BOUNDS FOR LINEAR AND NONLINEAR INITIAL VALUE PROBLEMS

by

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NOMENCLATURE

B_1 = bounding coefficient in the Tchebyshev series for x_1

B_u = bounding coefficient in the Tchebyshev series for x_u

$C[R]$ = the class of functions which are continuous on the region R.

$C^1[R]$ = the class of functions which are continuous with continuous first derivatives on the Region R.

$f(x)$ = spring rate which is function of x

$F(x)$ = spring rate

$g(\frac{dx}{dt})$ = damping characteristic

$G(\frac{dx}{dt})$ = damping characteristic

m = mass in spring-mass system

N = order of the Tchebyshev series

R = a region in $(x,\dot{x},t)$ space

R_1 = a region in $(x,\dot{x},t)$ space

t = time

T = maximum time over which the problem is considered

v_0 = initial velocity

x = dependent variable

$\dot{x}$ = first derivative of the dependent variable with respect to time or dimensionless time.

$\ddot{x}$ = second derivative of the dependent variable with respect to time or dimensionless time

x_1 = lower bound

x_{max} = maximum value of x on some interval

$\dot{x}_{max}$ = maximum value of $\dot{x}$ on some interval

x_{min} = minimum value of x on some interval

$\dot{x}_{min}$ = minimum value of $\dot{x}$ on some interval

x_0 = initial displacement

x_u = upper bound

α = linearity parameter in the spring-mass system

β = nonlinearity parameter in the spring-mass system

Δ_l = negative constant

Δ_{max} = maximum value of residual

Δ_{min} = minimum value of residual

Δ_u = positive constant

ε = nonlinearity parameter in the damper

τ = t/T, nondimensional time

ξ = linearity parameter in the damper

INTRODUCTION

The solution of differential equations is an important activity in the analysis of many engineering problems. There are many initial value and boundary value problems for which exact solutions are either impossible or impracticable to obtain using existing methods. With the development of high speed digital computers, attempts have been made to solve difficult problems by approximate techniques. Many of these techniques result in a good approximate solution; but unless the exact solution is known, the error associated with the approximate solution can not be precisely determined.

Error analysis, for the case of nonlinear differential equations is difficult and sometimes impracticable. Error estimates are available for some equations but no general conclusions concerning error analysis have been formulated for nonlinear differential equations. This is particularly unfortunate since uncertainty in approximate numerical solutions is usually more critical for nonlinear than linear equations. Sometimes the error analysis is more difficult and cumbersome than solving approximately the differential equation itself. It has been stated that, "In fact, the problem of error analysis in many instances overshadows the numerical technique." [6]*

In using most approximate methods, an attempt is made to obtain an "approximate solution" for the "exact problem", but the method used in this report is to determine an "exact solution" to an "approximate problem." In this work, an attempt has been made to obtain exact solutions to

* [ ] Numbers in brackets designate references at the end of report.

modified problems such that these solutions provide upper and lower bounding functions for the original problems. Since upper and lower bounds are obtained, further error analysis is unnecessary.

Bell [1] developed a new method for finding rigorous upper and lower bounds to the solution of a wide class of second order ordinary differential equations with initial conditions.

Specifically, he treated the following initial value problem.

$$\ddot{x} + f\,(t,x,\dot{x}) = 0 \qquad (1)$$

$$x(0) = x_0 \qquad (2)$$

$$\dot{x}(0) = v_0 \qquad (3)$$

in some time interval $I = [0,T]$, where $f(t,x(t),\dot{x}(t))$ is continuous and

$$\frac{\partial f}{\partial x} \geq 0 \qquad \text{for } t \varepsilon I \qquad (4)$$

The upper "dots" indicate differentiation with respect to t.

He has formulated and proved a theorem for upper and lower bounds for the displacement $x(t)$ and the velocity $\dot{x}(t)$. The theorem is given in Appendix A. Further, he also developed a numerical technique to find the bounding function x_u and x_1 such that

$$x_1 \leq x \leq x_u \qquad \text{for } t \varepsilon I \qquad (5)$$

and he applied this technique to a few problems.

The purpose of this work is to apply this technique to other problems having linear or nonlinear governing differential equations. The problems included in this report correspond to a nonlinear spring mass system with a linear damper and a nonlinear spring mass system with a nonlinear damper. The major objective of this work is to verify the method developed by Bell.

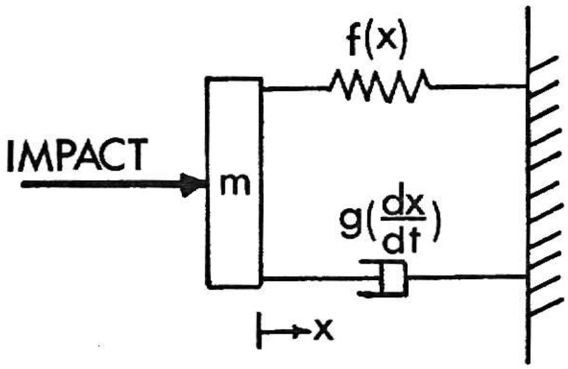
The work of Appl and Hung [2] involved finding bounds for the solutions of a similar problem. The difference is that in contrast to equation (4) of

Bell's work, $\frac{\partial f}{\partial x} < 0$ was used by Appl and Hung. The condition imposed by equation (4) results in a more difficult problem because of the oscillatory nature of solutions.

The bounding technique developed by Bell is applied to several problems which are obtained by changing parameters like spring coefficient, damping coefficient, etc. The bounding solutions were obtained in closed analytic form and, for the problems considered, yielded good bounds.

Definition of the Problem

Consider a system composed of a mass, damper and spring as shown in the diagram.



Where,

$f(x)$ is the spring rate which is a function of x.

$g(\frac{dx}{dt})$ is the damping characteristic which is a function of $\frac{dx}{dt}$

The differential equation of the motion of the mass is

$$m\frac{d^2x}{dt^2} + g\,(\frac{dx}{dt}) \cdot \frac{dx}{dt} + f(x) \cdot x = 0 \qquad (6)$$

with initial conditions

$$x(0) = x_0 \qquad (7)$$

$$\frac{dx}{dt}\,(0) = v_0 \qquad (8)$$

Dividing by m yields

$$\frac{d^2x}{dt^2} + G(\frac{dx}{dt}) \cdot \frac{dx}{dt} + F(x) \cdot x = 0 \qquad (9)$$

where

$$G(\frac{dx}{dt}) = \frac{g\,(\frac{dx}{dt})}{m} \qquad (10)$$

$$F(x) = \frac{f(x)}{m} \qquad (11)$$

The following substitution was used to nondimensionalize the time variable

$$\tau = \frac{t}{T} \tag{12}$$

The following notation is used:

$$\dot{x} = \frac{dx}{d\tau} = T \left(\frac{dx}{dt}\right) \tag{13}$$

$$\ddot{x} = \frac{d^2x}{d\tau^2} = T^2 \left(\frac{d^2x}{dt^2}\right) \tag{14}$$

The resulting equation is

$$\frac{\ddot{x}}{T^2} + G\left(\frac{\dot{x}}{T}\right) \cdot \frac{\dot{x}}{T} + F(x) \cdot x = 0 \tag{15}$$

with initial conditions

$$x(0) = x_0 \tag{16}$$

$$\dot{x}(0) = v_0 \tag{17}$$

This is the governing differential equation for a nonlinear spring mass system with damping.

The following functions were selected for $F(x)$ and $G(\dot{x})$

$$F(x) = \alpha + \beta x^2 \tag{18}$$

$$G\left(\frac{dx}{dt}\right) = \xi \left\{ 1 + \varepsilon \left(\frac{dx}{dt}\right)^2 \right\} \tag{19}$$

The differential equation then takes the form

$$\frac{d^2x}{dt^2} + \xi \left\{\frac{dx}{dt} + \varepsilon\left(\frac{dx}{dt}\right)^3\right\} + \alpha x + \beta x^3 = 0 \tag{20}$$

Equation (20) is the general governing differential equation for problems worked in this report. The parameters α, β, ξ and ε were alloted different values and bounds were obtained.

Initial Conditions

Initial conditions were selected for impact loads. Impact was so applied that initial displacement at time t = 0, was x_0 and velocity v_0.

In this work, we were particularly interested in analyzing the problems in the nonlinear range. So the initial magnitude of velocity was selected such that the range of the problem was nonlinear.

It was found that if the initial velocity was too small, it was difficult to plot the displacement and compare with other values obtained by changing parameters because the maximum displacement was too small. Initial velocity $v_0 = 5.0$ gave sufficiently large values to plot and compare with the other results. So the following values were selected for initial conditions.

$$x(0) = x_0 = 0.0 \tag{21}$$

$$\dot{x}(0) = v_0 = 5.0 \tag{22}$$

Method for Determining Upper and Lower Bounds for the Displacement

In this work, the technique and program developed by Bell were used. This program contains two function subroutines which had to be changed.

To illustrate the method, the following example is considered.

$$\frac{d^2x}{dt^2} + \xi \frac{dx}{dt} + \alpha x + \beta x^3 = 0 \tag{23}$$

The bounding differential equations are

$$\frac{d^2x_u}{dt^2} + \xi \frac{dx_u}{dt} + \alpha x_1 + \beta x_1^{\,3} = \Delta_u \tag{24}$$

$$\frac{d^2x_1}{dt^2} + \xi \frac{dx_1}{dt} + \alpha x_u + \beta x_u^{\,3} = \Delta_1 \tag{25}$$

with initial conditions

$$x_u(0) = x_1(0) = 0 \tag{26}$$

$$\frac{dx_u}{dt}(0) = \frac{dx_1}{dt}(0) = 5.0 \tag{27}$$

where $\Delta_u > 0$ (28)

and $\Delta_1 < 0$ (29)

The following substitution was used to nondimensionlize the time variable

$$\tau = {}^{t}/T, \tag{30}$$

$$\dot{x} = \frac{dx}{d\tau} = T \frac{dx}{dt} \tag{31}$$

$$\ddot{x} = \frac{d^2x}{d\tau^2} = T^2 \frac{d^2x}{dt^2} \tag{32}$$

This resulted in the bounding system

$$\ddot{x}_u + \xi T \dot{x}_u + T^2 \alpha x_1 + T^2 \beta x_1^3 = T^2 \Delta_u \tag{33}$$

$$\ddot{x}_1 + \xi T \dot{x}_1 + T^2 \alpha x_u + T^2 \beta x_u^3 = T^2 \Delta_1 \tag{34}$$

$$x_u(0) = x_1(0) = 0 \tag{35}$$

$$\dot{x}_u(0) = \dot{x}_1(0) = 5.0 \tag{36}$$

The bounding theorem used in this work involves the simultaneous solution of the two coupled bounding differential equations. Because of this and $\frac{\partial f}{\partial x} \geq 0$, there is inherent tendency for the difference $x_u(\tau) - x_1(\tau)$ to increase as t increases. Therefore it is required that Δ_u and Δ_1 be small for the bounding solution to remain close together as τ increases.

In theory Δ_u and Δ_1 can be made arbitrarily small, and in the limiting case, $\Delta_u = \Delta_1 = 0$. However, in the algorithm Δ_u and Δ_1 must be of sufficient initial magnitude so that the approximately obtained analytic forms for x_u and x_1, when substituted into the bounding differential equations, still satisfy the hypothesis $\Delta_u > 0$ and $\Delta_1 < 0$. The technique developed by Bell utilizes the Runge-Kutta-Nystrom method for numerical integration of differential equations and uses shifted Tchebyshev polynomials as approximate analytic forms for the points obtained by the Runge-Kutta-Nystrom method.

Therefore, the initial choices for Δ_u and Δ_1 depend on the degree of precision of the Runge-Kutta-Nystrom integration method, curve fitting procedures and computer precision.

The curve fitting procedure used by Bell consisted of approximating x_u and x_1 by linear combinations of shifted Tchebyshev polynomials of the first kind. In final form, x_u and x_1 are expressed as

$$x_u = \sum_{i=1}^{N+2} B_{ui} T_i^* (\tau) \tag{37}$$

$$x_1 = \sum_{i=1}^{N+2} B_{1i} T_i^* (\tau) \tag{38}$$

Where B_{ui}, B_{1i} are bounding coefficients and T_i^* are the Tchebyshev polynomials.

