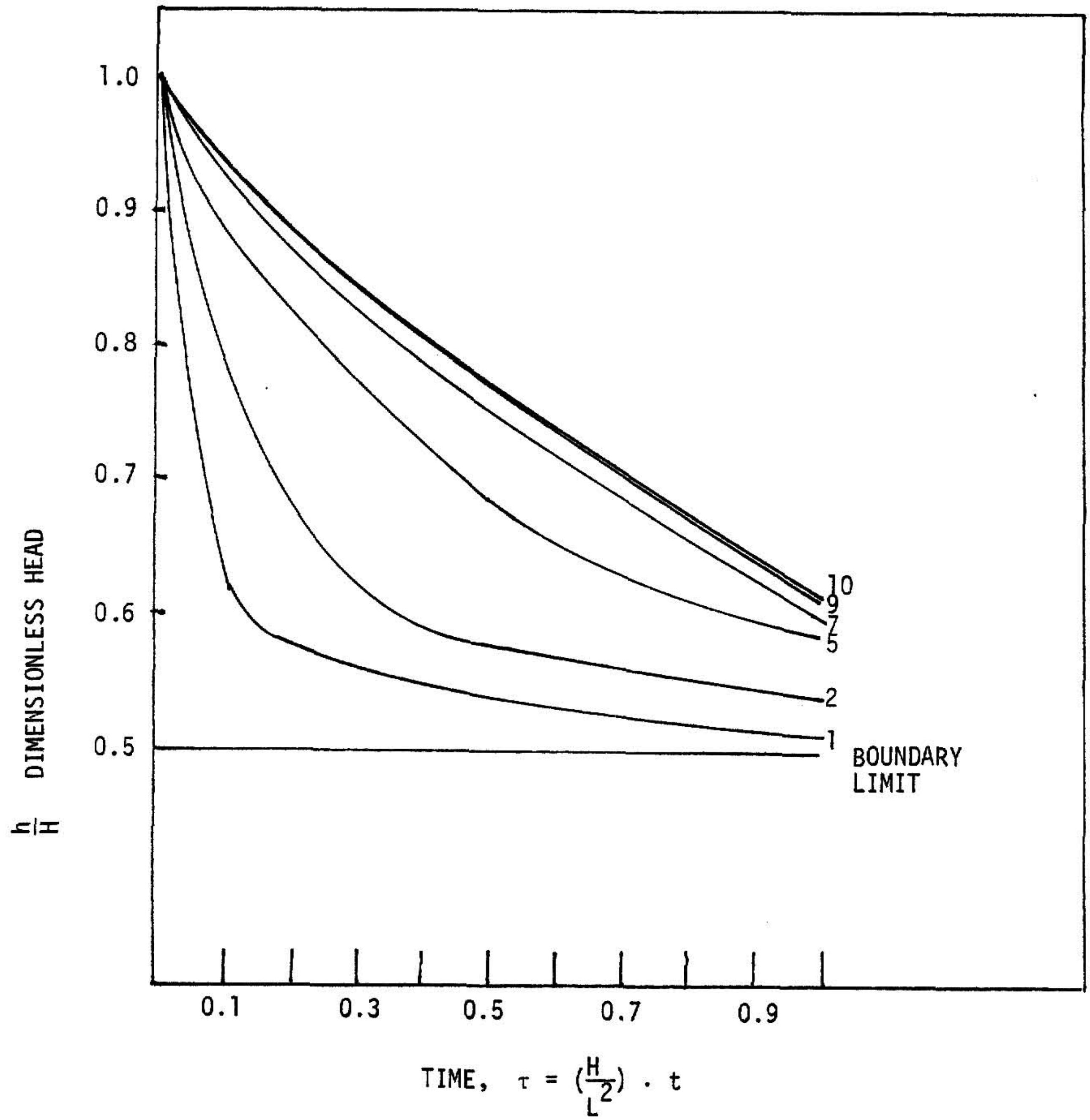

Fig. 3. Final relationship between water head and time period under different distance.

