LOGICAL DESCRIPTIONS

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INTRODUCTION

The theory of logical descriptions is an indispensable tool in the analysis of the formal structure of everyday mathematics. The word "description" will be used to indicate a name which by its own structure unequivocally identifies the object of which it is a name. When one speaks of "the derivative of $f(x)$," "the line through two points," "the square root of x," one is using descriptions. All particular constants or functions of mathematics are given by descriptions.

The purpose of this report is to present in a natural order some of the elementary properties of logical descriptions. To this end a fundamental knowledge of symbolic logic will be assumed on the part of the reader. Those theorems which are specifically relevant to this development are listed subsequently in the latter part of the introduction.

In constructing a description one ascertains a statement $F(x)$ which is true when x is the object in question and takes as a description of the object "the x such that $F(x)$." This is symbolized by "$\imath xF(x)$," (where $\imath$ denotes the Greek letter iota) which is read "the x such that $F(x)$," and which is called a description. The statement "there is exactly one x such that $F(x)$" is symbolized by "$(E_1x)F(x)$." With regard to this notation the following comment is in order. If $(E_1x)F(x)$, then $\imath xF(x)$ is a name of the unique object which makes $F(x)$ true. However, suppose $\sim(E_1x)F(x)$, so that there is no unique object which makes $F(x)$ true. With reference to this statement the following convention will be made. Choose an arbitrary, fixed object, say π, and agree that, if $(E_1x)F(x)$ then $\imath xF(x)$ is a name of the unique x which makes $F(x)$ true, and if $\sim(E_1x)F(x)$, then $\imath xF(x)$ is the name of π. The following principle of mathematical reasoning, known as "modus ponens," will frequently be used in this report: If "P" and "$P \supset Q$" are both proved, then one is entitled to infer that "Q" is proved. The logical analogue to the formal axiom of choice is the principle: "If $(Ex)F(x)$, then $F(y)$." This will be referred to as Rule C.

Finally, the prefixes "(x)" and "(Ex)" will denote "For all x," and "there exists an x such that" respectively.

The following is a list of definitions, axioms and propositions which will be referred to periodically.

<u>Definition 1</u>: $P_1, P_2, \ldots, P_n \vdash Q$ indicates that there is a sequence of statements $S_1, S_2, \ldots, S_s$ such that S_s is Q and for each S_i either:

(1) S_i is an axiom. (2) S_i is a P. (3) S_i is the same as some earlier S_j. (4) S_i is derived from two earlier S's by modus ponens.

<u>Definition 2</u>: $P_1, P_2, \ldots, P_n \vdash_G Q$ indicates that there is a sequence of statements $S_1, S_2, \ldots, S_s$, such that S_s is Q and for each S_i either:

(1) S_i is an axiom. (2) S_i is a P. (3) There is a j less than i such that S_i and S_j are the same. (4) There are j and k, each less than i, such that S_k is S_j S_i. (5) There is a variable x, which does not occur free in any of $P_1, P_2, \ldots, P_n$, and a j less than i such that S_i is $(x)S_j$.

<u>Definition 3</u>: $P_1, P_2, \ldots, P_n \vdash_C Q$ indicates that there is a sequence of statements $S_1, S_2, \ldots, S_s$, such that S_s is Q and for each S_i either:

(1) S_i is an axiom. (2) S_i is a P. (3) There is a j less than i such that S_i and S_j are the same. (4) There are j and k, each less than i, such that S_k is S_j S_i. (5) There is a variable x, which does not occur free in any of $P_1, P_2, \ldots, P_n$ or in any earlier step which is a result of Rule C, and there is a j less than i such that S_i is $(x)S_j$. (6) There are variables x and y, not necessarily distinct; such that y does not occur free in any of $P_1, P_2, \ldots, P_n$, or in any earlier step which is a result of Rule C, and there is a j less than i such that S_j is $(Ex)F(x)$ and S_i is $F(y)$. In this case we say that $F(y)$ is derived from $(Ex)F(x)$ by use of Rule C with y.

Definition 4: Let A denote either a variable or a description. Let P be a statement and Q be the result of replacing each free occurrence of x (if any) in P by an occurrence of A. Consider each variable y which has free occurrences in A. If some bound occurrence of y in Q is one of the free occurrences of y in an occurrence of A in Q which is the result of replacing x in P by an occurrence of A, then the replacement causes confusion. Otherwise the replacement causes no confusion. If P,Q,R, are statements, not necessarily distinct, and $x, x_1, x_2, \ldots, x_n$ are variables not necessarily distinct, then each of the following is an axiom:

Axiom scheme 1: $(x_1)(x_2)\ldots(x_n)(P \supset PP)$.

Axiom scheme 2: $(x_1)(x_2)\ldots(x_n)(PQ \supset P)$.

Axiom scheme 3: $(x_1)(x_2)\ldots(x_n)(P \supset Q . \supset . \sim(QR) \supset \sim(RP))$.

Axiom scheme 4: $(x_1)(x_2)\ldots(x_n)((x).P \supset Q : \supset : (x)P . \supset . (x)Q)$.

If $x, x_1, x_2, \ldots, x_n$ are variables, not necessarily distinct, and P is a statement with no free occurrences of x, then the following is an axiom:

Axiom scheme 5: $(x_1)(x_2)\ldots(x_n)(P \supset (x)P)$.

If $x, y, x_1, x_2, \ldots, x_n$ are variables, not necessarily distinct, then the following is an axiom:

Axiom scheme 6: $(x_1)(x_2)\ldots(x_n)((x)F(x,y) \supset F(y,y))$.

Let $x_1, x_2, \ldots, x_n, x, y, z$, be variables, of which x,y, and z are distinct, but of which $x_1, x_2, \ldots, x_n$ need not be distinct either from each other or from x,y or z. Let P be a statement which contains no bound occurrences of x or y. Let Q and R be the results of replacing all free occurrences of z in P by occurrences of x and y respectively. Then the following is an axiom:

Axiom scheme 7: $(x_1)(x_2)\ldots(x_n)(x,y) : x=y . \supset . Q \supset R$.

Let $x_1, x_2, \ldots, x_n, x$ be variables, not necessarily distinct. Then the following is an axiom:

Axiom scheme 8: $(x_1)(x_2)\ldots(x_n)(x)x=x$.

Proposition 1: If $P_1, \ldots, P_n \vdash Q$, then $P_1, \ldots, P_n, R_1, \ldots, R_n \vdash Q$.

Proposition 2: If $P_1, \ldots, P_n \vdash Q_1$ and $Q_1, \ldots, Q_n \vdash R$, then $P_1, \ldots, P_n, Q_2, \ldots, Q_n \vdash R$.

Proposition 3: If $P_1, \ldots, P_n \vdash Q$ and $R_1, \ldots, R_n \vdash Q \supset S$, then $P_1, \ldots, P_n, R_1, \ldots, R_n \vdash S$.

Proposition 4: If $\vdash Q_1$ and $Q_1, \ldots, Q_n \vdash R$, then $Q_2, \ldots, Q_n \vdash R$.

Proposition 5: If $\vdash Q_1$, $\vdash Q_2, \ldots, \vdash Q_n$, and $Q_1, \ldots, Q_n \vdash R$, then $\vdash R$.

Proposition 6: Let $P_1, P_2, \ldots, P_n$ be statements. Let X be a statement built up from $P_1, P_2, \ldots, P_n$ by use of $\&$ and $\sim$, using each P more than once if desired. Let X take the value T whatever sets of values T and F be assigned to $P_1, P_2, \ldots, P_n$. Then $\vdash X$.