The following tactic was used to obtain the bounding solutions. First, the computer analysis was performed with $\Delta_u = \Delta_1 = 0$. This was, of course, equivalent to standard numerical integration and curve fitting, since the bounding differential equations degenerated, under these conditions, to the differential equation of interest. These identical approximate solutions were substituted into the bounding differential equations to check the signs of the equal residuals, Δ_u and Δ_1. The residuals displayed a variation with τ. Since these residuals were equal to each other and not identically zero, they could not both satisfy the hypothesis of the bounding theorem. Specifically, they could not satisfy the requirement $\Delta_u > 0$, $\Delta_1 < 0$ for $t \varepsilon I$. However, the extent to which they failed to satisfy this hypothesis served as a guide for determination of trial values for Δ_u and Δ_1 for a subsequent attempt at finding approximate solutions which would satisfy the hypothesis of the theorem.

Example Problems

I The case of the nonlinear spring mass system with linear damping.

(i) The case of

$$\ddot{x} + \xi\dot{x} + \alpha x + \beta x^3 = 0 \qquad \tau\varepsilon I_1 \tag{39}$$

$$x(0) = 0 \tag{40}$$

$$\dot{x}(0) = 5.0 \tag{41}$$

$$\alpha = 2\beta = 4\pi^2 \tag{42}$$

$$T = 1.3 \tag{43}$$

was considered. For the various values of ξ, $\xi = 0.0, 1.0, 5.0, 10.0$ bounds were obtained and bounding curves for the average value of x as a function of τ are presented in Figure 1.

Bounds for the case $\xi = 0.0$, are presented in Table 1. The maximum spread in the bounds for $\xi = 0.0$, was 0.12528×10^{-4} which was about 0.00168% of the maximum of the absolute value of x_u.

Bounds were obtained for the case $\xi = 5.0$ and the results, in the form of the bounding coefficients, are presented in Table 2. The maximum spread in the bounds for $\xi = 5.0$, was 0.396×10^{-8} which was about 0.8388×10^{-6}% of the maximum of the absolute value of x_u.

(ii) The case of

$$\ddot{x} + \xi\dot{x} + \alpha x + \beta x^3 = 0 \qquad \tau\varepsilon I_1 \tag{44}$$

$$x(0) = 0 \tag{45}$$

$$\dot{x}(0) = 5.0 \tag{46}$$

$$\alpha = \beta = 4\pi^2 \tag{47}$$

$$T = 1.3 \tag{48}$$

was considered. For the various values of ξ, ξ = 0.0, 5.0, 10.0, bounds were obtained and bounding curves for the average value of x as a function of τ are presented in Figure 2.

Bounds were obtained for the case $\xi = 1.0$ and the results, in the form of bounding coefficients, are presented in Table 3. The maximum spread in the bounds for $\xi = 1.0$, was 0.69838×10^{-5} which was about 0.00107833% of the maximum of the absolute value of x_u.

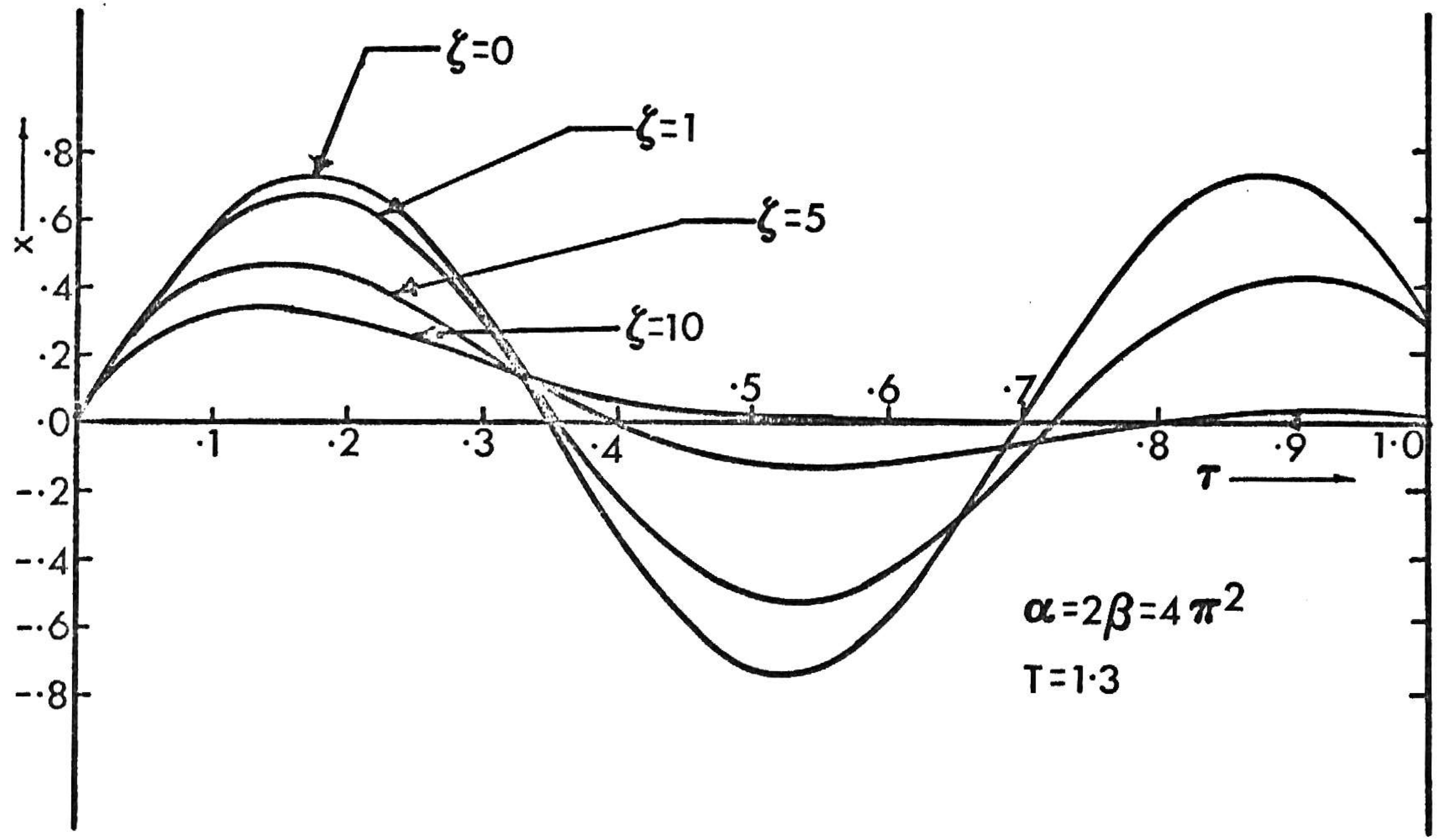


FIG. 1 Average values of the bounding functions for the case of

$$\ddot{x} + \xi\,\dot{x} + \alpha x + \beta x^3 = 0, \quad x(0) = 0, \quad \dot{x}(0) = 5.0$$

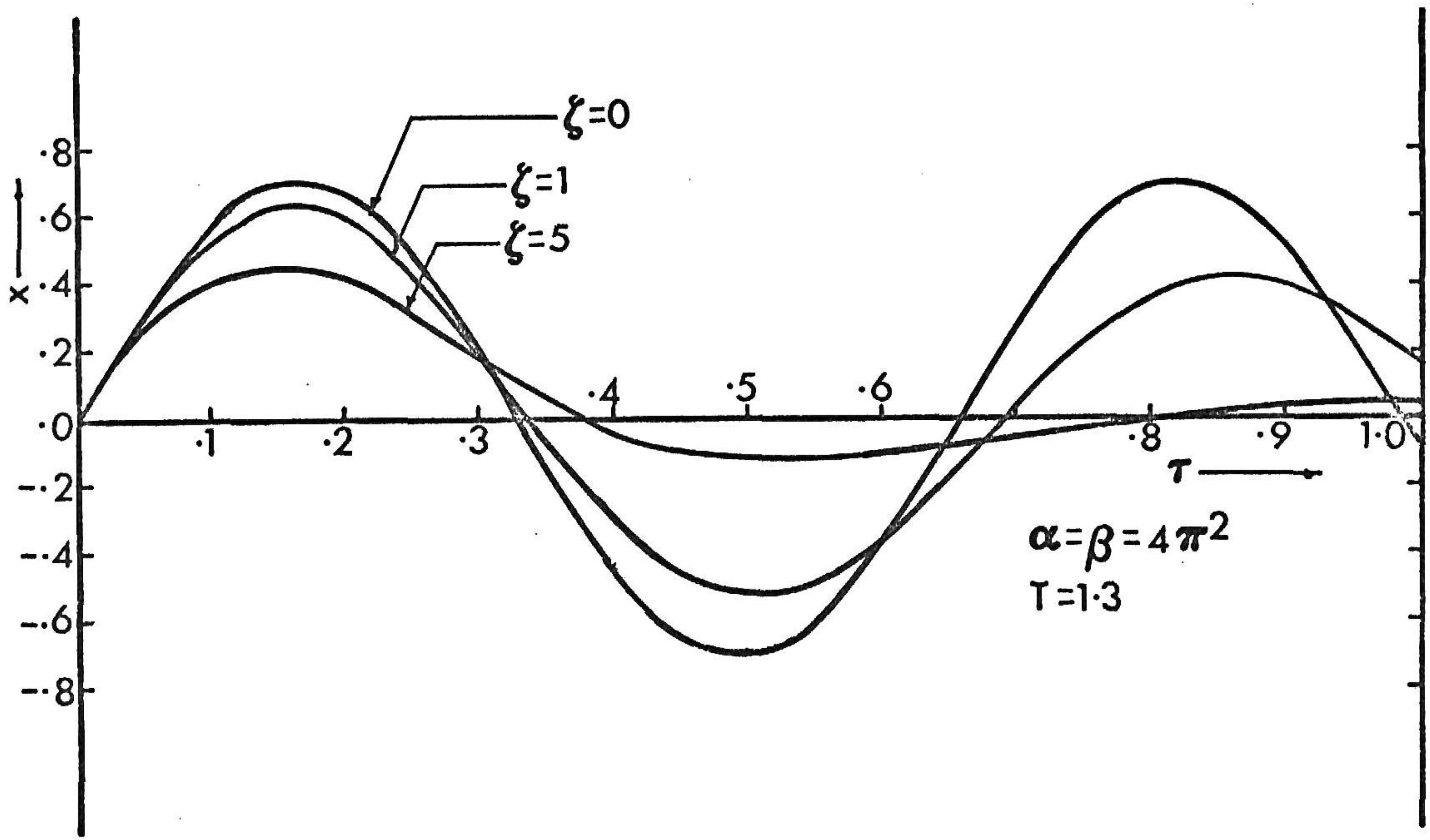


FIG. 2 Average values of the bounding functions for the case of

$\ddot{x} + \xi\,\dot{x} + \alpha x + \beta x^3 = 0, \quad x(0) = 0, \quad \dot{x}(0) = 5.0$

II The case of the nonlinear spring mass system with nonlinear damping.

(i) The case of

$$\ddot{x} + \xi\ \{\dot{x} + \varepsilon(\dot{x})^3\} + \alpha x + \beta x^3 = 0, \qquad \tau \varepsilon I_1 \tag{49}$$

$$x(0) = 0 \tag{50}$$

$$\dot{x}(0) = 5.0 \tag{51}$$

$$\alpha = 2\beta = 4\pi^2 \tag{52}$$

$$T = 1.3 \tag{53}$$

$$\varepsilon = 0.25 \tag{54}$$

was considered. For the various values of ξ, $\xi = 0.0, 0.2, 1.0, 2.0$, bounds were obtained and bounding curves for the average value of x as a function of τ are presented in Figure 3.

Bounds were obtained for the case $\xi = 0.2$ and the results, in the form of bounding coeficients, are presented in Table 4.

Bounds for the case $\xi = 1.0$, are presented in Table 5. The maximum spread in the bounds for $\xi = 1.0$, was 0.54915×10^{-5} which was about 0.01093% of the maximum of the absolute value of x_u.

(ii) The case of

$$\ddot{x} + \xi\ \{\dot{x} + \varepsilon(\dot{x})^3\} + \alpha x + \beta x^3 = 0, \qquad \tau \varepsilon I_1 \tag{55}$$

$$x(0) = 0 \tag{56}$$

$$\dot{x}(0) = 5.0 \tag{57}$$

$$\alpha = \beta = 4\pi^2 \tag{58}$$

$$T = 1.3 \tag{59}$$

$$\varepsilon = 0.25 \tag{60}$$

was considered. For the various values of ξ, $\xi = 0.0, 0.2, 1.0, 2.0$, bounds were obtained and bounding curves for the average value of x as a function of τ are presented in Figure 4.

Bounds for the case $\xi = 0.2$, are presented in Table 6. The maximum spread in the bounds for $\xi = 0.2$, was 0.25167×10^{-3} which was about 0.0390743% of the maximum of the absolute value of x_u.