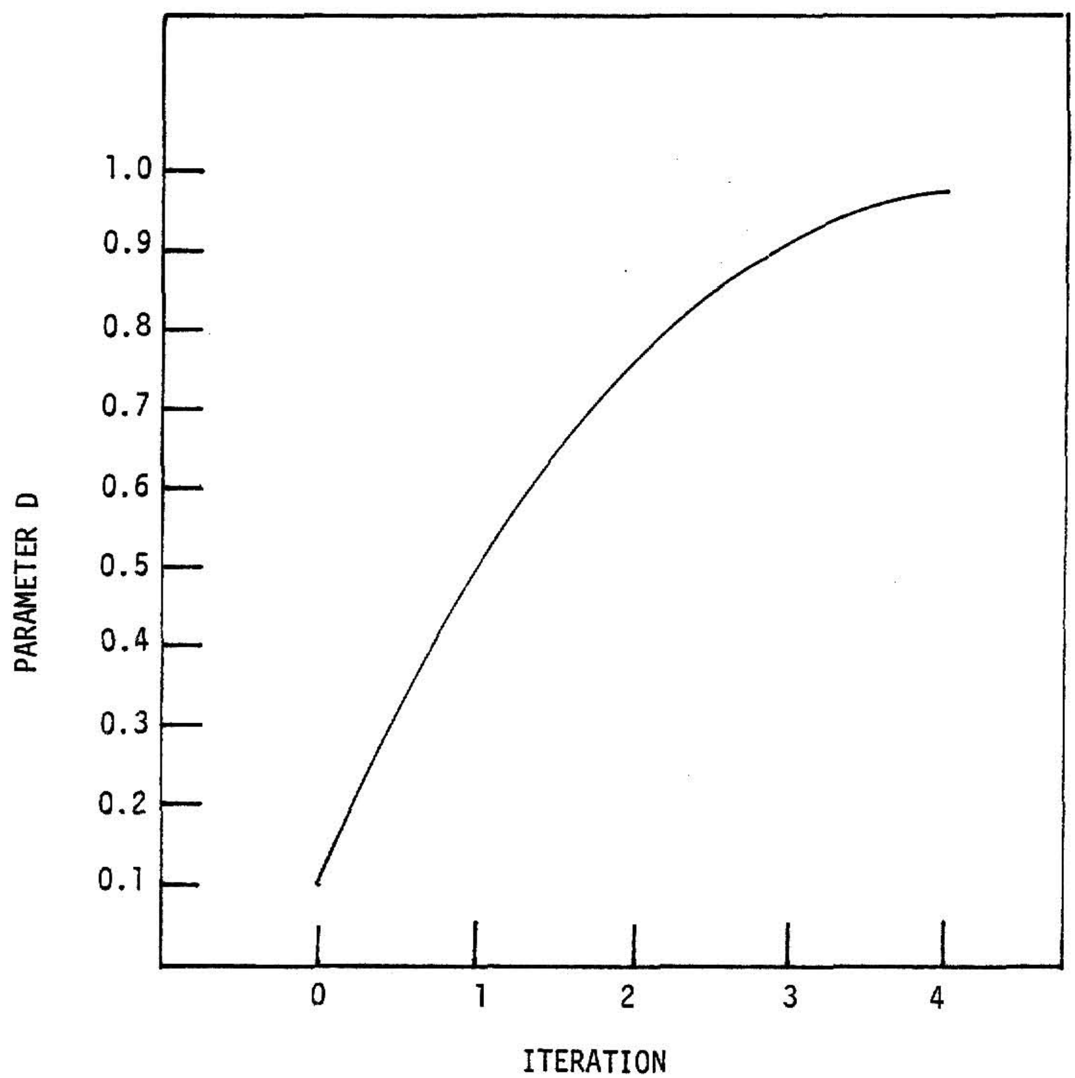

Fig. 4. The convergence rate of parameter D.

