A computationally efficient bootstrap-equivalent test for ANOVA in skewed populations with a large number of factor levels

by

Richard Opoku-Nsiah

M.S., Youngstown State University, 2011

AN ABSTRACT OF A DISSERTATION

submitted in partial fulfillment of the
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DOCTOR OF PHILOSOPHY

Department of Statistics
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Abstract

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In the first part of this dissertation, we start with describing suitable bootstrap statistics and procedures for hypothesis tests in one- and two-way ANOVA with a large number of levels and small sample sizes. We prove that the theoretical type I error-rate of Akritas and Papadatos (2004) and Wang and Akritas (2006) test statistics and the corresponding bootstrap versions have accuracy of order $O(1/\sqrt{a})$. We then modify their statistics to obtain asymptotically pivotal statistics in our current framework. We prove that the theoretical type I error-rate of the bootstrap version of the pivotal statistics is accurate up to order $O(1/\sqrt{a})$. In the second part of the dissertation, we propose a new test statistic in one-way ANOVA which is asymptotically pivotal in the current setting. We improve the accuracy of approximation of the distribution of the test statistic by deriving asymptotic expansion of the statistic under the current framework and define a new test rejection region through Cornish-Fisher expansion of quantiles. The type I error-rate of the new test has a faster convergence rate and is accurate up to order $O(1/a)$. Simulation studies show that our tests

performs better in terms of type I error-rate but comparable power with that of Akritas and Papadatos (2004) in the large a small n_i setting. The connection between our asymptotic expansions and bootstrap distribution in the large a, small n_i setting is discussed. Our proposed test based on asymptotic expansion and Cornish-Fisher expansion of quantiles have both the advantage of higher accuracy and computational efficiency due to no resampling is needed.

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Chapter 1

Bootstrap test for ANOVA with a large number of levels and skewed populations

1.1 Introduction

For scientific investigations in agricultural screening, many of the experiments collect molecular data using high throughput technologies such as micro-array and sequencing. The data collection from such experiments often arise in the form of one-way and crossed two-way designs with very small number of replications within each treatment combinations. In most cases, the data are skewed in distribution. We are interested in testing the hypothesis of no main treatment effect (one-way and two-way designs) and no interaction effect (two-way design) when the number of treatment combinations is large but with small replications within each treatment combination in the presence of extreme observations and heteroscedastic variances.

In this framework of large number of treatments with small replications within each treatment in the presence of extreme observations and heteroscedastic variances, pioneer studies in the literature include, cf., Akritas and Arnold (2000), Akritas and Papadatos (2004), Wang and Akritas (2004), Boos and Brownie (1995) etc. Bathke (2002), Wang and Akritas (2006), Wang and Akritas (2011) have also conducted research on two-way, three-way ANOVA and other multi-factor designs when the number of treatments is large. In these papers, they presented different test statistics and their asymptotic distributions. Even though the form of their statistics may be different they all give asymptotic normal or chi-square distribution to approximate the distribution of their test statistics. We prove in this dissertation that the error of their approximations is of order $O(a^{-1/2})$ (where a is the number of treatments). With this rate, the type I error of these tests converges slowly to the nominal level when the data are skewed.

It is well known that in the classical small number of treatments with large replications setting, the bootstrap tests and confidence intervals generally have better approximation accuracy. Efron (1979), Beran (1988) and Hinkley (1988) have studied the general bootstrap hypothesis test. Fisher and Hall (1990) used both asymptotic pivotal and non-pivotal statistics to provide a general idea for conducting bootstrap hypothesis test in heteroscedastic and unbalanced analysis of variance. They noted that their non-pivotal statistics have slower convergence rate while their asymptotic pivotal statistic

$$T_2 = \sum_{i=1}^{a} \left\{ n_i(n_i-1)(\overline{X}_{i\cdot} - \overline{X}_{\cdot\cdot})^2 / \sum_{j=1}^{n_i} (X_{ij} - \overline{X}_{i\cdot})^2 \right\},$$

which was proposed by James (1951), has faster convergence rate in the classical setting of large sample sizes. For computing the bootstrap version of their pivotal statistic, they defined another statistic

$$T_{02} = \sum_{i=1}^{a} \left\{ n_i(n_i-1)(\overline{Y}_{i\cdot} - \overline{Y}_{\cdot\cdot})^2 / \sum_{j=1}^{n_i} (Y_{ij} - \overline{Y}_{i\cdot})^2 \right\},$$

where $Y_{ij} \equiv X_{ij} - \mu_i$, $\overline{Y}_{i\cdot}$ and $\overline{Y}_{\cdot\cdot}$ are defined in the obvious manner. Under the null hypothesis of no treatment effect, the distributions of T_2 and T_{02} are identical. The statistic T_{02} provides an easy statistic for resample. Let $X^*_{i1}, \ldots, X^*_{i1}$ be a simple random sample of $\{X_{i1}, \cdots, X_{in_i}\}$ with replacement. Then the bootstrap version of T_{02} is computed as

$$T^*_{02} = \sum_{i=1}^{a} \left\{ n_i(n_i-1)(\overline{Y}^*_{i.} - \overline{Y}^*_{..})^2 / \sum_{j=1}^{n_i} (Y^*_{ij} - \overline{Y}^*_{i.})^2 \right\},$$

where $Y^*_{ij} \equiv X^*_{ij} - \overline{X}_i.$, $\overline{Y}^*_{i.} \equiv n_i^{-1}\sum_j Y^*_{ij}$, $\overline{Y}^*.. \equiv N^{-1}\sum\sum Y^*_{ij}$ and $N = n_1 + \cdots + n_a$. Fisher and Hall (1990) recommended to approximate the distribution of T_{02} under the null with the bootstrap distribution of T^*_{02}. They showed that the bootstrap distribution of T^*_{02} approximates the distribution of their pivotal statistic T_2 for large n_i. In the large number of treatment a and small replications n_i setting, Fisher and Hall (1990) bootstrap procedure fails. In the next paragraph, we give analytical description of why their bootstrap approach does not work in our current framework.

For a large a small n_i setting, suppose $\mathbf{X} = (X_{11}, \cdots, X_{1n_1}, \cdots, X_{a1}, \cdots, X_{an_a})'$. Then under the null hypothesis of no treatment effect we want to deduce that the distribution of T_2 does not approximate well the bootstrap distribution of T^*_{02}. Analytically, we note that T_2 and T^*_{02} are functions of $(\overline{X}_i. - \overline{X}..)^2$ and $(\overline{Y}^*_{i.} - \overline{Y}^*_{..})^2$, respectively. Then under the null, it can be shown that

$$\begin{aligned} E(\overline{X}_i. - \overline{X}..)^2 &= \mathrm{Var}(\overline{X}_i. - \overline{X}..) \\ &= \frac{\sigma_i^2}{n_i} + \frac{1}{a^2}\sum_{i=1}\frac{\sigma_i^2}{n_i} - 2\frac{\sigma_i^2}{N} \\ &\to \frac{\sigma_i^2}{n_i}, \end{aligned}$$

as $a \to \infty$ and n_i stays fixed. Similarly, we have

$$\begin{aligned} E[(\overline{Y}^*_{i.} - \overline{Y}^*_{..})^2|\mathbf{X}] &= [E(\overline{Y}^*_{i.} - \overline{Y}^*_{..})|\mathbf{X}]^2 + \mathrm{Var}[(\overline{Y}^*_{i.} - \overline{Y}^*_{..})|\mathbf{X}] \\ &= (\overline{X}_{i.} - \overline{X}_{..})^2 + \frac{\hat{\sigma}_i^2}{n_i} + \frac{1}{a^2}\sum_{i=1}^{a}\frac{\hat{\sigma}_i^2}{n_i} - 2\frac{\hat{\sigma}_i^2}{N} \\ &\to (\overline{X}_{i.} - \overline{X}_{..})^2 + \frac{\hat{\sigma}_i^2}{n_i}, \end{aligned}$$

as $a \to \infty$ and n_i stays fixed, where $\hat{\sigma}_i^2 = n_i^{-1}\sum_{j=1}^{n_i}(X_{ij} - \overline{X}_i.)^2$. Thus, the difference between $E(\overline{X}_i. - \overline{X}..)^2$ and $E[(\overline{Y}^*_{i.} - \overline{Y}^*_{..})^2|\mathbf{X}]$ is

$$E[(\overline{Y}^*_{i.} - \overline{Y}^*_{..})^2|\mathbf{X}] - E(\overline{X}_i. - \overline{X}..)^2 \to (\overline{X}_{i.} - \overline{X}_{..})^2 + \frac{\hat{\sigma}_i^2 - \sigma_i^2}{n_i}.$$

Since the difference does not approaches zero when n_i stays fixed, then analytically we can infer that $E(T_2) - E(T^*_{02}|\mathbf{X})$ will not approach zero and thus the centers of T_2 and T^*_{02} are not the same when the number of treatments is large. In the next paragraph, we illustrate with an example that for a large number of treatment levels and small n_i, the bootstrap distribution does not approximate the distribution of their test statistic well.

Using the Fisher and Hall (1990) bootstrap resampling approach outlined above, we present the following example to explain the limitation of their test statistic when the group sizes n_i's are small. We simulate data from a skewed Chi-square distribution with degrees of freedom 3, with number of treatment levels, $a = 20$, and small group sizes; 4, 4, 4, 4, 4, 4, 4, 4, 6, 6, 4, 4, 5, 4, 4, 4, 4, 4, 4, 5. The data satisfies the null hypothesis. We compute the test statistic T_2 presented above. The data generation and computation of the test statistic were repeated 5000 times to obtain the Monte Carlo probability density and cumulative distribution functions of the test statistic T_2. Figure 1 shows the plot of Monte Carlo cdf of the 5000 runs of T_2 and the empirical cumulative distribution function (ecdf) of 2000 bootstrap statistics T_{02} from one sample. The Monte Carlo pdf and kernel density estimate of the bootstrap statistics were also plotted in the right panels of Figure 1 (but in the scale of $x^{1/5}$ because their ranges are too different to be plotted in the original scale). In the setting of this example, it is observed from Figure 1 that neither the ecdf nor kernel density estimate of 2000 bootstrap statistics T_{02} approximates the Monte Carlo cdf and pdf of T_2 under the null, in our large a small number of replications setting. The bootstrap statistic and T_{02} obviously have drastically different support. Therefore, it is important to consider a test statistic well suited for the bootstrap methodology when the number of replications is small and the number of treatment levels is large.

In this dissertation, we study the type I error accuracy of the test statistics of Akritas and Papadatos (2004) and Wang and Akritas (2006) and their bootstrap versions when the number of treatments is large with small replications in the presence of heteroscedastic and non-normal data in one- and two-way ANOVA, respectively.

In section 1.2, we discuss analytically that Akritas and Papadatos (2004) test statistic is suitable for bootstrap hypothesis test in one-way ANOVA in our current framework. We

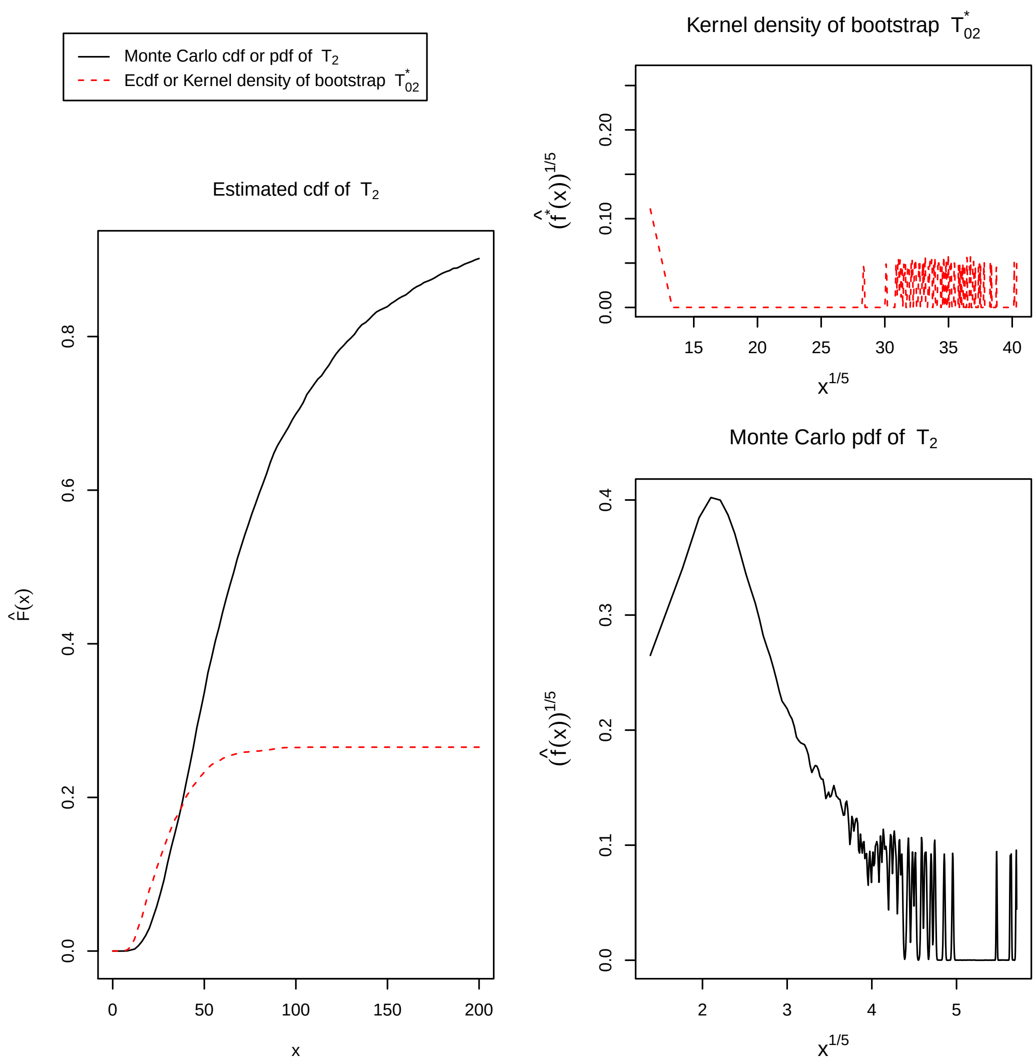

Figure 1.1: *Empirical cdf and kernel Density estimate of 2000 bootstrap statistics T_{02} from one sample vs. Monte Carlo pdf and cdf of T_2. The data contains 20 groups of χ^2_3 samples of group sizes 4, 4, 4, 4, 4, 4, 4, 4, 6, 6, 4, 4, 5, 4, 4, 4, 4, 4, 4, 5.*

study the theoretical type I error-rate of Akritas and Papadatos (2004) statistic and its corresponding bootstrap version. A modification of Akritas and Papadatos (2004) is also considered and we study the type I error-rate of its bootstrap version. Similarly, in section 1.3 we study the theoretical type I error-rate of Wang and Akritas (2006) statistic which is suitable for bootstrap hypothesis test in two-way ANOVA. We investigate the type I error-rate of its corresponding bootstrap version. We also discuss a modification of Wang and Akritas (2006) statistic and present the type I error-rate of its corresponding bootstrap version. Sections 1.4 and 1.5 will present simulation studies in one- and two-way ANOVA, respectively. The technical proofs for one- and two-way ANOVA are presented in sections 1.6 and 1.7 respectively.

1.2 Bootstrap Test for One-Way Analysis of Variance

In this section, we let X_{ij}, $j = 1, \cdots, n_i$ be independent observations from treatment i, $i = 1, \cdots, a$ with unknown mean μ_i and standard deviation σ_i. We study the bootstrap hypothesis test for testing the hypothesis of no treatment effect, i.e., $H_0 : \mu_i = \mu$ when the number of treatments a is large with small number of replications n_i within each treatment level in the presence of heteroscedastic and non-normal data. We discuss an appropriate test statistic suitable for the bootstrap procedure in the next paragraph.

In this large number of treatments and heteroscedastic setup, Akritas and Papadatos (2004) presented several test statistics and recommended to use the unweighted statistic T_a given by

$$T_a = a^{-1/2} \sum_{i=1}^{a} \left[n_i(\overline{X}_{i\cdot} - \overline{X}_{\cdot\cdot})^2 - \left(1 - \frac{n_i}{N}\right) S_i^2 \right] \tag{1.2.1}$$

where $\overline{X}_{i.} = n_i^{-1} \sum\limits_{j=1}^{n_i} X_{ij}$, $\overline{X}_{\cdot\cdot} = N^{-1} \sum_{i=1}^{a} \sum_{j=1}^{n_i} X_{ij}$, $S_i^2 = (n_i - 1)^{-1} \sum\limits_{j=1}^{n_i} (X_{ij} - \overline{X}_{i\cdot})^2$ and

$N = n_1 + \cdots + n_a$. Since T_a is a function of $(\overline{X}_{i\cdot} - \overline{X}_{\cdot\cdot})^2$ and S_i^2, then under the null,

$$
\begin{aligned}
E[n_i(\overline{X}_{i\cdot} - \overline{X}_{\cdot\cdot})^2 - S_i^2] &= n_i E(\overline{X}_{i\cdot} - \overline{X}_{\cdot\cdot})^2 - E(S_i^2) \\
&= \sigma_i^2 + \frac{n_i}{a^2}\sum_{i'=1}^{a}\frac{\sigma_{i'}^2}{n_{i'}} - \frac{2n_i}{N}\sigma_i^2 - \sigma_i^2 \\
&= \frac{n_i}{a^2}\sum_{i'=1}^{a}\frac{\sigma_{i'}^2}{n_{i'}} - \frac{2n_i}{N}\sigma_i^2 \\
&\to 0
\end{aligned}
$$

as $a \to \infty$ and n_i stays fixed. Now suppose $\mathbf{X}_i^* = \{X_{i1}^*, X_{i2}^*, \cdots, X_{in_i}^*\}'$ denote a sample drawn randomly with replacement from $\mathbf{X}_i = \{X_{i1}, X_{i2}, \cdots, X_{in_i}\}'$, where $\mathbf{X}_i$ is the collection of independent and identically distributed observations from treatment level i, $i = 1, 2, \cdots, a$. To construct the bootstrap version of the test statistic in (1.2.1), we consider the transformation $Y_{ij} = X_{ij} - \mu_i$ as used in Fisher and Hall (1990). Since μ_i is unknown, we use the resampled data to compute $Y_{ij}^* = X_{ij}^* - \overline{X}_{i\cdot}$. The bootstrap version of the test statistic T_a in (1.2.1) is T_a^*, which is computed from the resampled data as follows:

$$
T_a^* = a^{-1/2}\sum_{i=1}^{a}\left[n_i(\overline{Y}_{i.}^* - \overline{Y}^*_{\cdot\cdot})^2 - \left(1 - \frac{n_i}{N}\right)S_i^{2*}\right] \tag{1.2.2}
$$

where $\overline{Y}_{i.}^* = n_i^{-1}\sum_{j=1}^{n_i} Y_{ij}^*$, $\overline{Y}_{..}^* = N^{-1}\sum_{i=1}^{a}\sum_{j=1}^{n_i} Y_{ij}^*$, $N = \sum_{i=1}^{a} n_i$, and $S_i^{2*} = (n_i - 1)^{-1}\sum_{j=1}^{n_i}(Y_{ij}^* - \overline{Y}_{i.}^*)^2$. Since T_a^* depends on $(\overline{Y}_{i.}^* - \overline{Y}^*_{\cdot\cdot})^2$ and S_i^{2*} we compute the following

$$
\begin{aligned}
E[n_i(\overline{Y}_{i.}^* - \overline{Y}^*_{\cdot\cdot})^2|\mathbf{X}] &= n_i E[(\overline{Y}_{i.}^* - \overline{Y}^*_{\cdot\cdot})^2|\mathbf{X}] \\
&= n_i(\overline{X}_{i.} - \overline{X}_{..})^2 + \hat{\sigma}_i^2 + \frac{n_i}{a^2}\sum_{i=1}\frac{\hat{\sigma}_i^2}{n_i} - 2\frac{n_i}{N}\hat{\sigma}_i^2 \\
&\to n_i(\overline{X}_{i.} - \overline{X}_{..})^2 + \hat{\sigma}_i^2,
\end{aligned}
$$

as $a \to \infty$ and n_i stays fixed, where $\hat{\sigma}_i^2 = n_i^{-1} \sum_{j=1}^{n_i} (X_{ij} - \overline{X}_{i\cdot})^2$. Next we also compute

$$
\begin{aligned}
E[S_i^{2*}|\mathbf{X}] &= E\left[\frac{1}{n_i - 1} \sum_{j=1}^{n_i} (Y_{ij}^* - \overline{Y}_{i.}^*)^2 |\mathbf{X}\right] \\
&= \frac{1}{n_i - 1} \sum_{j=1}^{n_i} E\left[(Y_{ij}^* - \overline{Y}_{i.}^*)^2 |\mathbf{X}\right] \\
&= \frac{1}{n_i - 1} \sum_{j=1}^{n_i} \left\{[E(Y_{ij}^* - \overline{Y}_{i.}^*)|\mathbf{X}]^2 + \mathrm{Var}[(Y_{ij}^* - \overline{Y}_{i.}^*)|\mathbf{X}]\right\} \\
&\to \frac{1}{n_i - 1} \sum_{j=1}^{n_i} \left\{(X_{ij} - \overline{X}_{i.})^2 + \hat{\sigma}_i^2 - \frac{\hat{\sigma}_i^2}{n_i}\right\} \\
&= S_i^2 + \hat{\sigma}_i^2.
\end{aligned}
$$

Using the above computations, we approximate the center of T_a^* as follows:

$$
\begin{aligned}
E\left[a^{-1/2} \sum_{i=1}^{a} \{n_i(\overline{Y}_{i.}^* - \overline{Y}^*..)^2 - S_i^{2*}\} |\mathbf{X}\right] &= a^{-1/2} \sum_{i=1}^{a} \{E[n_i(\overline{Y}_{i.}^* - \overline{Y}^*..)^2|\mathbf{X}] - E[S_i^{2*}|\mathbf{X}]\} \\
&= a^{1/2}\left[\frac{1}{a} \sum_{i=1}^{a} \{n_i(\overline{X}_{i.} - \overline{X}_{..})^2 - S_i^2\}\right] \\
&\to N\left(0, \mathrm{Var}\left[\frac{1}{\sqrt{a}} \sum_{i=1}^{a} \{n_i(\overline{X}_{i.} - \overline{X}_{..})^2 - S_i^2\}\right]\right)
\end{aligned}
$$

Therefore, T_a^* is centered at 0 when n_i stays fixed and after dividing by its standard deviation it's distribution free and thus the statistic T_a can be used for the bootstrap.

1.2.1 Type I error accuracy of Akritas and Papadatos (2004) test

In this subsection, we investigate the type I error-rate of Akritas and Papadatos (2004) test in our current framework of large a small n_i.

Akritas and Papadatos (2004) showed that the limiting distribution for the unweighted statistic T_a is $N(0, \nu^2)$, where $\nu^2 = 2(s^4 + \gamma^4)$ with $\frac{1}{a}\sum_{i=1}^{a} \sigma_i^4 \to s^4$ and $\frac{1}{a}\sum_{i=1}^{a} \frac{\sigma_i^4}{n_i - 1} \to \gamma^4$ as $a \to \infty$ under the null. To study the type I error accuracy, we need more accurate approximation of the distribution. We give the asymptotic expansion of the distribution of T_a in the next paragraph.

By using projection method for quadratic forms, Akritas and Papadatos (2004) showed that T_a in (1.2.1) is asymptotically equivalent to $\widetilde{T}_a = \frac{1}{\sqrt{a}} \sum_i^a g_i$ under H_0, where

$$g_i = \left(1 - \frac{n_i}{N}\right) \frac{1}{n_i - 1} \sum_{j_1, j_2 = 1, j_2 \neq j_1}^{n_i} \epsilon_{ij_1} \epsilon_{ij_2},$$

with $\epsilon_{ij} = X_{ij} - E(X_{ij})$. We know g_i's, $i = 1, 2, \cdots, a$ are independent random variables. After some algebra, they obtained

$$\widetilde{T}_a - T_a = a^{-1/2} N^{-1} \sum_{i_1 \neq i_2, i_1, i_2 = 1}^{a} \sum_{j_1}^{n_{i_1}} \sum_{j_2}^{n_{i_2}} \epsilon_{i_1 j_1} \epsilon_{i_2 j_2}$$

with $E(\widetilde{T}_a - T_a) = 0$ and

$$E(\widetilde{T}_a - T_a)^2 = \frac{2}{aN^2} \left(\left(\sum_{i=1}^{a} n_i \sigma_i^2 \right)^2 - \sum_{i=1}^{a} n_i \sigma_i^4 \right) \leq \frac{2}{a} \max_{1 \leq i \leq a} \sigma_i^4 \to 0.$$

Thus, $\widetilde{T}_a - T_a = O_p(a^{-1/2})$. Note that $\widetilde{T}_a$ is the sum of independent random variables. Applying Corollary 19.4 of Bhattacharya and Rao (2010), we know that the distribution of $\widetilde{T}_a$ admits Edgeworth expansion. To obtain the Edgeworth expansion of the distribution of T_a, we write $P(T_a \leq x)$ as

$$P(T_a = \widetilde{T}_a + (T_a - \widetilde{T}_a) \leq x) = P(T_a = \widetilde{T}_a + a^{-\frac{1}{2}} \Delta_a \leq x)$$

where $\Delta_a = a^{1/2}(T_a - \widetilde{T}_a)$ satisfies $\Delta_a = O_p(1)$. In order to determine $P(T_a = \widetilde{T}_a + a^{-\frac{1}{2}} \Delta_a \leq x)$, we compute the first four cumulants of

$$T_a = \widetilde{T}_a + a^{-\frac{1}{2}} \Delta_a.$$

The first cumulant of T_a is computed as

$$K_1(T_a) = E(T_a) = E(\widetilde{T}_a) + a^{-\frac{1}{2}} E(\Delta_a) = K_1(\widetilde{T}_a) \tag{1.2.3}$$

since $E(\Delta_a) = 0$ and $K_1(\widetilde{T}_a) = E(\widetilde{T}_a)$ is the first cumulant of $\widetilde{T}_a$. Next, to compute the second cumulant $K_2(\widetilde{T}_a)$ we note that

$$\begin{aligned} E(T_a)^2 &= E(\widetilde{T}_a + a^{-\frac{1}{2}} \Delta_a)^2 \\ &= E(\widetilde{T}_a^{\,2}) + 2a^{-\frac{1}{2}} E(\widetilde{T}_a \Delta_a) + a^{-1} E(\Delta_a^{\,2}) \\ &= E(\widetilde{T}_a^{\,2}) + O(a^{-1}), \end{aligned}$$

since $E(\widetilde{T}_a\Delta_a) = 0$ (the proof is given in the Section 1.6.3). Therefore

$$\begin{aligned} K_2(T_a) &= E(T_a)^2 - (E(T_a))^2 \\ &= K_2(\widetilde{T}_a) + O(a^{-1}), \end{aligned} \tag{1.2.4}$$

where $K_2(\widetilde{T}_a) = E(\widetilde{T}_a^{\,2}) - (E(\widetilde{T}_a))^2$ is the second cumulant of $\widetilde{T}_a^{\,2}$. Now, in order to obtain the third cumulant $K_3(T_a)$ we have that

$$\begin{aligned} E(T_a)^3 &= E(\widetilde{T}_a + a^{-\frac{1}{2}}\Delta_a)^3 \\ &= E(\widetilde{T}_a^{\,3}) + 3a^{-\frac{1}{2}}E(\widetilde{T}_a^{\,2}\Delta_a) + 3a^{-1}E(\widetilde{T}_a\Delta_a^2) + a^{-\frac{3}{2}}E(\Delta_a{}^3) \\ &= E(\widetilde{T}_a^{\,3}) + O(a^{-1}), \end{aligned}$$

which used the fact that $E(\widetilde{T}_a^{\,2}\Delta_a) = 0$ (shown in Section 1.6.4), and $E(\widetilde{T}_a\Delta_a^2)$, $E(\Delta_a^3)$ are at most $O(1)$ with the Cramer's condition since $\widetilde{T}_a$ and Δ_a are $O_p(1)$.

$$\begin{aligned} K_3(T_a) &= E(T_a)^3 - 3E(T_a)^2E(T_a) + 2(E(T_a))^3 \\ &= E(\widetilde{T}_a^{\,3}) + O(a^{-1}) - 3(E(\widetilde{T}_a^{\,2}) + O(a^{-1}))E(\widetilde{T}_a) + 2(E(\widetilde{T}_a))^3 \\ &= E(\widetilde{T}_a^{\,3}) - 3E(\widetilde{T}_a^{\,2})E(\widetilde{T}_a) + 2(E(\widetilde{T}_a))^3 + O(a^{-1}) \\ &= K_3(\widetilde{T}_a) + O(a^{-1}), \end{aligned} \tag{1.2.5}$$

where $K_3(\widetilde{T}_a)$ is the third cumulant of $\widetilde{T}_a$. Next, we write

$$\begin{aligned} E(T_a)^4 &= E(\widetilde{T}_a + a^{-\frac{1}{2}}\Delta_a)^4 \\ &= E(\widetilde{T}_a^{\,4}) + 4a^{-\frac{1}{2}}E(\widetilde{T}_a^{\,3}\Delta_a) + 6a^{-1}E(\widetilde{T}_a^{\,2}\Delta_a^2) + 4a^{-\frac{3}{2}}E(\widetilde{T}_a\Delta_a^3) + a^{-2}E(\Delta_a{}^4) \\ &= E(\widetilde{T}_a^{\,4}) + O(a^{-1}), \end{aligned}$$

where the last equality is due to the fact that $E(\widetilde{T}_a^{\,3}\Delta_a) = 0$ (the proof is given in Section 1.6.5) and the rest of the terms are $O(1)$ for similar reason as explained for $E(T_a^3)$. Lastly, the fourth cumulant $K_4(T_a)$ is given by

$$\begin{aligned}
K_4(T_a) &= E(T_a)^4 - 4E(T_a)E(T_a)^3 - 3(E(T_a)^2)^2 + 12E(T_a)^2(E(T_a))^2 - 6(E(T_a))^4 \\
&= E(\widetilde{T_a}^4) + O(a^{-1}) - 4(E(\widetilde{T}_a))(E(\widetilde{T_a}^3) + O(a^{-1})) - 3(E(\widetilde{T_a}^2) + O(a^{-1}))^2 \\
&+ 12(E(\widetilde{T_a}^2) + O(a^{-1}))(E(\widetilde{T}_a))^2 - 6(E(\widetilde{T}_a))^4 \\
&= E(\widetilde{T_a}^4) - 4E(\widetilde{T}_a)E(\widetilde{T_a}^3) - 3(E(\widetilde{T_a}^2))^2 + 12E(\widetilde{T_a}^2)(E(\widetilde{T}_a))^2 - 6(E(\widetilde{T}_a))^4 + O(a^{-1}) \\
&= K_4(\widetilde{T}_a) + O(a^{-1}),
\end{aligned} \tag{1.2.6}$$

where $K_4(\widetilde{T}_a) = E(\widetilde{T_a}^4) - 4E(\widetilde{T}_a)E(\widetilde{T_a}^3) - 3(E(\widetilde{T_a}^2))^2 + 12E(\widetilde{T_a}^2)(E(\widetilde{T}_a))^2 - 6(E(\widetilde{T}_a))^4$ is the fourth cumulant of $\widetilde{T}_a$. As discussed in Hall (1992b) based on equations (3.30)-(3.32) to deduce equation (3.36), then using $K_1(T_a), K_2(T_a), K_3(T_a)$ and $K_4(T_a)$ in (1.2.3), (1.2.4), (1.2.5) and (1.2.6), respectively, we have that

$$P(T_a \le x) = P(\widetilde{T}_a \le x) + O(a^{-1}).$$

Therefore, the distributions of T_a and $\widetilde{T}_a$ have the same Edgeworth expansion up to order $O(a^{-1})$. Corollary 19.4 of Bhattacharya and Rao (2010) can be used to get the Edgeworth expansion of $S_a = T_a/\nu$ where

$$\nu = \sqrt{2(s^4 + \gamma^4)}. \tag{1.2.7}$$

Accordingly, under the regularity conditions in the section 1.6.1, $\widetilde{T}_a$ admits Edgeworth expansion of the form

$$F_{\widetilde{T}}(x) = P(\widetilde{T}_a \le x) = P\left(S_a \le \frac{x}{\nu}\right) = \Phi\left(\frac{x}{\nu}\right) + \sum_{k=1}^{s-2} \frac{1}{a^{k/2}} P_k\left(\frac{x}{\nu}\right)\phi\left(\frac{x}{\nu}\right) + O(a^{-(\frac{s-1}{2})}), \tag{1.2.8}$$

uniformly for $\forall x \in R$, where $\Phi(x)$ and $\phi(x)$ are the cdf and pdf of the standard normal distribution and $P_k(x)$ is a polynomial of degree $3k-1$, with coefficients that depend on the population moments of g_i, $i = 1, 2, \cdots, a$. T_a has Edgeworth expansion

$$F_T(x) = F_{\widetilde{T}}(x) + O(a^{-1}) = \Phi\left(\frac{x}{\nu}\right) + \frac{1}{a^{1/2}} P_1\left(\frac{x}{\nu}\right)\phi\left(\frac{x}{\nu}\right) + O(a^{-1}). \tag{1.2.9}$$

Even though the terms in $O(a^{-1})$ are omitted, they can be written as expansions to give

$$\begin{aligned} F_T(x) &= \Phi\left(\frac{x}{\nu}\right) + \frac{1}{a^{1/2}} P_1\left(\frac{x}{\nu}\right)\phi\left(\frac{x}{\nu}\right) + \sum_{k=2}^{s-2} \frac{1}{a^{k/2}} P_{1k}\left(\frac{x}{\nu}\right)\phi\left(\frac{x}{\nu}\right) + O(a^{-(\frac{s-1}{2})}) \\ &= \Phi\left(\frac{x}{\nu}\right) + \sum_{k=1}^{s-2} \frac{1}{a^{k/2}} P_{1k}\left(\frac{x}{\nu}\right)\phi\left(\frac{x}{\nu}\right) + O(a^{-(\frac{s-1}{2})}) \end{aligned} \tag{1.2.10}$$

where $P_{1k}(x) = P_1(x)$ and $P_{1k}(.)$ is polynomials of degree $3k-1$ with coefficients that depend on the population moments of g_i, $i = 1, 2, \cdots, a$. In practice, Akritas and Papadatos (2004) approximate the distribution of T_a with $N(0, \widehat{\nu}^2)$ where $\hat{\nu}^2 = \frac{2}{a}\sum_{i=1}^{a}\left(\frac{n_i}{n_i-1}\right)\widehat{\sigma_i^4}$ and $\widehat{\sigma_i^4}$ is an unbiased estimate of σ_i^4 given by the U-statistic

$$\widehat{\sigma_i^4} = \frac{1}{P_{n_i}^4} \sum_{j_1 \neq j_2 \neq j_3 \neq j_4}^{n_i} \frac{(x_{ij_1} - x_{ij_2})^2 (x_{ij_3} - x_{ij_4})^2}{4},$$

where $P_{n_i}^4 = n_i(n_i-1)(n_i-2)(n_i-3)$. Based on the expansion (1.2.8), we can see that the accuracy of approximating the distribution of T_a using $N(0, \widehat{\nu}^2)$ is only of order $O(a^{-1/2})$ since $F_T(x) - \Phi(\frac{x}{\hat{\nu}}) = O_p(a^{-1/2})$ due to the fact that

$$\widehat{\nu} - \nu = O_p(a^{-1/2}). \tag{1.2.11}$$

The proof of (1.2.11) is shown in the section 1.6.2. We discuss the type I error accuracy of Akritas and Papadatos (2004) in the next paragraph.

The test of Akritas and Papadatos (2004) is based on normal approximation. At α level of significance, the estimated quantile of the Akritas and Papadatos (2004) test is $\hat{\nu} z_\alpha$ where z_α is the quantile of the standard normal distribution and $\hat{\nu}$ is the estimate of ν given in (1.2.7). The type I error-rate of the Akritas and Papadatos (2004) test is $P(T_a > \hat{\nu} z_{1-\alpha})$, which can be written as

$$P(T_a > \hat{\nu} z_{1-\alpha}) = P(T_a - (\hat{\nu} - \nu) z_{1-\alpha} > \nu z_{1-\alpha}). \tag{1.2.12}$$

To compute $P(T_a - (\hat{\nu} - \nu) z_{1-\alpha} > \nu z_{1-\alpha})$ in (1.2.12), we need to know the Edgeworth expansion of the distribution of $T_a^T = T_a - (\hat{\nu} - \nu) z_{1-\alpha}$. Write

$$T_a^T = T_a - a^{-\frac{1}{2}} D_a^T = \widetilde{T}_a + a^{-\frac{1}{2}} \Delta_a - a^{-\frac{1}{2}} D_a \tag{1.2.13}$$

where $T_a = \widetilde{T}_a + a^{-\frac{1}{2}}\Delta_a$ and $D_a = a^{1/2}(\hat{\nu} - \nu)z_{1-\alpha}$. We note that $D_a = O_p(1)$ and by Taylor series expansion of $g(x) = \sqrt{x}$ around ν^2 we have

$$\hat{\nu} - \nu = g'(\nu^2)(\hat{\nu}^2 - \nu^2) + \frac{g''(\nu^2)}{2}(\hat{\nu}^2 - \nu^2)^2 + O_p(a^{-\frac{3}{2}})$$

where $g'(\nu^2) = 1/(2\sqrt{\nu^2})$ and $g''(\nu^2) = -1/(4\nu^3)$. Therefore $E(D_a) = -8^{-1}\nu^{-3}\sqrt{a}E[(\hat{\nu}^2 - \nu^2)^2] = O(1/\sqrt{a})$ because $\hat{\nu}^2 - \nu^2 = O_p(1/\sqrt{a})$. To obtain the Edgeworth expansion of T_a^T, we compute the first four cumulants of T_a^T. The first cumulant of T_a^T is computed as

$$K_1(T_a^T) = E(T_a^T) = E(T_a) - a^{-\frac{1}{2}}E(D_a) = K_1(T_a) + O(a^{-1}). \tag{1.2.14}$$

To obtain the second cumulant $K_2(T_a^T)$, we compute the second moment of T^T as

$$\begin{aligned} E(T_a^T)^2 &= E(\widetilde{T}_a + a^{-\frac{1}{2}}\Delta_a - a^{-\frac{1}{2}}D_a)^2 \\ &= E(\widetilde{T}_a + a^{-\frac{1}{2}}\Delta_a)^2 - 2a^{-\frac{1}{2}}E\left[(\widetilde{T}_a + a^{-\frac{1}{2}}\Delta_a)D_a\right] + a^{-1}E(D_a^2) \\ &= E(T_a^2) - 2a^{-\frac{1}{2}}E(\widetilde{T}_aD_a) - 2a^{-1}E(\Delta_aD_a) + a^{-1}E(D_a^2) \\ &= E(T_a^2) + u_a + O(a^{-1}), \end{aligned} \tag{1.2.15}$$

where $u_a = -2a^{-\frac{1}{2}}E(\widetilde{T}_aD_a) = -\frac{8z_{1-\alpha}g'(\nu^2)}{a^{\frac{3}{2}}}\sum_{i=1}^{a}\delta_i\left(\frac{n_i}{n_i-1}\right)\sigma_i^6(\gamma_i^2 - 2)$, $\gamma_i = E(\epsilon_{ij}^3)$ and $\delta_i = (1 - n_i/N)(1/(n_i - 1))$. The derivation of (1.2.15) is given in Section 1.6.6. Therefore the second cumulant is given by

$$\begin{aligned} K_2(T_a^T) &= E(T_a^T)^2 - (E(T_a^T))^2 \\ &= E(T_a^2) - \frac{8z_{1-\alpha}g'(\nu^2)}{a^{\frac{3}{2}}}\sum_{i=1}^{a}\delta_i\left(\frac{n_i}{n_i-1}\right)\sigma_i^6(\gamma_i^2 - 2) + O(a^{-1}) - (E(T_a) + O(a^{-1}))^2 \\ &= E(T_a^2) - (E(T_a))^2 - \frac{8z_{1-\alpha}g'(\nu^2)}{a^{\frac{3}{2}}}\sum_{i=1}^{a}\delta_i\left(\frac{n_i}{n_i-1}\right)\sigma_i^6(\gamma_i^2 - 2) + O(a^{-1}) \\ &= K_2(T_a) - 8\frac{z_{1-\alpha}g'(\nu^2)}{a^{\frac{3}{2}}}\sum_{i=1}^{a}\delta_i\left(\frac{n_i}{n_i-1}\right)\sigma_i^6(\gamma_i^2 - 2) + O(a^{-1}), \end{aligned} \tag{1.2.16}$$

where $K_2(T_a) = E(T_a^2) - (E(T_a))^2$ is the second cumulant of T_a. Next, we compute the third moment of T_a^T as

$$\begin{aligned}
E(T_a^T)^3 &= E(\widetilde{T}_a + a^{-\frac{1}{2}}\Delta_a - a^{-\frac{1}{2}}D_a)^3 \\
&= E(\widetilde{T}_a + a^{-\frac{1}{2}}\Delta_a)^3 - 3a^{-\frac{1}{2}}E\left[(\widetilde{T}_a + a^{-\frac{1}{2}}\Delta_a)^2 D_a\right] + 3a^{-1}E\left[(\widetilde{T}_a + a^{-\frac{1}{2}}\Delta_a)D_a^2\right] - a^{-\frac{3}{2}}E({D_a}^3) \\
&= E(T_a^3) - 3a^{-\frac{1}{2}}E(\widetilde{T}_a^2 D_a) + O(a^{-1}) \\
&= E(T_a^3) + O(a^{-1}),
\end{aligned}$$

where the third equality used the fact that $E(\widetilde{T}_a^2 D_a) = O(a^{-1/2})$, which is shown in section 1.6.7. Thus the third cumulant $K_3(T_a^T)$ is computed as

$$\begin{aligned}
K_3(T_a^T) &= E(T_a^T)^3 - 3E(T_a^T)^2 E(T_a^T) + 2(E(T_a^T))^3 \\
&= E(T_a^3) + O(a^{-1}) - 3\left[E(T_a^2) - \frac{8z_{1-\alpha}g'(\nu^2)}{a^{\frac{3}{2}}}\sum_{i=1}^{a}\delta_i\left(\frac{n_i}{n_i-1}\right)\sigma_i^6(\gamma_i^2-2) + O(a^{-1})\right] \\
&\quad [E(T_a) + O(a^{-1})] + 2(E(T_a) + O(a^{-1}))^3 \\
&= E(T_a^3) - 3E(T_a^2)E(T_a) + 24\frac{z_{1-\alpha}g'(\nu^2)}{a^{\frac{3}{2}}}E(T_a)\sum_{i=1}^{a}\delta_i\left(\frac{n_i}{n_i-1}\right)\sigma_i^6(\gamma_i^2-2) + 2(E(T_a))^3 \\
&\quad + O(a^{-1}) \\
&= K_3(T_a) + O(a^{-1}), \qquad (1.2.17)
\end{aligned}$$

where the last equality is because $E(T_a) = 0$ under H_0 and $K_3(T_a) = E(T_a^3) - 3E(T_a^2)E(T_a) + 2(E(T_a))^3$ is the third cumulant of T_a. Next, we compute the fourth moment of T_a^T as

$$\begin{aligned}
E(T_a^T)^4 &= E(\widetilde{T}_a + a^{-\frac{1}{2}}\Delta_a - a^{-\frac{1}{2}}D_a)^4 \\
&= E(\widetilde{T}_a + a^{-\frac{1}{2}}\Delta_a)^4 - 4a^{-\frac{1}{2}}E\left[(\widetilde{T}_a + a^{-\frac{1}{2}}\Delta_a)^3 D_a\right] + 6a^{-1}E\left[(\widetilde{T}_a + a^{-\frac{1}{2}}\Delta_a)^2 D_a^2\right] \\
&\quad - 4a^{-\frac{3}{2}}E\left[(\widetilde{T}_a + a^{-\frac{1}{2}}\Delta_a)D_a^3\right] + a^{-2}E({D_a}^4) \\
&= E(T_a^4) - 4a^{-\frac{1}{2}}E(\widetilde{T}_a^3 D_a) + O(a^{-1}),
\end{aligned}$$

where $E(\widetilde{T}_a^3 D_a) = 24\frac{z_{1-\alpha}g'(\nu^2)}{a^2}\sum_{i\neq i'}^{a}[\delta_i^2 n_i(n_i-1)\sigma_i^4]\left[\delta_{i'}\left(\frac{n_{i'}}{n_{i'}-1}\right)(\gamma_{i'}^2-2)\sigma_{i'}^6\right]$ is proved in section 1.6.8. Thus the fourth cumulant $K_4(T_a^T)$ is obtained as

$$
\begin{aligned}
K_4(T_a^T) &= E(T_a^T)^4 - 4E(T_a^T)E(T_a^T)^3 - 3(E(T_a^T)^2)^2 + 12E(T_a^T)^2(E(T_a^T))^2 - 6(E(T_a^T))^4 \\
&= E(T_a^4) - 96\frac{z_{1-\alpha}g'(\nu^2)}{a^{\frac{5}{2}}}\sum_{i\neq i'}^{a}[\delta_i^2 n_i(n_i-1)\sigma_i^4]\left[\delta_{i'}\left(\frac{n_{i'}}{n_{i'}-1}\right)(\gamma_{i'}^2-2)\sigma_{i'}^6\right] + O(a^{-1}) \\
&\quad -4(E(T_a)+O(a^{-1}))(E(T_a^3)+O(a^{-1})) \\
&\quad -3\left[E(T_a^2) - 8\frac{z_{1-\alpha}g'(\nu^2)}{a^{\frac{3}{2}}}\sum_{i=1}^{a}\delta_i\left(\frac{n_i}{n_i-1}\right)\sigma_i^6(\gamma_i^2-2) + O(a^{-1})\right]^2 \\
&\quad +12\left[E(T_a^2) - 8\frac{z_{1-\alpha}g'(\nu^2)}{a^{\frac{3}{2}}}\sum_{i=1}^{a}\delta_i\left(\frac{n_i}{n_i-1}\right)\sigma_i^6(\gamma_i^2-2) + O(a^{-1})\right](E(T_a)+O(a^{-1}))^2 \\
&\quad -6(E(T_a)+O(a^{-1}))^4 \\
&= E(T_a^4) - 96\frac{z_{1-\alpha}g'(\nu^2)}{a^{\frac{5}{2}}}\sum_{i\neq i'}^{a}[\delta_i^2 n_i(n_i-1)\sigma_i^4]\left[\delta_{i'}\left(\frac{n_{i'}}{n_{i'}-1}\right)(\gamma_{i'}^2-2)\sigma_{i'}^6\right] + O(a^{-1}) \\
&\quad {}_4E(T_a)E(T_a)^3 + O(a^{-1}) \\
&\quad -3\left[(E(T_a)^2)^2 - 16\nu^2\frac{z_{1-\alpha}g'(\nu^2)}{a^{\frac{3}{2}}}\sum_{i=1}^{a}\delta_i\left(\frac{n_i}{n_i-1}\right)\sigma_i^6(\gamma_i^2-2) + O(a^{-1})\right] \\
&\quad +12\left[E(T_a)^2(E(T_a))^2 - 8(E(T_a))^2\frac{z_{1-\alpha}g'(\nu^2)}{a^{\frac{3}{2}}}\sum_{i=1}^{a}\delta_i\left(\frac{n_i}{n_i-1}\right)\sigma_i^6(\gamma_i^2-2) + O(a^{-1})\right] \\
&\quad -6(E(T_a))^4 + O(a^{-1}) \\
&= E(T_a^4) - 4E(T_a)E(T_a^3) - 3(E(T_a^2))^2 + 12E({T_a}^2)(E(T_a))^2 - 6(E(T_a))^4 + O(a^{-1}) \\
&= K_4(T_a) + O(a^{-1}), \qquad (1.2.18)
\end{aligned}
$$

where the third and fourth equalities used the result $E(T_a^2) = \nu^2 + O(a^{-1}) = 2a^{-1}\sum_{i=1}^{a}\delta_i^2 n_i(n_i-1)\sigma_i^4 + O(a^{-1})$ and $K_4(T_a) = E(T_a^4) - 4E(T_a)E(T_a^3) - 3(E(T_a^2))^2 + 12E(T_a^2)(E(T_a))^2 - 6(E(T_a))^4$ is the fourth cumulant of T_a. Using $K_1(T_a^T), K_2(T_a^T), K_3(T_a^T)$ and $K_4(T_a^T)$ in (1.2.14), (1.2.16), (1.2.17) and (1.2.18), respectively, it has been shown in the Section 1.6.9 that

$$
\begin{aligned}
P(T_a^T > \nu z_{1-\alpha}) &= P(T_a > \nu z_{1-\alpha}) + \left[\frac{u_a}{2\nu^2}\nu z_{1-\alpha} + \frac{u_a K_3(T_a)}{6\nu^5}\{(\nu z_{1-\alpha})^3 - 3(\nu z_{1-\alpha})\}\right. \\
&\quad \left. + \frac{u_a K_4(T_a)}{24\nu^6}\{(\nu z_{1-\alpha})^5 - 10(\nu z_{1-\alpha})^3 + 15(\nu z_{1-\alpha})\}\right]\phi(\nu z_{1-\alpha}) \\
&\quad + O(a^{-1}), \qquad (1.2.19)
\end{aligned}
$$

where $u_a = -8\frac{z_{1-\alpha}g'(\nu^2)}{a^{\frac{3}{2}}}\sum_{i=1}^a \delta_i\left(\frac{n_i}{n_i-1}\right)\sigma_i^6(\gamma_i^2-2)$. Depending on the order of $K_3(T_a)$ and $K_4(T_a)$, the additional terms in (1.2.19) beyond $P(T_a > \nu z_{1-\alpha})$ is polynomial that contains only the odd power of $\nu z_{1-\alpha}$. Now we need to compute $P(T_a > \nu z_{1-\alpha})$ in (1.2.19). Based on (1.2.10), this probability can be written as

$$\begin{aligned} P(T_a > \nu z_{1-\alpha}) &= 1 - P(T_a \leq \nu z_{1-\alpha}) \\ &= 1 - F_T(\nu z_{1-\alpha}) \\ &= 1 - \Phi(z_{1-\alpha}) - \sum_{k=1}^{s-2}\frac{1}{a^{k/2}}P_k(z_{1-\alpha})\phi(z_{1-\alpha}) + O(a^{-(\frac{s-1}{2})}) \\ &= \alpha + \frac{1}{a^{1/2}}P_1(z_{1-\alpha})\phi(z_{1-\alpha}) + O(a^{-1}), \end{aligned}$$

where $P_1(z_{1-\alpha})$ is a quadratic function that only contains even power of $z_{1-\alpha}$, which would not cancel the odd power terms in (1.2.19). Therefore (1.2.19) becomes $P(T_a^T > \nu z_{1-\alpha}) = \alpha + O(a^{-1/2})$, since all the other terms are of order $O(a^{-1/2})$ or smaller. Thus, the theoretical type I error-rate of Akritas and Papadatos (2004) test is only accurate up to order $O(a^{-1/2})$.

1.2.2 Bootstrap Test with T_a and its type I error accuracy

In this subsection, we study the accuracy of bootstrap approximation to the distribution of the test statistic T_a given in (1.2.1). We also discuss the analogous bootstrap test for one-way ANOVA.

At α level of significance, the bootstrap statistic in (1.2.2) could be used to obtain the $1-\alpha$ analytical bootstrap quantile $\widehat{\omega}_{1-\alpha}^T$, which admits the following Cornish-Fisher expansion under regularity conditions.

$$\hat{\omega}_{1-\alpha}^T = \hat{\nu}z_{1-\alpha} + \hat{\nu}\sum_{k=1}^{s-2}\frac{1}{a^{k/2}}\hat{p}_{1k}^{cf}(z_{1-\alpha})$$

where $\hat{\nu}$ is an estimate of ν in (1.2.7) and $\hat{p}_{1k}^{cf}(.)$ is a function of the estimate of $P_{1k}(.)$ in (1.2.10). For example, $\hat{p}_1^{cf}(x) = -\hat{P}_1(x)$, $\hat{p}_{12}^{cf}(x) = \hat{P}_{11}(x)\hat{P'}_{11}(x) - 0.5x\hat{P}_{11}(x)^2 - \hat{P}_{12}(x)$. This analytical form of quantile $\hat{\omega}_{1-\alpha}^T$ is an approximation of the quantile of T_a.

In practice, computational approach is generally used to approximate the analytical bootstrap quantile $\hat{\omega}^T_{1-\alpha}$ as follows:

- For a very large B, we randomly draw B bootstrap samples $\mathbf{X}^*_{ib} = \left\{X^*_{i1b}, X^*_{i2b}, \cdots, X^*_{in_ib}\right\}'$, from the observed data $\mathbf{X}_i = \{X_{i1}, X_{i2}, \cdots, X_{in_i}\}'$ for $i = 1, 2, \cdots, a$, $b = 1, 2, \cdots, B$.

- For each sample, we compute $T^*_{a,b} = a^{-1/2} \sum_{i=1}^{a} \left[n_i(\overline{Y}^{*(b)}_{i.} - \overline{Y}^{*(b)}..)^2 - \left(1 - \frac{n_i}{N}\right) S^{2*(b)}_i\right]$, $b = 1, 2, \cdots, B$, where $\overline{Y}^{*(b)}_{i.}$, $\overline{Y}^{*(b)}$ and $S^{2*(b)}_i$ are the average of i^{th} sample, overall average and sample variance calculated from the b^{th} bootstrap sample, respectively.

- We rank the B bootstrap replications of T^*_a as $T^*_{a,(1)} \leq T^*_{a,(2)} \leq \cdots \leq T^*_{a,(B)}$. Then an estimate of the computational bootstrap quantile $\widehat{\omega}^T_{1-\alpha,B}$ is $T^*_{a,(k_B)}$, where $T^*_{a,(k_B)}$ has k_B observations smaller than or equal to it and $k_B = [(B+1)(1-\alpha)]$.

- At significance level α, the null hypothesis is rejected if $T_a > \widehat{\omega}^T_{1-\alpha,B}$.

To test for no treatment effect, the bootstrap test uses the bootstrap quantile $\widehat{\omega}^T_{1-\alpha,B}$ to define the rejection region. The theoretical type I error-rate of this bootstrap test is $P(T_a > \widehat{\omega}^T_{1-\alpha,B})$. From the computational procedure, we know that $\widehat{\omega}^T_{1-\alpha,B}$ is the sample quantile based on $T^*_{a,b}$, $b = 1, 2, \cdots, B$, which is a random sample from the bootstrap distribution conditional on the observed data. The bootstrap distribution admits the following Edgeworth expansion

$$\widehat{F}_T(x) = P(T^*_a \leq x|\underline{\mathbf{X}}) = \Phi\left(\frac{x}{\widehat{\nu}}\right) + \sum_{k=1}^{s-2} \frac{1}{a^{k/2}} \hat{P}_{1k}\left(\frac{x}{\widehat{\nu}}\right) \phi\left(\frac{x}{\widehat{\nu}}\right), \tag{1.2.20}$$

where $\widehat{\nu}$ is the plug-in estimate of ν in (1.2.7) and $\hat{P}_{1k}(x)$ is the plug-in estimate of $P_{1k}(x)$ in (1.2.10), in which the unknown population moments are replaced by their corresponding sample moments.

Since $T^*_{a,b}$, $b = 1, 2, \cdots, B$ is a random sample from $\widehat{F}_T(x)$, which has $1-\alpha$ quantile $\widehat{\omega}^T_{1-\alpha}$ conditional on the observed data, we can quantify the order of $\widehat{\omega}^T_{1-\alpha,B} - \widehat{\omega}^T_{1-\alpha}$. Specifically, conditional on the observed data $\widehat{\omega}^T_{1-\alpha,B}$ is approximately normal with mean $\widehat{\omega}^T_{1-\alpha}$ and variance $(1-\alpha)\alpha(B[\hat{f}(\widehat{F}_T(1-\alpha))]^2)^{-1}$ where $\widehat{F}_T(.)$ is given in (1.2.20) and $\hat{f}(.)$ is the corresponding pdf. Hence $\widehat{\omega}^T_{1-\alpha,B} - \widehat{\omega}^T_{1-\alpha} = O_p(B^{-1/2})$. This term can be made arbitrarily small by using a large B in computation. Then we can write $P(T_a > \widehat{\omega}^T_{1-\alpha,B})$ as

$$P(T_a - (\hat{\omega}^T_{1-\alpha,B} - \hat{\omega}^T_{1-\alpha}) - (\hat{\omega}^T_{1-\alpha} - \omega^T_{1-\alpha}) > \omega^T_{1-\alpha}) = P(T_a - (\hat{\omega}^T_{1-\alpha} - \omega^T_{1-\alpha}) > \omega^T_{1-\alpha}) + O(B^{-1/2}).$$

To quantify the order of $\widehat{\omega}^T_{1-\alpha} - \omega^T_{1-\alpha}$ we need to trace back to $S_a = T_a/\nu$, which was used to obtain the Edgeworth expansion of T_a. Let $\omega^S_{1-\alpha}$ and $\hat{\omega}^S_{1-\alpha}$ denote the true and estimated $1-\alpha$ quantiles of $S_a = T_a/\nu$. Based on Cornish-Fisher expansion of quantiles, $\omega^S_{1-\alpha}$ and $\hat{\omega}^S_{1-\alpha}$ can be written as (under the regularity conditions in section)

$$\omega^S_{1-\alpha} = z_{1-\alpha} + \sum_{k=1}^{s-2} \frac{1}{a^{k/2}} p^{cf}_{1k}(z_{1-\alpha}) + O(a^{-(\frac{s-1}{2})}) \tag{1.2.21}$$

and

$$\hat{\omega}^S_{1-\alpha} = z_{1-\alpha} + \sum_{k=1}^{s-2} \frac{1}{a^{k/2}} \hat{p}^{cf}_{1k}(z_{1-\alpha}) \tag{1.2.22}$$

where $p^{cf}_{1k}(.)$ and $\hat{p}^{cf}_{1k}(.)$ are functions of $P_{1k}(.)$ in (1.2.10) and $\hat{P}_{1k}(.)$ in (1.2.20) respectively. Then the true quantile $\omega^T_{1-\alpha}$ of T_a in (1.2.1) is $\omega^T_{1-\alpha} = \nu\omega^S_{1-\alpha}$. The analytical form of the bootstrap quantile $\hat{\omega}^T_{1-\alpha}$ is given as $\hat{\omega}^T_{1-\alpha} = \widehat{\nu}\hat{\omega}^S_{1-\alpha}$.

$$\begin{aligned}
\hat{\omega}^T_{1-\alpha} - \omega^T_{1-\alpha} &= \widehat{\nu}\hat{\omega}^S_{1-\alpha} - \nu\omega^S_{1-\alpha} \\
&= (\widehat{\nu} - \nu)z_{1-\alpha} + \widehat{\nu}\sum_{k=1}^{s-2} \frac{1}{a^{k/2}} \hat{p}^{cf}_{1k}(z_{1-\alpha}) - \nu\sum_{k=1}^{s-2} \frac{1}{a^{k/2}} p^{cf}_{1k}(z_{1-\alpha}) + O(a^{-(\frac{s-1}{2})}) \\
&= O_p(a^{-1/2}),
\end{aligned}$$

since $\widehat{\nu} - \nu = O_p(a^{-1/2})$. Thus, $\hat{\omega}^T_{1-\alpha}$ approximates the true $1-\alpha$ quantile $\omega^T_{1-\alpha}$ in the order of $O(a^{-1/2})$. Now using the above results, the type I error $P(T_a > \hat{\omega}^T_{1-\alpha,B})$ can be written as

$$
\begin{aligned}
& P(T_a > \omega^T_{1-\alpha} + \hat{\omega}^T_{1-\alpha,B} - \hat{\omega}^T_{1-\alpha} + \hat{\omega}^T_{1-\alpha} - \omega^T_{1-\alpha}) \\
= \ & P(T_a - (\hat{\omega}^T_{1-\alpha,B} - \hat{\omega}^T_{1-\alpha}) - (\hat{\omega}^T_{1-\alpha} - \omega^T_{1-\alpha}) > \omega^T_{1-\alpha}) \\
= \ & P(T_a - (\hat{\omega}^T_{1-\alpha} - \omega^T_{1-\alpha}) > \omega^T_{1-\alpha}) + O(B^{-1/2}) \\
= \ & P\left(T_a - \left((\widehat{\nu}-\nu)z_{1-\alpha} + \widehat{\nu}\sum_{k=1}^{s-2}\frac{1}{a^{k/2}}\hat{p}^{cf}_k(z_{1-\alpha}) - \nu\sum_{k=1}^{s-2}\frac{1}{a^{k/2}}p^{cf}_k(z_{1-\alpha})\right) > \omega^T_{1-\alpha}\right) + O(B^{-1/2}) \\
= \ & P\left(T_a - \left((\widehat{\nu}-\nu)z_{1-\alpha} + \nu\sum_{k=1}^{s-2}\frac{1}{a^{k/2}}[\hat{p}^{cf}_k(z_{1-\alpha}) - p^{cf}_k(z_{1-\alpha})] + O(a^{-1})\right) > \omega^T_{1-\alpha}\right) + O(B^{-1/2}) \\
= \ & P\left(T_a - \{(\widehat{\nu}-\nu)z_{1-\alpha} + O_p(a^{-1})\} > \omega^T_{1-\alpha}\right) + O(B^{-1/2}) \\
= \ & P(T_a - (\widehat{\nu}-\nu)z_{1-\alpha} > \omega^T_{1-\alpha}) + O(a^{-1}) + O(B^{-1/2}) \qquad (1.2.23)
\end{aligned}
$$

where (1.2.23) is as a result of the Delta method in Hall (1992b) section 2.7. Based on the results in (1.2.19), $P(T_a - (\widehat{\nu}-\nu)z_{1-\alpha} > \omega^T_{1-\alpha})$ in (1.2.23) is given by,

$$
\begin{aligned}
P(T_a - (\widehat{\nu}-\nu)z_{1-\alpha} > \omega^T_{1-\alpha}) \ = \ & P(T_a > \omega^T_{1-\alpha}) + \left[\frac{u_a}{2\nu^2}(\omega^T_{1-\alpha}) + \frac{u_a K_3(T_a)}{6\nu^5}\{(\omega^T_{1-\alpha})^3 - 3(\omega^T_{1-\alpha})\}\right. \\
& \left. + \frac{u_a K_4(T_a)}{24\nu^6}\{(\omega^T_{1-\alpha})^5 - 10(\omega^T_{1-\alpha})^3 + 15(\omega^T_{1-\alpha})\}\right]\phi(\omega^T_{1-\alpha}) + O(a^{-1}),
\end{aligned}
$$

where $u_a = -8\frac{z_{1-\alpha}g'(\nu^2)}{a^{\frac{3}{2}}}\sum_{i=1}^a \delta_i\left(\frac{n_i}{n_i-1}\right)\sigma_i^6(\gamma_i^2-2)$, $K_3(T_a)$ and $K_4(T_a)$ are given in (1.2.17) and (1.2.18), respectively. Therefore,

$$
\begin{aligned}
P(T_a > \widehat{\omega}^T_{1-\alpha,B}) \ = \ & \alpha + \left[\frac{u_a}{2\nu^2}(\omega^T_{1-\alpha}) + \frac{u_a K_3(T_a)}{6\nu^5}\{(\omega^T_{1-\alpha})^3 - 3(\omega^T_{1-\alpha})\}\right. \\
& \left. + \frac{u_a K_4(T_a)}{24\nu^6}\{(\omega^T_{1-\alpha})^5 - 10(\omega^T_{1-\alpha})^3 + 15(\omega^T_{1-\alpha})\}\right]\phi(\omega^T_{1-\alpha}) + O(a^{-1}) + O(B^{-1/2}),
\end{aligned}
$$

since $P(T_a > \omega^T_{1-\alpha}) = \alpha$. That is, the accuracy of the type I error-rate of the bootstrap test based on the test statistic T_a in (1.2.1) is of order $O(a^{-1/2}) + O(B^{-1/2})$.

Therefore, if B is large enough, which can be achieved in computation, the theoretical type I error-rate of both bootstrap test using T_a in (1.2.1) and the test of Akritas and Papadatos (2004) are the same and accurate up to order $O(a^{-1/2})$. This is analogous to the classical situation that bootstrapping a non-pivotal statistics does not improve the accuracy.

An advantage of the bootstrap test is that its quantiles $\widehat{\omega}^T_{1-\alpha}$ uses the sample moments to estimate the population skewness and kurtosis in the data, while the normal quantiles used in Akritas and Papadatos (2004) totally ignores the skewness and kurtosis. This could lead to slightly better numerical performance in applications for the bootstrap test (at the expense of more computational time).

1.2.3 Bootstrap Test with $T_a/\hat{\nu}$ and its type I error accuracy

It is noted that the test statistic T_a in (1.2.1) is not asymptotically pivotal. We consider an asymptotically pivotal statistic M_a defined as

$$M_a = \frac{T_a}{\hat{\nu}} \tag{1.2.24}$$

where $\hat{\nu}$ is the estimate of ν in (1.2.7). It can be shown that the limiting distribution of M_a is $N(0,1)$. To study the type I error accuracy of the test based on M_a in (1.2.24), we need more accurate approximation of the distribution. As in subsection 2.1, under the regularity conditions in 1.6.1 and under H_0, M_a admits Edgeworth expansion of the form

$$F_{M_a}(x) = P(M_a \le x) = \Phi(x) + \sum_{k=1}^{s-2} \frac{1}{a^{k/2}} Q_k(x)\phi(x) + O(a^{-(\frac{s-1}{2})}), \tag{1.2.25}$$

uniformly for $\forall x \in R$, where $\Phi(x)$ and $\phi(x)$ are the cdf and pdf of the standard normal distribution and $Q_k(x)$ is a polynomial of degree $3k-1$.