Bounds were obtained for the case $\xi = 2.0$ and the results, in the form of bounding coefficients, are presented in Table 7.

(iii) The case of

$$\ddot{x} + \xi \{\dot{x} - \varepsilon(\dot{x})^3\} + \alpha x + \beta x^3 = 0 \qquad \tau \varepsilon I_1 \tag{61}$$

$$x(0) = 0 \tag{62}$$

$$\dot{x}(0) = 5.0 \tag{63}$$

$$\alpha = \beta = 4\pi^2 \tag{64}$$

$$T = 1.3 \tag{65}$$

$$\varepsilon = 0.01 \tag{66}$$

was considered. For the various values of ξ, $\xi = 0.0, 1.0, 2.0, 5.0$, bounds were obtained and bounding curves for the average value of x as a function of τ are presented in Figure 5.

Bounds for the case $\xi = 2.0$, are presented in Table 8. The maximum spread in the bounds for $\xi = 2.0$, was 0.59415×10^{-6} which was about 0.00009735% of the maximum of the absolute value of x_u.

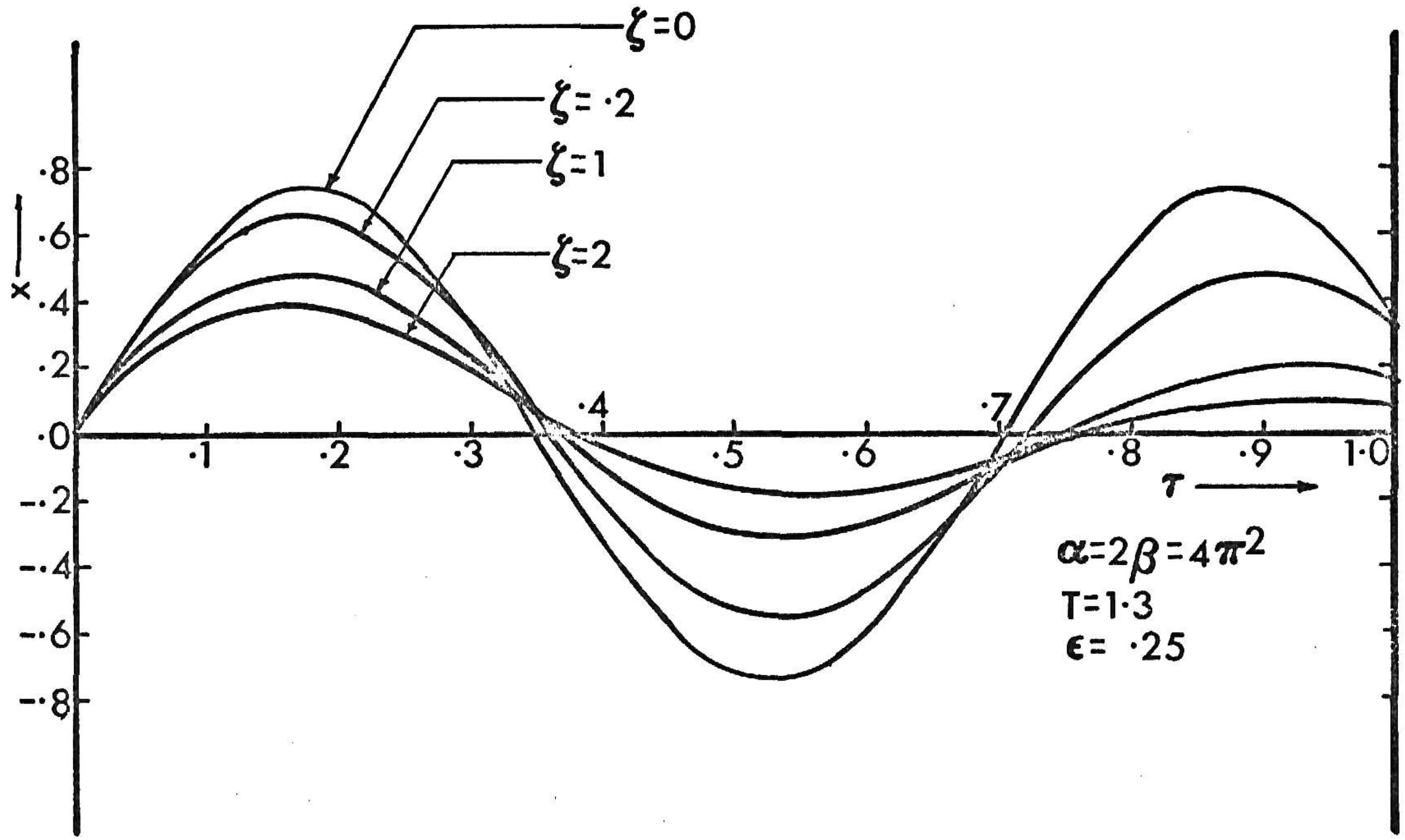


FIG. 3 Average values of the bounding functions for the case of

$\ddot{x} + \xi\{\dot{x} + \varepsilon\,(\dot{x})^3\} + \alpha x + \beta x^3 = 0, \quad x(0) = 0, \quad \dot{x}(0) = 5.0$

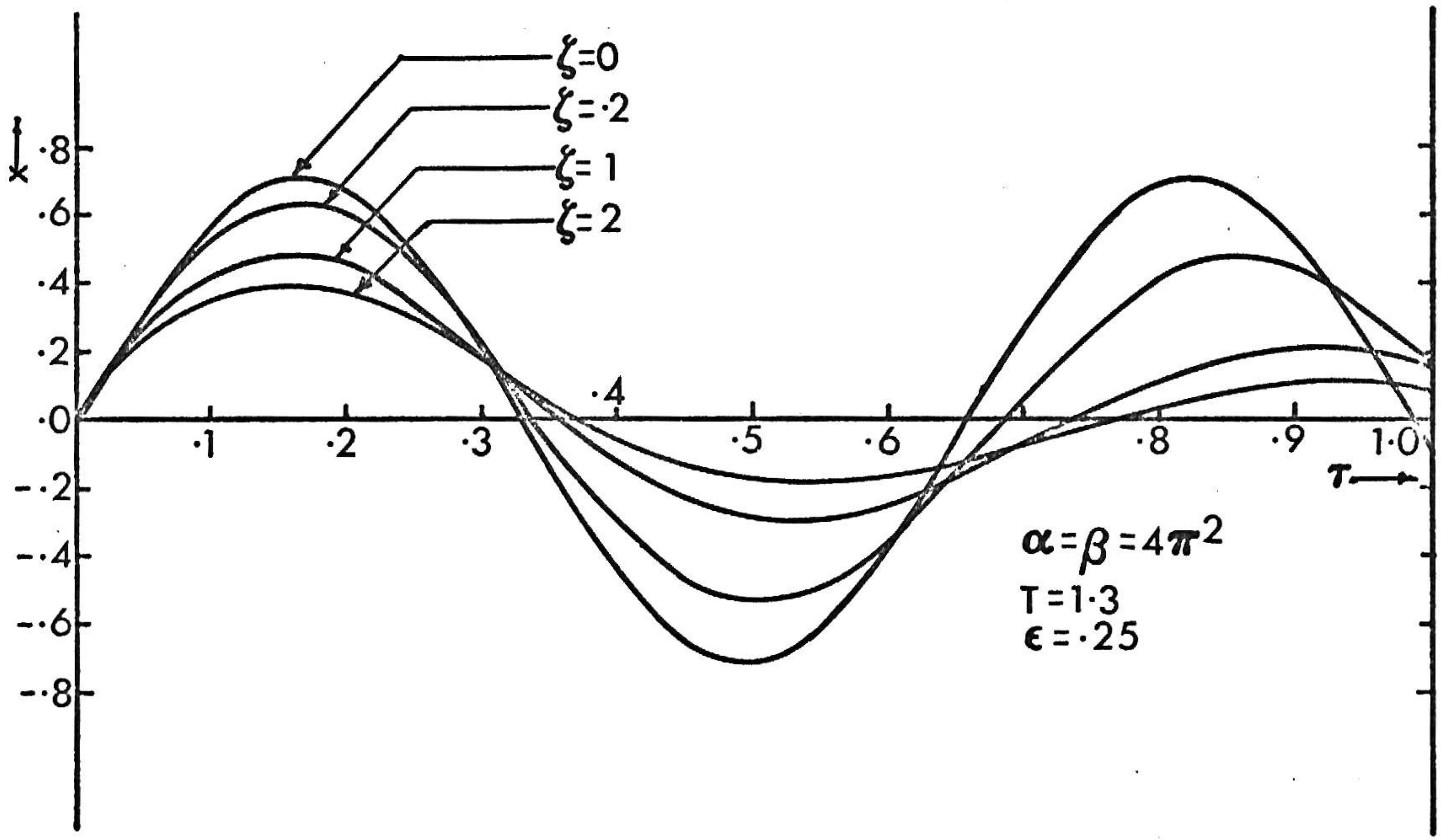


FIG. 4 Average values of the bounding functions for the case of

$\ddot{x} + \xi\{\dot{x} + \varepsilon\,(\dot{x})^3\} + \alpha x + \beta x^3 = 0, \quad x(0) = 0, \quad \dot{x}(0) = 5.0$

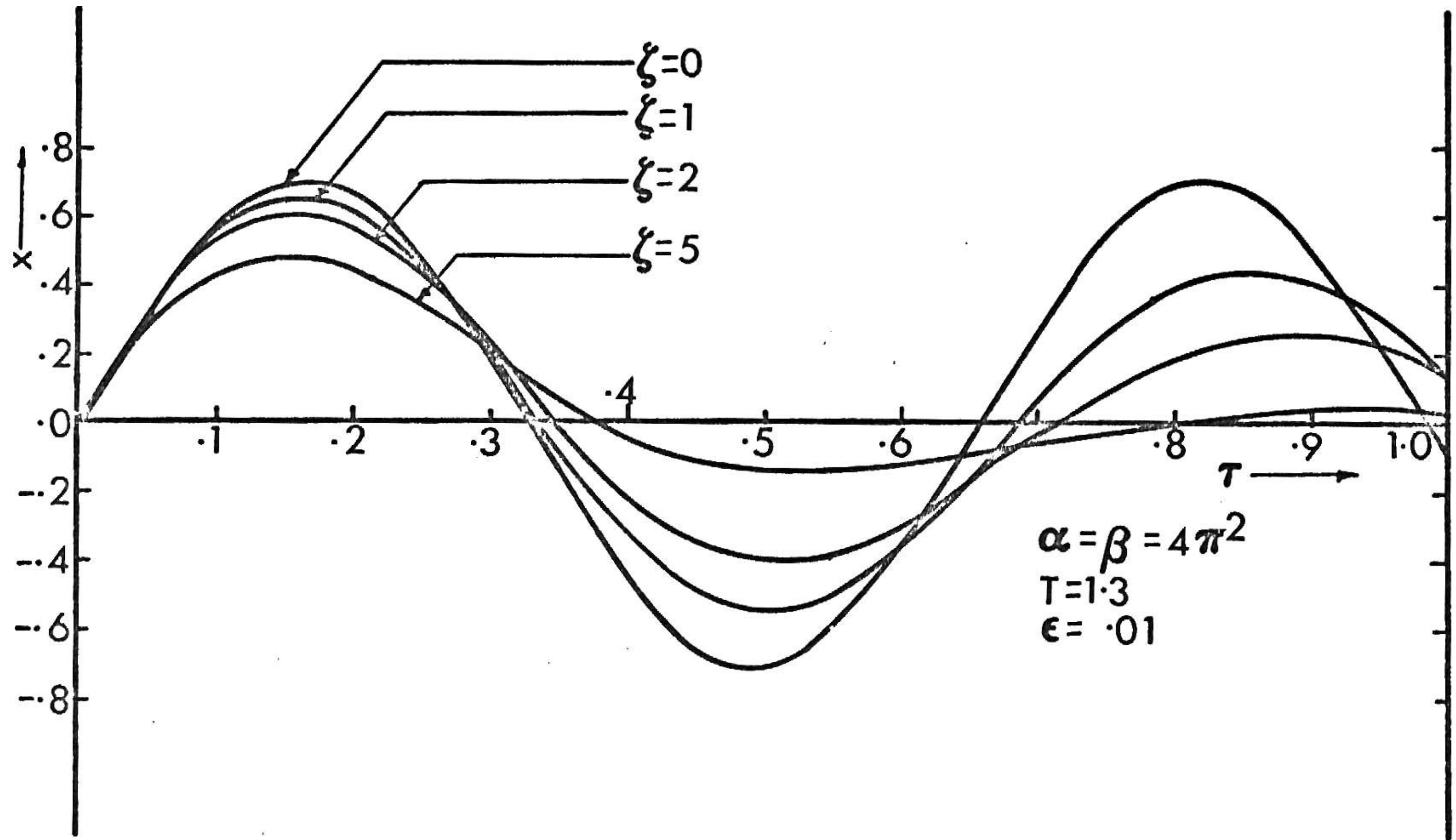


FIG. 5 Average values of the bounding functions for the case of

$$\ddot{x} + \xi\{\dot{x} - \varepsilon\,(\dot{x})^3\} + \alpha x + \beta x^3 = 0, \quad x(0) = 0, \quad \dot{x}(0) = 5.0$$

CONCLUSIONS

This report uses a newly developed method for obtaining bounds to the solution of a wide class of second order initial value problems. The method has been applied to several problems of engineering interest.