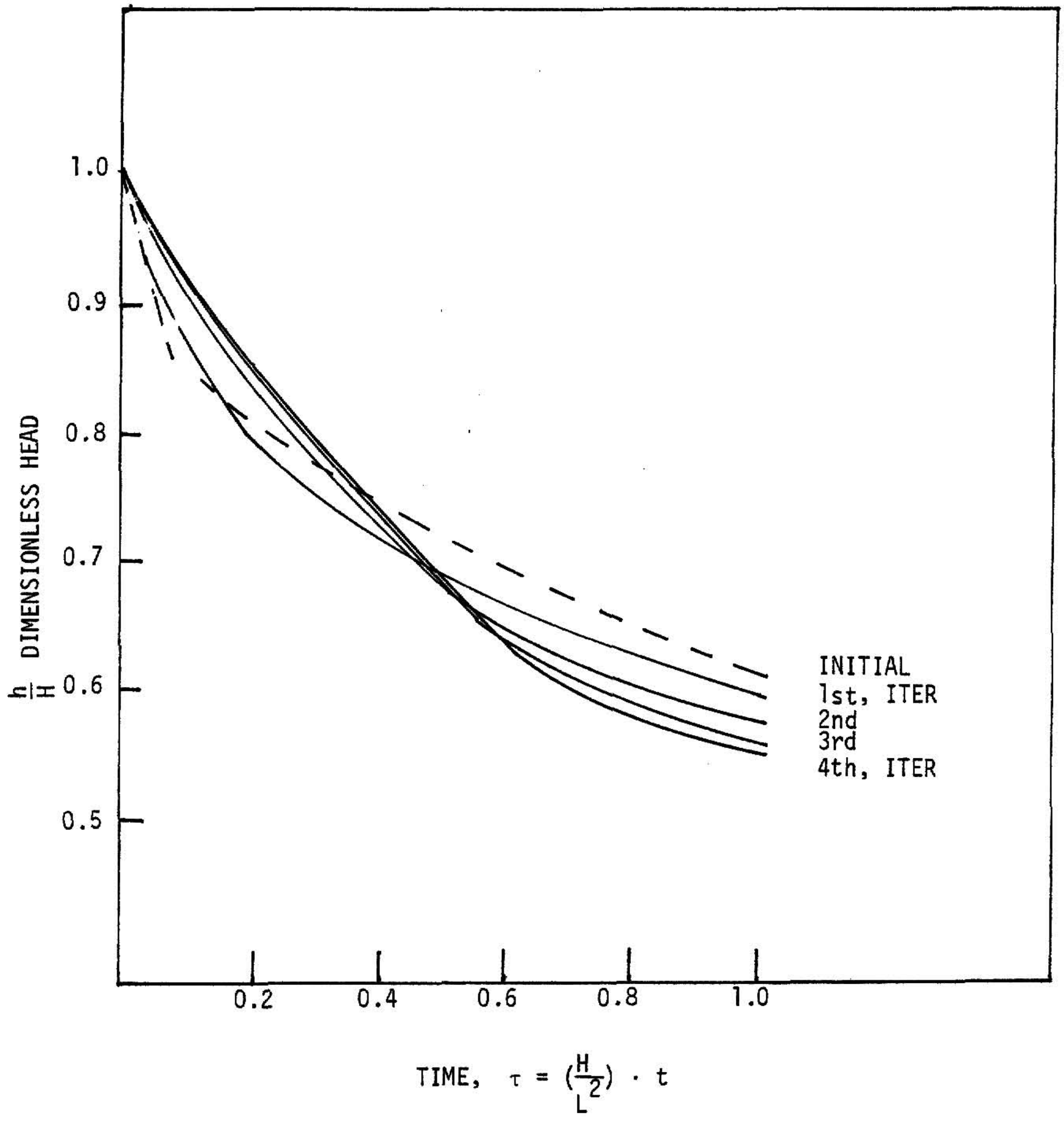

Fig. 5. Water head corresponding to time period under different iteration, 5th grid point be considered.