Based on the expansion (1.2.25), we note that the accuracy of approximation using $N(0,1)$ limiting distribution of M_a is only of order $O(a^{-1/2})$ due to the fact that $F_{M_a}(x) - \Phi(x) = O_p(a^{-1/2})$. In application, Akritas and Papadatos (2004) test is actually based on M_a using $N(0,1)$ approximation. At α level of significance, the estimated quantile of the Akritas and Papadatos (2004) test using $N(0,1)$ is z_α, which is the quantile of the standard normal distribution. The type I error-rate of this test is $P(M_a > z_{1-\alpha})$ which can be computed as $\alpha + O(a^{-1/2})$. Therefore, the type I error-rate of Akritas and Papadatos (2004) test based on $N(0,1)$ approximation to the distribution of M_a is only accurate up to order

$O(a^{-1/2})$. In the rest of this subsection, we study the type I error accuracy of bootstrap test based on M_a in (1.2.24).

Suppose $\mathbf{X}_i^* = \left\{X_{i1}^*, X_{i2}^*, \cdots, X_{in_i}^*\right\}'$ denote a sample drawn randomly with replacement from $\mathbf{X}_i = \{X_{i1}, X_{i2}, \cdots, X_{in_i}\}'$, where $\mathbf{X}_i$ is the collection of independent and identically distributed observations from treatment level i, $i = 1, 2, \cdots, a$. Then using the resampled data to compute $Y_{ij}^* = X_{ij}^* - \overline{X}_{i.}$, the bootstrap version of the test statistic M_a in (1.2.24) is M_a^*, which is computed from the resampled data as follows:

$$M_a^* = \frac{T_a^*}{\hat{\nu}^*} \tag{1.2.26}$$

where T_a^* is given in (1.2.2), $\hat{\nu}^* = \sqrt{\frac{2}{a}\sum_{i=1}^a \left(\frac{n_i}{n_i-1}\right)\widehat{\sigma_i^{4*}}}$, and $\widehat{\sigma_i^{4*}}$ is given by the u-statistic

$$\widehat{\sigma_i^{4*}} = \frac{1}{n_i(n_i-1)(n_i-2)(n_i-3)} \sum_{j_1 \neq j_2 \neq j_3 \neq j_4}^{n_i} \frac{(y_{ij_1}^* - y_{ij_2}^*)^2 (y_{ij_3}^* - y_{ij_4}^*)^2}{4}.$$

At α level of significance, the bootstrap statistic in (1.2.26) could be used to approximate the $1-\alpha$ analytical bootstrap quantile $\widehat{\omega}_{1-\alpha}^M$, which admits the following Cornish-Fisher expansion under regularity conditions.

$$\hat{\omega}_{1-\alpha}^M = z_{1-\alpha} + \sum_{k=1}^{s-2} \frac{1}{a^{k/2}} \hat{q}_k^{cf}(z_{1-\alpha}) \tag{1.2.27}$$

where $\hat{q}_k^{cf}(.)$ is a function of the plug-in estimate of $Q_k(.)$ in (1.2.25). This analytical form of quantile $\hat{\omega}_{1-\alpha}^M$ is an approximation of the quantile of M_a.

In practice, computational approach is generally used to approximate the analytical bootstrap quantile $\hat{\omega}_{1-\alpha}^M$ as follows:

- For a very large B, we randomly draw B bootstrap samples $\mathbf{X}_{ib}^* = \left\{X_{i1b}^*, X_{i2b}^*, \cdots, X_{in_ib}^*\right\}'$, from the observed data $\mathbf{X}_i = \{X_{i1}, X_{i2}, \cdots, X_{in_i}\}'$ for $i = 1, 2, \cdots, a$, $b = 1, 2, \cdots, B$.

- For each sample, we compute $M_{a,b}^* = \frac{T_{a,b}^*}{\hat{\nu}^{*(b)}}$, $b = 1, 2, \cdots, B$, where

$$T_{a,b}^* = a^{-1/2} \sum_{i=1}^a \left[n_i(\overline{Y}_{i.}^{*(b)} - \overline{Y}^{*(b)}..)^2 - \left(1 - \frac{n_i}{N}\right) S_i^{2*(b)} \right],$$

$\hat{\nu}^{*(b)} = \sqrt{\frac{2}{a}\sum_{i=1}^{a}\left(\frac{n_i}{n_i-1}\right)\widehat{\sigma_i^{4*(b)}}}$. $\widehat{\sigma_i^{4*(b)}}$ is given by u-statistic

$$\widehat{\sigma_i^{4*(b)}} = \frac{1}{n_i(n_i-1)(n_i-2)(n_i-3)}\sum_{j_1\neq j_2\neq j_3\neq j_4}^{n_i}\frac{(y_{ij_1}^{*(b)}-y_{ij_2}^{*(b)})^2(y_{ij_3}^{*(b)}-y_{ij_4}^{*(b)})^2}{4},$$

$\overline{Y}_{i.}^{*(b)}$, $\overline{Y}_{..}^{*(b)}$ and $S_i^{2*(b)}$ are the average of i^{th} sample, overall average and sample variance calculated from the b^{th} bootstrap sample, respectively.

- We rank the B bootstrap replications of M_a^* as $M_{a,(1)}^* \leq M_{a,(2)}^* \leq \cdots \leq M_{a,(B)}^*$. Then an estimate of the computational bootstrap quantile $\widehat{\omega}_{1-\alpha,B}^{M}$ is $M_{a,(l_B)}^*$, where $M_{a,(l_B)}^*$ has l_B observations smaller than or equal to it and $l_B = [(B+1)(1-\alpha)]$.

- At significance level α, the null hypothesis is rejected if $M_a > \widehat{\omega}_{1-\alpha,B}^{M}$.

The test for no treatment effect based on bootstrapping test statistic M_a uses the bootstrap quantile $\widehat{\omega}_{1-\alpha,B}^{M}$ to define the rejection region. The theoretical type I error-rate of this bootstrap test is $P(M_a > \widehat{\omega}_{1-\alpha,B}^{M})$. From the computational procedure, we know that $\widehat{\omega}_{1-\alpha,B}^{M}$ is the sample quantile based on $M_{a,b}^*$, $b = 1, 2, \cdots, B$, which is a random sample from the bootstrap distribution conditional on the observed data. The bootstrap distribution admits the following Edgeworth expansion

$$\widehat{F}_{M_a} = P(M_a^* \leq x|\underline{\mathbf{X}}) = \Phi(x) + \sum_{k=1}^{s-2}\frac{1}{a^{k/2}}\hat{Q}_k(x)\phi(x), \tag{1.2.28}$$

where $\hat{Q}_k(x)$ is the plug-in estimate of $Q_k(x)$ in (1.2.25), in which the unknown population moments are replaced by their corresponding sample moments. Similar to the argument between (1.2.20) and (1.2.21) in section 2.2, we know, $\widehat{\omega}_{1-\alpha,B}^{M}$ is approximately normal with mean $\widehat{\omega}_{1-\alpha}^{M}$ and variance $(1-\alpha)\alpha(B[\hat{f}(\widehat{F}_{M_a}(1-\alpha))]^2)^{-1}$ where $\widehat{F}_{M_a}(.)$ is given in (1.2.28) and $\hat{f}(.)$ is the corresponding pdf. Hence $\widehat{\omega}_{1-\alpha,B}^{M} - \widehat{\omega}_{1-\alpha}^{M} = O_p(B^{-1/2})$. This term can be made arbitrarily small by using a large B in computation. Then we can write $P(M_a > \widehat{\omega}_{1-\alpha,B}^{M})$ as $P(M_a - (\hat{\omega}_{1-\alpha,B}^{M} - \hat{\omega}_{1-\alpha}^{M}) - (\hat{\omega}_{1-\alpha}^{M} - \omega_{1-\alpha}^{M}) > \omega_{1-\alpha}^{M}) = P(M_a - (\hat{\omega}_{1-\alpha}^{M} - \omega_{1-\alpha}^{M}) > \omega_{1-\alpha}^{M}) + O(B^{-1/2})$.

To quantify the order of $\widehat{\omega}^M_{1-\alpha} - \omega^M_{1-\alpha}$, based on Cornish-Fisher expansion of quantiles, $\omega^M_{1-\alpha}$ can be written as (under the regularity conditions in 1.6.1)

$$\omega^M_{1-\alpha} = z_{1-\alpha} + \sum_{k=1}^{s-2} \frac{1}{a^{k/2}} q^{cf}_k(z_{1-\alpha}) + O(a^{-(\frac{s-1}{2})}) \tag{1.2.29}$$

where $q^{cf}_k(.)$ is a function of $Q_k(.)$ in (1.2.25). The estimate of $\omega^M_{1-\alpha}$ is $\widehat{\omega}^M_{1-\alpha}$ given in (1.2.27). Therefore we have

$$\begin{aligned} \hat{\omega}^M_{1-\alpha} - \omega^M_{1-\alpha} &= \sum_{k=1}^{s-2} \frac{1}{a^{k/2}} \left[\{\hat{q}^{cf}_k(z_{1-\alpha}) - E[\hat{q}^{cf}_k(z_{1-\alpha})]\} + \{E[\hat{q}^{cf}_k(z_{1-\alpha})] - q^{cf}_k(z_{1-\alpha})\} \right] \\ &= O_p(a^{-\frac{1}{2}}). \end{aligned}$$

The last equality is obtained by noting that $\hat{q}^{cf}_k$ is a polynomial with coefficients based on the sample moments while $q^{cf}_k(z_{1-\alpha})$ is the corresponding polynomial with coefficients based on the population moments. From central limit theorem for independent data, we have that $a^{1/2}(\hat{q}^{cf}_k(z_{1-\alpha}) - E[\hat{q}^{cf}_k(z_{1-\alpha})]) = O_p(1)$ while the bias part $E[\hat{q}^{cf}_k(z_{1-\alpha})] - q^{cf}_k(z_{1-\alpha}) = O_p(1)$, see Theorem 2.3.1 (to get a sense of the terms in $q^{cf}_k()$). Thus, $\hat{\omega}^M_{1-\alpha}$ approximates the true $1-\alpha$ quantile $\omega^T_{1-\alpha}$ in the order of $O_p(a^{-1/2})$. Now using the above results, the type I error $P(M_a > \hat{\omega}^M_{1-\alpha,B})$ can be written as

$$\begin{aligned} & P(M_a > \omega^M_{1-\alpha} + \hat{\omega}^M_{1-\alpha,B} - \hat{\omega}^M_{1-\alpha} + \hat{\omega}^M_{1-\alpha} - \omega^M_{1-\alpha}) \\ =\ & P(M_a - (\hat{\omega}^M_{1-\alpha,B} - \hat{\omega}^M_{1-\alpha}) - (\hat{\omega}^M_{1-\alpha} - \omega^M_{1-\alpha}) > \omega^M_{1-\alpha}) \\ =\ & P(M_a - (\hat{\omega}^M_{1-\alpha} - \omega^M_{1-\alpha}) > \omega^M_{1-\alpha}) + O(B^{-1/2}) \\ =\ & P(M_a + O_p(a^{-\frac{1}{2}}) > \omega^M_{1-\alpha}) + O(B^{-1/2}) \\ =\ & P\left(M_a > \omega^M_{1-\alpha}\right) + O(a^{-\frac{1}{2}}) + O(B^{-\frac{1}{2}}) \end{aligned} \tag{1.2.30}$$

where the last three equalities are due to the Delta method in Hall (1992b) section 2.7. This can also be computed in more detail using the same technique as in previous section. We give it in 1.6.10. Therefore, $P(M_a > \omega^M_{1-\alpha}) = \alpha + O(a^{-1/2}) + O(B^{-1/2})$. Thus the accuracy of the type I error-rate of the bootstrap test based on the test statistic M_a in (1.2.24) is of order $O(a^{-1/2}) + O(B^{-1/2})$.

Therefore, for a very large B, which can be achieved in computation, the theoretical type I error-rate of bootstrap test using asymptotically pivotal statistic M_a in (1.2.24) is $O(a^{-1/2})$, which has the same rate of convergence as the test of Akritas and Papadatos (2004) using $T_a/\hat{\nu}$ which is the statistics used in applications. Thus the type I error accuracy for bootstrap test based on asymptotically pivotal statistic is not better than the test of Akritas and Papadatos (2004) or bootstrap test based on non-pivotal statistics.

1.3 Bootstrap Test for Two-Way Analysis of Variance

With increasing advancement in technology especially in scientific research such as micro-array and agricultural screening experiments, large amount of data are collected. In this section, we are particularly interested in data from crossed two-way designs that have a large number of treatment combinations but a small number of replications within each treatment. The small replications used in the experiment are due to high cost of equipments for conducting the experiments. For examples see Dudoit et al. (2002), Wang and Akritas (2006), Wang and Akritas (2011). It is of interest to the researcher to examine which treatments have significant effects as well as the significance of the interaction effect.

The classical F-test is known to perform well under the classical conditions of small number of treatment combinations but with equal number of large replications within each cell of treatment combination and normality. For unequal number of replications and in the presence of heteroscedastic variances Ananda and Weerahandi (1997) discussed that results based on the classical F-test is not robust. They demonstrated that the type I error-rate of the classical F-test can be highly inflated under violation of unbalanced and heteroscedastic assumptions. Some studies in two-way crossed designs have been conducted in literature in the framework of a large number of treatment combinations with a small number of replications. Wang and Akritas (2006) proposed new test statistics for testing the main and interaction effects in a two-way heteroscedastic ANOVA based on original observations. Wang and Akritas (2004) provides results for two-way ANOVA based on ranks and Wang and Akritas (2011) provide test statistic for multi-way layout high dimensional ANOVA. The results of the aforementioned papers are based on the limiting distribution of the test statistic which converges slowly.

It is notable that under the classical settings with small number of treatment level combinations and large replications within each treatment level combination, some bootstrap tests and confidence intervals can provide better approximation accuracy. Fisher and Hall (1990) used the classical F-statistic to outline a general procedure for conducting a bootstrap hypothesis test in a two-way ANOVA. However their results are only valid under the classical setting. The conditions of Fisher and Hall (1990) bootstrap hypothesis testing do not apply to our current framework that has a large number of treatment level combinations with small replications in the presence of skewness and heteroscedastic variances.

In this section, we demonstrate the bootstrap hypothesis test for two-way analysis of variance when the number of rows is large but with fixed number of columns and small replications within each treatment level combination in the presence of heteroscedastic variances and extreme observations.

We consider observations X_{ijk}, $i = 1, 2, \cdots, a$, $j = 1, 2, \cdots, b$ and $k = 1, 2, \cdots, n_{ij}$ in a crossed two-way design structure. We assume that X_{ijk}'s are independent and identically distributed in each (i, j) treatment level combination. The observations were decomposed as $X_{ijk} = \mu + \alpha_i + \beta_j + \gamma_{ij} + \epsilon_{ijk}$, subject to the identifiability constraints $\sum_{i=1}^{a} \alpha_i = \sum_{j=1}^{b} \beta_j = \sum_{i=1}^{a} \gamma_{ij} = \sum_{j=1}^{b} \gamma_{ij} = 0$. The interest is to test for no main row effect, i.e., $H_0(\alpha)$: all $\alpha_i = 0$ and no interaction effect, i.e., $H_0(\gamma)$: all $\gamma_{ij} = 0$, when the number of rows, a, is large but with fixed number of columns, b, and with small replications n_{ij} in each treatment level combination. In this two-way setup, the following notations will be used: $\overline{X}_{ij.} = n_{ij}^{-1} \sum_{k=1}^{n_{ij}} X_{ijk}$, $\widetilde{X}_{i..} = b^{-1} \sum_{j=1}^{b} \overline{X}_{ij.}$, $\widetilde{X}_{.j.} = a^{-1} \sum_{i=1}^{a} \overline{X}_{ij.}$, $\widetilde{X}_{...} = (ab)^{-1} \sum_{i=1}^{a} \sum_{j=1}^{b} \overline{X}_{ij.}$, $N = \sum_{i=1}^{a} \sum_{j=1}^{b} n_{ij}$.

1.3.1 Type I error accuracy of Wang and Akritas (2006) test

In this two-way ANOVA model, we first consider the type I error accuracy of the test statistics proposed by Wang and Akritas (2006) for testing of no main row effect and no interaction effect. The test statistics are

$$T_A = (ab)^{-1/2} \sum_{i=1}^{a} \sum_{j=1}^{b} \left[(\widetilde{X}_{i..} - \widetilde{X}_{...})^2 - \frac{1}{b} \left(1 - \frac{1}{a} \right) \frac{S_{ij}^2}{n_{ij}} \right] \tag{1.3.1}$$

and

$$T_C = (ab)^{-1/2} \sum_{i=1}^{a} \sum_{j=1}^{b} \left[(\overline{X}_{ij.} - \widetilde{X}_{i..} - \widetilde{X}_{.j.} + \widetilde{X}_{...})^2 - \frac{(a-1)(b-1)}{ab} \frac{S_{ij}^2}{n_{ij}} \right], \tag{1.3.2}$$

for testing no main row effect and no interaction effect, respectively, where $S_{ij}^2 = (n_{ij} - 1)^{-1} \sum_{k=1}^{n_{ij}} (X_{ijk} - \overline{X}_{ij.})^2$. These statistics were particularly proposed for the current framework of large number of rows but fixed number of columns with small replications within each (i, j) cell under the presence of heteroscedastic and extreme observations. Under the null hypothesis of no main row effect, Wang and Akritas (2006) gave the asymptotic distribution of T_A as $N(0, \nu_A^2)$, as $a \to \infty$ and b is fixed, where $\nu_A^2 = 2(\phi^4 + b\eta^4)/b^2$ with

$$\phi^4 = \lim_{a \to \infty} (ab)^{-1} \sum_{i=1}^{a} \sum_{j=1}^{b} \sigma_{ij}^4 [n_{ij}(n_{ij} - 1)]^{-1} \tag{1.3.3}$$

and

$$\eta^4 = \lim_{a \to \infty} (ab^2)^{-1} \sum_{i=1}^{a} \sum_{j_1 \neq j_2}^{b} \frac{\sigma_{ij_1}^2}{n_{ij_1}} \frac{\sigma_{ij_2}^2}{n_{ij_2}}. \tag{1.3.4}$$

They also gave the null limiting distribution for T_C as $N(0, \nu_C^2)$ when $a \to \infty$ and b is fixed, where $\nu_C^2 = \lim_{a \to \infty} 2(b-1)^2 \phi^4 / b^2 + 2\eta^4 / b$. To study the type I error accuracy of these tests, we describe the asymptotic expansion of the distribution of both T_A and T_C in this section.

Based on Proposition 3.4 in Wang and Akritas (2006), the distributions of T_A and $\widetilde{T}_A$ are asymptotically equivalent, where $\widetilde{T}_A = \frac{1}{\sqrt{a}} \sum_{i=1}^{a} u_{Ai}$ with

$$u_{Ai} = \frac{a-1}{ab^{\frac{3}{2}}} \left[\left(\sum_{j=1}^{b} \overline{X}_{ij.} \right)^2 - \sum_{j=1}^{b} \frac{S_{ij}^2}{n_{ij}} \right], \tag{1.3.5}$$

$i = 1, 2, \cdots, a$ and without loss of generality assuming that $E(X_{ijk}) = 0$. This is because both the first and second moments of $T_A - \widetilde{T}_A$ go to zero as $a \to \infty$ while n_{ij} and b stay fixed under the null. We need to know the order of $T_A - \widetilde{T}_A$ in order to study the accuracy of their test. The test statistic T_A in (1.3.1) can be written as

$$
\begin{aligned}
T_A &= (ab)^{-1/2} \sum_{i=1}^{a} \left[\frac{a}{ab} \left(\sum_{j=1}^{b} \bar{\epsilon}_{ij.} \right)^2 - \left(\frac{a-1}{ab} \right) \sum_{j=1}^{b} \frac{S_{ij}^2}{n_{ij}} \right] - (ab)^{1/2} \left(\frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} \bar{\epsilon}_{ij.} \right)^2 \\
&\quad + (ab)^{-1/2} \sum_{i=1}^{a} \sum_{j=1}^{b} \left[2(\widetilde{\epsilon}_{i..} - \widetilde{\epsilon}_{...})(\bar{\mu}_{i.} - \bar{\mu}_{..}) + (\bar{\mu}_{..} - \bar{\mu}_{i.})^2 \right] \\
&= \sum_{i=1}^{a} \frac{a-1}{(ab)^{-\frac{3}{2}}} \left[\left(\sum_{j=1}^{b} \bar{\epsilon}_{ij.} \right)^2 - \sum_{j=1}^{b} \frac{S_{ij}^2}{n_{ij}} \right] - (ab)^{-3/2} \sum_{i_1 \neq i_2}^{a} \left(\sum_{j=1}^{b} \bar{\epsilon}_{i_1 j.} \right) \left(\sum_{j=1}^{b} \bar{\epsilon}_{i_2 j.} \right) \\
&\quad + (ab)^{-1/2} \sum_{i=1}^{a} \sum_{j=1}^{b} \left[2(\widetilde{\epsilon}_{i..} - \widetilde{\epsilon}_{...})(\bar{\mu}_{i.} - \bar{\mu}_{..}) + (\bar{\mu}_{..} - \bar{\mu}_{i.})^2 \right].
\end{aligned}
$$

The last equality can be written as

$$
T_A = \widetilde{T}_A - (ab)^{-3/2} \sum_{i_1 \neq i_2}^{a} \left(\sum_{j=1}^{b} \bar{\epsilon}_{i_1 j.} \right) \left(\sum_{j=1}^{b} \bar{\epsilon}_{i_2 j.} \right) + \Omega_a, \tag{1.3.6}
$$

where

$$
\widetilde{T_A} = \sum_{i=1}^{a} \frac{a-1}{(ab)^{3/2}} \left\{ \sum_{j \neq j_1}^{b} \bar{\epsilon}_{ij.} \bar{\epsilon}_{ij_1.} + \sum_{j=1}^{b} \sum_{k \neq k_1}^{n_{ij}} \frac{\epsilon_{ijk} \epsilon_{ijk_1}}{n_{ij}(n_{ij}-1)} \right\}, \tag{1.3.7}
$$

$$
\Omega_a = (ab)^{-1/2} \sum_{i=1}^{a} \sum_{j=1}^{b} \left[2(\widetilde{\epsilon}_{i..} - \widetilde{\epsilon}_{...}) \alpha_i + \alpha_i^2 \right] \tag{1.3.8}
$$

and $\alpha_i = \bar{\mu}_{i.} - \bar{\mu}_{..}$. Denote the test statistic T_A under H_0 as $T_A^{(0)}$. Thus under the null we have

$$
T_A^{(0)} = \widetilde{T}_A - (ab)^{-3/2} \sum_{i_1 \neq i_2}^{a} \left(\sum_{j=1}^{b} \bar{\epsilon}_{i_1 j.} \right) \left(\sum_{j=1}^{b} \bar{\epsilon}_{i_2 j.} \right). \tag{1.3.9}
$$

Then

$$
T_A^{(0)} - \widetilde{T}_A = -(ab)^{-3/2} \sum_{i_1 \neq i_2}^{a} \left(\sum_{j=1}^{b} \bar{\epsilon}_{i_1 j.} \right) \left(\sum_{j=1}^{b} \bar{\epsilon}_{i_2 j.} \right) = -\frac{b^{1/2}}{a^{3/2}} \sum_{i_1 \neq i_2}^{a} \widetilde{\epsilon}_{i_1..} \widetilde{\epsilon}_{i_2..},
$$

with $E(T_A^{(0)} - \widetilde{T}_A) = 0$ and

$$\begin{aligned} E(b^{\frac{1}{2}}(T_A^{(0)} - \widetilde{T}_A))^2 &= 2\left(\frac{b^2}{a^3}\right)\sum_{i_1\neq i_2}^{a} E[\widetilde{\epsilon}_{i_1..}^2]E[\widetilde{\epsilon}_{i_2..}^2] \\ &= 2\left(\frac{1}{a^3b^2}\right)\sum_{i_1\neq i_2}^{a}\sum_{j_1=1}^{b}\sum_{j_2=1}^{b}\frac{\sigma_{i_1j_1}^2}{n_{i_1j_1}}\frac{\sigma_{i_2j_2}^2}{n_{i_2j_2}} \\ &= O(a^{-1}). \end{aligned}$$

Therefore, we have that $T_A^{(0)} - \widetilde{T}_A = O_p(a^{-1/2}b^{-1/2})$. Noting that $\widetilde{T}_A$ is the sum of independent random variables, we can apply Corollary 19.4 in Bhattacharya and Rao (2010) to obtain the asymptotic expansion of the distribution of $\widetilde{T}_A$. Thus to derive the null asymptotic distribution of T_A in (1.3.1), we write $T_A^{(0)}$ as

$$T_A^{(0)} = \widetilde{T}_A + a^{-\frac{1}{2}}\Pi_a, \tag{1.3.10}$$

where

$$\Pi_a = a^{\frac{1}{2}}(T_A^{(0)} - \widetilde{T_A}) = a^{\frac{1}{2}}\left(-\frac{b^{1/2}}{a^{3/2}}\sum_{i_1\neq i_2}^{a}\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\right), \tag{1.3.11}$$

satisfies $\Pi_a = O_p(1)$. It is shown in Section 1.7.3 that

$$P(T_A^{(0)} \le x) = P(\widetilde{T}_A \le x) + O(a^{-1}). \tag{1.3.12}$$

That is, the order of accuracy in approximating the distribution of $T_A^{(0)}$ with that of $\widetilde{T}_A$ is $O(a^{-1})$. Now we can apply Corollary 19.4 in Bhattacharya and Rao (2010) to obtain the Edgeworth expansion of $S_A = \frac{T_A^{(0)}}{\nu_A}$, where

$$\nu_A = \sqrt{2(\phi^4 + b\eta^4)/b^2}. \tag{1.3.13}$$

Similar to the development of $F_T(x)$ in (1.2.10), under the regularity conditions in Section 1.7.1, $T_A^{(0)}$ also has Edgeworth expansion of the form

$$F_{T_A^{(0)}}(x) = P(T_A^{(0)} \le x) = P\left(S_A \le \frac{x}{\nu_A}\right) = \Phi\left(\frac{x}{\nu_A}\right) + \sum_{k=1}^{s-2}\frac{1}{a^{k/2}}P_k^A\left(\frac{x}{\nu_A}\right)\phi\left(\frac{x}{\nu_A}\right) + O(a^{-(\frac{s-1}{2})}), \tag{1.3.14}$$

for some $s \geq 3$ where $\Phi(.)$ and $\phi(.)$ are the cdf and pdf of standard normal random variable, $P_k^A(x)$ is a polynomial of degree $3k-1$, with coefficients that depend on the population moments of u_{Ai} given in (1.3.5).

Under the alternative hypothesis $H_1(\alpha)$: at least one $\alpha_i > 0$, the test statistic T_A in (1.3.6) can be written as

$$T_A^{(1)} = \widetilde{T}_A + a^{-\frac{1}{2}}\Pi_a + \Omega_a = T_A^{(0)} + \Omega_a, \tag{1.3.15}$$

(note the superscript (1) in $T_A^{(1)}$ is to emphasize that the data were from H_1), where $\widetilde{T}_A, \Omega_a$ and $T_A^{(0)}$ are given in (1.3.7), (1.3.8) and (1.3.10), respectively. Under $H_1(\alpha)$,

$$\Pi_a = a^{\frac{1}{2}}\left(-\frac{b^{1/2}}{a^{3/2}}\sum_{i_1 \neq i_2}^{a} \widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\right) = a^{\frac{1}{2}}(T_A^{(1)} - \widetilde{T}_A - \Omega_a), \tag{1.3.16}$$

satisfies $\Pi_a = O_p(1)$. We assume that the departure from the null hypothesis satisfies

$$\sqrt{b}\sum_{i=1}^{a}\alpha_i^2 = O(a^{\frac{1}{2}}). \tag{1.3.17}$$

To determine the distribution of $T_A^{(1)}$ we write $P(T_A^{(1)} \leq x)$ as

$$P(T_A^{(1)} \leq x) = P(T_A^{(0)} + \Omega_a \leq x).$$

It is shown in Section 1.7.7 that the above probability is

$$\begin{aligned} & P(T_A^{(1)} \leq x) \\ = \;& P(T_A^{(0)} \leq x) - \left[\frac{\kappa_2}{2\nu_A^2}\left(x - \frac{\kappa_1}{\nu_A}\right) + \frac{\kappa_3}{6\nu_A^3}\left\{\left(x - \frac{\kappa_1}{\nu_A}\right)^2 - 1\right\}\right]\phi\left(x - \frac{\kappa_1}{\nu_A}\right) + O(a^{-1}), \end{aligned} \tag{1.3.18}$$

where $P(T_A^{(0)} \leq x)$ is the distribution of $T_A^{(0)}$ given in (1.3.14),

$$\kappa_1 = (b/a)^{1/2}\sum_{i=1}^{a}\alpha_i^2, \tag{1.3.19}$$

$$\kappa_2 = \frac{4}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}\alpha_i^2\frac{\sigma_{ij}^2}{n_{ij}}, \tag{1.3.20}$$

and

$$\kappa_3 = \frac{8(a-1)^3}{a^{9/2}b^{7/2}} \sum_{i=1}^{a} \left[\sum_{j \neq j_1}^{b} \alpha_i \frac{\gamma_{ij}\sigma_{ij}^3}{n_{ij}^2} \frac{\sigma_{ij_1}^2}{n_{ij_1}} + \sum_{j=1}^{b} \alpha_i \frac{\gamma_{ij}\sigma_{ij}^5}{n_{ij}^2(n_{ij}-1)} \right]. \tag{1.3.21}$$

We observe that the difference in distributions of $T_A^{(0)}$ in (1.3.12) and $T_A^{(1)}$ in (1.3.18) is due to the moments of Ω_a in (1.3.8) and the moments of cross terms of Ω_a and $T_A^{(0)}$.

In practice, Wang and Akritas (2006) approximate the distribution of $T_A^{(0)}$ with $N(0, \hat{\nu}_A^2)$ where

$$\hat{\nu}_A^2 = \frac{2}{a} \sum_{i=1}^{a} \left[\frac{1}{b^3} \sum_{j=1}^{b} \frac{\widehat{\sigma_{ij}^4}}{n_{ij}(n_{ij}-1)} + \frac{1}{b^3} \sum_{j_1 \neq j_2}^{b} \frac{s_{ij_1}^2}{n_{ij_1}} \frac{s_{ij_2}^2}{n_{ij_2}} \right], \tag{1.3.22}$$

is an unbiased estimate of ν_A^2 with $\widehat{\sigma_{ij}^4}$ being an unbiased estimate of σ_{ij}^4 given by the $U-$statistic,

$$\begin{aligned} \widehat{\sigma_{ij}^4} &= \frac{1}{P_{n_{ij}}^4} \frac{1}{12} \sum_{k_1 \neq k_2 \neq k_3 \neq k_4}^{n_{ij}} \left[(x_{ijk_1} - x_{ijk_2})^2 (x_{ijk_3} - x_{ijk_4})^2 + (x_{ijk_1} - x_{ijk_3})^2 (x_{ijk_2} - x_{ijk_4})^2 \right. \\ &+ \left. (x_{ijk_1} - x_{ijk_4})^2 (x_{ijk_2} - x_{ijk_3})^2 \right] \end{aligned}$$

where $P_{n_{ij}}^4 = n_{ij}(n_{ij}-1)(n_{ij}-2)(n_{ij}-3)$, and s_{ij}^2 denotes the sample variance for the (i, j) cell. Comparing the CDF of $N(0, \hat{\nu}_A^2)$ with (1.3.4), we can see that the error of approximation using $N(0, \hat{\nu}_A^2)$ is of order $O(a^{-1/2})$ since $F_{T_A}(x) - \Phi(\frac{x}{\hat{\nu}_A}) = O_p(a^{-1/2})$ this is a result of Taylor expansion and the fact that

$$\widehat{\nu}_A - \nu_A = O_p(a^{-1/2}), \tag{1.3.23}$$

which can be shown similarly as in Section 1.6.2.

Similarly, for large number of row factor levels, a, and fixed number of column factor levels, b, we let $\widetilde{T}_C$ denote the projection of T_C where $\widetilde{T}_C = \frac{1}{\sqrt{a}} \sum_{i=1}^{a} u_{Ci}$ with

$$u_{Ci} = \frac{(a-1)(b-1)}{ab^{\frac{3}{2}}} \sum_{j=1}^{b} \left(\overline{X}_{ij.}^2 - \frac{S_{ij}^2}{n_{ij}} \right) - \frac{a-1}{(ab)^{\frac{3}{2}}} \sum_{j_1 \neq j_2}^{b} \overline{X}_{ij_1.} \overline{X}_{ij_2.}, \tag{1.3.24}$$

for $i = 1, 2, \cdots, a$ and without loss of generality assuming that $E(X_{ijk}) = 0$. Let $T_C^{(0)}$ the test statistic under the null for testing of no interaction effect. Then based on Proposition 3.4 in Wang and Akritas (2006), the distributions of $T_C^{(0)}$ and $\widetilde{T}_C$ are asymptotically equivalent. Since $\widetilde{T}_C$ is the sum of independent random variables, we can apply Corollary 19.4 in Bhattacharya and Rao (2010) to get the asymptotic expansion of the distribution of $\widetilde{T}_C$. Likewise, it can be shown that under the null $T_C^{(0)} - \widetilde{T}_C = O_p(a^{-1/2})$ and the $P(T_C^{(0)} \leq x) = P(\widetilde{T}_C \leq x) + O(a^{-1})$. Thus, we can apply Corollary 19.4 in Bhattacharya and Rao (2010) to obtain the Edgeworth expansion of $S_C = \frac{T_C}{\nu_C}$, where

$$\nu_C = \sqrt{2(b-1)^2\phi^4/b^2 + 2\eta^4/b}. \tag{1.3.25}$$

Therefore similar to the development of $F_T(x)$ in (1.2.10), under the regularity conditions in Section 1.7.1, $T_C^{(0)}$ admits Edgeworth expansion of the form

$$F_{T_C^{(0)}}(x) = P(T_C^{(0)} \leq x) = P\left(S_C \leq \frac{x}{\nu_C}\right) = \Phi\left(\frac{x}{\nu_C}\right) + \sum_{k=1}^{s-2} \frac{1}{a^{k/2}} P_k^C\left(\frac{x}{\nu_C}\right)\phi\left(\frac{x}{\nu_C}\right) + O(a^{-(\frac{s-1}{2})}), \tag{1.3.26}$$

for some $s \geq 3$ where $\Phi(.)$ and $\phi(.)$ are the cdf and pdf of standard normal random variable, $P_k^C(x)$ is a polynomial of degree $3k-1$, with coefficients that depend on the population moments of u_{Ci} given in (1.3.24). In practice, Wang and Akritas (2006) approximate the distribution of $T_C^{(0)}$ with $N(0, \hat{\nu}_C^2)$ where

$$\hat{\nu}_C^2 = \frac{2}{a}\sum_{i=1}^{a}\left[\frac{(b-1)^2}{b^3}\sum_{j=1}^{b}\frac{\widehat{\sigma_{ij}^4}}{n_{ij}(n_{ij}-1)} + \frac{1}{b^3}\sum_{j_1 \neq j_2}^{b}\frac{s_{ij_1}^2}{n_{ij_1}}\frac{s_{ij_2}^2}{n_{ij_2}}\right], \tag{1.3.27}$$

is an unbiased estimate of ν_A^2. Comparing the CDF of $N(0, \hat{\nu}_C^2)$ with (1.3.26), we can see that the error of approximation using $N(0, \hat{\nu}_C^2)$ is of order $O(a^{-1/2})$ since $F_{T_C}(x) - \Phi(\frac{x}{\hat{\nu}_C}) = O_p(a^{-1/2})$ due to the fact that it can be shown that

$$\widehat{\nu}_C - \nu_C = O_p(a^{-1/2}), \tag{1.3.28}$$

where $\widehat{\nu}_C$ is an estimate of ν_C in (1.3.25).

It is noted that in testing for no main row and no interaction effects, the test of Wang and Akritas (2006) is based on normal approximation. At α level of significance, to test for no main row effect, the estimated quantiles of the Wang and Akritas (2006) test is $\hat{\nu}_A z_\alpha$, where z_α is the quantile of the standard normal distribution and $\hat{\nu}_A$ is the estimate of ν_A

given in (1.3.13). The type I error-rate of the Wang and Akritas (2006) test for no main row effect is $P(T_A^{(0)} > \hat{\nu}_A z_{1-\alpha})$, which can be written as

$$P(T_A^{(0)} > \hat{\nu}_A z_{1-\alpha}) = P(T_A^{(0)} - (\hat{\nu}_A - \nu_A) z_{1-\alpha} > \nu_A z_{1-\alpha}).$$

Similar to the proof in the one-way case, the above probability can be shown to be $P(T_A^{(0)} > \nu_A z_{1-\alpha}) = \alpha + O(a^{-1/2})$. Therefore, the theoretical type I error-rate of no main row effect for Wang and Akritas (2006) test is only accurate up to order $O(a^{-1/2})$. It can also be shown that the theoretical type I error-rate for the test of no interaction effect based on Wang and Akritas (2006) test is also accurate up to order $O(a^{-1/2})$.

1.3.2 Bootstrap tests with T_A and T_C and their type I error accuracy

In this subsection, an analogous procedure for studying the bootstrap test for one-way analysis of variance in subsection 2.2 is employed to study the bootstrap test and its type I error accuracy for two-way analysis of variance based on the test statistics T_A and T_C in (1.3.1) and (1.3.2), respectively.

We denote by $\mathbf{X}_{ij}^* = \left\{X_{ij1}^*, X_{ij2}^*, \cdots, X_{ijn_{ij}}^*\right\}$, random samples drawn with replacement from the data $\mathbf{X}_{ij} = \left\{X_{ij1}, X_{ij2}, \cdots, X_{ijn_{ij}}\right\}$, where $\mathbf{X}_{ij}$ is the collection of independent and identically distributed observations from each treatment level combination (i,j), $i = 1, 2, \cdots, a$ and $j = 1, 2, \cdots, b$. To construct the bootstrap version of the test statistics for testing no main row and no interaction effects in (1.3.1) and (1.3.2), respectively, consider the transformation $Y_{ijk} = X_{ijk} - \mu_{ij}$. Since μ_{ij} is unknown, we use the resampled data to compute $Y_{ijk}^* = X_{ijk}^* - \overline{X}_{ij.}$. The following notations will be used: $\overline{Y}_{ij.}^* = n_{ij}^{-1} \sum_{k=1}^{n_{ij}} Y_{ijk}^*$, $\widetilde{Y}_{i..}^* = b^{-1} \sum_{j=1}^{b} \overline{Y}_{ij.}^*$, $\widetilde{Y}_{.j.}^* = a^{-1} \sum_{i=1}^{a} \overline{Y}_{ij.}^*$, $\widetilde{Y}_{...}^* = (ab)^{-1} \sum_{i=1}^{a} \sum_{j=1}^{b} \overline{Y}_{ij.}^*$, $S_{ij}^{2*} = (n_{ij} - 1)^{-1} \sum_{k=1}^{n_{ij}} (Y_{ijk}^* - \overline{Y}_{ij.}^*)^2$. The bootstrap version of the test statistic T_A in (1.3.1) is T_A^*, which is computed from the resampled data as

$$T_A^* = (ab)^{-1/2} \sum_{i=1}^{a} \sum_{j=1}^{b} \left[(\widetilde{Y}_{i..}^* - \widetilde{Y}_{...}^*)^2 - \frac{1}{b} \left(1 - \frac{1}{a}\right) \frac{S_{ij}^{2*}}{n_{ij}} \right]. \tag{1.3.29}$$

Likewise the bootstrap version of the test statistic T_C is T_C^* computed from the resampled data as

$$T_C^* = (ab)^{-1/2} \sum_{i=1}^{a} \sum_{j=1}^{b} \left[(\overline{Y}_{ij.}^* - \widetilde{Y}_{i..}^* - \widetilde{Y}_{.j.}^* + \widetilde{Y}_{...}^*)^2 - \frac{(a-1)(b-1)}{ab} \frac{S_{ij}^{2*}}{n_{ij}} \right]. \tag{1.3.30}$$

At α level of significance, the bootstrap statistics in (1.3.29) and (1.3.30) could be used to obtain the $1-\alpha$ analytical bootstrap quantiles $\widehat{\omega}_{1-\alpha}^{T_A}$ and $\widehat{\omega}_{1-\alpha}^{T_C}$, respectively, which admits the following Cornish-Fisher expansion under regularity condition.

$$\hat{\omega}_{1-\alpha}^{T_A} = \hat{\nu}_A z_{1-\alpha} + \hat{\nu}_A \sum_{k=1}^{s-2} \frac{1}{a^{k/2}} \hat{p}_k^A(z_{1-\alpha}) \tag{1.3.31}$$

and

$$\hat{\omega}_{1-\alpha}^{T_C} = \hat{\nu}_C z_{1-\alpha} + \hat{\nu}_C \sum_{k=1}^{s-2} \frac{1}{a^{k/2}} \hat{p}_k^C(z_{1-\alpha}) \tag{1.3.32}$$

where $\hat{\nu}_A$ is an estimate of ν_A in (1.3.3), $\hat{\nu}_C$ is an estimate of ν_C in (1.3.6), $\hat{p}_k^A(.)$ and $\hat{p}_k^C(.)$ are functions of the estimates of $P_k^A(.)$ in (1.3.4) and $P_k^C(.)$ in (1.3.17), respectively. These analytical form of quantiles $\hat{\omega}_{1-\alpha}^{T_A}$ and $\hat{\omega}_{1-\alpha}^{T_A}$ are an approximation of the quantiles of T_A and T_C, respectively.

To obtain (1.3.28) and (1.3.29) in practice, computational approach is generally used to approximate the analytical bootstrap quantiles $\hat{\omega}_{1-\alpha}^{T_A}$ and $\hat{\omega}_{1-\alpha}^{T_C}$ as follows:

- For a very large M, we randomly draw M bootstrap samples $\mathbf{X}_{ijm}^* = \left\{ X_{ij1m}^*, X_{ij2m}^*, \cdots, X_{ijn_{ij}m}^* \right\}'$, from the observed data $\mathbf{X}_{ij} = \left\{ X_{ij1}, X_{ij2}, \cdots, X_{ijn_{ij}} \right\}'$ for $i = 1, 2, \cdots, a$, $j = 1, 2, \cdots, b$ and $m = 1, 2, \cdots, M$.

- For each sample, we compute

$$T_{A,m}^* = (ab)^{-1/2} \sum_{i=1}^{a} \sum_{j=1}^{b} \left[(\widetilde{Y}_{i..}^{*(m)} - \widetilde{Y}_{...}^{*(m)})^2 - \frac{1}{b}\left(1 - \frac{1}{a}\right) \frac{S_{ij}^{2*(m)}}{n_{ij}} \right]$$

and

$$T_C^{*(m)} = (ab)^{-1/2} \sum_{i=1}^{a} \sum_{j=1}^{b} \left[(\overline{Y}_{ij.}^{*(m)} - \widetilde{Y}_{i..}^{*(m)} - \widetilde{Y}_{.j.}^{*(m)} + \widetilde{Y}_{...}^{*(m)})^2 - \frac{(a-1)(b-1)}{ab} \frac{S_{ij}^{2*(m)}}{n_{ij}} \right],$$

where $m = 1, 2, \cdots, M$, $\overline{Y}_{ij.}^{*(m)}$, $\widetilde{Y}_{i..}^{*(m)}$, $\widetilde{Y}_{.j.}^{*(m)}$, $\widetilde{Y}_{...}^{*(m)}$, and $S_{ij}^{2*(m)}$ are the weighted average of $(ij)^{th}$ sample, unweighted average of i^{th} sample, unweighted mean of j^{th}, overall unweighted mean and sample variance calculated from the m^{th} bootstrap sample, respectively.

- After ranking the M bootstrap replications of T_A^* and T_C^* as $T_{A,(1)}^* \leq T_{A,(2)}^* \leq \cdots \leq T_{A,(M)}^*$ and $T_{C,(1)}^* \leq T_{C,(2)}^* \leq \cdots \leq T_{C,(M)}^*$ respectively, then an estimate of the computational bootstrap quantiles $\widehat{\omega}_{1-\alpha,M}^{T_A}$ and $\widehat{\omega}_{1-\alpha,M}^{T_C}$ are $T_{A,(\kappa_M)}^*$ and $T_{C,(\kappa_M)}^*$, respectively. Where $T_{A,(\kappa_M)}^*$ has $\kappa_M \approx [(M+1)(1-\alpha)]$ observations smaller than or equal to it and $T_{C,(\eta_M)}^*$ has $\eta \approx [(M+1)(1-\alpha)]$ observations smaller than or equal to it.

- At α level of significance, the null hypotheses of no main row and no interaction effects are rejected if $T_A > \widehat{\omega}_{1-\alpha,M}^{T_A}$ and $T_C > \widehat{\omega}_{1-\alpha,M}^{T_C}$.

In testing for no main row and no interaction effects, the bootstrap test uses the bootstrap quantile $\widehat{\omega}_{1-\alpha,M}^{T_A}$ and $\widehat{\omega}_{1-\alpha,M}^{T_C}$ to define the rejection regions. The theoretical type I error-rates of these bootstrap tests to test for no main row and no interaction effects are $P(T_A > \widehat{\omega}_{1-\alpha,M}^{T_A})$ and $P(T_C > \widehat{\omega}_{1-\alpha,M}^{T_C})$. From the computational procedure, we note that $\widehat{\omega}_{1-\alpha,M}^{T_A}$ and $\widehat{\omega}_{1-\alpha,M}^{T_C}$ are the sample quantiles based on $T_{A,m}^*$ and $T_{C,m}^*$, $m = 1, 2, \cdots, M$ respectively, which is a random sample from the bootstrap distribution conditional on the observed data. The bootstrap distributions of T_A^* and T_C^* admit Edgeworth expansions

$$\widehat{F}_{T_A}(x) = P(T_A^* \leq x|\underline{\mathbf{X}}) = \Phi\left(\frac{x}{\widehat{\nu_A}}\right) + \sum_{k=1}^{s-2} \frac{1}{a^{k/2}} \hat{P}_k^A\left(\frac{x}{\widehat{\nu_A}}\right) \phi\left(\frac{x}{\widehat{\nu_A}}\right) \tag{1.3.33}$$

and

$$\widehat{F}_{T_C}(x) = P(T_C^* \leq x|\underline{\mathbf{X}}) = \Phi\left(\frac{x}{\widehat{\nu_C}}\right) + \sum_{k=1}^{s-2} \frac{1}{a^{k/2}} \hat{P}_k^C\left(\frac{x}{\widehat{\nu_C}}\right) \phi\left(\frac{x}{\widehat{\nu_C}}\right) \tag{1.3.34}$$

where $\widehat{\nu_A}$ is the plugin estimate of ν_A in (1.3.3), $\widehat{\nu_C}$ is the plugin estimate of ν_C in (1.3.6), and $\hat{P}_k^A(x)$ and $\hat{P}_k^C(x)$ are the plugin estimates of $P_k^A(x)$ in (1.3.4) and $P_k^C(x)$ in (1.3.7), respectively, in which the unknown population moments are replaced by their corresponding sample moments.

As discussed in subsection 2.2, we note that conditional on the observed data, $\widehat{\omega}^{T_A}_{1-\alpha,M}$ is approximately normal with mean $\widehat{\omega}^{T_A}_{1-\alpha}$ and variance $(1-\alpha)\alpha(M[\hat{f}(\widehat{F}_{T_A}(1-\alpha))]^2)^{-1}$ where $\widehat{F}_{T_A}(.)$ is given in (1.3.13) and $\hat{f}(.)$ is the corresponding pdf. Thus, the order of $\widehat{\omega}^{T_A}_{1-\alpha,M} - \widehat{\omega}^{T_A}_{1-\alpha}$ can be quantified as $O_p(M^{-1/2})$. Similarly, $\widehat{\omega}^{T_C}_{1-\alpha,M} - \omega^{T_C}_{1-\alpha} = \widehat{\omega}^{T_C}_{1-\alpha,M} - \widehat{\omega}^{T_C}_{1-\alpha} + \widehat{\omega}^{T_C}_{1-\alpha} - \omega^{T_C}_{1-\alpha}$ with $\widehat{\omega}^{T_C}_{1-\alpha,M} - \widehat{\omega}^{T_C}_{1-\alpha} = O_p(M^{-1/2})$. These terms can be made arbitrarily small by using a large M in computation. Then we write the theoretical type I error-rate of the bootstrap test for main row effect $P(T_A > \hat{\omega}^{T_A}_{1-\alpha,M})$ as $P(T_A > \omega^{T_A}_{1-\alpha} + \hat{\omega}^{T_A}_{1-\alpha,M} - \hat{\omega}^{T_A}_{1-\alpha} + \hat{\omega}^{T_A}_{1-\alpha} - \omega^{T_A}_{1-\alpha}) = P(T_A - (\hat{\omega}^{T_A}_{1-\alpha} - \omega^{T_A}_{1-\alpha}) > \omega^{T_A}_{1-\alpha}) + O(M^{-1/2})$. In order to compute $P(T_A - (\hat{\omega}^{T_A}_{1-\alpha} - \omega^{T_A}_{1-\alpha}) > \omega^{T_A}_{1-\alpha})$, we let $\omega^{S_A}_{1-\alpha}$ and $\hat{\omega}^{S_A}_{1-\alpha}$ denote the true and estimated $1-\alpha$ quantiles of $S_A = \frac{T_A}{\nu_A}$. Based on Cornish-Fisher expansion of quantiles, $\omega^{S_A}_{1-\alpha}$ and $\hat{\omega}^{S_A}_{1-\alpha}$ can be written as (under the regularity conditions in section)

$$\omega^{S_A}_{1-\alpha} = z_{1-\alpha} + \sum_{k=1}^{s-2} \frac{1}{a^{k/2}} p_k^A(z_{1-\alpha}) \tag{1.3.35}$$

and

$$\hat{\omega}^{S_A}_{1-\alpha} = z_{1-\alpha} + \sum_{k=1}^{s-2} \frac{1}{a^{k/2}} \hat{p}_k^A(z_{1-\alpha}) \tag{1.3.36}$$

where $p_k^A(.)$ and $\hat{p}_k^A(.)$ are functions of $P_k^A(.)$ in (1.3.4) and $\hat{P}_k^A(.)$ in (1.3.13), respectively. Then the true quantile $\omega^{T_A}_{1-\alpha}$ of T_A in (1.3.1) is $\omega^{T_A}_{1-\alpha} = \nu_A \omega^{S_A}_{1-\alpha}$. The analytical form of the bootstrap quantile $\hat{\omega}^{T_A}_{1-\alpha}$ is given in (1.3.31). Therefore we have

$$\begin{aligned}
\hat{\omega}^{T_A}_{1-\alpha} - \omega^{T_A}_{1-\alpha} &= \widehat{\nu_A}\hat{\omega}^{S_A}_{1-\alpha} - \nu_A \omega^{S_A}_{1-\alpha} \\
&= (\widehat{\nu_A} - \nu_A) z_{1-\alpha} + \widehat{\nu_A} \sum_{k=1}^{s-2} \frac{1}{a^{k/2}} \hat{p}_k^A(z_{1-\alpha}) - \nu_A \sum_{k=1}^{s-2} \frac{1}{a^{k/2}} p_k^A(z_{1-\alpha}) \\
&= O_p(a^{-1/2}),
\end{aligned}$$

since $\widehat{\nu_A} - \nu_A = O_p(a^{-1/2})$. Thus, $\hat{\omega}^{T_A}_{1-\alpha}$ approximates the true bootstrap quantile $\omega^{T_A}_{1-\alpha}$ in the order of $O(a^{-1/2})$. Now using the above results, $P(T_A > \hat{\omega}^{T_A}_{1-\alpha,M})$ can be written as

$$\begin{aligned}
& P(T_A > \omega^{T_A}_{1-\alpha} + \hat{\omega}^{T_A}_{1-\alpha,M} - \hat{\omega}^{T_A}_{1-\alpha} + \hat{\omega}^{T_A}_{1-\alpha} - \omega^{T_A}_{1-\alpha}) \\
= \ & P(T_A - (\hat{\omega}^{T_A}_{1-\alpha,M} - \hat{\omega}^{T_A}_{1-\alpha}) - (\hat{\omega}^{T_A}_{1-\alpha} - \omega^{T_A}_{1-\alpha}) > \omega^{T_A}_{1-\alpha}) \\
= \ & P(T_A - (\hat{\omega}^{T_A}_{1-\alpha} - \omega^{T_A}_{1-\alpha}) > \omega^{T_A}_{1-\alpha}) + O(M^{-1/2}) \\
= \ & \alpha + O(a^{-1/2}) + O(M^{-1/2}), \qquad (1.3.37)
\end{aligned}$$

where (1.3.17) follows as discussed in the one-way case. Similarly, it can be shown that $P(T_C > \hat{\omega}^{T_C}_{1-\alpha,M}) = \alpha + O(a^{-1/2}) + O(M^{-1/2})$. That is, the accuracy of the type I error-rates of the bootstrap tests based on T_A in (1.3.1) and T_C (1.3.2) are of the same order $O(a^{-1/2}) + O(M^{-1/2})$.

Therefore, if M is large enough, which can be achieved in computation, the theoretical type I error-rate of both bootstrap test and the test of Wang and Akritas (2006) have the same accuracy of up to order $O(a^{-1/2})$ based on the test statistics T_A and T_C given in (1.3.1) and (1.3.2), respectively. An advantage of the bootstrap quantiles $\widehat{\omega}^{T_A}_{1-\alpha}$ and $\widehat{\omega}^{T_C}_{1-\alpha}$ uses the sample moments to estimate the population skewness and kurtosis in the data, while the normal quantiles used in Wang and Akritas (2006) totally ignores the skewness and kurtosis. This could lead to slightly better numerical performance in applications for the bootstrap tests.

1.3.3 Bootstrap tests with $T_A/\hat{\nu}_A$ and $T_C/\hat{\nu}_C$ and their type I error accuracy

As discussed in subsections 3.1 and 3.2, the test of Wang and Akritas (2006) and bootstrap test based on the test statistics T_A and T_C given in (1.3.1) and (1.3.2), respectively, have the same type I error accuracy of up to order $O(a^{-1/2})$. This is due to the fact that both T_A and T_C are asymptotically non-pivotal statistics. It is therefore important to consider test statistic which is asymptotically pivotal. In this subsection, we discuss bootstrap test based on pivotal statistic similar to the one-way case in subsection 2.3.

We denote by M_A and M_C the studentized versions of the statistics T_A and T_C, respectively, such that

$$M_A = \frac{T_A}{\hat{\nu}_A}, \tag{1.3.38}$$

and

$$M_C = \frac{T_C}{\hat{\nu}_C}, \tag{1.3.39}$$

where $\hat{\nu}_A$ and $\hat{\nu}_C$ are the estimates of ν_A and ν_C given in (1.3.22) and (1.3.27) respectively. We note that both M_A and M_C have limiting distribution of $N(0,1)$ under the null. Since we are interested in the type I error accuracy of the test based on M_A and M_C, we consider an asymptotic expansion of distribution of M_A and M_C. Under the regularity conditions in section 1.7.1 and 1.7.2, respectively, M_A and M_C admit Edgeworth expansion of the forms

$$F_{M_A}(x) = P(M_A \leq x) = \Phi(x) + \sum_{k=1}^{s-2} \frac{1}{a^{k/2}} Q_k^A(x)\phi(x) + O(a^{-(\frac{s-1}{2})}), \tag{1.3.40}$$

and

$$F_{M_C}(x) = P(M_C \leq x) = \Phi(x) + \sum_{k=1}^{s-2} \frac{1}{a^{k/2}} Q_k^C(x)\phi(x) + O(a^{-(\frac{s-1}{2})}), \tag{1.3.41}$$

respectively, uniformly for $\forall x \in R$, where $\Phi(x)$ and $\phi(x)$ are the cdf and pdf of the standard normal distribution, $Q_k^A(x)$ and $Q_k^C(x)$ are polynomials of degree $3k-1$, with coefficients that depend on the population moments of u_{Ai} and u_{Ci}, given in (1.3.5) and (1.3.24), respectively, for $i = 1, 2, \cdots, a$.

Since we want to study the bootstrap test with M_A and M_C, we consider $Y_{ijk}^* = X_{ijk}^* - \overline{X}_{ij.}$ which has been discussed in the second paragraph of subsection 3.2. We denote by M_A^* and M_C^* the bootstrap versions of M_A and M_C given in (1.3.38) and (1.3.39), respectively such that

$$M_A^* = \frac{T_A^*}{\hat{\nu}_A^*} \tag{1.3.42}$$

$$M_C^* = \frac{T_C^*}{\hat{\nu}_C^*} \tag{1.3.43}$$

where T_A^* and T_C^* are given in (1.3.29) and (1.3.30), respectively,

$$\hat{\nu}_A^* = \sqrt{\frac{2}{a}\sum_{i=1}^{a}\left[\frac{1}{b^3}\sum_{j=1}^{b}\frac{\widehat{\sigma_{ij}^{4*}}}{n_{ij}(n_{ij}-1)} + \frac{1}{b^3}\sum_{j_1\neq j_2}^{b}\frac{s_{ij_1}^{2*}}{n_{ij_1}}\frac{s_{ij_2}^{2*}}{n_{ij_2}}\right]},$$

and

$$\hat{\nu}_C^* = \sqrt{\frac{2}{a}\sum_{i=1}^{a}\left[\frac{(b-1)^2}{b^3}\sum_{j=1}^{b}\frac{\widehat{\sigma_{ij}^{4*}}}{n_{ij}(n_{ij}-1)} + \frac{1}{b^3}\sum_{j_1\neq j_2}^{b}\frac{s_{ij_1}^{2*}}{n_{ij_1}}\frac{s_{ij_2}^{2*}}{n_{ij_2}}\right]},$$

with $\widehat{\sigma_{ij}^{4*}}$ given by the $U-$statistic,

$$\begin{aligned}\widehat{\sigma_{ij}^{4*}} &= \frac{1}{P_{n_{ij}}^4}\frac{1}{12}\sum_{k_1\neq k_2\neq k_3\neq k_4}^{n_{ij}}\left[(y_{ijk_1}^* - y_{ijk_2}^*)^2(y_{ijk_3}^* - y_{ijk_4}^*)^2 + (y_{ijk_1}^* - y_{ijk_3}^*)^2(y_{ijk_2}^* - y_{ijk_4}^*)^2\right.\\ &+ \left.(y_{ijk_1}^* - y_{ijk_4}^*)^2(y_{ijk_2}^* - y_{ijk_3}^*)^2\right].\end{aligned}$$

Theoretically, the quantiles of M_A^* and M_C^* are used to approximate the $1-\alpha$ analytical bootstrap quantiles $\widehat{\omega}_{1-\alpha}^{M_A}$ and $\widehat{\omega}_{1-\alpha}^{M_C}$, respectively, at α level of significance. Under the regularity conditions in section 1.7.1 and 1.7.2, $\widehat{\omega}_{1-\alpha}^{M_A}$ and $\widehat{\omega}_{1-\alpha}^{M_C}$ admit Cornish-Fisher expansion of the forms

$$\hat{\omega}_{1-\alpha}^{M_A} = z_{1-\alpha} + \sum_{k=1}^{s-2}\frac{1}{a^{k/2}}\hat{q}_k^A(z_{1-\alpha}) \tag{1.3.44}$$

and

$$\hat{\omega}_{1-\alpha}^{M_C} = z_{1-\alpha} + \sum_{k=1}^{s-2}\frac{1}{a^{k/2}}\hat{q}_k^C(z_{1-\alpha}), \tag{1.3.45}$$

respectively, where $\hat{q}_k^A$ and $\hat{q}_k^A$ are functions of the plug-in estimates of Q_k^A and Q_k^C given in (1.3.40) and (1.3.41) respectively. Using the computational procedure outlined in subsection 3.2, we approximate $\widehat{\omega}_{1-\alpha}^{M_A}$ and $\widehat{\omega}_{1-\alpha}^{M_C}$ with the bootstrap quantiles $\widehat{\omega}_{1-\alpha,B}^{M_A}$ and $\widehat{\omega}_{1-\alpha,B}^{M_C}$, respectively, where B is the number of bootstrap replications of M_A^* and M_C^* denoted by $M_{A,(1)}^* \leq M_{A,(2)}^* \leq \cdots \leq M_{A,(B)}^*$ and $M_{C,(1)}^* \leq M_{C,(2)}^* \leq \cdots \leq M_{C,(B)}^*$, respectively. We note that the B bootstrap replications, $M^*_{A,b}$, $b = 1, \cdots, B$, is a random sample from the bootstrap distribution conditional on the observed data. Thus the bootstrap distribution for testing no main row effect admits Edgeworth expansion of the form

$$\widehat{F}_{M_A} = P(M^*_A \leq x|\underline{\mathbf{X}}) = \Phi(x) + \sum_{k=1}^{s-2} \frac{1}{a^{k/2}} \hat{Q}^A_k(x)\phi(x), \tag{1.3.46}$$

where $\hat{Q}^A_k(x)$ is the plug-in estimate of $Q^A_k(x)$ in (1.3.40), in which the unknown population moments are replaced by their corresponding sample moments.

To test for no main row treatment effect, at α level of significance, $H_0(\alpha)$ is rejected if $M_A > \widehat{\omega}^{M_A}_{1-\alpha}$. In theory, the type I error of the bootstrap test is $P(M_A > \widehat{\omega}^{M_A}_{1-\alpha})$. This probability can be written as

$$\begin{aligned} P(M_A > \widehat{\omega}^{M_A}_{1-\alpha}) &= P(M_A - (\hat{\omega}^{M_A}_{1-\alpha,B} - \hat{\omega}^{M_A}_{1-\alpha}) - (\hat{\omega}^{M_A}_{1-\alpha} - \omega^{M_A}_{1-\alpha}) > \omega^{M_A}_{1-\alpha}) \\ &= P(M_A - (\hat{\omega}^{M_A}_{1-\alpha} - \omega^{M_A}_{1-\alpha}) > \omega^{M_A}_{1-\alpha}) + O(B^{-1/2}), \end{aligned} \tag{1.3.47}$$

where the equality is due to the fact that $\hat{\omega}^{M_A}_{1-\alpha,B} - \hat{\omega}^{M_A}_{1-\alpha} = O_p(B^{-1/2})$ which follows from the discussion in subsection 2.2. $\omega^{M_A}_{1-\alpha}$ is the true quantile of M_A given in (1.3.38). Under the regularity conditions in section 1.7.1, $\omega^{M_A}_{1-\alpha}$ admits the following Cornish-Fisher expansion

$$\omega^{M_A}_{1-\alpha} = z_{1-\alpha} + \sum_{k=1}^{s-2} \frac{1}{a^{k/2}} q^A_k(z_{1-\alpha}) + O(a^{-(\frac{s-1}{2})}), \tag{1.3.48}$$

where $q^A_k(.)$ is a function of $Q^A_k(.)$ in (1.3.40). To quantify the order of $\hat{\omega}^{M_A}_{1-\alpha} - \omega^{M_A}_{1-\alpha}$ we have

$$\begin{aligned} \hat{\omega}^{M_A}_{1-\alpha} - \omega^{M_A}_{1-\alpha} &= \sum_{k=1}^{s-2} \frac{1}{a^{k/2}} \left[\{\hat{q}^A_k(z_{1-\alpha}) - E[\hat{q}^A_k(z_{1-\alpha})]\} + \{E[\hat{q}^A_k(z_{1-\alpha})] - q^A_k(z_{1-\alpha})\} \right] \\ &= O_p(a^{-\frac{1}{2}}), \end{aligned}$$

where the equality is due to the fact that by central limit theorem (Lyapounov condition) $a^{1/2}(\hat{q}^A_k(z_{1-\alpha}) - E[\hat{q}^A_k(z_{1-\alpha})]) = O_p(1)$ and the bias part $E[\hat{q}^A_k(z_{1-\alpha})] - q^A_k(z_{1-\alpha}) = O_p(1)$. Thus, $\hat{\omega}^{M_A}_{1-\alpha}$ approximates the true $1-\alpha$ quantile $\omega^{M_A}_{1-\alpha}$ in the order of $O_p(a^{-1/2})$. Using the Delta method of Hall (1992b), the probability in (1.3.47) is equal to

$$P(M_A > \widehat{\omega}^{M_A}_{1-\alpha}) = P(M_A > \omega^{M_A}_{1-\alpha}) + O_p(a^{-\frac{1}{2}}) + O(B^{-1/2}) = \alpha + O_p(a^{-\frac{1}{2}}) + O(B^{-1/2}).$$

Similarly, it can be shown that the type I error rate for test of no interaction effect is equal to $\alpha + O_p(a^{-1/2}) + O(B^{-1/2})$. Therefore, the accuracy of the bootstrap test based on M_A and M_C is of order $O_p(a^{-1/2}) + O(B^{-1/2})$.