Proposition 7: "Deduction theorem:" If $P_1, P_2, \ldots, P_n, Q \vdash R$; then $P_1, P_2, \ldots, P_n \vdash Q \supset R$.

Proposition 8: "Equivalence theorem:" Let $P_1, P_2, \ldots, P_n, A, B$ be statements and $x_1, x_2, \ldots, x_a$ be variables. W is built up out of some or all of the P's and A and be by means of $\&$, $\sim$, and (x), where each time (x) is used, x is one of $x_1, x_2, \ldots, x_a$, and where one may use each P or each x or A or B more than once if desired. V is the result of replacing some or none of the A's in W by B's. Let $y_1, y_2, \ldots, y_b$ be variables such that there are no free occurrences of any of the x's in $(y_1)(y_2)\ldots(y_b)(A \equiv B)$. Then $\vdash (y_1)(y_2)\ldots(y_b)(A \equiv B) . \supset . W \equiv V$.

Proposition 9: "Substitution theorem:" Assume the hypothesis of the equivalence theorem. If $\vdash A \equiv B$ and $\vdash W$, then $\vdash V$.

Proposition 10: "Corollary to Duality Theorem:" Let $P_1, P_2, \ldots, P_n$ be statements and let W and V be built up out of some or all of the P's by use of $\sim$, $\&$, $\vee$, (x), and (Ex), where each P is used as often as desired, and whatever variables in (x) and (Ex) are used as often as desired. Let X and Y be the results of replacing $\&$ by $\vee$, $\vee$ by $\&$, (x) by (Ex), and (Ex) by (x) in W and V, respectively. If $\vdash W \equiv V$, and if this would continue to hold if we replace each P_i by $\sim P_i$, then $\vdash X \equiv Y$.

Proposition 11: Suppose that $P_1, P_2, \ldots, P_n \vdash_c Q$. Let $y_1, y_2, \ldots, y_n$ be the y's with which Rule C is used in the given demonstration of $P_1, P_2, \ldots, P_n \vdash_c Q$. If none of these occur free in Q, then $P_1, P_2, \ldots, P_n \vdash Q$.

Proposition 12: If $P_1, P_2, \ldots, P_n, Q$ are statements, not necessarily distinct, and x is a variable which has no free occurrences in any of $P_1, P_2, \ldots, P_n$, and if $P_1, P_2, \ldots, P_n \vdash Q$, then $P_1, P_2, \ldots, P_n \vdash (x)Q$.

Proposition 13: "Generalization Principle:" If P is a statement and x is a variable and $\vdash P$, then $\vdash (x)P$.

Proposition 14: $P \supset Q,\ Q \supset R \vdash P \supset R$.

Proposition 15: $\vdash P \supset (Q \supset PQ)$.

Proposition 16: $P \supset R,\ Q \supset R \vdash P \vee Q \supset R$.

Proposition 17: $\vdash P \supset . Q \supset R : \equiv : Q \supset . P \supset R$.

Proposition 18: $\vdash PQ \equiv QP$.

Proposition 19: $\vdash P \supset Q . P \supset R : \equiv : P \supset QR$.

Proposition 20: $\vdash P \supset \sim Q . \equiv . Q \supset \sim P$.

Proposition 21: $\vdash PQ \supset R : \equiv : P \supset . Q \supset R$.

Proposition 22: $\vdash P . Q \vee R : \equiv : PQ \vee PR$.

Proposition 23: $\vdash (x)P \supset P$.

Proposition 24: $\vdash (x)F(x) \equiv (y)F(y)$.

Proposition 25: $\vdash (x)(y)P \equiv (y)(x)P$.

Proposition 26: $\vdash (x) . PQ . \equiv : (x)P . (x)Q$.

Proposition 27: $\vdash (x) . P \supset Q : \equiv : (Ex)P . \supset Q$ if no free occurrences of x in Q.

Proposition 28: $\vdash F(y,y) \supset (Ex)F(x,y)$.

Proposition 29: $\vdash (Ex)F(x) \equiv (Ey)F(y)$.

Proposition 30: $\vdash (x)(y) : x=y \supset y=x$.

Proposition 31: $\vdash (x,y,z) : x=y . y=z . \supset . x=z$.

Proposition 32: $\vdash (x,y) : x=y . \supset . F(x) \supset F(y)$.

Proposition 33: $\vdash (z)(E_1x)x=z.$

Proposition 34: $(E_1\alpha)F(\alpha):\equiv:(E_1x).K(x).F(x).$

AXIOMS FOR DESCRIPTIONS

The axiom schemes for i and proofs of subsequent theorems will be stated in agreement with introductory conventions in order to avoid explicit clarification of necessary assumptions concerning the absence of confusion of bound variables.

Axiom scheme 9: Let $x_1,x_2,\ldots,x_n,x,y$ be variables, not necessarily distinct. Let $F(x),F(iyQ),Q$ be statements. Then $(x_1,x_2,\ldots,x_n):(x)F(x).\supset.F(iyQ)$ is an axiom.

Axiom scheme 10: Let $x_1,x_2,\ldots,x_n,x$ be variables, not necessarily distinct. Let P and Q be statements. Then $(x_1,x_2,\ldots,x_n):(x).P\equiv Q.\supset.ixP=ixQ$ is an axiom.

Axiom scheme 11: Let $x_1,x_2,\ldots,x_n,x,y$ be variables, not necessarily distinct. Let $F(x)$ and $F(y)$ be statements. Then $(x_1,x_2,\ldots,x_n).ixF(x)=iyF(y)$ is an axiom.

Axiom scheme 12: Let $x_1,x_2,\ldots,x_n,x$ be variables, not necessarily distinct. Let P be a statement. Then $(x_1,x_2,\ldots,x_n):.(E_1x)P:\supset:(x):ixP=x.\equiv.P$ is an axiom.

Axiom scheme 9 in conjunction with Axiom scheme 6 says that if A is an object (i.e. a description or variable), then $(x)F(x)\supset F(A)$. As no restrictions are put on iyQ, this means that iyQ is to be interpreted as an object even in the situation where $\sim(E_1y)Q$ and where iyQ has no meaning. Objections to this convention can be resolved by interpreting iyQ as a name for π in all such cases.

Axiom scheme 10 indicates that if P and Q are equivalent for all x, then ixP and ixQ are names of the same object. The question as to what sense this makes if $\sim(E_1x)P$ can be answered similarly as for Axiom scheme 9.

Axiom scheme 11 provides a means of changing bound variables to bound variables as long as no confusion of bound variables is caused. An immediate consequence of Axiom scheme 12 is that if $(E_1x)P$, then ixP is the unique x which makes P true. The proof of this assertion will be postponed until the next section.

GENERAL PROPERTIES OF DESCRIPTIONS

Theorem 1: $\vdash ixP=ixP$

Proof: Let $F(x)$ be $x=x$, then by Axiom scheme 9, $\vdash (x)x=x. \supset .ixP=ixP$. Hence by Axiom scheme 8 and proposition 3 of the statement calculus the theorem follows.

Theorem 2: If $F(x)$ is a statement such that there is no confusion of bound variables in $F(ixF(x))$, then $\vdash (E_1x)F(x) \supset F(ixF(x))$.