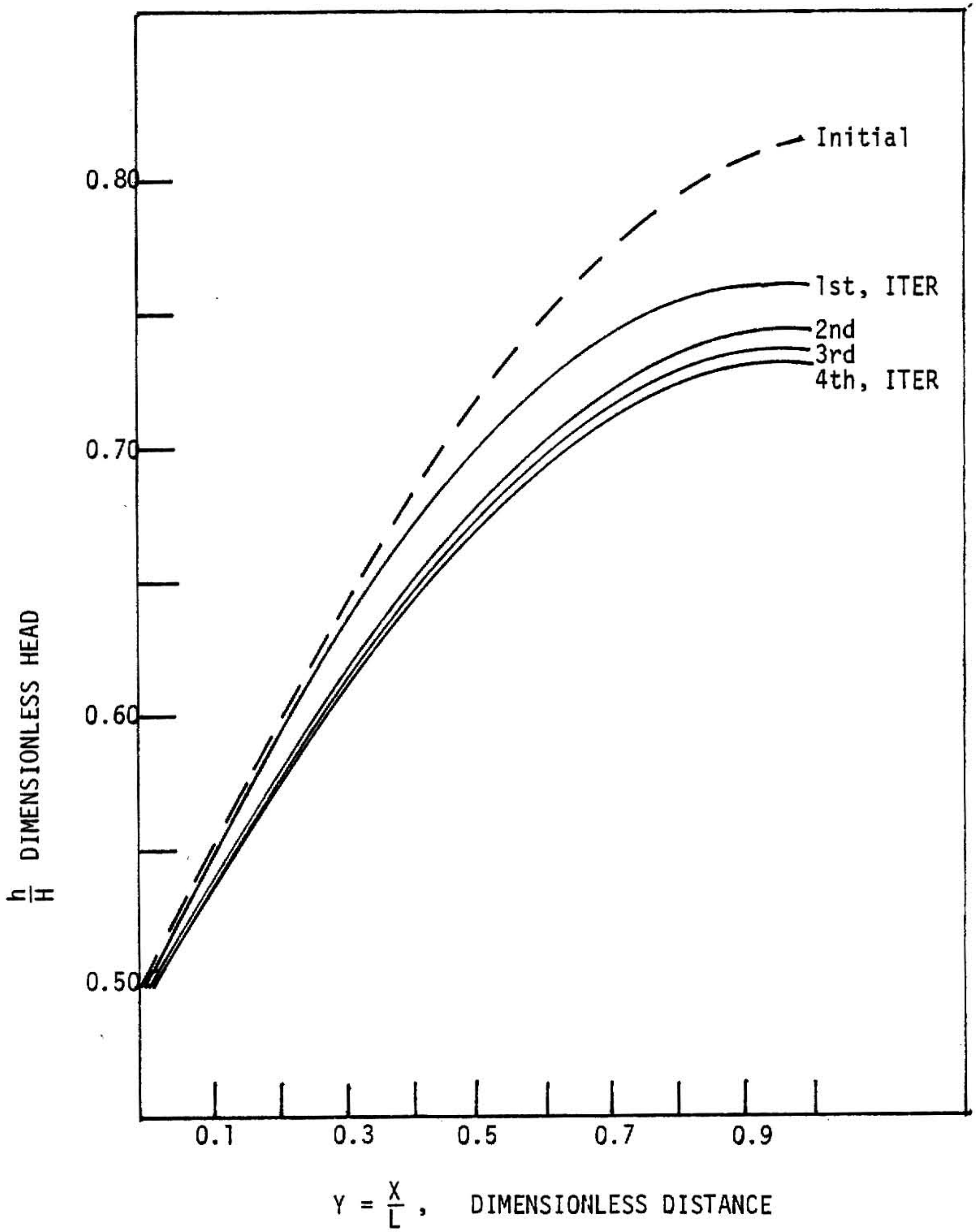

Fig. 6. Water head corresponding to distance under different iteration, 6th time period is considered.

Initial

DIST / TIME	0	1	2	3	4	5	6	7	8	9	10
0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
1	0.500000	0.627000	0.683000	0.776000	0.796000	0.835000	0.874000	0.886000	0.886999	0.896999	0.906999
2	0.500000	0.614000	0.666000	0.752000	0.772000	0.810000	0.848000	0.862000	0.864000	0.874000	0.884000
3	0.500000	0.601000	0.649000	0.7288000	0.748000	0.785000	0.822000	0.838000	0.841000	0.851000	0.861000
4	0.500000	0.588000	0.632000	0.704000	0.724000	0.760000	0.796000	0.814000	0.818000	0.828000	0.838000
5	0.500000	0.575000	0.615000	0.68000	0.700000	0.735000	0.770000	0.790000	0.795000	0.805000	0.815000
6	0.500000	0.562000	0.598000	0.565000	0.676000	0.710000	0.744000	0.766000	0.772000	0.782000	0.792000
7	0.500000	0.549000	0.581000	0.632000	0.652000	0.685000	0.718000	0.742000	0.749000	0.759000	0.769000
8	0.500000	0.536000	0.564000	0.608000	0.628000	0.660000	0.692000	0.718000	0.726000	0.736000	0.746000
9	0.500000	0.523000	0.547000	0.584000	0.604000	0.635000	0.666000	0.694000	0.703000	0.713000	0.723000
10	0.500000	0.510000	0.530000	0.560000	0.580000	0.610000	0.640000	0.670000	0.680000	0.690000	0.700000

Table 1. Initial Condition for θ.

j	r_j	$\theta_5^*(t_j)$
1	0	1.00000
2	0.1	0.90518
3	0.2	0.82802
4	0.3	0.77215
5	0.4	0.72803
6	0.5	0.69229
7	0.6	0.66296
8	0.7	0.63867
9	0.8	0.61842
10	0.9	0.60142
11	1.0	0.58707

Table 2. Observed Values of the Dimensionless Head at the Fifth Discretized Point.

First

ITER. / TIME	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	0.500000	0.623311	0.711564	0.786200	0.827502	0.854462	0.901452	0.968448	0.957445	0.964752	0.964752
0.2	0.500000	0.596945	0.674085	0.750084	0.780045	0.821545	0.861438	0.948523	0.884505	0.886922	0.886922
0.3	0.500000	0.573951	0.622345	0.708205	0.730765	0.775465	0.824100	0.840145	0.854104	0.855242	0.855242
0.4	0.500000	0.564452	0.618042	0.684722	0.716440	0.735528	0.788705	0.815100	0.815442	0.823441	0.823441
0.5	0.500000	0.5510082	0.607121	0.650984	0.695842	0.711075	0.754482	0.781054	0.785404	0.802352	0.802352
0.6	0.500000	0.549805	0.580024	0.621440	0.664902	0.689452	0.710248	0.746558	0.752802	0.754516	0.754516
0.7	0.500000	0.531004	0.570544	0.608675	0.634856	0.654854	0.683458	0.721058	0.713644	0.721482	0.721482
0.8	0.500000	0.528402	0.558420	0.585542	0.608457	0.632784	0.657820	0.689334	0.689544	0.705144	0.705144
0.9	0.50000	0.521850	0.542844	0.572285	0.590657	0.621275	0.644758	0.653845	0.672580	0.657845	0.657845
1.0	0.500000	0.521109	0.531001	0.557604	0.574640	0.690422	0.620596	0.642248	0.650445	0.646185	0.646185

Table 3. Convergence Rate of θ, First Iteration.