In conclusion, for a very large B, bootstrap test based on asymptotically pivotal statistics have the same rate of convergence compared to that of non-pivotal statistics and the test of Wang and Akritas (2006) based on $T_A/\hat{\nu}_A$ and $T_C/\hat{\nu}_C$ for the test of no main and interaction effects, respectively.

1.4 Simulation Studies for One-Way ANOVA

In this section, we conduct simulation studies to investigate the performance of the bootstrap test in one-way ANOVA. We compare the type I error-rate of Akritas and Papadatos (2004) test in subsection 1.2.1 and the bootstrap test based on both T_a^* and M_a^* in subsections 1.2.2 and 1.2.3, respectively.

1.4.1 Simulation setting

We consider heteroscedastic data generated from chi-square distribution with 3 degrees of freedom, chi-square distribution with 8 degrees of freedom, and normal (0, 1) as follows:

- D1: $Y_{ij} = i\tau/a + log(i+1)\epsilon_{ij}$, where ϵ_{ij} are i.i.d. $N(0,1)$.
- D2: $Y_{ij} = 8(i\tau/a)^2 + 8log(i+1)(X_{ij} - 8)$, where X_{ij} are i.i.d. χ^2_8.
- D3: $Y_{ij} = 3(i\tau/a)^2 + 3log(i+1)(X_{ij} - 3)$, where X_{ij} are i.i.d. χ^2_3.

$D2$ and $D3$ have larger heteroscedastic variances while $D1$ has small heteroscedastic variance. In addition, the distribution $D3$ is more skewed than $D2$ and $D1$. The data generation and tests were repeated 2000 times under H_0 i.e. $\tau = 0$. The proportion of rejections from the 2000 runs are reported as type I error.

We report the type I error-rate results for $D1 - D3$ for the number of treatment levels a = 10, 15, 20, 25, 50, 75 and 100 at nominal $\alpha = 0.05$. We use the following group sizes and number of treatment levels in the data generation. For $a = 10$, the group sizes are 4, 5, 4, 6, 5, 6, 4, 5, 4, 4; for a =15, $n_i = 5, 4, 4, 4, 4, 4, 6, 4, 4, 5, 5, 4, 4, 5, 4$ and for $a = 20$, n_i is equal to 4, 4, 4, 4, 4, 4, 4, 4, 6, 6, 4, 4, 5, 4, 4, 4, 4, 4, 5. When $a = 25$, 50, 75 and 100 we use the group sizes $6, 4, 5, 4, 5, 4, 4, 4, 5, 4, 6, 4, 4, 5, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 6, \cdots, 6$, where all omitted n_i's are equal to 6.

1.4.2 Simulation results

The estimated type I error-rate for $D1, D2$ and $D3$ are displayed in Table 1 for Akritas and Papadatos (2004) test labeled as AP, the bootstrap tests in subsections 1.2.2 and 1.2.3 labeled as T_a.boot and M_a.boot, respectively. We notice in Table 1.1 that the AP test is more liberal while the bootstrap test based on asymptotically pivotal statistic is conservative when the data comes from the more skewed distribution $D3$ but stable type I error-rate for $D1$ and $D2$ when the number of treatment levels a is large. The bootstrap test based on non-pivotal statistic T_a has inflated type I error for normal or the less skewed data ($D1$ and $D2$) when a is small. For $a = 100$, the bootstrap test based on T_a has good type I error for these two distributions. For the more skewed distribution $D3$, the bootstrap test with T_a has the best type I error control.

In general both T_a.boot and M_a.boot perform better than the T_a test for large number of treatments when the data comes from symmetric $D1$ or light skewed distributions $D2$ with heteroscedastic variances. On the other hand, for the more skewed data $D3$ the type I error-rate of M_a.boot is conservative for large number of treatments (i.e. $a > 25$). This is due to the fact that the bootstrap approach uses biased estimates in estimating the population parameters such as the skewness. To achieve a better convergence type I error accuracy, we need to consider a higher order asymptotic expansions of the test statistic based on Edgeworth and Cornish-Fisher expansions.

Distribution	a	AP	Ma.boot	Ta.boot
	10	8.6	2.8	8.3
	15	10.0	2.8	8.6
	20	7.8	2.9	7.8
$D1$	25	9.3	3.7	8.5
	50	6.9	4.0	6.3
	75	6.8	4.8	6.3
	100	6.3	3.9	5.8
	10	9.9	3.0	8.6
	15	9.4	2.6	7.8
	20	8.3	2.9	7.0
$D2$	25	8.2	2.8	7.1
	50	7.6	4.0	6.5
	75	6.5	3.9	4.7
	100	5.9	4.2	4.9
	10	9.5	2.2	7.2
	15	8.0	2.0	5.7
	20	7.8	1.6	5.5
$D3$	25	8.3	1.7	5.3
	50	6.4	2.6	3.8
	75	5.8	2.3	2.7
	100	5.1	3.4	3.0

Table 1.1: *Percent of rejection under H_0 for $D1$, $D2$, $D3$ at $\alpha = 0.05$*

1.5 Simulation Studies for Two-Way ANOVA

We conduct numerical studies to examine the performance of the bootstrap test in two-way ANOVA with a large number of factor levels under heteroscedastic variances for both symmetric and skewed data. We compare the estimated type I error-rate and power for the main row effect for the test of Wang and Akritas (2006) and the bootstrap test based on T_A and M_A in subsections 1.3.2 and 1.3.3 respectively. For the interaction effect we compare the estimated type I error-rate for the test of Wang and Akritas (2006) and the bootstrap tests based on T_C and M_C.

1.5.1 Simulation Setting

In the two-way ANOVA, we generate data from heteroscedastic $N(0,1)$, χ^2_8, χ^2_3 and mixed distribution from gamma and normal distributions as follows:

- E1: $Y_{ijk} = i\tau/a + (1 + i/4 + j/4)X_{ijk}$, where X_{ijk} are i.i.d. $N(0,1)$.
- E2: $Y_{ijk} = 8(i\tau/a)^2 + (log(i+1) + 2log_2(j))(X_{ijk} - 8)$, where X_{ijk} are i.i.d. χ^2_8.
- E3: $Y_{ijk} = 3(i\tau/a)^2 + (log(i+1) + 2log_2(j))(X_{ijk} - 3)$, where X_{ijk} are i.i.d. χ^2_3.
- E4:

$$Y_{ijk} \sim \begin{cases} \text{Gamma(shape} = 0.02i, \text{rate} = 0.1i/4) & \text{if } j = 1; \\ \text{Normal}(0.08 + i\tau/a, i/a) & \text{if } j = 2. \end{cases}$$

It should be noted that the data generation setting for $E1 - E4$ are under H_0 for both tests of main row effect and interaction effect. In setting E4, the data for $j = 1$ has mean 0.8 while those for $j = 2$ has mean 0.08 when $\tau = 0$. This difference is the main column effect. However, we are only testing for the main row effect and interaction effect. The gamma distribution used in E4 has skewness parameter $2/\sqrt{0.02i}$, which ranges from 4.472136 to 1.632993 for a increases from 10 to 75. The chi-square distribution used in E3 and E2 has skewness parameter $\sqrt{8/df}$, which is equal to 1 for E2 and 1.632993 for E3. The data generation and tests were repeated 2000 times. When $\tau = 0$ the data is generated under H_0; otherwise it's generated under the alternative hypothesis. In the data generation, we consider fixed number of column factor levels $b = 2$ and small group sizes depending on the number of row factor levels used. The group sizes and their corresponding row factor levels are as follows: for $i = 1, \cdots, a$,

$$\begin{array}{ll}
n_{i1} = (4,5,4,6,5,6,4,5,4,4) & \text{if } a = 10; \\
n_{i2} = (4,4,4,4,5,4,4,5,6,5) & \text{if } a = 10; \\
n_{i1} = (5,4,4,4,4,4,6,4,4,5,5,4,4,5,4) & \text{if } a = 15; \\
n_{i2} = (4,4,4,4,4,4,7,4,4,4,4,4,4,5,4) & \text{if } a = 15; \\
n_{i1} = (4,4,4,4,4,4,4,4,6,6,4,4,5,4,4,4,4,4,4,5) & \text{if } a = 20; \\
n_{i2} = (4,4,4,5,4,6,4,5,4,5,4,4,4,4,5,4,4,4,4,4) & \text{if } a = 20; \\
n_{i1} = (6,4,5,4,5,4,4,4,5,4,6,4,4,5,4,4,4,4,4,4,4,4,4,4,5,\cdots,5) & \text{if } a = 25, 50 \text{ or } 75; \\
n_{i2} = (6,4,6,4,4,4,4,4,6,6,6,4,4,4,4,4,5,4,4,4,4,4,6,4,5,4,\cdots,4) & \text{if } a = 25, 50 \text{ or } 75,
\end{array}$$

where all omitted n_{i1}'s and n_{i2}'s are equal to 5 and 4, respectively. The proportion of rejections from the 2000 runs are reported as either type I error or power depending on whether the data is generated under the null or alternative hypothesis, respectively. For the test of no main row and interaction effects, we report the estimated type I error-rate for $E1 - E4$ with $a = 10, 15, 20, 25, 50$ and 75. We report the achieved power for only main row effect for $E3$ with $a = 25$ and 50. The different values of τ are specified in the table.

1.5.2 Simulation Results

Table 1.2 displays the estimated type I error-rate for main row and interaction effects for the test of Wang and Akritas (2006) and the bootstrap tests in subsections 1.3.2 and 1.3.3. We denote by TA, TA.boot and MA.boot for the test of Wang and Akritas (2006) and the bootstrap tests in subsections 1.3.2 and 1.3.3, respectively, for the main row effect. Their corresponding interaction effects are denoted as TC, TC.boot and MC.boot.

As shown in Table 1.2, the type I error-rate for TA and TC are liberal for $E1 - E3$. The bootstrap test based on asymptotically pivotal statistic MA.boot and MC.boot have better type I error-rate for symmetric data with heteroscedastic variances $E1$ but conservative for heteroscedastic skewed distributions $E2$ -$E4$. TA.boot and TC.boot have better type I error-rate for light skewed heteroscedastic data $E2$ with large a but liberal with smaller a or for symmetric heteroscedastic data $E1$. They have a better type I error-rate for the more skewed data $E3$ for smaller a but become conservative for $a > 50$. For the mixed

Distribution	a	TA	MA.boot	TA.boot	TC	MC.boot	TC.boot
	10	8.5	4.7	8.8	7.8	4.0	8.5
	15	7.8	4.1	8.2	8.2	4.0	8.8
	20	7.7	3.8	8.3	7.2	3.4	7.8
$E1$	25	6.6	2.6	7.6	6.9	4.0	8.0
	50	7.6	4.9	8.3	8.3	5.5	9.2
	75	7.1	4.9	8.2	6.5	4.2	7.3
	10	8.3	2.5	7.2	8.3	2.5	7.2
	15	9.7	2.0	8.2	9.7	2.0	8.2
	20	8.8	2.4	7.1	8.8	2.4	7.1
$E2$	25	8.3	2.6	6.9	8.3	2.6	6.9
	50	7.5	2.7	6.0	7.5	2.7	6.0
	75	7.5	3.3	5.9	7.5	3.3	5.9
	10	7.3	1.5	5.9	7.3	1.5	5.9
	15	8.5	1.6	6.6	8.5	1.6	6.6
$E3$	20	7.6	1.7	5.2	7.6	1.7	5.2
	25	8.0	2.0	5.9	8.0	2.0	5.9
	50	6.0	1.9	4.5	6.0	1.9	4.5
	75	6.2	1.8	3.5	6.2	1.8	3.5
	10	5.4	1.6	2.8	5.2	1.6	3.2
	15	4.6	1.4	2.3	4.5	1.3	2.3
$E4$	20	4.8	1.5	2.3	4.5	1.2	2.3
	25	4.9	2.1	2.5	4.9	1.8	2.3
	50	5.2	1.9	2.0	5.6	2.1	2.3
	75	4.5	2.3	2.2	4.9	2.2	2.1

Table 1.2: *Percent of rejection under H_0 for $E1$- $E4$ at $\alpha = 0.05$ level*

distribution setting $E4$, the TA and TC have well controlled type I error for all a values but all bootstrap tests are conservative.

Next, the percent of achieved power for more skewed data $E3$ with $a = 25$ and 50 is displayed in Table 1.3. We observed that the achieved power based on TA is inflated while that of MA.boot is conservative. TA.boot has a reasonable achieved power.

a	tau	TA	TA.boot	MA.boot
25	0.0	8.0	5.9	2.0
25	1.0	34.1	25.2	12.0
25	1.2	67.5	55.3	31.4
25	1.4	96.4	90.8	72.0
25	1.6	100.0	99.0	94.2
25	1.8	100.0	100.0	99.0
50	0.0	6.0	4.5	1.9
50	1.0	23.4	16.5	8.2
50	1.2	55.8	43.2	27.0
50	1.4	92.7	84.1	67.7
50	1.6	99.8	99.0	96.0
50	1.8	100.0	100.0	99.8

Table 1.3: *Percent of achieved power for E3 at* $\alpha = 0.05$

1.6 Technical Proofs for One-Way Layout

1.6.1 Regularity conditions

Let $u_i = \left(1 - \frac{n_i}{N}\right) \frac{1}{n_i - 1} \sum_{j_1, j_2 = 1, j_2 \neq j_1}^{n_i} \epsilon_{ij_1} \epsilon_{ij_2}$, $i = 1, 2, \cdots, a$, where $\epsilon_{ij} = X_{ij} - \mu_i$ are the error terms of independent random variables X_{ij}. For asymptotic expansion of T_a, we assume the following regularity conditions (see Corollary 19.4 of Bhattacharya and Rao (2010)):

- A1: For some $\delta > 0$, $E(\epsilon_{ij})^{4+2\delta} < \infty$.

- A2: Let $\rho_s = a^{-1} \sum_{i=1}^{a} E|u_i|^s$, then $\sup_a \rho_s < \infty$

- A3: Define $g_m (m \geq 0)$ such that $g_m(t) = \prod_{j=m+1}^{m+p} |E(exp\{itu_j\})|$ for some integer p, satisfy $\sup_{m \geq 0} \int g_m(t) dt < \infty$ and $\sup\{g_m(t) : |t| > z, m \geq 0\} < 1$ $(z > 0)$.

Even though three conditions are given, only conditions $A1$ and $A3$ are sufficient. I first explain below that under condition $A1$ the random variables u_i's have finite absolute s

moments. I then further explain that knowing $A1$ ensures that $A2$ hold.

$$
\begin{aligned}
E\left|\left(1-\frac{n_i}{N}\right)\frac{1}{n_i-1}\sum_{j_1,j_2=1,j_2\neq j_1}^{n_i}\epsilon_{ij_1}\epsilon_{ij_2}\right|^s &= \left(1-\frac{n_i}{N}\right)^s\frac{1}{(n_i-1)^s}E\left|\sum_{j_1,j_2=1,j_2\neq j_1}^{n_i}\epsilon_{ij_1}\epsilon_{ij_2}\right|^s \\
&\leq \left(1-\frac{n_i}{N}\right)^s\frac{n_i^{s-1}(n_i-1)^{s-1}}{(n_i-1)^s}\sum_{j_1,j_2=1,j_2\neq j_1}^{n_i}E\left|\epsilon_{ij_1}\epsilon_{ij_2}\right|^s \\
&\leq \left(1-\frac{n_i}{N}\right)^s\frac{n_i^{s-1}}{(n_i-1)}\sum_{j_1,j_2=1,j_2\neq j_1}^{n_i}E(|\epsilon_{ij_1}|^s)E(|\epsilon_{ij_2}|^s) \\
&= \left(1-\frac{n_i}{N}\right)^s n_i^s\left[E(\epsilon_{ij_1}^s)\right]^2<\infty,
\end{aligned}
$$

where the last inequality is due to $A1$. Therefore, u_i has finite absolute moments, i.e., $E|u_i|^s<\infty$ for some $s\geq 3$. Next

$$
\begin{aligned}
\sup_a\frac{1}{a}\sum_{i=1}^{a}E\left|\left(1-\frac{n_i}{N}\right)\frac{1}{n_i-1}\sum_{j_1,j_2=1,j_2\neq j_1}^{n_i}\epsilon_{ij_1}\epsilon_{ij_2}\right|^s &\leq \sup_a\frac{1}{a}\sum_{i=1}^{a}\left(1-\frac{n_i}{N}\right)^s n_i^s\left[E(\epsilon_{ij_1}^s)\right]^2 \\
&< \infty.
\end{aligned}
$$

where the last inequality is due to $A1$. Thus, $A2$ hold.

1.6.2 Proof of (1.2.11).

To prove (1.2.5), we first show that under condition $A1$ in section 1.6.1

$$
\sqrt{a}(\hat{\nu}^2-\nu^2)\to N\left(0,\lim_{a\to\infty}\frac{4}{a}\sum_{i=1}^{a}\left(\frac{n_i}{n_i-1}\right)^2\mathrm{Var}(\widehat{\sigma_i^4})\right), \tag{1.6.1}
$$

where $\nu^2=\frac{2}{a}\sum_{i=1}^a\left(\frac{n_i}{n_i-1}\right)\sigma_i^4$ and $\hat{\nu}^2=\frac{2}{a}\sum_{i=1}^a\left(\frac{n_i}{n_i-1}\right)\widehat{\sigma_i^4}$. $\widehat{\sigma_i^4}$ is an unbiased estimate of σ_i^4 given by U-statistic

$$
\widehat{\sigma_i^4}=\frac{1}{n_i(n_i-1)(n_i-2)(n_i-3)}\sum_{j_1\neq j_2\neq j_3\neq j_4}^{n_i}\frac{(\epsilon_{ij_1}-\epsilon_{ij_2})^2(\epsilon_{ij_3}-\epsilon_{ij_4})^2}{4} \tag{1.6.2}
$$

and $\mathrm{Var}(\widehat{\sigma_i^4})$ can be computed using the variance of u-statistics. We let

$$
\hat{\nu}^2=\sum_{i=1}^{a}d_i, \tag{1.6.3}
$$

where $d_i = \frac{2}{a}\left(\frac{n_i}{n_i-1}\right)\widehat{\sigma_i^4}$, $i = 1, 2, \cdots, a$ and $d_1, d_2, \cdots, d_a$ are independent but not identically distributed with $E(d_i) = \frac{2}{a}\left(\frac{n_i}{n_i-1}\right)\sigma_i^4$. To show (1.6.1), it remains to verify Lyapounov's condition for (1.6.3) is satisfied, i.e.

$$\lim_{a\to\infty} L(a) = 0, \tag{1.6.4}$$

where $L(a) = \sum_{i=1}^a E[|d_i - E(d_i)|^{2+\delta}] = \left(\frac{2}{a}\right)^{2+\delta}\sum_{i=1}^a\left(\frac{n_i}{n_i-1}\right)^{2+\delta} E\left[|\widehat{\sigma_i^4} - \sigma_i^4|^{2+\delta}\right]$ for some $\delta > 0$. It follows that

$$E\left[|\widehat{\sigma_i^4} - \sigma_i^4|^{2+\delta}\right] = E\left|\frac{1}{n_i(n_i-1)(n_i-2)(n_i-3)}\sum_{j_1\neq j_2\neq j_3\neq j_4}^{n_i}\frac{(\epsilon_{ij_1}-\epsilon_{ij_2})^2(\epsilon_{ij_3}-\epsilon_{ij_4})^2}{4} - \sigma_i^4\right|^{2+\delta}.$$

Using the inequality

$$|\sum_{i=1}^m z_i|^p \le m^{p-1}\sum_{i=1}^m |z_i|^p, m \ge 1, p \ge 1, \tag{1.6.5}$$

$E\left[|\widehat{\sigma_i^4} - \sigma_i^4|^{2+\delta}\right]$ can be written as

$$\begin{aligned}
E\left[|\widehat{\sigma_i^4} - \sigma_i^4|^{2+\delta}\right] &\le 2^{1+\delta}\left[\frac{E|\sum_{j_1\neq j_2\neq j_3\neq j_4}^{n_i}(\epsilon_{ij_1}-\epsilon_{ij_2})^2(\epsilon_{ij_3}-\epsilon_{ij_4})^2|^{2+\delta}}{(4n_i(n_i-1)(n_i-2)(n_i-3))^{2+\delta}} + |\sigma_i^4|^{2+\delta}\right]\\
&\le 2^{1+\delta}\left[\frac{\sum_{j_1\neq j_2\neq j_3\neq j_4}^{n_i} E|(\epsilon_{ij_1}-\epsilon_{ij_2})^2(\epsilon_{ij_3}-\epsilon_{ij_4})^2|^{2+\delta}}{4^{2+\delta}(n_i(n_i-1)(n_i-2)(n_i-3))} + |\sigma_i^4|^{2+\delta}\right]\\
&= 2^{1+\delta}\left[\frac{[E(\epsilon_{ij_1}-\epsilon_{ij_2})^{4+2\delta}]^2}{4^{2+\delta}} + |\sigma_i^4|^{2+\delta}\right].
\end{aligned}$$

Next, we apply the moments assumptions; $E(\epsilon_{ij_1}-\epsilon_{ij_2})^{4+2\delta} \le M_1 < \infty$ and $\sigma_i^4 \le M_2 < \infty$, we have that

$$E\left[|\widehat{\sigma_i^4} - \sigma_i^4|^{2+\delta}\right] \le 2^{1+\delta}\left[\frac{M_1^2}{4^{2+\delta}} + M_2^{2+\delta}\right] < \infty.$$

Substituting the above expression for $E\left[|\widehat{\sigma_i^4} - \sigma_i^4|^{2+\delta}\right]$ in $L(a)$ in (1.6.4), we obtain $L(a) \le \frac{1}{a^{1+\delta}}M_3 \to 0$ as $a \to \infty$. Hence, (1.6.1) holds. Next, using (1.6.1) and applying delta method, the desired result (1.2.5) is achieved by noting that

$$\sqrt{a}(\hat{\nu} - \nu) \to N\left(0, \frac{\frac{1}{a}\sum_{i=1}^a\left(\frac{n_i}{n_i-1}\right)^2 \mathrm{Var}(\widehat{\sigma_i^4})}{\nu^2}\right). \tag{1.6.6}$$

1.6.3 Proof of $E(\widetilde{T}_a \Delta_a) = 0$.

We let $\delta_i = \left(1 - \frac{n_i}{N}\right)\left(\frac{1}{n_i - 1}\right)$.

$$
\begin{aligned}
E(\widetilde{T}_a \Delta_a) &= a^{\frac{1}{2}} E[\widetilde{T}_a(\widetilde{T}_a - T_a)] \\
&= a^{\frac{1}{2}} E\left[\left(\frac{1}{\sqrt{a}} \sum_{i=1}^{a} \delta_i \sum_{j_1 \neq j_2}^{n_i} \epsilon_{ij_1}\epsilon_{ij_2}\right)\left(\frac{1}{\sqrt{a}N} \sum_{i_1 \neq i_2}^{a} \sum_{j_3=1}^{n_{i_1}} \sum_{j_4=1}^{n_{i_2}} \epsilon_{i_1 j_3}\epsilon_{i_2 j_4}\right)\right] \\
&= \frac{1}{\sqrt{a}N} \sum_{i=1}^{a} \sum_{i_1 \neq i_2}^{a} \delta_i \sum_{j_1 \neq j_2}^{n_i} \sum_{j_3=1}^{n_{i_1}} \sum_{j_4=1}^{n_{i_2}} E(\epsilon_{ij_1}\epsilon_{ij_2}\epsilon_{i_1 j_3}\epsilon_{i_2 j_4}) \\
&= 0,
\end{aligned}
$$

where the last equality is due to the fact that when $i_1 \neq i_2$, $\epsilon_{i_1 j}$ and $\epsilon_{i_2 j'}$ are independent for any j, j'.

1.6.4 Proof of $E(\widetilde{T}_a^2 \Delta_a) = 0$.

$$
\begin{aligned}
E(\widetilde{T}_a^2 \Delta_a) &= a^{\frac{1}{2}} E[\widetilde{T}_a^2(\widetilde{T}_a - T_a)] \\
&= a^{\frac{1}{2}} E\left[\left(\frac{1}{\sqrt{a}} \sum_{i=1}^{a} \delta_i \sum_{j_1 \neq j_2}^{n_i} \epsilon_{ij_1}\epsilon_{ij_2}\right)^2 \left(\frac{1}{\sqrt{a}N} \sum_{i_1 \neq i_2}^{a} \sum_{j_3=1}^{n_{i_1}} \sum_{j_4=1}^{n_{i_2}} \epsilon_{i_1 j_3}\epsilon_{i_2 j_4}\right)\right] \\
&= \frac{1}{aN} E\left[\left(\sum_{i=1}^{a} \delta_i \sum_{j_1 \neq j_2}^{n_i} \epsilon_{ij_1}\epsilon_{ij_2}\right)\left(\sum_{i_1=1}^{a} \delta_{i_1} \sum_{j_3 \neq j_4}^{n_{i_1}} \epsilon_{i_1 j_3}\epsilon_{i_1 j_4}\right)\left(\sum_{i_2 \neq i_3}^{a} \sum_{j_5=1}^{n_{i_2}} \sum_{j_6=1}^{n_{i_3}} \epsilon_{i_2 j_5}\epsilon_{i_3 j_6}\right)\right] \\
&= \frac{1}{aN} \sum_{i=1}^{a} \sum_{i_1=1}^{a} \sum_{i_2 \neq i_3}^{a} \delta_i \delta_{i_1} \sum_{j_1 \neq j_2}^{n_i} \sum_{j_3 \neq j_4}^{n_{i_1}} \sum_{j_5=1}^{n_{i_2}} \sum_{j_6=1}^{n_{i_3}} E(\epsilon_{ij_1}\epsilon_{ij_2}\epsilon_{i_1 j_3}\epsilon_{i_1 j_4}\epsilon_{i_2 j_5}\epsilon_{i_3 j_6}) \\
&= \frac{2}{aN} \sum_{i_1 \neq i_2}^{a} \delta_{i_1}\delta_{i_2} \sum_{j_1 \neq j_2}^{n_{i_1}} \sum_{j_3 \neq j_4}^{n_{i_2}} \sum_{j_5=1}^{n_{i_1}} \sum_{j_6=1}^{n_{i_2}} E(\epsilon_{i_1 j_1}\epsilon_{i_1 j_2}\epsilon_{i_2 j_3}\epsilon_{i_2 j_4}\epsilon_{i_1 j_5}\epsilon_{i_2 j_6}) \\
&= 0.
\end{aligned}
$$

1.6.5 Proof of $E(\widetilde{T}_a^3\Delta_a) = 0$.

$$
\begin{aligned}
E(\widetilde{T}_a^3\Delta_a) &= a^{\frac{1}{2}}E[\widetilde{T}_a^3(\widetilde{T}_a - T_a)] \\
&= a^{\frac{1}{2}}E\left[\left(\frac{1}{\sqrt{a}}\sum_{i=1}^{a}\delta_i\sum_{j_1\neq j_2}^{n_i}\epsilon_{ij_1}\epsilon_{ij_2}\right)^3\left(\frac{1}{\sqrt{a}N}\sum_{i_1\neq i_2}^{a}\sum_{j_3=1}^{n_{i_1}}\sum_{j_4=1}^{n_{i_2}}\epsilon_{i_1j_3}\epsilon_{i_2j_4}\right)\right] \\
&= \frac{1}{a^{\frac{3}{2}}N}E\left[\left(\sum_{i=1}^{a}\delta_i\sum_{j_1\neq j_2}^{n_i}\epsilon_{ij_1}\epsilon_{ij_2}\right)\left(\sum_{i_1=1}^{a}\delta_{i_1}\sum_{j_3\neq j_4}^{n_{i_1}}\epsilon_{i_1j_3}\epsilon_{i_1j_4}\right)\left(\sum_{i_2=1}^{a}\delta_{i_2}\sum_{j_5\neq j_6}^{n_{i_2}}\epsilon_{i_2j_5}\epsilon_{i_2j_6}\right)\right. \\
&\qquad \left.\left(\sum_{i_3\neq i_4}^{a}\sum_{j_7=1}^{n_{i_3}}\sum_{j_8=1}^{n_{i_4}}\epsilon_{i_3j_7}\epsilon_{i_4j_8}\right)\right] \\
&= \frac{1}{a^{\frac{3}{2}}N}\sum_{i=1}^{a}\sum_{i_1=1}^{a}\sum_{i_2=1}^{a}\sum_{i_3\neq i_4}^{a}\delta_i\delta_{i_1}\delta_{i_2}\sum_{j_1\neq j_2}^{n_i}\sum_{j_3\neq j_4}^{n_{i_1}}\sum_{j_5\neq j_6}^{n_{i_2}}\sum_{j_7=1}^{n_{i_3}}\sum_{j_8=1}^{n_{i_4}}E(\epsilon_{ij_1}\epsilon_{ij_2}\epsilon_{i_1j_3}\epsilon_{i_1j_4}\epsilon_{i_2j_5}\epsilon_{i_2j_6}\epsilon_{i_3j_7}\epsilon_{i_4j_8}) \\
&= \frac{2}{a^{\frac{3}{2}}N}\sum_{i_1\neq i_2}^{a}\delta_{i_1}\delta_{i_2}^2\sum_{j_1\neq j_2}E(\epsilon_{i_1j_1}^2\epsilon_{i_1j_2}\epsilon_{i_2j_1}^2\epsilon_{i_2j_2}^2) \\
&= 0.
\end{aligned}
$$

1.6.6 Proof of $E(\widetilde{T}_aD_a) = 4z_{1-\alpha}g'(\nu^2)a^{-1}\sum_{i=1}^a\delta_i\left(\frac{n_i}{n_i-1}\right)\sigma_i^6(\gamma_i^2-2) + O(a^{-1/2})$.

$$
\begin{aligned}
& E(\widetilde{T}_aD_a) \\
={}& E\left[\left(\frac{1}{\sqrt{a}}\sum_{i=1}^{a}\delta_i\sum_{j_1\neq j_2}^{n_i}\epsilon_{ij_1}\epsilon_{ij_2}\right)\left(a^{\frac{1}{2}}(\hat{\nu}-\nu)z_{1-\alpha}\right)\right] \\
={}& z_{1-\alpha}E\left[\sum_{i=1}^{a}\delta_i\sum_{j_1\neq j_2}^{n_i}\epsilon_{ij_1}\epsilon_{ij_2}(\hat{\nu}-\nu)\right] \\
={}& z_{1-\alpha}E\left[\sum_{i=1}^{a}\delta_i\sum_{j_1\neq j_2}^{n_i}\epsilon_{ij_1}\epsilon_{ij_2}\left\{g'(\nu^2)(\hat{\nu}^2-\nu^2)+\frac{g''(\nu^2)}{2}(\hat{\nu}^2-\nu^2)^2+O_p(a^{-\frac{3}{2}})\right\}\right] \\
={}& z_{1-\alpha}g'(\nu^2)E\left[\sum_{i=1}^{a}\delta_i\sum_{j_1\neq j_2}^{n_i}\epsilon_{ij_1}\epsilon_{ij_2}(\hat{\nu}^2-\nu^2)\right]+z_{1-\alpha}\frac{g''(\nu^2)}{2}E\left[\sum_{i=1}^{a}\delta_i\sum_{j_1\neq j_2}^{n_i}\epsilon_{ij_1}\epsilon_{ij_2}(\hat{\nu}^2-\nu^2)^2\right]+O(a^{-\frac{1}{2}}).
\end{aligned}
$$

Now considering the first term in (1.6.7) we have

$$
\begin{aligned}
E\left[\sum_{i=1}^{a}\delta_i\sum_{j_1\neq j_2}^{n_i}\epsilon_{ij_1}\epsilon_{ij_2}(\hat{\nu}^2-\nu^2)\right] &= E\left[\left(\sum_{i=1}^{a}\delta_i\sum_{j_1\neq j_2}^{n_i}\epsilon_{ij_1}\epsilon_{ij_2}\right)\left(\frac{2}{a}\sum_{i_1=1}^{a}\left(\frac{n_{i_1}}{n_{i_1}-1}\right)(\widehat{\sigma^4_{i_1}}-\sigma^4_{i_1})\right)\right] \\
&= \frac{2}{a}\sum_{i=1}^{a}\sum_{i_1=1}^{a}\delta_i\left(\frac{n_{i_1}}{n_{i_1}-1}\right)\sum_{j_1\neq j_2}^{n_i}E[\epsilon_{ij_1}\epsilon_{ij_2}(\widehat{\sigma^4_{i_1}}-\sigma^4_{i_1})] \\
&= \frac{2}{a}\sum_{i=1}^{a}\sum_{i_1=1}^{a}\delta_i\left(\frac{n_{i_1}}{n_{i_1}-1}\right)\sum_{j_1\neq j_2}^{n_i}E[\epsilon_{ij_1}\epsilon_{ij_2}\widehat{\sigma^4_{i_1}}] \\
&= \frac{1}{2a}\sum_{i=1}^{a}\delta_i\left(\frac{n_i}{n_i-1}\right)\left(\frac{1}{{}^{n_i}P_4}\right)\sum_{j_1\neq j_2}^{n_i}\sum_{j_3\neq j_4\neq j_5\neq j_6}^{n_i}E[\epsilon_{ij_1}\epsilon_{ij_2}(\epsilon_{ij_3}-\epsilon_{ij_4})^2 \\
&\quad (\epsilon_{ij_5}-\epsilon_{ij_6})^2] \\
&= \frac{1}{2a}\sum_{i=1}^{a}\delta_i\left(\frac{n_i}{n_i-1}\right)\left(\frac{1}{{}^{n_i}P_4}\right)\sum_{j_1\neq j_2\neq j_3\neq j_4}^{n_i}\{4E[\epsilon_{ij_1}\epsilon_{ij_2}(\epsilon_{ij_1}-\epsilon_{ij_2})^2 \\
&\quad (\epsilon_{ij_3}-\epsilon_{ij_4})^2]+8E[\epsilon_{ij_1}\epsilon_{ij_3}(\epsilon_{ij_1}-\epsilon_{ij_2})^2(\epsilon_{ij_3}-\epsilon_{ij_4})^2]\} \\
&= \frac{1}{2a}\sum_{i=1}^{a}\delta_i\left(\frac{n_i}{n_i-1}\right)\{4E[\epsilon_{ij_1}\epsilon_{ij_2}(\epsilon_{ij_1}-\epsilon_{ij_2})^2]E[(\epsilon_{ij_3}-\epsilon_{ij_4})^2] \\
&\quad +8E[\epsilon_{ij_1}(\epsilon_{ij_1}-\epsilon_{ij_2})^2]E[\epsilon_{ij_3}(\epsilon_{ij_3}-\epsilon_{ij_4})^2]\} \\
&= \frac{1}{2a}\sum_{i=1}^{a}\delta_i\left(\frac{n_i}{n_i-1}\right)\{4(-2\sigma_i^4)(2\sigma_i^2)+8(\sigma_i^3\gamma_i)(\sigma_i^3\gamma_i)\} \\
&= \frac{4}{a}\sum_{i=1}^{a}\delta_i\left(\frac{n_i}{n_i-1}\right)\{\sigma_i^6(\gamma_i^2-2)\}. \qquad (1.6.7)
\end{aligned}
$$

Next we consider the second term in (1.6.7),

$$
\begin{aligned}
E\left[\sum_{i=1}^{a}\delta_i\sum_{j_1\neq j_2}^{n_i}\epsilon_{ij_1}\epsilon_{ij_2}(\hat{\nu}^2-\nu^2)^2\right] &= E\left[\left(\sum_{i=1}^{a}\delta_i\sum_{j_1\neq j_2}^{n_i}\epsilon_{ij_1}\epsilon_{ij_2}\right)\left(\frac{2}{a}\sum_{i_1=1}^{a}\left(\frac{n_{i_1}}{n_{i_1}-1}\right)(\widehat{\sigma^4_{i_1}}-\sigma^4_{i_1})\right)\right. \\
&\quad \left.\left(\frac{2}{a}\sum_{i_2=1}^{a}\left(\frac{n_{i_2}}{n_{i_2}-1}\right)(\widehat{\sigma^4_{i_2}}-\sigma^4_{i_2})\right)\right] \\
&= \frac{4}{a^2}\sum_{i=1}^{a}\sum_{i_1=1}^{a}\sum_{i_2=1}^{a}\delta_i\left(\frac{n_{i_1}}{n_{i_1}-1}\right)\left(\frac{n_{i_2}}{n_{i_2}-1}\right)\sum_{j_1\neq j_2}^{n_i}E[\epsilon_{ij_1}\epsilon_{ij_2}(\widehat{\sigma^4_{i_1}}-\sigma^4_{i_1}) \\
&\quad (\widehat{\sigma^4_{i_2}}-\sigma^4_{i_2})] \\
&= \frac{4}{a^2}\sum_{i=1}^{a}\delta_i\left(\frac{n_i}{n_i-1}\right)^2\sum_{j_1\neq j_2}^{n_i}E[\epsilon_{ij_1}\epsilon_{ij_2}(\widehat{\sigma^4_{i_1}}-\sigma^4_{i_1})^2] \\
&= O(a^{-1}). \qquad (1.6.8)
\end{aligned}
$$

Substituting (1.6.7) and (1.6.8) into (1.6.7), we have

$$E(\widehat{T}_a D_a) = 4\frac{z_{1-\alpha}g'(\nu^2)}{a}\sum_{i=1}^{a}\delta_i\left(\frac{n_i}{n_i-1}\right)\{\sigma_i^6(\gamma_i^2-2)\}+O(a^{-1/2}).$$

1.6.7 Proof of $E(\widetilde{T}_a^2 D_a) = O(a^{-1/2})$.

$$\begin{aligned}
E(\widetilde{T}_a^2 D_a) &= E\left[\left(\frac{1}{\sqrt{a}}\sum_{i=1}^{a}\delta_i\sum_{j_1\neq j_2}^{n_i}\epsilon_{ij_1}\epsilon_{ij_2}\right)^2\left(a^{\frac{1}{2}}(\hat{\nu}-\nu)z_{1-\alpha}\right)\right]\\
&= \frac{z_{1-\alpha}}{a^{\frac{1}{2}}}E\left[\left(\sum_{i=1}^{a}\delta_i\sum_{j_1\neq j_2}^{n_i}\epsilon_{ij_1}\epsilon_{ij_2}\right)\left(\sum_{i_1=1}^{a}\delta_{i_1}\sum_{j_3\neq j_4}^{n_{i_1}}\epsilon_{i_1j_3}\epsilon_{i_1j_4}\right)(\hat{\nu}-\nu)\right]\\
&= \frac{z_{1-\alpha}}{a^{\frac{1}{2}}}E\left[\sum_{i=1}^{a}\sum_{i_1=1}^{a}\delta_i\delta_{i_1}\sum_{j_1\neq j_2}^{n_i}\sum_{j_3\neq j_4}^{n_{i_1}}\epsilon_{ij_1}\epsilon_{ij_2}\epsilon_{i_1j_3}\epsilon_{i_1j_4}\right.\\
&\qquad\left.\left(g'(\nu^2)(\hat{\nu}^2-\nu^2)+\frac{g''(\nu^2)}{2}(\hat{\nu}^2-\nu^2)^2+\frac{g'''(\nu^2)}{3!}(\hat{\nu}^2-\nu^2)^3\right)+O_p(a^{-\frac{1}{2}})\right]\\
&= \frac{z_{1-\alpha}g'(\nu^2)}{a^{\frac{1}{2}}}E\left[\sum_{i=1}^{a}\sum_{i_1=1}^{a}\delta_i\delta_{i_1}\sum_{j_1\neq j_2}^{n_i}\sum_{j_3\neq j_4}^{n_{i_1}}\epsilon_{ij_1}\epsilon_{ij_2}\epsilon_{i_1j_3}\epsilon_{i_1j_4}(\hat{\nu}^2-\nu^2)\right]\\
&+ \frac{z_{1-\alpha}g''(\nu^2)}{2a^{\frac{1}{2}}}E\left[\sum_{i=1}^{a}\sum_{i_1=1}^{a}\delta_i\delta_{i_1}\sum_{j_1\neq j_2}^{n_i}\sum_{j_3\neq j_4}^{n_{i_1}}\epsilon_{ij_1}\epsilon_{ij_2}\epsilon_{i_1j_3}\epsilon_{i_1j_4}(\hat{\nu}^2-\nu^2)^2\right]\\
&+ \frac{z_{1-\alpha}g'''(\nu^2)}{6a^{\frac{1}{2}}}E\left[\sum_{i=1}^{a}\sum_{i_1=1}^{a}\delta_i\delta_{i_1}\sum_{j_1\neq j_2}^{n_i}\sum_{j_3\neq j_4}^{n_{i_1}}\epsilon_{ij_1}\epsilon_{ij_2}\epsilon_{i_1j_3}\epsilon_{i_1j_4}(\hat{\nu}^2-\nu^2)^3\right]+O(a^{-\frac{1}{2}})
\end{aligned} \tag{1.6.9}$$

We compute the first term in (1.6.9) as

$$\begin{aligned}
&= \frac{z_{1-\alpha}g'(\nu^2)}{a^{\frac{1}{2}}}E\left[\sum_{i=1}^{a}\sum_{i_1=1}^{a}\delta_i\delta_{i_1}\sum_{j_1\neq j_2}^{n_i}\sum_{j_3\neq j_4}^{n_{i_1}}\epsilon_{ij_1}\epsilon_{ij_2}\epsilon_{i_1j_3}\epsilon_{i_1j_4}\left(\frac{2}{a}\sum_{i_2=1}^{a}\left(\frac{n_{i_2}}{n_{i_2}-1}\right)(\widehat{\sigma_{i_2}^4}-\sigma_{i_2}^4)\right)\right]\\
&= 2\frac{z_{1-\alpha}g'(\nu^2)}{a^{\frac{3}{2}}}\sum_{i=1}^{a}\sum_{i_1=1}^{a}\sum_{i_2=1}\delta_i\delta_{i_1}\left(\frac{n_{i_2}}{n_{i_2}-1}\right)\sum_{j_1\neq j_2}^{n_i}\sum_{j_3\neq j_4}^{n_{i_1}}E[\epsilon_{ij_1}\epsilon_{ij_2}\epsilon_{i_1j_3}\epsilon_{i_1j_4}(\widehat{\sigma_{i_2}^4}-\sigma_{i_2}^4)]\\
&= 2\frac{z_{1-\alpha}g'(\nu^2)}{a^{\frac{3}{2}}}\sum_{i=1}^{a}\delta_i^2\left(\frac{n_i}{n_i-1}\right)\sum_{j_1\neq j_2}^{n_i}\sum_{j_3\neq j_4}^{n_i}E[\epsilon_{ij_1}\epsilon_{ij_2}\epsilon_{ij_3}\epsilon_{ij_4}(\widehat{\sigma_i^4}-\sigma_i^4)]\\
&= O(a^{-\frac{1}{2}}).
\end{aligned} \tag{1.6.10}$$

Next the second term in (1.6.9) is obtained as follows

$$
\begin{aligned}
&= \frac{z_{1-\alpha}g''(\nu^2)}{2a^{\frac{1}{2}}}E\left[\sum_{i=1}^{a}\sum_{i_1=1}^{a}\delta_i\delta_{i_1}\sum_{j_1\neq j_2}^{n_i}\sum_{j_3\neq j_4}^{n_{i_1}}\epsilon_{ij_1}\epsilon_{ij_2}\epsilon_{i_1j_3}\epsilon_{i_1j_4}\left(\frac{2}{a}\sum_{i_2=1}^{a}\left(\frac{n_{i_2}}{n_{i_2}-1}\right)(\widehat{\sigma^4_{i_2}}-\sigma^4_{i_2})\right)\right.\\
&\quad\left.\left(\frac{2}{a}\sum_{i_3=1}^{a}\left(\frac{n_{i_3}}{n_{i_3}-1}\right)(\widehat{\sigma^4_{i_3}}-\sigma^4_{i_3})\right)\right]\\
&= 2\frac{z_{1-\alpha}g''(\nu^2)}{a^{\frac{5}{2}}}\sum_{i=1}^{a}\sum_{i_1=1}^{a}\sum_{i_2=1}^{a}\sum_{i_3=1}^{a}\delta_i\delta_{i_1}\left(\frac{n_{i_2}}{n_{i_2}-1}\right)\left(\frac{n_{i_3}}{n_{i_3}-1}\right)\sum_{j_1\neq j_2}^{n_i}\sum_{j_3\neq j_4}^{n_{i_1}}E[\epsilon_{ij_1}\epsilon_{ij_2}\epsilon_{i_1j_3}\epsilon_{i_1j_4}\\
&\quad(\widehat{\sigma^4_{i_2}}-\sigma^4_{i_2})(\widehat{\sigma^4_{i_3}}-\sigma^4_{i_3})]\\
&= O(a^{-\frac{1}{2}}). \qquad (1.6.11)
\end{aligned}
$$

We now compute the third term in (1.6.9) as follows

$$
\begin{aligned}
&= \frac{z_{1-\alpha}g'''(\nu^2)}{6a^{\frac{1}{2}}}E\left[\sum_{i=1}^{a}\sum_{i_1=1}^{a}\delta_i\delta_{i_1}\sum_{j_1\neq j_2}^{n_i}\sum_{j_3\neq j_4}^{n_{i_1}}\epsilon_{ij_1}\epsilon_{ij_2}\epsilon_{i_1j_3}\epsilon_{i_1j_4}\left(\frac{2}{a}\sum_{i_2=1}^{a}\left(\frac{n_{i_2}}{n_{i_2}-1}\right)(\widehat{\sigma^4_{i_2}}-\sigma^4_{i_2})\right)\right.\\
&\quad\left.\left(\frac{2}{a}\sum_{i_3=1}^{a}\left(\frac{n_{i_3}}{n_{i_3}-1}\right)(\widehat{\sigma^4_{i_3}}-\sigma^4_{i_3})\right)\left(\frac{2}{a}\sum_{i_4=1}^{a}\left(\frac{n_{i_4}}{n_{i_4}-1}\right)(\widehat{\sigma^4_{i_4}}-\sigma^4_{i_4})\right)\right]\\
&= \frac{4z_{1-\alpha}g'''(\nu^2)}{3a^{\frac{7}{2}}}\sum_{i=1}^{a}\sum_{i_1=1}^{a}\sum_{i_2=1}^{a}\sum_{i_3=1}^{a}\sum_{i_4=1}^{a}\delta_i\delta_{i_1}\left(\frac{n_{i_2}}{n_{i_2}-1}\right)\left(\frac{n_{i_3}}{n_{i_3}-1}\right)\left(\frac{n_{i_4}}{n_{i_4}-1}\right)\sum_{j_1\neq j_2}^{n_i}\sum_{j_3\neq j_4}^{n_{i_1}}\\
&\quad E[\epsilon_{ij_1}\epsilon_{ij_2}\epsilon_{i_1j_3}\epsilon_{i_1j_4}(\widehat{\sigma^4_{i_2}}-\sigma^4_{i_2})(\widehat{\sigma^4_{i_3}}-\sigma^4_{i_3})(\widehat{\sigma^4_{i_4}}-\sigma^4_{i_4})]\\
&= O(a^{-\frac{3}{2}}). \qquad (1.6.12)
\end{aligned}
$$

Therefore substituting (1.6.10), (1.6.11) and (1.6.11) into (1.6.9) the desired result is obtain. That is $E(\widetilde{T}_a^2 D_a) = O(a^{-1/2})$.

1.6.8 Proof of $E(\widetilde{T}_a^3 D_a) = 24\frac{z_{1-\alpha}g'(\nu^2)}{a^2}\sum_{i\neq i'}^a [\delta_i^2 n_i(n_i-1)\sigma_i^4]\left[\delta_{i'}\left(\frac{n_{i'}}{n_{i'}-1}\right)(\gamma_{i'}^2-2)\sigma_{i'}^6\right]$.

$$
\begin{aligned}
& E(\widetilde{T}_a^3 D_a) \\
= \;& E\left[\left(\frac{1}{\sqrt{a}}\sum_{i=1}^a \delta_i \sum_{j_1\neq j_2}^{n_i}\epsilon_{ij_1}\epsilon_{ij_2}\right)^3\left(a^{\frac{1}{2}}(\hat{\nu}-\nu)z_{1-\alpha}\right)\right] \\
= \;& \frac{z_{1-\alpha}}{a}E\left[\left(\sum_{i=1}^a \delta_i \sum_{j_1\neq j_2}^{n_i}\epsilon_{ij_1}\epsilon_{ij_2}\right)\left(\sum_{i_1=1}^a \delta_{i_1} \sum_{j_3\neq j_4}^{n_{i_1}}\epsilon_{i_1j_3}\epsilon_{i_1j_4}\right)\left(\sum_{i_2=1}^a \delta_{i_2} \sum_{j_5\neq j_6}^{n_{i_2}}\epsilon_{i_2j_5}\epsilon_{i_2j_6}\right)(\hat{\nu}-\nu)\right] \\
= \;& \frac{z_{1-\alpha}}{a}E\left[\sum_{i=1}^a\sum_{i_1=1}^a\sum_{i_2=1}^a \delta_i\delta_{i_1}\delta_{i_2}\sum_{j_1\neq j_2}^{n_i}\sum_{j_3\neq j_4}^{n_{i_1}}\sum_{j_5\neq j_6}^{n_{i_2}}\epsilon_{ij_1}\epsilon_{ij_2}\epsilon_{i_1j_3}\epsilon_{i_1j_4}\epsilon_{i_2j_5}\epsilon_{i_2j_6}\sum_{k=1}^4\frac{g^{(k)}(\nu^2)}{k!}(\hat{\nu}^2-\nu^2)^k\right] \\
+ \;& O_p(a^{-\frac{1}{2}}). \qquad (1.6.13)
\end{aligned}
$$

We compute the first term in (1.6.13) with $k=1$ as follows:

$$
\begin{aligned}
= \;& \frac{z_{1-\alpha}g'(\nu^2)}{a}E\left[\sum_{i=1}^a\sum_{i_1=1}^a\sum_{i_2=1}^a \delta_i\delta_{i_1}\delta_{i_2}\sum_{j_1\neq j_2}^{n_i}\sum_{j_3\neq j_4}^{n_{i_1}}\sum_{j_5\neq j_6}^{n_{i_2}}\epsilon_{ij_1}\epsilon_{ij_2}\epsilon_{i_1j_3}\epsilon_{i_1j_4}\epsilon_{i_2j_5}\epsilon_{i_2j_6}\frac{2}{a}\sum_{i_3=1}^a\left(\frac{n_{i_3}}{n_{i_3}-1}\right)(\widehat{\sigma_{i_3}^4}-\sigma_{i_3}^4)\right] \\
= \;& \frac{2z_{1-\alpha}g'(\nu^2)}{a^2}\sum_{i=1}^a\sum_{i_1=1}^a\sum_{i_2=1}^a\sum_{i_3=1}^a \delta_i\delta_{i_1}\delta_{i_2}\left(\frac{n_{i_3}}{n_{i_3}-1}\right)\sum_{j_1\neq j_2}^{n_i}\sum_{j_3\neq j_4}^{n_{i_1}}\sum_{j_5\neq j_6}^{n_{i_2}}E[\epsilon_{ij_1}\epsilon_{ij_2}\epsilon_{i_1j_3}\epsilon_{i_1j_4}\epsilon_{i_2j_5}\epsilon_{i_2j_6}(\widehat{\sigma_{i_3}^4}-\sigma_{i_3}^4)] \\
= \;& \frac{6z_{1-\alpha}g'(\nu^2)}{a^2}\sum_{i\neq i_1}^a \delta_i^2\delta_{i_1}\left(\frac{n_{i_1}}{n_{i_1}-1}\right)\sum_{j_1\neq j_2}^{n_i}\sum_{j_3\neq j_4}^{n_i}\sum_{j_5\neq j_6}^{n_{i_1}}E[\epsilon_{ij_1}\epsilon_{ij_2}\epsilon_{ij_3}\epsilon_{ij_4}\epsilon_{i_1j_5}\epsilon_{i_1j_6}(\widehat{\sigma_{i_1}^4}-\sigma_{i_1}^4)] \\
= \;& \frac{6z_{1-\alpha}g'(\nu^2)}{a^2}\sum_{i\neq i_1}^a \delta_i^2\delta_{i_1}\left(\frac{n_{i_1}}{n_{i_1}-1}\right)\sum_{j_1\neq j_2}^{n_i}\sum_{j_3\neq j_4}^{n_i}\sum_{j_5\neq j_6}^{n_{i_1}}E[\epsilon_{ij_1}\epsilon_{ij_2}\epsilon_{ij_3}\epsilon_{ij_4}]E[\epsilon_{i_1j_5}\epsilon_{i_1j_6}(\widehat{\sigma_{i_1}^4}-\sigma_{i_1}^4)] \\
= \;& \frac{6z_{1-\alpha}g'(\nu^2)}{a^2}\sum_{i\neq i_1}^a \delta_i^2\delta_{i_1}\left(\frac{n_{i_1}}{n_{i_1}-1}\right)\sum_{j_1\neq j_2}^{n_i}\sum_{j_3\neq j_4}^{n_i}E[\epsilon_{ij_1}\epsilon_{ij_2}\epsilon_{ij_3}\epsilon_{ij_4}]\sum_{j_5\neq j_6}^{n_{i_1}}E[\epsilon_{i_1j_5}\epsilon_{i_1j_6}(\widehat{\sigma_{i_1}^4}-\sigma_{i_1}^4)] \\
= \;& \frac{12z_{1-\alpha}g'(\nu^2)}{a^2}\sum_{i\neq i_1}^a \delta_i^2\delta_{i_1}\left(\frac{n_{i_1}}{n_{i_1}-1}\right)\sum_{j_1\neq j_2}^{n_i}E[\epsilon_{ij_1}^2\epsilon_{ij_2}^2]\sum_{j_5\neq j_6}^{n_{i_1}}E[\epsilon_{i_1j_5}\epsilon_{i_1j_6}(\widehat{\sigma_{i_1}^4}-\sigma_{i_1}^4)] \\
= \;& \frac{12z_{1-\alpha}g'(\nu^2)}{a^2}\sum_{i\neq i_1}^a \delta_i^2\delta_{i_1}\left(\frac{n_{i_1}}{n_{i_1}-1}\right)[n_i(n_i-1)\sigma_i^4][2\sigma_{i_1}^6(\gamma_{i_1}^2-2)] \\
= \;& \frac{24z_{1-\alpha}g'(\nu^2)}{a^2}\sum_{i\neq i_1}^a \delta_i^2[n_i(n_i-1)\sigma_i^4]\delta_{i_1}\left(\frac{n_{i_1}}{n_{i_1}-1}\right)[\sigma_{i_1}^6(\gamma_{i_1}^2-2)]. \qquad (1.6.14)
\end{aligned}
$$

The second term in (1.6.13) with $k=2$ is obtained as

$$
\begin{aligned}
&= \frac{z_{1-\alpha} g''(\nu^2)}{2a} E\left[\sum_{i=1}^{a}\sum_{i_1=1}^{a}\sum_{i_2=1}^{a} \delta_i \delta_{i_1}\delta_{i_2} \sum_{j_1\neq j_2}^{n_i}\sum_{j_3\neq j_4}^{n_{i_1}}\sum_{j_5\neq j_6}^{n_{i_2}} \epsilon_{ij_1}\epsilon_{ij_2}\epsilon_{i_1j_3}\epsilon_{i_1j_4}\epsilon_{i_2j_5}\epsilon_{i_2j_6}\right. \\
&\quad \left.\left(\frac{2}{a}\sum_{i_3=1}^{a}\left(\frac{n_{i_3}}{n_{i_3}-1}\right)(\widehat{\sigma^4_{i_3}}-\sigma^4_{i_3})\right)\left(\frac{2}{a}\sum_{i_3=1}^{a}\left(\frac{n_{i_3}}{n_{i_3}-1}\right)(\widehat{\sigma^4_{i_3}}-\sigma^4_{i_3})\right)\right] \\
&= \frac{2z_{1-\alpha} g''(\nu^2)}{a^3}\sum_{i=1}^{a}\sum_{i_1=1}^{a}\sum_{i_2=1}^{a}\sum_{i_3=1}^{a}\sum_{i_4=1}^{a} \delta_i\delta_{i_1}\delta_{i_2}\left(\frac{n_{i_3}}{n_{i_3}-1}\right)\left(\frac{n_{i_4}}{n_{i_4}-1}\right)\sum_{j_1\neq j_2}^{n_i}\sum_{j_3\neq j_4}^{n_{i_1}}\sum_{j_5\neq j_6}^{n_{i_2}} E[\epsilon_{ij_1}\epsilon_{ij_2} \\
&\quad \epsilon_{i_1j_3}\epsilon_{i_1j_4}\epsilon_{i_2j_5}\epsilon_{i_2j_6}(\widehat{\sigma^4_{i_3}}-\sigma^4_{i_3})(\widehat{\sigma^4_{i_4}}-\sigma^4_{i_4})] \\
&= O(a^{-1}). \qquad (1.6.15)
\end{aligned}
$$

The terms with $k=3$ and 4 are of smaller order than $O(a^{-1})$. Putting these into (1.6.13), the proof is complete.

1.6.9 Proof of (1.2.19).

We let $u_a = -8\frac{z_{1-\alpha}g'(\nu^2)}{a^{\frac{3}{2}}}\sum_{i=1}^{a}\delta_i\left(\frac{n_i}{n_i-1}\right)\sigma_i^6(\gamma_i^2-2)$. Using $K_1(T_a^T), K_2(T_a^T), K_3(T_a^T)$ and $K_4(T_a^T)$ in (1.2.14), (1.2.16), (1.2.17) and (1.2.18), respectively, we can derive $\chi_{T_a^T/\nu}$ the characteristic function of T_a^T in (1.2.13). Under condition A1 in Section 1.6.1, the characteristic function $\chi_{T_a^T/\nu}$ can be written as:

$$
\begin{aligned}
\chi_{T_a^T/\nu} &= \exp\left\{K_1(T_a^T)\frac{(it)}{\nu} + K_2(T_a^T)\frac{(it)^2}{2\nu^2} + K_3(T_a^T)\frac{(it)^3}{6\nu^3} + K_4(T_a^T)\frac{(it)^4}{24\nu^4}\right\} + O(a^{-1}) \\
&= \exp\left\{K_1(T_a)\frac{(it)}{\nu} + [K_2(T_a)+u_a]\frac{(it)^2}{2\nu^2} + K_3(T_a)\frac{(it)^3}{6\nu^3} + K_4(T_a)\frac{(it)^4}{24\nu^4} + O(a^{-1})\right\} \\
&= \exp(-\frac{t^2}{2})\exp\left\{K_1(T_a)\frac{(it)}{\nu} + [(K_2(T_a)-1)+u_a]\frac{(it)^2}{2\nu^2} + K_3(T_a)\frac{(it)^3}{6\nu^3} + K_4(T_a)\frac{(it)^4}{24\nu^4}\right. \\
&\quad \left. + O(a^{-1})\right\} \\
&= \exp(-\frac{t^2}{2})\exp\left\{K_1(T_a)\frac{(it)}{\nu} + (K_2(T_a)-1)\frac{(it)^2}{2\nu^2} + \frac{u_a}{\nu^2}\frac{(it)^2}{2} + K_3(T_a)\frac{(it)^3}{6\nu^3}\right. \\
&\quad \left. + K_4(T_a)\frac{(it)^4}{24\nu^4} + O(a^{-1})\right\}. \qquad (1.6.16)
\end{aligned}
$$

Note that $K_2(T_a) = \nu^2 + O(a^{-1})$, $K_1(T_a) = 0$ under H_0 but will not varnish under H_a. So we keep this symbolic notation in it. Further, we know $u_a = O(a^{-1/2})$. Applying Taylor series expansion to (1.6.16), we get

$$\begin{aligned}\chi_{T_a^T/\nu} &= \exp(-\frac{t^2}{2})\Big[1 + K_1(T_a)\frac{(it)}{\nu} + (K_2(T_a)-1)\frac{(it)^2}{2\nu^2} + \frac{u_a}{\nu^2}\frac{(it)^2}{2} + K_3(T_a)\frac{(it)^3}{6\nu^3} \\ &\quad + K_4(T_a)\frac{(it)^4}{24\nu^4} + \frac{u_a K_3(T_a)}{6\nu^5}(it)^5 + \frac{u_a K_4(T_a)}{\nu^6}\frac{(it)^6}{24} + \frac{K_3(T_a)K_4(T_a)}{72\nu^7}(it)^7\Big] \\ &\quad + O(a^{-1}). \end{aligned} \tag{1.6.17}$$

By Applying the inverse Fourier transform, we obtain the pdf of T_a^T under condition A3 in Section 1.6.1 as follows,

$$\begin{aligned} f_{T_a^T/\nu}(x) &= \int_{-\infty}^{\infty} e^{-itx}\chi_{T_a^T/\nu}(t)dt \\ &= \int_{-\infty}^{\infty} e^{-itx}\exp(-\frac{t^2}{2})\Big[1 + K_1(T_a)\frac{(it)}{\nu} + (K_2(T_a)-1)\frac{(it)^2}{2\nu^2} + \frac{u_a}{\nu^2}\frac{(it)^2}{2} + K_3(T_a)\frac{(it)^3}{6\nu^3} \\ &\quad + K_4(T_a)\frac{(it)^4}{24\nu^4} + \frac{u_a K_3(T_a)}{6\nu^5}(it)^5 + \frac{u_a K_4(T_a)}{\nu^6}\frac{(it)^6}{24} + \frac{K_3(T_a)K_4(T_a)}{72\nu^7}(it)^7\Big]\, dt + O(a^{-1}) \\ &= \phi(x) + \Big[K_1(T_a)\frac{H_1(x)}{\nu} + (K_2(T_a)-1)\frac{H_2(x)}{2\nu^2} + \frac{u_a}{\nu^2}\frac{H_2(x)}{2} + K_3(T_a)\frac{H_3(x)}{6\nu^3} \\ &\quad + K_4(T_a)\frac{H_4(x)}{24\nu^4} + \frac{u_a K_3(T_a)}{6\nu^5}H_5(x) + \frac{u_a K_4(T_a)}{\nu^6}\frac{H_6(x)}{24} + \frac{K_3(T_a)K_4(T_a)}{72\nu^7}H_7(x)\Big]\phi(x) \\ &\quad + O(a^{-1}), \end{aligned}$$

where $H_0(x) = 1$, $H_1(x) = x$, $H_2(x) = x^2 - 1$, $H_3(x) = x^3 - 3x$, $H_4(x) = x^4 - 6x^2 + 3$, $H_5(x) = x^5 - 10x^3 + 15x$, $H_6(x) = x^6 - 15x^4 + 45x^2 - 15$, and $H_7(x) = x^7 - 20x^5 + 105x^3 - 105x$

are Hermite polynomials. We now obtain the cdf of T_a^T as;

$$
\begin{aligned}
F_{T_a^T/\nu}(x) &= \int_{-\infty}^{x} f_g(u)du \\
&= \Phi(x) - \left[K_1(T_a)\frac{H_0(x)}{\nu} + (K_2(T_a)-1)\frac{H_1(x)}{2\nu^2} + \frac{u_a}{\nu^2}\frac{H_1(x)}{2} + K_3(T_a)\frac{H_2(x)}{6\nu^3}\right. \\
&\quad \left. + K_4(T_a^T)\frac{H_3(x)}{24\nu^4} + \frac{u_a K_3(T_a)}{6\nu^5}H_4(x) + \frac{u_a K_4(T_a)}{\nu^6}\frac{H_5(x)}{24} + \frac{K_3(T_a)K_4(T_a)}{72\nu^7}H_6(x)\right]\phi(x) \\
&\quad + O(a^{-1}) \\
&= \Phi(x) - \left[K_1(T_a)\frac{H_0(x)}{\nu} + (K_2(T_a)-1)\frac{H_1(x)}{2\nu^2} + K_3(T_a)\frac{H_2(x)}{6\nu^3} + K_4(T_a)\frac{H_3(x)}{24\nu^4}\right. \\
&\quad \left. + \frac{K_3(T_a)K_4(T_a)}{72\nu^7}H_6(x)\right]\phi(x) - \left[\frac{u_a}{\nu^2}\frac{H_1(x)}{2} + \frac{u_a K_3(T_a)}{3\nu^5}H_4(x) + \frac{u_a K_4(T_a)}{\nu^6}\frac{H_5(x)}{24}\right]\phi(x) \\
&\quad + O(a^{-1}) \\
&= P(T_a \le x) - \left[\frac{u_a}{\nu^2}\frac{H_1(x)}{2} + \frac{u_a K_3(T_a)}{3\nu^5}H_4(x) + \frac{u_a K_4(T_a)}{\nu^6}\frac{H_5(x)}{24}\right]\phi(x) + O(a^{-1}),
\end{aligned}
$$

where

$$
\begin{aligned}
P(T_a \le x) &= \Phi(x) - \left[K_1(T_a)\frac{H_0(x)}{\nu} + (K_2(T_a)-1)\frac{H_1(x)}{2\nu^2} + K_3(T_a)\frac{H_2(x)}{6\nu^3}\right. \\
&\quad \left. + K_4(T_a)\frac{H_3(x)}{24\nu^4} + \frac{K_3(T_a)K_4(T_a)}{72\nu^7}H_6(x)\right]\phi(x).
\end{aligned}
$$

Thus $P(T_a^T > x) = P(T_a > x) - \left[\frac{u_a}{2\nu^2}(x) + \frac{u_a K_3(T_a)}{3\nu^5}(x^3-3x) + \frac{u_a K_4(T_a)}{24\nu^6}(x^5 - 8x^3 + 11x)\right]\phi(x) + O(a^{-1})$.