Proof: By Axiom scheme 9,

$\vdash (x):ixF(x)=x. \equiv F(x): \supset :ixF(x)=ixF(x). \equiv F(ixF(x))$. By truth values,

$\vdash (x):ixF(x)=x. \equiv F(x): \supset :ixF(x)=ixF(x). \equiv F(ixF(x)) :. \supset .:(x):ixF(x)=x.$

$\equiv F(x): \supset :ixF(x)=ixF(x). \supset F(ixF(x)$. Then

$\vdash (x):ixF(x)=x. \equiv F(x): \supset :ixF(x)=ixF(x). \supset F(ixF(x))$ by the statement calculus. Hence $\vdash ixF(x)=ixF(x):. \supset .:(x):ixF(x)=x. \equiv F(x): \supset :F(ixF(x))$ by Proposition 17 and the Substitution Theorem. So by Theorem 1 and the statement calculus,

$\vdash (x):ixF(x)=x. \equiv F(x): \supset :F(ixF(x))$. Now by Axiom scheme 12

$\vdash (E_1x)F(x): \supset :(x):ixF(x)=x. \equiv F(x)$. Hence by Propositions 14 and 5 of the statement calculus, $\vdash (E_1x)F(x) \supset F(ixF(x))$ which is the theorem.

Theorem 3: If $\vdash (E_1x)F(x)$, then $ixF(x)$ is the unique x which makes $F(x)$ true.

Proof: It follows directly from Theorem 2 that $ixF(x)$ is one of the x's which makes $F(x)$ true. It remains to be shown that $ixF(x)$ is unique. Assume $\vdash F(z)$. By Axiom scheme 12, $\vdash (E_1x)F(x): \supset :(x):ixF(x)=x. \equiv F(x)$. By hypothesis $\vdash (E_1x)F(x)$, hence by modus ponens and Proposition 24 we conclude $\vdash (z):ixF(x)=z. \equiv F(z)$. By Axiom scheme 6, $\vdash ixF(x)=z. \equiv F(z)$, whence $\vdash F(z). \supset .ixF(x)=z$ by the definition of equivalence, and so $\vdash z=ixF(x)$ by modus ponens and the symmetric property of equality.

Theorem 4: $\vdash (y).ix(x=y)=y$, where x and y are distinct variables.

Proof: Choose y, a variable distinct from x and take $F(x)$ to be $x=y$. Then by Proposition 33 $\vdash (y)(E_1x)x=y$, and so $\vdash (E_1x)x=y$ by Proposition 23. Hence by Theorem

2 $\vdash F(\imath x F(x))$. That is, $\vdash \imath x(x=y)=y$. So by the Generalization Principle of the restricted predicate calculus, $\vdash (y).\imath x(x=y)=y$.

Lemma 1: $\vdash (x,y):x=y. \equiv y=x$.

Proof: By Proposition 30 $\vdash (x,y).x=y \supset y=x$. Replacing x by y and y by x gives $\vdash (y,x).y=x \supset x=y$. Then by Proposition 25 $\vdash (x,y).y=x \supset x=y$. By Propositions 15 and 3 of the statement calculus, $\vdash (x,y).x=y \supset y=x:(x,y).y=x \supset x=y$. Hence $\vdash (x,y):x=y. \equiv .y=x$ by Proposition 26 and the definition of equivalence.

Theorem 5: $\vdash (y):\imath xP=y. \equiv .y=\imath xP$.

Proof: Choose a variable z which does not occur in $\imath xP$. Then by Lemma 1, $\vdash (x,z):x=z. \equiv .z=x$. So by Axiom scheme 9, $\vdash (z):\imath xP=z. \equiv .z=\imath xP$. Then by Axiom scheme 6, $\vdash \imath xP=y. \equiv .y=\imath xP$, since it is permitted that there be free occurrences of y in $\imath xP$. The theorem follows by the Generalization Principle.

Corollary 1: $\vdash (y).y=\imath x(x=y)$.

Proof: Let P be $x=y$, then by Theorem 5, $\vdash (y):\imath x(x=y)=y. \equiv .y=\imath x(x=y)$. By Proposition 6 and the definition of equivalence,
$\vdash (y):\imath x(x=y)=y. \supset .y=\imath x(x=y):.(y):y=\imath x(x=y). \supset .\imath x(x=y)=y$. Hence by Axiom scheme 2 $\vdash (y):\imath x(x=y)=y. \supset y=\imath x(x=y)$. Then by Axiom scheme 4 and Theorem 4 the corollary follows.

Corollary 2: $\vdash \imath xP=\imath yQ. \equiv .\imath yQ=\imath xP$.

Proof: By Theorem 5 $\vdash (y):\imath xP=y. \equiv .y=\imath xP$. Hence by Axiom scheme 9, $\vdash \imath xP=\imath yQ. \equiv .\imath yQ=\imath xP$.

Theorem 6a: $\vdash (y,z):\imath xP=y.y=z. \supset .\imath xP=z$.

Proof: Choose variables u,w which do not occur in $\imath xP$. Then by Proposition 31 $\vdash (x,u,w):x=u.u=w. \supset x=w$. So by Axiom scheme 9, $\vdash (u,w):\imath xP=u.u=w. \supset .\imath xP=w$. Then by two applications of Axiom scheme 6, $\vdash \imath xP=y.y=z. \supset .\imath xP=z$, where there are permissible free occurrences of y and z in $\imath xP$. The theorem is then a consequence of two applications of the Generalization Principle.

Theorem 6b: $\vdash (x,z):x=\imath yQ.\imath yQ=z.\supset.x=z.$

Proof: The proof is analogous to the proof of Theorem 6a.

Theorem 6c: $\vdash (z):\imath xP=\imath yQ.\imath yQ=z.\supset \imath xP=z.$

Proof: Choose a variable v which does not occur in $\imath yQ$. Then by Theorem 6a $\vdash (y,v):\imath xP=y.y=v.\supset.\imath xP=v.$ So by Axiom scheme 9, $\vdash (v):\imath xP=\imath yQ.\imath yQ=v.\supset.\imath xP=v.$ Then by Axiom scheme 6, $\vdash \imath xP=\imath yQ.\imath yQ=z.\supset.\imath xP=z.$ The theorem follows by the Generalization Principle.

Corollary 6c: $\vdash \imath xP=\imath yQ.\imath yQ=\imath zR.\supset.\imath xP=\imath zR.$

Proof: The corollary follows directly from the theorem by Axiom scheme 9.

Theorem 6d: $\vdash (y):\imath xP=y.y=\imath zR.\supset.\imath xP=\imath zR.$

Proof: Using Proposition 25 of the restricted predicate calculus and Theorem 6a, the proof is analogous to that of Theorem 6c.