Second

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	0.500000	0.622514	0.721565	0.802752	0.857563	0.889456	0.922405	0.958672	0.961042	0.968250	0.968250
0.2	0.500000	0.588452	0.676352	0.746405	0.780485	0.827583	0.867435	0.914225	0.908225	0.906042	0.906042
0.3	0.500000	0.570623	0.620274	0.698824	0.728456	0.772445	0.826220	0.842285	0.859472	0.861404	0.861404
0.4	0.500000	0.558225	0.610548	0.668057	0.708544	0.731548	0.780242	0.802540	0.815260	0.821052	0.821052
0.5	0.500000	0.544304	0.590842	0.640825	0.680854	0.702543	0.742914	0.768425	0.775032	0.785764	0.785764
0.6	0.500000	0.546725	0.578648	0.617524	0.657508	0.678448	0.708425	0.722655	0.730805	0.738605	0.738605
0.7	0.500000	0.528036	0.570028	0.608412	0.630483	0.641086	0.673446	0.702428	0.702545	0.712950	0.712950
0.8	0.500000	0.524485	0.554240	0.581452	0.605438	0.622427	0.648722	0.664525	0.668072	0.685624	0.685624
0.9	0.500000	0.521645	0.542584	0.567534	0.589742	0.612160	0.624035	0.638427	0.648062	0.630112	0.630112
1.0	0.500000	0.521148	0.536045	0.557284	0.573258	0.587400	0.608735	0.619252	0.625072	0.610542	0.610542

Table 4. Convergence Rate of θ, Second Iteration

Third

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	0.500000	0.621150	0.730254	0.812045	0.878240	0.905046	0.931024	0.951185	0.962245	0.969114	0.969114
0.2	0.500000	0.586585	0.678426	0.744628	0.780705	0.827850	0.876258	0.902452	0.910050	0.908272	0.908272
0.3	0.500000	0.566712	0.618540	0.685811	0.727260	0.770845	0.826402	0.844265	0.861502	0.862802	0.862902
0.4	0.500000	0.551540	0.602275	0.660524	0.702142	0.727846	0.778780	0.797960	0.815161	0.820943	0.820943
0.5	0.500000	0.543856	0.587240	0.635282	0.672440	0.692244	0.732252	0.750062	0.772142	0.771069	0.771069
0.6	0.500000	0.538205	0.574321	0.616543	0.647513	0.662854	0.699530	0.718242	0.729046	0.735612	0.735612
0.7	0.50000	0.527012	0.566924	0.584205	0.624585	0.638514	0.664215	0.692787	0.693645	0.707145	0.707145
0.8	0.500000	0.522536	0.552252	0.577624	0.604524	0.620272	0.641254	0.658240	0.659428	0.675232	0.675232
0.9	0.500000	0.521622	0.542532	0.568225	0.589732	0.602545	0.614850	0.622472	0.624765	0.627954	0.627854
1.0	0.500000	0.521164	0.539524	0.557112	0.571610	0.586946	0.596748	0.605132	0.615622	0.606245	0.606245

Table 5. Convergence Rate of θ, Third Iteration

Final

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	0.500000	0.621146	0.730165	0.812076	0.879461	0.904072	0.931452	0.951170	0.962362	0.969114	0.969114
0.2	0.500000	0.586584	0.678345	0.742204	0.780887	0.824850	0.876616	0.894521	0.910050	0.911040	0.911040
0.3	0.500000	0.566734	0.61827	0.685805	0.727258	0.771896	0.826402	0.844326	0.861514	0.862905	0.862905
0.4	0.500000	0.551342	0.601769	0.655946	0.697438	0.727842	0.778793	0.797456	0.815161	0.820943	0.820943
0.5	0.500000	0.543820	0.585647	0.632475	0.672435	0.692145	0.732046	0.750062	0.770143	0.771069	0.771069
0.6	0.500000	0.537479	0.574321	0.611420	0.647503	0.662824	0.699530	0.718148	0.728906	0.733516	0.733516
0.7	0.500000	0.527007	0.564157	0.592109	0.624573	0.638502	0.663319	0.687651	0.693639	0.706943	0.706943
0.8	0.500000	0.522479	0.551320	0.573562	0.604260	0.618267	0.640078	0.658124	0.659343	0.675123	0.675123
0.9	0.500000	0.521632	0.542564	0.564721	0.589730	0.601274	0.614752	0.621740	0.623751	0.627951	0.627951
1.0	0.500000	0.521174	0.539591	0.557008	0.571570	0.586844	0.596596	0.604810	0.605423	0.606145	0.606145