1.6.10 Proof of (1.2.30).

To compute $P\left(M_a - \frac{1}{a^{1/2}}\left[\hat{q}_1^{cf}(z_{1-\alpha}) - q_1^{cf}(z_{1-\alpha}))\right] > \omega_{1-\alpha}^M\right)$ in (1.2.30), we need to know the Edgeworth expansion of the distribution of

$$M_a^M = M_a - a^{-1}\Delta_a^M, \tag{1.6.18}$$

where $\Delta_a^M = a^{1/2}\left[\hat{q}_1^{cf}(z_{1-\alpha}) - q_1^{cf}(z_{1-\alpha}))\right]$. We note that $\Delta_a^M = O_p(1)$. To obtain the Edgeworth expansion of M_a^M, we compute the first four cumulants of M_a^M. We compute the first cumulant of M_a^M as

$$K_1(M_a^M) = E(M_a^M) = E(M_a) - a^{-1}E(\Delta_a^M) = K_1(M_a) + O(a^{-1}), \tag{1.6.19}$$

where $K_1(M_a)$ is the first cumulant of M_a in (1.2.24). To get the second cumulant $K_2(M_a^M)$, we need to obtain the second moment of M_a^M:

$$\begin{aligned}
E(M_a^{M^2}) &= E(M_a - a^{-1}\Delta_a^M)^2 \\
&= E(M_a^2) - 2a^{-1}E(M_a\Delta_a^M) + a^{-2}E(\Delta_a^{M^2}) \\
&= E(M_a^2) + O(a^{-1}).
\end{aligned}$$

Thus the second cumulant is equal to

$$\begin{aligned}
K_2(M_a^M) &= E(M_a^{M^2}) - (E(M_a^M))^2 \\
&= E(M_a^2) + O(a^{-1}) - (E(M_a) + O(a^{-1}))^2 \\
&= E(M_a^2) - (E(M_a))^2 + O(a^{-1}) \\
&= K_2(M_a) + O(a^{-1}), \qquad (1.6.20)
\end{aligned}$$

where $K_2(M_a) = E(M_a^2) - (E(M_a))^2$ is the second cumulant of M_a. Next, we compute the third moment as follows:

$$\begin{aligned}
E(M_a^{M^3}) &= E(M_a - a^{-1}\Delta_a^M)^3 \\
&= E(M_a^3) - 3a^{-1}E(M_a^2\Delta_a^M) + 3a^{-2}E(M_a\Delta_a^{M^2}) - a^{-3}E(\Delta_a^{M^3}) \\
&= E(M_a^3) + O(a^{-1})
\end{aligned}$$

The third cumulant $K_3(M_a^M)$ is obtained as

$$\begin{aligned}
K_3(M_a^M) &= E(M_a^{M^3}) - 3E(M_a^{M^2})E(M_a^M) + 2(E(M_a^M))^3 \\
&= E(M_a^3) + O(a^{-1}) - 3(E(M_a) + O(a^{-1}))(E(M_a) + O(a^{-1})) + 2(E(M_a) + O(a^{-1}))^3 \\
&= E(M_a^3) - 3E(M_a^2)E(M_a) + 2(E(M_a))^3 + O(a^{-1}) \\
&= K_3(M_a) + O(a^{-1}), \qquad (1.6.21)
\end{aligned}$$

where $K_3(M_a)$ is the third cumulant of M_a. Lastly, to obtain the fourth cumulant $K_4(M_a^M)$ we compute the fourth moment as follows

$$\begin{aligned}
E(M_a^{M^4}) &= E(M_a - a^{-1}\Delta_a^M)^4 \\
&= E(M_a^4) - 4a^{-1}E(M_a^3\Delta_a^M) + 6a^{-2}E(M_a^2\Delta_a^{M^2}) - 4a^{-3}E(M_a\Delta_a^{M^3}) + a^{-4}E(\Delta_a^{M^4}) \\
&= E(M_a^4) + O(a^{-1}).
\end{aligned}$$

Thus the fourth cumulant is given by

$$\begin{aligned}
K_4(M_a^M) &= E(M_a^{M^4}) - 4E(M_a^M)E(M_a^{M^3}) - 3(E(M_a^{M^2}))^2 + 12E(M_a^{M^2})(E(M_a^M))^2 - 6(E(M_a^M))^4 \\
&= E(M_a^4) + O(a^{-1}) - 4(E(M_a) + O(a^{-1}))(E(M_a^3) + O(a^{-1})) - 3(E(M_a^2) + O(a^{-1}))^2 \\
&+ 12(E(M_a^2) + O(a^{-1}))^2(E(M_a) + O(a^{-1}))^2 - 6(E(M_a) + O(a^{-1}))^4 \\
&= E(M_a^4) - 4E(M_a)E(M_a^3) - 3(E(M_a^2))^2 + 12E({M_a}^2)(E(M_a))^2 - 6(E(M_a))^4 + O(a^{-1}) \\
&= K_4(M_a) + O(a^{-1}), \qquad (1.6.22)
\end{aligned}$$

where $K_4(M_a) = E(M_a^4) - 4E(M_a)E(M_a^3) - 3(E(M_a^2))^2 + 12E(M_a^2)(E(M_a))^2 - 6(E(M_a))^4$ is the fourth cumulant of M_a. Then as discussed in Hall (1992b) based on equations (3.30)-(3.32) to deduce equation (3.36), using $K_1(M_a^M), K_2(M_a^M), K_3(M_a^M)$ and $K_4(M_a^M)$ in (1.6.19), (1.6.20), (1.6.21) and (1.6.22), respectively, we obtain

$$P(M_a^M > \omega_{1-\alpha}^M) = P(M_a > \omega_{1-\alpha}^M) + O(a^{-1}). \qquad (1.6.23)$$

1.7 Technical Proofs for Two-Way Layout

1.7.1 Regularity conditions for testing of no main row effect

Let $u_{Ai} = \frac{a-1}{ab^{\frac{3}{2}}}\left[\left(\sum_{j=1}^b \bar{\epsilon}_{ij.}\right)^2 - \sum_{j=1}^b \frac{S_{ij}^2}{n_{ij}}\right]$, $i = 1, 2, \cdots, a$. Then u_{Ai}, $i = 1, 2, \cdots, a$ are independent random variables with 0 mean. We assume the following regularity conditions hold (see Bhattacharya and Rao (2010)):

B1: For some $\delta \geq 1$, $E(\epsilon_{ijk})^{4+2\delta} < \infty$.

B2: Define $g_{mA}(m \geq 0)$ such that $g_{mA}(t) = \prod_{j=m+1}^{m+p} |E(exp\{itu_{Aj}\})|$ for some integer p, satisfies $\sup_{m\geq 0} \int g_{mA}(t)dt < \infty$ and $\sup\{g_{mA}(t) : |t| > z, m \geq 0\} < 1$ $(z > 0)$.

1.7.2 Regularity conditions for testing of no interaction effect

Let $u_{Ci} = \frac{(a-1)(b-1)}{ab^{\frac{3}{2}}} \sum_{j=1}^{b} \left(\bar{\epsilon}_{ij.}^2 - \frac{S_{ij}^2}{n_{ij}}\right) - \frac{a-1}{(ab)^{\frac{3}{2}}} \sum_{j_1 \neq j_2}^{b} \bar{\epsilon}_{ij_1.} \bar{\epsilon}_{ij_2.}$, $i = 1, 2, \cdots, a$. Then u_{Ci}, $i = 1, 2, \cdots, a$ are independent random variables with 0 mean. We assume that the following regularity conditions hold (see Bhattacharya and Rao (2010)):

C1: For some $\delta \geq 1$, $E(\epsilon_{ijk})^{4+2\delta} < \infty$.

C2: Define $g_{mC}(m \geq 0)$ such that $g_{mC}(t) = \prod_{j=m+1}^{m+p} |E(exp\{itu_{Cj}\})|$ for some integer p, satisfies $\sup_{m\geq 0} \int g_{mC}(t)dt < \infty$ and $\sup\{g_{mC}(t) : |t| > z, m \geq 0\} < 1$ $(z > 0)$.

1.7.3 Proof of (1.3.12).

To prove (1.3.12) we note that under the null, the test statistic can be written as

$$T_A^{(0)} = \widetilde{T_A} - \frac{b^{1/2}}{a^{3/2}} \sum_{i_1 \neq i_2}^{a} \widetilde{\epsilon}_{i_1..} \widetilde{\epsilon}_{i_2..}, \tag{1.7.1}$$

where

$$\widetilde{T_A} = \sum_{i=1}^{a} \frac{a-1}{(ab)^{3/2}} \left\{ \sum_{j \neq j_1}^{b} \bar{\epsilon}_{ij.} \bar{\epsilon}_{ij_1.} + \sum_{j=1}^{b} \sum_{k \neq k_1}^{n_{ij}} \frac{\epsilon_{ijk}\epsilon_{ijk_1}}{n_{ij}(n_{ij}-1)} \right\}. \tag{1.7.2}$$

It can be seen that $T_A^{(0)} - \widetilde{T_A} = O_p(1/\sqrt{a})$ because $E(T_A^{(0)} - \widetilde{T_A}) = 0$ and $E[b(T_A^{(0)} - \widetilde{T_A})^2] = O(1/a)$. We write $P(T_A^{(0)} \leq x)$ as

$$P(\widetilde{T_A} + (T_A^{(0)} - \widetilde{T_A}) \leq x) = P(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a \leq x) \tag{1.7.3}$$

where $\Pi_a = a^{1/2}(T_A^{(0)} - \widetilde{T_A})$ satisfies $\Pi_a = O_p(1)$. In order to determine $P(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a \leq x)$, we compute the first four cumulants of

$$T_A^{(0)} = \widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a. \tag{1.7.4}$$

The first cumulant of $T_A^{(0)}$ is computed as

$$K_1(T_A^{(0)}) = E(T_A^{(0)}) = E(\widetilde{T_A}) + a^{-\frac{1}{2}}E(\Pi_a) = K_1(\widetilde{T_A}) \tag{1.7.5}$$

because $E(\Pi_a) = 0$ and $K_1(\widetilde{T_A}) = E(\widetilde{T_A})$ is the first cumulant of $\widetilde{T_A}$. Next, to compute the second cumulant $K_2(T_A^{(0)})$ we note that

$$\begin{aligned} E(T_A^{(0)})^2 &= E(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a)^2 \\ &= E(\widetilde{T_A}^2) + 2a^{-\frac{1}{2}}E(\widetilde{T_A}\Pi_a) + a^{-1}E(\Pi_a{}^2) \\ &= E(\widetilde{T_A}^2) + 2a^{-\frac{1}{2}}E(\widetilde{T_A}\Pi_a) + O(a^{-1}) \\ &= E(\widetilde{T_A}^2) + O(a^{-1}), \end{aligned} \tag{1.7.6}$$

since $E(\widetilde{T_A}\Pi_a) = 0$ (see section 1.7.4 for its proof). Therefore the second cumulant of $T_A^{(0)}$ is obtained as

$$K_2(T_A^{(0)}) = E(T_A^{(0)})^2 - (E(T_A^{(0)}))^2 = K_2(\widetilde{T_A}) + O(a^{-1}), \tag{1.7.7}$$

where $K_2(\widetilde{T_A}) = E(\widetilde{T_A}^2) - (E(\widetilde{T_A}))^2$ is the second cumulant of $\widetilde{T}_A$. Now, in order to obtain the third cumulant $K_3(T_A^{(0)})$ we obtain the third moment of $T_A^{(0)}$ as

$$\begin{aligned} E(T_A^{(0)})^3 &= E(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a)^3 \\ &= E(\widetilde{T_A}^3) + 3a^{-\frac{1}{2}}E(\widetilde{T_A}^2\Pi_a) + 3a^{-1}E(\widetilde{T_A}\Pi_a^2) + a^{-\frac{3}{2}}E(\Pi_a{}^3) \\ &= E(\widetilde{T_A}^3) + O(a^{-1}), \end{aligned} \tag{1.7.8}$$

where the last equality is due to the fact that $E(\widetilde{T_A}^2\Pi_a) = 0$ (the proof is given in Section 1.7.5), and $E(\widetilde{T_A}\Pi_a^2)$, $E(\Pi_a^3)$ are at most $O(1)$ with the Cramer's condition since $\widetilde{T_A}$ and Π_a are $O_p(1)$.

Therefore the third cumulant $K_3(T_A^{(0)})$ is equal to

$$\begin{aligned} K_3(T_A^{(0)}) &= E(T_A^{(0)})^3 - 3E(T_A^{(0)})^2E(T_A^{(0)}) + 2(E(T_A^{(0)}))^3 \\ &= E(\widetilde{T_A}^3) + O(a^{-1}) - 3(E(\widetilde{T_A}^2) + O(a^{-1}))E(\widetilde{T_A}) + 2(E(\widetilde{T_A}))^3 \\ &= E(\widetilde{T_A}^3) - 3E(\widetilde{T_A}^2)E(\widetilde{T}_A) + 2(E(\widetilde{T_A}))^3 + O(a^{-1}) \\ &= K_3(\widetilde{T_A}) + O(a^{-1}), \end{aligned} \tag{1.7.9}$$

where $K_3(\widetilde{T_A})$ is the third cumulant of $\widetilde{T_A}$. Next, we compute the fourth moment of $T_A^{(0)}$ as

$$\begin{aligned}
E(T_A^{(0)})^4 &= E(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a)^4 \\
&= E(\widetilde{T_A}^4) + 4a^{-\frac{1}{2}}E(\widetilde{T_A}^3\Pi_a) + 6a^{-1}E(\widetilde{T_A}^2\Pi_a^2) + 4a^{-\frac{3}{2}}E(\widetilde{T_A}\Pi_a^3) \\
&\quad + a^{-2}E(\Pi_a{}^4) \\
&= E(\widetilde{T_A}^4) + O(a^{-1}), \qquad (1.7.10)
\end{aligned}$$

where the last equality is due to the fact that $E(\widetilde{T_A}^3\Pi_a) = 0$ (the proof is given in Section 1.7.6) and the rest of the terms are $O(1)$ for similar reason as explained for $E(T_A^{(0)})^3$. Lastly, the fourth cumulant $K_4(T_A^{(0)})$ is given by

$$\begin{aligned}
K_4(T_A^{(0)}) &= E(T_A^{(0)})^4 - 4E(T_A^{(0)})E(T_A^{(0)})^3 - 3(E(T_A^{(0)})^2)^2 + 12E(T_A^{(0)})^2(E(T_A^{(0)}))^2 - 6(E(T_A^{(0)}))^4 \\
&= E(\widetilde{T_A}^4) + O(a^{-1}) - 4(E(\widetilde{T_A}))(E(\widetilde{T_A}^3) + O(a^{-1})) - 3(E(\widetilde{T_A}^2) + O(a^{-1}))^2 \\
&\quad + 12(E(\widetilde{T_A}^2) + O(a^{-1}))(E(\widetilde{T_A}))^2 - 6(E(\widetilde{T_A}))^4 \\
&= E(\widetilde{T_A}^4) - 4E(\widetilde{T_A})E(\widetilde{T_A}^3) - 3(E(\widetilde{T_A}^2))^2 + 12E(\widetilde{T_A}^2)(E(\widetilde{T_A}))^2 - 6(E(\widetilde{T_A}))^4 + O(a^{-1}) \\
&= K_4(\widetilde{T_A}) + O(a^{-1}), \qquad (1.7.11)
\end{aligned}$$

where

$$\begin{aligned}
K_4(\widetilde{T_A}) &= E(\widetilde{T_A}^4) - 4E(\widetilde{T_A})E(\widetilde{T_A}^3) - 3(E(\widetilde{T_A}^2))^2 + 12E(\widetilde{T_A}^2)(E(\widetilde{T}_a))^2 - 6(E(\widetilde{T_A}))^4 \\
&= E(\widetilde{T_A}^4) - 3(E(\widetilde{T_A}^2))^2
\end{aligned}$$

(since $\mathrm{E}(\widetilde{T_A}) = 0$) is the fourth cumulant of $\widetilde{T_A}$. As discussed in section 3.5 of Hall (1992b), then using $K_1(T_A^{(0)}), K_2(T_A^{(0)}), K_3(T_A^{(0)})$ and $K_4(T_A^{(0)})$ in (1.7.5), (1.7.7), (1.7.9) and (1.7.11), respectively, the proof of (1.3.12) can be completed.

1.7.4 Proof of $E(\widetilde{T_A}\Pi_a) = 0$.

Using (1.7.2) and (1.7.1), we have

$$
\begin{aligned}
E(\widetilde{T_A}\Pi_a) &= a^{1/2}E[\widetilde{T_A}(T_A^{(0)} - \widetilde{T_A})] \\
&= a^{1/2}E\left[\left(\sum_{i=1}^{a}\frac{a-1}{(ab)^{3/2}}\left\{\sum_{j\neq j_1}^{b}\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.} + \sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\frac{\epsilon_{ijk}\epsilon_{ijk_1}}{n_{ij}(n_{ij}-1)}\right\}\right)\left(-\frac{b^{1/2}}{a^{3/2}}\sum_{i_1\neq i_2}^{a}\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\right)\right] \\
&= -\frac{a-1}{a^{5/2}b}\sum_{i=1}^{a}\sum_{i_1\neq i_2}^{a}\left[\sum_{j\neq j_1}^{b}E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}] + \sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\frac{E[\epsilon_{ijk}\epsilon_{ijk_1}\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}]}{n_{ij}(n_{ij}-1)}\right] \\
&= -\frac{2(a-1)}{a^{5/2}b}\sum_{i\neq i'}^{a}\left[\sum_{j\neq j_1}^{b}E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\widetilde{\epsilon}_{i..}]E[\widetilde{\epsilon}_{i'..}] + \sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\frac{E[\epsilon_{ijk}\epsilon_{ijk_1}\widetilde{\epsilon}_{i..}]E[\widetilde{\epsilon}_{i_2..}]}{n_{ij}(n_{ij}-1)}\right] \\
&= 0
\end{aligned}
$$

1.7.5 Proof of $E(\widetilde{T}_A^2\Pi_a) = 0$.

To show that $E(\widetilde{T}_A^2\Pi_a) = 0$, we first compute $\widetilde{T}_A^2$. Using (1.7.2) we have that

$$
\begin{aligned}
\widetilde{T}_A^2 &= \left(\sum_{i=1}^{a}\frac{a-1}{(ab)^{3/2}}\left\{\sum_{j\neq j_1}^{b}\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.} + \sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\frac{\epsilon_{ijk}\epsilon_{ijk_1}}{n_{ij}(n_{ij}-1)}\right\}\right)^2 \\
&= \frac{(a-1)^2}{(ab)^3}(\Lambda_1 + \Lambda_2 + 2\Lambda_3), \qquad (1.7.12)
\end{aligned}
$$

where

$$
\Lambda_1 = \left(\sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\right)^2, \qquad (1.7.13)
$$

$$
\Lambda_2 = \left(\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\frac{\epsilon_{ijk}\epsilon_{ijk_1}}{n_{ij}(n_{ij}-1)}\right)^2, \qquad (1.7.14)
$$

and

$$
\Lambda_3 = \sum_{i=1}^{a}\sum_{i_1=1}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2=1}^{b}\sum_{k\neq k_1}^{n_{i_1j_2}}\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\epsilon_{i_1j_2k}\epsilon_{i_1j_2k_1}. \qquad (1.7.15)
$$

We now express $E(\widetilde{T}_A^2 \Pi_a)$ in terms of Λ_1, Λ_2 and Λ_3 in (1.7.13), (1.7.14) and (1.7.15), respectively.

$$\begin{aligned} E(\widetilde{T}_A^2 \Pi_a) &= a^{1/2} E[\widetilde{T}_A^2 (T_A^{(0)} - \widetilde{T}_A)] \\ &= a^{1/2} E\left[\frac{(a-1)^2}{(ab)^3}(\Lambda_1 + \Lambda_2 + 2\Lambda_3)(T_A^{(0)} - \widetilde{T}_A)\right] \\ &= \frac{(a-1)^2}{a^{5/2} b^3}[E(\Lambda_1(T_A^{(0)} - \widetilde{T}_A)) + E(\Lambda_2(T_A^{(0)} - \widetilde{T}_A)) + 2E(\Lambda_3(T_A^{(0)} - \widetilde{T}_A))]. \end{aligned} \tag{1.7.16}$$

Using (1.7.13) and (1.7.1) we compute the first term in (1.7.16) as.

$$\begin{aligned} E(\Lambda_1(T_A^{(0)} - \widetilde{T}_A)) &= E\left[\left(\sum_{i=1}^{a}\sum_{j\neq j_1}^{b} \bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\right)^2 \left(-\frac{b^{1/2}}{a^{3/2}}\sum_{i_1\neq i_2}^{a} \widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\right)\right] \\ &= -\frac{b^{1/2}}{a^{3/2}} \sum_{i_1\neq i_2}^{a}\sum_{i_3,i_4}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2\neq j_3}^{b} E[\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\bar{\epsilon}_{i_3 j.}\bar{\epsilon}_{i_3 j_1.}\bar{\epsilon}_{i_4 j_2.}\bar{\epsilon}_{i_4 j_3.}] \\ &= -2\frac{b^{1/2}}{a^{3/2}} \sum_{i_1\neq i_2}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2\neq j_3}^{b} E[\widetilde{\epsilon}_{i_1..}\bar{\epsilon}_{i_1 j.}\bar{\epsilon}_{i_1 j_1.}]E[\widetilde{\epsilon}_{i_2..}\bar{\epsilon}_{i_2 j_2.}\bar{\epsilon}_{i_2 j_3.}] \\ &= 0, \end{aligned}$$

since $j \neq j_1$ and $j_2 \neq j_3$. Next we use (1.7.14) and (1.7.1) to compute the second term in (1.7.16).

$$\begin{aligned} E(\Lambda_2(T_A^{(0)} - \widetilde{T}_A)) &= E\left[\left(\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}} \frac{\epsilon_{ijk}\epsilon_{ijk_1}}{n_{ij}(n_{ij}-1)}\right)^2 \left(-\frac{b^{1/2}}{a^{3/2}}\sum_{i_1\neq i_2}^{a} \widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\right)\right] \\ &= -\frac{b^{1/2}}{a^{3/2}} \sum_{i_1\neq i_2}^{a}\sum_{i_3,i_4}^{a}\sum_{j_1,j_2}^{b}\sum_{k\neq k_1}^{n_{i_3j_1}}\sum_{k_2\neq k_3}^{n_{i_4j_2}} \frac{E[\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\epsilon_{i_3j_1k}\epsilon_{i_3j_2k_1}\epsilon_{i_4j_2k_2}\epsilon_{i_4j_3k_3}]}{n_{i_3j_1}(n_{i_3j_1}-1)n_{i_4j_2}(n_{i_4j_2}-1)} \\ &= -2\frac{b^{1/2}}{a^{3/2}} \sum_{i_1\neq i_2}^{a}\sum_{j_1,j_2}^{b}\sum_{k\neq k_1}^{n_{i_1j_1}}\sum_{k_2\neq k_3}^{n_{i_2j_2}} \frac{E[\widetilde{\epsilon}_{i_1..}\epsilon_{i_1j_1k}\epsilon_{i_1j_1k_1}]E[\widetilde{\epsilon}_{i_2..}\epsilon_{i_2j_2k_2}\epsilon_{i_2j_2k_3}]}{n_{i_1j_1}(n_{i_1j_1}-1)n_{i_2j_2}(n_{i_2j_2}-1)} \\ &= 0, \end{aligned}$$

where the last equality is due to the fact that $\epsilon_{i_1j_1k}$ and $\epsilon_{i_1j_1k_1}$ are independent when $k \neq k_1$. Lastly we use (1.7.15) and (1.7.1) to compute the third term in (1.7.16).

$$\begin{aligned}
E(\Lambda_3(T_A^{(0)} - \widetilde{T}_A)) &= E\left[\left(\sum_{i=1}^{a}\sum_{i_1=1}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2=1}^{b}\sum_{k\neq k_1}^{n_{i_1j_2}} \frac{\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.}\epsilon_{i_1j_2k}\epsilon_{i_1j_2k_1}}{n_{i_1j_2}(n_{i_1j_2}-1)}\right)\left(-\frac{b^{1/2}}{a^{3/2}}\sum_{i_2\neq i_3}^{a}\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{i_3..}\right)\right] \\
&= -\frac{b^{1/2}}{a^{3/2}}\sum_{i=1}^{a}\sum_{i_1=1}^{a}\sum_{i_2\neq i_3}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2=1}^{b}\sum_{k\neq k_1}^{n_{i_1j_2}} \frac{E[\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.}\epsilon_{i_1j_2k}\epsilon_{i_1j_2k_1}\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{i_3..}]}{n_{i_1j_2}(n_{i_1j_2}-1)} \\
&= -2\frac{b^{1/2}}{a^{3/2}}\sum_{i_1\neq i_2}^{a}\sum_{j_1,j_2}^{b}\sum_{k\neq k_1}^{n_{i_1j_1}}\sum_{k_2\neq k_3}^{n_{i_2j_2}} \frac{E[\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.}\widetilde{\epsilon}_{i..}]E[\epsilon_{i_1j_2k}\epsilon_{i_1j_2k_1}\widetilde{\epsilon}_{i_1..}]}{n_{i_1j_2}(n_{i_1j_2}-1)} \\
&= 0.
\end{aligned}$$

The last equality is also due to the fact that $\epsilon_{i_1j_1k}$ and $\epsilon_{i_1j_1k_1}$ are independent when $k \neq k_1$. Thus combining the three terms we have that $E(\widetilde{T}_A^2\Pi_a) = 0$.

1.7.6 Proof of $E(\widetilde{T}_A^3\Pi_a) = 0$.

In order to prove that $E(\widetilde{T}_A^3\Pi_a) = 0$, we write $\widetilde{T}_A^3$ as follows.

$$\begin{aligned}
\widetilde{T}_A^3 &= \left(\sum_{i=1}^{a}\frac{a-1}{(ab)^{3/2}}\left\{\sum_{j\neq j_1}^{b}\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.} + \sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\frac{\epsilon_{ijk}\epsilon_{ijk_1}}{n_{ij}(n_{ij}-1)}\right\}\right)^3 \\
&= \frac{(a-1)^3}{(ab)^{9/2}}(\Lambda_4 + \Lambda_5 + 3\Lambda_6 + 3\Lambda_7), \qquad (1.7.17)
\end{aligned}$$

where

$$\Lambda_4 = \left(\sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.}\right)^3, \qquad (1.7.18)$$

$$\Lambda_5 = \left(\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\frac{\epsilon_{ijk}\epsilon_{ijk_1}}{n_{ij}(n_{ij}-1)}\right)^3, \qquad (1.7.19)$$

$$\Lambda_6 = \sum_{i=1}^{a}\sum_{i_1,i_2}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2,j_3}^{b}\sum_{k\neq k_1}^{n_{i_1j_2}}\sum_{k_2\neq k_3}^{n_{i_2j_3}}\frac{\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.}\epsilon_{i_1j_2k}\epsilon_{i_1j_2k_1}\epsilon_{i_2j_3k_2}\epsilon_{i_2j_3k_3}}{n_{i_1j_2}(n_{i_1j_2}-1)n_{i_2j_3}(n_{i_2j_3}-1)}, \qquad (1.7.20)$$

and

$$\Lambda_7 = \sum_{i=1}^{a}\sum_{i_1,i_2}^{a}\sum_{j}^{b}\sum_{j_1\neq j_2}^{b}\sum_{j_3\neq j_4}^{b}\sum_{k\neq k_1}^{n_{ij}} \frac{\epsilon_{ijk}\epsilon_{ijk_1}\overline{\epsilon}_{i_1j_1.}\overline{\epsilon}_{i_1j_2.}\overline{\epsilon}_{i_2j_3.}\overline{\epsilon}_{i_2j_4.}}{n_{ij}(n_{ij}-1)}. \tag{1.7.21}$$

We now express $E(\widetilde{T}_A^3\Pi_a)$ in terms of Λ_4, Λ_5, Λ_6 and Λ_7 in (1.7.18), (1.7.19), (1.7.20) and (1.7.21), respectively.

$$\begin{aligned}
E(\widetilde{T}_A^3\Pi_a) &= a^{1/2}E[\widetilde{T}_A^3(T_A^{(0)}-\widetilde{T}_A)] \\
&= a^{1/2}E\left[\frac{(a-1)^3}{(ab)^{9/2}}(\Lambda_4+\Lambda_5+3\Lambda_6+3\Lambda_7)(T_A^{(0)}-\widetilde{T}_A)\right] \\
&= \frac{(a-1)^3}{a^4b^{9/2}}[E(\Lambda_4(T_A^{(0)}-\widetilde{T}_A))+E(\Lambda_5(T_A^{(0)}-\widetilde{T}_A))+3E(\Lambda_6(T_A^{(0)}-\widetilde{T}_A)) \\
&\quad + 3E(\Lambda_6(T_A^{(0)}-\widetilde{T}_A))].
\end{aligned} \tag{1.7.22}$$

Using (1.7.18) and (1.7.1) we compute the first term in (1.7.22) as follows.

$$\begin{aligned}
& E(\Lambda_4(T_A^{(0)}-\widetilde{T}_A)) \\
=\ & E\left[\left(\sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.}\right)^3\left(-\frac{b^{1/2}}{a^{3/2}}\sum_{i_1\neq i_2}^{a}\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\right)\right] \\
=\ & -\frac{b^{1/2}}{a^{3/2}}\sum_{i_1\neq i_2}^{a}\sum_{i_3,i_4,i_5}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2\neq j_3}^{b}\sum_{j_4\neq j_5}^{b} E[\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\overline{\epsilon}_{i_3j.}\overline{\epsilon}_{i_3j_1.}\overline{\epsilon}_{i_4j_2.}\overline{\epsilon}_{i_4j_3.}\overline{\epsilon}_{i_5j_4.}\overline{\epsilon}_{i_5j_5.}] \\
=\ & -6\frac{b^{1/2}}{a^{3/2}}\sum_{i_1\neq i_2}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2\neq j_3}^{b}\sum_{j_4\neq j_5}^{b} E[\widetilde{\epsilon}_{i_1..}\overline{\epsilon}_{i_1j.}\overline{\epsilon}_{i_1j_1.}\overline{\epsilon}_{i_1j_2.}\overline{\epsilon}_{i_1j_3.}]E[\widetilde{\epsilon}_{i_2..}\overline{\epsilon}_{i_2j_4.}\overline{\epsilon}_{i_2j_5.}] \\
=\ & 0,
\end{aligned}$$

where the last equality is due to the independence of $\bar{\epsilon}_{i_2j_4.}$ and $\bar{\epsilon}_{i_2j_5.}$ when $j_4 \neq j_5$. Next we use (1.7.19) and (1.7.1) to compute the second term in (1.7.22).

$$\begin{aligned}
&E(\Lambda_5(T_A^{(0)} - \widetilde{T}_A)) \\
= \;& E\left[\left(\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}} \frac{\epsilon_{ijk}\epsilon_{ijk_1}}{n_{ij}(n_{ij}-1)}\right)^3 \left(-\frac{b^{1/2}}{a^{3/2}}\sum_{i_1\neq i_2}^{a} \widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\right)\right] \\
= \;& -\frac{b^{1/2}}{a^{3/2}}\sum_{i_1\neq i_2}^{a}\sum_{i_3,i_4,i_5}^{a}\sum_{j_1,j_2,j_3}^{b}\sum_{k\neq k_1}^{n_{i_3j_1}}\sum_{k_2\neq k_3}^{n_{i_4j_2}}\sum_{k_4\neq k_5}^{n_{i_5j_3}} \frac{E[\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\epsilon_{i_3j_1k}\epsilon_{i_3j_2k_1}\epsilon_{i_4j_2k_2}\epsilon_{i_4j_3k_3}\epsilon_{i_5j_3k_4}\epsilon_{i_5j_3k_5}]}{n_{i_3j_1}(n_{i_3j_1}-1)n_{i_4j_2}(n_{i_4j_2}-1)n_{i_5j_3}(n_{i_5j_3}-1)} \\
= \;& -6\frac{b^{1/2}}{a^{3/2}}\sum_{i_1\neq i_2}^{a}\sum_{j_1,j_2}^{b}\sum_{k\neq k_1}^{n_{i_1j_1}}\sum_{k_2\neq k_3}^{n_{i_1j_2}}\sum_{k_4\neq k_5}^{n_{i_2j_2}} \frac{E[\widetilde{\epsilon}_{i_1..}\epsilon_{i_1j_1k}\epsilon_{i_1j_1k_1}\epsilon_{i_1j_1k_2}\epsilon_{i_1j_1k_3}]E[\widetilde{\epsilon}_{i_2..}\epsilon_{i_2j_2k_4}\epsilon_{i_2j_2k_5}]}{n_{i_1j_1}^2(n_{i_1j_1}-1)^2 n_{i_2j_2}(n_{i_2j_2}-1)} \\
= \;& 0,
\end{aligned}$$

where the last equality is due to the independence of $\epsilon_{i_2j_2k_4}$ and $\epsilon_{i_2j_2k_5}$ when $k_4 \neq k_5$. Next we use (1.7.20) and (1.7.1) to compute the third term in (1.7.22).

$$\begin{aligned}
&E(\Lambda_6(T_A^{(0)} - \widetilde{T}_A)) \\
= \;& E\left[\left(\sum_{i,i_1,i_2}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2,j_3}^{b}\sum_{k\neq k_1}^{n_{i_1j_2}}\sum_{k_2\neq k_3}^{n_{i_2j_3}} \frac{\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\epsilon_{i_1j_2k}\epsilon_{i_1j_2k_1}\epsilon_{i_2j_3k_2}\epsilon_{i_2j_3k_3}}{n_{i_1j_2}(n_{i_1j_2}-1)n_{i_2j_3}(n_{i_2j_3}-1)}\right)\left(-\frac{b^{1/2}}{a^{3/2}}\sum_{i_3\neq i_4}^{a}\widetilde{\epsilon}_{i_3..}\widetilde{\epsilon}_{i_4..}\right)\right] \\
= \;& -\frac{b^{1/2}}{a^{3/2}}\sum_{i,i_1,i_2}^{a}\sum_{i_3\neq i_4}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2,j_3}^{b}\sum_{k\neq k_1}^{n_{i_1j_2}}\sum_{k_2\neq k_3}^{n_{i_2j_3}} \frac{E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\epsilon_{i_1j_2k}\epsilon_{i_1j_2k_1}\epsilon_{i_2j_3k_2}\epsilon_{i_2j_3k_3}\widetilde{\epsilon}_{i_3..}\widetilde{\epsilon}_{i_4..}]}{n_{i_1j_2}(n_{i_1j_2}-1)n_{i_2j_3}(n_{i_2j_3}-1)} \\
= \;& 0.
\end{aligned}$$

Lastly we use (1.7.21) and (1.7.1) to compute the fourth term in (1.7.22).

$$\begin{aligned}
&E(\Lambda_7(T_A^{(0)} - \widetilde{T}_A)) \\
= \;& E\left[\left(\sum_{i,i_1,i_2}^{a}\sum_{j}^{b}\sum_{j_1\neq j_2}^{b}\sum_{j_3\neq j_4}^{b}\sum_{k\neq k_1}^{n_{ij}} \frac{\epsilon_{ijk}\epsilon_{ijk_1}\bar{\epsilon}_{i_1j_1.}\bar{\epsilon}_{i_1j_2.}\bar{\epsilon}_{i_2j_3.}\bar{\epsilon}_{i_2j_4.}}{n_{ij}(n_{ij}-1)}\right)\left(-\frac{b^{1/2}}{\sqrt{a}}\sum_{i_3\neq i_4}^{a}\widetilde{\epsilon}_{i_3..}\widetilde{\epsilon}_{i_4..}\right)\right] \\
= \;& -\frac{b^{1/2}}{a^{3/2}}\sum_{i,i_1,i_2}^{a}\sum_{i_3\neq i_4}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2,j_3}^{b}\sum_{k\neq k_1}^{n_{i_1j_2}}\sum_{k_2\neq k_3}^{n_{i_2j_3}} \frac{E[\epsilon_{ijk}\epsilon_{ijk_1}\bar{\epsilon}_{i_1j_1.}\bar{\epsilon}_{i_1j_2.}\bar{\epsilon}_{i_2j_3.}\bar{\epsilon}_{i_2j_4.}\widetilde{\epsilon}_{i_3..}\widetilde{\epsilon}_{i_4..}]}{n_{ij}(n_{ij}-1)} \\
= \;& 0.
\end{aligned}$$

Therefore, combining the four terms we end up with $E(\widetilde{T}_A^3\Pi_a) = 0$.

1.7.7 Proof of (1.3.18).

As given in (1.3.17) we assume the departure from the null hypothesis satisfies

$$\frac{\sqrt{b}}{\sqrt{a}}\sum_{i=1}^{a}\alpha_i^2 < \infty.$$

To determine the distribution of $T_A^{(1)}$ in (1.3.15) we write $P(T_A^{(1)} \leq x)$ as

$$P(T_A^{(1)} \leq x) = P(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a + \Omega_a \leq x) = P(T_A^{(0)} + \Omega_a \leq x), \tag{1.7.23}$$

where Ω_a is given in (1.3.8). To determine this probability, we compute the first four cumulants of $T_A^{(1)}$. The first cumulant $K_1(T_A^{(1)})$ is obtained as

$$K_1(T_A^{(1)}) = E(T_A^{(1)}) = K_1(T_A^{(0)}) + \kappa_1, \tag{1.7.24}$$

where $K_1(T_A^{(0)}) = E(\widetilde{T_A})$ is the first cumulant of $T_A^{(0)}$ given in (1.7.5) and

$$\kappa_1 = (b/a)^{1/2}\sum_{i=1}^{a}\alpha_i^2. \tag{1.7.25}$$

The second cumulant $K_2(T_A^{(1)})$ can be computed as

$$\begin{aligned} K_2(T_A^{(1)}) = \mathrm{Var}(T_A^{(1)}) &= E[T_A^{(0)} + (\Omega_a - E(\Omega_a))]^2 \\ &= E(T_A^{(0)})^2 + 2E[(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a)(\Omega_a - E(\Omega_a))] \\ &\quad + E(\Omega_a - E(\Omega_a))^2. \end{aligned} \tag{1.7.26}$$

The term $E(T_A^{(0)})^2$ was given in (1.7.6). In order to obtain the second term in (1.7.26), we note that

$$\Omega_a - E(\Omega_a) = 2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i=1}^{a}[(\widetilde{\epsilon}_{i..} - \widetilde{\epsilon}_{...})\alpha_i], \tag{1.7.27}$$

which gives

$$E[(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a)(\Omega_a - E(\Omega_a))] = E[\widetilde{T_A}(\Omega_a - E(\Omega_a))] + a^{-\frac{1}{2}}E[\Pi_a(\Omega_a - E(\Omega_a))].$$

Using (1.3.7) and (1.7.27) we compute $E[\widetilde{T_A}(\Omega_a - E(\Omega_a))]$ in the above equation as follows:

$$\begin{aligned}
& E[\widetilde{T_A}(\Omega_a - E(\Omega_a))] \\
= \; & E\left[\sum_{i=1}^{a} \frac{a-1}{(ab)^{3/2}}\left\{\sum_{j\neq j_1}^{b} \bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.} + \sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}} \frac{\epsilon_{ijk}\epsilon_{ijk_1}}{n_{ij}(n_{ij}-1)}\right\}\left\{2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i=1}^{a}[(\widetilde{\epsilon}_{i..} - \widetilde{\epsilon}_{...})\alpha_i]\right\}\right] \\
= \; & \frac{2(a-1)}{a^2 b}\sum_{i,i_1}^{a}\left[\sum_{j\neq j_1}^{b} \alpha_{i_1} E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}(\widetilde{\epsilon}_{i_1..} - \widetilde{\epsilon}_{...})] + \sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}} \alpha_{i_1}\frac{E[\epsilon_{ijk}\epsilon_{ijk_1}(\widetilde{\epsilon}_{i_1..} - \widetilde{\epsilon}_{...})]}{n_{ij}(n_{ij}-1)}\right] \\
= \; & \frac{2(a-1)}{a^2 b}\sum_{i}^{a}\left[\sum_{j\neq j_1}^{b} \alpha_{i} E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}(\widetilde{\epsilon}_{i..} - \widetilde{\epsilon}_{...})] + \sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}} \alpha_{i}\frac{E[\epsilon_{ijk}\epsilon_{ijk_1}(\widetilde{\epsilon}_{i..} - \widetilde{\epsilon}_{...})]}{n_{ij}(n_{ij}-1)}\right] \\
= \; & 0
\end{aligned}$$

since $j \neq j_1$ and $k \neq k_1$. Next we use (1.3.16) and (1.7.27) to compute $E[\Pi_a(\Omega_a - E(\Omega_a))]$.

$$\begin{aligned}
E[\Pi_a(\Omega_a - E(\Omega_a))] &= a^{\frac{1}{2}} E\left[\left(-\frac{b^{1/2}}{a^{3/2}}\sum_{i_1\neq i_2}^{a} \widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\right)\left(2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i=1}^{a}[(\widetilde{\epsilon}_{i..} - \widetilde{\epsilon}_{...})\alpha_i]\right)\right] \\
&= -\frac{2b}{a^{\frac{3}{2}}}\sum_{i_1\neq i_2}^{a}\sum_{i=1}^{a} \alpha_i E[\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}(\widetilde{\epsilon}_{i..} - \widetilde{\epsilon}_{...})] \\
&= -\frac{4b}{a^{\frac{3}{2}}}\sum_{i\neq i_1}^{a} \alpha_i E[\widetilde{\epsilon}_{i_1..}]E[\widetilde{\epsilon}_{i..}(\widetilde{\epsilon}_{i..} - \widetilde{\epsilon}_{...})] = 0
\end{aligned}$$

since $E[\widetilde{\epsilon}_{i_1..}] = 0$. We get the second term as $E[(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a)(\Omega_a - E(\Omega_a))] = 0$. Using (1.7.27) we proceed to compute the third term in (1.7.26).

$$\begin{aligned}
E[(\Omega_a - E(\Omega_a))^2] &= E\left[2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i=1}^{a}[(\widetilde{\epsilon}_{i..} - \widetilde{\epsilon}_{...})\alpha_i]\right]^2 \\
&= \frac{4b}{a}\sum_{i,i_1}^{a} \alpha_i\alpha_{i_1} E[(\widetilde{\epsilon}_{i..} - \widetilde{\epsilon}_{...})(\widetilde{\epsilon}_{i_1..} - \widetilde{\epsilon}_{...})] \\
&= \frac{4b}{a}\left[\sum_{i=1}^{a} \alpha_i^2 E[\widetilde{\epsilon}_{i..}^2 - 2\widetilde{\epsilon}_{i..}\widetilde{\epsilon}_{...} + \widetilde{\epsilon}_{...}^2] + \sum_{i\neq i_1}^{a} \alpha_i\alpha_{i_1} E[\widetilde{\epsilon}_{i..}\widetilde{\epsilon}_{i_1..} - \widetilde{\epsilon}_{i..}\widetilde{\epsilon}_{...} - \widetilde{\epsilon}_{...}\widetilde{\epsilon}_{i_1..} + \widetilde{\epsilon}_{...}^2]\right] \\
&= \frac{4b}{a}\left[\sum_{i=1}^{a} \alpha_i^2 E[\widetilde{\epsilon}_{i..}^2 - 2\widetilde{\epsilon}_{i..}\widetilde{\epsilon}_{...} + \widetilde{\epsilon}_{...}^2] + \sum_{i=1}^{a}(-\alpha_i^2) E[-2\widetilde{\epsilon}_{i..}\widetilde{\epsilon}_{...} + \widetilde{\epsilon}_{...}^2]\right] \\
&= \frac{4b}{a}\sum_{i=1}^{a} \alpha_i^2 E[\widetilde{\epsilon}_{i..}^2] = \frac{4}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b} \alpha_i^2 \frac{\sigma_{ij}^2}{n_{ij}}
\end{aligned}$$

since it can be shown that

$$E(\widetilde{\epsilon}_{i..}^2) = \frac{1}{b^2}\sum_{j=1}^{b}\frac{\sigma_{ij}^2}{n_{ij}}, \quad E(\widetilde{\epsilon}_{i..}\widetilde{\epsilon}_{...}) = \frac{1}{ab^2}\sum_{j=1}^{b}\frac{\sigma_{ij}^2}{n_{ij}}, \quad E(\widetilde{\epsilon}_{...}^2) = \frac{1}{(ab)^2}\sum_{i=1}^{a}\sum_{j=1}^{b}\frac{\sigma_{ij}^2}{n_{ij}}. \tag{1.7.28}$$

Therefore the second cumulant is given by

$$K_2(T_A^{(1)}) \quad = \quad K_2(T_A^{(0)}) + \kappa_2, \tag{1.7.29}$$

where $K_2(T_A^{(0)})$ is the second cumulant of $T_A^{(0)}$ under the null given in (1.7.7) and

$$\kappa_2 = \frac{4}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}\alpha_i^2\frac{\sigma_{ij}^2}{n_{ij}}. \tag{1.7.30}$$

Next we compute the third cumulant $K_3(T_A^{(1)})$ as follows:

$$\begin{aligned} K_3(T_A^{(1)}) &= E[(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a) + (\Omega_a - E(\Omega_a))]^3 \\ &= E(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a)^3 + 3E[(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a)^2(\Omega_a - E(\Omega_a))] \\ &\quad + 3E[(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a)(\Omega_a - E(\Omega_a))^2] + E(\Omega_a - E(\Omega_a))^3. \end{aligned} \tag{1.7.31}$$

To get the result for $K_3(T_A^{(1)})$ we need to compute each term in (1.7.31) separately. The first term in (1.7.31) is the third cumulant of $T_A^{(0)}$ given in (1.7.9). We compute the second term in (1.7.31) as follows:

$$\begin{aligned} E[(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a)^2(\Omega_a - E(\Omega_a))] &= E[(\widetilde{T_A}^2 + 2a^{-\frac{1}{2}}\widetilde{T_A}\Pi_a + a^{-1}\Pi_a^2)(\Omega_a - E(\Omega_a))] \\ &= E[\widetilde{T_A}^2(\Omega_a - E(\Omega_a))] + 2a^{-\frac{1}{2}}E[\widetilde{T_A}\Pi_a(\Omega_a - E(\Omega_a))] \\ &\quad + a^{-1}E[\Pi_a^2(\Omega_a - E(\Omega_a))] \\ &= E[\widetilde{T_A}^2(\Omega_a - E(\Omega_a))] + 2a^{-\frac{1}{2}}E[\widetilde{T_A}\Pi_a(\Omega_a - E(\Omega_a))] + O(a^{-1}), \end{aligned}$$

where the last equality is due to the fact that $E[\Pi_a^2(\Omega_a - E(\Omega_a))]$ is at most O(1) since $\Pi_a = O_p(1)$ and $\Omega_a - E(\Omega_a) = O_p(1)$. Using the result in (1.7.12), $E[\widetilde{T_A}^2(\Omega_a - E(\Omega_a))]$ is expressed as

$$E[\widetilde{T_A}^2(\Omega_a - E(\Omega_a))] = \frac{(a-1)^2}{(ab)^3}[E\{\Lambda_1(\Omega_a - E(\Omega_a))\} + E\{\Lambda_2(\Omega_a - E(\Omega_a))\} + 2E\{\Lambda_3(\Omega_a - E(\Omega_a))\}],$$

where Λ_1, Λ_2 and Λ_3 are given in (1.7.13), (1.7.14) and (1.7.15), respectively. We compute the three terms in the above equation as follows:

$$
\begin{aligned}
E\{\Lambda_1(\Omega_a - E(\Omega_a))\} &= E\left[\left(\sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\right)^2\left(2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i=1}^{a}[(\widetilde{\epsilon}_{i..}-\widetilde{\epsilon}_{...})\alpha_i]\right)\right] \\
&= 2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i,i_1,i_2}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2\neq j_3}^{b}\alpha_{i_2}E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\bar{\epsilon}_{i_1j_2.}\bar{\epsilon}_{i_1j_3.}(\widetilde{\epsilon}_{i_2..}-\widetilde{\epsilon}_{...})] \\
&= 2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2\neq j_3}^{b}\alpha_{i}E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\bar{\epsilon}_{ij_2.}\bar{\epsilon}_{ij_3.}(\widetilde{\epsilon}_{i..}-\widetilde{\epsilon}_{...})] \\
&= 4\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\alpha_{i}E[\bar{\epsilon}^2_{ij.}\bar{\epsilon}^2_{ij_1.}(\widetilde{\epsilon}_{i..}-\widetilde{\epsilon}_{...})] \\
&= 4\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\alpha_{i}\{E[\bar{\epsilon}^2_{ij.}\bar{\epsilon}^2_{ij_1.}\widetilde{\epsilon}_{i..}]-E[\bar{\epsilon}^2_{ij.}\bar{\epsilon}^2_{ij_1.}\widetilde{\epsilon}_{...}]\} \\
&= 4\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\alpha_{i}\left\{E\left[\bar{\epsilon}^2_{ij.}\bar{\epsilon}^2_{ij_1.}\left(\frac{1}{b}\sum_{j_2=1}^{b}\bar{\epsilon}_{ij_2.}\right)\right]\right. \\
&\quad \left. - E\left[\bar{\epsilon}^2_{ij.}\bar{\epsilon}^2_{ij_1.}\left(\frac{1}{ab}\sum_{i_1=1}^{a}\sum_{j_2=1}^{b}\bar{\epsilon}_{i_1j_2.}\right)\right]\right\} \\
&= 8\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\alpha_{i}\left\{\frac{1}{b}E[\bar{\epsilon}^3_{ij.}]E[\bar{\epsilon}^2_{ij_1.}]-\frac{1}{ab}E[\bar{\epsilon}^3_{ij.}]E[\bar{\epsilon}^2_{ij_1.}]\right\} \\
&= \frac{8(a-1)}{a^{3/2}b^{1/2}}\sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\alpha_i\frac{\gamma_{ij}\sigma^3_{ij}}{n^2_{ij}}\frac{\sigma^2_{ij_1}}{n_{ij_1}}.
\end{aligned}
$$

$$
\begin{aligned}
& E\{\Lambda_2(\Omega_a - E(\Omega_a))\} \\
= \;& E\left[\left(\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}} \frac{\epsilon_{ijk}\epsilon_{ijk_1}}{n_{ij}(n_{ij}-1)}\right)^2 \left(2\left(\frac{b}{a}\right)^{\frac{1}{2}} \sum_{i=1}^{a}[(\widetilde{\epsilon}_{i..} - \widetilde{\epsilon}_{...})\alpha_i]\right)\right] \\
= \;& 2\left(\frac{b}{a}\right)^{\frac{1}{2}} \sum_{i,i_1,i_2}^{a}\sum_{j,j_1}^{b}\sum_{k\neq k_1}^{n_{ij}}\sum_{k_2\neq k_3}^{n_{i_1j_1}} \alpha_{i_2}\frac{E[\epsilon_{ijk}\epsilon_{ijk_1}\epsilon_{i_1j_1k_2}\epsilon_{i_1j_1k_3}(\widetilde{\epsilon}_{i_2..} - \widetilde{\epsilon}_{...})]}{n_{ij}(n_{ij}-1)n_{i_1j_1}(n_{i_1j_1}-1)} \\
= \;& 2\left(\frac{b}{a}\right)^{\frac{1}{2}} \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\sum_{k_2\neq k_3}^{n_{ij}} \alpha_i\frac{E[\epsilon_{ijk}\epsilon_{ijk_1}\epsilon_{ijk_2}\epsilon_{ijk_3}(\widetilde{\epsilon}_{i..} - \widetilde{\epsilon}_{...})]}{n_{ij}^2(n_{ij}-1)^2} \\
= \;& 4\left(\frac{b}{a}\right)^{\frac{1}{2}} \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}} \alpha_i\frac{E[\epsilon_{ijk}^2\epsilon_{ijk_1}^2(\widetilde{\epsilon}_{i..} - \widetilde{\epsilon}_{...})]}{n_{ij}^2(n_{ij}-1)^2} \\
= \;& 4\left(\frac{b}{a}\right)^{\frac{1}{2}} \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}} \alpha_i\frac{[E(\epsilon_{ijk}^2\epsilon_{ijk_1}^2\widetilde{\epsilon}_{i..}) - E(\epsilon_{ijk}^2\epsilon_{ijk_1}^2\widetilde{\epsilon}_{...})]}{n_{ij}^2(n_{ij}-1)^2} \\
= \;& 4\left(\frac{b}{a}\right)^{\frac{1}{2}} \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}} \alpha_i E\left[\epsilon_{ijk}^2\epsilon_{ijk_1}^2\left(\frac{1}{b}\sum_{j_1=1}^{b}\sum_{k_2=1}^{n_{ij_1}}\frac{\epsilon_{ij_1k_2}}{n_{ij_1}}\right)\right] n_{ij}^{-2}(n_{ij}-1)^{-2} \\
& - 4\left(\frac{b}{a}\right)^{\frac{1}{2}} \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}} \alpha_i E\left[\epsilon_{ijk}^2\epsilon_{ijk_1}^2\left(\frac{1}{ab}\sum_{i_1=1}^{a}\sum_{j_1=1}^{b}\sum_{k_2=1}^{n_{ij_1}}\frac{\epsilon_{i_1j_1k_2}}{n_{i_1j_1}}\right)\right] n_{ij}^{-2}(n_{ij}-1)^{-2} \\
= \;& 8\left(\frac{1}{ab}\right)^{\frac{1}{2}} \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}} \alpha_i\frac{E[\epsilon_{ijk}^3]E[\epsilon_{ijk_1}^2]}{n_{ij}^3(n_{ij}-1)^2} - \;8\left(\frac{1}{a^{3/2}b^{1/2}}\right)\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}} \alpha_i\frac{E[\epsilon_{ijk}^3]E[\epsilon_{ijk_1}^2]}{n_{ij}^3(n_{ij}-1)^2} \\
= \;& \frac{8(a-1)}{a^{3/2}b^{1/2}}\sum_{i=1}^{a}\sum_{j=1}^{b}\alpha_i\frac{\gamma_{ij}\sigma_{ij}^5}{n_{ij}^2(n_{ij}-1)},
\end{aligned}
$$

and

$$
\begin{aligned}
& E\{\Lambda_3(\Omega_a - E(\Omega_a))\} \\
= \;& E\left[\left(\sum_{i=1}^{a}\sum_{i_1=1}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2=1}^{b}\sum_{k\neq k_1}^{n_{i_1j_2}} \frac{\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.}\epsilon_{i_1j_2k}\epsilon_{i_1j_2k_1}}{n_{i_1j_2}(n_{i_1j_2}-1)}\right)\left(2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i=1}^{a}[(\widetilde{\epsilon}_{i..} - \widetilde{\epsilon}_{...})\alpha_i]\right)\right] \\
= \;& 2\left(\frac{b}{a}\right)^{\frac{1}{2}} \sum_{i,i_1,i_2}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2=1}^{b}\sum_{k\neq k_1}^{n_{i_ij_2}} \alpha_{i_2}\frac{E[\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.}\epsilon_{i_1j_2k}\epsilon_{i_1j_2k_1}(\widetilde{\epsilon}_{i_2..} - \widetilde{\epsilon}_{...})]}{n_{i_1j_2}(n_{i_1j_2}-1)} \\
= \;& 4\left(\frac{b}{a}\right)^{\frac{1}{2}} \sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\sum_{k\neq k_1}^{n_{ij}} \alpha_i\frac{E[\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.}\epsilon_{ijk}\epsilon_{ijk_1}(\widetilde{\epsilon}_{i..} - \widetilde{\epsilon}_{...})]}{n_{ij}(n_{ij}-1)} \\
= \;& 0,
\end{aligned}
$$

where the last equality is due to the fact that $j \neq j_1$ and $k \neq k_1$. Therefore combining the terms for $E\{\Lambda_1(\Omega_a - E(\Omega_a))\}, E\{\Lambda_2(\Omega_a - E(\Omega_a))\}$ and $E\{\Lambda_3(\Omega_a - E(\Omega_a))\}$ we have

$$\begin{aligned}
E[\widetilde{T_A}^2(\Omega_a - E(\Omega_a))] &= \frac{(a-1)^2}{(ab)^3}\left[\frac{8(a-1)}{a^{3/2}b^{1/2}}\sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\alpha_i\frac{\gamma_{ij}\sigma_{ij}^3}{n_{ij}^2}\frac{\sigma_{ij_1}^2}{n_{ij_1}} + O(a^{-\frac{1}{2}})\right.\\
&\quad\left.+\frac{8(a-1)}{a^{3/2}b^{1/2}}\sum_{i=1}^{a}\sum_{j=1}^{b}\alpha_i\frac{\gamma_{ij}\sigma_{ij}^5}{n_{ij}^2(n_{ij}-1)}\right]\\
&= \frac{8(a-1)^3}{a^{9/2}b^{7/2}}\sum_{i=1}^{a}\left[\sum_{j\neq j_1}^{b}\alpha_i\frac{\gamma_{ij}\sigma_{ij}^3}{n_{ij}^2}\frac{\sigma_{ij_1}^2}{n_{ij_1}} + \sum_{j=1}^{b}\alpha_i\frac{\gamma_{ij}\sigma_{ij}^5}{n_{ij}^2(n_{ij}-1)}\right].
\end{aligned}$$

Using (1.3.7), (1.3.16) and (1.7.27) we compute $E[\widetilde{T_A}\Pi_a(\Omega_a - E(\Omega_a))]$ as follows:

$$\begin{aligned}
&E[\widetilde{T_A}\Pi_a(\Omega_a - E(\Omega_a))]\\
&= E\left[\sum_{i=1}^{a}\frac{a-1}{(ab)^{3/2}}\left\{\sum_{j\neq j_1}^{b}\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.} + \sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\frac{\epsilon_{ijk}\epsilon_{ijk_1}}{n_{ij}(n_{ij}-1)}\right\}a^{\frac{1}{2}}\left(-\frac{b^{1/2}}{a^{3/2}}\sum_{i_1\neq i_2}^{a}\overline{\epsilon}_{i_1..}\overline{\epsilon}_{i_2..}\right)\right.\\
&\quad\left.\left(2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i_3=1}^{a}[(\widetilde{\epsilon}_{i_3..} - \widetilde{\epsilon}_{...})\alpha_{i_3}]\right)\right]\\
&= -\frac{2(a-1)}{a^3b^{1/2}}\sum_{i=1}^{a}\sum_{i_1\neq i_2}^{a}\sum_{i_3=1}^{a}\left[\sum_{j\neq j_1}^{b}\alpha_{i_3}E[\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.}\overline{\epsilon}_{i_1..}\overline{\epsilon}_{i_2..}(\widetilde{\epsilon}_{i_3..} - \widetilde{\epsilon}_{...})]\right.\\
&\quad\left.+\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\alpha_{i_3}\frac{E[\epsilon_{ijk}\epsilon_{ijk_1}\overline{\epsilon}_{i_1..}\overline{\epsilon}_{i_2..}(\widetilde{\epsilon}_{i_3..} - \widetilde{\epsilon}_{...})]}{n_{ij}(n_{ij}-1)}\right]\\
&= -\frac{4(a-1)}{a^3b^{1/2}}\sum_{i_1\neq i_2}^{a}\left[\sum_{j\neq j_1}^{b}\alpha_{i_1}E[\overline{\epsilon}_{i_1..}(\widetilde{\epsilon}_{i_3..} - \widetilde{\epsilon}_{...})]E[\overline{\epsilon}_{i_2j.}\overline{\epsilon}_{i_2j_1.}\overline{\epsilon}_{i_2..}]\right.\\
&\quad\left.+\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\alpha_{i_1}\frac{E[\overline{\epsilon}_{i_1..}(\widetilde{\epsilon}_{i_3..} - \widetilde{\epsilon}_{...})]E[\epsilon_{i_2jk}\epsilon_{i_2jk_1}\overline{\epsilon}_{i_2..}]}{n_{i_2j}(n_{i_2j}-1)}\right]\\
&= 0
\end{aligned}$$

since $j \neq j_1$ and $k \neq k_1$. Combining the results for $E[\widetilde{T_A}^2(\Omega_a - E(\Omega_a))]$ and $E[\widetilde{T_A}\Pi_a(\Omega_a - E(\Omega_a))]$, the second term in (1.7.31) is given by

$$\begin{aligned}
&E[(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a)^2(\Omega_a - E(\Omega_a))]\\
&= \frac{8(a-1)^3}{a^{9/2}b^{7/2}}\sum_{i=1}^{a}\left[\sum_{j\neq j_1}^{b}\alpha_i\frac{\gamma_{ij}\sigma_{ij}^3}{n_{ij}^2}\frac{\sigma_{ij_1}^2}{n_{ij_1}} + \sum_{j=1}^{b}\alpha_i\frac{\gamma_{ij}\sigma_{ij}^5}{n_{ij}^2(n_{ij}-1)}\right]. \qquad (1.7.32)
\end{aligned}$$

We now consider the third term in (1.7.31).

$$E[(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a)(\Omega_a - E(\Omega_a))^2] = E[(\widetilde{T_A}(\Omega_a - E(\Omega_a))^2] + a^{-\frac{1}{2}}E[\Pi_a(\Omega_a - E(\Omega_a))^2].$$

We use (1.3.7) and (1.7.27) to compute $E[\widetilde{T_A}(\Omega_a - E(\Omega_a))^2]$ as follows:

$$\begin{aligned}
& E[\widetilde{T_A}(\Omega_a - E(\Omega_a))^2] \\
= \; & E\left[\sum_{i=1}^{a}\frac{a-1}{(ab)^{3/2}}\left\{\sum_{j\neq j_1}^{b}\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.} + \sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\frac{\epsilon_{ijk}\epsilon_{ijk_1}}{n_{ij}(n_{ij}-1)}\right\}\left(2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i_1=1}^{a}[(\widetilde{\epsilon}_{i_1..} - \widetilde{\epsilon}_{...})\alpha_{i_1}]\right)^2\right] \\
= \; & \frac{4(a-1)}{a^{5/2}b^{1/2}}\sum_{i,i_1,i_2}^{a}\left[\sum_{j\neq j_1}^{b}\alpha_{i_1}\alpha_{i_2}E[(\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.})(\widetilde{\epsilon}_{i_1..} - \widetilde{\epsilon}_{...})(\widetilde{\epsilon}_{i_2..} - \widetilde{\epsilon}_{...})]\right. \\
& \left. + \sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\alpha_{i_1}\alpha_{i_2}\frac{E[(\epsilon_{ijk}\epsilon_{ijk_1})(\widetilde{\epsilon}_{i_1..} - \widetilde{\epsilon}_{...})(\widetilde{\epsilon}_{i_2..} - \widetilde{\epsilon}_{...})]}{n_{ij}(n_{ij}-1)}\right] \\
= \; & \frac{4(a-1)}{a^{5/2}b^{1/2}}\sum_{i,i_1,i_2}^{a}\left[\sum_{j\neq j_1}^{b}\alpha_{i_1}\alpha_{i_2}E[\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.}(\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..} - \widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{...} - \widetilde{\epsilon}_{...}\widetilde{\epsilon}_{i_2..} + \widetilde{\epsilon}^2_{...})]\right. \\
& \left. + \sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\alpha_{i_1}\alpha_{i_2}\frac{E[\epsilon_{ijk}\epsilon_{ijk_1}(\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..} - \widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{...} - \widetilde{\epsilon}_{...}\widetilde{\epsilon}_{i_2..} + \widetilde{\epsilon}^2_{...})]}{n_{ij}(n_{ij}-1)}\right].
\end{aligned}$$

It is shown in Section 1.7.8 that

$$\begin{aligned}
& \sum_{i,i_1,i_2}^{a}\left[\sum_{j\neq j_1}^{b}\alpha_{i_1}\alpha_{i_2}E[\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.}(\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..} - \widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{...} - \widetilde{\epsilon}_{...}\widetilde{\epsilon}_{i_2..} + \widetilde{\epsilon}^2_{...})]\right. \\
& \left. + \sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\alpha_{i_1}\alpha_{i_2}\frac{E[\epsilon_{ijk}\epsilon_{ijk_1}(\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..} - \widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{...} - \widetilde{\epsilon}_{...}\widetilde{\epsilon}_{i_2..} + \widetilde{\epsilon}^2_{...})]}{n_{ij}(n_{ij}-1)}\right] \\
= \; & O\left(\left(1 - \frac{2}{a} + \frac{1}{a^2}\right)\frac{\sqrt{a}}{\sqrt{b}}\right). \qquad (1.7.33)
\end{aligned}$$

Therefore, $E[\widetilde{T_A}(\Omega_a - E(\Omega_a))^2] = O(a^{-1}b^{-1})$.