Theorem 7: Let F(x,y) be a statement and let F($\imath xP$,y) and F(y,y) be the results of replacing all free occurrences of x in F(x,y) by occurrences of $\imath xP$ and y respectively, and suppose these replacements cause no confusion of bound variables. Then, $\vdash (y):\imath xP=y.\supset.F(\imath xP,y)\equiv F(y,y).$

Proof: Let z be a variable which does not occur in F(x,y) or $\imath xP$, and let F(x,z) and F(z,z) denote the results of replacing all free occurrences of y in F(x,y) and F(y,y) by occurrences of z. Also let F($\imath xP$,z) be the result of replacing all free occurrences of x in F(x,z) by occurrences of $\imath xP$. Clearly there is no confusion of bound variables in F($\imath xP$,z) since there is none in F($\imath xP$,y). By Axiom scheme 7, $\vdash (x,z):x=z.\supset.F(x,z)\supset F(z,z).$ So by Axiom scheme 9, $\vdash (z):\imath xP=z.\supset.F(\imath xP,z)\supset F(z,z).$ Hence by Axiom scheme 6, $\vdash \imath xP=y.\supset.F(\imath xP,y)\supset F(y,y).$

By Axiom scheme 7 $\vdash (z,x):z=x.\supset.F(z,z)\supset F(x,z)$, and so $\vdash z=x.\supset.F(z,z)\supset F(x,z)$ by two uses of Proposition 23. Then let x=z be R_1 in Proposition 1 of the statement calculus; hence $x=z \vdash z=x.\supset.F(z,z)\supset F(x,z).$ By the Deduction Theorem

$\vdash x=z:\supset:z=x.\supset.F(z,z)\supset F(x,z)$. By truth values $\vdash P\supset.Q\supset R:\equiv:P\supset Q.\supset P\supset R$, and so $\vdash x=z:\supset:z=x.\supset.F(z,z)\supset F(x,z).:\equiv:.x=z\supset z=x.\supset.x=z.\supset F(z,z)\supset F(x,z)$. By the definition of equivalence, Proposition 18 and Axiom scheme 2 we conclude $\vdash x=z\supset z=x:\supset:x=z.\supset.F(z,z)\supset F(x,z)$. Hence by Proposition 30 $\vdash x=z.\supset.F(z,z)\supset F(x,z)$, and so by two uses of the Generalization Principle $\vdash(x,z):x=z.\supset.F(z,z)\supset F(x,z)$. By Axiom scheme 9 $\vdash(z):\imath xP=z.\supset.F(z,z)\supset F(\imath xP,z)$. Then by Axiom scheme 6 $\vdash\imath xP=y.\supset.F(y,y)\supset F(\imath xP,y)$. Consequently, $\vdash\imath xP=y.\supset.F(\imath xP,y)\equiv F(y,y)$ by Proposition 19 and the definition of equivalence. The theorem follows by the Generalization Principle.

<u>Theorem 8</u>: $\vdash F(\imath yQ)\supset(Ex)F(x)$.

Proof: By Axiom scheme 9, $\vdash(x)\sim F(x)\supset\sim F(\imath yQ)$. So by Proposition 20 $\vdash F(\imath yQ)\supset\sim(x)\sim F(x)$. Hence $\vdash F(\imath yQ)\supset(Ex)F(x)$ by the definition of $(Ex)F(x)$.

<u>Lemma 2</u>: $\vdash(y):F(y).\equiv.(Ex).x=y.F(x)$.

Proof: By Proposition 28 $\vdash y=y.F(y).\supset.(Ex).x=y.F(x)$. So by Proposition 21 $\vdash y=y\supset:F(y).\supset.(Ex).x=y.F(x)$. Then by Axiom scheme 8, $\vdash F(y).\supset.(Ex).x=y.F(x)$.

Now by Proposition 32, $\vdash(x,y):x=y.F(x).\supset.F(y)$. So $\vdash(x):x=y.F(x).\supset.F(y)$. by Propositions 23 and 25. Then $\vdash(Ex).x=y.F(x):\supset:F(y)$ by Proposition 27 since there are no free occurrences of x in $F(y)$. Hence $\vdash F(y).\equiv.(Ex).x=y.F(x)$ and the lemma follows by the Generalization Principle.

<u>Theorem 9</u>: $\vdash F(\imath xP).\equiv.(Ex).x=\imath xP.F(x)$.

Proof: Choose a variable z which does not occur in $\imath xP$. Then by Lemma 2, $\vdash(x):F(x).\equiv.(Ez).z=x.F(z)$. Hence by Axiom scheme 9, $\vdash F(\imath xP).\equiv.(Ez).z=\imath xP.F(z)$. So by Proposition 29, $\vdash F(\imath xP).\equiv.(Ex).x=\imath xP.F(x)$ which is the theorem.

<u>Lemma 3</u>: $\vdash(y):F(y).\equiv.(x).x=y\supset F(x)$.

Proof: Replace $F(x)$ by $\sim F(x)$ in Lemma 2 and use the Corollary to the Duality Theorem.

Theorem 10: $\vdash F(\imath xP) :\equiv: (x) : x = \imath xP . \supset . F(x).$

Proof: The proof is analogous to the proof of Theorem 9.

DEFINITION BY CASES

Definitions by cases are very common in the schemata of mathematical dialogue. The requisite circumstances pertaining to such a definition are the existence of two or more mutually exclusive conditions on a variable x; for example, "x is rational" and "x is irrational", and the desire to define f(x) for each x covered by one of the conditions, with different definitions according to which condition x satisfies.

A condition on x is generally a statement involving x. The statement $\sim(P_i P_j)$ is to be interpreted as the statement that the conditions P_i and P_j are mutually exclusive. The assertion that several conditions $P_1, P_2, \ldots, P_n$ be mutually exclusive is the logical product of all statements $\sim(P_i P_j)$ with $1 \leq i < j \leq n$.

Lemma 4: $\vdash (x) . P \equiv Q : \supset : (E_1 x) P \equiv (E_1 x) Q.$

Proof: Take A to be P, B to be Q, W to be $(E_1 x)P$ and V to be $(E_1 x)Q$ in the hypothesis of the Equivalence Theorem (Proposition 8). The lemma is then a direct consequence of the theorem.

Lemma 5: Let $P_1, P_2, \ldots, P_n, R_1, R_2, \ldots, R_n$ be any statements; let Q be the logical product of $\sim(P_i P_j)$ for every i and j with $1 \leq i < j \leq n$. Then for $1 \leq k \leq n$,

$\vdash QP_k : \supset : R_1 P_1 \vee R_2 P_2 \vee \ldots \vee R_n P_n . \equiv . R_k.$

Proof: Assume $QP_k : \supset : R_1 P_1 \vee R_2 P_2 \ldots \vee R_n P_n . \equiv . R_k$ is not universally valid; that is, there exists a set of truth values for $P_1, \ldots, P_n, Q$, and $R_1, \ldots, R_n$ which makes $QP_k : \supset : R_1 P_1 \vee R_2 P_2 \vee \ldots \vee R_n P_n . \equiv . R_k$ take the value F. Inspection of a truth value table for "$\supset$" indicates that this can occur only if QP_k has truth value T and $P_1 R_1 \vee P_2 R_2 \vee \ldots \vee P_n R_n . \equiv . R_k$ has truth value F. The truth value T for QP_k implies both Q and P_k have truth value T. But the truth value T for $\sim(P_i P_j)$ implies

P_iP_j has truth value F for every i and j with $1 \leq i < j \leq n$. Hence P_iP_k has truth value F for $i < k$ and P_kP_i has truth value F for $i > k$ and so P_i has truth value F for all $i \neq k$.

The truth value F for $P_1R_1 \vee P_2R_2 \vee \ldots \vee P_nR_n \equiv R_k$, however, implies either R_k has truth value F and $P_1R_1 \vee P_2R_2 \vee \ldots \vee P_nR_n$ has truth value T or R_k has truth value T and $P_1R_1 \vee P_2R_2 \vee \ldots \vee P_nR_n$ has truth value F.

Case I. The truth value T for $P_1R_1 \vee P_2R_2 \vee \ldots \vee P_nR_n$ and truth value F for R_k implies P_iR_i has the truth value T for some $i \neq k$. Hence P_i has truth value T for some $i \neq k$. But this contradicts the statements that P_i has truth value F for all $i \neq k$.

Case II. The truth value F for $P_1R_1 \vee P_2R_2 \vee \ldots \vee P_nR_n$ implies P_iR_i has truth value F for all i. Then P_kR_k has truth value F and hence P_k has truth value F since R_k has truth value T, which is also a contradiction.