Table 6. Convergence Rate of θ, Fourth Iteration

j	r_j	Final Numerical Results	Observation Data
1	0	1.000000	1.000000
2	0.1	0.904072	0.90518
3	0.2	0.824850	0.82802
4	0.3	0.771896	0.77215
5	0.4	0.727842	0.72803
6	0.5	0.692145	0.69229
7	0.6	0.662824	0.66296
8	0.7	0.638502	0.63867
9	0.8	0.618267	0.61842
10	0.9	0.601274	0.60142
11	1.0	0.586844	0.58707

Table 7. Comparison between the numerical results and experimental observation data.

ITER	0	1	2	3	4
D	0.100000	0.485724	0.781152	0.896472	0.999527

Table 8. Convergence Rates of Parameter D.

j	r_j	Present Numerical Results	Yeh's Work
1	0	1.000000	1.000000
2	0.1	0.904072	0.905060
3	0.2	0.824850	0.82789
4	0.3	0.771896	0.77200
5	0.4	0.727842	0.72787
6	0.5	0.692145	0.69213
7	0.6	0.662824	0.66280
8	0.7	0.638502	0.63852
9	6.8	0.618267	0.61827
10	0.9	0.601274	0.60127
11	1.0	0.586844	0.58694

Table 9. Comparison between the present result and Yeh's work.