Using (1.3.7) and (1.7.27) we compute $E[\Pi_a(\Omega_a - E(\Omega_a))^2]$ as follows:

$$\begin{aligned}
E[\Pi_a(\Omega_a - E(\Omega_a))^2] &= E\left[a^{\frac{1}{2}}\left(-\frac{b^{1/2}}{a^{3/2}}\sum_{i_1\neq i_2}^{a}\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\right)\left(2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i_3=1}^{a}[(\widetilde{\epsilon}_{i_3..}-\widetilde{\epsilon}_{...})\alpha_{i_3}]\right)^2\right] \\
&= -\frac{4b^{\frac{3}{2}}}{a^2}\sum_{i_1\neq i_2}^{a}\sum_{i_3,i_4}^{a}\alpha_{i_3}\alpha_{i_4}E[\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}(\widetilde{\epsilon}_{i_3..}-\widetilde{\epsilon}_{...})(\widetilde{\epsilon}_{i_4..}-\widetilde{\epsilon}_{...})] \\
&= -\frac{8b^{\frac{3}{2}}}{a^2}\sum_{i_1\neq i_2}^{a}\alpha_{i_1}\alpha_{i_2}E[\widetilde{\epsilon}^2_{i_1..}]E[\widetilde{\epsilon}^2_{i_2..}] \\
&= -\frac{8}{a^2 b^{\frac{1}{2}}}\sum_{i_1\neq i_2}^{a}\alpha_{i_1}\alpha_{i_2}\left(\frac{1}{b}\sum_{j_1=1}^{b}\frac{\sigma^2_{i_1j_1}}{n_{i_1j_1}}\right)\left(\frac{1}{b}\sum_{j_1=1}^{b}\frac{\sigma^2_{i_2j_2}}{n_{i_2j_2}}\right). \qquad (1.7.34)
\end{aligned}$$

It can be seen that

$$\frac{1}{b}\sum_{j_1=1}^{b}\frac{\sigma^2_{i_1j_1}}{n_{i_1j_1}} \leq M_2, \ \forall i, \qquad (1.7.35)$$

for some finite M_2 since $E(\epsilon^4_{ijk}) < \infty$. Therefore

$$\begin{aligned}
E[\Pi_a(\Omega_a - E(\Omega_a))^2] &\leq \frac{8}{a^2 b^{\frac{1}{2}}}\sum_{i_1\neq i_2}^{a}|\alpha_{i_1}\alpha_{i_2}|M_2^2 \\
&\leq \frac{4}{a^2 b^{\frac{1}{2}}}M_2^2\sum_{i_1\neq i_2}^{a}(\alpha^2_{i_1}+\alpha^2_{i_2}) \\
&= O(a^{-1}b^{-1}).
\end{aligned}$$

where the last equality is due to condition (1.3.17). Thus combining $E[\widetilde{T_A}(\Omega_a - E(\Omega_a))^2]$ and $E[\Pi_a(\Omega_a - E(\Omega_a))^2]$, the third term in (1.7.31) is

$$3E[(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a)(\Omega_a - E(\Omega_a))^2] = O(a^{-1}b^{-1}). \qquad (1.7.36)$$

We proceed to compute the fourth term in (1.7.31). Using (1.7.27) we compute $E(\Omega_a -$

$E(\Omega_a))^3$ as follows:

$$\begin{aligned}
E(\Omega_a - E(\Omega_a))^3 &= E\left[\left(2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i=1}^{a}[(\widetilde{\epsilon}_{i\cdot\cdot} - \widetilde{\epsilon}_{\cdots})\alpha_i]\right)^3\right] \\
&= 8\left(\frac{b}{a}\right)^{\frac{3}{2}}\sum_{i,i_1,i_2}^{a}\alpha_i\alpha_{i_1}\alpha_{i_2}E[(\widetilde{\epsilon}_{i\cdot\cdot} - \widetilde{\epsilon}_{\cdots})(\widetilde{\epsilon}_{i_1\cdot\cdot} - \widetilde{\epsilon}_{\cdots})(\widetilde{\epsilon}_{i_2\cdot\cdot} - \widetilde{\epsilon}_{\cdots})] \\
&= 8\left(\frac{b}{a}\right)^{\frac{3}{2}}\left[\sum_{i}^{a}\alpha_i^3E[(\widetilde{\epsilon}_{i\cdot\cdot} - \widetilde{\epsilon}_{\cdots})^3] + 2\sum_{i\neq i_1}^{a}\alpha_i^2\alpha_{i_1}E[(\widetilde{\epsilon}_{i\cdot\cdot} - \widetilde{\epsilon}_{\cdots})^2(\widetilde{\epsilon}_{i_1\cdot\cdot} - \widetilde{\epsilon}_{\cdots})]\right. \\
&\quad \left. + \sum_{i\neq i_1\neq i_2}^{a}\alpha_i\alpha_{i_1}\alpha_{i_2}E[(\widetilde{\epsilon}_{i\cdot\cdot} - \widetilde{\epsilon}_{\cdots})(\widetilde{\epsilon}_{i_1\cdot\cdot} - \widetilde{\epsilon}_{\cdots})(\widetilde{\epsilon}_{i_2\cdot\cdot} - \widetilde{\epsilon}_{\cdots})]\right]. \qquad (1.7.37)
\end{aligned}$$

The first term satisfies

$$|\sum_{i}^{a}\alpha_i^3E[(\widetilde{\epsilon}_{i\cdot\cdot} - \widetilde{\epsilon}_{\cdots})^3]| \leq \sum_{i}^{a}\alpha_i^2|[\alpha_iE(\widetilde{\epsilon}_{i\cdot\cdot} - \widetilde{\epsilon}_{\cdots})^3]| \leq \sum_{i}^{a}\alpha_i^2\frac{M_3}{b^2} = O\left(\frac{\sqrt{a}}{b^{\frac{5}{2}}}\right),$$

where the last inequality is a result of condition (1.3.17), $E(\epsilon_{ijk}^4) < \infty$ and the fact that α_i is bounded for all i, which lead to the existence of some finite M_3 such that

$$\frac{1}{b}\sum_{j=1}^{b}\frac{\gamma_{ij}\sigma_{ij}^3}{n_{ij}^2} \leq M_3,\ \forall i. \qquad (1.7.38)$$

The second term satisfies

$$
\begin{aligned}
\sum_{i\neq i_1}^{a}\alpha_i^2\alpha_{i_1}E[(\widetilde{\epsilon}_{i..}-\widetilde{\epsilon}_{...})^2(\widetilde{\epsilon}_{i_1..}-\widetilde{\epsilon}_{...})] &= \sum_{i\neq i_1}^{a}\alpha_i^2\alpha_{i_1}[E(\widetilde{\epsilon}_{i..}^2)E(\widetilde{\epsilon}_{i_1..})-E(\widetilde{\epsilon}_{i..}^2\widetilde{\epsilon}_{...})+E(\widetilde{\epsilon}_{...}^2\widetilde{\epsilon}_{i_1..}) \\
&\quad + E(\widetilde{\epsilon}_{...}^3)-2E(\widetilde{\epsilon}_{i..}\widetilde{\epsilon}_{...}\widetilde{\epsilon}_{i_1..})+2E(\widetilde{\epsilon}_{i..}\widetilde{\epsilon}_{...}^2)] \\
&= \sum_{i\neq i_1}^{a}\alpha_i^2\alpha_{i_1}\left[-\frac{1}{a}E(\widetilde{\epsilon}_{i..}^3)+\frac{1}{a^2}E(\widetilde{\epsilon}_{i_1..}^3)+E(\widetilde{\epsilon}_{...}^3)+\frac{2}{a^2}E(\widetilde{\epsilon}_{i..}^3)\right] \\
&= \sum_{i\neq i_1}^{a}\alpha_i^2\alpha_{i_1}\left(-\frac{1}{a}+\frac{3}{a^2}\right)E(\widetilde{\epsilon}_{i..}^3)+\sum_{i\neq i_1}^{a}\alpha_i^2\alpha_{i_1}E(\widetilde{\epsilon}_{...}^3) \\
&= \sum_{i=1}^{a}\alpha_i^3\left(\frac{1}{a}-\frac{3}{a^2}\right)E(\widetilde{\epsilon}_{i..}^3)-\frac{1}{a^3}\sum_{i=1}^{a}\alpha_i^3\sum_{i_1=1}^{a}E(\widetilde{\epsilon}_{i_1..}^3) \\
&\leq \sum_{i=1}^{a}|\alpha_i^3|\left\{\left|(\frac{1}{a}-\frac{3}{a})E(\widetilde{\epsilon}_{i..}^3)\right|+\frac{1}{a^3}\sum_{i_1=1}^{a}|E(\widetilde{\epsilon}_{i_1..}^3)|\right\} \\
&\leq \sum_{i=1}^{a}|\alpha_i^3|\left(\frac{1}{a}-\frac{3}{a^2}+\frac{2}{a^2}\right)\frac{M_3}{b^2} \\
&= \sum_{i=1}^{a}\alpha_i^2|\alpha_i|\left(\frac{1}{a}-\frac{3}{a^2}+\frac{2}{a^2}\right)\frac{M_3}{b^2} \\
&= O\left(\frac{1}{\sqrt{a}b^{\frac{5}{2}}}\right).
\end{aligned}
$$

where the last inequality is a due to the fact that $\sum_{i=1}^{a}\alpha_i^2|\alpha_i|\leq O\left(\frac{\sqrt{a}}{\sqrt{b}}\right)$ with condition (1.3.17) since α_i is bounded. Similarly, the third term also satisfies

$$
\begin{aligned}
&\sum_{i\neq i_1\neq i_2}^{a}\alpha_i\alpha_{i_1}\alpha_{i_2}E[(\widetilde{\epsilon}_{i..}-\widetilde{\epsilon}_{...})(\widetilde{\epsilon}_{i_1..}-\widetilde{\epsilon}_{...})(\widetilde{\epsilon}_{i_2..}-\widetilde{\epsilon}_{...})] \\
=\ &\sum_{i\neq i_1\neq i_2}^{a}\alpha_i\alpha_{i_1}\alpha_{i_2}E[3\widetilde{\epsilon}_{i..}\widetilde{\epsilon}_{...}^2-\widetilde{\epsilon}_{...}^3] \\
=\ &\sum_{i\neq i_1\neq i_2}^{a}\alpha_i\alpha_{i_1}\alpha_{i_2}\left[\frac{3}{a^2}E(\widetilde{\epsilon}_{i..}^3)-\frac{1}{a^3}\sum_{i_3=1}^{a}E(\widetilde{\epsilon}_{i_3..}^3)\right] \\
=\ &\frac{3}{a^2}\sum_{i=1}^{a}E(\widetilde{\epsilon}_{i..}^3)\sum_{i_2\neq i}[\alpha_{i_2}(-\alpha_i-\alpha_{i_2})]-\frac{2}{a^3}\sum_{i=1}^{a}\alpha_i^3\sum_{i_3=1}^{a}E(\widetilde{\epsilon}_{i_3..}^3) \qquad (1.7.39) \\
=\ &O\left(\frac{\sqrt{a}}{b^{\frac{5}{2}}}\right), \qquad (1.7.40)
\end{aligned}
$$

where (1.7.39) is due to

$$\begin{aligned}\sum_{i\neq i_1\neq i_2}^{a}\alpha_i\alpha_{i_1}\alpha_{i_2} &= \sum_{i\neq i_1}^{a}\alpha_i\alpha_{i_1}(0-\alpha_i-\alpha_{i_1})\\ &= -\sum_{i\neq i_1}^{a}\alpha_i^2\alpha_{i_1}-\sum_{i\neq i_1}^{a}\alpha_i\alpha_{i_1}^2\\ &= \sum_{i=1}^{a}\alpha_i^3+\sum_{i_1=1}^{a}\alpha_{i_1}^3\\ &= 2\sum_{i=1}^{a}\alpha_i^3,\end{aligned}$$

and (1.7.40) is because

$$\begin{aligned}\sum_{i\neq i_1\neq i_2}^{a}\alpha_i\alpha_{i_1}\alpha_{i_2}\frac{3}{a^2}E(\tilde{\epsilon}_{i..}^3) &= \sum_{i=1}^{a}\alpha_i E(\tilde{\epsilon}_{i..}^3)\sum_{i_2\neq i}^{a}\alpha_{i_2}(-\alpha_i-\alpha_{i_2})\\ &= \sum_{i=1}^{a}\alpha_i\frac{3}{a^2}E(\tilde{\epsilon}_{i..}^3)\left[\alpha_i^2-\sum_{i_2\neq i}^{a}\alpha_{i_2}^2\right]\\ &= \frac{3}{a^2}\sum_{i=1}^{a}\left[\alpha_i^3-\alpha_i\sum_{i_2\neq i}^{a}\alpha_{i_2}^2\right]E(\tilde{\epsilon}_{i..}^3)\\ &\leq \frac{3}{a^2}\sum_{i=1}^{a}\left[|\alpha_i^3|+|\alpha_i|\sum_{i_2\neq i}^{a}\alpha_{i_2}^2\right]|E(\tilde{\epsilon}_{i..}^3)|\\ &\leq \frac{3}{a^2}\sum_{i=1}^{a}\left[\alpha_i^2+\sum_{i_2\neq i}^{a}\alpha_{i_2}^2\right]M_\alpha\frac{M_3}{b^2}\\ &= O\left(\frac{1}{\sqrt{a}b^{\frac{5}{2}}}\right),\end{aligned}$$

where the last equality uses conditions (1.3.17) and the fact that α_i is bounded for all i. Putting them together gives

$$E(\Omega_a-E(\Omega_a))^3=\left(\frac{b}{a}\right)^{\frac{3}{2}}O\left(\frac{\sqrt{a}}{b^{\frac{5}{2}}}\right)=O(a^{-1}b^{-1}). \qquad (1.7.41)$$

Replacing the first term in (1.7.31) with $K_3(T_A^{(0)})$ and putting (1.7.32), (1.7.36) and (1.7.41) into (1.7.31) we give the third cumulant of $T_A^{(1)}$ as follows:

$$\begin{aligned} K_3(T_A^{(1)}) &= K_3(T_A^{(0)}) + \frac{8(a-1)^3}{a^{9/2}b^{7/2}} \sum_{i=1}^{a} \left[\sum_{j \neq j_1}^{b} \alpha_i \frac{\gamma_{ij}\sigma_{ij}^3}{n_{ij}^2} \frac{\sigma_{ij_1}^2}{n_{ij_1}} + \sum_{j=1}^{b} \alpha_i \frac{\gamma_{ij}\sigma_{ij}^5}{n_{ij}^2(n_{ij}-1)} \right] \\ &= K_3(T_A^{(0)}) + \kappa_3, \end{aligned} \tag{1.7.42}$$

where $K_3(T_A^{(0)})$ is the third cumulant of $T_A^{(0)}$ under the null given in (1.7.9), and

$$\kappa_3 = \frac{8(a-1)^3}{a^{9/2}b^{7/2}} \sum_{i=1}^{a} \left[\sum_{j \neq j_1}^{b} \alpha_i \frac{\gamma_{ij}\sigma_{ij}^3}{n_{ij}^2} \frac{\sigma_{ij_1}^2}{n_{ij_1}} + \sum_{j=1}^{b} \alpha_i \frac{\gamma_{ij}\sigma_{ij}^5}{n_{ij}^2(n_{ij}-1)} \right]. \tag{1.7.43}$$

Lastly we compute the fourth cumulant $K_4(T_A^{(1)})$ as follows:

$$\begin{aligned} &K_4(T_A^{(1)}) \\ &= E[(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a) + (\Omega_a - E(\Omega_a))]^4 - 3[\mathrm{Var}(T_A^{(1)})]^2 \\ &= E(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a)^4 + 4E[(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a)^3(\Omega_a - E(\Omega_a))] + 6E[(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a)^2(\Omega_a - E(\Omega_a))^2] \\ &\quad + 4E[(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a)(\Omega_a - E(\Omega_a))^3] + E(\Omega_a - E(\Omega_a))^4 - 3[\mathrm{Var}(T_A^{(1)})]^2. \end{aligned} \tag{1.7.44}$$

In order to obtain the result for $K_4(T_A^{(1)})$ we need to compute each term in (1.7.44) separately. The first term in (1.7.44) is given in (1.7.10) as $E(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a)^4 = E(\widetilde{T_A}^4) + O(a^{-1})$. Next we compute the second term in (1.7.44) as follows:

$$\begin{aligned} E[(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a)^3(\Omega_a - E(\Omega_a))] &= E[(\widetilde{T_A}^3 + 3a^{-\frac{1}{2}}\widetilde{T_A}^2\Pi_a + 3a^{-1}\widetilde{T_A}\Pi_a^2 + a^{-\frac{3}{2}}\Pi_a^3)(\Omega_a - E(\Omega_a))] \\ &= E[\widetilde{T_A}^3(\Omega_a - E(\Omega_a))] + 3a^{-\frac{1}{2}}E[\widetilde{T_A}^2\Pi_a(\Omega_a - E(\Omega_a))] \\ &\quad + O(a^{-1}), \end{aligned} \tag{1.7.45}$$

where the last equality is due to the Cramer's condition and the fact that $E[\widetilde{T_A}\Pi_a^2(\Omega_a - E(\Omega_a))]$ and $E[\Pi_a^3(\Omega_a - E(\Omega_a))]$ are at most $O(1)$. Using the result in (1.7.17) we express $E[\widetilde{T_A}^3(\Omega_a - E(\Omega_a))]$ as

$$\begin{aligned} E[\widetilde{T_A}^3(\Omega_a - E(\Omega_a))] &= E\left[\frac{(a-1)^3}{(ab)^{9/2}}(\Lambda_4 + \Lambda_5 + 3\Lambda_6 + 3\Lambda_7)(\Omega_a - E(\Omega_a))\right] \\ &= \frac{(a-1)^3}{(ab)^{9/2}}[E(\Lambda_4(\Omega_a - E(\Omega_a))) + E(\Lambda_5(\Omega_a - E(\Omega_a))) \\ &\quad + 3E(\Lambda_6(\Omega_a - E(\Omega_a))) + 3E(\Lambda_7(\Omega_a - E(\Omega_a)))], \end{aligned}$$

where $\Lambda_4, \Lambda_5, \Lambda_6$ and Λ_7 are presented in (1.7.18), (1.7.19), (1.7.20) and (1.7.21), respectively. We compute the four terms in the above equation as follows:

$$
\begin{aligned}
& E\{\Lambda_4(\Omega_a - E(\Omega_a))\} \\
= \;& E\left[\left(\sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\right)^3\left(2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i=1}^{a}[(\widetilde{\epsilon}_{i..}-\widetilde{\epsilon}_{...})\alpha_i]\right)\right] \\
= \;& 2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i,i_1,i_2,i_3}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2\neq j_3}^{b}\sum_{j_4\neq j_5}^{b}\alpha_{i_3}E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\bar{\epsilon}_{i_1j_2.}\bar{\epsilon}_{i_1j_3.}\bar{\epsilon}_{i_2j_4.}\bar{\epsilon}_{i_2j_5.}(\widetilde{\epsilon}_{i_3..}-\widetilde{\epsilon}_{...})] \\
= \;& 2\left(\frac{b}{a}\right)^{\frac{1}{2}}\left[\sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2\neq j_3}^{b}\sum_{j_4\neq j_5}^{b}\alpha_i E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\bar{\epsilon}_{ij_2.}\bar{\epsilon}_{ij_3.}\bar{\epsilon}_{ij_4.}\bar{\epsilon}_{ij_5.}\widetilde{\epsilon}_{i..}]\right. \\
& \left.+\,2\sum_{i\neq i_1}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2\neq j_3}^{b}\sum_{j_4\neq j_5}^{b}\alpha_{i_1}E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\bar{\epsilon}_{ij_4.}\bar{\epsilon}_{ij_5.}]E[\bar{\epsilon}_{i_1j_2.}\bar{\epsilon}_{i_1j_3.}\widetilde{\epsilon}_{i_1..}]\right] \\
= \;& O(a^{\frac{1}{2}}b^{\frac{7}{2}}),
\end{aligned}
$$

where the last equality is because the second term is zero since $\bar{\epsilon}_{i_1j_2.}$ and $\bar{\epsilon}_{i_1j_3.}$ are independent when $j_2 \neq j_3$. Next,

$$
\begin{aligned}
& E\{\Lambda_5(\Omega_a - E(\Omega_a))\} \\
= \;& E\left[\left(\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\frac{\epsilon_{ijk}\epsilon_{ijk_1}}{n_{ij}(n_{ij}-1)}\right)^3\left(2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i=1}^{a}[(\widetilde{\epsilon}_{i..}-\widetilde{\epsilon}_{...})\alpha_i]\right)\right] \\
= \;& 2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i,i_1,i_2,i_3}^{a}\sum_{j,j_1,j_2}^{b}\sum_{k\neq k_1}^{n_{ij}}\sum_{k_2\neq k_3}^{n_{i_1j_1}}\sum_{k_4\neq k_5}^{n_{i_2j_2}}\alpha_{i_3}\frac{E[\epsilon_{ijk}\epsilon_{ijk_1}\epsilon_{i_1j_1k_2}\epsilon_{i_1j_1k_3}\epsilon_{i_2j_2k_4}\epsilon_{i_2j_2k_5}(\widetilde{\epsilon}_{i_3..}-\widetilde{\epsilon}_{...})]}{n_{ij}(n_{ij}-1)n_{i_1j_1}(n_{i_1j_1}-1)n_{i_2j_2}(n_{i_2j_2}-1)} \\
= \;& 2\left(\frac{b}{a}\right)^{\frac{1}{2}}\left[\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\sum_{k_2\neq k_3}^{n_{ij}}\sum_{k_4\neq k_5}^{n_{ij}}\alpha_i\frac{E[\epsilon_{ijk}\epsilon_{ijk_1}\epsilon_{ijk_2}\epsilon_{ijk_3}\epsilon_{ijk_4}\epsilon_{ijk_5}\widetilde{\epsilon}_{i..}]}{n_{ij}^3(n_{ij}-1)^3}\right. \\
& \left.+\,2\sum_{i\neq i_1}^{a}\sum_{j=1}^{b}\sum_{j_1=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\sum_{k_2\neq k_3}^{n_{i_1j}}\sum_{k_4\neq k_5}^{n_{ij}}\alpha_{i_1}\frac{E[\epsilon_{ijk}\epsilon_{ijk_1}\epsilon_{ijk_4}\epsilon_{ijk_5}]E[\epsilon_{i_1j_1k_2}\epsilon_{i_1j_1k_3}\widetilde{\epsilon}_{i_1..}]}{n_{ij}^2(n_{ij}-1)^2n_{i_1j_1}(n_{i_1j_1}-1)}\right] \\
= \;& O(a^{\frac{1}{2}}b^{\frac{3}{2}}),
\end{aligned}
$$

where the last equality is due to the fact that the second term is zero for $k_2 \neq k_3$, $\epsilon_{i_1jk_2}$ and $\epsilon_{i_1jk_3}$ are independent.

Next

$$
\begin{aligned}
&E\{\Lambda_6(\Omega_a - E(\Omega_a))\}\\
= \;& E\left[\sum_{i,i_1,i_2}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2,j_3}^{b}\sum_{k\neq k_1}^{n_{i_1j_2}}\sum_{k_2\neq k_3}^{n_{i_2j_3}} \frac{\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\epsilon_{i_1j_2k}\epsilon_{i_1j_2k_1}\epsilon_{i_2j_3k_2}\epsilon_{i_2j_3k_3}}{n_{i_1j_2}(n_{i_1j_2}-1)n_{i_2j_3}(n_{i_2j_3}-1)}\left(2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i_3=1}^{a}[(\widetilde{\epsilon}_{i_3..}-\widetilde{\epsilon}_{...})\alpha_{i_3}]\right)\right]\\
= \;& 2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i,i_1,i_2,i_3}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2,j_3}^{b}\sum_{k\neq k_1}^{n_{i_1j_2}}\sum_{k_2\neq k_3}^{n_{i_2j_3}}\alpha_{i_3}\frac{E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\epsilon_{i_1j_2k}\epsilon_{i_1j_2k_1}\epsilon_{i_2j_3k_2}\epsilon_{i_2j_3k_3}(\widetilde{\epsilon}_{i_3..}-\widetilde{\epsilon}_{...})]}{n_{i_1j_2}(n_{i_1j_2}-1)n_{i_2j_3}(n_{i_2j_3}-1)}\\
= \;& O(a^{\frac{1}{2}}b^{\frac{5}{2}}) + 4\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i\neq i_1}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2,j_3}^{b}\sum_{k\neq k_1}^{n_{i_1j_2}}\sum_{k_2\neq k_3}^{n_{ij_3}}\alpha_{i_1}\frac{E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\epsilon_{ij_3k_2}\epsilon_{ij_3k_3}]E[\epsilon_{i_1j_2k}\epsilon_{i_1j_2k_1}\widetilde{\epsilon}_{i_3..}]}{n_{ij_3}(n_{ij_3}-1)n_{i_1j_2}(n_{i_1j_2}-1)}\\
= \;& O(a^{\frac{1}{2}}b^{\frac{5}{2}})
\end{aligned}
$$

since for $k\neq k_1$, $\epsilon_{i_1j_2k}$ and $\epsilon_{i_1j_2k_1}$ are independent. Similarly,

$$
\begin{aligned}
&E\{\Lambda_7(\Omega_a - E(\Omega_a))\}\\
= \;& E\left[\sum_{i,i_1,i_2}^{a}\sum_{j}^{b}\sum_{j_1\neq j_2}^{b}\sum_{j_3\neq j_4}^{b}\sum_{k\neq k_1}^{n_{ij}}\frac{\epsilon_{ijk}\epsilon_{ijk_1}\bar{\epsilon}_{i_1j_1.}\bar{\epsilon}_{i_1j_2.}\bar{\epsilon}_{i_2j_3.}\bar{\epsilon}_{i_2j_4.}}{n_{ij}(n_{ij}-1)}\left(2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i_3=1}^{a}[(\widetilde{\epsilon}_{i_3..}-\widetilde{\epsilon}_{...})\alpha_{i_3}]\right)\right]\\
= \;& 2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i,i_1,i_2,i_3}^{a}\sum_{j}^{b}\sum_{j_1\neq j_2}^{b}\sum_{j_3\neq j_4}^{b}\sum_{k\neq k_1}^{n_{ij}}\alpha_{i_3}\frac{E[\epsilon_{ijk}\epsilon_{ijk_1}\bar{\epsilon}_{i_1j_1.}\bar{\epsilon}_{i_1j_2.}\bar{\epsilon}_{i_2j_3.}\bar{\epsilon}_{i_2j_4.}(\widetilde{\epsilon}_{i_3..}-\widetilde{\epsilon}_{...})]}{n_{ij}(n_{ij}-1)}\\
= \;& O(a^{\frac{1}{2}}b^{\frac{7}{2}}) + 2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i\neq i_1}^{a}\sum_{j}^{b}\sum_{j_1\neq j_2}^{b}\sum_{j_3\neq j_4}^{b}\sum_{k\neq k_1}^{n_{ij}}\alpha_{i_1}\{E[\epsilon_{ijk}\epsilon_{ijk_1}\bar{\epsilon}_{ij_3.}\bar{\epsilon}_{ij_4.}]E[(\widetilde{\epsilon}_{i_1..}\bar{\epsilon}_{i_1j_1.}\bar{\epsilon}_{i_1j_2.}]\\
&+ E[\epsilon_{ijk}\epsilon_{ijk_1}(\widetilde{\epsilon}_{i..}]E[\bar{\epsilon}_{i_1j_1.}\bar{\epsilon}_{i_1j_2.}\bar{\epsilon}_{i_1j_3.}\bar{\epsilon}_{i_1j_4.}]\}\frac{1}{n_{ij}(n_{ij}-1)}\\
= \;& O(a^{\frac{1}{2}}b^{\frac{7}{2}}),
\end{aligned}
$$

where the last equality is obtained because for $j_1\neq j_2$, $\bar{\epsilon}_{i_1j_1.}$ and $\bar{\epsilon}_{i_1j_2.}$ are independent, and for $k\neq k_1$, ϵ_{ijk} and ϵ_{ijk_1} are independent. Therefore combining the results for the four terms $E\{\Lambda_4(\Omega_a-E(\Omega_a))\}, E\{\Lambda_5(\Omega_a-E(\Omega_a))\}, E\{\Lambda_6(\Omega_a-E(\Omega_a))\}$ and $E\{\Lambda_7(\Omega_a-E(\Omega_a))\}$, we get

$$
E[\widetilde{T_A}^3(\Omega_a - E(\Omega_a))] = \frac{(a-1)^3}{(ab)^{9/2}}O(a^{\frac{1}{2}}b^{\frac{7}{2}}) = O(a^{-1}b^{-1}).
$$

Next we compute $E[\widetilde{T_A}^2\Pi_a(\Omega_a - E(\Omega_a))]$ in (1.7.45). Using the result in (1.7.12), we express

$E[\widetilde{T_A}^2\Pi_a(\Omega_a - E(\Omega_a))]$ as

$$\begin{aligned} E[\widetilde{T_A}^2\Pi_a(\Omega_a - E(\Omega_a))] &= \frac{(a-1)^2}{(ab)^3}[E\{\Lambda_1\Pi_a(\Omega_a - E(\Omega_a))\} + E\{\Lambda_2\Pi_a(\Omega_a - E(\Omega_a))\} \\ &\quad + 2E\{\Lambda_3\Pi_a(\Omega_a - E(\Omega_a))\}], \end{aligned}$$

where Λ_1, Λ_2 and Λ_3 are presented in (1.7.13), (1.7.14) and (1.7.15), respectively. We compute the three terms in the above equation as follows:

$$\begin{aligned} & E\{\Lambda_1\Pi_a(\Omega_a - E(\Omega_a))\} \\ = & E\left[\left(\sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.}\right)^2 a^{\frac{1}{2}}\left(-\frac{b^{1/2}}{a^{3/2}}\sum_{i_1\neq i_2}^{a}\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\right)\left(2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i_3=1}^{a}[(\widetilde{\epsilon}_{i_3..} - \widetilde{\epsilon}_{...})\alpha_{i_3}]\right)\right] \\ = & -2\left(\frac{b}{a^{\frac{3}{2}}}\right)\sum_{i_1\neq i_2}^{a}\sum_{i_3,i_4,i_5}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2\neq j_3}^{b}\alpha_{i_5}E[\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\overline{\epsilon}_{i_3j.}\overline{\epsilon}_{i_3j_1.}\overline{\epsilon}_{i_4j_2.}\overline{\epsilon}_{i_4j_3.}(\widetilde{\epsilon}_{i_5..} - \widetilde{\epsilon}_{...})] \\ = & -2\left(\frac{b}{a^{\frac{3}{2}}}\right)\sum_{i_1\neq i_2}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2\neq j_3}^{b}\alpha_{i_1}\{2E[\overline{\epsilon}_{i_1..}\overline{\epsilon}_{i_1j.}\overline{\epsilon}_{i_1j_1.}\overline{\epsilon}_{i_1j_2.}\overline{\epsilon}_{i_1j_3.}]E[\overline{\epsilon}_{i_2..}\widetilde{\epsilon}_{i_2..}] \\ & + 4E[\widetilde{\epsilon}_{i_1..}\overline{\epsilon}_{i_1j.}\overline{\epsilon}_{i_1j_1.}]E[\widetilde{\epsilon}_{i_2..}\overline{\epsilon}_{i_2j_2.}\overline{\epsilon}_{i_2j_3.}\widetilde{\epsilon}_{i_2..}]\} \\ = & O(a^{\frac{1}{2}}b^3), \end{aligned}$$

$$\begin{aligned} & E\{\Lambda_2\Pi_a(\Omega_a - E(\Omega_a))\} \\ = & E\left[\left(\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\frac{\epsilon_{ijk}\epsilon_{ijk_1}}{n_{ij}(n_{ij}-1)}\right)^2 a^{\frac{1}{2}}\left(-\frac{b^{1/2}}{a^{3/2}}\sum_{i_1\neq i_2}^{a}\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\right)\left(2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i=1}^{a}[(\widetilde{\epsilon}_{i..} - \widetilde{\epsilon}_{...})\alpha_i]\right)\right] \\ = & -2\left(\frac{b}{a^{\frac{3}{2}}}\right)\sum_{i_1\neq i_2}^{a}\sum_{i_3,i_4,i_5}^{a}\sum_{j,j_1}^{b}\sum_{k\neq k_1}^{n_{i_3j}}\sum_{k_2\neq k_3}^{n_{i_4j_1}}\alpha_{i_5}\frac{E[\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\epsilon_{i_3jk}\epsilon_{i_3jk_1}\epsilon_{i_4j_1k_2}\epsilon_{i_4j_1k_3}(\widetilde{\epsilon}_{i_5..} - \widetilde{\epsilon}_{...})]}{n_{i_3j}(n_{i_3j}-1)n_{i_4j_1}(n_{i_4j_1}-1)} \\ = & -2\left(\frac{b}{a^{\frac{3}{2}}}\right)\sum_{i_1\neq i_2}^{a}\sum_{j=1}^{b}\sum_{j_1=1}^{b}\sum_{k\neq k_1}^{n_{i_1j}}\sum_{k_2\neq k_3}^{n_{i_1j}}\alpha_{i_2}\left\{2\frac{E[\widetilde{\epsilon}_{i_1..}\epsilon_{i_1jk}\epsilon_{i_1jk_1}\epsilon_{i_1jk_2}\epsilon_{i_1jk_3}]E[\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{i_2..}]}{n_{i_1j}^2(n_{i_1j}-1)^2}\right. \\ & \left. + 4\frac{E[\widetilde{\epsilon}_{i_1..}\epsilon_{i_1jk}\epsilon_{i_1jk_1}]E[\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{i_2..}\epsilon_{i_2j_1k}\epsilon_{i_2j_1k_1}]}{n_{i_1j}(n_{i_1j}-1)n_{i_2j_1}(n_{i_2j_1}-1)}\right\} \\ = & O(a^{\frac{1}{2}}b^3), \end{aligned}$$

and

$$
\begin{aligned}
& E\{\Lambda_3\Pi_a(\Omega_a - E(\Omega_a))\} \\
= \;& E\left[\left(\sum_{i,i_1=1}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2=1}^{b}\sum_{k\neq k_1}^{n_{i_1j_2}} \frac{\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.}\epsilon_{i_1j_2k}\epsilon_{i_1j_2k_1}}{n_{i_1j_2}(n_{i_1j_2}-1)}\right) a^{\frac{1}{2}} \left(-\frac{b^{1/2}}{a^{3/2}}\sum_{i_2\neq i_3}^{a}\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{i_3..}\right)\right. \\
& \left.\left(2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i_4=1}^{a}[(\widetilde{\epsilon}_{i_4..}-\widetilde{\epsilon}_{...})\alpha_{i_4}]\right)\right] \\
= \;& -2\left(\frac{b}{a^{\frac{3}{2}}}\right)\sum_{i,i_1=1}^{a}\sum_{i_2\neq i_3}^{a}\sum_{i_4=1}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2=1}^{b}\sum_{k\neq k_1}^{n_{i_1j_2}}\alpha_{i_4}\frac{E[\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.}\epsilon_{i_1j_2k}\epsilon_{i_1j_2k_1}\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{i_3..}(\widetilde{\epsilon}_{i_4..}-\widetilde{\epsilon}_{...})]}{n_{i_1j_2}(n_{i_1j_2}-1)} \\
= \;& O(a^{\frac{1}{2}}b^3).
\end{aligned}
$$

Combining the three terms $E\{\Lambda_1\Pi_a(\Omega_a - E(\Omega_a))\}$, $E\{\Lambda_2\Pi_a(\Omega_a - E(\Omega_a))\}$ and $E\{\Lambda_3\Pi_a(\Omega_a - E(\Omega_a))\}$ we get

$$
E[\widetilde{T_A}^2\Pi_a(\Omega_a - E(\Omega_a))] = \frac{(a-1)^2}{(ab)^3}O(a^{\frac{1}{2}}b^3) = O(a^{-\frac{1}{2}}).
$$

Thus the second term in (1.7.45) is equal to

$$
E[(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a)^3(\Omega_a - E(\Omega_a))] = O(a^{-1}). \tag{1.7.46}
$$

We now proceed to obtain the result for the third term $E[(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a)^2(\Omega_a - E(\Omega_a))^2]$ in (1.7.44) as follows:

$$
\begin{aligned}
& E[(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a)^2(\Omega_a - E(\Omega_a))^2] \\
= \;& E[(\widetilde{T_A}^2 + 2a^{-\frac{1}{2}}\widetilde{T_A}\Pi_a + a^{-1}\Pi_a^2)(\Omega_a - E(\Omega_a))^2] \\
= \;& E[\widetilde{T_A}^2(\Omega_a - E(\Omega_a))^2] + 2a^{-\frac{1}{2}}E[\widetilde{T_A}\Pi_a(\Omega_a - E(\Omega_a))^2] + \; a^{-1}E[\Pi_a^2(\Omega_a - E(\Omega_a))^2] \\
= \;& E[\widetilde{T_A}^2(\Omega_a - E(\Omega_a))^2] + 2a^{-\frac{1}{2}}E[\widetilde{T_A}\Pi_a(\Omega_a - E(\Omega_a))^2] + O(a^{-1}),
\end{aligned}
$$

where the last equality is due to the fact that $E[\Pi_a^2(\Omega_a - E(\Omega_a))^2]$ is at most O(1). Using the result in (1.7.12), $E[\widetilde{T_A}^2(\Omega_a - E(\Omega_a))^2]$ is expressed as

$$
\begin{aligned}
& E[\widetilde{T_A}^2(\Omega_a - E(\Omega_a))^2] \\
= \;& \frac{(a-1)^2}{(ab)^3}[E\{\Lambda_1(\Omega_a - E(\Omega_a))^2\} + E\{\Lambda_2(\Omega_a - E(\Omega_a))^2\} + 2E\{\Lambda_3(\Omega_a - E(\Omega_a))^2\}],
\end{aligned}
$$

where Λ_1, Λ_2 and Λ_3 are given in (1.7.13), (1.7.14) and (1.7.15), respectively. We compute the three terms in the above equation one by one.

$$
\begin{aligned}
& E\{\Lambda_1(\Omega_a - E(\Omega_a))^2\} \\
= \;& E\left[\left(\sum_{i=1}^{a}\sum_{j\neq j_1}^{b} \bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\right)^2 \left(2\left(\frac{b}{a}\right)^{\frac{1}{2}} \sum_{i=1}^{a} [(\widetilde{\epsilon}_{i..} - \widetilde{\epsilon}_{...})\alpha_i]\right)^2\right] \\
= \;& 4\left(\frac{b}{a}\right) \sum_{i,i_1,i_2,i_3}^{a} \sum_{j\neq j_1}^{b} \sum_{j_2\neq j_3}^{b} \alpha_{i_2}\alpha_{i_3} E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\bar{\epsilon}_{i_1j_2.}\bar{\epsilon}_{i_1j_3.}(\widetilde{\epsilon}_{i_2..} - \widetilde{\epsilon}_{...})(\widetilde{\epsilon}_{i_3..} - \widetilde{\epsilon}_{...})] \\
= \;& 4\left(\frac{b}{a}\right) \sum_{i,i_1,i_2,i_3}^{a} \sum_{j\neq j_1}^{b} \sum_{j_2\neq j_3}^{b} \alpha_{i_2}\alpha_{i_3} E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\bar{\epsilon}_{i_1j_2.}\bar{\epsilon}_{i_1j_3.}(\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{i_3..} - \widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{...} - \widetilde{\epsilon}_{i_3..}\widetilde{\epsilon}_{...} + \widetilde{\epsilon}^2_{...})] \\
= \;& 8\left(\frac{b}{a}\right) \sum_{i,i_1,i_2,i_3}^{a} \sum_{j\neq j_1}^{b} \alpha_{i_2}\alpha_{i_3} \left[E(\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\bar{\epsilon}_{i_1j.}\bar{\epsilon}_{i_1j_1.}\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{i_3..}) - 2E(\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\bar{\epsilon}_{i_1j.}\bar{\epsilon}_{i_1j_1.}\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{...})\right. \\
& \left. + E(\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\bar{\epsilon}_{i_1j.}\bar{\epsilon}_{i_1j_1.}\widetilde{\epsilon}^2_{...})\right]. \qquad (1.7.47)
\end{aligned}
$$

The first term in (1.7.47) satisfies

$$
\begin{aligned}
& 8\left(\frac{b}{a}\right) \sum_{i,i_1,i_2,i_3}^{a} \sum_{j\neq j_1}^{b} \alpha_{i_2}\alpha_{i_3} E(\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\bar{\epsilon}_{i_1j.}\bar{\epsilon}_{i_1j_1.}\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{i_3..}) \\
= \;& 8\left(\frac{b}{a}\right) \sum_{i=1}^{a}\sum_{j\neq j_1}^{b} \alpha_i^2 E(\bar{\epsilon}^2_{ij.}\bar{\epsilon}^2_{ij_1.}\widetilde{\epsilon}^2_{i..}) + 8\left(\frac{b}{a}\right) \sum_{i\neq i_1}^{a}\sum_{j\neq j_1}^{b} [\alpha_{i_1}^2 E(\bar{\epsilon}^2_{ij.})E(\bar{\epsilon}^2_{ij_1.})E(\widetilde{\epsilon}^2_{i_1..}) \\
& + 2\alpha_i\alpha_{i_1} E(\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\widetilde{\epsilon}_{i..})E(\bar{\epsilon}_{i_1j.}\bar{\epsilon}_{i_1j_1.}\widetilde{\epsilon}_{i_1..})] \\
= \;& 8\left(\frac{b}{a}\right) \sum_{i=1}^{a} \alpha_i^2 \left(\frac{1}{b^2}\right) \sum_{j\neq j_1}^{b} \left[2E(\bar{\epsilon}^3_{ij.}\bar{\epsilon}^3_{ij_1.}) + \sum_{j_3=1}^{b} E(\bar{\epsilon}^2_{ij.}\bar{\epsilon}^2_{ij_1.}\bar{\epsilon}^2_{ij_3.}) + 2E(\bar{\epsilon}^4_{ij.}\bar{\epsilon}^2_{ij_1.})\right] \\
& + 8\left(\frac{b}{a}\right) \sum_{i\neq i_1}^{a}\sum_{j\neq j_1}^{b} \alpha_{i_1}^2 E(\bar{\epsilon}^2_{ij.})E(\bar{\epsilon}^2_{ij_1.})E(\widetilde{\epsilon}^2_{i_1..}) \\
= \;& 8\left(\frac{b}{a}\right) \sum_{i=1}^{a} \alpha_i^2 M_5 + 8\left(\frac{b}{a}\right) \sum_{i\neq i_1}^{a}\sum_{j\neq j_1}^{b} \alpha_{i_1}^2 E(\bar{\epsilon}^2_{ij.})E(\bar{\epsilon}^2_{ij_1.})E(\widetilde{\epsilon}^2_{i_1..}) \\
= \;& O\left(\frac{\sqrt{b}}{\sqrt{a}}\right) + 8\left(\frac{1}{ab}\right) \sum_{i\neq i_1}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2=1}^{b} \alpha_{i_1}^2 \frac{\sigma^2_{ij}}{n_{ij}}\frac{\sigma^2_{ij_1}}{n_{ij_1}}\frac{\sigma^2_{i_1j_2}}{n_{i_1j_2}},
\end{aligned}
$$

where the last equality is due to condition (1.3.17), and $E(\epsilon^4_{ijk}) < \infty$ which assures that there exists some finite M_5 such that,

$$
\begin{aligned}
&\left(\frac{1}{b^2}\right)\sum_{j\neq j_1}^{b}\left[E(\bar{\epsilon}^3_{ij.}\bar{\epsilon}^3_{ij_1.})+\sum_{j_3=1}^{b}E(\bar{\epsilon}^2_{ij.}\bar{\epsilon}^2_{ij_1.}\bar{\epsilon}^2_{ij_3.})\right]\\
=\;&\frac{1}{b^2}\sum_{j\neq j_1}\frac{\sigma^6_{ij}}{n^4_{ij}}[\gamma^2_{ij}+2\tau_{ij}+2n_{ij}(n_{ij-1})]+\frac{1}{b^3}\sum_{j\neq j_1\neq j_2}\frac{\sigma^2_{ij}}{n_{ij}}\frac{\sigma^2_{ij_1}}{n_{ij_1}}\frac{\sigma^2_{ij_2}}{n_{ij_2}}\leq M_5,\ \forall i.
\end{aligned}
$$

The second term in (1.7.47) satisfies

$$
-16\left(\frac{b}{a}\right)\sum_{i,i_1,i_2,i_3}^{a}\sum_{j\neq j_1}^{b}\alpha_{i_2}\alpha_{i_3}E(\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\bar{\epsilon}_{i_1j.}\bar{\epsilon}_{i_1j_1.}\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{...})=0,
$$

since $\sum_{i_3=1}^{a}\alpha_{i_3}=0$. Similarly, the third term in (1.7.47) also satisfies

$$
8\left(\frac{b}{a}\right)\sum_{i,i_1,i_2,i_3}^{a}\sum_{j\neq j_1}^{b}\alpha_{i_2}\alpha_{i_3}E(\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\bar{\epsilon}_{i_1j.}\bar{\epsilon}_{i_1j_1.}\widetilde{\epsilon}^2_{...})=0,
$$

since $\sum_{i_2=1}^{a}\alpha_{i_2}=0$ and $\sum_{i_3=1}^{a}\alpha_{i_3}=0$. Putting the terms together gives (1.7.47) as

$$
E\{\Lambda_1(\Omega_a-E(\Omega_a))^2\}=8\left(\frac{1}{ab}\right)\sum_{i\neq i_1}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2=1}^{b}\alpha^2_{i_1}\frac{\sigma^2_{ij}}{n_{ij}}\frac{\sigma^2_{ij_1}}{n_{ij_1}}\frac{\sigma^2_{i_1j_2}}{n_{i_1j_2}}+O\left(\frac{\sqrt{b}}{\sqrt{a}}\right).
$$

Next

$$
\begin{aligned}
&E\{\Lambda_2(\Omega_a-E(\Omega_a))^2\}\\
=\;&E\left[\left(\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\frac{\epsilon_{ijk}\epsilon_{ijk_1}}{n_{ij}(n_{ij}-1)}\right)^2\left(2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i=1}^{a}[(\widetilde{\epsilon}_{i..}-\widetilde{\epsilon}_{...})\alpha_i]\right)^2\right]\\
=\;&4\left(\frac{b}{a}\right)\sum_{i,i_1,i_2,i_3}^{a}\sum_{j,j_1}^{b}\sum_{k\neq k_1}^{n_{ij}}\sum_{k_2\neq k_3}^{n_{i_1j_1}}\alpha_{i_2}\alpha_{i_3}\frac{E[\epsilon_{ijk}\epsilon_{ijk_1}\epsilon_{i_1j_1k_2}\epsilon_{i_1j_1k_3}(\widetilde{\epsilon}_{i_2..}-\widetilde{\epsilon}_{...})(\widetilde{\epsilon}_{i_3..}-\widetilde{\epsilon}_{...})]}{n_{ij}(n_{ij}-1)n_{i_1j_1}(n_{i_1j_1}-1)}\\
=\;&4\left(\frac{b}{a}\right)\sum_{i,i_1,i_2,i_3}^{a}\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\sum_{k_2\neq k_3}^{n_{i_1j}}\alpha_{i_2}\alpha_{i_3}\frac{E[\epsilon_{ijk}\epsilon_{ijk_1}\epsilon_{i_1jk_2}\epsilon_{i_1jk_3}(\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{i_3..}-\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{...}-\widetilde{\epsilon}_{i_3..}\widetilde{\epsilon}_{...}+\widetilde{\epsilon}^2_{...})]}{n_{ij}(n_{ij}-1)n_{i_1j}(n_{i_1j}-1)}\\
=\;&4\left(\frac{b}{a}\right)\sum_{i,i_1,i_2,i_3}^{a}\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\sum_{k_2\neq k_3}^{n_{i_1j}}\alpha_{i_2}\alpha_{i_3}\left[E(\epsilon_{ijk}\epsilon_{ijk_1}\epsilon_{i_1jk_2}\epsilon_{i_1jk_3}\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{i_3..})-2E(\epsilon_{ijk}\epsilon_{ijk_1}\epsilon_{i_1jk_2}\epsilon_{i_1jk_3}\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{...})\right.\\
&\left.+E(\epsilon_{ijk}\epsilon_{ijk_1}\epsilon_{i_1jk_2}\epsilon_{i_1jk_3}\widetilde{\epsilon}^2_{...})\right]\frac{1}{n_{ij}(n_{ij}-1)n_{i_1j}(n_{i_1j}-1)}.
\end{aligned}
\tag{1.7.48}
$$

The first term in (1.7.48) satisfies

$$
\begin{aligned}
& 4\left(\frac{b}{a}\right) \sum_{i,i_1,i_2,i_3}^{a} \sum_{j=1}^{b} \sum_{k\neq k_1}^{n_{ij}} \sum_{k_2\neq k_3}^{n_{i_1 j}} \alpha_{i_2}\alpha_{i_3} \frac{E(\epsilon_{ijk}\epsilon_{ijk_1}\epsilon_{i_1jk_2}\epsilon_{i_1jk_3}\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{i_3..})}{n_{ij}(n_{ij}-1)n_{i_1j}(n_{i_1j}-1)} \\
= \;& 8\left(\frac{b}{a}\right) \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k\neq k_1}^{n_{ij}} \alpha_i^2 \frac{E(\epsilon_{ijk}^2\epsilon_{ijk_1}^2\widetilde{\epsilon}_{i..}^2)}{n_{ij}^2(n_{ij}-1)^2} + 8\left(\frac{b}{a}\right) \sum_{i\neq i_1}^{a} \sum_{j=1}^{b} \sum_{k\neq k_1}^{n_{ij}} \left[\frac{\alpha_{i_1}^2 E(\epsilon_{ijk}^2)E(\epsilon_{ijk_1}^2)E(\widetilde{\epsilon}_{i_1..}^2)}{n_{ij}^2(n_{ij}-1)^2}\right. \\
& \left. + 2\alpha_i\alpha_{i_1} \frac{E(\epsilon_{ijk}\epsilon_{ijk_1}\widetilde{\epsilon}_{i..})E(\epsilon_{i_1jk}\epsilon_{i_1jk_1}\widetilde{\epsilon}_{i_1..})}{n_{ij}(n_{ij}-1)n_{i_1j}(n_{i_1j}-1)}\right] \\
= \;& 8\left(\frac{b}{a}\right) \sum_{i=1}^{a} \alpha_i^2 \left(\frac{1}{b^2}\right) \sum_{j=1}^{b} \sum_{k\neq k_1}^{n_{ij}} \left[\frac{2}{n_{ij}^2} E(\epsilon_{ijk}^3\epsilon_{ijk_1}^3) + \sum_{k_2=1}^{n_{ij}} \frac{1}{n_{ij}^2} E(\epsilon_{ijk}^2\epsilon_{ijk_1}^2\epsilon_{ijk_2}^2)\right] \frac{1}{n_{ij}^2(n_{ij}-1)^2} \\
& + 8\left(\frac{b}{a}\right) \sum_{i\neq i_1}^{a} \sum_{j=1}^{b} \sum_{k\neq k_1}^{n_{ij}} \alpha_{i_1}^2 \frac{E(\epsilon_{ijk}^2)E(\epsilon_{ijk_1}^2)E(\widetilde{\epsilon}_{i_1..}^2)}{n_{ij}^2(n_{ij}-1)^2} \\
= \;& \left[8\left(\frac{1}{ab}\right) \sum_{i=1}^{a} \alpha_i^2 M_{5i} + 8\left(\frac{b}{a}\right) \sum_{i\neq i_1}^{a} \sum_{j=1}^{b} \sum_{k\neq k_1}^{n_{ij}} \alpha_{i_1}^2 E(\epsilon_{ijk}^2)E(\epsilon_{ijk_1}^2)E(\widetilde{\epsilon}_{i_1..}^2)\right] \frac{1}{n_{ij}^2(n_{ij}-1)^2} \\
= \;& \frac{8}{ab} \sum_{i\neq i_1}^{a} \sum_{j=1}^{b} \sum_{j_1=1}^{b} \alpha_{i_1}^2 \frac{\sigma_{ij}^4}{n_{ij}(n_{ij}-1)} \frac{\sigma_{i_1j_1}^2}{n_{i_1j_1}} + \frac{8}{a} \sum_{i=1}^{a} \alpha_i^2 M_{5i} \\
= \;& \frac{8}{ab} \sum_{i\neq i_1}^{a} \sum_{j=1}^{b} \sum_{j_1=1}^{b} \alpha_{i_1}^2 \frac{\sigma_{ij}^4}{n_{ij}(n_{ij}-1)} \frac{\sigma_{i_1j_1}^2}{n_{i_1j_1}} + O\left(\frac{1}{\sqrt{ab}}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
M_{5i} &= \frac{1}{b} \sum_{j=1}^{b} \sum_{k\neq k_1}^{n_{ij}} \left[\frac{1}{n_{ij}^2} E(\epsilon_{ijk}^3\epsilon_{ijk_1}^3) + \sum_{k_2=1}^{n_{ij}} \frac{1}{n_{ij}^2} E(\epsilon_{ijk}^2\epsilon_{ijk_1}^2\epsilon_{ijk_2}^2)\right] \frac{1}{n_{ij}^2(n_{ij}-1)^2} \\
&= \frac{1}{b} \sum_{j=1}^{b} \left(\frac{1}{n_{ij}^3(n_{ij}-1)}\right) \sigma_{ij}^6 [\gamma_{ij}^2 + 2\tau_{ij} + n_{ij} - 2], \qquad (1.7.49)
\end{aligned}
$$

is bounded with the assumption of $E(\epsilon_{ijk}^4) < \infty$. The second term in (1.7.48) satisfies

$$
-8\left(\frac{b}{a}\right) \sum_{i,i_1,i_2,i_3}^{a} \sum_{j=1}^{b} \sum_{k\neq k_1}^{n_{ij}} \sum_{k_2\neq k_3}^{n_{i_1 j}} \alpha_{i_2}\alpha_{i_3} \frac{E(\epsilon_{ijk}\epsilon_{ijk_1}\epsilon_{i_1jk_2}\epsilon_{i_1jk_3}\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{...})}{n_{ij}(n_{ij}-1)n_{i_1j}(n_{i_1j}-1)} = 0
$$

since $\sum_{i_3=1}^{a} \alpha_{i_3} = 0$. Similarly, the remaining term in (1.7.48) satisfies

$$
4\left(\frac{b}{a}\right) \sum_{i,i_1,i_2,i_3}^{a} \sum_{j=1}^{b} \sum_{k\neq k_1}^{n_{ij}} \sum_{k_2\neq k_3}^{n_{i_1 j}} \alpha_{i_2}\alpha_{i_3} \frac{E(\epsilon_{ijk}\epsilon_{ijk_1}\epsilon_{i_1jk_2}\epsilon_{i_1jk_3}\widetilde{\epsilon}_{...}^2)}{n_{ij}(n_{ij}-1)n_{i_1j}(n_{i_1j}-1)} = 0
$$

since $\sum_{i_2=1}^{a} \alpha_{i_2} = 0$ and $\sum_{i_3=1}^{a} \alpha_{i_3} = 0$. Putting the terms together into (1.7.48) gives

$$E\{\Lambda_2(\Omega_a - E(\Omega_a))^2\} = 8\left(\frac{1}{ab}\right)\sum_{i\neq i_1}^{a}\sum_{j=1}^{b}\sum_{j_1=1}^{b}\alpha_{i_1}^2\frac{\sigma_{ij}^4}{n_{ij}(n_{ij}-1)}\frac{\sigma_{i_1j_1}^2}{n_{i_1j_1}} + O\left(\frac{1}{\sqrt{ab}}\right).$$

Now

$$\begin{aligned}
& E\{\Lambda_3(\Omega_a - E(\Omega_a))^2\} \\
= \;& E\left[\left(\sum_{i=1}^{a}\sum_{i_1=1}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2=1}^{b}\sum_{k\neq k_1}^{n_{i_1j_2}}\frac{\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.}\epsilon_{i_1j_2k}\epsilon_{i_1j_2k_1}}{n_{i_1j_2}(n_{i_1j_2}-1)}\right)\left(2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i=1}^{a}[(\widetilde{\epsilon}_{i..} - \widetilde{\epsilon}_{...})\alpha_i]\right)^2\right] \\
= \;& 4\left(\frac{b}{a}\right)\sum_{i,i_1,i_2,i_3}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2=1}^{b}\sum_{k\neq k_1}^{n_{i_ij_2}}\alpha_{i_2}\alpha_{i_3}\frac{E[\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.}\epsilon_{i_1j_2k}\epsilon_{i_1j_2k_1}(\widetilde{\epsilon}_{i_2..} - \widetilde{\epsilon}_{...})(\widetilde{\epsilon}_{i_3..} - \widetilde{\epsilon}_{...})]}{n_{i_1j_2}(n_{i_1j_2}-1)} \\
= \;& 8\left(\frac{b}{a}\right)\sum_{i,i_1,i_2,i_3}^{a}\sum_{j\neq j_1}^{b}\sum_{k\neq k_1}^{n_{i_ij}}\alpha_{i_2}\alpha_{i_3}\frac{E[\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.}\epsilon_{i_1jk}\epsilon_{i_1jk_1}(\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{i_3..} - \widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{...} - \widetilde{\epsilon}_{i_3..}\widetilde{\epsilon}_{...} + \widetilde{\epsilon}^2_{...})]}{n_{i_1j}(n_{i_1j}-1)} \\
= \;& 8\left(\frac{b}{a}\right)\sum_{i,i_1,i_2,i_3}^{a}\sum_{j\neq j_1}^{b}\sum_{k\neq k_1}^{n_{i_ij}}\alpha_{i_2}\alpha_{i_3}[E(\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.}\epsilon_{i_1jk}\epsilon_{i_1jk_1}\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{i_3..}) - 2E(\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.}\epsilon_{i_1jk}\epsilon_{i_1jk_1}\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{...}) \\
& + E(\overline{\epsilon}_{ij.}\overline{\epsilon}_{ij_1.}\epsilon_{i_1jk}\epsilon_{i_1jk_1}\widetilde{\epsilon}^2_{...})]\frac{1}{n_{i_1j}(n_{i_1j}-1)}.
\end{aligned}$$

The summation of the second and third terms in the above equations is zero because

$\sum_{i_3=1}^{a}\alpha_{i_3}=0$. The first term in the above equation satisfies

$$
\begin{aligned}
& 8\left(\frac{b}{a}\right)\sum_{i,i_1,i_2,i_3}^{a}\sum_{j\neq j_1}^{b}\sum_{k\neq k_1}^{n_{i_1j}}\alpha_{i_2}\alpha_{i_3}\frac{E(\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\epsilon_{i_1jk}\epsilon_{i_1jk_1}\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{i_3..})}{n_{i_1j}(n_{i_1j}-1)} \\
= \; & 8\left(\frac{b}{a}\right)\sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\sum_{k\neq k_1}^{n_{ij}}\alpha_i^2\frac{E(\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\epsilon_{ijk}\epsilon_{ijk_1}\widetilde{\epsilon}^2_{i..})}{n_{i_1j}(n_{i_1j}-1)}+8\left(\frac{b}{a}\right)\sum_{i\neq i_1}^{a}\sum_{j\neq j_1}^{b}\sum_{k\neq k_1}^{n_{i_1j}}\Bigg[\alpha_{i_1}^2\frac{E(\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\epsilon_{ijk}\epsilon_{ijk_1})E(\widetilde{\epsilon}^2_{i_1..})}{n_{i_1j}(n_{i_1j}-1)} \\
& +2\alpha_i\alpha_{i_1}\frac{E(\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\widetilde{\epsilon}_{i..})E(\epsilon_{i_1jk}\epsilon_{i_1jk_1}\widetilde{\epsilon}_{i_1..})}{n_{i_1j}(n_{i_1j}-1)}\Bigg] \\
= \; & 16\left(\frac{b}{a}\right)\sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\sum_{k\neq k_1}^{n_{ij}}\alpha_i^2\left(\frac{1}{b^2}\right)\frac{E(\bar{\epsilon}^2_{ij.}\epsilon_{ijk}\epsilon_{ijk_1})E(\bar{\epsilon}^2_{ij_1.})}{n_{ij}(n_{ij}-1)} \\
= \; & 32\left(\frac{b}{a}\right)\sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\sum_{k\neq k_1}^{n_{ij}}\alpha_i^2\left(\frac{1}{b^2}\right)\left(\frac{1}{n_{ij}^2}\right)\frac{E(\epsilon^2_{ijk})E(\epsilon^2_{ijk_1})E(\bar{\epsilon}^2_{ij_1.})}{n_{ij}(n_{ij}-1)} \\
= \; & \frac{32}{ab}\sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\alpha_i^2\frac{\sigma_{ij}^4}{n_{ij}^2}\frac{\sigma_{ij_1}^2}{n_{ij_1}} \\
\leq \; & \frac{32}{ab}\sum_{i=1}^{a}\alpha_i^2\left(\sum_{j=1}^{b}\frac{\sigma_{ij}^4}{n_{ij}^2}\right)\left(\sum_{j_1=1}^{b}\frac{\sigma_{ij_1}^2}{n_{ij_1}}\right) \\
= \; & O\left(\frac{\sqrt{b}}{\sqrt{a}}\right),
\end{aligned}
$$

where the last equality uses condition (1.3.17). Putting the three terms together gives

$$E\{\Lambda_3(\Omega_a-E(\Omega_a))^2\}=O\left(\frac{\sqrt{b}}{\sqrt{a}}\right).$$

Therefore combining the results for the terms $E\{\Lambda_1(\Omega_a-E(\Omega_a))^2\}, E\{\Lambda_2(\Omega_a-E(\Omega_a))^2\}$ and $E\{\Lambda_3(\Omega_a-E(\Omega_a))^2\}$, we get

$$
\begin{aligned}
E[\widetilde{T_A}^2(\Omega_a-E(\Omega_a))^2] = \; & \frac{8(a-1)^2}{(ab)^4}\sum_{i\neq i_1}^{a}\left[\sum_{j\neq j_1}^{b}\sum_{j_2=1}^{b}\alpha_{i_1}^2\frac{\sigma_{ij}^2}{n_{ij}}\frac{\sigma_{ij_2}^2}{n_{ij_2}}\frac{\sigma_{i_1j_1}^2}{n_{i_1j_1}}+\sum_{j=1}^{b}\sum_{j_1=1}^{b}\alpha_{i_1}^2\frac{\sigma_{ij}^4}{n_{ij}(n_{ij}-1)}\frac{\sigma_{i_1j_1}^2}{n_{i_1j_1}}\right] \\
& +O(a^{-\frac{3}{2}}b^{-\frac{5}{2}}).
\end{aligned}
$$

Next using (1.3.7), (1.3.16) and (1.7.27) we compute $E[\widetilde{T_A}\Pi_a(\Omega_a - E(\Omega_a))^2]$ as follows:

$$
\begin{aligned}
& E[\widetilde{T_A}\Pi_a(\Omega_a - E(\Omega_a))^2] \\
= \; & E\left[\sum_{i=1}^{a} \frac{a-1}{(ab)^{3/2}} \left\{ \sum_{j \neq j_1}^{b} \bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.} + \sum_{j=1}^{b}\sum_{k \neq k_1}^{n_{ij}} \frac{\epsilon_{ijk}\epsilon_{ijk_1}}{n_{ij}(n_{ij}-1)} \right\} \right. \\
& \left. a^{\frac{1}{2}} \left(-\frac{b^{1/2}}{a^{3/2}} \sum_{i_1 \neq i_2}^{a} \widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..} \right) \left(2\left(\frac{b}{a}\right)^{\frac{1}{2}} \sum_{i_3=1}^{a} [(\widetilde{\epsilon}_{i_3..} - \widetilde{\epsilon}_{...})\alpha_{i_3}] \right)^2 \right] \\
= \; & -\frac{4(a-1)}{a^{7/2}} \sum_{i_1 \neq i_2}^{a} \sum_{i_3, i_4, i_5}^{a} \left[\sum_{j \neq j_1}^{b} \alpha_{i_4}\alpha_{i_5} E[\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\bar{\epsilon}_{i_3j.}\bar{\epsilon}_{i_3j_1.}(\widetilde{\epsilon}_{i_4..} - \widetilde{\epsilon}_{...})(\widetilde{\epsilon}_{i_5..} - \widetilde{\epsilon}_{...})] \right. \\
& \left. + \sum_{j=1}^{b}\sum_{k \neq k_1}^{n_{i_3j}} \alpha_{i_4}\alpha_{i_5} \frac{E[\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\epsilon_{i_3jk}\epsilon_{i_3jk_1}(\widetilde{\epsilon}_{i_4..} - \widetilde{\epsilon}_{...})(\widetilde{\epsilon}_{i_5..} - \widetilde{\epsilon}_{...})]}{n_{i_3j}(n_{i_3j}-1)} \right] \\
= \; & -\frac{4(a-1)}{a^{7/2}} \sum_{i_1 \neq i_2}^{a} \sum_{i_3, i_4, i_5}^{a} \left[\sum_{j \neq j_1}^{b} \alpha_{i_4}\alpha_{i_5} E[\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\bar{\epsilon}_{i_3j.}\bar{\epsilon}_{i_3j_1.}(\widetilde{\epsilon}_{i_4..}\widetilde{\epsilon}_{i_5..} - 2\widetilde{\epsilon}_{i_4..}\widetilde{\epsilon}_{...} + \widetilde{\epsilon}^2_{...})] \right. \\
& \left. + \sum_{j=1}^{b}\sum_{k \neq k_1}^{n_{i_3j}} \alpha_{i_4}\alpha_{i_5} \frac{E[\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\epsilon_{i_3jk}\epsilon_{i_3jk_1}(\widetilde{\epsilon}_{i_4..}\widetilde{\epsilon}_{i_5..} - 2\widetilde{\epsilon}_{i_4..}\widetilde{\epsilon}_{...} + \widetilde{\epsilon}^2_{...})]}{n_{i_3j}(n_{i_3j}-1)} \right] \\
= \; & -\frac{4(a-1)}{a^{7/2}} \sum_{i_1 \neq i_2}^{a} \sum_{i_3, i_4, i_5}^{a} \alpha_{i_4}\alpha_{i_5} \left[\sum_{j \neq j_1}^{b} E[\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\bar{\epsilon}_{i_3j.}\bar{\epsilon}_{i_3j_1.}\widetilde{\epsilon}_{i_4..}\widetilde{\epsilon}_{i_5..}] + \sum_{j=1}^{b}\sum_{k \neq k_1}^{n_{i_3j}} \frac{E[\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\epsilon_{i_3jk}\epsilon_{i_3jk_1}\widetilde{\epsilon}_{i_4..}\widetilde{\epsilon}_{i_5..}]}{n_{i_3j}(n_{i_3j}-1)} \right] \\
= \; & -\frac{16(a-1)}{a^{7/2}} \sum_{i_1 \neq i_2}^{a} \alpha_{i_1}\alpha_{i_2} \left[\sum_{j \neq j_1}^{b} E[\widetilde{\epsilon}_{i_1..}\bar{\epsilon}_{i_1j.}\bar{\epsilon}_{i_1j_1.}\widetilde{\epsilon}_{i_1..}] E[\widetilde{\epsilon}^2_{i_2..}] + \sum_{j=1}^{b}\sum_{k \neq k_1}^{n_{i_1j}} \frac{E[\widetilde{\epsilon}_{i_1..}\epsilon_{i_1jk}\epsilon_{i_1jk_1}\widetilde{\epsilon}_{i_1..}] E[\widetilde{\epsilon}^2_{i_2..}]}{n_{i_1j}(n_{i_1j}-1)} \right]. \quad (1.7.50)
\end{aligned}
$$