Hence $QP_k : \supset : R_1P_1 \vee R_2P_2 \vee \ldots \vee R_nP_n \equiv R_k$ never has the value F. As it must take either the value T or F in each case, it must take the value T in all cases. Therefore $QP_k : \supset : P_1R_1 \vee P_2R_2 \vee \ldots \vee P_nR_n \equiv R_k$ is a tautology and by the Truth Value Theorem (Proposition 6) the lemma follows.

Lemma 6: If $\vdash P \supset . Q \equiv R$ and $\vdash R$, then $\vdash P \supset Q$.

Proof: By truth values, $\vdash P \supset . Q \equiv R : \supset : P \supset . R \equiv Q$. Hence $\vdash P \supset . R \equiv Q$ by modus ponens and so $P \vdash R \supset Q$. $Q \supset R$ by the definition of equivalence and Proposition 2. Then $\vdash P \supset . R \supset Q$ by Axiom Scheme 2 and the Deduction Theorem. Hence $\vdash R \supset . P \supset Q$ by Proposition 17. The lemma follows by modus ponens with $\vdash R$.

Theorem 11: Let $P_1, P_2, \ldots, P_n$ be statements and let Q be the logical product of all statements $\sim(P_iP_j)$ for all i and j with $1 \leq i < j \leq n$. Let y be a variable. For each i, $1 \leq i \leq n$, let A_i be a variable different from y or a description not containing free occurrences of y. Then for $1 \leq k \leq n$:

I. $\vdash (y).QP_k :\supset: (E_1 y) : y=A_1 . P_1 . \vee . y=A_2 P_2 . \vee . \cdots . \vee . y=A_n P_n .$

II. $\vdash (y).QP_k :\supset: \imath y(y=A_1 . P_1 . \vee . y=A_2 P_2 . \vee . \cdots . \vee . y=A_n P_n)=A_k .$

Proof: By Lemma 5, $\vdash QP_k :\supset: R_1 P_1 \vee R_2 P_2 \vee \ldots \vee R_n P_n . \equiv . R_k .$ Taking R_i to be $y=A_i$ and $F(y)$ to be $y=A_1 . P_1 . \vee . y=A_2 . P_2 . \vee . \cdots . \vee . y=A_n . P_n$ gives $\vdash QP_k :\supset: F(y) . \equiv . y=A_k .$ So by Rule C and Axiom scheme 4, $\vdash (y).QP_k :\supset: (y) : F(y) . \equiv . y=A_k .$ Now by Lemma 4, $\vdash (y) : F(y) . \equiv . y=A_k . :\supset: . (E_1 y)F(y) . \equiv . (E_1 y)F(y) . \equiv . (E_1 y) y=A_k$, where P is F(y) and Q is $y=A_k$. If A_k is a variable, then $\vdash (E_1 y)\ y=A_k$ from Proposition 33 and Axiom scheme 6. If A_k is a description, then $\vdash (E_1 y) y=A_k$ from Proposition 33 and Axiom scheme 9. In either case, $\vdash (E_1 y) y=A_k$, and so $\vdash (y) : F(y) . \equiv . y=A_k . :\supset: . (E_1 y)F(y)$ by Lemma 6. Hence $\vdash (y).QP_k :\supset: (E_1 y)F(y)$ by Propositions 14 and 5, which is part I of the theorem.

Now by Axiom scheme 10, $\vdash (y) : F(y) . \equiv . y=A_k . :\supset: . \imath yF(y)=\imath y(y=A_k)$. Then by Theorem 4 and whichever of Axiom schemes 6 or 9 is appropriate, $\vdash \imath y(y=A_k)=A_k$. By Theorem 6c and Proposition 23, $\vdash \imath yF(y)=\imath y(y=A_k) . \imath y(y=A_k)=A_k . \supset . \imath yF(y)=A_k$, and so by Proposition 21, $\vdash \imath yF(y)=\imath y(y=A_k) . \supset . \imath y(y=A_k)=A_k \supset \imath yF(y)=A_k$. Then by Proposition 17, $\vdash \imath y(y=A_k)=A_k . \supset . \imath yF(y)=\imath y(y=A_k) \supset \imath yF(y)=A_k$. Hence by modus ponens with $\vdash \imath y(y=A_k)=A_k$ one gets $\vdash \imath yF(y)=\imath y(y=A_k) . \supset . \imath yF(y)=A_k$. So by two uses of Propositions 14 and 5, $\vdash (y).QP_k :\supset: \imath yF(y)=A_k$ which is the second part of the theorem.

In practical mathematical discussions the amount of generality of the preceeding theorem is usually more than desirable or useful. In most instances when making a definition by cases the P's will be conditions on x and consequently contain free occurrences of the variable x. However, y can generally be taken as a variable which does not occur in the P's, giving a special case of the theorem. This special case is the one commonly used.

<u>Theorem 12</u>: Let $P_1, P_2, \ldots, P_n$ be statements and let Q be the logical product of all statements $\sim(P_i P_j)$ with $1 \leq i < j \leq n$. Let y be a variable not occurring in any of the P's. For each i, $1 \leq i \leq n$, let A_i be a variable different from y or a description not containing free occurrences of y. Then:

I. $\vdash Q(P_1 \vee P_2 \vee \ldots \vee P_n) : \supset : (E_1 y) : y{=}A_1 . P_1 . \vee . y{=}A_2 . P_2 . \vee . \cdots . \vee . y{=}A_n . P_n .$ Also for $1 \leq k \leq n$:

II. $\vdash QP_k : \supset : \text{i}y(y{=}A_1 . P_1 . \vee . y{=}A_2 . P_2 . \vee . \cdots . \vee . y{=}A_n . P_n){=}A_k .$

Proof: Take F(y) to be $y{=}A_1 . P_1 . \vee . y{=}A_2 . P_2 . \vee . \cdots . \vee . y{=}A_n . P_n$; then by Theorem 11,

a. $\vdash (y) . QP_k : \supset : (E_1 y)F(y).$

b. $\vdash (y) . QP_k : \supset : \text{i}yF(y){=}A_k .$

From part a, $\vdash QP_k : \supset : (E_1 y)F(y)$ by Axiom scheme 6. Hence for $k{=}1,2,\ldots,n$ $\vdash P_k \supset : Q . \supset . (E_1 y)F(y)$ by Proposition 21 and the Substitution Theorem. Then by repeated applications of Propositions 16 and 5, $\vdash P_1 \vee P_2 \vee \ldots \vee P_n : \supset : Q . \supset . (E_1 y)F(y)$ and so, $\vdash Q(P_1 \vee P_2 \vee \ldots \vee P_n) . \supset . (E_1 y)F(y)$ which is part I of the theorem.

Since there are no occurrences of y in any P_i, there are no occurrences of y in QP_k. So by Proposition 23 and Axiom scheme 5, $\vdash (y)QP_k \supset QP_k$ and $\vdash QP_k \supset (y)QP_k$. Then by Proposition 15, $\vdash QP_k \supset (y)QP_k : \supset : (y)QP_k \supset QP_k . \supset . QP_k \supset (y)QP_k . (y)QP_k \supset QP_k .$ Hence $\vdash (y) . QP_k : \equiv : QP_k$ by modus ponens and the definition of equivalence. Then $\vdash QP_k : \supset : \text{i}yF(y){=}A_k$ by the Substitution Theorem and part b. This is the second part of the theorem.