The results obtained, using an IBM 360/50 computer with 16 place arithmetic were good as shown by the maximum spread in the bounds. It appears that the method can be used successfully for many engineering analysis problems; the limitations being those of the theorem and the accuracy of machine computations. An important feature of this method is the elimination of "Error Analysis".

Improved results for a nonlinear system can be obtained by increasing the number of steps in the Runge-Kutta-Nystrom integration process and the number of terms in the Tchebyshev polynomials. However, the computer time and cost become a practical limitation in some cases. It is, therefore, concluded that the use of a larger and faster computer is essential for the success of this procedure when applied to nonlinear problems.

The time required to complete a single case was approximately 11 minutes on the IBM 360/50 computer.

TABLE 1

Upper and Lower Bounds for the case $\ddot{x} + \xi\dot{x} + \alpha x + \beta x^3 = 0$, $x(0) = 0$, $\dot{x}(0) = 5.0$, $\alpha = 2\beta = 4\pi^2$, $\xi = 0.0$, $T = 1.3$, $N = 50$, Number of integration steps = 2000, $\Delta_u = 0.8 \times 10^{-7}$, $\Delta_\ell = -0.8 \times 10^{-7}$

TAU	X UPPER	X LOWER	X AVG.	X DIF.
0.0	0.0	0.0	0.0	0.0
0.05	0.315903274612	0.315903274269	0.315903274441	0.1715×10^{-9}
0.1	0.576269930415	0.576269928968	0.576269929692	0.7232×10^{-9}
0.15	0.725830515403	0.725830511780	0.725830513591	0.1811×10^{-8}
0.2	0.725965089992	0.725965082393	0.725965086192	0.3799×10^{-8}
0.25	0.576636129000	0.576636114465	0.576636121733	0.7267×10^{-8}
0.3	0.316413572861	0.316413547090	0.316413559975	0.1288×10^{-7}
0.35	0.000557382261	0.000557339570	0.000557360916	0.2134×10^{-7}
0.4	-0.315392792344	-0.315392860052	-0.315392826198	0.3385×10^{-7}
0.45	-0.575903354491	-0.575903461529	-0.575903408010	0.5351×10^{-7}
0.5	-0.725695403754	-0.725695578524	-0.725695491139	0.8738×10^{-7}
0.55	-0.726099060667	-0.726099356975	-0.726099208821	0.1481×10^{-6}
0.6	-0.577001730271	-0.577002237467	-0.577001983869	0.2535×10^{-6}
0.65	-0.316923259908	-0.316924105112	-0.316923682510	0.4226×10^{-6}
0.7	-0.001114047473	-0.001115395602	-0.001114721538	0.6740×10^{-6}
0.75	0.314883258444	0.314881172681	0.314882215562	0.1042×10^{-5}
0.8	0.575538178642	0.575534935294	0.575536556968	0.1621×10^{-5}
0.85	0.725562639244	0.725557398637	0.725560018940	0.2620×10^{-5}
0.9	0.726237296731	0.726228465988	0.726232881360	0.4415×10^{-5}
0.95	0.577375048073	0.577359983609	0.577367515841	0.7532×10^{-5}
1.0	0.317446169768	0.317421113762	0.317433641765	0.1252×10^{-4}

TABLE 2

Bounding coefficients for the case $\ddot{x} + \xi \dot{x} + \alpha x + \beta x^3 = 0$, $x(0) = 0$, $\dot{x}(0) = 5.0$, $\alpha = 2\beta = 4\pi^2$,

$T = 1.3$, $\xi = 5.0$, $N = 50$

Number of integration steps = 2000, $\Delta_u = 0.8 \times 10^{-9}$, $\Delta_\ell = -0.8 \times 10^{-9}$

i	B_u	B_L	i	B_u	B_L
1	$0.95401745 \times 10^{-1}$	$0.95401743 \times 10^{-1}$	27	$-0.19212973 \times 10^{-8}$	$-0.19212973 \times 10^{-8}$
2	$-0.14542412 \times 10^{0}$	$-0.14542412 \times 10^{0}$	28	$-0.78854319 \times 10^{-8}$	$-0.78854319 \times 10^{-9}$
3	$0.53489186 \times 10^{-1}$	$0.53489185 \times 10^{-1}$	29	$0.82669024 \times 10^{-9}$	$0.82669024 \times 10^{-10}$
4	0.13030168×10^{0}	0.13030168×10^{0}	30	$-0.29384526 \times 10^{-9}$	$-0.29384526 \times 10^{-9}$
5	$-0.14515128 \times 10^{0}$	$-0.14515128 \times 10^{0}$	31	$-0.31925200 \times 10^{-11}$	$-0.31925200 \times 10^{-11}$
6	$0.39644838 \times 10^{-1}$	$0.39644838 \times 10^{-1}$	32	$0.62914609 \times 10^{-10}$	$0.62914609 \times 10^{-10}$
7	$0.10558233 \times 10^{-1}$	$0.10558233 \times 10^{-1}$	33	$-0.36386619 \times 10^{-10}$	$-0.36386619 \times 10^{-10}$
8	$-0.88189346 \times 10^{-2}$	$-0.88189346 \times 10^{-2}$	34	$0.77520946 \times 10^{-11}$	$0.77520947 \times 10^{-11}$
9	$0.20019173 \times 10^{-2}$	$0.20019173 \times 10^{-2}$	35	$0.35635751 \times 10^{-11}$	$0.35635751 \times 10^{-11}$
10	$0.25704069 \times 10^{-5}$	$0.25704072 \times 10^{-5}$	36	$-0.38163621 \times 10^{-11}$	$-0.38163622 \times 10^{-11}$
11	$-0.29698954 \times 10^{-3}$	$-0.29698954 \times 10^{-3}$	37	$0.14016681 \times 10^{-11}$	$0.14016682 \times 10^{-11}$
12	$0.25931192 \times 10^{-3}$	$0.25931192 \times 10^{-3}$	38	$0.38112079 \times 10^{-13}$	$0.38112002 \times 10^{-13}$
13	$-0.10498398 \times 10^{-3}$	$-0.10498398 \times 10^{-3}$	39	$-0.32272575 \times 10^{-12}$	$-0.32272580 \times 10^{-12}$
14	$-0.14144831 \times 10^{-4}$	$-0.14144831 \times 10^{-4}$	40	$0.17376070 \times 10^{-12}$	$0.17376082 \times 10^{-12}$
15	$0.39265692 \times 10^{-4}$	$0.39265692 \times 10^{-4}$	41	$-0.30163913 \times 10^{-13}$	$-0.30163926 \times 10^{-13}$
16	$-0.18247118 \times 10^{-4}$	$-0.18247118 \times 10^{-4}$	42	$-0.19644218 \times 10^{-13}$	$-0.19644286 \times 10^{-13}$
17	$0.80607631 \times 10^{-6}$	$0.80607631 \times 10^{-6}$	43	$0.17781573 \times 10^{-13}$	$0.17781586 \times 10^{-13}$
18	$0.31955571 \times 10^{-5}$	$0.31955571 \times 10^{-5}$	44	$-0.60634775 \times 10^{-14}$	$-0.60634682 \times 10^{-14}$
19	$-0.17381112 \times 10^{-5}$	$-0.17381112 \times 10^{-5}$	45	$-0.35483119 \times 10^{-15}$	$-0.35482424 \times 10^{-15}$
20	$0.33541944 \times 10^{-6}$	$0.33541944 \times 10^{-6}$	46	$0.15331653 \times 10^{-14}$	$0.15331955 \times 10^{-14}$

TABLE 2 (Cont'd)

Bounding coefficients for the case $\ddot{x} + \xi \dot{x} + \alpha x + \beta x^3 = 0$, $x(0) = 0$, $\dot{x}(0) = 5.0$, $\alpha = 2\beta = 4\pi^2$,

$T = 1.3$, $\xi = 5.0$, $N = 50$

Number of integration steps = 2000, $\Delta_u = 0.8 \times 10^{-9}$, $\Delta_\ell = -0.8 \times 10^{-9}$

i	B_u	B_L	i	B_u	B_L
21	$0.13215109 \times 10^{-6}$	$0.13215109 \times 10^{-6}$	47	$-0.80613431 \times 10^{-15}$	$-0.80621706 \times 10^{-15}$
22	$-0.14855422 \times 10^{-6}$	$-0.14855422 \times 10^{-6}$	48	$0.12414256 \times 10^{-15}$	$0.12414979 \times 10^{-15}$
23	$0.64685248 \times 10^{-7}$	$0.64685248 \times 10^{-7}$	49	$0.10139137 \times 10^{-15}$	$0.10149248 \times 10^{-15}$
24	$-0.50295555 \times 10^{-8}$	$-0.50295555 \times 10^{-8}$	50	$-0.74677527 \times 10^{-16}$	$-0.74727110 \times 10^{-16}$
25	$-0.12689167 \times 10^{-7}$	$-0.12689167 \times 10^{-7}$	51	$0.19758797 \times 10^{-16}$	$0.19725037 \times 10^{-16}$
26	$0.85522358 \times 10^{-8}$	$0.85522358 \times 10^{-8}$	52	$-0.15384208 \times 10^{-17}$	$-0.15170871 \times 10^{-17}$

TABLE 3

Bounding coefficient for the case $\ddot{x} + \xi\dot{x} + \alpha x + \beta x^3 = 0$, $x(0) = 0$, $\dot{x}(0) = 5.0$, $\alpha = \beta = 4\pi^2$, $T = 1.3$, $\xi = 1.0$, $N = 50$

Number of integration steps = 2000, $\Delta_u = 0.8 \times 10^{-7}$, $\Delta_\ell = -0.8 \times 10^{-7}$

i	B_u	B_L	i	B_u	B_L
1	0.16128710×10^{0}	0.16128437×10^{0}	27	$0.16472691 \times 10^{-5}$	$0.16472692 \times 10^{-5}$
2	$-0.23223287 \times 10^{-1}$	$-0.23228092 \times 10^{-1}$	28	$-0.35466736 \times 10^{-6}$	$-0.35466721 \times 10^{-6}$
3	0.21217301×10^{0}	0.21216970×10^{0}	29	$-0.40574828 \times 10^{-6}$	$-0.40574829 \times 10^{-6}$
4	$0.92809961 \times 10^{-1}$	$0.92808128 \times 10^{-1}$	30	$0.61036113 \times 10^{-7}$	$0.61036056 \times 10^{-7}$
5	$-0.37569849 \times 10^{0}$	$-0.37569934 \times 10^{0}$	31	$0.13323727 \times 10^{-6}$	$0.13323726 \times 10^{-6}$
6	$0.23455098 \times 10^{-1}$	$0.23454752 \times 10^{-1}$	32	$0.21644997 \times 10^{-8}$	$0.21645184 \times 10^{-8}$
7	$0.88923417 \times 10^{-1}$	$0.88923304 \times 10^{-1}$	33	$-0.61533461 \times 10^{-7}$	$-0.61533459 \times 10^{-7}$
8	$-0.14764903 \times 10^{-1}$	$-0.14764915 \times 10^{-1}$	34	$0.15100554 \times 10^{-8}$	$0.15100500 \times 10^{-8}$
9	$-0.64495641 \times 10^{-2}$	$-0.64495442 \times 10^{-2}$	35	$0.25161447 \times 10^{-7}$	$0.25161446 \times 10^{-7}$
10	$0.92528238 \times 10^{-3}$	$0.92529764 \times 10^{-3}$	36	$-0.35159575 \times 10^{-8}$	$-0.35159560 \times 10^{-8}$
11	$0.11808419 \times 10^{-2}$	$0.11808440 \times 10^{-2}$	37	$-0.80043629 \times 10^{-8}$	$-0.80043624 \times 10^{-8}$
12	$0.13249637 \times 10^{-2}$	$0.13249605 \times 10^{-2}$	38	$0.16464601 \times 10^{-8}$	$0.16464596 \times 10^{-8}$
13	$-0.22266806 \times 10^{-2}$	$-0.22266824 \times 10^{-2}$	39	$0.22827444 \times 10^{-8}$	$0.22827442 \times 10^{-8}$
14	$-0.31939493 \times 10^{-3}$	$-0.31939501 \times 10^{-3}$	40	$-0.37912381 \times 10^{-9}$	$-0.37912367 \times 10^{-9}$
15	$0.12598661 \times 10^{-2}$	$0.12598603 \times 10^{-2}$	41	$-0.77636772 \times 10^{-9}$	$-0.77636765 \times 10^{-9}$
16	$-0.13263851 \times 10^{-3}$	$-0.13263853 \times 10^{-3}$	42	$0.60273080 \times 10^{-10}$	$0.60273038 \times 10^{-10}$
17	$-0.35999109 \times 10^{-3}$	$-0.35999116 \times 10^{-3}$	43	$0.31759144 \times 10^{-9}$	$0.31759142 \times 10^{-9}$
18	$0.80850657 \times 10^{-4}$	$0.80850667 \times 10^{-4}$	44	$-0.29418072 \times 10^{-10}$	$-0.29418061 \times 10^{-10}$
19	$0.65964489 \times 10^{-4}$	$0.65964533 \times 10^{-4}$	45	$-0.12149321 \times 10^{-9}$	$-0.12149320 \times 10^{-9}$
20	$-0.94707604 \times 10^{-5}$	$-0.94707486 \times 10^{-5}$	46	$0.20178891 \times 10^{-10}$	$0.20178888 \times 10^{-10}$