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APPENDIX 1

THE COMPUTER PROGRAM USED IN CHAPTER 3

```
C
C           THIS PROGRAM SOLVES A SET OF
            NONLINEAR DIFFERENTIAL EQU. AND
            UNKNOWN PARAMETERS ARE TO BE IDENTIFIED
C           USING THE SUPERPOSITION
            PRINCIPLE AND RUNGE-KUTTA TECHNIQUE
C
            DIMENSION Y (11), OO (11,11), PP (11,11),
            SITA (11,11), SITAK (11), C (11,11), F(80)
            COMMON/VAR 4/MN, MN 3
            COMMON/VAR 3/SITA
            COMMON/VAR 2/MN 1, MN 2
            COMMON/VAR 1/IK, DO, DATAY
            COMMON/VAR/NUM
C
C           OBSERVATION DATA
C
            SITAK(1) = 1.00000
            SITAK(2) = 0.90518
            SITAK(3) = 0.72802
            SITAK(4) = 0.77215
            SITAK(5) = 0.72803
            SITAK(6) = 0.69229
            SITAK(7) = 0.66296
            SITAK(8) = 0.63867
            SITAK(9) = 0.61842
```

```
      SITAK(10) = 0.60142
      SITAK(11) = 0.58707
C
      EPSLO = 0.000001
      MN = 11
      MN1 = MN-1
      MN2 = MN-2
      MN3 = MN-3
C
      IT = 0
      DATAY = 0.1
      DATAT = 0.1
      N = 9
C
C     THE INITIAL VALUE OF SITA AND D
C
      DO  418  J = 2, MN
      SITA(J,1) = 0.5
  418 CONTINUE
      DO  417  J = 1, MN
      SITA (1,J) = 0
  417 CONTINUE
      SITA(11,2) = 0.51
      SITA(11,3) = 0.53
      SITA(11,4) = 0.56
      SITA(11,5) = 0.58
```

```
          SITA(11,6) = 0.61
          SITA(11,7) = 0.64
          SITA(11,8) = 0.67
          SITA(11,9) = 0.68
          SITA(11,10) = 0.69
          SITA(11,11) = 0.70

          DO  419          J = 1,  MN2
          SITA(11-J,2) = SITA(12-J, 2) + 0.013
          SITA(11-J,3) = SITA(12-J,3) + 0.017
          SITA(11-J,4) = S(TA(12-3,4) + 0.024
          SITA(11-J,5) = SITA(12-J,5) + 0.024
          SITA(11-J,6) = SITA(12-J,6) + 0.025
          SITA(11-J,7) = SITA(12-J,7) + 0.026
          SITA(11-J,8) = SITA(12-J,8) + 0.024
          SITA(11-J,9) = SITA(12-J,9) + 0.023
          SITA(11-J,10) = SITA(12-J,10) + 0.023
          SITA(11-J,11) = SITA(12-J,11) + 0.023
  419     CONTINUE
C
C
C         PARTICULAR SOLUTION OF P
C
  111     IT = IT + 1
          NUM = 1
          IK = 0
```

```
            INT = 1
            DO 11  J = 2, MN1
            Y(J-1) = 0
            PP(1,J) = 0
   11       CONTINUE
            PP(1, MN) = 0
            PP(1,1) = 0
   15       IK = IK + 1
            CALL  RKG4 (INT, DATAY, N, Y, F, L,
            MM, JJ)
            DO  19  J = 2, MN 1
            PP (IK,J) = Y(J-1)
   19       CONTINUE
            INT = INT + 1
            IF (IK. LE. 10)  GO TO 15
            DO  333  I = 2, MN
            PP(I, 1) = 0.5
            PP(I, MN) = PP(I,MN 1)
  333       CONTINUE
C
C           PARTICULAR SOLUTION OF Q
C
            INT = 1
            IK = 0
            NUM = 2
            DO  31  J = 2, MN1
```

```
          QQ(1, J) = 1.0
          Y(J-1) = 1.0
   31     CONTINUE
          QQ(1, MN) = 1
          QQ(1, 1) = 1
   35     IK = IK + 1
          CALL RKG4 (INT, DATAY, N, Y, F, L,
          MM, JJ)
          DO 39  J = 2, MN 1
          QQ (IK, J) = Y (J-1)
   39     CONTINUE
          INT = INT + 1
          IF (IK, LE. 10)  GO TO 35
          DO 334  I = 2, MN
          QQ(I, 1) = 0.5
          QQ(I, MN) = QQ(1, MN 1)
  334     CONTINUE
C
C         FIND D1 FROM KNOWN SITA
          ASUM = 0
          BSUM = 0
          DO 61  J = 1, MN
          ASUM = ASUM + ((SITAK(J) * PP(J,6)
          - QQ(J,6) ** PP(J,6)) / 1)
          BSUM = BSUM + ((PP(J,6) ** 2) / 1)
  61      CONTINUE
```

```
          D1 = ASUM/BSUM
C
C         SUBSTITUTE D1 INTO SITA = D * P + Q
C
          DO  606  I = 2, MN
          DO  606  J = 2, MN 1
          C (I,J) = D1 * PP(I,J) + QQ(I,J)
          SITA (I,J) = ((I,J)
    606   CONTINUE
          DO 607  I = 1, MN
          C (1, I) = 1.0
          SITA (1, I) = ((1,I)
    607   CONTINUE
          DO  608  J = 2, MN
          C (J, 1) = 0.5
          C (J, MN) = C (J, MN 1)
          SITA (J, MN) = C (J, MN)
    608   CONTINUE
          PRINT 9, IT, D1
      9   FORMAT ('0',' ITERATION = ', I2,
          ' D = ',  F 14, 7)
          DO  295      i = 1, MN
          PRINT 1, I, (C (I,J), J = 1, MN)
    295   CONTINUE
C
C         TEST WHEATHER IT IS LESS THE ACCURACY WE NEED
```

```
C
          TSUM = 0
          DO  27   J = 1,  MN
          TSUM = TSUM + CCCC(J, 6) - SITAK(J)
          ) ** 2) / 1)
   27     CONTINUE