DESCRIPTIONS WITH RESTRICTED QUANTIFICATION

The natural interpretation of $\text{i}\alpha F(\alpha)$ when α is restricted to the range $K(\alpha)$ would seem to be $\text{i}x(K(x).F(x))$; however, in case $\sim(E_1 x).K(x).F(x)$. this definition is not adequate to prove $\vdash K(\text{i}\alpha F(\alpha))$. In order to resolve this

dilemma a definition of $i\alpha F(\alpha)$ by cases is generally adopted. First choose a fixed object denoted by A satisfying the restriction $K(x)$, so that $\vdash K(A)$. Then define $i\alpha F(\alpha)$ to be $ix(K(x).F(x))$ in case $(E_1x).K(x).F(x)$, and to be A in case $\sim(E_1x).K(x).F(x)$. That is, $i\alpha F(\alpha)$ is defined to be:
$iy(y=ix(K(x).F(x)):(E_1x):K(x).F(x).:\vee:.y=A.\sim(E_1x).K(x).F(x))$, where y is a variable not occurring in A or $K(x)F(x)$.

<u>Theorem 13</u>: If α is subject to the restriction $K(\alpha)$ and A is the fixed object chosen for use in defining $i\alpha F(\alpha)$, then:

I. $\vdash(E_1\alpha)F(\alpha).\supset.i\alpha F(\alpha)=ix(K(x)F(x))$.

II. $\vdash\sim(E_1\alpha)F(\alpha).\supset.i\alpha F(\alpha)=A$.

Proof: Taking P_1 to be $(E_1x).K(x).F(x)$, P_2 to be $\sim(E_1x).K(x).F(x)$, and Q to be $\sim(P_1P_2)$ gives $\vdash QP_1\equiv P_1$ and $\vdash QP_2\equiv P_2$ since $\vdash\sim(P_1\sim P_1).P_1\equiv P_1$ and $\vdash\sim(P_1\sim P_1).P_2\equiv P_2$ by truth values. Hence by Theorem 12, part II and the Substitution Theorem,
$\vdash(E_1x).K(x).F(x):\supset:iy(y=ix(K(x).F(x)):(E_1x).K(x).F(x).:\vee:.y=A.\sim(E_1x).K(x).F(x))=ix(K(x).F(x))$. Hence by the definition of $i\alpha F(\alpha)$ and Proposition 34
$\vdash(E_1\alpha)F(\alpha).\supset.i\alpha F(\alpha)=ix(K(x)F(x))$ which is the first part of the theorem.

Also $\vdash\sim(E_1x).K(x).F(x):\supset:iy(y=ix(K(x).F(x)):E_1(x):K(x).F(x).:\vee:.y=A.\sim(E_1x).K(x)F(x))=A$ by Theorem 12, part II. Thus $\vdash\sim(E_1\alpha)F(\alpha).\supset.i\alpha F(\alpha)=A$ by reasoning similar to that in part I.

<u>Theorem 14</u>: If α is subject to the restriction $K(\alpha)$ and z does not occur in $i\alpha F(\alpha)$, then $\vdash(Ez).i\alpha F(\alpha)=z.K(z)$.

Proof: By Theorem 1 and the definition of $i\alpha F(\alpha)$, $\vdash i\alpha F(\alpha)=i\alpha F(\alpha)$. Then by Theorems 8 and 5 $\vdash i\alpha F(\alpha)=i\alpha F(\alpha).\supset.(Ez).i\alpha F(\alpha)=z$. So $\vdash(Ez).i\alpha F(\alpha)=z$ by modus ponens. Hence $\vdash_C i\alpha F(\alpha)=z$ by Rule C. The completion of the proof is based on the logical principle of "proof by cases."

Case I. If $(E_1\alpha)F(\alpha)$, then $(E_1\alpha)F(\alpha)\vdash \imath\alpha F(\alpha)=\imath x(K(x)F(x))$ by Theorem 13, part I, and Proposition 2. Then by Theorem 6c, $\vdash \imath x(K(x).F(x))=\imath\alpha F(\alpha).\imath\alpha F(\alpha)=z.\supset.\imath x(K(x).F(x))=z$. Hence $(E_1\alpha)F(\alpha)\vdash_c \imath x(K(x).F(x))=z$ by the definition of $\vdash_c$, Theorem 5, corollary 2, Proposition 21 and two uses of Proposition 3. By Proposition 34 $\vdash(E_1\alpha)F(\alpha):\equiv:(E_1x).K(x).F(x)$. Hence $(E_1\alpha)F(\alpha)\vdash(E_1x).K(x).F(x)$ and so, $(E_1\alpha)F(\alpha)\vdash(x):\imath x(K(x).F(x)=x.\equiv K(x).F(x)$ by Axiom scheme 12 and Proposition 3. Then $(E_1\alpha)F(\alpha)\vdash \imath x(K(x).F(x)=z.\equiv K(z).F(z)$ by Axiom scheme 6. So $(E_1\alpha)F(\alpha)\vdash \imath x(K(x).F(x))=z.\supset K(z).F(z)$ by the definition of equivalence and Axiom scheme 2. Consequently, $(E_1\alpha)F(\alpha)\vdash_c K(z).F(z)$ by the definition of $\vdash_c$ and Proposition 3 with $(E_1\alpha)F(\alpha)\vdash_c \imath x(K(x)F(x))=z$. Hence $(E_1\alpha)F(\alpha)\vdash_c K(z)$. Now $\vdash \imath\alpha F(\alpha)=z.\supset.K(z)\supset \imath\alpha F(\alpha)=z.K(z)$ by Proposition 15. So $(E_1\alpha)F(\alpha)\vdash_c \imath\alpha F(\alpha)=z.K(z)$ by two uses of Proposition 3 and the definition of $\vdash_c$. Finally, $(E_1\alpha)F(\alpha)\vdash_c (Ez).\imath\alpha F(\alpha)=z.K(z)$ by Proposition 28. Hence $(E_1\alpha)F(\alpha)\vdash(Ez).\imath\alpha F(\alpha)=z.K(z)$ by Proposition 11.

Case II. Suppose $\sim(E_1\alpha)F(\alpha)$, then $\sim(E_1\alpha)F(\alpha)\vdash \imath\alpha F(\alpha)=A$ by Theorem 13, part II, and Proposition 2. So by reasoning analogous to that in Case I, $\sim(E_1\alpha)F(\alpha)\vdash_c A=z$. But $\vdash A=z.\supset.K(A)\equiv K(z)$ by Theorem 7, and so $\vdash A=z.\supset.K(A)\supset K(z)$. However in the definition of $\imath\alpha F(\alpha)$, A was chosen such that $\vdash K(A)$. Hence $\sim(E_1\alpha)F(\alpha)\vdash_c K(z)$ by the definition of $\vdash_c$ and two uses of Propositions 3. Therefore $\sim(E_1\alpha)F(\alpha)\vdash(Ez).\imath\alpha F(\alpha)=z.K(z)$ by reasoning similar to that in Case I.

Now by Case I and Case II, from the Deduction Theorem we get: $\vdash(E_1\alpha)F(\alpha)\supset E(z).\imath\alpha F(\alpha)=z.K(z)$ and $\vdash\sim(E_1\alpha)F(\alpha)\supset E(z).\imath\alpha F(\alpha)=z.K(z)$. Hence by Propositions 16 and 5, $\vdash(E_1\alpha)F(\alpha)\vee\sim(E_1\alpha)F(\alpha).\supset.E(z).\imath\alpha F(\alpha)=z.K(z)$. But by truth values, $\vdash(E_1\alpha)F(\alpha)\vee\sim(E_1\alpha)F(\alpha)$, and so $\vdash(Ez).\imath\alpha F(\alpha)=z.K(z)$ by modus ponens, which is the theorem.