TABLE 3 (Cont'd)

Bounding coefficient for the case $\ddot{x} + \xi\dot{x} + \alpha x + \beta x^3 = 0$, $x(0) = 0$, $\dot{x}(0) = 5.0$, $\alpha = \beta = 4\pi^2$, $T = 1.3$, $\xi = 1.0$, $N = 50$

Number of integration steps = 2000, $\Delta_u = 0.8 \times 10^{-7}$, $\Delta_\ell = -0.8 \times 10^{-7}$

i	B_u	B_L	i	B_u	B_L
21	$-0.19500033 \times 10^{-4}$	$-0.19500045 \times 10^{-4}$	47	$0.39139802 \times 10^{-10}$	$0.39139799 \times 10^{-10}$
22	$-0.37903212 \times 10^{-5}$	$-0.37903281 \times 10^{-5}$	48	$-0.86844948 \times 10^{-11}$	$-0.86844940 \times 10^{-11}$
23	$0.11810519 \times 10^{-4}$	$0.11810521 \times 10^{-4}$	49	$-0.10477444 \times 10^{-10}$	$-0.10477443 \times 10^{-10}$
24	$0.68382741 \times 10^{-6}$	$0.68382936 \times 10^{-6}$	50	$0.25238279 \times 10^{-11}$	$0.25238277 \times 10^{-11}$
25	$-0.54060567 \times 10^{-5}$	$-0.54060569 \times 10^{-5}$	51	$0.17773576 \times 10^{-11}$	$0.17773574 \times 10^{-11}$
26	$0.65924584 \times 10^{-6}$	$0.65924537 \times 10^{-6}$	52	$-0.44008808 \times 10^{-12}$	$-0.44008806 \times 10^{-12}$

TABLE 4

Bounding coefficient for the case $\ddot{x} + \xi\{\dot{x} + \varepsilon(\dot{x})^3\} + \alpha x + \beta x^3 = 0$, $x(0) = 0$, $\dot{x}(0) = 5.0$, $\alpha = 2\beta = 4\pi^2$, $T = 1.3$, $\xi = 0.2$, $N = 50$, $\Delta_u = 0.4 \times 10^{-6}$, $\Delta_\ell = -0.4 \times 10^{-6}$

Number of integration steps = 2000

i	B_u	B_L	i	B_u	B_L
1	0.18688592×10^{0}	0.18687842×10^{0}	27	$-0.62209889 \times 10^{-6}$	$-0.62209975 \times 10^{-6}$
2	$0.93102559 \times 10^{-2}$	$0.92971674 \times 10^{-2}$	28	$0.14341214 \times 10^{-6}$	$0.14341566 \times 10^{-6}$
3	0.26738746×10^{0}	0.26737863×10^{0}	29	$0.38258977 \times 10^{-7}$	$0.38260029 \times 10^{-7}$
4	0.18142744×10^{0}	0.18142262×10^{0}	30	$0.21893544 \times 10^{-6}$	$0.21893427 \times 10^{-6}$
5	$-0.36765605 \times 10^{0}$	$-0.36765831 \times 10^{0}$	31	$-0.50011747 \times 10^{-7}$	$-0.50012374 \times 10^{-7}$
6	$-0.29027014 \times 10^{-1}$	$-0.29027995 \times 10^{-1}$	32	$-0.14883546 \times 10^{-6}$	$-0.14883513 \times 10^{-6}$
7	$0.79962488 \times 10^{-1}$	$0.79962091 \times 10^{-1}$	33	$0.53640921 \times 10^{-7}$	$0.53641253 \times 10^{-7}$
8	$-0.32352314 \times 10^{-2}$	$-0.32353390 \times 10^{-2}$	34	$0.49427735 \times 10^{-7}$	$0.49427668 \times 10^{-7}$
9	$-0.65831750 \times 10^{-2}$	$-0.65831474 \times 10^{-2}$	35	$-0.23434800 \times 10^{-7}$	$-0.23434964 \times 10^{-7}$
10	$0.17631992 \times 10^{-4}$	$0.17680313 \times 10^{-4}$	36	$-0.86497938 \times 10^{-8}$	$-0.86497971 \times 10^{-8}$
11	$-0.38983593 \times 10^{-3}$	$-0.38981716 \times 10^{-3}$	37	$0.23756785 \times 10^{-8}$	$0.23757488 \times 10^{-8}$
12	$0.21704630 \times 10^{-2}$	$0.21704611 \times 10^{-2}$	38	$0.10481686 \times 10^{-8}$	$0.10481825 \times 10^{-8}$
13	$-0.14463294 \times 10^{-3}$	$-0.14463650 \times 10^{-3}$	39	$0.31560607 \times 10^{-8}$	$0.31560346 \times 10^{-8}$
14	$-0.13534175 \times 10^{-2}$	$-0.13534177 \times 10^{-2}$	40	$-0.11252628 \times 10^{-8}$	$-0.11252736 \times 10^{-8}$
15	$0.30669926 \times 10^{-3}$	$0.30669969 \times 10^{-3}$	41	$-0.21799857 \times 10^{-8}$	$-0.21799775 \times 10^{-8}$
16	$0.40591820 \times 10^{-3}$	$0.40591776 \times 10^{-3}$	42	$0.98175183 \times 10^{-9}$	$0.98175823 \times 10^{-9}$
17	$-0.13314908 \times 10^{-3}$	$-0.13314971 \times 10^{-3}$	43	$0.72754411 \times 10^{-9}$	$0.72754205 \times 10^{-9}$
18	$-0.60045664 \times 10^{-4}$	$-0.60045730 \times 10^{-4}$	44	$-0.40968952 \times 10^{-9}$	$-0.40969283 \times 10^{-9}$
19	$0.12236530 \times 10^{-4}$	$0.12236781 \times 10^{-4}$	45	$-0.13376639 \times 10^{-9}$	$-0.13376613 \times 10^{-9}$
20	$0.12630652 \times 10^{-5}$	$0.12631748 \times 10^{-5}$	46	$0.45282837 \times 10^{-10}$	$0.45284326 \times 10^{-10}$

TABLE 4 (Cont'd)

Bounding coefficient for the case $\ddot{x} + \xi\{\dot{x} + \varepsilon(\dot{x})^3\} + \alpha x + \beta x^3 = 0$, $x(0) = 0$, $\dot{x}(0) = 5.0$, $\alpha = 2\beta = 4\pi^2$, $T = 1.3$, $\xi = 0.2$, $N = 50$, $\Delta_u = 0.4 \times 10^{-6}$, $\Delta_\ell = -0.4 \times 10^{-6}$

Number of integration steps = 2000

i	B_u	B_L	i	B_u	B_L
21	$0.17531945 \times 10^{-4}$	$0.17531895 \times 10^{-4}$	47	$0.23620428 \times 10^{-10}$	$0.23620549 \times 10^{-10}$
22	$-0.23662781 \times 10^{-5}$	$-0.23663194 \times 10^{-5}$	48	$0.51379077 \times 10^{-10}$	$0.51378477 \times 10^{-10}$
23	$-0.11677027 \times 10^{-4}$	$-0.11677016 \times 10^{-4}$	49	$-0.16163536 \times 10^{-10}$	$-0.16163639 \times 10^{-10}$
24	$0.33911919 \times 10^{-5}$	$0.33912088 \times 10^{-5}$	50	$-0.36366472 \times 10^{-10}$	$-0.36355260 \times 10^{-10}$
25	$0.37848653 \times 10^{-5}$	$0.37848630 \times 10^{-5}$	51	$0.57433319 \times 10^{-11}$	$0.57433601 \times 10^{-11}$
26	$-0.15323110 \times 10^{-5}$	$-0.15323196 \times 10^{-5}$	52	$0.93842401 \times 10^{-11}$	$0.93841940 \times 10^{-11}$

TABLE 5

Upper and Lower Bounds for the case $\ddot{x} + \xi\{\dot{x} + \varepsilon(\dot{x})^3\} + \alpha x + \beta x^3 = 0$, $x(0) = 0$, $\dot{x}(0) = 5.0$, $\alpha = 2\beta = 4\pi^2$, $\xi = 1.0$, $T = 1.3$, $N = 50$, Number of integration steps = 2000, $\Delta_u = 0.4 \times 10^{-5}$, $\Delta_\ell = -0.4 \times 10^{-5}$

TAU	X UPPER	X LOWER	X AVG.	X DIF.
0.0	0.0	0.0	0.0	0.0
0.05	0.263247210812	0.263247197711	0.263247204261	0.6550×10^{-8}
0.1	0.427641857621	0.427641807763	0.427641832692	0.2492×10^{-7}
0.15	0.499551941250	0.499551820821	0.499551881036	0.6021×10^{-7}
0.2	0.477433982198	0.477433737005	0.477433859601	0.1225×10^{-6}
0.25	0.374097029044	0.374096591218	0.374096810131	0.2189×10^{-6}
0.3	0.222254538372	0.222253846971	0.222254192671	0.3457×10^{-6}
0.35	0.058722013227	0.058721012120	0.058721512673	0.5005×10^{-6}
0.4	−0.089507657862	−0.089509050794	−0.089508354328	0.6964×10^{-6}
0.45	−0.205156334141	−0.205158269032	−0.205157301587	0.9674×10^{-6}
0.5	−0.276725806663	−0.276728555566	−0.276727181115	0.1374×10^{-5}
0.55	−0.297498029891	−0.297502051209	−0.297500040550	0.2010×10^{-5}
0.6	−0.267597163548	−0.267603146009	−0.267600154778	0.2991×10^{-5}
0.65	−0.196148562599	−0.196157408762	−0.196152985681	0.4423×10^{-5}
0.7	−0.099726594339	−0.099739391631	−0.997329989985	0.6398×10^{-5}
0.75	−0.002831210743	−0.002813086473	−0.002822148608	0.9062×10^{-5}
0.8	0.094954867852	0.094929425419	0.094942146635	0.1272×10^{-4}
0.85	0.164016536170	0.163980614962	0.163998575566	0.1760×10^{-4}
0.9	0.201596346147	0.201544855404	0.201570600776	0.2574×10^{-4}
0.95	0.204010842059	0.203935894728	0.203973368394	0.3747×10^{-4}
1.0	0.173186794584	0.173076965069	0.173131879826	0.5491×10^{-4}

TABLE 6

Upper and Lower Bounds for the case $\ddot{x} + \xi\{\dot{x} + \varepsilon(\dot{x})^3\} + \alpha x + \beta x^3 = 0$, $x(0) = 0$, $\dot{x}(0) = 5.0$, $\alpha = \beta = 4\pi^2$, $\xi = 0.2$, $\varepsilon = 0.25$, $T = 1.3$, $N = 50$,
Number of integration steps = 2000, $\Delta_u = 0.3 \times 10^{-5}$, $\Delta_\ell = -0.3 \times 10^{-5}$