          IF (TSUM. LE. EPSLO)  GO TO 125
C
C         FIND THE CURRENT SITA FROM
          THE PREVIOUS SITA
          DO = D1
          IT = IT + 1
          GO TO 111
   125    STOP
          END
C
C
C         DIFFERENTIAL EQUATION DEFINED
C
C         DIFFERENTIAL EQA. OF P
C
          SUBROUTINE DFY L (X, DX)
          DIMENSION DX (11), X (11), SITA (11, 11)
          COMMON/VAR 1/ IK, DO, DATAY
          COMMON/VAR 2/ MN1, MN2
          COMMON/VAR 3/SITA
```

```
      COMMON/VAR 4/MN, MN 3
      DX(1) = X(1) * (DO/(2 * (DATAY) ** 2)
      * (-4 * SITA (IK, 2)) + X(2)* (DO/C 2 * (DATAY)
      1 ** 2) * (2 * SITA(IK,3))) + 1/(2*(DATAY) ** 2) * (SITA (IK,3)
        ** 2 - 2 * SITA (IK, 2) 2 ** 2 + SITA (IK, 1) ** 2)
      DX(MN2) = X(MN2) * (DO/(2 * DATAY ** 2) * (- 2 * SITA (IK, MN1)))
        + X(MN3) * (DO/
      1 (2 * DATAY ** 2) * 2 * SITA(IK, MN2)) + 1/(2 * DATAY ** 2) *
        (-SITA(IK, MN1) **2 +
      2 SITA (IK, MN2) ** 2/
        DO 37 J = 3, MN2
      DX (J-1) = X(J-1) * (DØ(2 * (DATAY) ** 2) * (-4 * SITA (IK, J)))
        + X(J) * (DO/ (2 *
      1 (DATAY) ** 2) * (2 * SITA (IK, J+1))) + X(J-2) * (DO/(2 *
        (DATAY) ** 2) * (2 *
      2 SITA (IK, J-1))) + 1/(2 * (DATAY) ** 2) * (SITA (IK, J+1) ** 2
        - 2 * SITA (IK, J) ** 2 +
      3 SITA (IK, J-1) ** 2)
   37 CONTINUE
      RETURN
      END
C
C     DIFFERENTIAL EQUATION OF Q
C
      SUBROUTINE DFY2(X, DX)
      DIENSION DX(11), X(11), SITA(11,11)
```

```
          COMMON/VAR1/IK, DO, DATAY
          COMMON/VAR2/MN1, MN2
          COMMON/VAR3/SITA
          COMMON/VAR4/MN, MN3
          DX(1) = (X(1) - SITA(IK, 2)) * (DO/(2 * (DATAY ** 2)) * (- 4 *
            SITA (IK, 2))) + (X(2)
          1 - SITA (IK, 3)) * (DO/(2 * (DATAY) ** 2) * (2 * SITA (IK,
            J+1))) + (X(J-2)
          2 - SITA(IK, J-1)) * (DO/(2 * (DATAY ** 2)) * (2 * SITA (IK,
            J-1)))
    43    CONTINUE
          RETURN
          END
          SUBROUTINE RKG4 (INT, DT, N, Y, F
          L, M, J)
C         DT   TIME INTERVAL
C         INT  INTEGRAATION TIME INDEX, TIME = (INT-1) * DT
C   N     NUMBERS OF O. D. EQNS
C   Y     INTEGRATION VALUE
C   F     DUMMY STORAGE
C   L     DUMMY INDEX
C   MM    DUMMY INDEX
C   J     DUMMY INDEX
C
          DIMENSION DY (11), Y(11), F(80)
          COMMON/VAR/NUM
```

```
       T = (INT - 1) * DT
       IF (INT. (TT.1)  GO TO 450
410    L = 3
       M = 0
450    CONTINUE
       GO TO (160, 110, 300), L
100    GO TO (101, 110), IG
101    J = 1
       L = 2
       DO  106  K = 1, N
       K 1 = K + 3 * N
       K 2 = K1 + N
       K 3 = N + K
       F (K1) = Y(K)
       F (K3) = F(K1)
106    F (K2) = DY(K)
       GO TO 406
110    DO 140  K = 1, N
       K1 = K
       K2 = K + 5 * N
       K3 = K2 + N
       K4 = K + N
       GO TO (111, 112, 113, 114), J
111    F(K1) = DY(K) * DT
       Y(K) = F(K4) + 0.5 * F(K1)
       GO TO 140
```

```
112     F(K2) = DY(K) * DT
        GO TO 124
113     Y(K) = F(K4) + (I (K1) + 2.0 * (F (K2) + F(K3)) + DY (K) * DT)/6.0
        GO TO 140
124     Y(K) = 0.5 * F(K2)
        Y(K) = Y(K) + F(K4)
        GO TO 140
134     Y(K) = F(K4) + F(K3)
140     CONTINUE
        GO TO (170, 180, 170, 180), J
170     T = T + 0.5 * DT
180     J = J+1
        IF (J-4) 404, 404, 299
299     M = 1
        GO TO 406
300     IG = 1
        GO TO 405
404     IG = 2
405     L = 1
406     CONTINUE
        IF (M-1) 475, 410, 475
475     GO TO (500, 600, 600) L.
C
C       TRANSFORM LOT
C
500     CONTINUE
```

QUASILINEARIZATION APPLIED TO OPTIMAL IDENTIFICATION OF AQUIFER DIFFUSIVITY IN STREAM INTERACTION SYSTEM

by

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ABSTRACT

A computational procedure based on observation wells is presented for solving the problem of identifying aquifer parameters. This procedure, quasilinearization, was developed by Bellman and Kalaba. In the problem solved, the governing equation is a second order nonlinear partial differential equation subject to time varying boundary conditions for which no closed form solution exists. The equation is first transformed into a set of nonlinear ordinary differential equations by discretizing the time variable. The resulting equations form a two-point boundary value problem which can be linearized by quasilinearization. The equations can then be solved and, at the same time, the aquifer parameter is identified. The results showed that the computational procedure is fairly efficient.