Corollary 1: If α is subject to the restriction $K(\alpha)$ and there is no confusion of bound variables in $K(i\alpha F(\alpha))$, then $\vdash K(i\alpha F(\alpha))$.

Proof: By Lemmas 1 and 2 and the Substitution Theorem, $\vdash (y):K(y).\equiv(Ez).y=z.K(z)$, and so $\vdash K(i\alpha F(\alpha)):\equiv:(Ez).i\alpha F(\alpha)=z.K(z)$ by Axiom scheme 9. Hence $\vdash K(i\alpha F(\alpha))$ by the definition of equivalence and Theorem 14.

Theorem 15: If α and β are subject to the restriction $K(\alpha)$, and z does not occur in $i\beta G(\beta)$, then $\vdash (\alpha)F(\alpha).\supset.F(i\beta G(\beta))$.

Proof: By Theorem 14, $\vdash (Ez).i\beta G(\beta)=z.K(z)$ and so $\vdash_C i\beta G(\beta)=z.K(z)$ by Rule C. Hence $\vdash_C K(z)$ and $\vdash_C i\beta G(\beta)=z$ by Axiom scheme 2 and Proposition 18. By Proposition 23, $\vdash (z).K(z)\supset F(z):\supset:K(z)\supset F(z)$. Using restricted quantification this is; $\vdash(\alpha)F(\alpha).\supset.K(z)\supset F(z)$. Then $\vdash K(z)\supset.(\alpha)F(\alpha)\supset F(z)$ by Proposition 17. Hence $\vdash_C (\alpha)F(\alpha).\supset.F(z)$ by modus ponens and the definition of $\vdash_C$. By Theorem 7 and the definition of $i\beta G(\beta)$, $\vdash i\beta G(\beta)=z.\supset.F(i\beta G(\beta)\equiv F(z)$ and so, $\vdash_C F(i\beta G(\beta)\equiv F(z)$ by modus ponens. Hence $\vdash (\alpha)F(\alpha)\supset F(i\beta G(\beta))$ by the Substitution Theorem and Proposition 11 since z does not occur in $i\beta G(\beta)$.

Theorem 16: If α is subject to the restriction $K(\alpha)$ and A is the fixed object chosen in defining $i\alpha F(\alpha)$, then $\vdash (\alpha).F(\alpha)\equiv G(\alpha):\supset:i\alpha F(\alpha)=i\alpha G(\alpha)$.

Proof: Assume $(\alpha).F(\alpha)\equiv G(\alpha)$. That is, $(x):K(x).\supset.F(x)\equiv G(x)$, and so $K(x).\supset.F(x)\equiv G(x)$ by Axiom scheme 6. Then $K(x).\supset.F(x)\equiv G(x):\supset:K(x)F(x)\equiv K(x)G(x)$ by truth values. So $K(x)F(x)\equiv K(x)G(x)$ by modus ponens. Hence $(\alpha).F(\alpha)\equiv G(\alpha)\vdash K(x)F(x)\equiv K(x)G(x)$ by the definition of $\vdash$. So $(\alpha).F(\alpha)\equiv G(\alpha)\vdash(x):K(x)F(x)\equiv K(x)G(x)$ by Proposition 12.

Case I: Suppose $(E_1\alpha)F(\alpha)$; that is, $(E_1x).K(x).F(x)$. By Lemma 4, $\vdash(x):K(x)F(x)\equiv K(x)G(x):\supset:(E_1x).K(x).F(x)\equiv(E_1x).K(x).G(x)$. Hence $(\alpha).F(\alpha)\equiv G(\alpha)\vdash(E_1x).K(x).F(x)\equiv(E_1x).K(x).G(x)$ by Proposition 3. So by the definition of equivalence, Axiom scheme 2 and Proposition 3, $(\alpha).F(\alpha)\equiv G(\alpha),(E_1\alpha)F(\alpha)\vdash(E_1x)K(x).G(x)$. That is,

$(\alpha).F(\alpha)\equiv G(\alpha)$, $(E_1\alpha)F(\alpha)\vdash(E_1\alpha)G(\alpha)$. Then by Theorem 13, part I and Proposition 2 $(E_1\alpha)F(\alpha)\vdash i\alpha F(\alpha)=ix(K(x).F(x))$ and $(\alpha).F(\alpha)\equiv G(\alpha)$, $(E_1\alpha)F(\alpha)\vdash i\alpha G(\alpha)=ix(K(x).G(x))$ using Proposition 3. However, by Axiom scheme 10, $\vdash(x):K(x).F(x)\equiv K(x).G(x):\supset:ix(K(x).F(x))=ix(K(x).G(x))$. So $(\alpha).F(\alpha)\equiv G(\alpha)\vdash ix(K(x).F(x))=ix(K(x).G(x))$ by Proposition 3. Then by corollary 6c, $\vdash i\alpha F(\alpha)=ix(K(x)F(x)).ix(K(x).F(x))=ix(K(x).G(x)):\supset:i\alpha F(\alpha)=ix(K(x).G(x))$. Hence $(\alpha).F(\alpha)\equiv G(\alpha),(E_1\alpha)F(\alpha)\vdash i\alpha F(\alpha)=ix(K(x).G(x))$ by Proposition 21 and two uses of Proposition 3. Finally $(\alpha).F(\alpha)\equiv G(\alpha)$, $(E_1\alpha)F(\alpha)\vdash i\alpha F(\alpha)=i\alpha G(\alpha)$ by Theorem 5, corollary 2, the transitive property of descriptions and Propositions 3 and 21.

Case II. Suppose $\sim(E_1\alpha)F(\alpha)$. Then by Lemma 4 and reasoning similar to that in Case I, $(\alpha).F(\alpha)\equiv G(\alpha)$, $\sim(E_1\alpha)F(\alpha)\vdash(E_1x)K(x).G(x))$. That is, $(\alpha)F(\alpha)\equiv G(\alpha)$, $\sim(E_1\alpha)F(\alpha)\vdash\sim(E_1\alpha)G(\alpha)$. Then by Theorem 13, part II, $\vdash\sim(E_1\alpha)F(\alpha).\supset.i\alpha F(\alpha)=A$ and $\vdash\sim(E_1\alpha)G(\alpha).\supset.i\alpha G(\alpha)=A$. Hence $\sim(E_1\alpha)F(\alpha)\vdash i\alpha F(\alpha)=A$ by Proposition 2 and $(\alpha).F(\alpha)\equiv G(\alpha)$, $\sim(E_1\alpha)F(\alpha)\vdash i\alpha G(\alpha)=A$ by Propositions 3 and 2. Then $\vdash i\alpha F(\alpha)=A.A=i\alpha G(\alpha).\supset.i\alpha F(\alpha)=i\alpha G(\alpha)$ by corollary 6c and so $(\alpha).F(\alpha)\equiv G(\alpha)$, $\sim(E_1\alpha)F(\alpha)\vdash i\alpha F(\alpha)=i\alpha G(\alpha)$ by Theorem 5, corollary 2, Proposition 21 and two uses of Proposition 3.