The first term in (1.7.50) satisfies

$$\begin{aligned}
& -\frac{16(a-1)}{a^{7/2}}\sum_{i_1\neq i_2}^{a}\alpha_{i_1}\alpha_{i_2}\sum_{j\neq j_1}^{b}E[\widetilde{\epsilon}_{i_1..}\overline{\epsilon}_{i_1j.}\overline{\epsilon}_{i_1j_1.}\widetilde{\epsilon}_{i_1..}]E[\widetilde{\epsilon}^2_{i_2..}] \\
= & -\frac{16(a-1)}{a^{7/2}}\sum_{i_1\neq i_2}^{a}\alpha_{i_1}\alpha_{i_2}\sum_{j\neq j_1}^{b}\left[\frac{2}{b^2}E(\overline{\epsilon}^2_{i_1j.})E(\overline{\epsilon}^2_{i_1j_1.})\right]\left[\frac{1}{b^2}\sum_{j_2=1}^{b}E(\overline{\epsilon}^2_{i_2j_2.})\right] \\
= & -\frac{32(a-1)}{a^{7/2}}\sum_{i_1\neq i_2}^{a}\alpha_{i_1}\alpha_{i_2}\sum_{j\neq j_1}^{b}\left[\frac{1}{b^2}\frac{\sigma^2_{i_1j}}{n_{i_1j}}\frac{\sigma^2_{i_1j_1}}{n_{i_1j_1}}\right]\left[\frac{1}{b^2}\sum_{j_2=1}^{b}\frac{\sigma^2_{i_2j_2}}{n_{i_2j_2}}\right] \\
\leq & \ \frac{32(a-1)}{a^{7/2}}\frac{1}{b}\sum_{i_1\neq i_2}^{a}|\alpha_{i_1}\alpha_{i_2}|M_7 \\
\leq & \ \frac{16(a-1)}{a^{7/2}}\frac{1}{b}\sum_{i_1\neq i_2}^{a}(\alpha^2_{i_1}+\alpha^2_{i_2})M_7 \\
= & \ O\left(\frac{1}{ab^{\frac{3}{2}}}\right),
\end{aligned}$$

where the last equality is due to condition (1.3.17), and the finite fourth central moment condition, which assures that for some finite M_7,

$$\sum_{j\neq j_1}^{b}\left[\frac{1}{b^2}\frac{\sigma^2_{i_1j}}{n_{i_1j}}\frac{\sigma^2_{i_1j_1}}{n_{i_1j_1}}\right]\left[\frac{1}{b}\sum_{j_2=1}^{b}\frac{\sigma^2_{i_2j_2}}{n_{i_2j_2}}\right]\leq M_7, \forall i.$$

Similarly, it can be shown that the second term in (1.7.50) satisfies

$$-\frac{24(a-1)}{a^{7/2}}\sum_{i_1\neq i_2}^{a}\alpha_{i_1}\alpha_{i_2}\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{i_1j}}\frac{E[\widetilde{\epsilon}_{i_1..}\epsilon_{i_1jk}\epsilon_{i_1jk_1}\widetilde{\epsilon}_{i_1..}]E[\widetilde{\epsilon}^2_{i_2..}]}{n_{i_1j}(n_{i_1j}-1)}=O\left(\frac{1}{ab^{\frac{5}{2}}}\right).$$

Putting them together gives

$$E[\widetilde{T_A}\Pi_a(\Omega_a-E(\Omega_a))^2]=O\left(\frac{1}{ab^{\frac{3}{2}}}\right).$$

Therefore combining the results for $E[\widetilde{T_A}^2(\Omega_a-E(\Omega_a))^2]$ and $E[\widetilde{T_A}\Pi_a(\Omega_a-E(\Omega_a))^2]$, the third term in (1.7.44) is equal to

$$\begin{aligned}
& E[(\widetilde{T_A}+a^{-\frac{1}{2}}\Pi_a)^2(\Omega_a-E(\Omega_a))^2] \\
= & \ \frac{8(a-1)^2}{(ab)^4}\sum_{i\neq i_1}^{a}\left[\sum_{j\neq j_1}^{b}\sum_{j_2=1}^{b}\alpha^2_{i_1}\frac{\sigma^2_{ij}}{n_{ij}}\frac{\sigma^2_{ij_2}}{n_{ij_2}}\frac{\sigma^2_{i_1j_1}}{n_{i_1j_1}}+\sum_{j=1}^{b}\sum_{j_1=1}^{b}\alpha^2_{i_1}n_{ij}(n_{ij}-1)\sigma^4_{ij}\frac{\sigma^2_{i_1j_1}}{n_{i_1j_1}}\right] \\
& +O(a^{-1}).
\end{aligned} \tag{1.7.51}$$

Next we compute the fourth term $E[(\widetilde{T_A}+a^{-\frac{1}{2}}\Pi_a)(\Omega_a-E(\Omega_a))^3]$ in (1.7.44):

$$E[(\widetilde{T_A}+a^{-\frac{1}{2}}\Pi_a)(\Omega_a-E(\Omega_a))^3]=E[(\widetilde{T_A}(\Omega_a-E(\Omega_a))^3]+a^{-\frac{1}{2}}E[\Pi_a(\Omega_a-E(\Omega_a))^3].$$

Using (1.3.7) and (1.7.27) we compute $E[\widetilde{T_A}(\Omega_a-E(\Omega_a))^3]$ as follows:

$$\begin{aligned}
&E[\widetilde{T_A}(\Omega_a-E(\Omega_a))^3]\\
=\;& E\left[\sum_{i=1}^{a}\frac{a-1}{(ab)^{3/2}}\left\{\sum_{j\neq j_1}^{b}\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}+\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\frac{\epsilon_{ijk}\epsilon_{ijk_1}}{n_{ij}(n_{ij}-1)}\right\}\left(2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i_1=1}^{a}[(\widetilde{\epsilon}_{i_1..}-\widetilde{\epsilon}_{...})\alpha_{i_1}]\right)^3\right]\\
=\;& \frac{8(a-1)}{a^3}\sum_{i,i_1,i_2,i_3}^{a}\left[\sum_{j\neq j_1}^{b}\alpha_{i_1}\alpha_{i_2}\alpha_{i_3}E[(\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.})(\widetilde{\epsilon}_{i_1..}-\widetilde{\epsilon}_{...})(\widetilde{\epsilon}_{i_2..}-\widetilde{\epsilon}_{...})(\widetilde{\epsilon}_{i_3..}-\widetilde{\epsilon}_{...})]\right.\\
&\left.+\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\alpha_{i_1}\alpha_{i_2}\alpha_{i_3}\frac{E[(\epsilon_{ijk}\epsilon_{ijk_1})(\widetilde{\epsilon}_{i_1..}-\widetilde{\epsilon}_{...})(\widetilde{\epsilon}_{i_2..}-\widetilde{\epsilon}_{...})(\widetilde{\epsilon}_{i_3..}-\widetilde{\epsilon}_{...})]}{n_{ij}(n_{ij}-1)}\right]\\
=\;& \frac{8(a-1)}{a^3}\sum_{i,i_1,i_2,i_3}^{a}\alpha_{i_1}\alpha_{i_2}\alpha_{i_3}\left[\sum_{j\neq j_1}^{b}E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{i_3..}]+\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\frac{E[\epsilon_{ijk}\epsilon_{ijk_1}\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{i_3..}]}{n_{ij}(n_{ij}-1)}\right].
\end{aligned}$$

The first term in the above equation can be simplified as

$$\begin{aligned}
&\frac{8(a-1)}{a^3}\sum_{i,i_1,i_2,i_3}^{a}\sum_{j\neq j_1}^{b}\alpha_{i_1}\alpha_{i_2}\alpha_{i_3}E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{i_3..}]\\
=\;& \frac{8(a-1)}{a^3}\sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\alpha_i^3E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\widetilde{\epsilon}_{i..}^3]\\
=\;& \frac{16(a-1)}{a^3}\sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\alpha_i^3E\left[\bar{\epsilon}_{ij.}\left(\frac{1}{b}\bar{\epsilon}_{ij.}\right)\bar{\epsilon}_{ij_1.}\left(\frac{1}{b}\bar{\epsilon}_{ij_1.}\right)^2\right]\\
=\;& \frac{16(a-1)}{(ab)^3}\sum_{i=1}^{a}\alpha_i^3\sum_{j\neq j_1}^{b}E(\bar{\epsilon}_{ij.}^2)E(\bar{\epsilon}_{ij_1.}^3)\\
\leq\;& \frac{16(a-1)}{a^3b}\sum_{i=1}^{a}\alpha_i^2M_\alpha M_8\\
=\;& O(a^{-\frac{3}{2}}b^{-\frac{3}{2}}),
\end{aligned}$$

where the last equality is due to condition (1.3.17), and M_α is a finite upper bound of α_i, $\forall i$ and M_8 is an upper bound of $\frac{\sigma_{ij}^2}{n_{ij}}\frac{\gamma_{ij_1}\sigma_{ij_1}^3}{n_{ij_1}^2}$, which exists since $E(\epsilon_{ijk}^4)<\infty$, $\forall i,j$. Similarly, the second term can be shown to satisfy

$$\frac{8(a-1)}{a^3}\sum_{i,i_1,i_2,i_3}^{a}\sum_{j\neq j_1}^{b}\alpha_{i_1}\alpha_{i_2}\alpha_{i_3}\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\frac{E[\epsilon_{ijk}\epsilon_{ijk_1}\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{i_3..}]}{n_{ij}(n_{ij}-1)}=O(a^{-\frac{3}{2}}b^{-\frac{5}{2}}).$$

Putting them together gives

$$E[\widetilde{T_A}(\Omega_a-E(\Omega_a))^3]=O(a^{-\frac{3}{2}}b^{-\frac{3}{2}}).$$

Next we use (1.3.16) and (1.7.27) to compute $E[\Pi_a(\Omega_a-E(\Omega_a))^3]$:

$$\begin{aligned}
E[\Pi_a(\Omega_a-E(\Omega_a))^3] &= E\left[a^{\frac{1}{2}}\left(-\frac{b^{1/2}}{a^{3/2}}\sum_{i_1\neq i_2}^{a}\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\right)\left(2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i_3=1}^{a}[(\widetilde{\epsilon}_{i_3..}-\widetilde{\epsilon}_{...})\alpha_{i_3}]\right)^3\right]\\
&= -\frac{8b}{a^{\frac{5}{2}}}\sum_{i_1\neq i_2}^{a}\sum_{i_3,i_4,i_5}^{a}\alpha_{i_3}\alpha_{i_4}\alpha_{i_5}E[\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}(\widetilde{\epsilon}_{i_3..}-\widetilde{\epsilon}_{...})(\widetilde{\epsilon}_{i_4..}-\widetilde{\epsilon}_{...})(\widetilde{\epsilon}_{i_5..}-\widetilde{\epsilon}_{...})]\\
&= -\frac{8b}{a^{\frac{5}{2}}}\sum_{i_1\neq i_2}^{a}\sum_{i_3,i_4,i_5}^{a}\alpha_{i_3}\alpha_{i_4}\alpha_{i_5}E[\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{i_3..}\widetilde{\epsilon}_{i_4..}\widetilde{\epsilon}_{i_5..}]\\
&= -\frac{48}{a^{\frac{5}{2}}b^2}\sum_{i_1\neq i_2}^{a}\alpha_{i_1}\alpha_{i_2}^2\left(\frac{1}{b^2}\sum_{j_1=1}^{b}\sum_{j_2=1}^{b}\frac{\sigma_{i_1j_1}^2}{n_{i_1j_1}}\frac{\gamma_{i_2j_2}\sigma_{i_2j_2}^3}{n_{i_2j_2}^2}\right)\\
&\leq \frac{48}{a^{\frac{5}{2}}b^2}\sum_{i_1\neq i_2}^{a}|\alpha_{i_1}\alpha_{i_2}^2|M_8 \quad\leq \frac{24}{a^{\frac{5}{2}}b^2}\sum_{i_1\neq i_2}^{a}(\alpha_{i_1}^2+\alpha_{i_2}^4)M_8\\
&= O(a^{-\frac{3}{2}}b^{-\frac{5}{2}}).
\end{aligned}$$

Thus combining $E[\widetilde{T_A}(\Omega_a-E(\Omega_a))^3]$ and $E[\Pi_a(\Omega_a-E(\Omega_a))^3]$, the fourth term in (1.7.44) is

$$E[(\widetilde{T_A}+a^{-\frac{1}{2}}\Pi_a)(\Omega_a-E(\Omega_a))^3]=O(a^{-\frac{3}{2}}b^{-\frac{3}{2}}). \tag{1.7.52}$$

Using (1.7.27) we now compute the fifth term $E(\Omega_a - E(\Omega_a))^4$ in (1.7.44) as follows:

$$
\begin{aligned}
& E(\Omega_a - E(\Omega_a))^4 \\
= \; & E\left[\left(2\left(\frac{b}{a}\right)^{\frac{1}{2}}\sum_{i=1}^{a}[(\widetilde{\epsilon}_{i..} - \widetilde{\epsilon}_{...})\alpha_i]\right)^4\right] \\
= \; & 16\left(\frac{b}{a}\right)^2 \sum_{i,i_1,i_2,i_3}^{a} \alpha_i\alpha_{i_1}\alpha_{i_2}\alpha_{i_3} E[(\widetilde{\epsilon}_{i..} - \widetilde{\epsilon}_{...})(\widetilde{\epsilon}_{i_1..} - \widetilde{\epsilon}_{...})(\widetilde{\epsilon}_{i_2..} - \widetilde{\epsilon}_{...})(\widetilde{\epsilon}_{i_3..} - \widetilde{\epsilon}_{...})] \\
= \; & 16\left(\frac{b}{a}\right)^2 \sum_{i,i_1,i_2,i_3}^{a} \alpha_i\alpha_{i_1}\alpha_{i_2}\alpha_{i_3} E[\widetilde{\epsilon}_{i..}\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}\widetilde{\epsilon}_{i_3..}] \\
= \; & 16\left(\frac{b}{a}\right)^2 \left\{\sum_{i=1}^{a}\alpha_i^4 E(\widetilde{\epsilon}_{i..}^4) + 2\sum_{i\neq i_1}^{a}\alpha_i^2\alpha_{i_1}^2 E(\widetilde{\epsilon}_{i..}^2)E(\widetilde{\epsilon}_{i_1..}^2)\right\} \\
= \; & 16\left(\frac{1}{a}\right)^2 \left\{\sum_{i=1}^{a}\alpha_i^4\frac{1}{b^2}\left(\sum_{j=1}^{b}\frac{\sigma_{ij}^4}{n_{ij}^3}(\tau_{ij} + n_{ij} - 1) + 2\sum_{j_1\neq j_2}^{b}\frac{\sigma_{ij_1}^2}{n_{ij_1}}\frac{\sigma_{ij_2}^2}{n_{ij_2}}\right)\right. \\
& \left. + 2\sum_{i\neq i_1}^{a}\alpha_i^2\alpha_{i_1}^2\left(\frac{1}{b^2}\sum_{j_2=1}^{b}\sum_{j_3=1}^{b}\frac{\sigma_{ij_2}^2}{n_{ij_2}}\frac{\sigma_{i_1j_3}^2}{n_{i_1j_3}}\right)\right\} \\
\leq \; & 16\left(\frac{1}{a}\right)^2 \left\{\sum_{i=1}^{a}\alpha_i^4 M_9 + 3\sum_{i\neq i_1}^{a}\alpha_i^2\alpha_{i_1}^2 M_2\right\} \\
= \; & O(a^{-\frac{3}{2}}b^{-\frac{1}{2}}) + O(a^{-1}b^{-1}),
\end{aligned}
\tag{1.7.53}
$$

where the last equality is due to condition (1.3.17), M_2 is given in (1.7.35) and the condition that $E(\epsilon_{ijk}) < \infty$ assures that there exists some finite M_9 such that

$$
\frac{1}{b^2}\left(\sum_{j=1}^{b}\frac{\sigma_{ij}^4}{n_{ij}^3}(\tau_{ij} + n_{ij} - 1) + 3\sum_{j_1\neq j_2}^{b}\frac{\sigma_{ij_1}^2}{n_{ij_1}}\frac{\sigma_{ij_2}^2}{n_{ij_2}}\right) \leq M_{10}, \forall i.
$$

Lastly we compute $[\mathrm{Var}(T_A^{(1)})]^2$ in (1.7.44). Using the result in (1.7.29) we have

$$
\begin{aligned}
& [\mathrm{Var}(T_A^{(1)})]^2 \\
= \; & \left[E(\widetilde{T_A}^2) + O(a^{-1}) + \frac{4}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}\alpha_i^2\frac{\sigma_{ij}^2}{n_{ij}}\right]^2 \\
= \; & [E(\widetilde{T_A}^2)]^2 + 8\left(\frac{1}{ab}\right)E(\widetilde{T_A}^2)\sum_{i=1}^{a}\sum_{j=1}^{b}\alpha_i^2\frac{\sigma_{ij}^2}{n_{ij}} + 16\left(\frac{1}{ab}\right)^2\left[\sum_{i=1}^{a}\sum_{j=1}^{b}\alpha_i^2\frac{\sigma_{ij}^2}{n_{ij}}\right]\left[\sum_{i_1=1}^{a}\sum_{j_1=1}^{b}\alpha_{i_1}^2\frac{\sigma_{i_1j_1}^2}{n_{i_1j_1}}\right] \\
& + O(a^{-1}).
\end{aligned}
$$

The third term is $O(a^{-1}b^{-1})$. Therefore,

$$[\text{Var}(T_A^{(1)})]^2 = [E(\widetilde{T_A}^2)]^2 + 8\left(\frac{1}{ab}\right) E(\widetilde{T_A}^2) \sum_{i=1}^{a}\sum_{j=1}^{b} \alpha_i^2 \frac{\sigma_{ij}^2}{n_{ij}} + O(a^{-1}).$$

To finalize the result for $[\text{Var}(T_A^{(1)})]^2$, we need to compute $E(\widetilde{T_A}^2)$. Using the result in (1.7.12) we compute the $E(\widetilde{T}_A^2)$ as follows:

$$E(\widetilde{T}_A^2) = \frac{(a-1)^2}{(ab)^3}[E(\Lambda_1) + E(\Lambda_2) + 2E(\Lambda_3)].$$

Using (1.7.13) we compute $E(\Lambda_1)$.

$$\begin{aligned}
E(\Lambda_1) &= E\left(\sum_{i=1}^{a}\sum_{j\neq j_1}^{b} \bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\right)^2 \\
&= \sum_{i,i_1}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2\neq j_3}^{b} E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\bar{\epsilon}_{i_1j_2.}\bar{\epsilon}_{i_1j_3.}] \\
&= 2\sum_{i=1}^{a}\sum_{j\neq j_1}^{b} E[\bar{\epsilon}_{ij.}^2]E[\bar{\epsilon}_{ij_1.}^2] \\
&= 2\sum_{i=1}^{a}\sum_{j\neq j_1}^{b} \frac{\sigma_{ij}^2}{n_{ij}}\frac{\sigma_{ij_1}^2}{n_{ij_1}}.
\end{aligned}$$

Next we use (1.7.14) to compute $E(\Lambda_2)$.

$$\begin{aligned}
E(\Lambda_2) &= E\left(\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}} \frac{\epsilon_{ijk}\epsilon_{ijk_1}}{n_{ij}(n_{ij}-1)}\right)^2 \\
&= \sum_{i,i_1}^{a}\sum_{j,j_1}^{b}\sum_{k\neq k_1}^{n_{ij}}\sum_{k_2\neq k_3}^{n_{i_1j_1}} \frac{E[\epsilon_{ijk}\epsilon_{ijk_1}\epsilon_{i_1j_1k_2}\epsilon_{i_1j_1k_3}]}{n_{ij}(n_{ij}-1)n_{i_1j_1}(n_{i_1j_1}-1)} \\
&= \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}}\sum_{k_2\neq k_3}^{n_{i_1j_1}} \frac{E[\epsilon_{ijk}\epsilon_{ijk_1}\epsilon_{ijk_2}\epsilon_{ijk_3}]}{n_{ij}^2(n_{ij}-1)^2} \\
&= 2\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k\neq k_1}^{n_{ij}} \frac{E[\epsilon_{ijk}^2]E[\epsilon_{ijk_1}^2]}{n_{ij}^2(n_{ij}-1)^2} \\
&= 2\sum_{i=1}^{a}\sum_{j=1}^{b} \frac{\sigma_{ij}^4}{n_{ij}(n_{ij}-1)}.
\end{aligned}$$

Lastly we use (1.7.15) to compute $E(\Lambda_3)$.

$$\begin{aligned}
E(\Lambda_3) &= E\left[\sum_{i=1}^{a}\sum_{i_1=1}^{a}\sum_{j\neq j_1}^{b}\sum_{j_2=1}^{b}\sum_{k\neq k_1}^{n_{i_1j_2}} \frac{\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\epsilon_{i_1j_2k}\epsilon_{i_1j_2k_1}}{n_{i_1j_2}(n_{i_1j_2}-1)}\right] \\
&= 2\sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\sum_{k\neq k_1}^{n_{ij}} \frac{E[\bar{\epsilon}_{ij.}\epsilon_{ijk}\epsilon_{ijk_1}]E[\bar{\epsilon}_{ij_1.}]}{n_{ij}(n_{ij}-1)} \\
&= 0.
\end{aligned}$$

Therefore combining the terms $E(\Lambda_1), E(\Lambda_2)$ and $E(\Lambda_3)$ we get

$$E(\widetilde{T}_A^2) = 2\left(\frac{(a-1)^2}{(ab)^3}\right)\sum_{i=1}^{a}\left(\sum_{j\neq j_1}^{b}\frac{\sigma_{ij}^2}{n_{ij}}\frac{\sigma_{ij_1}^2}{n_{ij_1}} + \sum_{j=1}^{b}\frac{\sigma_{ij}^4}{n_{ij}(n_{ij}-1)}\right).$$

Thus $[\mathrm{Var}(T_A^{(1)})]^2$ is equal to

$$\begin{aligned}
&[\mathrm{Var}(T_A^{(1)})]^2 \\
= \;& [E(\widetilde{T_A}^2)]^2 + 16\left(\frac{(a-1)^2}{(ab)^4}\right)\sum_{i=1}^{a}\left[\sum_{j\neq j_1}^{b}\frac{\sigma_{ij}^2}{n_{ij}}\frac{\sigma_{ij_1}^2}{n_{ij_1}} + \sum_{j=1}^{b}\frac{\sigma_{ij}^4}{n_{ij}(n_{ij}-1)}\right]\left[\sum_{i_2=1}^{a}\sum_{j_2=1}^{b}\alpha_{i_2}^2\frac{\sigma_{i_2j_2}^2}{n_{i_2j_2}}\right] \\
& + O(a^{-1}). \qquad (1.7.54)
\end{aligned}$$

Based on (1.7.51) and (1.7.54) we note that

$$6E[(\widetilde{T_A} + a^{-\frac{1}{2}}\Pi_a)^2(\Omega_a - E(\Omega_a))^2] - 3[\mathrm{Var}(T_A^{(1)})]^2 = -3[E(\widetilde{T_A}^2)]^2 + O(a^{-1}). \quad (1.7.55)$$

Thus, substituting the results in (1.7.10), (1.7.46), (1.7.51), (1.7.52), (1.7.53) and (1.7.54) into (1.7.44), we obtain the fourth cumulant as

$$\begin{aligned}
K_4(T_A^{(1)}) &= E(\widetilde{T_A}^4) - 3[E(\widetilde{T_A}^2)]^2 + O(a^{-1}) \\
&= K_4(T_A^{(0)}) + O(a^{-1}), \qquad (1.7.56)
\end{aligned}$$

where $K_4(T_A^{(0)})$ is the fourth cumulant of $T_A^{(0)}$ under the null.

In summary, the cumulants of the test statistic under H_1 are related to those under H_0 in the following manner:

$$K_j(T_A^{(1)}) = K_j(T_A^{(0)}) + \kappa_j + O(a^{-1}), j = 1, 2, 3, 4,$$

where $\kappa_1 = (b/a)^{1/2}\sum_{i=1}^{a}\alpha_i^2$, $\kappa_2 = \frac{4}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}\alpha_i^2\frac{\sigma_{ij}^2}{n_{ij}}$, $\kappa_3 = \frac{8(a-1)^3}{a^{9/2}b^{7/2}}\sum_{i=1}^{a}\left[\sum_{j\neq j_1}^{b}\alpha_i\frac{\gamma_{ij}\sigma_{ij}^3}{n_{ij}^2}\frac{\sigma_{ij_1}^2}{n_{ij_1}}+\sum_{j=1}^{b}\alpha_i\frac{\gamma_{ij}\sigma_{ij}^5}{n_{ij}^2(n_{ij}-1)}\right]$ and $\kappa_4 = 0$. These are given in (1.7.24), (1.7.29), (1.7.42) and (1.7.56), respectively. Now we can derive $\chi_{T_A^{(1)}/\nu_A}$, the characteristic function of $T_A^{(1)}$ in (1.2.13). Under the conditions (B1) in Section 1.7.1, the characteristic function $\chi_{T_A^{(1)}/\nu_A}$ can be written as:

$$\begin{aligned}
\chi_{T_A^{(1)}/\nu_A}(t) &= \exp\left\{K_1(T_A^{(1)})\frac{(it)}{\nu_A}+K_2(T_A^{(1)})\frac{(it)^2}{2\nu_A^2}+K_3(T_A^{(1)})\frac{(it)^3}{6\nu_A^3}+K_4(T_A^{(1)})\frac{(it)^4}{24\nu_A^4}+O(a^{-1})\right\}\\
&= \exp\left\{[K_1(T_A^{(0)})+\kappa_1]\frac{(it)}{\nu_A}+[K_2(T_A^{(0)})+\kappa_2]\frac{(it)^2}{2\nu_A^2}+[K_3(T_A^{(0)})+\kappa_3]\frac{(it)^3}{6\nu_A^3}\right.\\
&\quad\left.+K_4(T_A^{(0)})\frac{(it)^4}{24\nu_A^4}+O(a^{-1})\right\}\\
&= \exp\left(-\frac{t^2}{2}\right)\exp\left(\frac{\kappa_1}{\nu_A}it\right)\exp\left\{K_1(T_A^{(0)})\frac{(it)}{\nu_A}+\left[\left(\frac{K_2(T_A^{(0)})}{\nu_A^2}-1\right)+\frac{\kappa_2}{\nu_A^2}\right]\frac{(it)^2}{2}\right.\\
&\quad\left.+[K_3(T_A^{(0)})+\kappa_3]\frac{(it)^3}{6\nu_A^3}+K_4(T_A^{(0)})\frac{(it)^4}{24\nu_A^4}+O(a^{-1})\right\}\\
&= \exp\left(-\frac{t^2}{2}\right)\exp\left(\frac{\kappa_1}{\nu_A}it\right)\exp\left\{\frac{K_1(T_A^{(0)})}{\nu_A}(it)+\left(\frac{K_2(T_A^{(0)})}{\nu_A^2}-1\right)\frac{(it)^2}{2}+\frac{\kappa_2}{2\nu_A^2}(it)^2\right.\\
&\quad\left.+\frac{[K_3(T_A^{(0)})+\kappa_3]}{6\nu_A^3}(it)^3+\frac{K_4(T_A^{(0)})}{24\nu_A^4}(it)^4+O(a^{-1})\right\}. \qquad (1.7.57)
\end{aligned}$$

Applying Taylor series expansion to (1.7.57), we get

$$\begin{aligned}
&\chi_{T_A^{(1)}/\nu_A}(t)\\
=\ &\exp\left(-\frac{t^2}{2}\right)\exp\left(\frac{\kappa_1}{\nu_A}it\right)\left[1+\frac{K_1(T_A^{(0)})}{\nu_A}(it)+\left(\frac{K_2(T_A^{(0)})}{\nu_A^2}-1\right)\frac{(it)^2}{2}+\frac{\kappa_2}{2\nu_A^2}(it)^2\right.\\
&\left.+\frac{[K_3(T_A^{(0)})+\kappa_3]}{6\nu_A^3}(it)^3+\frac{K_4(T_A^{(0)})}{24\nu_A^4}(it)^4\right]+\ O(a^{-1}). \qquad (1.7.58)
\end{aligned}$$

By Applying the inverse Fourier transform, we obtain the pdf of $T_A^{(1)}$ under conditions (B1) and (B2) in Section 1.7.1 as follows:

$$
\begin{aligned}
f_{T_A^{(1)}/\nu_A}(x) &= \int_{-\infty}^{\infty} e^{-itx}\chi_{T_A^{(1)}/\nu_A}(t)dt \\
&= \int_{-\infty}^{\infty} e^{-itx}\exp\left(-\frac{t^2}{2}\right)\exp\left(\frac{\kappa_1}{\nu_A}it\right)\left[1+\frac{K_1(T_A^{(0)})}{\nu_A}(it)+\left(\frac{K_2(T_A^{(0)})}{\nu_A^2}-1\right)\frac{(it)^2}{2}\right. \\
&\quad \left.+\frac{\kappa_2}{2\nu_A^2}(it)^2+\frac{[K_3(T_A^{(0)})+\kappa_3]}{6\nu_A^3}(it)^3+\frac{K_4(T_A^{(0)})}{24\nu_A^4}(it)^4\right]+\ O(a^{-1})dt \\
&= \int_{-\infty}^{\infty} e^{-it\left(x-\frac{\kappa_1}{\nu_A}\right)}e^{\left(-\frac{t^2}{2}\right)}\left[1+\frac{K_1(T_A^{(0)})}{\nu_A}(it)+\left(\frac{K_2(T_A^{(0)})}{\nu_A^2}-1\right)\frac{(it)^2}{2}\right. \\
&\quad \left.+\frac{\kappa_2}{2\nu_A^2}(it)^2+\frac{[K_3(T_A^{(0)})+\kappa_3]}{6\nu_A^3}(it)^3+\frac{K_4(T_A^{(0)})}{24\nu_A^4}(it)^4\right]+O(a^{-1})dt \\
&= \phi\left(x-\frac{\kappa_1}{\nu_A}\right)+\left[\frac{K_1(T_A^{(0)})}{\nu_A}H_1\left(x-\frac{\kappa_1}{\nu_A}\right)+\frac{1}{2}\left(\frac{K_2(T_A^{(0)})}{\nu_A^2}-1\right)H_2\left(x-\frac{\kappa_1}{\nu_A}\right)\right. \\
&\quad +\frac{\kappa_2}{2\nu_A^2}H_2\left(x-\frac{\kappa_1}{\nu_A}\right)+\frac{[K_3(T_A^{(0)})+\kappa_3]}{6\nu_A^3}H_3\left(x-\frac{\kappa_1}{\nu_A}\right) \\
&\quad \left.+\frac{K_4(T_A^{(0)})}{24\nu_A^4}H_4\left(x-\frac{\kappa_1}{\nu_A}\right)\right]\phi\left(x-\frac{\kappa_1}{\nu_A}\right)+O(a^{-1}),
\end{aligned}
$$

where $H_0(x)=1$, $H_1(x)=x$, $H_2(x)=x^2-1$, $H_3(x)=x^3-3x$, and $H_4(x)=x^4-6x^2+3$ are

Hermite polynomials. We can then obtain the cdf of $T_A^{(1)}$. For any $y \in R$, let $t = \left(y - \frac{\kappa_1}{\nu_A}\right)$

$$
\begin{aligned}
F_{T_A^{(1)}/\nu_A}(y) &= \int_{-\infty}^{y} f_g(u)du \\
&= \Phi(t) - \left[\frac{K_1(T_A^{(0)})}{\nu_A} H_0(t) + \frac{1}{2}\left(\frac{K_2(T_A^{(0)})}{\nu_A^2} - 1\right) H_1(t) + \frac{\kappa_2}{2\nu_A^2} H_1(t)\right. \\
&\quad \left. + \frac{[K_3(T_A^{(0)}) + \kappa_3]}{6\nu_A^3} H_2(t) + \frac{K_4(T_A^{(0)})}{24\nu_A^4} H_3(t)\right] \phi(t) + O(a^{-1}) \\
&= \Phi(t) - \left[\frac{K_1(T_A^{(0)})}{\nu_A} H_0(t) + \frac{1}{2}\left(\frac{K_2(T_A^{(0)})}{\nu_A^2} - 1\right) H_1(t) + \frac{K_3(T_A^{(0)})}{6\nu_A^3} H_2(t)\right. \\
&\quad \left. + \frac{K_4(T_A^{(0)})}{24\nu_A^4} H_3(t)\right] \phi(t) - \left[\frac{\kappa_2}{2\nu_A^2} H_1(t) + \frac{\kappa_3}{6\nu_A^3} H_2(t)\right] \phi(t) + O(a^{-1}) \\
&= P(T_A^{(0)} \le t) - \left[\frac{\kappa_2}{2\nu_A^2}(t) + \frac{\kappa_3}{6\nu_A^3}(t^2 - 1)\right] \phi(t) + O(a^{-1}) \\
&= P(T_A^{(0)} \le y - \frac{\kappa_1}{\nu_A}) - \left[\frac{\kappa_2}{2\nu_A^2}\left(y - \frac{\kappa_1}{\nu_A}\right) + \frac{\kappa_3}{6\nu_A^3}\left(\left(y - \frac{\kappa_1}{\nu_A}\right)^2 - 1\right)\right] \phi\left(y - \frac{\kappa_1}{\nu_A}\right) \\
&\quad + O(a^{-1}).
\end{aligned}
$$

where

$$
\begin{aligned}
P(T_A^{(0)} \le t) &= \Phi(t) - \left[\frac{K_1(T_A^{(0)})}{\nu_A} H_0(t) + \frac{1}{2}\left(\frac{K_2(T_A^{(0)})}{\nu_A^2} - 1\right) H_1(t) + \frac{K_3(T_A^{(0)})}{6\nu_A^3} H_2(t)\right. \\
&\quad \left. + \frac{K_4(T_A^{(0)})}{24\nu_A^4} H_3(t)\right] \phi(t)
\end{aligned}
$$

is the Edgeworth expansion of $T_A^{(0)}$ under H_0.

1.7.8 Proof of (1.7.33).

To prove (1.7.33), we first consider

$$
\begin{aligned}
& \sum_{i,i_1,i_2}^{a} \sum_{j \ne j_1}^{b} \alpha_{i_1}\alpha_{i_2} E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}(\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..} - \widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{...} - \widetilde{\epsilon}_{...}\widetilde{\epsilon}_{i_2..} + \widetilde{\epsilon}^2_{...})] \\
= & \sum_{i,i_1,i_2}^{a} \sum_{j \ne j_1}^{b} \alpha_{i_1}\alpha_{i_2} \left\{E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}] - \ 2E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{...}] + E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\widetilde{\epsilon}^2_{...}]\right\}.
\end{aligned}
$$

We compute each term in the above equation as follows:

$$\begin{aligned}
\sum_{i,i_1,i_2}^{a}\sum_{j\neq j_1}^{b}\alpha_{i_1}\alpha_{i_2}E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}] &= 2\sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\alpha_i^2\frac{1}{b^2}E[\bar{\epsilon}_{ij.}^2]E[\bar{\epsilon}_{ij_1.}^2] \\
&= 2\sum_{i=1}^{a}\alpha_i^2\left(\frac{1}{b^2}\sum_{j\neq j_1}^{b}\frac{\sigma_{ij}^2}{n_{ij}}\frac{\sigma_{ij_1}^2}{n_{ij_1}}\right).
\end{aligned}$$

$$\begin{aligned}
\sum_{i,i_1,i_2}^{a}\sum_{j\neq j_1}^{b}\alpha_{i_1}\alpha_{i_2}E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{...}] &= -2\sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\alpha_i^2\frac{1}{ab^2}E[\bar{\epsilon}_{ij.}^2]E[\bar{\epsilon}_{ij_1.}^2] \\
&= -2\frac{1}{a}\sum_{i=1}^{a}\alpha_i^2\left(\frac{1}{b^2}\sum_{j\neq j_1}^{b}\frac{\sigma_{ij}^2}{n_{ij}}\frac{\sigma_{ij_1}^2}{n_{ij_1}}\right).
\end{aligned}$$

$$\begin{aligned}
\sum_{i,i_1,i_2}^{a}\sum_{j\neq j_1}^{b}\alpha_{i_1}\alpha_{i_2}E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}\widetilde{\epsilon}_{...}^2] &= -2\sum_{i=1}^{a}\sum_{j\neq j_1}^{b}\alpha_i^2\frac{1}{a^2b^2}E[\bar{\epsilon}_{ij.}^2]E[\bar{\epsilon}_{ij_1.}^2] \\
&= -2\frac{1}{a^2}\sum_{i=1}^{a}\alpha_i^2\left(\frac{1}{b^2}\sum_{j\neq j_1}^{b}\frac{\sigma_{ij}^2}{n_{ij}}\frac{\sigma_{ij_1}^2}{n_{ij_1}}\right).
\end{aligned}$$

Putting them together we get

$$\begin{aligned}
&\sum_{i,i_1,i_2}^{a}\sum_{j\neq j_1}^{b}\alpha_{i_1}\alpha_{i_2}E[\bar{\epsilon}_{ij.}\bar{\epsilon}_{ij_1.}(\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{i_2..}-\widetilde{\epsilon}_{i_1..}\widetilde{\epsilon}_{...}-\widetilde{\epsilon}_{...}\widetilde{\epsilon}_{i_2..}+\widetilde{\epsilon}_{...}^2)] \\
&= 2\left(1+\frac{2}{a}-\frac{1}{a^2}\right)\sum_{i=1}^{a}\alpha_i^2\left(\frac{1}{b^2}\sum_{j\neq j_1}^{b}\frac{\sigma_{ij}^2}{n_{ij}}\frac{\sigma_{ij_1}^2}{n_{ij_1}}\right) \\
&\leq 2\left(1+\frac{2}{a}-\frac{1}{a^2}\right)\sum_{i=1}^{a}\alpha_i^2 M_1 \\
&= O\left(\left(1-\frac{2}{a}+\frac{1}{a^2}\right)\frac{\sqrt{a}}{\sqrt{b}}\right),
\end{aligned}$$

where the last equality uses condition (1.3.17) and M_1 is a finite upper bound for $\frac{\sigma_{ij}^2}{n_{ij}}\frac{\sigma_{ij_1}^2}{n_{ij_1}}$ for all i,j due to the fact that $E(\epsilon_{ijk}^4)<\infty$, which assures that

$$\frac{1}{b^2}\sum_{j\neq j_1}^{b}\frac{\sigma_{ij}^2}{n_{ij}}\frac{\sigma_{ij_1}^2}{n_{ij_1}}\leq M_1, \forall i.$$

Following the same procedure it can be shown that

$$\sum_{i,i_1,i_2}^{a} \sum_{j=1}^{b} \sum_{k \neq k_1}^{n_{ij}} \alpha_{i_1} \alpha_{i_2} E[\epsilon_{ijk} \epsilon_{ijk_1} (\widetilde{\epsilon}_{i_1..} \widetilde{\epsilon}_{i_2..} - \widetilde{\epsilon}_{i_1..} \widetilde{\epsilon}_{...} - \widetilde{\epsilon}_{...} \widetilde{\epsilon}_{i_2..} + \widetilde{\epsilon}^{2}_{...})] = O\left(\left(1 - \frac{2}{a} + \frac{1}{a^2}\right) \frac{\sqrt{a}}{b^{\frac{3}{2}}}\right).$$

Thus the proof of (1.7.33) is complete.

Chapter 2

Asymptotic Expansions in One-way ANOVA with a large number of levels and skewed populations

2.1 Motivation and existing methods

For scientific investigations in agricultural screening experiments, researchers often want to compare a large number of cultivars (or genotypes) with very small number of replications within each cultivar. The choice of small replications is due to cost concern since many of the experiments collect molecular data using high throughput technologies such as micro-array and sequencing. Data from such experiments are often found to be skewed. We are interested in testing the hypothesis of no main treatment effect when the number of treatment levels is large while the sample sizes are small. We are particularly interested in the case that there are extreme observations and heteroscedasticity in presence.

The set up of this data setting can be described as follows. Suppose there are a treatments (cultivars) and independent observations X_{ij}, $j = 1, \cdots, n_i$, observed from treatment i, $i = 1, \cdots, a$. The distribution of X_{ij} is unknown with mean μ_i and standard deviation σ_i. We are interested in testing the hypothesis of no treatment effect, i.e., $H_0 : \mu_i = \mu$ versus H_1 : at least one μ_i is different from μ, for some constant μ. The number of treatments a is large while n_i's are small. Pioneer studies in the literature include, cf., Akritas and

Arnold (2000), Akritas and Papadatos (2004), Wang and Akritas (2004), Boos and Brownie (1995) etc. Bathke (2002), Wang and Akritas (2006), Wang and Akritas (2011) have also conducted research on two-way, three-way ANOVA and other multifactor designs when the number of treatments is large. In these papers, they presented different test statistics and their asymptotic distributions. Even though the form of their statistics may be different they all give asymptotic normal or chisquare distribution to approximate the distribution of their test statistics. The error of their approximations is of order $O(a^{-1/2})$ in the sense that the difference between the true distribution and the approximate distribution of the test statistic is $O(a^{-1/2})$. With this rate, the type I error of these tests converges slowly to the nominal level when the data are skewed.

In the next paragraph, we first review three articles. Fujikoshi et al. (1999), Yanagihara (2000) and Harrar and Gupta (2007). All of them studied bootstrap tests or asymptotic expansions in ANOVA setting. We will explain that the bootstrap methods in the first three references fail to work in the large a small n_i setting. The result of Harrar and Gupta (2007) works in large a small n_i case if the variance is constant. None of them applies to large a, small n_i and heteroscedastic settings.

Fujikoshi et al. (1999) provided a higher order asymptotic expansion of the limiting null distribution of the regular F-statistic for one-way ANOVA up to order $1/a$ when the variances are constant. Their result is based on the assumption that Huber's condition (Huber (1973)), $n/n_i = O(1)$ is satisfied. Therefore Fujikoshi et al. (1999) expansion requires that the number of replications to be large. Their result does not apply to the setting of small n_i's and heteroscedastic variances.

Yanagihara (2000) also derived the asymptotic expansion of the null distribution of the test statistic T_2 proposed by James (1951) suited for one-way ANOVA under heteroscedastic and unbalanced situations. His result however, is also based on the assumption that Huber's condition (Huber (1973)), $n/n_i = O(1)$ is satisfied. This requires the number of replications n_i's to be large.

Harrar and Gupta (2007) improved the approximation of the limiting distribution of the regular ANOVA $F-$statistic, by deriving the asymptotic expansion of the null distribution of the classical F-statistic for one-way ANOVA under non-normality. Their approach works for small number of replications and large number of treatment levels. However, their results require homoscedastic variance and the $F-$statistic is not asymptotically pivotal in heteroscedastic case. As mentioned in Fisher and Hall (1990), the classical F statistic has a complicated limiting distribution that depends on the population parameters σ_i and skewness in the unbalanced case and these parameters cannot be estimated with order $O_p(1/\sqrt{a})$ when n_i's are small. In practice, population parameters are unknown, to apply the Edgeworth expansion of Harrar and Gupta (2007), estimated population parameters need to be used, leading to an overall accuracy of $O(1/\sqrt{a})$ for their unbalanced case. In addition, it's difficult to assess the common variance assumption when the group sizes are small. It's therefore necessary to develop a test with better type I error accuracy for skewed populations with a large number of treatment levels and small sample size per treatment level.

In this chapter, we propose asymptotically pivotal statistic in the setting of large number of treatments with heteroscedastic variance and non-normal data. Our statistic is suitable for both small and large number of replications in unbalanced one-way ANOVA. We give a higher order approximation of the limiting distribution by providing asymptotic expansion of the test statistic up to order $O(1/a)$. We will then develop a test using the Cornish-Fisher expansion of the distribution. We prove that the new test has better type I error accuracy and power for data from skewed populations.

The proposed test statistic is introduced in section 2.2. In section 2.3, we give the Edgeworth and Cornish-Fisher expansions of the test statistic. We discuss the bootstrap distribution of our proposed test statistic and its connection with Edgeworth expansion in section 2.4. In section 2.5, we present the new test. The theoretical type I error is presented in section 2.6. Section 2.7 is devoted to the power of the proposed test. In section 2.8, numerical results will be presented. Finally, the technical proofs are presented in section 2.9.

2.2 The proposed test statistic

To describe how our test statistic is proposed, we give a brief review of the development of the test in this large a small n_i setting. Under homoscedasticity, Boos and Brownie (1995) and Akritas and Arnold (2000) derived the asymptotic null distribution of the classical F statistic for balanced case in this large a and small n_i setting Scheffé (1959) reported that the classical F test is sensitive to departures from homoscedastic assumption, particularly in unbalanced case. Akritas and Papadatos (2004) presented results to show that the classical F-statistic is sensitive to departures from homoscedastic in both balanced and unbalanced case. They suggested to use $a^{1/2}(F-1)$ as the test statistic when the number of treatment levels is large, where $F = MST/MSE$ is the classical F-statistic, for both balanced and unbalanced homoscedastic variances. In the next paragraph, we discuss some statistics that are suited for heteroscedastic variances and unbalaced case.

In this large number of treatments and heteroscedatic setup, Akritas and Papadatos (2004) proposed an unweighted statistic T_a given by $T_a = a^{-1/2}\sum_{i=1}^{a}[n_i(\overline{X}_{i\cdot}-\overline{X}_{\cdot\cdot})^2-(1-\frac{n_i}{N})S_i^2]$ where $S_i^2 = (n_i-1)^{-1}\sum_{j=1}^{n_i}(X_{ij}-\overline{X}_{i\cdot})^2$ and $N = n_1+\cdots+n_a$. They showed that the limiting distribution for the unweighted statistic T_a is normal under both null and local alternatives for the cases when group sizes are either small or large. In addition, Akritas and Papadatos (2004) also considered the generalized or weighted least squares statistics $\widehat{T}_W$ given by $\widehat{T}_W = \sum_{i=1}^{a}\frac{n_i}{S_i^2}\overline{X}_{i\cdot}^2 - \frac{1}{\sum_{i=1}^{a} n_i/S_i^2}\left(\sum_{i=1}^{a}\frac{n_i}{S_i^2}\overline{X}_{i\cdot}\right)^2$. They showed that the limiting distribution for the weighted least squares statistic $\widehat{T}_W$ is also normal under both null and local alternatives for the case when group sizes is large. They noted that, the weighted least squares statistic $\widehat{T}_W$ doesn't work well for small sample sizes. Akritas and Papadatos (2004) found that, the asymptotic properties of the unweighted statistic T_a are preferable to those of the classical

F-statistic even in the homoscedastic case, and they recommended T_a over the classical F-statistic in all unbalanced situations (homoscedastic or not).

Considering the same setup, Wang and Akritas (2004) constructed a different test statistic, that is suitable for both small and large group sizes in unbalanced ANOVA regardless of whether the variances are heteroscedastic or constant. We will give the form of the statistic shortly. They derived the limiting null distribution of the rank version of the statistic to be normal.

The test statistics used by Akritas and Papadatos (2004), Wang and Akritas (2004), Harrar and Gupta (2007) are not asymptotically pivotal. Their limiting results are only accurate up to order $O(1/\sqrt{a})$. As noted in Fisher and Hall (1990) and also discussed by Hall (1992a), asymptotically pivotal statistics have a faster rate of convergence and better accuracy compared to non-pivotal statistics. The objective of this paper is to construct asymptotically pivotal statistic and derive a better approximation to the limiting distribution of the asymptotically pivotal test statistic. In the rest of this section, we will derive our asymptotically pivotal test statistic by modifying the test statistic proposed by Wang and Akritas (2004), suitable for our large a with small n_i, under both homoscedastic and heteroscedastic settings.

For the setting with a large number of treatment levels and small group sizes in the heteroscedastic unbalanced setup, we start with the mean squares for treatment and mean squares for error constructed by Wang and Akritas (2004) based on the original observations. Let $\overline{X}_i. = n_i^{-1}\sum_{j=1}^{n_i} X_{ij}$, $\widetilde{X}.. = a^{-1}\sum_{i=1}^{a}\overline{X}_i.$, $\overline{X}.. = N^{-1}\sum_{i=1}^{a}\sum_{j=1}^{n_i} X_{ij}$, where $N = \sum n_i$. We define

$$\widetilde{MST}(\underline{\mathbf{X}}) = \frac{1}{a-1}\sum_{i=1}^{a}(\overline{X}_i. - \widetilde{X}..)^2, \qquad \widetilde{MSE^{(2)}}(\underline{\mathbf{X}}) = \frac{1}{a}\sum_{i=1}^{a}\frac{1}{n_i}S_{X,i}^2 \tag{2.2.1}$$

where $\underline{\mathbf{X}} = (X_{11}, \cdots, X_{1n_i}, \cdots, X_{a1}, \cdots, X_{an_a})'$ and $S_i^2 = (n_i - 1)^{-1}\sum_{j=1}^{n_i}(X_{ij} - \overline{X}_i.)^2$. They noted that, the test statistic based on the ratio of $\widetilde{MST}(\underline{\mathbf{X}})$ and $\widetilde{MSE^{(2)}}(\underline{\mathbf{X}})$ is also suitable for large number of replications within each treatment level. Considering the test of no treatment effect, i.e. $H_0 : \mu_i = \mu$, we have that $\widetilde{MST}(\underline{\mathbf{X}}) = \widetilde{MST}(\underline{\epsilon}) = \frac{1}{a-1}\sum_{i=1}^{a}(\bar{\epsilon}_{i\cdot} - \widetilde{\epsilon}_{\cdot\cdot})^2$, where $\underline{\epsilon} = (\epsilon_{11}, \cdots, \epsilon_{1n_i}, \cdots, \epsilon_{a1}, \cdots, \epsilon_{an_a})'$ and $\epsilon_{ij} = X_{ij} - \mu_i$. Under the null hypothesis of no treatment effect, $E\left(\widetilde{MST}(\underline{\mathbf{X}})\right) = E\left(\widetilde{MSE^{(2)}}(\underline{\mathbf{X}})\right)$, thus, it is reasonable to compare $\widetilde{MST}(\underline{\mathbf{X}})$ with $\widetilde{MSE^{(2)}}(\underline{\mathbf{X}})$ for the test of no treatment effect. Note also, that for the test of no treatment effect, we have

$$\widetilde{MST}(\underline{\mathbf{X}}) - \widetilde{MSE^{(2)}}(\underline{\mathbf{X}}) = \frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)} - \frac{1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}. \tag{2.2.2}$$

The proof of (2.2.2) is given in section 2.9.1. The mean and variance of $\sqrt{a}\left(\widetilde{MST}(\underline{\mathbf{X}}) - \widetilde{MSE^{(2)}}(\underline{\mathbf{X}})\right)$ under the null are given by

$$E\left[\sqrt{a}\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)} - \frac{1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right] = 0 \tag{2.2.3}$$

and

$$Var\left[\sqrt{a}\left(\widetilde{MST}(\underline{\mathbf{X}}) - \widetilde{MSE^{(2)}}(\underline{\mathbf{X}})\right)\right] = \frac{2}{a(a-1)^2}\sum_{i\neq i'}^{a}\frac{\sigma_i^2\sigma_{i'}^2}{n_i n_{i'}} + \frac{2}{a}\sum_{i=1}^{a}\frac{\sigma_i^4}{n_i(n_i-1)} \tag{2.2.4}$$

An unbiased estimate of $\sigma_i^2\sigma_{i'}^2$ is $S_i^2S_{i'}^2$ when $i \neq i'$ and S_i^2 is the sample variance. An unbiased estimate of σ_i^4 is $\widehat{\sigma_i^4}$ based on U-statistic given by

$$\widehat{\sigma_i^4} = \frac{1}{n_i(n_i-1)(n_i-2)(n_i-3)}\sum_{j_1\neq j_2\neq j_3\neq j_4}^{n_i}\frac{(x_{ij_1} - x_{ij_2})^2(x_{ij_3} - x_{ij_4})^2}{4} \tag{2.2.5}$$

Thus $\widehat{var} = \frac{2}{a(a-1)^2}\sum_{i\neq i'}^{a}\frac{S_i^2S_{i'}^2}{n_in_{i'}} + \frac{2}{a}\sum_{i=1}^{a}\frac{\widehat{\sigma_i^4}}{n_i(n_i-1)}$ is an unbiased estimate of the variance. Finally, the asymptotic pivotal statistic W_a is defined as

$$M_a(\underline{\mathbf{X}}) = \frac{\sqrt{a}\left(\widetilde{MST}(\underline{\mathbf{X}}) - \widetilde{MSE^{(2)}}(\underline{\mathbf{X}})\right)}{\sqrt{\widehat{var}}}. \tag{2.2.6}$$

2.3 Edgeworth expansion and Cornish-Fisher expansion of the test statistic

In this section, we present higher order approximation of the null distribution and quantiles of the test statistic $M_a(\underline{\mathbf{X}})$ presented in (2.2.6).

Throughout the rest of this chapter, we assume $\epsilon_{ij} = X_{ij} - \mu_i$, $j = 1, \cdots, n_i$, are independent observations from some unknown distribution F_i say, with unknown means μ_i's and unknown standard deviations σ_i's, for $i = 1, \cdots, a$. We define the following averages:

$$Y_{1a} = \frac{1}{a}\sum_{i=1}^{a}\sum_{j \neq j'} \frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}, \qquad Y_{2a} = \frac{-1}{a(a-1)}\sum_{i \neq i'} \bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot},$$

$$Y_{3a} = \frac{1}{a(a-1)}\sum_{i \neq i'} \frac{S_i^2 S_{i'}^2}{n_i n_{i'}}, \qquad Y_{4a} = \frac{1}{a}\sum_{i=1}^{a} \frac{\widehat{\sigma_i^4}}{n_i(n_i-1)}.$$

Let $\underline{\mathbf{Y}} = (Y_{1a}, Y_{2a}, Y_{3a}, Y_{4a})'$ and $\mathbf{u}$ be its mean i.e.,

$$\begin{aligned}
\mathbf{u} &= E(\underline{\mathbf{Y}}) = (E(Y_{1a}), E(Y_{2a}), E(Y_{3a}), E(Y_{4a}))' \\
&= (u_1, u_2, u_3, u_4)' \\
&= \left(0, \quad 0, \quad \frac{1}{a(a-1)}\sum_{i \neq i'}^{a} \frac{\sigma_i^2\sigma_{i'}^2}{n_i n_{i'}}, \qquad \frac{1}{a}\sum_{i=1}^{a} \frac{\sigma_i^4}{n_i(n_i-1)}\right)'
\end{aligned}$$

The test statistic $M_a(\underline{\mathbf{X}})$ defined in (2.2.6) can be written as

$$M_a(\underline{\mathbf{X}}) \quad \cong \quad W_a(\underline{\mathbf{Y}}) \quad = \quad \frac{\sqrt{a}(Y_{1a} + Y_{2a})}{h(\underline{\mathbf{Y}})} \quad = \quad \sqrt{a} g_a(\underline{\mathbf{Y}})$$

where

$$h(\underline{\mathbf{Y}}) \quad = \quad \sqrt{\frac{2}{a-1} Y_{3a} + 2Y_{4a}} \quad ; \quad g_a(\underline{\mathbf{Y}}) \quad = \quad \frac{Y_{1a} + Y_{2a}}{h(\underline{\mathbf{Y}})}.$$

Note that Y_{1a} and Y_{4a} are averages of non-iid terms. Also, Y_{2a} and Y_{3a} are quadratic forms of non-iid terms. If they were iid, then Bhattacharya et al. (2016) Theorems 11.2 - 11.4 on pages 285 - 288 can be applied. But in our case, the summand for different i's are not iid. By Taylor series expansion of $g_a(\underline{\mathbf{Y}})$ at $\mathbf{u}$, we obtain

$$\begin{aligned}
g_a(\underline{\mathbf{Y}}) &= g_a(\mathbf{u}) + \frac{\partial g_a(\mathbf{u})}{\partial \mathbf{u}}(\mathbf{Y} - \mathbf{u})' + \frac{1}{2}(\mathbf{Y} - \mathbf{u})' \frac{\partial^2 g_a(\mathbf{u})}{\partial \mathbf{u}^2}(\mathbf{Y} - \mathbf{u}) + O_p(\|\mathbf{Y} - \mathbf{u}\|^3) \\
&= \frac{1}{h(\mathbf{u})}(Y_{1a}+Y_{2a}) - \frac{h^{-3}(\mathbf{u})}{a-1}(Y_{3a}-u_3)(Y_{1a}+Y_{2a}) - h^{-3}(\mathbf{u})(Y_{4a}-u_4)(Y_{1a}+Y_{2a}) + O_p(a^{\frac{-3}{2}}).
\end{aligned}$$

Therefore we can write

$$W_a(\underline{\mathbf{Y}}) = \sqrt{a} g_a(\underline{\mathbf{Y}}) = g_1(\underline{\mathbf{Y}}) + g_2(\underline{\mathbf{Y}}) + g_3(\underline{\mathbf{Y}}) + O_p(a^{-1})$$

where

$$g_1(\underline{\mathbf{Y}}) = \frac{\sqrt{a}}{h(\mathbf{u})}(Y_{1a} + Y_{2a}),$$

$$g_2(\underline{\mathbf{Y}}) = \frac{-\sqrt{a}h^{-3}(\mathbf{u})}{a-1}(Y_{3a} - u_3)(Y_{1a} + Y_{2a}) = O_p(a^{-2}),$$

$$g_3(\underline{\mathbf{Y}}) = -\sqrt{a}h^{-3}(\mathbf{u})(Y_{4a} - u_4)(Y_{1a} + Y_{2a}).$$

Therefore,

$$M_a(\mathbf{X}) = W_a(\underline{\mathbf{Y}}) = \sqrt{a}g_a(\underline{\mathbf{Y}}) = g_1(\underline{\mathbf{Y}}) + g_3(\underline{\mathbf{Y}}) + O_p(a^{-1}).$$

Before we state the Theorem, we state the assumptions.

$$K1: \qquad E(X_{ij} - \mu_i)^6 < \infty$$

$$K2: \qquad \limsup_{||\mathbf{t}|| \to \infty} |E(\exp[i(t_1 Y_{1a} + t_2 Y_{2a} + t_3 Y_{3a} + t_4 Y_{4a})])| < 1, \forall a > 1,$$

where $\mathbf{t} = (t_1, t_2, t_3, t_4)$ and $||t|| = (t_1^2 + t_2^2 + t_3^2 + t_4^2)^{1/2}$. $K2$ corresponds to the Cramer's condition used on page 544 of Harrar and Gupta (2007).

Theorem 2.3.1. *Suppose $n_i \geq 4$ are fixed for all i. Then under $H_0 : \mu_i = \mu, \forall i$ and regularity conditions $K1$ and $K2$, the distribution of the test statistic $M_a(\underline{\mathbf{X}})$ given in (2.2.6) has the following expansion*

$$F_M(x) = P(M_a \leq x) = \Phi(x) + \frac{1}{\sqrt{a}}Q_1(x)\phi(x) + O(a^{-1}), \tag{2.3.1}$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the cumulative distribution and probability density functions of the standard normal distribution and

$$Q_1(x) = -\kappa_{11}^g - \frac{\kappa_{33}^g}{6}(x^2 - 1) \tag{2.3.2}$$

with

$$\kappa_{11}^g = \frac{-2}{h(\mathbf{u})^3 a}\sum_{i=1}^{a}\frac{\sigma_i^6(\gamma_i^2 - 2)}{n_i^2(n_i-1)^2}, \quad \kappa_{33}^g = \frac{2}{h(\mathbf{u})^3 a}\sum_{i=1}^{a}\frac{\sigma_i^6(5\gamma_i^2 + 4n_i - 14)}{n_i^2(n_i-1)^2},$$

$$\text{and} \quad h(\mathbf{u}) = \sqrt{\frac{2}{a(a-1)^2}\sum_{i \neq i'}^{a}\frac{\sigma_i^2\sigma_{i'}^2}{n_i n_{i'}} + \frac{2}{a}\sum_{i=1}^{a}\frac{\sigma_i^4}{n_i(n_i-1)}},$$

and γ_i is a measure of the population skewness computed as $\gamma_i = E\left[\left(\frac{X_{ij}-\mu_i}{\sigma_i}\right)^3\right]$.

The proof of Theorem 2.3.1 is given in section 2.9.1. When $\sigma_i^6(\gamma_i^2-2)$ and $\sigma_i^6(5\gamma_i^2+4n_i-14)$ are estimated with unbiased estimates, the $\widehat{\kappa_{11}^g}$ and $\widehat{\kappa_{33}^g}$ estimates of κ_{11}^g and κ_{33}^g have accuracy of order $O_p(a^{-1/2})$. Such estimators can be obtained with Jacknife procedure or U-statistics. The latter can be written without trouble when $n_i \geq 10$. The Jacknife estimate can be obtained with any sample size ≥ 3. Denote $\widetilde{Q_1(x)}$ the estimate of $Q_1(x)$ when unbiased estimators $\widehat{\kappa_{11}^g}$ and $\widehat{\kappa_{33}^g}$ are used to replace κ_{11}^g and κ_{33}^g. The resulting estimate $\hat{F}_M(x)$ will have accuracy of order $O(a^{-1})$. This provides a better order of approximation compared to the asymptotic distribution of the classical F test.

Next, we give the percentiles of the test statistic $M_a(\underline{\mathbf{X}})$ based on Cornish-Fisher expansion.

Corollary 2.3.2. *Denote ω_α the α-level quantile of the test statistic M_a. Then, based on Cornish-Fisher expansion, ω_α admits an expansion of the form in (2.3.3) below:*

$$\omega_\alpha = z_\alpha + \frac{1}{\sqrt{a}} q_1(z_\alpha) + O_p(a^{-1}) \tag{2.3.3}$$

where z_α is the α-level quantile of the standard normal distribuion and $q_1(z_\alpha) = -Q_1(z_\alpha)$ with Q_1 given in equation (2.3.2).

The proof of Corollary 2.3.2 is given in section 2.9.3.

The distribution of the test statistic $M_a(\underline{\mathbf{X}})$ given in (2.3.1) is the first-order Edgeworth Expansion. Hall (1992a) discussed that, under more stringent conditions with all moments finite, the full Edgeworth Expansion of sum of iid variables has the form

$$F(x) = \Phi(x) + \frac{1}{\sqrt{a}} Q_1^{'}(x)\phi(x) + \frac{1}{a} Q_2^{'}(x)\phi(x) + \cdots . \tag{2.3.4}$$

where $Q_k^{'}(x)$ is a polynomial of degree $3k-1$, with coefficients that depend on the population moments. In our case, we could also achieve higher order expansions by using higher order Taylor expansion of $g_a(Y)$. But in terms of formulating the test rejection region with ω_α, it's not helpful to consider higher order expansions.

2.4 Bootstrap distribution of the test statistic and its connection with the Edgeworth Expansion

Consider the observed data $\underline{\mathbf{X}} = (X_{11}, \cdots, X_{1n_1}, \cdots, X_{a1}, \cdots, X_{an_a})^{'}$. In this section, we consider resamples from $\underline{\mathbf{X}}$, and introduce the test statistic M_a^*, where M_a^* is the bootstrap version of the test statistic M_a given in (2.2.6). We will present the bootstrap distribution of the test statistic M_a^* and discuss the approximation of the bootstrap distribution of M_a^* to $F_M(x)$ presented in (2.3.4).