TAU	X UPPER	X LOWER	X AVG.	X DIF.
0.0	0.0	0.0	0.0	0.0
0.05	0.302339835984	0.302339824028	0.302339830006	0.5977×10^{-8}
0.1	0.530193979430	0.530193930808	0.530193955119	0.2431×10^{-7}
0.15	0.639423373457	0.639423250392	0.639423311925	0.6153×10^{-7}
0.2	0.602010013377	0.602009745663	0.602009879520	0.1338×10^{-6}
0.25	0.433988395671	0.433987875285	0.433988135478	0.2601×10^{-6}
0.3	0.190433622943	0.190432728548	0.190433175745	0.4471×10^{-6}
0.35	-0.070458719279	-0.070460109657	-0.070459414468	0.6951×10^{-6}
0.4	-0.304147957816	-0.304150024885	-0.304148991850	0.1034×10^{-6}
0.45	-0.472989331898	-0.472992472706	-0.422990902302	0.1570×10^{-5}
0.5	-0.543551402573	-0.543556474662	-0.543553938618	0.2536×10^{-5}
0.55	-0.499405974219	-0.499414567903	-0.499410271061	0.4296×10^{-5}
0.6	-0.355531817633	-0.355546260119	-0.355539038876	0.7221×10^{-5}
0.65	-0.151938721738	-0.151961705837	-0.151950213788	0.1149×10^{-4}
0.7	0.067478520818	0.067443996615	0.067461258716	0.1726×10^{-4}
0.75	0.265581022886	0.265530441457	0.265555732171	0.2529×10^{-4}
0.8	0.410031946629	0.409956309220	0.409994127924	0.3781×10^{-4}
0.85	0.473379373339	0.473260472323	0.473319922831	0.5945×10^{-4}
0.9	0.442024488275	0.441829736349	0.441927112312	0.9737×10^{-4}
0.95	0.325773064737	0.325454020820	0.325613542778	0.1595×10^{-3}
1.0	0.154433236004	0.153929889072	0.154181562538	0.2516×10^{-3}

TABLE 7

Bounding coefficients for the case $\ddot{x} + \xi\{\dot{x} + \varepsilon(\dot{x})^3\} + \alpha x + \beta x^3 = 0$, $x(0) = 0$, $\dot{x}(0) = 5$, $\alpha = \beta = 4\pi^2$, $\xi = 2.0$, $T = 1.3$, $N = 50$, $\Delta_u = 0.4 \times 10^{-4}$, $\Delta_\ell = -0.4 \times 10^{-4}$, Number of integration steps = 2000

i	B_u	B_L	i	B_u	B_L
1	$0.87452391 \times 10^{-1}$	$0.87292144 \times 10^{-1}$	27	$0.34599133 \times 10^{-6}$	$0.34604194 \times 10^{-6}$
2	$-0.84893795 \times 10^{-1}$	$-0.85163204 \times 10^{-1}$	28	$-0.46189885 \times 10^{-6}$	$-0.46190861 \times 10^{-6}$
3	$0.93256260 \times 10^{-1}$	$0.93092066 \times 10^{-1}$	29	$-0.65030057 \times 10^{-7}$	$-0.65025820 \times 10^{-7}$
4	0.10526144×10^{0}	0.10518401×10^{0}	30	$0.43754538 \times 10^{-6}$	$0.43754696 \times 10^{-6}$
5	$-0.15107274 \times 10^{0}$	$-0.15110271 \times 10^{0}$	31	$-0.40291319 \times 10^{-6}$	$-0.40292467 \times 10^{-6}$
6	$0.25171829 \times 10^{-1}$	$0.25161521 \times 10^{-1}$	32	$0.18112215 \times 10^{-6}$	$0.18112904 \times 10^{-6}$
7	$0.15963244 \times 10^{-1}$	$0.15959436 \times 10^{-1}$	33	$-0.13853822 \times 10^{-7}$	$-0.13851186 \times 10^{-7}$
8	$-0.47404898 \times 10^{-2}$	$-0.47415780 \times 10^{-2}$	34	$-0.34712449 \times 10^{-7}$	$-0.34716614 \times 10^{-7}$
9	$-0.21615786 \times 10^{-3}$	$-0.21590581 \times 10^{-3}$	35	$0.10893818 \times 10^{-7}$	$0.10894967 \times 10^{-7}$
10	$0.89452156 \times 10^{-3}$	$0.89482998 \times 10^{-3}$	36	$0.23424741 \times 10^{-7}$	$0.23425293 \times 10^{-7}$
11	$-0.18362527 \times 10^{-2}$	$-0.18362415 \times 10^{-2}$	37	$-0.34485371 \times 10^{-7}$	$-0.34486193 \times 10^{-7}$
12	$0.16213391 \times 10^{-2}$	$0.16212819 \times 10^{-2}$	38	$0.23179108 \times 10^{-7}$	$0.23179768 \times 10^{-7}$
13	$-0.40526884 \times 10^{-2}$	$-0.40529275 \times 10^{-3}$	39	$-0.70500671 \times 10^{-8}$	$-0.70502308 \times 10^{-8}$
14	$-0.26955116 \times 10^{-3}$	$-0.26954109 \times 10^{-3}$	40	$-0.15123816 \times 10^{-8}$	$-0.15126613 \times 10^{-8}$
15	$0.17206591 \times 10^{-3}$	$0.17207800 \times 10^{-3}$	41	$0.18923250 \times 10^{-8}$	$0.18926051 \times 10^{-8}$
16	$0.40392669 \times 10^{-4}$	$0.40386635 \times 10^{-4}$	42	$0.85110295 \times 10^{-9}$	$0.85105874 \times 10^{-9}$
17	$-0.99869547 \times 10^{-4}$	$-0.99874939 \times 10^{-4}$	43	$-0.26410735 \times 10^{-8}$	$-0.26411653 \times 10^{-8}$
18	$0.81343873 \times 10^{-4}$	$0.81346513 \times 10^{-4}$	44	$0.24335689 \times 10^{-8}$	$0.24336528 \times 10^{-8}$
19	$-0.41634131 \times 10^{-4}$	$-0.41632815 \times 10^{-4}$	45	$-0.11871150 \times 10^{-8}$	$-0.11871449 \times 10^{-8}$
20	$0.34060588 \times 10^{-5}$	$0.34057764 \times 10^{-5}$	46	$0.14859550 \times 10^{-9}$	$0.14858288 \times 10^{-9}$

TABLE 7 (Cont'd)

Bounding coefficients for the case $\ddot{x} + \xi\{\dot{x} + \varepsilon(\dot{x})^3\} + \alpha x + \beta x^3 = 0$, $x(0) = 0$, $\dot{x}(0) = 5$, $\alpha = \beta = 4\pi^2$, $\xi = 2.0$, $T = 1.3$, $N = 50$, $\Delta_u = 0.4 \times 10^{-4}$, $\Delta_\ell = -0.4 \times 10^{-4}$, Number of integration steps = 2000

i	B_u	B_L	i	B_u	B_L
21	$0.11050342 \times 10^{-4}$	$0.11050096 \times 10^{-4}$	47	$0.17508420 \times 10^{-9}$	$0.17510998 \times 10^{-9}$
22	$-0.41427990 \times 10^{-5}$	$-0.41431295 \times 10^{-5}$	48	$-0.16866242 \times 10^{-10}$	$-0.16878919 \times 10^{-10}$
23	$-0.48201081 \times 10^{-5}$	$-0.48199280 \times 10^{-5}$	49	$-0.18086826 \times 10^{-9}$	$-0.18087340 \times 10^{-9}$
24	$0.65612726 \times 10^{-5}$	$0.65614910 \times 10^{-5}$	50	$0.26083566 \times 10^{-9}$	$0.26084388 \times 10^{-9}$
25	$-0.39452304 \times 10^{-5}$	$-0.39453829 \times 10^{-5}$	51	$-0.22091030 \times 10^{-9}$	$-0.22091186 \times 10^{-9}$
26	$0.11277009 \times 10^{-5}$	$0.11276632 \times 10^{-5}$	52	$0.80036092 \times 10^{-10}$	$0.80035299 \times 10^{-10}$

TABLE 8

Upper and Lower Bounds for the case $\ddot{x} + \xi\{\dot{x} - \varepsilon(\dot{x})^3\} + \alpha x + \beta x^3 = 0$, $x(0) = 0$, $\dot{x}(0) = 5.0$, $\alpha = \beta = 4\pi^2$, $\xi = 0.2$, $\varepsilon = 0.01$, $T = 1.3$, $N = 50$,
Number of integration steps = 2000, $\Delta_u = 0.15 \times 10^{-7}$, $\Delta_\ell = -0.15 \times 10^{-7}$

TAU	X UPPER	X LOWER	X AVG.	X DIF.
0.0	0.0	0.0	0.0	0.0
0.05	0.300462432007	0.300462431943	0.300462431975	0.3169×10^{-10}
0.1	0.517773905016	0.517773904754	0.517773904885	0.1309×10^{-9}
0.15	0.608864235923	0.608864235281	0.608864235602	0.3212×10^{-9}
0.2	0.562054153996	0.562054152681	0.562054153338	0.6577×10^{-9}
0.25	0.407338671146	0.407338668708	0.407338669927	0.1218×10^{-8}
0.3	0.195660256912	0.195660252753	0.195660254833	0.2079×10^{-8}
0.35	-0.025252332401	-0.025252339017	-0.025252335709	0.3307×10^{-8}
0.4	-0.217895194680	-0.217895204763	-0.217895199722	0.5041×10^{-8}
0.45	-0.352830948510	-0.352830963727	-0.352830956118	0.7608×10^{-8}
0.5	-0.410512179219	-0.410512202467	-0.410512190843	0.1162×10^{-7}
0.55	-0.387678760064	-0.387678796034	-0.387678778049	0.1798×10^{-7}
0.6	-0.299069825397	-0.299069880927	-0.299069853162	0.2776×10^{-7}
0.65	-0.170490612760	-0.170490696940	-0.170490654850	0.4208×10^{-7}
0.7	-0.029575325101	-0.029575449635	-0.029575387368	0.6226×10^{-7}
0.75	0.099502155643	0.099501974814	0.099502065228	0.9041×10^{-7}
0.8	0.197697875812	0.197697614836	0.197697745324	0.1304×10^{-6}
0.85	0.252347697464	0.252347318966	0.252347508215	0.1892×10^{-6}
0.9	0.259039090283	0.259038536787	0.259038813535	0.2767×10^{-6}
0.95	0.222382627063	0.222381814559	0.222382220811	0.4062×10^{-6}
1.0	0.154167550567	0.154166362260	0.154166956413	0.5941×10^{-6}

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APPENDIX A

This appendix contains the bounding theorem:

Theorem:

If:

$$\ddot{x} + f\ (t,x,\dot{x}) = 0 \tag{67}$$

$$\ddot{x}_1 + f\ (t,x_u,\dot{x}_1) = \Delta_1 \tag{68}$$

$$\ddot{x}_u + f\ (t,x_1,\dot{x}_u) = \Delta_u \tag{69}$$

$$x(0) = x_1(0) = x_u(0) = x_0 \tag{70}$$

$$\dot{x}(0) = \dot{x}_1(0) = \dot{x}_u(0) = v_0 \tag{71}$$

where

$$t \varepsilon I = [0,T] \tag{72}$$

$$f \varepsilon C^1[R_1] \tag{73}$$

where R_1 is the Region and $C^1[R_1]$ is the class of continuous functions with continuous first derivatives on R_1.

$$R_1 = \{(x,\dot{x},t):\ x\varepsilon[x_{1\ min},\ x_{1\ max},],\ x\varepsilon[\dot{x}_{1\ min},\ \dot{x}_{1\ max}],\ t\varepsilon I\}, \tag{74}$$

with

$$x_{1\ min} = \min\ \{x_u, x_1\},\ t\varepsilon I, \tag{75}$$

$$x_{1\ max} = \max\ \left\{ \begin{matrix} x_u \\ x_1 \end{matrix} \right\},\ \ t\varepsilon I, \tag{76}$$

$$\dot{x}_{1\ min} = \min\ \left\{ \begin{matrix} \dot{x}_u \\ \dot{x}_1 \end{matrix} \right\},\ \ t\varepsilon I, \tag{77}$$

$$\dot{x}_{1\ max} = \max\ \left\{ \begin{matrix} \dot{x}_u \\ \dot{x}_1 \end{matrix} \right\},\ \ t\varepsilon I, \tag{78}$$

and where

$$\Delta_1 < 0 \tag{79}$$

$$\Delta_u > 0 \tag{80}$$

$$\frac{\partial f}{\partial x} \geq 0 \quad \text{on } R_1 \tag{81}$$

then

$$\dot{x}_1 < \dot{x} < \dot{x}_u \quad \text{for} \quad t \varepsilon I \tag{82}$$

and

$$x_1 < x < x_u \quad \text{for} \quad t \varepsilon I. \tag{83}$$

APPENDIX B

This appendix contains a listing of a typical digital computer program and it was used for the solution of

$$\frac{d^2x}{dt^2} + \xi \{\frac{dx}{dt} - \varepsilon (\frac{dx}{dt})^3\} + \alpha x + \beta x^3 = 0$$

with initial conditions $x(0) = 0$ and $\frac{dx}{dt}(0) = 5.0$.