Now from Case I and Case II, by two uses of the Deduction Theorem and Proposition 21, one gets: $\vdash(\alpha).F(\alpha)\equiv G(\alpha).(E_1\alpha)F(\alpha).\supset.i\alpha F(\alpha)=i\alpha G(\alpha)$ and $\vdash(\alpha).F(\alpha)\equiv G(\alpha).\sim(E_1\alpha)F(\alpha).\supset.i\alpha F(\alpha)=i\alpha G(\alpha)$. Hence by Propositions 16 and 3 $\vdash(\alpha).F(\alpha)\equiv G(\alpha).(E_1\alpha)F(\alpha)\vee(\alpha).F(\alpha)\equiv G(\alpha).\sim(E_1\alpha)F(\alpha):\supset:i\alpha F(\alpha)=i\alpha G(\alpha)$. Hence, $(\alpha).F(\alpha)\equiv G(\alpha)\vdash(E_1\alpha)F(\alpha)\vee\sim(E_1\alpha)F(\alpha).\supset.i\alpha F(\alpha)=i\alpha G(\alpha)$. by Propositions 22, 21 and 2. But by truth values, $\vdash(E_1\alpha)F(\alpha)\vee\sim(E_1\alpha)F(\alpha)$ and so $(\alpha).F(\alpha)\equiv G(\alpha)\vdash i\alpha F(\alpha)=i\alpha G(\alpha)$ by Proposition 3. The theorem then follows by the Deduction Theorem.

Theorem 17: If α and β are subject to the restriction $K(\alpha)$, then $\vdash i\alpha F(\alpha)=i\beta F(\beta)$.

Proof: The theorem follows from the definition of $i\alpha F(\alpha)$ and $i\beta F(\beta)$ by Axiom scheme 11.

Lemma 7: If $F(x)$, $F(ixP)$, $F(iyQ)$ are statements interpreted by introductory c·ventions, then $\vdash ixP=iyQ. \supset .F(ixP)\equiv F(iyQ)$.

Proof: By Theorem 7, $\vdash (y):ixP=y. \supset .F(ixP,y)\equiv F(y,y)$. Hence the lemma follows by Axiom scheme 9.

Theorem 18: If there are no free occurrences of x in $i\alpha F(\alpha)$, then $\vdash (E_1\alpha)F(\alpha):\supset:(x):i\alpha F(\alpha)=x.\equiv K(x).F(x)$.

Proof: Suppose $(E_1\alpha)F(\alpha)$. Then by Theorem 13, part I, and Proposition 2, $(E_1\alpha)F(\alpha)\vdash i\alpha F(\alpha)=ix(K(x).F(x))$. However by Axiom scheme 12, $(E_1\alpha)F(\alpha)\vdash(x):ix(K(x).F(x))=x.\equiv.K(x)F(x)$. Hence by Theorem 5, corollary 2, Lemma 7, and Proposition 3, $(E_1\alpha)F(\alpha)\vdash(x):i\alpha F(\alpha)=x.\equiv.K(x).F(x)$. The theorem follows by the Deduction Theorem.

Theorem 19: If α and β are subject to the restriction $K(\alpha)$ and there are no free occurrences of β in $i\alpha F(\alpha)$, then $\vdash(E_1\alpha)F(\alpha):\supset:(\beta):i\alpha F(\alpha)=\beta.\equiv.F(\beta)$.

Proof: Assume $(E_1\alpha).F(\alpha)$. Then by Theorem 18 and Proposition 2 $(E_1\alpha)F(\alpha)\vdash(x):i\alpha F(\alpha)=x.\equiv.K(x)F(x)$. So $(E_1\alpha)F(\alpha)\vdash i\alpha F(\alpha)=x.\equiv K(x)F(x)$ by Proposition 23. By truth values,

$\vdash i\alpha F(\alpha)=x.\equiv.K(x)F(x).:\supset:.K(x):\supset:i\alpha F(\alpha)=x.\equiv.F(x)$. Hence

$(E_1\alpha)F(\alpha)\vdash(x):.K(x):\supset:i\alpha F(\alpha)=x.\equiv.F(x)$ by Propositions 3 and 12, or $(E_1\alpha)F(\alpha)\vdash(\beta):i\alpha F(\alpha)=\beta.\equiv.F(\beta)$. The theorem follows by the Deduction Theorem.

The last theorem is analogous to Theorem 4 for unrestricted quantification.

Theorem 20: If α and β are subject to the restriction $K(\alpha)$, then $\vdash(\beta).i\alpha(\alpha=\beta)=\beta$.

Proof: Let $F(x)$ be $x=y$. By Proposition 30, $\vdash y=x \equiv F(x)$. Then by Proposition 3 and the Deduction Theorem, $\vdash K(x).\supset.y=x\equiv F(x)$, and so $\vdash (x):K(x).\supset.y=x\equiv F(x)$ by the Generalization Principle. Since $\vdash (Ey)K(y)$, we get $\vdash_C K(y)$ by Rule C. Hence $\vdash_C K(y):(x):K(x).\supset.y=x\equiv F(x)$ by Proposition 15, the definition of $\vdash_C$, and two uses of modus ponens. So $\vdash (Ey):K(y):(x):K(x).\supset.y=x\equiv F(x)$ by Proposition 28 and Proposition 11. Using restricted quantification this is $\vdash (E_1\alpha)F(\alpha)$. By Theorem 13, part I, $\vdash (E_1\alpha)F(\alpha).\supset.\imath\alpha F(\alpha)=\imath x(K(x)F(x))$. So $\vdash \imath x(K(x)F(x))=\imath\alpha F(\alpha)$ by modus ponens and Theorem 5, corollary 2. Then $\vdash F(\imath x(K(x)F(x))\equiv F(\imath\alpha F(\alpha))$ by Lemma 7. But $\vdash (E_1x)F(x)K(x)\supset F(\imath xK(x)F(x)).K(\imath xF(x)K(x))$ by Theorem 2. Hence $\vdash F(\imath xK(x)F(x))$ by Proposition 34, modus ponens and Axiom scheme 2. Then $\vdash F(\imath\alpha F(\alpha))$ by the definition of equivalence and modus ponens. That is, $\vdash \imath\alpha(\alpha=y)=y$. So $\vdash K(y).\supset.\imath\alpha(\alpha=y)=y$ by Proposition 3 and the Deduction Theorem. Hence $\vdash (y):K(y).\supset.\imath\alpha(\alpha=y)=y$ by the Generalization Principle. Using restricted quantification this is $\vdash (\beta).\imath\alpha(\alpha=\beta)=\beta$.

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LOGICAL DESCRIPTIONS

by

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Descriptions are so numerous in everyday mathematics that their formal structure warrants mention in any logical framework which purports to be adequate for the types of intuitive reasoning used by mathematicians in their mathematical thinking.

It is the purpose of this paper to present some of the fundamental properties of logical descriptions. The first section is simply a statement of the axiom schemes for i and some intuitively obvious consequences.

In proving theorems about descriptions, many of the results are not dependent upon any special axioms about descriptions, but upon the description itself. In practical considerations of $ixF(x)$, if one can prove $(E_1x)F(x)$, and hence infer $F(ixF(x))$, then practically all theorems about $ixF(x)$ follow from this result.

Definitions by cases are common in mathematics, hence a theorem is proved which permits such definitions.

When expressions such as "For all x, F(x)," or "The x such that F(x)," occur in mathematical dialogue it is tacitly understood that there are certain restrictions on the x, and that what is meant is something like, "For all real numbers, x, F(x)," or "The prime number, x, such that F(x)." Hence the last section deals with verifying theorems about logical descriptions with restricted quantification analogous to those for unrestricted quantification.