Consider the independent observations X_{ij}, $j = 1, 2, \cdots, n_i$, from some unknown distribution F_i say, with $i = 1, 2, \cdots, a$ with unknown means $\mu_i' s$ and unknown standard deviations $\sigma_i' s$. We use the bootstrap resampling idea discussed in Fisher and Hall (1990). Let $\mathbf{X}_i^* = \left\{X_{i1}^*, X_{i2}^*, \cdots, X_{in_i}^*\right\}$, denote a resample drawn by sampling randomly with replacement, from $\mathbf{X}_i = \{X_{i1}, X_{i2}, \cdots, X_{in_i}\}$, where $\mathbf{X}_i$ is the collection of independent and identically distributed observations from each treatment level i, $i = 1, 2, \cdots, a$. To construct the bootstrap version of the test statistic, consider the transformation $Y_{ij} = X_{ij} - \mu_i$ as used in Fisher and Hall (1990). Since μ_i is unknown, we use the resampled data to compute $Y_{ij}^* = X_{ij}^* - \overline{X}_{i.}$. The bootstrap version of the test statistic $M_a(\underline{\mathbf{X}})$ defined in (2.2.6) is $M_a^*(Y^*)$, which is computed from the resampled data as follows:

$$M_a^* = \frac{\sqrt{a}\left\{\frac{1}{a-1}\sum_{i=1}^{a}(\overline{Y}_{i.}^* - \widetilde{Y}^*..)^2 - \frac{1}{a}\sum_{i=1}^{a}\sum_{j=1}^{n_i}\frac{(Y_{ij}^* - \overline{Y}_{i.}^*)^2}{n_i(n_i-1)}\right\}}{\sqrt{\frac{2}{a(a-1)^2}\sum_{i\neq i'}^{a}\frac{S_i^{2*}S_{i'}^{2*}}{n_i n_{i'}} + \frac{2}{a}\sum_{i=1}^{a}\frac{\widehat{\sigma_i^{4*}}}{n_i(n_i-1)}}} \tag{2.4.1}$$

where $\overline{Y}_{i.}^* = n_i^{-1}\sum_{j=1}^{n_i} Y_{ij}^*$, $\widetilde{Y}^*.. = a^{-1}\sum_{i=1}^{a}\overline{Y}_{i.}^*$, $\overline{Y}^*.. = N^{-1}\sum_{i=1}^{a}\sum_{j=1}^{n_i} Y_{ij}^*$, where $N = \sum n_i$, $S_i^{2*} = (n_i-1)^{-1}\sum_{j=1}^{n_i}(Y_{ij}^* - \overline{Y}_{i.}^*)^2$ and $\widehat{\sigma_i^{4*}}$ is given by the U-statistic
$\widehat{\sigma_i^{4*}} = \frac{1}{n_i(n_i-1)(n_i-2)(n_i-3)}\sum_{j_1\neq j_2\neq j_3\neq j_4}^{n_i}\frac{(y_{ij_1}^* - y_{ij_2}^*)^2(y_{ij_3}^* - y_{ij_4}^*)^2}{4}$. We now proceed to present the bootstrap distribution of M_a^* given in (2.4.1).

The bootstrap distribution of the statistic M_a^* presented in (2.4.1) is the distribution of

M_a^* conditional on the observed data, $\underline{\mathbf{X}}$. It is equivalent to treat the observed data $\mathbf{X}_i$ as the population and sample from it randomly. Therefore, the distribution of M_a^* conditional on the observed data, admits Edgeworth expansion of the form

$$\widehat{F}_M^2(x) = P(M_a^* \leq x|\underline{\mathbf{X}}) = \Phi(x) + \frac{1}{\sqrt{a}}\hat{Q}_1(x)\phi(x) + \frac{1}{a}\hat{Q}_2(x)\phi(x) + \cdots . \tag{2.4.2}$$

where $\hat{Q}_k(x)$ is an estimate of $Q_k^{'}(x)$ in (2.3.4), in which the unknown population moments are replaced by their corresponding sample moments. This is analagous to the relationship of bootstrap distribution with Edgeworth expansion in sum of iid variables in Hall (1986) and Chapter 3 of Hall (1992b). If we use $\widehat{F}_M^2(x)$ presented in (2.4.2) to approximate $F_M(x)$ in (2.3.4), we have that

$$\begin{aligned}
\widehat{F}_M^{(2)}(x) - F_W(x) &= \frac{1}{\sqrt{a}}(\hat{Q}_1(x) - Q_1^{'}(x))\phi(x) + \cdots \\
&= \frac{1}{\sqrt{a}}[\{\hat{Q}_1(x) - E(\hat{Q}_1)\} + \{E(\hat{Q}_1) - Q_1^{'}(x)\}]\phi(x) + \cdots \\
&= O_p(a^{-1/2}),
\end{aligned}$$

where the last equality is due to the fact that by central limit theorem (Lyapounov's condition) $\hat{Q}_1(x) - E(\hat{Q}_1) = O_p(a^{-1/2})$. The bias part $E(\hat{Q}_1) - Q_1^{'}(x) = O_p(1)$. The reason being that the bootstrap uses sample moments to estimate the population parameters such as the skewness. Moreover, observing the form of $Q_1(x)$ in (2.3.2), it's realized that by Jensen's inequality $E(\hat{\gamma}_i^2) \geq (E(\hat{\gamma}_i))^2$ and $E(\hat{\sigma}_i^6) \geq (E(\hat{\sigma}_i^2))^3$ resulting in $E(\hat{\gamma}_i^2\hat{\sigma}_i^6) \geq (E(\hat{\sigma}_i^2))^3(E(\hat{\gamma}_i))^2$. Thus, the error of approximation is of order $O_p(a^{-1/2})$.

Next, we want to discuss the connection between $\widehat{F}_M^{(2)}(x)$ and $\hat{F}_M(x)$ an estimate of the cdf of M_a given in (2.3.1) with $\hat{\kappa}_{11}^g$ and $\hat{\kappa}_{33}^g$ being $a^{-1/2}$ consistent unbiased estimate for κ_{11}^g and κ_{33}^g respectively. The form of $\hat{F}_M(x)$ is

$$\hat{F}_M(x) = \Phi(x) + \frac{1}{\sqrt{a}}\widetilde{Q}_1(x)\phi(x), \tag{2.4.3}$$

where

$$\widetilde{Q}_1(x) = -\hat{\kappa}_{11}^g - \frac{\hat{\kappa}_{33}^g}{6}(x^2 - 1). \tag{2.4.4}$$

with

$$\hat{\kappa}_{11}^{g} = \frac{-2}{\hat{h}(u)^3 a}\sum_{i=1}^{a}\frac{\widehat{\sigma_i^6(\gamma_i^2-2)}}{n_i^2(n_i-1)^2} \quad , \quad \hat{\kappa}_{33}^{g} = \frac{2}{\hat{h}(u)^3 a}\sum_{i=1}^{a}\frac{\widehat{\sigma_i^6(5\gamma_i^2+4n_i-14)}}{n_i^2(n_i-1)^2},$$

$$\hat{h}(u) = \sqrt{\frac{2}{a(a-1)^2}\sum_{i\neq i'}\frac{S_i^2 S_{i'}^2}{n_i n_{i'}} + \frac{2}{a}\sum_{i=1}^{a}\frac{\widehat{\sigma_i^4}}{n_i(n_i-1)}},\ \hat{\gamma}_i = \frac{n_i}{(n_i-1)(n_i-2)}\sum_{j=1}^{n_i}\left\{\frac{X_{ij}-\overline{X}_{i\cdot}}{S_i}\right\}^3.$$

where $\widehat{\sigma_i^6(\gamma_i^2-2)}$ and $\widehat{\sigma_i^6(5\gamma_i^2+4n_i-14)}$ are unbiased estimates of $\sigma_i^6(\gamma_i^2-2)$ and $\sigma_i^6(5\gamma_i^2+4n_i-14)$, respectively. Hall (1986), considered coverage probabilities of confidence intervals and showed that bootstrap approximation to the distribution of a pivotal statistic is assymptotically equivalent to that of estimated first-order Edgeworth expansion approximation. Using transformation of pivotal statistic, Abramovitch and Singh (1985) also showed related result. Hall (1986) and Abramovitch and Singh (1985) results pertain to iid data and the classical large n_i case. In our current setting of large a small n_i, the theoretical results show that the estimated first-order Edgeworth expansion $\hat{F}_M(x)$ in (2.4.3) has a better approximation to $F_M(x)$ in (2.3.4) than bootstrap approximation $\widehat{F}_M^{(2)}(x)$ in (2.4.2). We give the following examples to demonstrate this.

We simulate data from a skewed population Chi-square distribution with degrees of freedom 3, with $a = 20$ and small group sizes; 4, 4, 4, 4, 4, 4, 4, 4, 6, 6, 4, 4, 5, 4, 4, 4, 4, 4, 4, 5. The data satisfies the null hypothesis. We compute the test statistic $M_a(\underline{X})$ presented in (2.2.6). The data generation and computation of the test statistic were repeated 5000 times to obtain the Monte Carlo density and distribution functions of the test statistic $M_a(\underline{X})$ presented in (2.2.6). We compare the approximations of $\widehat{F}_M^{(2)}(x)$ and $\hat{F}_M(x)$ given in (2.4.2) and (2.4.3) respectively, to the Monte Carlo cdf of the test statistic $M_a(\underline{X})$. The bootstrap density and Edgeworth expansion of density can also be compared. The estimate of the density function of (2.3.1) is given by

$$\hat{f}_M(x) = \phi(x) + \frac{1}{\sqrt{a}}\left[\hat{\kappa}_{11}^{g}x + \hat{\kappa}_{33}^{g}\left(\frac{x^3-3x}{6}\right)\right]\phi(x) + O(a^{-1}), \tag{2.4.5}$$

where $\hat{\kappa}_{11}^{g}$ and $\hat{\kappa}_{33}^{g}$ are given in (2.4.4). We compute the kernel density estimate of 2000 bootstrap statistics M_a^* and $\widehat{f}_M(x)$ presented in (2.4.5) to the Monte Carlo pdf of the test statistic $M_a(\underline{X})$. The plot of $\widehat{F}_M^{(2)}(x)$ (in red), $\hat{F}_M(x)$ (in blue) and the Monte Carlo cdf of the 5000 runs of $M_a(\underline{X})$ (in black) is shown on the left panel in Figure 2. The plot on the right panel in Figure 2 shows the kernel density estimate of 2000 bootstrap statistics M_a^* (in red), $\widehat{f}_M(x)$ presented in (2.4.5) (in blue) and the Monte Carlo pdf of the test statistic $M_a(\underline{X})$ (in black). The estimated pdf plot in Figure 2 also supports that the probability density functions for the bootstrap and first-order Edgeworth expansions are good approximations to the Monte Carlo pdf of the 5000 runs of M_a. In our next example, we again generate data for 20 groups (treatments), each from the Chi-square distribution with degrees of freedom 3. This time we consider moderate group sizes of n_i being 10, 10, 10, 10, 10, 10, 10, 10, 12, 12, 10, 10, 11, 10, 10, 10, 10, 10, 10, 11. The data satisfies the null hypothesis. Figure 3 displays the probability density and cummulative distribution curves of the bootstrap, first-order Edgeworth expansion and the Monte Carlo pdf and cdf of $M_a(\mathbf{X})$. The Monte Carlo pdf and cdf of $M_a(\mathbf{X})$, kernel density estimate of bootstrap statistics from one sample and our first-order Edgeworth expansion $\hat{F}_M(x)$ are computed similarly as in small n_i case. The plot on the right panel of Figure 3 shows the kernel density estimate of 2000 bootstrap statistics M_a^* (in red), $\widehat{f}_M(x)$ presented in (2.4.5) (in blue) and the Monte Carlo pdf of the test statistic $M_a(\underline{X})$ (in black). Figure 3 below, displays the probability density curves of the bootstrap, first-order Edgeworth expansion and the Monte Carlo pdf of $M_a(\underline{\mathbf{X}})$. From the estimated cdf plot in Figure 3, we observe that $\widehat{F}_M^{(2)}(x)$ and $\hat{F}_M(x)$ provide better approximations to the Monte Carlo cdf of the 5000 runs of $M_a(\underline{\mathbf{X}})$ for moderate group sizes than previous cas with small n_i's. The estimated pdf plot in Figure 3 also supports that the probability density functions for the bootstrap and first-order Edgeworth expansions are close and both are good approximations to the Monte Carlo pdf of the 5000 runs of M_a for moderate group sizes. Comparing Figures 2 and 3, we see that the approximations to Monte Carlo cdf of the 5000 runs of $M_a(\underline{\mathbf{X}})$ becomes better as the sample sizes n_i's increases. In the setting for our examples, the proposed statistic can be used for both small and moderate sample sizes. Both $\widehat{F}_M^{(2)}(x)$ and $\hat{F}_M(x)$ presented in (2.4.2) and (2.4.3) respectively, approximate $F_M(x)$

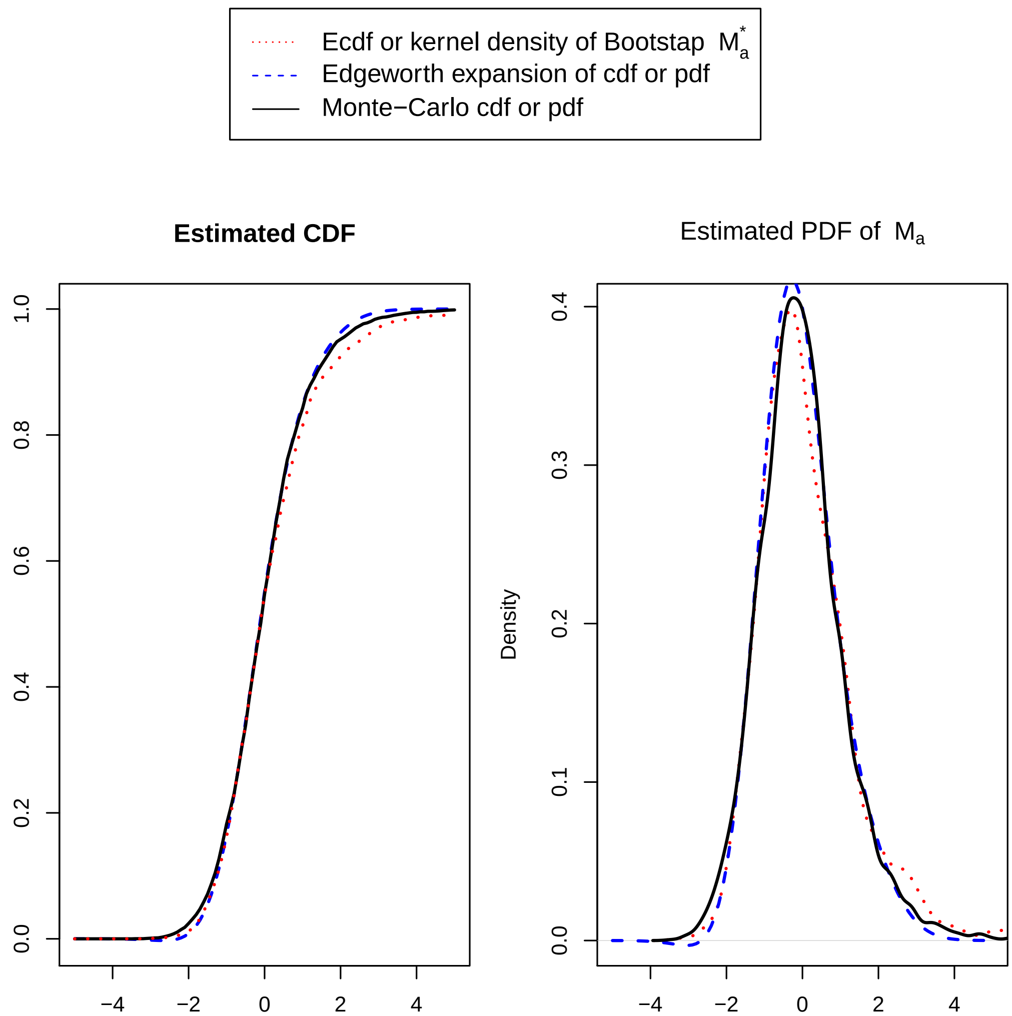

Figure 2.1: *Empirical cdf and kernel Density estimate of 2000 bootstrap statistics M_a^* from one sample vs. Monte Carlo pdf and cdf of M_a vs. First order Edgeworth expansion cdf (2.4.3) and pdf (2.4.5). The data contains 20 groups of χ_3^2 samples of group sizes 4, 4, 4, 4, 4, 4, 4, 4, 6, 6, 4, 4, 5, 4, 4, 4, 4, 4, 4, 5.*

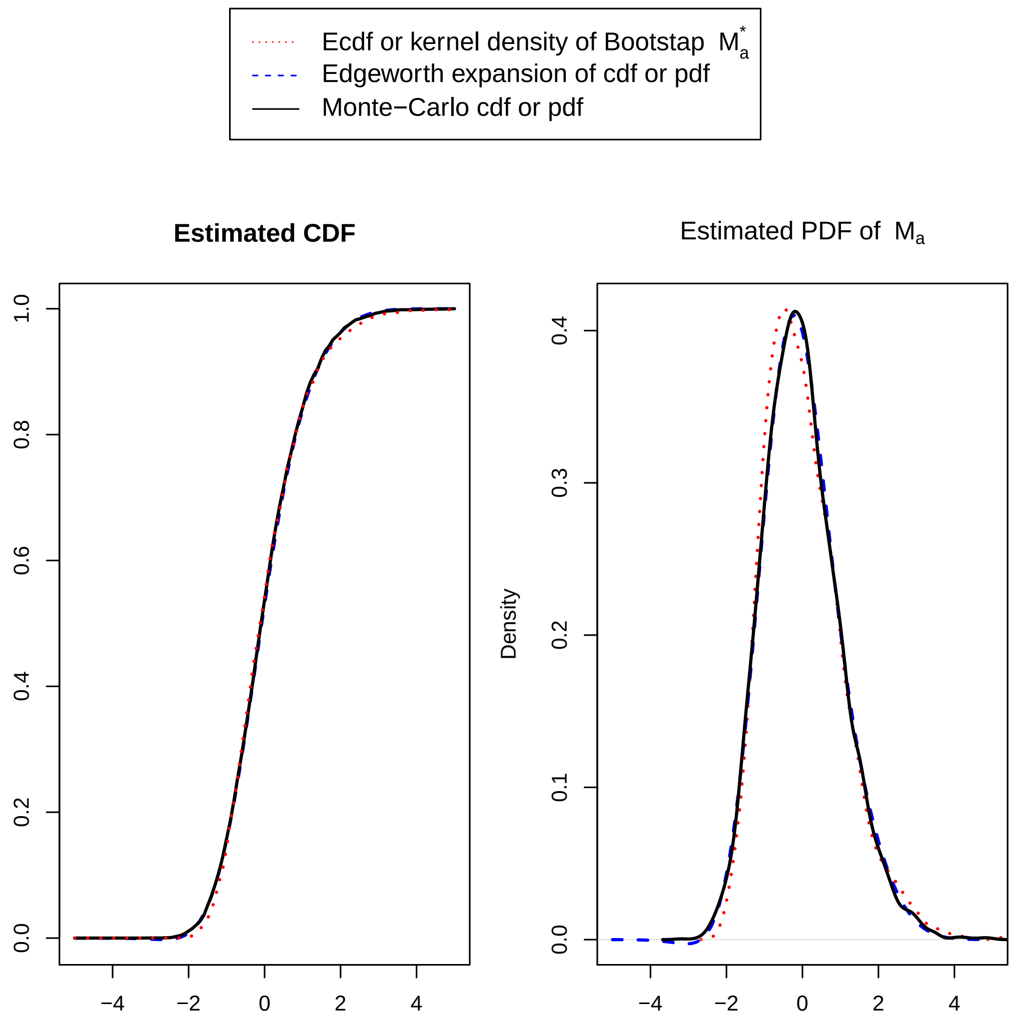

Figure 2.2: *Empirical cdf and kernel Density estimate of 2000 bootstrap statistics M_a^* from one sample vs. Monte Carlo pdf and cdf of M_a vs. First order Edgeworth expansion cdf (2.4.3) and pdf (2.4.5). The data contains 20 groups of χ_3^2 samples of group sizes 10, 10, 10, 10, 10, 10, 10, 10, 12, 12, 10, 10, 11, 10, 10, 10, 10, 10, 10, 11.*

in (2.3.4) well. They are much better than the bootstrap approximation of Fisher and Hall (1990) to their statistic.

2.5 A new test and its connection with the Bootstrap test

In this section, we specify a new rejection region to test the null hypothesis of no treatment effect.

We define $\hat{\omega}_\alpha$, the estimated quantile of ω_α presented in equation (2.3.3) as

$$\hat{\omega}_\alpha = z_\alpha + \frac{1}{\sqrt{a}}\hat{q}_1(z_\alpha) \tag{2.5.1}$$

where

$$\hat{q}_1(z_\alpha) = -\widetilde{Q}_1(z_\alpha) = \hat{\kappa}^g_{11} + \frac{\hat{\kappa}^g_{33}}{6}(z^2_\alpha - 1). \tag{2.5.2}$$

$\hat{\kappa}^g_{11}$ and $\hat{\kappa}^g_{33}$ are given in (2.4.4). Now, we define the new test rejection region based on the estimated first-order Cornish-Fisher expansion of the quantile in (2.5.1). To test the hypothesis of no treatment effect, i.e. $H_0 : \mu_i = \mu$ versus H_1 : at least one μ_i is different from μ, for some constant μ, we define the rejection region as

$$M_a(\underline{\mathbf{X}}) \geq \hat{\omega}_{1-\alpha} \tag{2.5.3}$$

i.e. the null hypothesis is rejected if the observed value of the test statistic M_a is more extreme than the critical values based on the estimated first-order Cornish-Fisher expansion of the quantiles. We consider one-sided test because under the alternative hypothesis, $H_a = E(X_{ij}) = \mu_i = \mu + \alpha_i$ where $\alpha_i \neq 0$, we have that $\widetilde{MST}(\underline{\mathbf{X}}) = \widetilde{MST}(\underline{\epsilon}) + 2\sqrt{a}(a-1)^{-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot} + c_a$ where $c_a = \frac{\sqrt{a}}{a-1}\sum_{i=1}^{a}\alpha_i^2$. Therefore under H_a, $E\left(\widetilde{MST}(\underline{\mathbf{X}}) - \widetilde{MSE}(\underline{\mathbf{X}})\right) \geq 0$.

Suppose we denote the analytical bootstrap quantile by $\hat{\omega}^b_\alpha$ which is an approximation of the true quantile of M_a, such that

$$\hat{\omega}^b_\alpha = z_\alpha + \frac{1}{\sqrt{a}}\hat{q}^b_1(z_\alpha) + \frac{1}{a}\hat{q}^b_2(z_\alpha) + \cdots,$$

where $\hat{q}^b_k(.)$ are functions of the plug-in estimates $\hat{Q}_k(.)$ in (2.4.2). As an example $\hat{q}^b_1(z_\alpha) = -\hat{Q}_1(z_\alpha)$. Then as discussed in section 1.2.3 the theoretical type I error-rate of the bootstrap test is accurate up to order $O(a^{-1/2})$. To have an idea what the estimated type I error-rate will be for the bootstrap test in practice, analytically, consider the form of $\widetilde{Q}_1(z_\alpha)$ in (2.4.4). $\hat{\kappa}^g_{11}$ and $\hat{\kappa}^g_{33}$ contain the term $\widehat{\sigma}^6_i \hat{\gamma}^2_i$. Since bootstrap method uses plug-in estimates, the estimation of $\widehat{\sigma}^6_i \hat{\gamma}^2_i$ results in $\hat{\kappa}^g_{11}$ underestimated and $\hat{\kappa}^g_{33}$ being inflated (notice the negative sign in $\hat{\kappa}^g_{11}$). Thus the bootstrap quantile becomes inflated which leads to the bootstrap test being conservative.

2.6 Type I error-rate of the proposed test

In this section, we derive the accuracy of the type I error rate of the test in (2.5.3). At a significance level of α, the probability of type I error is given in (2.6.1) below

$$P(M_a \geq \hat{\omega}_{1-\alpha}) \tag{2.6.1}$$

We can write $\hat{\omega}_\alpha$ as $\hat{\omega}_\alpha = \omega_\alpha + \hat{\omega}_\alpha - \omega_\alpha$, where ω_α is the true quantile of the distribution of M_a. We know that

$$\omega_\alpha = z_\alpha + \frac{1}{\sqrt{a}} q_1(z_\alpha) + \cdots ,$$

where $q_1(z_\alpha) = -Q_1(z_\alpha)$. Hence

$$\hat{\omega}_\alpha - \omega_\alpha = \frac{1}{\sqrt{a}}(\hat{q}_1(z_\alpha) - q_1(z_\alpha)) + O_p(a^{-1}) = O_p(a^{-1}),$$

since from (2.5.2) we know $\hat{q}_1(z_\alpha)$ is an unbiased estimate of $q_1(z_\alpha)$, thus $\hat{q}_1(z_\alpha) - q_1(z_\alpha) = O_p(a^{-1})$. Now using the above results, (2.6.1) can be written as

$$\begin{aligned} & P(M_a > \omega_{1-\alpha} + \hat{\omega}_{1-\alpha} - \omega_{1-\alpha}) \qquad (2.6.2) \\ = \; & P(M_a - (\hat{\omega}_{1-\alpha} - \omega_{1-\alpha}) > \omega_{1-\alpha}) \\ = \; & P(M_a > \omega_{1-\alpha}) + O(a^{-1}), \end{aligned}$$

since M_a and $M_a - (\hat{\omega}_\alpha - \omega_\alpha)$ all have the same first order Edgeworth expansion as a result of the Delta method in Hall (1992b) section 2.7. We need to compute $P(M_a > \omega_{1-\alpha})$. Write

$$P(M_a > \omega_{1-\alpha}) = 1 - P(M_a \leq \omega_{1-\alpha}) \quad = \quad 1 - F_M(\omega_{1-\alpha})$$

where $F_M(x)$ is the distribution of M_a presented in Theorem 2.3.1.

$$\begin{aligned} P(M_a > \omega_{1-\alpha}) &= 1 - \Phi\left(z_{1-\alpha} + \frac{q_1(z_{1-\alpha})}{\sqrt{a}} + O(1/a)\right) - \frac{1}{\sqrt{a}} Q_1\left(z_{1-\alpha} + \frac{q_1(z_{1-\alpha})}{\sqrt{a}} + O(1/a)\right) \\ &* \quad \phi\left(z_{1-\alpha} + \frac{q_1(z_{1-\alpha})}{\sqrt{a}} + O(1/a)\right) + O(a^{-1}). \end{aligned} \tag{2.6.3}$$

Next, we apply Taylor expansion to the expressions $\Phi\left(z_{1-\alpha} + \frac{q_1(z_{1-\alpha})}{\sqrt{a}} + O(1/a)\right)$, $Q_1\left(z_{1-\alpha} + \frac{q_1(z_{1-\alpha})}{\sqrt{a}} + O(1/a)\right)$ and $\phi\left(z_{1-\alpha} + \frac{q_1(z_{1-\alpha})}{\sqrt{a}} + O(1/a)\right)$ at $z_{1-\alpha}$. We have

$$\Phi\left(z_{1-\alpha} + \frac{q_1(z_{1-\alpha})}{\sqrt{a}} + O(1/a)\right) = \Phi(z_{1-\alpha}) + \frac{1}{\sqrt{a}} q_1(z_{1-\alpha})\phi(z_{1-\alpha}) + O(a^{-1}) \tag{2.6.4}$$

$$Q_1\left(z_{1-\alpha} + \frac{q_1(z_{1-\alpha})}{\sqrt{a}} + O(1/a)\right) = Q_1(z_{1-\alpha}) + \frac{1}{\sqrt{a}} q_1(z_{1-\alpha})Q_1^{'}(z_{1-\alpha}) + O(a^{-1}) \tag{2.6.5}$$

$$\phi\left(z_{1-\alpha} + \frac{q_1(z_{1-\alpha})}{\sqrt{a}} + O(1/a)\right) = \phi(z_{1-\alpha}) + \frac{1}{\sqrt{a}} q_1(z_{1-\alpha})\phi^{'}(z_{1-\alpha}) + O(a^{-1}) \tag{2.6.6}$$

Substituting (2.6.4), (2.6.5) and (2.6.6) into (2.6.3), we have

$$P(M_a > \omega_{1-\alpha}) = 1 - \Phi(z_{1-\alpha}) - \frac{1}{\sqrt{a}}\left[q_1(z_{1-\alpha}) + Q_1(z_{1-\alpha})\right]\phi(z_{1-\alpha}) + O(a^{-1}). \tag{2.6.7}$$

We know that $\Phi(z_{1-\alpha}) = 1 - \alpha$ and $q_1(z_{1-\alpha}) = -Q_1(z_{1-\alpha})$. Therefore

$$P(M_a > \omega_{1-\alpha}) = \alpha + O(a^{-1}). \tag{2.6.8}$$

Combine this with (2.6.2) we get

$$P(M_a \geq \hat{\omega}_{1-\alpha}) = \alpha + O(a^{-1}).$$

That is the accuracy of the type I error-rate of the test in (2.5.3) is of order $O(a^{-1})$.

2.7 Power of the proposed test

In this section, we present the distribution of the test statistic under the local alternative. We will also present the theoretical power of our new test.

We consider the decomposition $E(X_{ij}) = \mu_i = \mu + \alpha_i$ where $\sum_{i=0}^{a} \alpha_i = 0$. Then our hypothesis of $H_0 : \mu_i = \mu$ versus H_1 : at least one μ_i is different from μ, for some constant μ, is reformulated as $H_0 : \alpha_i = 0$ versus H_a : at least one $\alpha_i > 0$, $i = 1, \cdots, a$. Under the alternative hypothesis, we have that

$$\widetilde{MST}(\underline{\mathbf{X}}) = \widetilde{MST}(\underline{\epsilon}) + 2\sqrt{a}(a-1)^{-1} \sum_{i=1}^{a} \alpha_i \bar{\epsilon}_{i\cdot} + c_a \tag{2.7.1}$$

where

$$c_a = \frac{\sqrt{a}}{a-1} \sum_{i=1}^{a} \alpha_i^2.$$

We consider the case that c_a converges to a constant under the alternative hypothesis. More specifically, we assume the departure from the null hypothesis is of order

$$\alpha_i = O(a^{-\frac{1}{4}}), \text{for all } i = 1, \cdots, a. \tag{2.7.2}$$

In this case, the test statistic $M_a(\underline{\mathbf{X}})$ can be written as

$$\begin{aligned} M_a(\underline{\mathbf{X}}) &= \frac{\sqrt{a}(\widetilde{MST}(\underline{\mathbf{X}}) - \widetilde{MSE}^{(2)}(\underline{\mathbf{X}}))}{\sqrt{\frac{2}{a(a-1)^2} \sum_{i \neq i'} \frac{S_i^2 S_{i'}^2}{n_i n_{i'}} + \frac{2}{a} \sum_{i=1}^{a} \frac{\widehat{\sigma_i^4}}{n_i(n_i-1)}}} \\ &= \frac{\sqrt{a}(\widetilde{MST}(\underline{\epsilon}) - \widetilde{MSE}^{(2)}(\underline{\mathbf{X}})) + 2\sqrt{a}(a-1)^{-1} \sum_{i=1}^{a} \alpha_i \bar{\epsilon}_{i\cdot} + c_a}{\sqrt{\frac{2}{a(a-1)^2} \sum_{i \neq i'} \frac{S_i^2 S_{i'}^2}{n_i n_{i'}} + \frac{2}{a} \sum_{i=1}^{a} \frac{\widehat{\sigma_i^4}}{n_i(n_i-1)}}} \end{aligned} \tag{2.7.3}$$

To present the distribution of the test statistic $M_a(\underline{\mathbf{X}})$ in (2.7.3) we define the following averages;

$$Y_{1a} = \frac{1}{a} \sum_{i=1}^{a} \sum_{j \neq j'} \frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}, \qquad Y_{2a} = \frac{-1}{a(a-1)} \sum_{i \neq i'} \bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot},$$

$$Y_{3a} = \frac{1}{a(a-1)} \sum_{i \neq i'} \frac{S_i^2 S_{i'}^2}{n_i n_{i'}}, \qquad Y_{4a} = \frac{1}{a} \sum_{i=1}^{a} \frac{\widehat{\sigma_i^4}}{n_i(n_i-1)}, \qquad Y_{5a} = \frac{1}{a-1} \sum_{i=1}^{a} \alpha_i \bar{\epsilon}_{i\cdot}.$$

Let $\underline{\mathbf{Y}} = (Y_{1a}, Y_{2a}, Y_{3a}, Y_{4a}, Y_{5a})^{'}$ and $\mathbf{u}$ be its mean i.e.,

$$\begin{aligned}
\mathbf{u} &= E(\mathbf{Y}) = (E(Y_{1a}), E(Y_{2a}), E(Y_{3a}), E(Y_{4a}), E(Y_{5a}))^{'} \\
&= (u_1, u_2, u_3, u_4, u_5)^{'} \\
&= \left(0, \quad 0, \quad \frac{1}{a(a-1)}\sum_{i\neq i'}^{a}\frac{\sigma_i^2\sigma_{i'}^2}{n_i n_{i'}}, \qquad \frac{1}{a}\sum_{i=1}^{a}\frac{\sigma_i^4}{n_i(n_i-1)}, \quad 0\right)^{'}
\end{aligned}$$

The test statistic $M_a(\underline{\mathbf{X}})$ under the local alternative in (2.7.3) can be written as

$$M_a(\underline{\mathbf{X}}) \quad \cong \quad W_a(\underline{\mathbf{Y}}) \quad = \quad \frac{\sqrt{a}(Y_{1a}+Y_{2a}+Y_{5a})+c_a}{h(\underline{\mathbf{Y}})} \quad = \quad \sqrt{a}g_{H_a}(\underline{\mathbf{Y}}) + g^*(\underline{\mathbf{Y}})$$

where

$$h(\underline{\mathbf{Y}}) \quad = \quad \sqrt{\frac{2}{a-1}Y_{3a}+2Y_{4a}} \quad ; \quad g_{H_a}(\underline{\mathbf{Y}}) \quad = \quad \frac{Y_{1a}+Y_{2a}+Y_{5a}}{h(\underline{\mathbf{Y}})} \quad ; g^*(\underline{\mathbf{Y}}) = \frac{c_a}{h(\underline{\mathbf{Y}})}.$$

By Taylor series expansion of $g_{H_a}(\underline{\mathbf{Y}})$ at $\mathbf{u}$, we write

$$\begin{aligned}
g_{H_a}(\underline{\mathbf{Y}}) &= g_{H_a}(\mathbf{u}) + \frac{\partial g_{H_a}(\mathbf{u})}{\partial \mathbf{u}}(\mathbf{Y}-\mathbf{u})^{'} + \frac{1}{2}(\mathbf{Y}-\mathbf{u})^{'}\frac{\partial^2 g_{H_a}(\mathbf{u})}{\partial \mathbf{u}^2}(\mathbf{Y}-\mathbf{u}) + O_p(\|\mathbf{Y}-\mathbf{u}\|^3) \\
&= \frac{1}{h(\mathbf{u})}(Y_{1a}+Y_{2a}+Y_{5a}) - \frac{h^{-3}(\mathbf{u})}{a-1}(Y_{3a}-u_3)(Y_{1a}+Y_{2a}+Y_{5a}) \\
&\quad - h^{-3}(\mathbf{u})(Y_{4a}-u_4)(Y_{1a}+Y_{2a}+Y_{5a}) + O_p(a^{\frac{-3}{2}}).
\end{aligned}$$

We have that,

$$\sqrt{a}g_{H_a}(\underline{\mathbf{Y}}) = G_1(\underline{\mathbf{Y}}) + G_2(\underline{\mathbf{Y}}) + G_3(\underline{\mathbf{Y}}) + O_p(a^{-1})$$

where

$$G_1(\underline{\mathbf{Y}}) = \frac{\sqrt{a}}{h(\mathbf{u})}(Y_{1a}+Y_{2a}+Y_{5a})$$

$$G_2(\underline{\mathbf{Y}}) = \frac{-\sqrt{a}h^{-3}(\mathbf{u})}{a-1}(Y_{3a}-u_3)(Y_{1a}+Y_{2a}+Y_{5a}) = O_p(a^{-2})$$

and

$$G_3(\underline{\mathbf{Y}}) = -\sqrt{a}h^{-3}(\mathbf{u})(Y_{4a}-u_4)(Y_{1a}+Y_{2a}+Y_{5a}).$$

Next, we apply Taylor series expansion to $g^*(\underline{\mathbf{Y}})$ at $\mathbf{u}$ and write

$$\begin{aligned} g^*(\underline{\mathbf{Y}}) &= g^*(\mathbf{u}) + \frac{\partial g^*(\mathbf{u})}{\partial \mathbf{u}}(\underline{\mathbf{Y}} - \mathbf{u})' + O_p(\|\underline{\mathbf{Y}} - \mathbf{u}\|^2) \\ &= \frac{c_a}{h(\mathbf{u})} + G_4(\underline{\mathbf{Y}}) + O_p(a^{-1}) \end{aligned}$$

where

$$G_4(\underline{\mathbf{Y}}) = -c_a h^{-3}(\mathbf{u})(Y_{4a} - u_4).$$

Therefore, the test statistic $W_a(\underline{\mathbf{Y}})$ under the local alternatives is written as

$$W_a(\underline{\mathbf{Y}}) = G_1(\underline{\mathbf{Y}}) + G_3(\underline{\mathbf{Y}}) + G_4(\underline{\mathbf{Y}}) + \frac{c_a}{h(\mathbf{u})} + O_p(a^{-1}).$$

To state the result, we state Cramer's condition in this case as

$$K3: \qquad \limsup_{||\mathbf{t}|| \to \infty} |E[\exp\{i(t_1Y_{1a} + t_2Y_{2a} + t_3Y_{3a} + t_4Y_{4a} + t_5Y_{5a})\}]| < 1, \forall a > 1,$$

where $\mathbf{t} = (t_1, t_2, t_3, t_4, t_5)$ and $||t|| = (t_1^2 + t_2^2 + t_3^2 + t_4^2 + t_5^2)^{1/2}$.

Theorem 2.7.1. *Suppose $n_i \geq 4$ are fixed for all i. Then under the local alternative hypothesis of order in (2.7.2) and regularity conditions $K1$ and $K3$, the distribution of the test statistic M_a given in (2.7.3) has the following asymptotic expansion*

$$F_M(x) = P(M_a \leq x) = \Phi(x) + \frac{1}{\sqrt{a}} Q_1^{alt}(x)\phi(x) + O(a^{-1}) \tag{2.7.4}$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the cumulative distribution and probability density functions of the standard normal distribution and

$$Q_1^{alt}(x) = -(\kappa_{11}^g + \kappa_{11}^{g_1}) - \frac{1}{2}[\kappa_{22}^{g_2} + 2c_a h^{-1}(\mathbf{u})(\kappa_{11}^g + \kappa_{11}^{g_1})]x - \frac{1}{6}(\kappa_{33}^g - 15\kappa_{11}^{g_1})(x^2 - 1) \tag{2.7.5}$$

with

$$\kappa_{11}^{g_1} = \frac{-4}{(a-1)h(\mathbf{u})^3} \sum_{i=1}^{a} \frac{\alpha_i \gamma_i \sigma_i^5}{n_i^2(n_i - 1)} \text{ and } \kappa_{22}^{g_2} = \frac{4a^{3/2}}{(a-1)^2 h(\mathbf{u})^2} \sum_{i=1}^{a} \frac{\alpha_i^2 \sigma_i^2}{n_i}$$

and κ_{11}^g, κ_{33}^g and $h(\mathbf{u})$ were defined in equation (2.3.2).

The proof of Theorem 2.7.1 is given in section 2.9.4.

2.8 Simulation Studies

In this section, we conduct simulation studies to investigate the performance of our proposed test. We compare both type I error-rate and power of our proposed test in (2.5.3) to that of the asymptotic test proposed by Akritas and Papadatos (2004) based on the test statistic T_a. We also compare with the test based on the asymptotic expansion of the null distribution of the classical F-statistic in Harrar and Gupta (2007), the Fisher and Hall (1990) bootstrap test based on their pivotal statistic T_2, and the Bootstrap test based on statistic (2.4.1). Again we compare to $\widehat{F}_M(x)$ in (2.4.3) based on the p-value.

2.8.1 Simulation setting

The simulations are based on 2000 replications and the data were generated from the following three distributions; chi-square with 3 degrees of freedom, chi-square with 8 degrees of freedom, and normal (0, 1). The three distribution has skewness parameter 1.63299, 1 and 0, respectively. All three distributions were used in Harrar and Gupta (2007). In the data generation, we consider both constant variance and heteroscedastic cases. For heteroscedastic cases, we consider both small and large heteroscedastic variances.

The simulation studies were conducted for small group sizes and consider when the variances are homoscedastic and heteroscedastic cases. The small group sizes n_i for the number of treatment levels a = 10, is 4, 5, 4, 6, 5, 6, 4, 5, 4, 4; for a =15, n_i = 5, 4, 4, 4, 4, 4, 6, 4, 4, 5, 5, 4, 4, 5, 4; for a = 20, n_i = 4, 4, 4, 4, 4, 4, 4, 4, 6, 6, 4, 4, 5, 4, 4, 4, 4, 4, 5. When a = 25, 50, 75 and 100, we use the group sizes $6, 4, 5, 4, 5, 4, 4, 4, 5, 4, 6, 4, 4, 5, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 6, \cdots, 6)$, where all omitted n_i's are equal to 6. Under the null, for homoscedastic variance case, we generate our data as follow:

- D1: Standard normal N(0,1).
- D2: χ^2 with 3 degrees of freedom.
- D3: χ^2 with 8 degrees of freedom.

Considering small heteroscedastic variances, we generate the data as follows:

- D4: $Y_{ij} = i * \tau/a + log(i+1) * \epsilon_{ij}$, where ϵ_{ij} are i.i.d. $N(0,1)$.
- D5: $Y_{ij} = 3 * (i * \tau/a)^2 + log(i+1) * (X_{ij} - 3)$, where X_{ij} are i.i.d. χ_3^2.
- D6: $Y_{ij} = 8 * (i * \tau/a)^2 + log(i+1) * (X_{ij} - 8)$, where X_{ij} are i.i.d. χ_8^2.

For the large heteroscedastic variances the data generation are as follows:

- D7: $Y_{ij} = 3 * (i * \tau/a)^2 + 3 * log(i+1) * (X_{ij} - 3)$, where X_{ij} are i.i.d. χ_3^2.
- D8: $Y_{ij} = 8 * (i * \tau/a)^2 + 8 * log(i+1) * (X_{ij} - 8)$, where X_{ij} are i.i.d. χ_8^2.

We report the type I error-rate results for the number of treatment levels $a =$ 10, 15, 20, 25, 50, 75 and 100 with nominal $\alpha = 0.05$ for only the homoscedastic data $D1 - D3$. For heteroscedastic data we report the type I error-rate results for the number of treatment levels $a =$ 10, 15, 20, 25, 50, 75, 100, 150 and 200 with nominal $\alpha = 0.05$. Since the signal to noise ratio are the same for $D5$ and $D7$, and, $D6$ and $D8$, the type I error-rate for $D5$ and $D7$ are the same and that of $D6$ and $D8$ are also the same. We report the achieved power of our simulation studies for the number of treatment levels $a =$ 75, 100, 150 and 200, at nominal level $\alpha = 0.05$ for $D6 - D8$. When $\tau = 0$, the data is under the null. For the alternative hypothesis, we let τ take value in (0, 4). The values of τ are specified in the tables.

2.8.2 Simulation results

Table 2.1 shows the estimated type I error-rate for homoscedastic cases for the asymptotic test in Akritas and Papadatos (2004) labeled as AP, the asymptotic expansion in Harrar and Gupta (2007) labeled as EHG and our test in (2.5.3) labeled as CF. It is clear from Table 2.1 that for more skewed distribution χ_3^2 with constant variances both our test CF and EHG have empirical type I error converges to the nominal level faster than the AP test. Under heteroscedastic variances, the estimated type I error-rates are given in Table 2.2 for the EHG, AP test, Fisher and Hall (1990) bootstrap test based on their pivotal statistic T_2 based on both quantile (HF.reject) and p-value approach (HF.p), our Bootstrap test based on statistic (2.4.1) using quantile (Boot.CF.reject) and p-value approach (Boot.CF.p). We also listed a test WR, which uses p-value computed from $\widehat{F}_M(x)$ in (2.4.3) to make a conclusion. The results show that, HF.reject and HF.p did not reject any test. This consistent to the fact demonstrated in Figure 1.1 which shows that the values of the bootstrap statistic T_{02}^* are much larger than that of T_2. This leads to a large bootstrap quantile than the value of the test statistic T_2, thus leading to almost no rejections. Our test CF performed better than the bootstrap tests, EHG and AP in the settings of large number of treatments with small replications under the presence of heteroscedastic and skewed data. We can see that CF approached the desired nominal level of 0.05 faster than other tests. The WR using p-value also reached the nominal level for large a but had very liberal type I error for smaller a. This is because the approximation of cdf for smaller a might have abnormal behaviour at both ends of the cdf.

Next, we assess the power achieved using the heteroscedastic data $D6$, $D7$ and $D8$ described in the simulation setting section. The left panels of Figure 2.3 through Figure 2.14 display the power achieved by our test CF, our bootstrap test (BootCF), the test in Akritas and Papadatos (2004) and the test of Harrar and Gupta (2007). The right panels of Figures 2.3 through 2.14 plot the differences in power for BootCF - CF (red), AP - CF (green) and EHG - CF (blue). We observe that CF and AP have comparable power but better better than that of EHG and BootCF in the presence of heteroscedastic and skewed data.

In summary, the numerical results provided in the above simulation studies show that for large number of treatments, our test based on asymptotic expansion of our proposed test statistic is satisfactory for skewed data and even symmetric data under both homoscedastic and heteroscedastic variances.

Test	a	$D1$	$D2$	$D3$
AP	10	10.0	7.4	7.3
	15	8.0	7.0	8.0
	20	8.8	8.1	8.0
	25	9.2	6.6	7.5
	50	8.0	7.2	6.1
	75	5.6	6.5	5.1
	100	5.8	5.8	6.0
EHG	10	5.2	4.7	4.9
	15	5.4	4.1	5.0
	20	5.0	5.1	4.8
	25	6.6	4.6	5.2
	50	5.6	5.4	4.8
	75	4.4	5.5	4.0
	100	4.2	4.4	5.6
CF	10	7.4	6.7	6.8
	15	7.0	5.8	5.5
	20	7.6	6.1	5.1
	25	8.0	5.6	5.0
	50	5.2	6.4	5.1
	75	4.6	5.8	4.4
	100	5.0	5.1	5.0

Table 2.1: *Percent of rejection for homoscedastic (D1, D2, D3) cases,* $\alpha = 0.05$.

Distr.	a	HF_p	HF.reject	Boot.CF.p	Boot.CF.reject	EHG	AP	WR	CF
$D4$	10	0.0	0.0	3.3	3.1	5.9	9.0	13.2	8.2
	15	0.0	0.0	4.2	3.9	7.3	10.0	12.4	9.2
	20	0.0	0.0	3.8	3.5	7.0	10.5	10.9	9.1
	25	0.0	0.0	3.3	3.1	6.2	7.8	8.5	6.6
	50	0.0	0.0	4.3	4.0	4.0	7.1	7.3	7.2
	75	0.0	0.0	5.8	5.8	5.1	7.3	7.0	7.0
	100	0.0	0.0	4.8	4.8	4.3	7.3	7.2	7.0
	150	0.0	0.0	4.7	4.7	3.8	6.7	6.0	6.0
	200	0.0	0.0	3.9	4.1	3.3	5.8	5.2	5.2
$D5$ and $D7$	10	0.0	0.0	2.3	2.1	6.2	8.2	19.3	7.5
	15	0.0	0.0	1.4	1.4	5.7	7.3	18.8	5.7
	20	0.0	0.0	1.9	1.8	6.0	7.6	21.5	5.1
	25	0.0	0.0	2.4	2.3	9.3	9.8	24.9	6.2
	50	0.0	0.0	2.5	2.4	3.2	5.7	9.1	5.5
	75	0.0	0.0	2.8	2.9	3.1	5.9	5.5	5.5
	100	0.0	0.0	3.7	3.9	3.7	6.4	6.0	6.0
	150	0.0	0.0	4.3	4.4	4.0	6.1	5.7	5.7
	200	0.0	0.0	3.4	3.6	3.1	5.3	4.9	4.9
$D6$ and $D8$	10	0.0	0.0	3.0	2.9	7.2	9.1	16.7	7.4
	15	0.0	0.0	2.4	2.3	6.5	9.0	15.3	7.1
	20	0.0	0.0	2.5	2.4	6.8	9.4	15.8	7.1
	25	0.0	0.0	2.4	2.4	8.6	8.4	16.4	6.8
	50	0.0	0.0	3.8	3.8	3.9	7.0	7.2	5.4
	75	0.0	0.0	3.6	3.8	3.0	5.9	5.9	5.9
	100	0.0	0.0	4.3	4.4	3.4	6.6	6.1	6.1
	150	0.0	0.0	4.1	4.3	3.3	6.0	5.6	5.6
	200	0.0	0.0	4.1	4.2	3.6	5.6	5.0	5.0

Table 2.2: *Percent of rejection under H_0 for $D4$, $D5(D7)$ and $D6(D8)$ with $\tau = 0$ at $\alpha = 0.05$.*

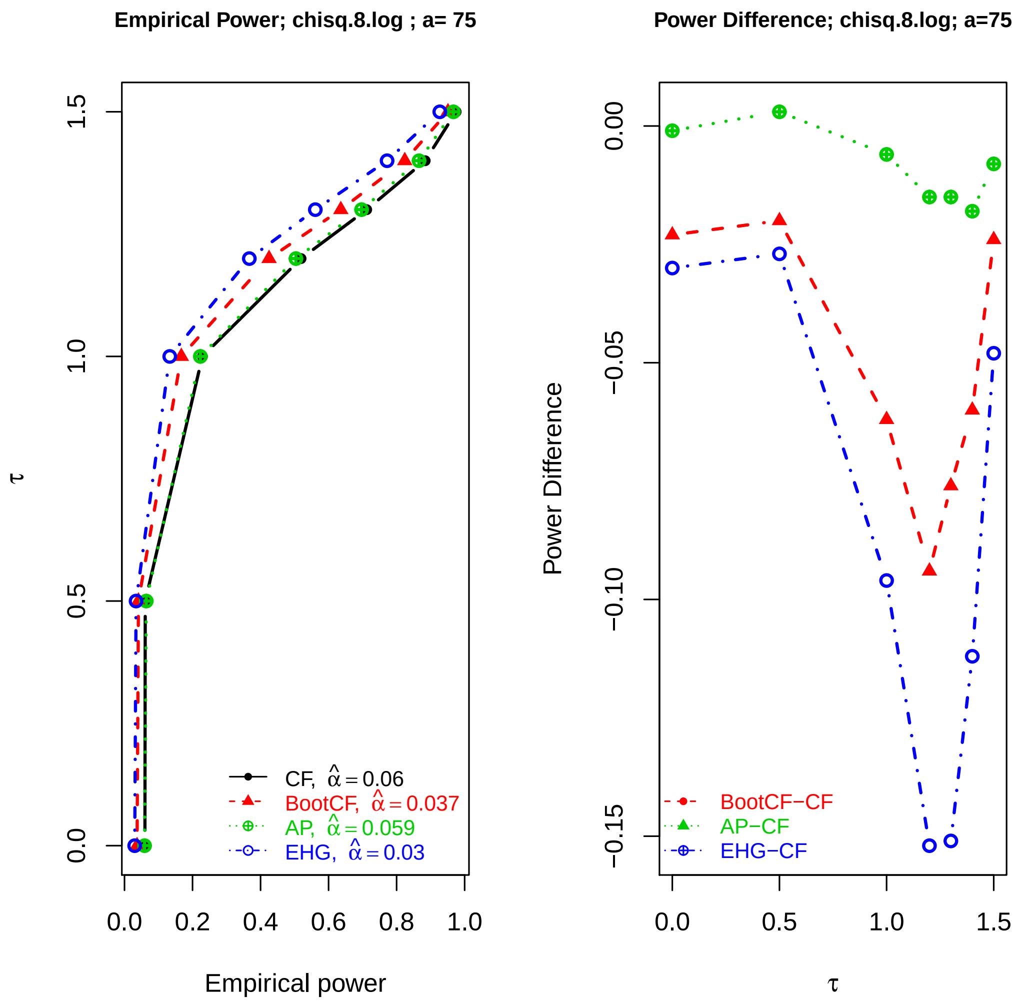

Figure 2.3: *Achieved Power for heteroscedastic χ^2_8 data D6, $a = 75$, $\alpha = 0.05$.*

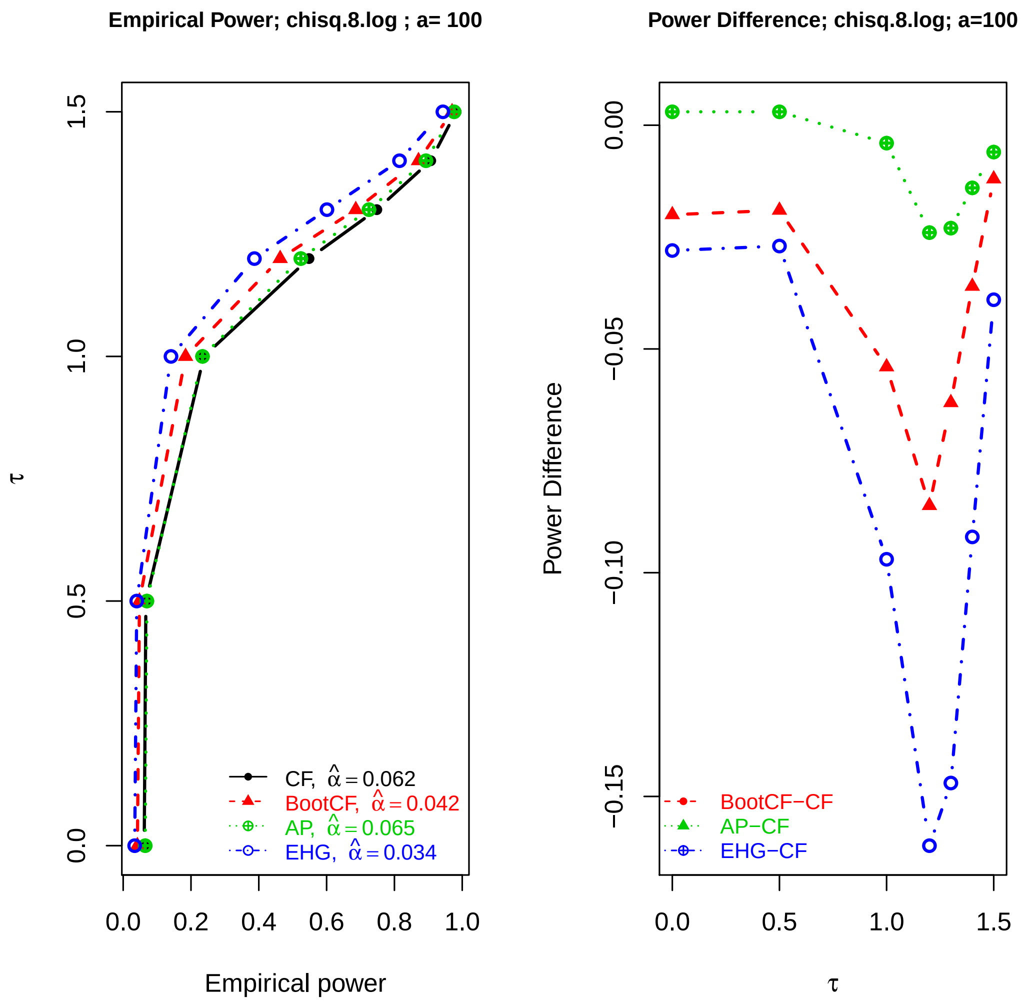

Figure 2.4: *Achieved Power for heteroscedastic* χ^2_8 *data D6, a =* 100*,* $\alpha = 0.05$.

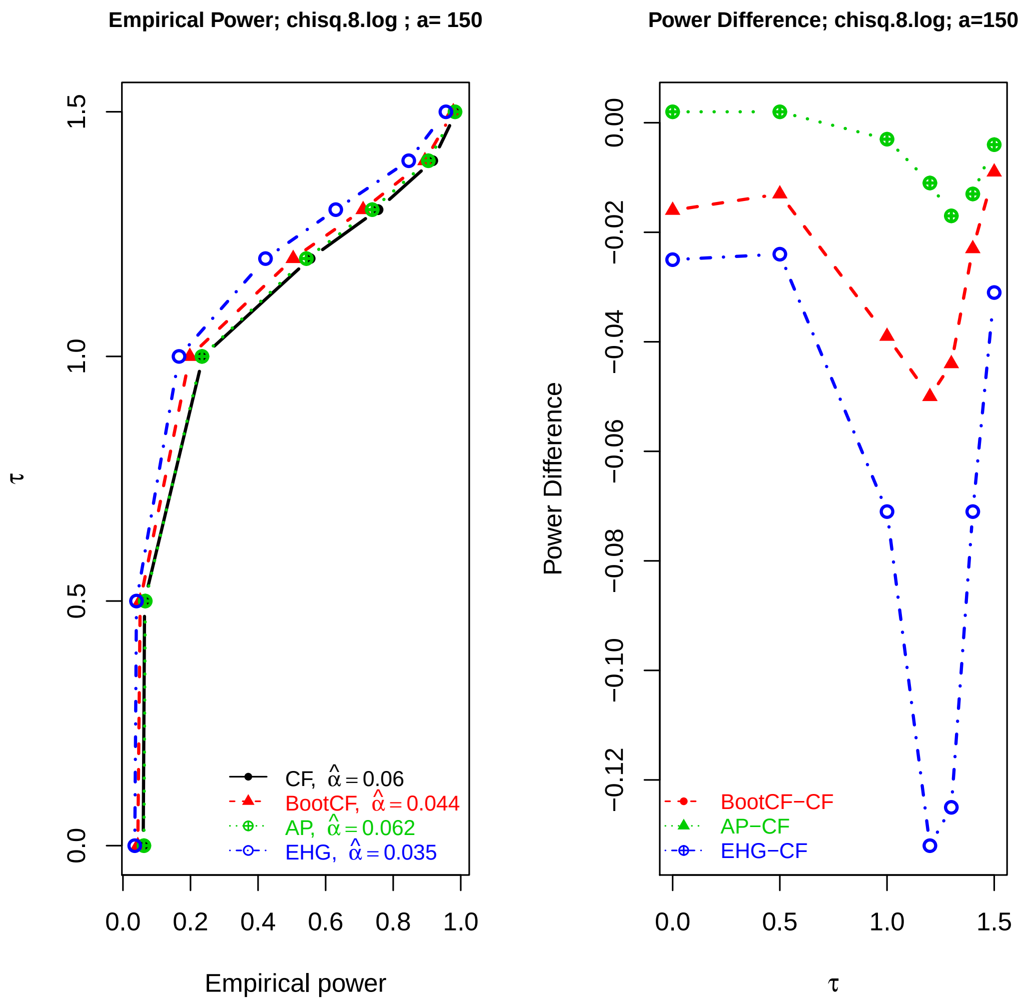

Figure 2.5: *Achieved Power for heteroscedastic* χ^2_8 *data D6, a =* 150, $\alpha = 0.05$.

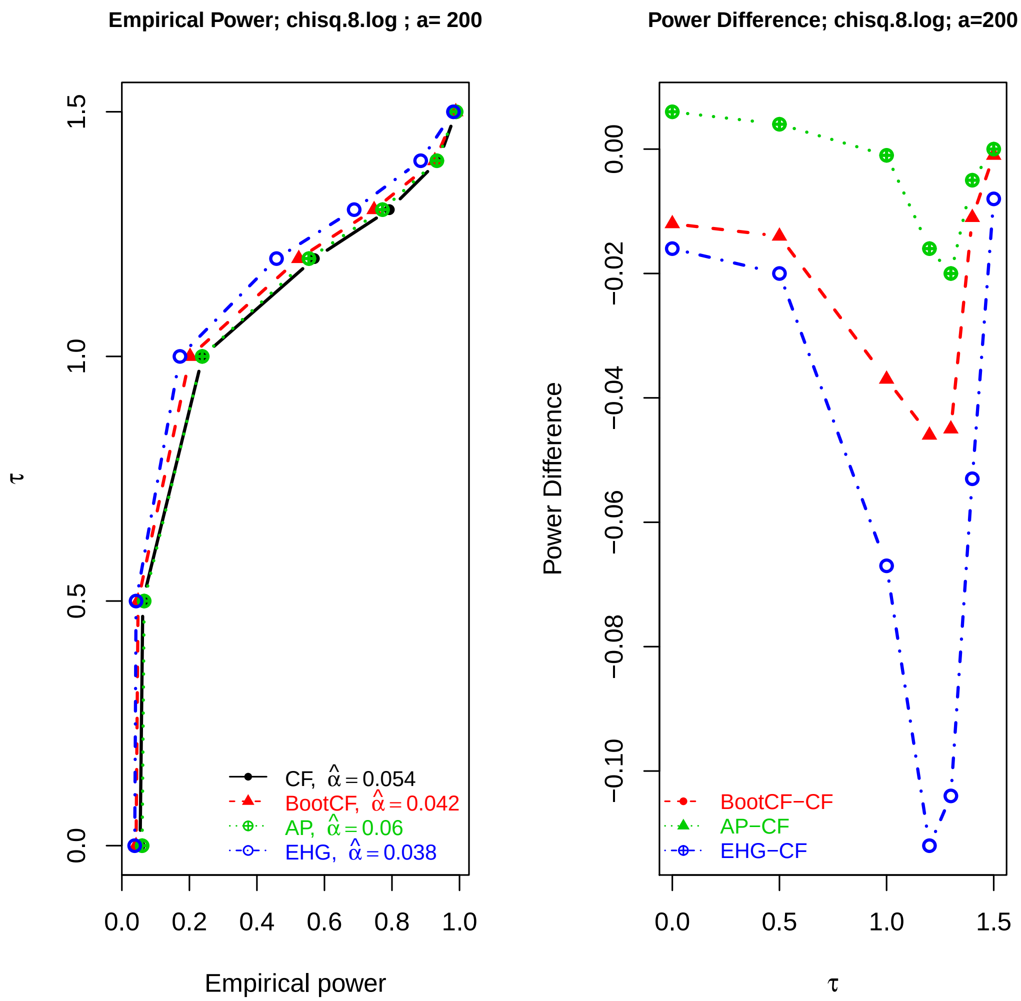

Figure 2.6: *Achieved Power for heteroscedastic χ^2_8 data D6, $a =$ 200, $\alpha = 0.05$.*

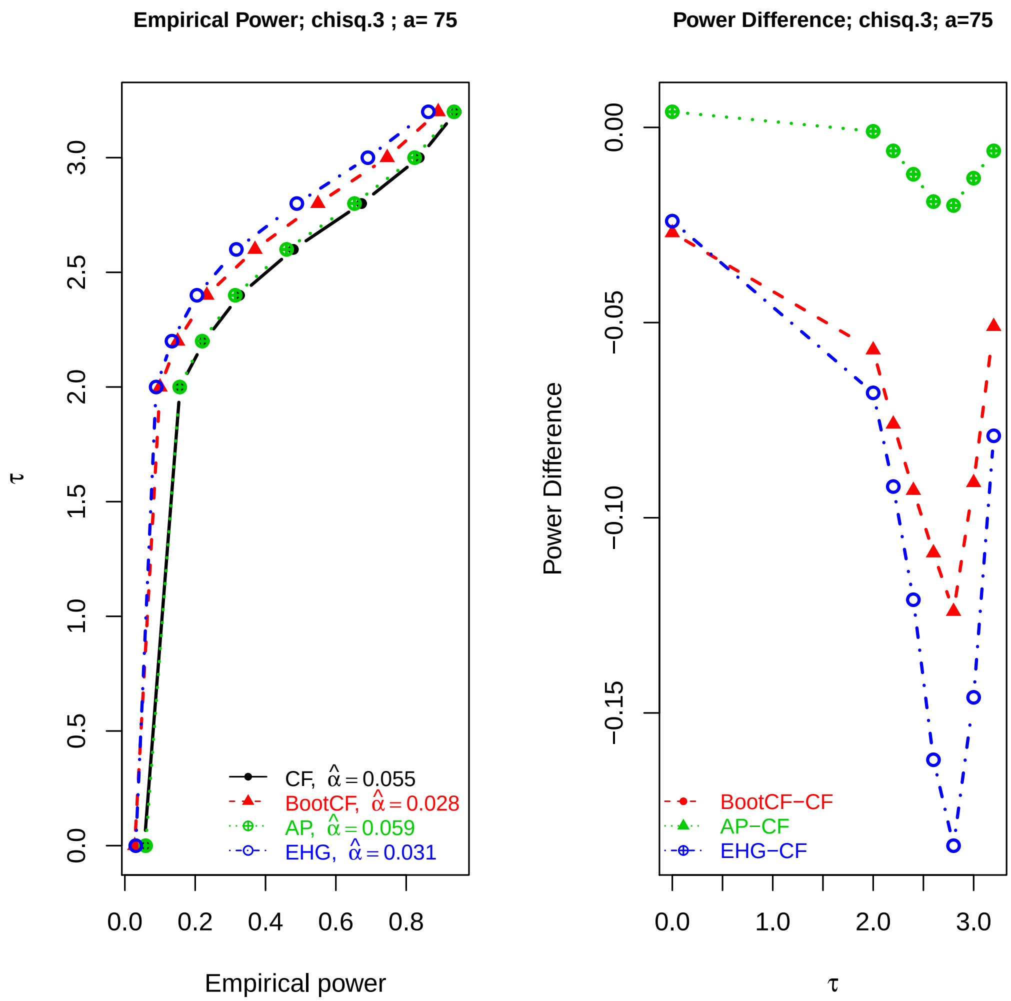

Figure 2.7: *Achieved Power for heteroscedastic* χ^2_3 *data D7,* $a = 75$*,* $\alpha = 0.05$.