ILLEGIBLE DOCUMENT

THE FOLLOWING DOCUMENT(S) IS OF POOR LEGIBILITY IN THE ORIGINAL

THIS IS THE BEST COPY AVAILABLE

```
            C      ROUNDS 5,    ******************************************************
0001               IMPLICIT REAL*8(A-H,O-Z)
0002               DIMENSION TAU(2001)
0003               DIMENSION X(2001)
0004               DIMENSION XL(2001)
0005               DIMENSION XDOT(2001)
0006               DIMENSION XDOTL(2001)
0007               DIMENSION XDD(63)
0008               DIMENSION XLDD(63)
0009               DIMENSION A(63)
0010               DIMENSION AL(63)
0011               DIMENSION B(63)
0012               DIMENSION BL(63)
0013               DIMENSION D(63)
0014               DIMENSION DS(63)
0015               DIMENSION TAUCOL(63)
0016               DIMENSION XCOL(63)
0017               DIMENSION XCOLL(63)
0018               DIMENSION CH(63),CHD(63),CHDD(63)
0019               DIMENSION CHO(63),CHDO(63),CHDDO(63)
0020               DIMENSION XDOTC(63)
0021               DIMENSION XDOTCL(63)
0022               DIMENSION TAUORT(63)
0023               DIMENSION XORT(63)
0024               DIMENSION XLORT(63)
0025               DIMENSION XDOTO(63)
0026               DIMENSION XDOTLO(63)
0027               DIMENSION CHORT(63,63)
0028               DIMENSION XA(2001),XLA(2001)
0029               DIMENSION XAVG(2001)
0030               DIMENSION XDIFR(2001),FRACER(2001)
0031               INTEGER STEP,STEP1,STEP2,STEPC
0032            10 FORMAT(1H ,4D16.8)
0033            11 FORMAT(1H ,6D12.4)
0034            12 FORMAT (I5,2D18.8)
0035            13 FORMAT('0   TMAX          XO              VO              ALP             BET
                  I       DEL            N')
0036            15 FORMAT(1H0,5E12.4,I9)
0037            16 FORMAT(' ',6D12.4,I5)
0038            17 FORMAT(1H0)
0039            18 FORMAT('0NUMBER OF RUNGE KUTTA POINTS IS ',I5)
0040            21 FORMAT(37H      N          A(N)               AL(N))
0041            22 FORMAT (1H ,7D16.8)
0042            23 FORMAT ('0ORTHOGONAL SUMMATION POINTS')
0043            24 FORMAT('    TAU',13X,'X UPPER',9X,'X LOWER',9X,'XDOT UPPER',6X,'XDO
                 1T LOWER',6X,'XDOTDOT UPPER',3X,'XDOTDOT LOWER')
0044            25 FORMAT ('0RUNGE KUTTA POINTS')
0045            26 FORMAT('    TAUCOL',6X,'XCOL',8X,'XCOLL',7X,'XT',10X,'XDIF',8X,'XDI
                 1FL')
0046            27 FORMAT('0THE MAX X DIF IS',E12.5,', OCCURING AT TAU = ',F7.4)
0047            29 FORMAT(1H ,7E12.4)
0048            30 FORMAT (1H1,'    N',12X,'B',28X,'BL')
0049            31 FORMAT (I5,2D28.18)
0050            32 FORMAT('0TAU ORTHOG     XDOTDOT ERROR       X ERROR')
0051            33 FORMAT (1H ,D10.4,2D18.8)
0052            34 FORMAT('  TAU ORTHOG    XDD RK',18X,'XDD CURVE FIT',11X,'X CURVE FI
                 1T DD')
0053            35 FORMAT(1H ,D12.4,3D24.16)
```

```
   36 FORMAT ('   TAU ORTHOG     X RK',20X,'X CURVE FIT')
   37 FORMAT (1H ,D12.4,2D24.16)
   38 FORMAT ('   TAU ORTHOG     XDD CF - XCF DD',9X,'XDD RK - XCF DD')
   39 FORMAT(5D24.16)
   40 FORMAT(' X0',22X,'C2',22X,'VO*TMAX',17X,'C1',22X,'XDD(0)')
   41 FORMAT(' DEL(0) =',D24.16)
   42 FORMAT(' TU',22X,'X',23X,'X DOT',19X,'X DOTDOT',16X,'FUN1')
   43 FORMAT(' TAU',9X,'X DOTDOT',16X,'X DOT',19X,'X')
   44 FORMAT(1H1,'    TAU',5X,'X UPPER',17X,'X LOWER',17X,'X AVG',19X,
     1'X DIF',19X,'FRACTIONAL ERROR')
   45 FORMAT (1H ,F7.5,5D24.16)
  500 FORMAT (1H ,6F11.3)
  545 FORMAT('        TAU        DEL UPPER    DEL LOWER',
     1'     X UPPER      X LOWER       X DIF')
      PI=3.14159265358979324D0
      TMAX=1.3D0
      X0=0.D0
      V0=5.D0
      ALP=4.D0*PI*PI
      BET=ALP
      ZETA=6.D0
      EPSN=-0.01D0
      DEL=0.75D-9
      DELL=-0.5D-10
      N=50
      N2=N+2
      LN=N-3
      LS=LN
      LNN=N-2
      STEP=2.D0
      STEPC=STEP+1
      NSTEP=(STEPC-1)/10
      STEP1=STEP/LS
      H=1.D0/STEP
      WRITE(3,17)
      WRITE(3,13)
      WRITE(3,16)TMAX,X0,V0,ALP,BET,DEL,N
      WRITE(3,19) STEP
      TAU(1)=0.D0
      X(1)=X0
      XL(1)=X0
      XDOT(1)=V0*TMAX
      XDOTL(1)=V0*TMAX
      STEP2=STEP+1
      DO 607 K=1,STEP
      CALL FUN(F,TAU(K),XL(K),XDOT(K),ALP,BET,DEL,TMAX)
      CALL FUN(FL,TAU(K),X(K),XDOTL(K),ALP,BET,DELL,TMAX)
      SK1=-0.5D0*H*  F
      SK1L=-0.5D0*H*  FL
      Y=X(K)+0.5D0*H*XDOT(K)+0.25D0*SK1*H
      YL=XL(K)+0.5D0*H*XDOTL(K)+0.25D0*SK1L*H
      TAU1=TAU(K)+0.5D0*H
      TAU2=TAU(K)+H
      XDOT1=XDOT(K)+SK1
      XDOT1L=XDOTL(K)+SK1L
      CALL FUN(F,TAU1,YL,XDOT1,ALP,BET,DEL,TMAX)
      CALL FUN(FL,TAU1,Y,XDOT1L,ALP,BET,DELL,TMAX)
      SK2=-0.5D0*H*  F
```

```
      SK2L =-0.5D0*H* FL
      XDOT1=XDOT(K)+SK2
      XDOT1L=XDOTL(K)+SK2L
      CALL FUN(F,TAU1,YL,XDOT1,ALP,BET,DEL,TMAX)
      CALL FUN(FL,TAU1,Y,XDOT1L,ALP,BET,DELL,TMAX)
      SK3 =-0.5D0*H* F
      SK3L =-0.5D0*H* FL
      Y=X(K)+H*XDOT(K)+SK3*H
      YL=XL(K)+H*XDOTL(K)+SK3L*H
      XDOT1=XDOT(K)+2.0D0*SK3
      XDOT1L=XDOTL(K)+2.0D0*SK3L
      CALL FUN(F,TAU2,YL,XDOT1,ALP,BET,DEL,TMAX)
      CALL FUN(FL,TAU2,Y,XDOT1L,ALP,BET,DELL,TMAX)
      SK4=-0.5D0*H* F
      SK4L=-0.5D0*H* FL
      TAU(K+1)=H*K
      X(K+1)=X(K)+H*XDOT(K)+(SK1+SK2+SK3)*H/3.D0
      XL(K+1)=XL(K)+H*XDOTL(K)+(SK1L+SK2L+SK3L)*H/3.D0
      XDOT(K+1)=XDOT(K)+(SK1+2.0D0*(SK2+SK3)+SK4)/(3.0D0 )
  607 XDOTL(K+1)=XDOTL(K)+(SK1L+2.0D0*(SK2L+SK3L)+SK4L)/(3.0D0 )
      ARG1=PI/N
      ARG2=PI/(2.D0*N)
      DO 600 I=1,N
  600 TAUORT(I)=0.5D0*DCOS(ARG1*I-ARG2)+0.5D0
      WRITE(3,23)
      WRITE(3,24)
      FSTEP=STEP
      DO 601 I=1,N
      TORT=FSTEP*TAUORT(I)
      ITORT=TORT
      K=ITORT+1
      H=(TORT-ITORT)/FSTEP
      CALL FUN(F,TAU(K),XL(K),XDOT(K),ALP,BET,DEL,TMAX)
      CALL FUN(FL,TAU(K),X(K),XDOTL(K),ALP,BET,DELL,TMAX)
      SK1=-0.5D0*H* F
      SK1L=-0.5D0*H* FL
      Y=X(K)+0.5D0*H*XDOT(K)+0.25D0*SK1*H
      YL=XL(K)+0.5D0*H*XDOTL(K)+0.25D0*SK1L*H
      TAU1=TAU(K)+0.5D0*H
      TAU2=TAU(K)+H
      XDOT1=XDOT(K)+SK1
      XDOT1L=XDOTL(K)+SK1L
      CALL FUN(F,TAU1,YL,XDOT1,ALP,BET,DEL,TMAX)
      CALL FUN(FL,TAU1,Y,XDOT1L,ALP,BET,DELL,TMAX)
      SK2=-0.5D0*H* F
      SK2L=-0.5D0*H* FL
      XDOT1=XDOT(K)+SK2
      XDOT1L=XDOTL(K)+SK2L
      CALL FUN(F,TAU1,YL,XDOT1,ALP,BET,DEL,TMAX)
      CALL FUN(FL,TAU1,Y,XDOT1L,ALP,BET,DELL,TMAX)
      SK3 =-0.5D0*H* F
      SK3L=-0.5D0*H* FL
      Y=X(K)+H*XDOT(K)+SK3*H
      YL=XL(K)+H*XDOTL(K)+SK3L*H
      XDOT1=XDOT(K)+2.0D0*SK3
      XDOT1L=XDOTL(K)+2.0D0*SK3L
      CALL FUN(F,TAU2,YL,XDOT1,ALP,BET,DEL,TMAX)
      CALL FUN(FL,TAU2,Y,XDOT1L,ALP,BET,DELL,TMAX)
```

```
      SK4=-0.5D0*H*  F
      SK4L=-0.5D0*H* FL
      XORT(I)=X(K)+H*XDOT(K)+(SK1+SK2+SK3)*H/3.D0
      XLORT(I)=XL(K)+H*XDOTL(K)+(SK1L+SK2L+SK3L)*H/3.D0
      XDOTO(I)=XDOT(K)+(SK1+SK2+SK2+SK3+SK3+SK4)/(3.0D0  )
      XDOTLO(I)=XDOTL(K)+(SK1L+SK2L+SK2L+SK3L+SK3L+SK4L)/(3.0D0 )
      CALL FUN (F,TAUORT(I),XLORT(I),XDOTO(I),ALP,BET,DEL,TMAX)
      CALL FUN (FL,TAUORT(I),XORT(I),XDOTLO(I),ALP,BET,DELL,TMAX)
      XDD(I)=-F
      XLDD(I)=-FL
      WRITE(3,22) TAUORT(I),XORT(I),XLORT(I),XDOTO(I),XDOTLO(I)
     1,XDD(I),XLDD(I)
  601 CONTINUE
      WRITE(3,25)
      WRITE(3,26)
      DO 310 K=1,LS
      TAUCOL(K)=TAU(K*STEP1+1)
      XCOL(K)=X(K*STEP1+1)
      XCOLL(K)=XL(K*STEP1+1)
      XDOTC(K)=XDOT(K*STEP1+1)
      XDOTCL(K)=XDOTL(K*STEP1+1)
      XT=0.D0
      XDIF=XT-XCOL(K)
      XDIFL=XT-XCOLL(K)
      WRITE(3,11)TAUCOL(K),XCOL(K),XCOLL(K),XT,XDIF,XDIFL
  310 CONTINUE
  553 CALL FUN(FZERO,0.0D0,XL(1),XDOT(1),ALP,BET,DEL,TMAX)
      CALL FUN(FLZERO,0.0D0,X(1),XDOTL(1),ALP,BET,DELL,TMAX)
      CALL CHEB(CHO,0.0D0,N2)
      CALL CHEBD(CHDO,CHO,0.0D0,N2)
      CALL CHEBDD(CHDDO,CHDO,0.0D0,N2)
      DO 602 J=1,N
      CALL CHEB(CH,TAUORT(J),N2)
      DO 602 I=1,N2
  602 CHORT(I,J)=CH(I)
      CONST1=2.D0/N
      DO 603 I=1,N
      SUM=0.D0
      SUML=0.D0
      DO 604 J=1,N
      SUM=SUM+XDD(J)*CHORT(I,J)
  604 SUML=SUML+XLDD(J)*CHORT(I,J)
      A(I)=CONST1*SUM
  603 AL(I)=CONST1*SUML
      A(1)=.5D0*A(1)
      AL(1)=.5D0*AL(1)
      DO 610 J=1,4
      A(N+J)=0.D0
  610 AL(N+J)=0.D0
 2012 CONTINUE
      WRITE(3,17)
      WRITE(3,21)
      WRITE(3,12)  (I,A(I),AL(I),I=1,N)
      D(1)=0.D0
      DS(1)=0.D0
      D (2)=(1.D0/32.D0)*A (4)
      DS(2)=(1.D0/32.D0)*AL(4)
      D (3)=(1.D0/96.D0)*A (5)
```

```
0225          DS(3)=(1.D0/96.D0)*AL(5)
0226          D (4)=(-1.D0/64.D0)*A (4)+(1.D0/192.D0)*A (6)
0227          DS(4)=(-1.D0/64.D0)*AL(4)+(1.D0/192.D0)*AL(6)
0228          D (5)=(-1.D0/120.D0)*A (5)+(1.D0/320.D0)*A (7)
0229          DS(5)=(-1.D0/120.D0)*AL(5)+(1.D0/320.D0)*AL(7)
0230          DO 611 J=6,N2
0231          D (J)=(A (J-2)/((J-2)*(J-1))-2.D0*A (J)/((J-2)*J)+A (J+2)/
             1((J-1)*J))/16.D0
0232      611 DS(J)=(AL(J-2)/((J-2)*(J-1))-2.D0*AL(J)/((J-2)*J)+AL(J+2)/
             1((J-1)*J))/16.D0
0233          SUM =-A (4)*CHO(3)/2.D0-A (5)*CHO(4)/3.D0
0234          SUML=-AL(4)*CHO(3)/2.D0-AL(5)*CHO(4)/3.D0
0235          DO 613 J=4,N
0236          SUM =SUM +(A (J)-A (J+2))*CHO(J+1)/J
0237      613 SUML=SUML+(AL(J)-AL(J+2))*CHO(J+1)/J
0238          C1=V0*TMAX-.25D0*SUM
0239          CL1=V0*TMAX-.25D0*SUML
0240          SUM =A (4)*CHO(2)/2.D0+A (5)*CHO(3)/6.D0+(-A (4)/4.D0+A (6)
             1/12.D0)*CHO(4)+(-A (5)/7.5D0+A (7)/20.D0)*CHO(5)
0241          SUML=AL(4)*CHO(2)/2.D0+AL(5)*CHO(3)/6.D0+(-AL(4)/4.D0+AL(6)
             1/12.D0)*CHO(4)+(-AL(5)/7.5D0+AL(7)/20.D0)*CHO(5)
0242          DO 614 J=6,N2
0243          SUML=SUML+(AL(J-2)/((J-2)*(J-1))-2.D0*AL(J)/((J-2)*J)
             1+AL(J+2)/((J-1)*J))*CHO(J)
0244      614 SUM =SUM +(A (J-2)/((J-2)*(J-1))-2.D0*A (J)/((J-2)*J)
             1+A (J+2)/((J-1)*J))*CHO(J)
0245          C2=X0-SUM/16.D0
0246          CL2=X0-SUML/16.D0
0247          B (1)=.5D0*C1 +C2 +3.D0*A (1)/16.D0-A (2)/12.D0-3.D0*A (3)/
             164.D0+D(1)
0248          BL(1)=.5D0*CL1+CL2+3.D0*AL(1)/16.D0-AL(2)/12.D0-3.D0*AL(3)/
             164.D0+DS(1)
0249          B (2)=.5D0*C1 +.25D0*A (1)-3.D0*A (2)/32.D0-A (3)/12.D0+D (2)
0250          BL(2)=.5D0*CL1+.25D0*AL(1)-3.D0*AL(2)/32.D0-AL(3)/12.D0+DS(2)
0251          B (3)=A (1)/16.D0-A (3)/24.D0+D (3)
0252          BL(3)=AL(1)/16.D0-AL(3)/24.D0+DS(3)
0253          B (4)=A (2)/96.D0+D (4)
0254          BL(4)=AL(2)/96.D0+DS(4)
0255          B (5)=A (3)/192.D0+D (5)
0256          BL(5)=AL(3)/192.D0+DS(5)
0257          DO 615 J=6,N2
0258          B(J)=D(J)
0259      615 BL(J)=DS(J)
0260          WRITE(3,30)
0261          WRITE(3,31) ((J,B(J),BL(J)),J=1,N2)
0262          DLMAX=-1000000.0
0263          DLMIN= 1000000.0
0264          DLLMAX=-1000000.0
0265          DLLMIN= 1000000.0
0266          XUMAX=-1000000.
0267          XLMIN= 1000000.
0268          XUMIN= 1000000.
0269          XLMAX=-1000000.
0270          STEPO=STEPC-1
0271          XDIFMX=0.D0
0272          WRITE(3,44)
0273          DO 390 K=1,STEPC
0274          TU=TAU(K)
```

```
      XX=0.0D0
      XXL=0.0D0
      XX1=0.0D0
      XX1L=0.0D0
      XX2=0.0D0
      XX2L=0.0D0
      CALL CHEB(CH,TU,N2)
      CALL CHEBD(CHD,CH,TU,N2)
      CALL CHEBDD(CHDD,CHD,TU,N2)
      DO 551 J=1,N2
      XX=XX+B(J)*CH(J)
      XXL=XXL+BL(J)*CH(J)
      XX1=XX1+B(J)*CHD(J)
      XX1L=XX1L+BL(J)*CHD(J)
      XX2=XX2+B(J)*CHDD(J)
  551 XX2L=XX2L+BL(J)*CHDD(J)
      XA(K)=XX
      XLA(K)=XXL
      CALL FUN1(F,TU,XXL,XX1,ALP,BET,TMAX)
      CALL FUN1(FL,TU,XX,XX1L,ALP,BET,TMAX)
      DL=XX2/(TMAX*TMAX)+F
      DLL=XX2L/(TMAX*TMAX)+FL
      IF(DL.GT.DLMAX) TUMAX=TU
      IF(DL.GT.DLMAX) DLMAX=DL
      IF(DL.LT.DLMIN) TUMIN=TU
      IF(DL.LT.DLMIN) DLMIN=DL
      IF(DLL.GT.DLLMAX) TULMAX=TU
      IF(DLL.GT.DLLMAX) DLLMAX=DLL
      IF(DLL.LT.DLLMIN) TULMIN=TU
      IF(DLL.LT.DLLMIN) DLLMIN=DLL
      IF(XX.GT.XUMAX) TXMAX=TU
      IF(XX.GT.XUMAX) XUMAX=XX
      IF(XXL.LT.XLMIN) TXMIN=TU
      IF(XXL.LT.XLMIN) XLMIN=XXL
      IF(XX.LT.XUMIN) XUMIN=XX
      IF(XXL.GT.XLMAX) XLMAX=XXL
      XAVG(K)=.5D0*(XX+XXL)
      XDIF=.5D0*(XX-XXL)
      XDIFR(K)=XDIF
      IF(XDIF.GT.XDIFMX) TUXMAX=TU
      IF(XDIF.GT.XDIFMX) XDIFMX=XDIF
  390 CONTINUE
      XSTAR=DABS(XUMIN)
      IF(DABS(XLMAX).GT.XSTAR) XSTAR=DABS(XLMAX)
      DO 400 K=1,STEPC
  400 FRACER(K)=XDIFR(K)/XSTAR
      WRITE(3,45) (TAU(K),XA(K),XLA(K),XAVG(K),XDIFR(K),FRACER(K)
     1,K=1,STEPC)
      WRITE(3,17)
      WRITE(3,10) TUMAX,DLMAX
      WRITE(3,10) TUMIN,DLMIN
      WRITE(3,10) TULMAX,DLLMAX
      WRITE(3,10) TULMIN,DLLMIN
      WRITE(3,27) XDIFMX,TUXMAX
      WRITE(3,10) TXMAX,XUMAX
      WRITE(3,10) TXMIN,XLMIN
      STOP
      END
```

```
FORTRAN IV G LEVEL  21                FUN              DATE = 73101        00/58/24              PAGE 0001

 0001              SUBROUTINE FUN (F,TAU,X,XDOT,ALP,BET,DFL,TMAX)
 0002              IMPLICIT REAL*8(A-H,O-Z)
 0003              ZETA=5.D0
 0004              EPSN=-0.01D0
 0005              F=(ZETA*(XDOT/TMAX+EPSN*(XDOT/TMAX)**3)+ALP*X+BET*X*X*X-DEL)
                  1*TMAX*TMAX
 0006              RETURN
 0007              END

FORTRAN IV G LEVEL  21                FUN1             DATE = 73101        00/58/24              PAGE 0001

 0001              SUBROUTINE FUN1(F,TAU,X,XDOT,ALP,BET,TMAX)
 0002              IMPLICIT REAL*8(A-H,O-Z)
 0003              ZETA=5.D0
 0004              EPSN=-0.C1D0
 0005              F= ZETA*(XDOT/TMAX+EPSN*(XDOT/TMAX)**3)+ALP*X+BET*X*X*X
 0006              RETURN
 0007              END

FORTRAN IV G LEVEL  21                CHEB             DATE = 73101        00/58/24              PAGE 0001

 0001              SUBROUTINE CHEB(CHS,TAUS,N)
 0002              IMPLICIT REAL*8(A-H,O-Z)
 0003              DIMENSION CHS(N)
 0004              CHS(1)=1.0D0
 0005              CHS(2)=2.0D0*TAUS-1.0D0
 0006              DO 510 IS=3,N
 0007          510 CHS(IS)=(4.0D0*TAUS-2.0D0)*CHS(IS-1)-CHS(IS-2)
 0008              RETURN
 0009              END

FORTRAN IV G LEVEL  21                CHEBD            DATE = 73101        00/58/24              PAGE 0001

 0001              SUBROUTINE CHEBD(CHDS,CHS,TAUS,N)
 0002              IMPLICIT REAL*8(A-H,O-Z)
 0003              DIMENSION CHDS(N),CHS(N)
 0004              CHDS(1)=0.0D0
 0005              CHDS(2)=2.0D0
 0006              DO 520 IS=3,N
 0007          520 CHDS(IS)=4.0D0*CHS(IS-1)+(4.0D0*TAUS-2.0D0)*CHDS(IS-1)-CHDS(IS-2)
 0008              RETURN
 0009              END

FORTRAN IV G LEVEL  21                CHEBDD           DATE = 73101        00/58/24              PAGE 0001

 0001              SUBROUTINE CHEBDD(CHDDS,CHDS,TAUS,N)
 0002              IMPLICIT REAL*8(A-H,O-Z)
 0003              DIMENSION CHDDS(N),CHDS(N)
 0004              CHDDS(1)=0.0D0
 0005              CHDDS(2)=0.0D0
 0006              DO 530 IS=3,N
 0007          530 CHDDS(IS)=(4.D0*TAUS-2.D0)*CHDDS(IS-1)+8.D0*CHDS(IS-1)-CHDDS(IS-2)
 0008              RETURN
 0009              END
```

BOUNDS FOR LINEAR AND NONLINEAR
INITIAL VALUE PROBLEMS

by

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ABSTRACT

A newly developed theorem and technique for finding upper and lower bounds, applicable to the following class of initial value problems has been applied to solve several problems of engineering interest.

$$\ddot{x}(t) + f(t,x,\dot{x}) = 0$$

$$x(0) = x_0$$

$$\dot{x}(0) = v_0$$

where f is continuous with continuous first derivatives and $\frac{\partial f}{\partial x} \geq 0$.

Bounding functions are obtained in analytic form. The technique is systematic and appears to be well adapted to analysis using high speed digital computers.

Example problems are solved for a linear and nonlinear spring mass damper arrangement. Numerical results were obtained and tabulated.