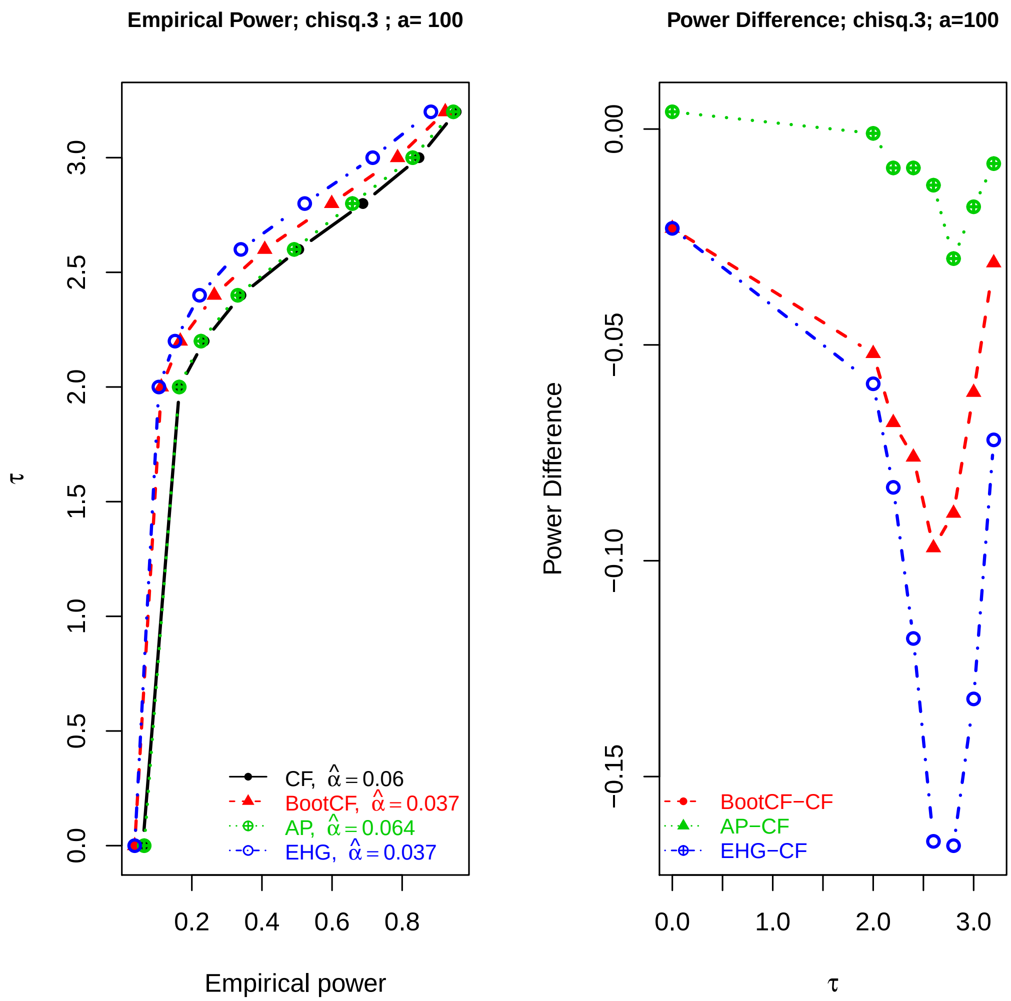

Figure 2.8: *Achieved Power for heteroscedastic χ^2_3 data D7, a = 100, $\alpha = 0.05$.*

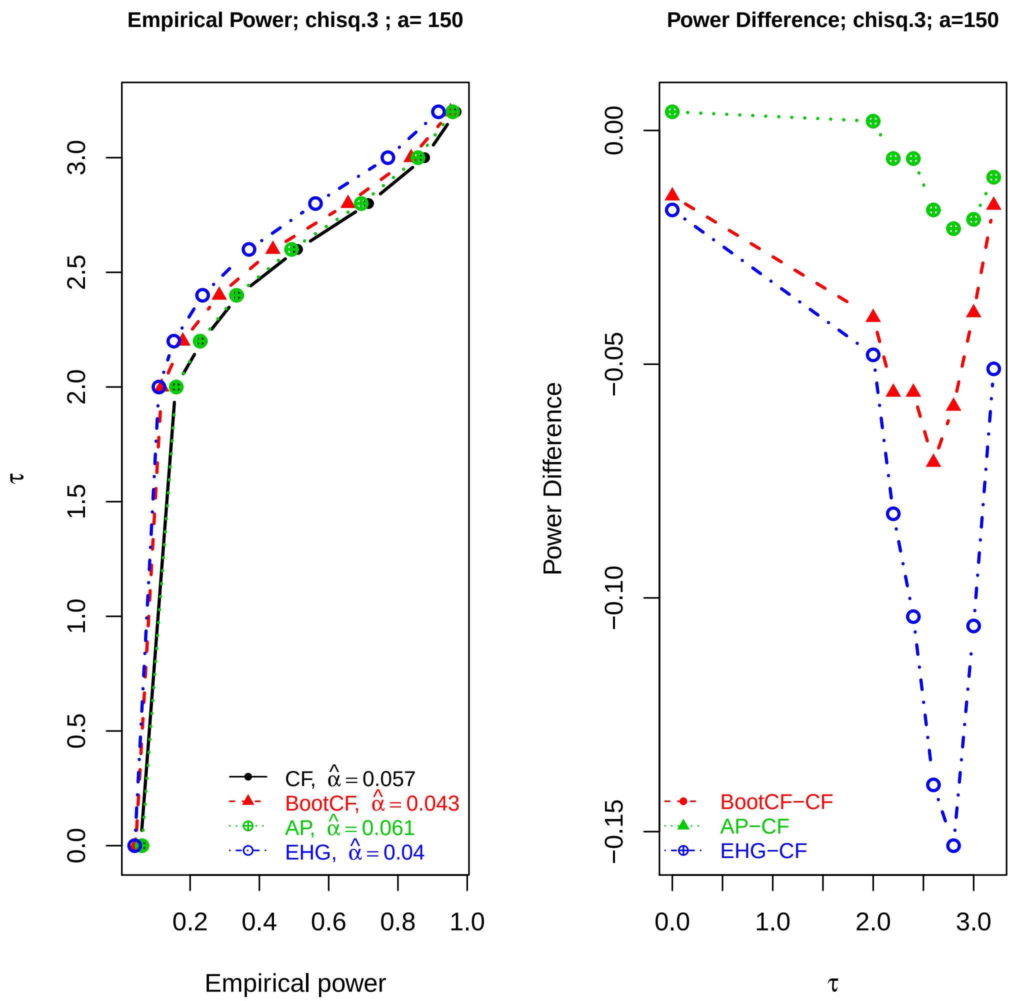

Figure 2.9: *Achieved Power for heteroscedastic* χ^2_3 *data D7, a = 150,* $\alpha = 0.05$.

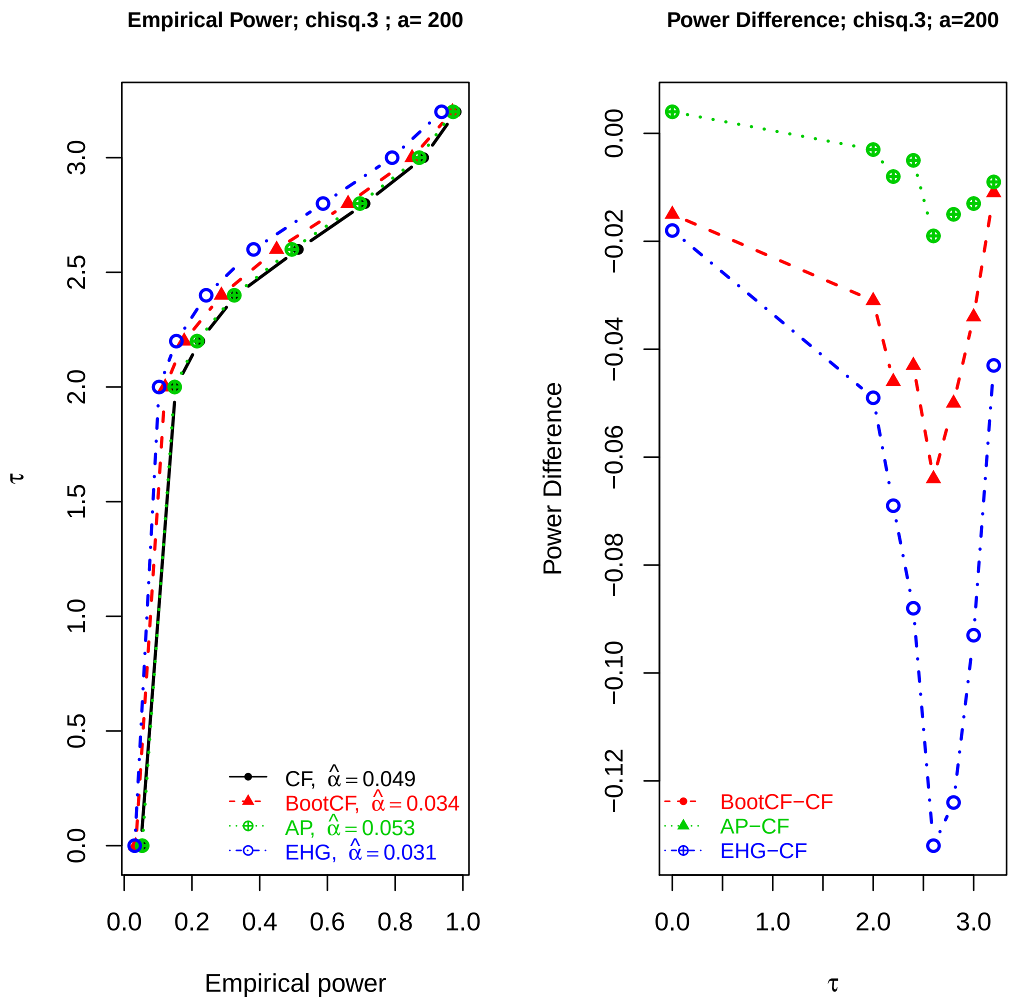

Figure 2.10: *Achieved Power for heteroscedastic* χ^2_3 *data D7, a =* 200*,* $\alpha = 0.05$*.*

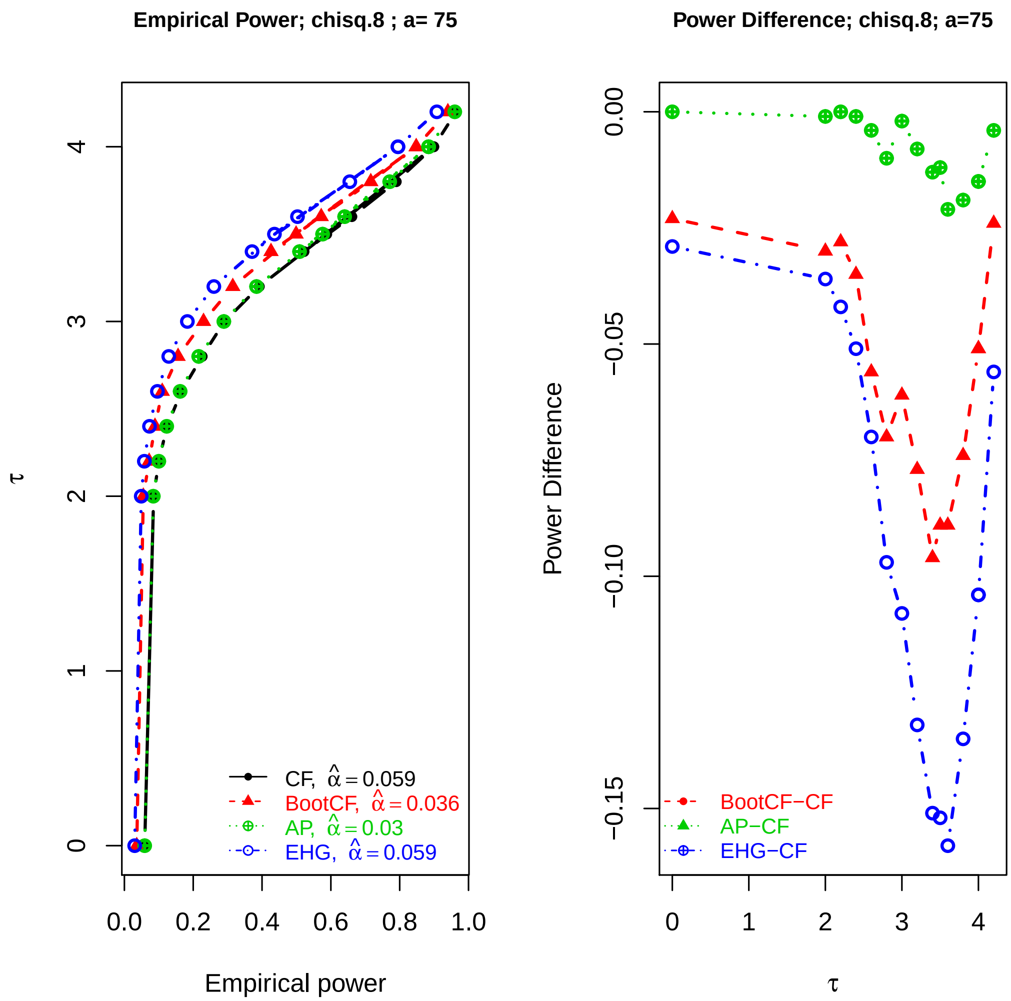

Figure 2.11: *Achieved Power for heteroscedastic* χ^2_8 *data D8,* $a = 75$*,* $\alpha = 0.05$.

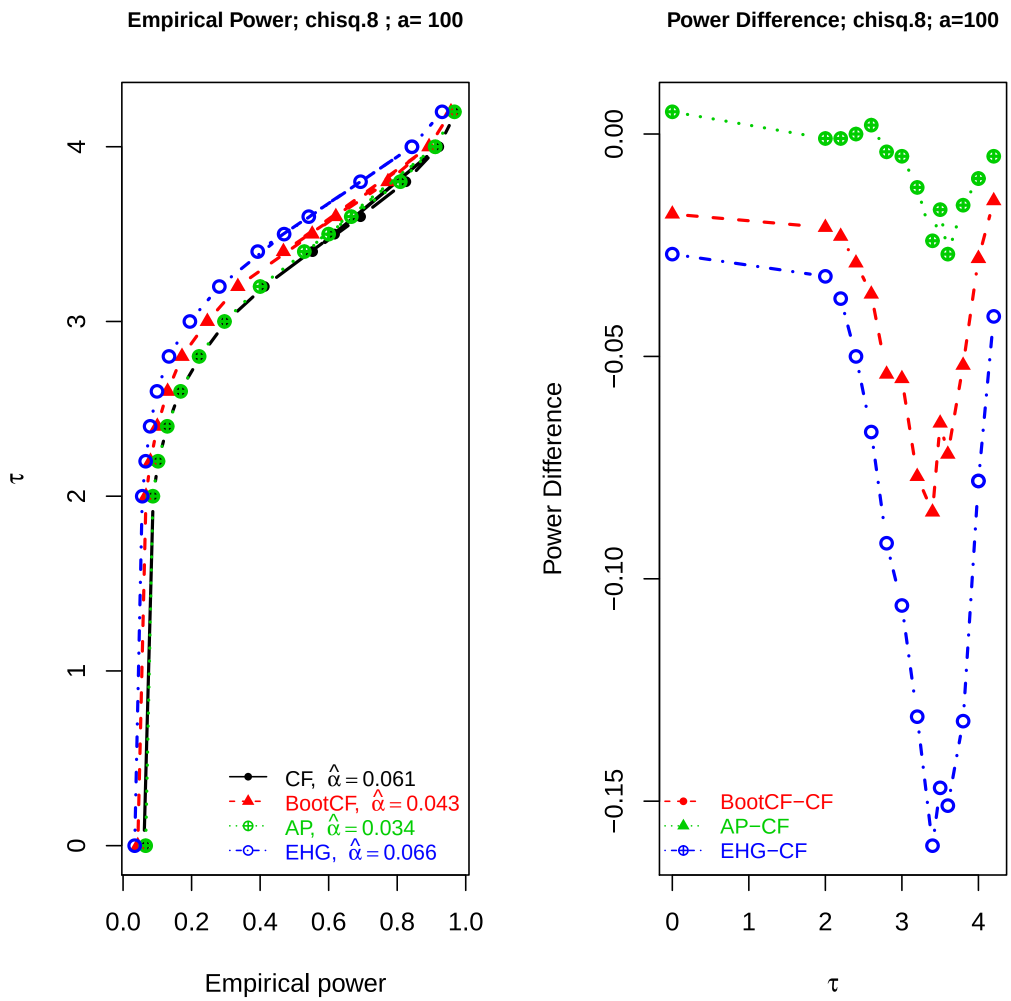

Figure 2.12: *Achieved Power for heteroscedastic* χ^2_8 *data D8, a =* 100*,* $\alpha = 0.05$.

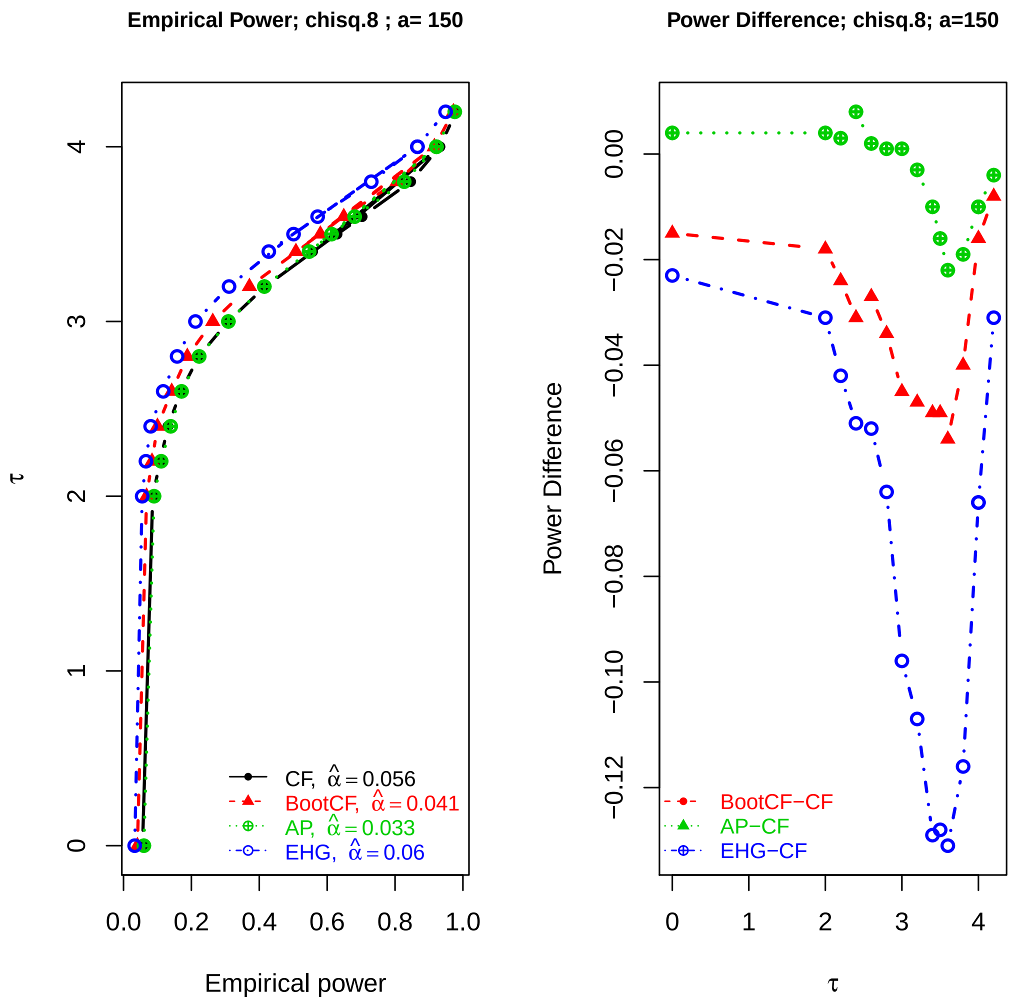

Figure 2.13: *Achieved Power for heteroscedastic* χ^2_8 *data D8,* $a =$ *150,* $\alpha = 0.05$.

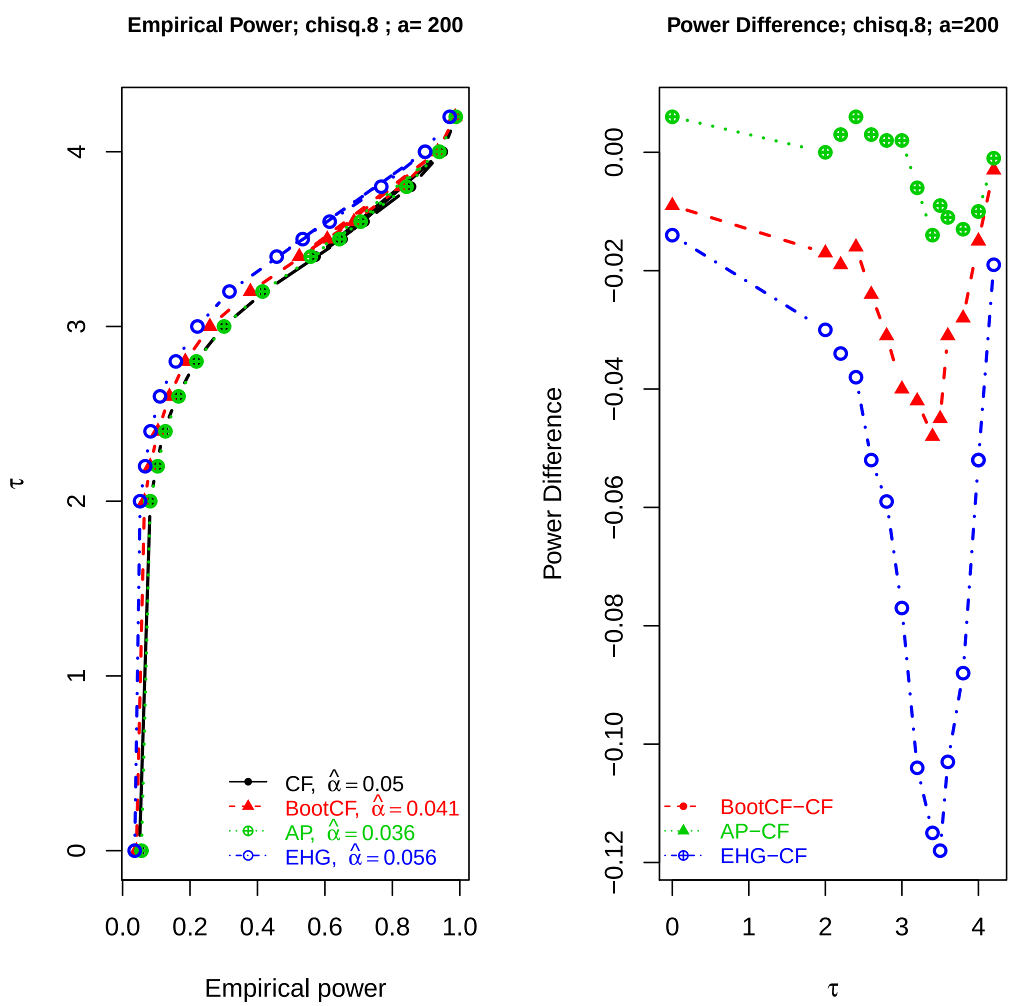

Figure 2.14: *Achieved Power for heteroscedastic* χ^2_8 *data D8,* $a =$ *200,* $\alpha = 0.05$.

2.9 Technical proofs

2.9.1 Proof of (2.2.2)

$$\widetilde{MST}(\underline{\mathbf{X}}) - \widetilde{MSE^{(2)}}(\underline{\mathbf{X}}) = \frac{1}{a-1}\sum_{i=1}^{a}(\overline{X}_{i\cdot} - \widetilde{X}_{\cdot\cdot})^2 - \frac{1}{a}\sum_{i=1}^{a}\sum_{j=1}^{n_i}\frac{(X_{ij}-\overline{X}_{i\cdot})^2}{n_i(n_i-1)}$$

Under the null hypothesis of no treatment effect, we have that

$$\begin{aligned}
\widetilde{MST}(\underline{\mathbf{X}}) - \widetilde{MSE^{(2)}}(\underline{\mathbf{X}}) &= \frac{1}{a-1}\sum_{i=1}^{a}(\overline{\epsilon}_{i\cdot} - \widetilde{\epsilon}_{\cdot\cdot})^2 - \frac{1}{a}\sum_{i=1}^{a}\sum_{j=1}^{n_i}\frac{(\epsilon_{ij}-\overline{\epsilon}_{i\cdot})^2}{n_i(n_i-1)} \\
&= \frac{1}{a-1}\sum_{i=1}^{a}(\overline{\epsilon}^2_{i\cdot} - a\widetilde{\epsilon}^2_{\cdot\cdot}) - \frac{1}{a}\sum_{i=1}^{a}\left(\sum_{j=1}^{n_i}\frac{\epsilon^2_{ij} - n_i\overline{\epsilon}^2_{i\cdot}}{n_i(n_i-1)}\right)
\end{aligned}$$

we expand the right terms on the right hand side as

$$\begin{aligned}
&= \frac{1}{a-1}\sum_{i=1}^{a}\overline{\epsilon}^2_{i\cdot} - \frac{a}{a-1}\widetilde{\epsilon}^2_{\cdot\cdot} - \frac{1}{a}\sum_{i=1}^{a}\sum_{j=1}^{n_i}\frac{\epsilon^2_{ij}}{n_i(n_i-1)} + \frac{1}{a}\sum_{i=1}^{a}\frac{\overline{\epsilon}^2_{i\cdot}}{n_i-1} \\
&= \frac{1}{a-1}\sum_{i=1}^{a}\overline{\epsilon}^2_{i\cdot} - \frac{1}{a(a-1)}\left(\sum_{i=1}^{a}\overline{\epsilon}_{i\cdot}\right)^2 - \frac{1}{a}\sum_{i=1}^{a}\sum_{j=1}^{n_i}\frac{\epsilon^2_{ij}}{n_i(n_i-1)} + \frac{1}{a}\sum_{i=1}^{a}\frac{\left(\sum_{j=1}^{n_i}\epsilon_{ij}\right)^2}{n_i^2(n_i-1)} \\
&= \frac{1}{a-1}\sum_{i=1}^{a}\overline{\epsilon}^2_{i\cdot} - \frac{1}{a(a-1)}\sum_{i\neq i'}\overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot} - \frac{1}{a(a-1)}\sum_{i=1}^{a}\overline{\epsilon}^2_{i\cdot} - \frac{1}{a}\sum_{i=1}^{a}\sum_{j=1}^{n_i}\frac{\epsilon^2_{ij}}{n_i(n_i-1)} \\
&+ \frac{1}{a}\sum_{i=1}^{a}\sum_{j=1}^{n_i}\frac{\epsilon^2_{ij}}{n_i^2(n_i-1)} + \frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i^2(n_i-1)} \\
&= \frac{1}{a-1}\sum_{i=1}^{a}\frac{n_i\overline{\epsilon}^2_{i\cdot}}{n_i-1} - \frac{1}{a(a-1)}\sum_{i\neq i'}\overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot} - \frac{1}{a}\sum_{i=1}^{a}\sum_{j=1}^{n_i}\frac{\epsilon^2_{ij}}{n_i(n_i-1)}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{a}\sum_{i=1}^{a}\frac{\left(\sum_{j=1}^{n_i}\epsilon_{ij}\right)^2}{n_i(n_i-1)} - \frac{1}{a(a-1)}\sum_{i\neq i'}\overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot} - \frac{1}{a}\sum_{i=1}^{a}\sum_{j=1}^{n_i}\frac{\epsilon^2_{ij}}{n_i(n_i-1)} \\
&= \frac{1}{a}\sum_{i=1}^{a}\sum_{j=1}^{n_i}\frac{\epsilon^2_{ij}}{n_i(n_i-1)} + \frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)} - \frac{1}{a(a-1)}\sum_{i\neq i'}\overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot} \\
&- \frac{1}{a}\sum_{i=1}^{a}\sum_{j=1}^{n_i}\frac{\epsilon^2_{ij}}{n_i(n_i-1)}
\end{aligned}$$

From the last equality, we see that the first and last terms cancels out to arrive at equation (2.2.2).

2.9.2 Proof of Theorem 2.3.1

To prove Theorem 2.3.1 we define the following averages;

$$Y_{1a} = \frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}, \qquad Y_{2a} = \frac{-1}{a(a-1)}\sum_{i\neq i'}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.,$$
$$Y_{3a} = \frac{1}{a(a-1)}\sum_{i\neq i'}\frac{S_i^2 S_{i'}^2}{n_i n_{i'}}, \qquad Y_{4a} = \frac{1}{a}\sum_{i=1}^{a}\frac{\widehat{\sigma_i^4}}{n_i(n_i-1)}.$$

Let $\underline{\mathbf{Y}} = (Y_{1a}, Y_{2a}, Y_{3a}, Y_{4a})'$ and $\underline{\mathbf{u}}$ be its mean i.e.,

$$\begin{aligned}
\underline{\mathbf{u}} &= E(\underline{\mathbf{Y}}) = (E(Y_{1a}), E(Y_{2a}), E(Y_{3a}), E(Y_{4a}))' \\
&= (u_1, u_2, u_3, u_4)' \\
&= \left(0, \quad 0, \quad \frac{1}{a(a-1)}\sum_{i\neq i'}^{a}\frac{\sigma_i^2\sigma_{i'}^2}{n_i n_{i'}}, \qquad \frac{1}{a}\sum_{i=1}^{a}\frac{\sigma_i^4}{n_i(n_i-1)}\right)'
\end{aligned}$$

The test statistic $M_a(\underline{\mathbf{X}})$ defined in (2.2.6) can be written as

$$M_a(\underline{\mathbf{X}}) \quad \cong \quad W_a(\underline{\mathbf{Y}}) \quad = \quad \frac{\sqrt{a}(Y_{1a}+Y_{2a})}{h(\underline{\mathbf{Y}})} \quad = \quad \sqrt{a}g_a(\underline{\mathbf{Y}})$$

where

$$h(\underline{\mathbf{Y}}) \quad = \quad \sqrt{\frac{2}{a-1}Y_{3a} + 2Y_{4a}} \quad ; \quad g_a(\underline{\mathbf{Y}}) \quad = \quad \frac{Y_{1a}+Y_{2a}}{h(\underline{\mathbf{Y}})}.$$

By Taylor series expansion of $g_a(\underline{\mathbf{Y}})$ at $\mathbf{u}$, we obtain

$$\begin{aligned}
g_a(\underline{\mathbf{Y}}) &= g_a(\mathbf{u}) + \frac{\partial g_a(\mathbf{u})}{\partial \mathbf{u}}(\mathbf{Y}-\mathbf{u})' + \frac{1}{2}(\mathbf{Y}-\mathbf{u})'\frac{\partial^2 g_a(\mathbf{u})}{\partial \mathbf{u}^2}(\mathbf{Y}-\mathbf{u}) + O_p(\|\mathbf{Y}-\mathbf{u}\|^3) \\
&= \frac{1}{h(\mathbf{u})}(Y_{1a}+Y_{2a}) - \frac{h^{-3}(\mathbf{u})}{a-1}(Y_{3a}-u_3)(Y_{1a}+Y_{2a}) - h^{-3}(\mathbf{u})(Y_{4a}-u_4)(Y_{1a}+Y_{2a}) + O_p(a^{\frac{-3}{2}}).
\end{aligned}$$

Therefore we can write

$$W_a(\underline{\mathbf{Y}}) = \sqrt{a}g_a(\underline{\mathbf{Y}}) = g_1(\underline{\mathbf{Y}}) + g_2(\underline{\mathbf{Y}}) + g_3(\underline{\mathbf{Y}}) + O_p(a^{-1})$$

where

$$g_1(\underline{\mathbf{Y}}) = \frac{\sqrt{a}}{h(\mathbf{u})}(Y_{1a} + Y_{2a}),$$
$$g_2(\underline{\mathbf{Y}}) = \frac{-\sqrt{a}h^{-3}(\mathbf{u})}{a-1}(Y_{3a} - u_3)(Y_{1a} + Y_{2a}) = O_p(a^{-2}),$$
$$g_3(\underline{\mathbf{Y}}) = -\sqrt{a}h^{-3}(\mathbf{u})(Y_{4a} - u_4)(Y_{1a} + Y_{2a}).$$

We end up with

$$W_a(\underline{\mathbf{Y}}) = \sqrt{a}g_a(\underline{\mathbf{Y}}) = g_1(\underline{\mathbf{Y}}) + g_3(\underline{\mathbf{Y}}) + O_p(a^{-1}).$$

Now, $W_a(\underline{\mathbf{Y}})$ is written as

$$W_a(\underline{\mathbf{Y}}) = g(\underline{\mathbf{Y}}) + O_p(a^{-1})$$

where

$$g(\underline{\mathbf{Y}}) = g_1(\underline{\mathbf{Y}}) + g_3(\underline{\mathbf{Y}}).$$

We now obtain the first four moments of $g(Y)$ as follows: The first moment of $g(\underline{\mathbf{Y}})$ is given by

$$E[g(\underline{\mathbf{Y}})] = E[g_1(\underline{\mathbf{Y}})] + E[g_3(\underline{\mathbf{Y}})].$$

$$E[g_1(\underline{\mathbf{Y}})] = \frac{\sqrt{a}}{h(\mathbf{u})}E[Y_{1a} + Y_{2a}] = 0$$

since

$$E[Y_{1a}] = E[Y_{2a}] = 0.$$

$$\begin{aligned} E[g_3(\underline{\mathbf{Y}})] &= -\sqrt{a}h^{-3}(\mathbf{u})E[(Y_{4a} - u_4)(Y_{1a} + Y_{2a})] \\ &= -\sqrt{a}h^{-3}(\mathbf{u})\left\{E[(Y_{4a} - u_4)Y_{1a}] + E[(Y_{4a} - u_4)Y_{2a}]\right\} \\ &= -h^{-3}(\mathbf{u})\sum_{i=1}^{a}\left[\frac{2}{a^{3/2}}\frac{\sigma_i^6(\gamma_i^2 - 2)}{n_i^2(n_i - 1)^2}\right], \end{aligned}$$

since

$$E[(Y_{4a} - u_4)Y_{1a}] = \frac{2}{a^2} \sum_{i=1}^{a} \frac{\sigma_i^6(\gamma_i^2 - 2)}{n_i^2(n_i - 1)^2}$$

and

$$E[(Y_{4a} - u_4)Y_{2a}] = 0.$$

Therefore, the first moment of $G(\underline{\mathbf{Y}})$ is

$$E[G(\underline{\mathbf{Y}})] = -\frac{2h^{-3}(\mathbf{u})}{a^{3/2}} \sum_{i=1}^{a} \left[\frac{\sigma_i^6(\gamma_i^2 - 2)}{n_i^2(n_i - 1)^2}\right].$$

Next, we obtain the second moment of $g(\underline{\mathbf{Y}})$ given by

$$\begin{aligned} E[g^2(\underline{\mathbf{Y}})] &= E[(g_1(\underline{\mathbf{Y}}) + g_3(\underline{\mathbf{Y}}))^2] \\ &= E[g_1^2(Y)] + E[g_3^2(\underline{\mathbf{Y}})] + 2E[g_1(\underline{\mathbf{Y}})g_3(\underline{\mathbf{Y}})]. \end{aligned}$$

$$E[g_1^2(\underline{\mathbf{Y}})] = \frac{\sqrt{a}}{h^2(\mathbf{u})} E[Y_{1a}^2 + 2Y_{1a}Y_{2a} + Y_{2a}^2].$$

$$E[Y_{1a}^2] = E\left[\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'} \frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right]^2 = \frac{2}{a^2}\sum_{i=1}^{a} \frac{\sigma_i^4}{n_i(n_i-1)}.$$

$$E[Y_{1a}Y_{2a}] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'} \frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\left(\frac{-1}{a(a-1)}\sum_{i\neq i'} \bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right] = 0.$$

$$E[Y_{2a}^2] = E\left[\frac{-1}{a(a-1)}\sum_{i\neq i'} \bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right]^2 = O(a^{-2}).$$

We have,

$$E[g_1^2(\underline{\mathbf{Y}})] = \frac{2}{ah^2(\mathbf{u})}\sum_{i=1}^{a}\left[\frac{\sigma_i^4}{n_i(n_i-1)}\right] + O(a^{-1}).$$

$$\begin{aligned} E[g_3^2(\underline{\mathbf{Y}})] &= ah^{-6}(\mathbf{u})E[\{(Y_{4a} - u_4)(Y_{1a} + Y_{2a})\}^2] \\ &= ah^{-6}(\mathbf{u})E[(Y_{4a} - u_4)^2(Y_{1a}^2 + 2Y_{1a}Y_{2a} + Y_{2a}^2)]. \end{aligned}$$

$$E[(Y_{4a} – u_4)^2 Y_{1a}^2] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4} - E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right] = O(a^{-2}).$$

$$\begin{aligned} E[(Y_{4a} - u_4)^2 Y_{1a} Y_{2a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4} - E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right. \\ &* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot}\right)\right] \\ &= O(a^{-2}). \end{aligned}$$

$$E[(Y_{4a} - u_4)^2 Y_{2a}^2] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4} - E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot}\right)^2\right] = O(a^{-3}).$$

We obtain

$$E[g_3^2(\underline{\mathbf{Y}})] = O(a^{-1}).$$

$$\begin{aligned} E[g_1(\underline{\mathbf{Y}})g_3(\underline{\mathbf{Y}})] &= ah^{-4}(\mathbf{u})E[\{(Y_{4a} - u_4)(Y_{1a} + Y_{2a})^2\}] \\ &= ah^{-4}(\mathbf{u})E[(Y_{4a} - u_4)(Y_{1a}^2 + 2Y_{1a}Y_{2a} + Y_{2a}^2)]. \end{aligned}$$

$$E[(Y_{4a} - u_4) Y_{1a}^2] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4} - E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right) \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right] = O(a^{-2}).$$

$$\begin{aligned} E[(Y_{4a} - u_4) Y_{1a} Y_{2a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4} - E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right) * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right. \\ &* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot}\right)\right] \\ &= O(a^{-2}). \end{aligned}$$

$$E[(Y_{4a} - u_4) Y_{2a}^2] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4} - E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right) \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}\overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot}\right)^2\right] = O(a^{-3}).$$

We end up with

$$E[g_1(\underline{\mathbf{Y}})g_3(\underline{\mathbf{Y}})] = O(a^{-1}).$$

Thus, the second moment of $E[g^2(\underline{\mathbf{Y}})]$ is given by

$$E[g^2(\underline{\mathbf{Y}})] = \frac{2}{ah^2(\mathbf{u})}\sum_{i=1}^{a}\left[\frac{\sigma_i^4}{n_i(n_i-1)}\right] + O(a^{-1}).$$

We now proceed to derive the third moment of $g(\underline{\mathbf{Y}})$.

$$\begin{aligned} E[g^3(\underline{\mathbf{Y}})] &= E[(g_1(\underline{\mathbf{Y}}) + g_3(\underline{\mathbf{Y}}))^3] \\ &= E[g_1^3(\underline{\mathbf{Y}})] + E[g_3^3(\underline{\mathbf{Y}})] + 3E[g_1^2(\underline{\mathbf{Y}})g_3(\underline{\mathbf{Y}})] + 3E[g_1(\underline{\mathbf{Y}})g_3^2(\underline{\mathbf{Y}})]. \end{aligned}$$

$$\begin{aligned} E[g_1^3(\underline{\mathbf{Y}})] &= \frac{a^{3/2}}{h^3(\mathbf{u})}E[(Y_{1a}+Y_{2a})^3] \\ &= \frac{a^{3/2}}{h^3(\mathbf{u})}E[Y_{1a}^3 + 3Y_{1a}^2Y_{2a} + 3Y_{1a}Y_{2a}^2 + Y_{2a}^3]. \end{aligned}$$

$$E[Y_{1a}^3] = E\left[\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right]^3 = \frac{4}{a^3}\sum_{i=1}^{a}\frac{\sigma_i^6[\gamma_i^2+2(n_i-2)]}{n_i^2(n_i-1)^2}.$$

$$E[Y_{1a}^2Y_{2a}] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right] = 0.$$

$$E[Y_{1a}Y_{2a}^2] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^2\right] = O(a^{-3}).$$

$$E[Y_{2a}^3] = E\left[\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right]^3 = O(a^{-3}).$$

Therefore we have,

$$E[g_1^3(\underline{\mathbf{Y}})] = \frac{4}{a^{3/2}h^3(\mathbf{u})}\sum_{i=1}^{a}\left[\frac{\sigma_i^6[\gamma_i^2+2(n_i-2)]}{n_i^2(n_i-1)^2}\right] + O(a^{-1}).$$

$$\begin{aligned}E[g_3^3(\underline{\mathbf{Y}})] &= -a^{3/2}h^{-9}(\mathbf{u})E[\{(Y_{4a}-u_4)(Y_{1a}+Y_{2a})\}^3] \\ &= -a^{3/2}h^{-9}(\mathbf{u})E[(Y_{4a}-u_4)^3(Y_{1a}^3+3Y_{1a}^2Y_{2a}+3Y_{1a}Y_{2a}^2+Y_{2a}^3)].\end{aligned}$$

$$E[(Y_{4a}-u_4)^3Y_{1a}^3] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^3\right] = O(a^{-3}).$$

$$\begin{aligned}E[(Y_{4a}-u_4)^3Y_{1a}^2Y_{2a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right. \\ &* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right] \\ &= O(a^{-4}).\end{aligned}$$

$$\begin{aligned}E[(Y_{4a}-u_4)^3Y_{1a}Y_{2a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right. \\ &* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^2\right] \\ &= O(a^{-4}).\end{aligned}$$

$$E[(Y_{4a}-u_4)^3Y_{2a}^3] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^3\right] = O(a^{-5}).$$

Therefore we obtain

$$E[g_3^3(\underline{\mathbf{Y}})] = O(a^{-3/2}).$$

$$\begin{aligned}E[g_1^2(\underline{\mathbf{Y}})g_3(\underline{\mathbf{Y}})] &= -ah^{-5}(\mathbf{u})E[(Y_{4a}-u_4)(Y_{1a}+Y_{2a})^3] \\ &= -ah^{-1}(\mathbf{u})E[(Y_{4a}-u_4)(Y_{1a}^3+3Y_{1a}^2Y_{2a}+3Y_{1a}Y_{2a}^2+Y_{2a}^3)].\end{aligned}$$

$$E[(Y_{4a}-u_4)Y_{1a}^3] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^3\right] = O(a^{-2}).$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{1a}^2Y_{2a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right) * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)\right] \\
&= O(a^{-3}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{1a}Y_{2a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right) * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)^2\right] \\
&= O(a^{-3}).
\end{aligned}
$$

$$
E[(Y_{4a}-u_4)Y_{2a}^3] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)^3\right] = O(a^{-4}).
$$

Therefore,

$$
E[g_1^2(\underline{\mathbf{Y}})g_3(\underline{\mathbf{Y}})] = O(a^{-1}).
$$

$$
\begin{aligned}
E[g_1(\underline{\mathbf{Y}})g_3^2(\underline{\mathbf{Y}})] &= a^{3/2}h^{-7}(\mathbf{u})E[(Y_{4a}-u_4)^2(Y_{1a}+Y_{2a}+Y_{5a})^3] \\
&= a^{3/2}h^{-7}(\mathbf{u})E[(Y_{4a}-u_4)^2(Y_{1a}^3+3Y_{1a}^2Y_{2a}+3Y_{1a}Y_{2a}^2+Y_{2a}^3)].
\end{aligned}
$$

$$
E[(Y_{4a}-u_4)^2Y_{1a}^3] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^3\right] = O(a^{-3}).
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2Y_{1a}^2Y_{2a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)\right] \\
&= O(a^{-3}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2 Y_{1a} Y_{2a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right. \\
&* \quad \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^2\right] \\
&= O(a^{-3}).
\end{aligned}
$$

$$
E[(Y_{4a}-u_4)^2 Y_{2a}^3] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^3\right] = O(a^{-4}).
$$

Therefore we obtain,

$$
E[g_1(\underline{\mathbf{Y}})g_3^2(\underline{\mathbf{Y}})] = O(a^{-3/2}).
$$

Thus, the third moment of $g(\underline{\mathbf{Y}})$, $E[g^3(\underline{\mathbf{Y}})]$ is given by

$$
E[g^3(\underline{\mathbf{Y}})] = \frac{4}{a^{3/2}h^3(\mathbf{u})}\sum_{i=1}^{a}\left[\frac{\sigma_i^6[\gamma_i^2+2(n_i-2)]}{n_i^2(n_i-1)^2}\right] + O(a^{-1}).
$$

Next, we derive the fourth moment of $g(\underline{\mathbf{Y}})$.

$$
\begin{aligned}
E[g^4(\underline{\mathbf{Y}})] &= E[(g_1(\underline{\mathbf{Y}})+g_3(\underline{\mathbf{Y}}))^4] \\
&= E[g_1^4(\underline{\mathbf{Y}})] + 4E[g_1^3(\underline{\mathbf{Y}})g_3(\underline{\mathbf{Y}})] + 6E[g_1^2(\underline{\mathbf{Y}})g_3^2(\underline{\mathbf{Y}})] + 4E[g_1(\underline{\mathbf{Y}})g_3^3(\underline{\mathbf{Y}})] + E[g_3(\underline{\mathbf{Y}})^4].
\end{aligned}
$$

$$
\begin{aligned}
E[g_1^4(\underline{\mathbf{Y}})] &= \frac{a^2}{h^4(\mathbf{u})}E[(Y_{1a}+Y_{2a})^4] \\
&= \frac{a^2}{h^4(\mathbf{u})}E[Y_{1a}^4 + 4Y_{1a}^3Y_{2a} + 6Y_{1a}^2Y_{2a}^2 + 4Y_{1a}Y_{2a}^3 + Y_{2a}^4].
\end{aligned}
$$

$$
E[Y_{1a}^4] = E\left[\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right]^4 = \frac{12}{a^4}\sum_{i\neq i'}^{a}\frac{\sigma_i^4\sigma_{i'}^4}{n_i(n_i-1)n_{i'}(n_{i'}-1)} + O(a^{-3}).
$$

$$
E[Y_{1a}^3Y_{2a}] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^3\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right] = O(a^{-3}).
$$

$$E[Y_{1a}^2 Y_{2a}^2] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot}\right)^2\right] = O(a^{-3}).$$

$$E[Y_{1a} Y_{2a}^3] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot}\right)^3\right] = O(a^{-4}).$$

$$E[Y_{2a}^4] = E\left[\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot}\right]^4 = O(a^{-4}).$$

Therefore, we end up with

$$E[g_1^4(\underline{\mathbf{Y}})] = \frac{12}{a^2 h^4(\mathbf{u})}\sum_{i\neq i'}^{a}\left[\frac{\sigma_i^4\sigma_{i'}^4}{n_i(n_i-1)n_{i'}(n_{i'}-1)}\right] + O(a^{-1}).$$

$$\begin{aligned}
E[g_1^3(\underline{\mathbf{Y}})g_3(\underline{\mathbf{Y}})] &= -a^2h^{-6}(\mathbf{u})E[(Y_{4a}-u_4)(Y_{1a}+Y_{2a})^4] \\
&= -a^2h^{-6}(\mathbf{u})E[(Y_{4a}-u_4)(Y_{1a}^4+4Y_{1a}^3Y_{2a}+6Y_{1a}^2Y_{2a}^2+4Y_{1a}Y_{2a}^3+Y_{2a}^4)].
\end{aligned}$$

$$E[(Y_{4a}-u_4)Y_{1a}^4] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^4\right] = O(a^{-3}).$$

$$\begin{aligned}
E[(Y_{4a}-u_4)Y_{1a}^3Y_{2a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right) * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{a}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^3\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot}\right)\right] \\
&= O(a^{-3}).
\end{aligned}$$

$$\begin{aligned}
E[(Y_{4a}-u_4)Y_{1a}^2Y_{2a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right) * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot}\right)^2\right] \\
&= O(a^{-4}).
\end{aligned}$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{1a}Y_{2a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)*\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right.\\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)^3\right]\\
&= O(a^{-4}).
\end{aligned}
$$

$$
E[(Y_{4a}-u_4)Y_{2a}^4] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)^4\right] = O(a^{-5}).
$$

Therefore, we end up with

$$
E[g_1^3(\underline{\mathbf{Y}})g_3(\underline{\mathbf{Y}})] = O(a^{-1}).
$$

$$
\begin{aligned}
E[g_1^2(\underline{\mathbf{Y}})g_3^2(\underline{\mathbf{Y}})] &= a^2h^{-8}(\mathbf{u})E[(Y_{4a}-u_4)^2(Y_{1a}+Y_{2a})^4]\\
&= -a^2h^{-8}(\mathbf{u})E[(Y_{4a}-u_4)^2(Y_{1a}^4+4Y_{1a}^3Y_{2a}+6Y_{1a}^2Y_{2a}^2+4Y_{1a}Y_{2a}^3+Y_{2a}^4)].
\end{aligned}
$$

$$
E[(Y_{4a}-u_4)^2Y_{1a}^4] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^4\right] = O(a^{-3}).
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2Y_{1a}^3Y_{2a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2*\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^3\right.\\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)\right]\\
&= O(a^{-4}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2Y_{1a}^2Y_{2a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2*\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right.\\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)^2\right]\\
&= O(a^{-4}).
\end{aligned}
$$

$$\begin{aligned}E[(Y_{4a}-u_4)^2Y_{1a}Y_{2a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right.\\ &* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\overline{\epsilon}_{i}.\overline{\epsilon}_{i'}.\right)^3\right]\\ &= O(a^{-5}).\end{aligned}$$

$$E[(Y_{4a}-u_4)^2Y_{2a}^4] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\overline{\epsilon}_{i}.\overline{\epsilon}_{i'}.\right)^4\right] = O(a^{-5}).$$

Therefore, we end up with

$$E[g_1^2(\underline{\mathbf{Y}})g_3^2(\underline{\mathbf{Y}})] = O(a^{-1}).$$

$$\begin{aligned}E[g_1(\underline{\mathbf{Y}})g_3^3(\underline{\mathbf{Y}})] &= -a^2h^{-10}(\mathbf{u})E[(Y_{4a}-u_4)^3(Y_{1a}+Y_{2a})^4]\\ &= -a^2h^{-10}(\mathbf{u})E[(Y_{4a}-u_4)^3(Y_{1a}^4+4Y_{1a}^3Y_{2a}+6Y_{1a}^2Y_{2a}^2+4Y_{1a}Y_{2a}^3+Y_{2a}^4)].\end{aligned}$$

$$E[(Y_{4a}-u_4)^3Y_{1a}^4] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^4\right] = O(a^{-4}).$$

$$\begin{aligned}E[(Y_{4a}-u_4)^3Y_{1a}^3Y_{2a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^3\right.\\ &* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\overline{\epsilon}_{i}.\overline{\epsilon}_{i'}.\right)\right]\\ &= O(a^{-4}).\end{aligned}$$

$$\begin{aligned}E[(Y_{4a}-u_4)^3Y_{1a}^2Y_{2a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right.\\ &* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}\overline{\epsilon}_{i}.\overline{\epsilon}_{i'}.\right)^2\right]\\ &= O(a^{-5}).\end{aligned}$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3 Y_{1a} Y_{2a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right.\\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^3\right]\\
&= O(a^{-5}).
\end{aligned}
$$

$$
E[(Y_{4a}-u_4)^3 Y_{2a}^4] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^4\right] = O(a^{-5}).
$$

Therefore, we end up with

$$
E[g_1(\underline{\mathbf{Y}})g_3^3(\underline{\mathbf{Y}})] = O(a^{-2}).
$$

$$
\begin{aligned}
E[g_3^4(\underline{\mathbf{Y}})] &= a^2 h^{-12}(\mathbf{u})E[(Y_{4a}-u_4)^4(Y_{1a}+Y_{2a})^4]\\
&= a^2 h^{-12}(\mathbf{u})E[(Y_{4a}-u_4)^4(Y_{1a}^4+4Y_{1a}^3Y_{2a}+6Y_{1a}^2Y_{2a}^2+4Y_{1a}Y_{2a}^3+Y_{2a}^4)].
\end{aligned}
$$

$$
E[(Y_{4a}-u_4)^4 Y_{1a}^4] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^4\right] = O(a^{-4}).
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4 Y_{1a}^3 Y_{2a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^3\right.\\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right]\\
&= O(a^{-5}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4 Y_{1a}^2 Y_{2a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right.\\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^2\right]\\
&= O(a^{-5}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4 Y_{1a} Y_{2a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}\bar{\epsilon}_{i.}\bar{\epsilon}_{i'.}\right)^3\right] \\
&= O(a^{-6}).
\end{aligned}
$$

$$
E[(Y_{4a}-u_4)^4 Y_{2a}^4] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i.}\bar{\epsilon}_{i'.}\right)^4\right] = O(a^{-6}).
$$

Therefore, we end up with

$$
E[g_3^4(\underline{\mathbf{Y}})] = O(a^{-2}).
$$

Thus, the fourth moment of $g(\underline{\mathbf{Y}})$, $E[g^4(\underline{\mathbf{Y}})]$ is given by

$$
E[g^4(\underline{\mathbf{Y}})] = \frac{12}{a^2 h^4(\mathbf{u})}\sum_{i\neq i'}^{a}\left[\frac{\sigma_i^4\sigma_{i'}^4}{n_i(n_i-1)n_{i'}(n_{i'}-1)}\right] + O(a^{-1}).
$$

Let κ_{1a}^g, κ_{2a}^g, κ_{3a}^g and κ_{4a}^g be the first four cumulants of $g(\underline{\mathbf{Y}})$. Then using the first four moments, we obtain the cumulants as follows:

$$
\kappa_{1a}^g \quad = \quad E[g(\underline{\mathbf{Y}})] \quad = \quad \frac{1}{\sqrt{a}}\kappa_{11}^g
$$

where κ_{11}^g is defined in equation (2.3.2).

$$
\kappa_{2a}^g = E[g^2(\underline{\mathbf{Y}})] - \{E[g(\underline{\mathbf{Y}})]\}^2 = 1 + O(a^{-1}).
$$

$$
\kappa_{3a}^g = E[g^3(\underline{\mathbf{Y}})] - 3E[g^2(\underline{\mathbf{Y}})]E[g(\underline{\mathbf{Y}})] + \{E[g(\underline{\mathbf{Y}})]\}^3 = \frac{1}{\sqrt{a}}\kappa_{33}^g + O(a^{-1})
$$

where κ_{33}^g is defined in equation (2.3.2).

$$
\begin{aligned}
\kappa_{4a}^g &= E[g^4(\underline{\mathbf{Y}})] - 4E[g^3(\underline{\mathbf{Y}})]E[g(\underline{\mathbf{Y}})] - 3\left\{E[g^2(\underline{\mathbf{Y}})]\right\}^2 + 12E[g^2(\underline{\mathbf{Y}})]\left\{E[g(\underline{\mathbf{Y}})]\right\}^2 \\
&- 6\left\{E[g(\underline{\mathbf{Y}})]\right\}^4 \\
&= O(a^{-1}).
\end{aligned}
$$

Using the cumulants, we now proceed to obtain the characteristic function of $g(\underline{\mathbf{Y}})$. Let χ_g be the characteristic function of $g(\underline{\mathbf{Y}})$. Then,

$$\begin{aligned}
\chi_g(t) &= exp\left\{\kappa^g_{1a}(it) + \kappa^g_{2a}\frac{(it)^2}{2} + \kappa^g_{3a}\frac{(it)^3}{6} + \kappa^g_{4a}\frac{(it)^4}{24}\right\} \\
&= exp[\frac{1}{\sqrt{a}}\kappa^g_{11}(it) + \frac{(it)^2}{2} + \frac{1}{\sqrt{a}}\kappa^g_{33}\frac{(it)^3}{6} + O(a^{-1})] \\
&= exp(-\frac{t^2}{2})exp[\frac{1}{\sqrt{a}}(\kappa^g_{11}(it) + \kappa^g_{33}\frac{(it)^3}{6}) + O(a^{-1})].
\end{aligned}$$

By Taylor expansion, we obtain

$$\chi_g(t) = exp(-\frac{t^2}{2})exp[1 + \frac{1}{\sqrt{a}}(\kappa^g_{11}(it) + \kappa^g_{33}\frac{(it)^3}{6}) + O(a^{-1})].$$

Applying the inverse Fourier transform, we obtain the pdf of g as,

$$\begin{aligned}
f_g(x) &= \int_{-\infty}^{\infty} e^{-itx}\chi_g(t)dt \\
&= \int_{-\infty}^{\infty} e^{-itx}exp(-\frac{t^2}{2})exp[1 + \frac{1}{\sqrt{a}}(\kappa^g_{11}(it) + \kappa^g_{33}\frac{(it)^3}{6}) + O(a^{-1})]dt \\
&= \phi(x) + \frac{1}{\sqrt{a}}[\kappa^g_{11}H_1(x) + \kappa^g_{33}\frac{H_3(x)}{6}]\phi(x) + O(a^{-1}),
\end{aligned}$$

where $H_0(x) = 1$, $H_1(x) = x$, $H_2(x) = x^2 - 1$, and $H_3(x) = x^3 - 3x$ are Hermite polynomials. We now obtain the cdf of g as;

$$\begin{aligned}
F_g(x) &= \int_{-\infty}^{x} f_g(u)du \\
&= \Phi(x) - \frac{1}{\sqrt{a}}[\kappa^g_{11}H_0(x) + \kappa^g_{33}\frac{H_2(x)}{6}]\phi(x) + O(a^{-1}) \\
&= \Phi(x) + \frac{1}{\sqrt{a}}Q_1(x)\phi(x) + O(a^{-1})
\end{aligned}$$

where

$$Q_1(x) = -[\kappa^g_{11} + \frac{1}{6}\kappa^g_{33}(x^2 - 1)].$$

Since $W_a(\underline{\mathbf{Y}}) = g(\underline{\mathbf{Y}}) + O_p(a^{-1})$ then by the delta method of Hall (1992b), $F_g(x)$ is also an approximated cdf of W_a in the order of $O(a^{-1})$. Thus,

$$F_{M_a}(x) = \Phi(x) + \frac{1}{\sqrt{a}}Q_1(x)\phi(x) + O(a^{-1}).$$

Hence the proof.

2.9.3 Proof of Corollary 2.3.2

Let ω_α be the solution to

$$P(M_a \leq \omega_\alpha) = \alpha,$$

for a given value of $\alpha \in (0, 1)$. Then, as discussed in Hall (1992b) section 2.5, we may invert the above expression to obtain ω_α as

$$\omega_\alpha = z_\alpha + \sum_{i=1}^{\infty} \frac{q_i(z_\alpha)}{(\sqrt{a})^i},$$

where z_α is the $\alpha-$level quantile of the standard normal distribution. Using the distribution of M_a presented in (2.3.1) and evaluating at ω_α, we have that

$$F_M(\omega_\alpha) = \Phi\left(z_\alpha + \sum_{i=1}^{\infty} \frac{q_i(z_\alpha)}{(\sqrt{a})^i}\right) + \frac{1}{\sqrt{a}} Q_1\left(z_\alpha + \sum_{i=1}^{\infty} \frac{q_i(z_\alpha)}{(\sqrt{a})^i}\right) \phi\left(z_\alpha + \sum_{i=1}^{\infty} \frac{q_i(z_\alpha)}{(\sqrt{a})^i}\right) + O(a^{-1}) \tag{2.9.1}$$

Next, we apply Taylor expansion of the expressions $\Phi\left(z_\alpha + \sum_{i=1}^{\infty} \frac{q_i(z_\alpha)}{(\sqrt{a})^i}\right)$, $Q_1\left(z_\alpha + \sum_{i=1}^{\infty} \frac{q_i(z_\alpha)}{(\sqrt{a})^i}\right)$ and $\phi\left(z_\alpha + \sum_{i=1}^{\infty} \frac{q_i(z_\alpha)}{(\sqrt{a})^i}\right)$ at z_α. Applying the Taylor expansion, we have

$$\Phi\left(z_\alpha + \sum_{i=1}^{\infty} \frac{q_i(z_\alpha)}{(\sqrt{a})^i}\right) = \Phi(z_\alpha) + \frac{1}{\sqrt{a}} q_i(z_\alpha)\phi(z_\alpha) + O(a^{-1}) \tag{2.9.2}$$

$$Q_1\left(z_\alpha + \sum_{i=1}^{\infty} \frac{q_i(z_\alpha)}{(\sqrt{a})^i}\right) = Q_1(z_\alpha) + \frac{1}{\sqrt{a}} q_i(z_\alpha)Q_1^{'}(z_\alpha) + O(a^{-1}) \tag{2.9.3}$$

$$\phi\left(z_\alpha + \sum_{i=1}^{\infty} \frac{q_i(z_\alpha)}{(\sqrt{a})^i}\right) = \phi(z_\alpha) + \frac{1}{\sqrt{a}} q_i(z_\alpha)\phi^{'}(z_\alpha) + O(a^{-1}) \tag{2.9.4}$$

Substituting (2.9.2), (2.9.3) and (2.9.4) into (2.9.1), we have

$$F_M(\omega_\alpha) = \Phi_(z_\alpha) + \frac{1}{\sqrt{a}} \left[q_1(z_\alpha) + Q_1(z_\alpha)\right] \phi q_1(z_\alpha) + O(a^{-1}).$$

We know that $F_M(\omega_\alpha) = \alpha$ and $\Phi_(z_\alpha) = \alpha$. Therefore, we have that

$$q_1(z_\alpha) = -Q_1(z_\alpha).$$

Hence the proof.

2.9.4 Proof of Theorem 2.7.1

To prove theorem 2.7.1, we define the following averages;

$$Y_{1a} = \frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}, \qquad Y_{2a} = \frac{-1}{a(a-1)}\sum_{i\neq i'}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.,$$

$$Y_{3a} = \frac{1}{a(a-1)}\sum_{i\neq i'}\frac{S_i^2 S_{i'}^2}{n_i n_{i'}}, \qquad Y_{4a} = \frac{1}{a}\sum_{i=1}^{a}\frac{\widehat{\sigma_i^4}}{n_i(n_i-1)}, \qquad Y_{5a} = \frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}..$$

Let $\underline{\mathbf{Y}} = (Y_{1a}, Y_{2a}, Y_{3a}, Y_{4a}, Y_{5a})'$ and $\underline{\mathbf{u}}$ be its mean i.e.,

$$\begin{aligned}
\underline{\mathbf{u}} &= E(\underline{\mathbf{Y}}) = (E(Y_{1a}), E(Y_{2a}), E(Y_{3a}), E(Y_{4a}), E(Y_{5a}))' \\
&= (u_1, u_2, u_3, u_4, u_5)' \\
&= \left(0, \quad 0, \quad \frac{1}{a(a-1)}\sum_{i\neq i'}^{a}\frac{\sigma_i^2\sigma_{i'}^2}{n_i n_{i'}}, \qquad \frac{1}{a}\sum_{i=1}^{a}\frac{\sigma_i^4}{n_i(n_i-1)}, \quad 0\right)'
\end{aligned}$$

The test statistic $M_a(\underline{\mathbf{X}})$ under the local alternative presented in (2.7.3) can be written as

$$M_a(\underline{\mathbf{X}}) \quad \cong \quad W_a(\underline{\mathbf{Y}}) \quad = \quad \frac{\sqrt{a}(Y_{1a}+Y_{2a}+Y_{5a})+c_a}{h(\underline{\mathbf{Y}})} \quad = \quad \sqrt{a}g_{H_a}(\underline{\mathbf{Y}}) + g^*(\underline{\mathbf{Y}})$$

where

$$h(\underline{\mathbf{Y}}) \quad = \quad \sqrt{\frac{2}{a-1}Y_{3a}+2Y_{4a}} \quad ; \quad g_a(\underline{\mathbf{Y}}) \quad = \quad \frac{Y_{1a}+Y_{2a}}{h(\underline{\mathbf{Y}})} \quad ; g^*(\underline{\mathbf{Y}}) = \frac{c_a}{h(\underline{\mathbf{Y}})}.$$

We first apply Taylor series expansion to $g_{H_a}(\underline{\mathbf{Y}})$ at $\mathbf{u}$. We obtain

$$\begin{aligned}
g_{H_a}(\underline{\mathbf{Y}}) &= g_{H_a}(\mathbf{u}) + \frac{\partial g_{H_a}(\mathbf{u})}{\partial \mathbf{u}}(\mathbf{Y}-\mathbf{u})' + \frac{1}{2}(\mathbf{Y}-\mathbf{u})'\frac{\partial^2 g_{H_a}(\mathbf{u})}{\partial \mathbf{u}^2}(\mathbf{Y}-\mathbf{u}) + O_p(\|\mathbf{Y}-\mathbf{u}\|^3) \\
&= \frac{1}{h(\mathbf{u})}(Y_{1a}+Y_{2a}+Y_{5a}) - \frac{h^{-3}(\mathbf{u})}{a-1}(Y_{3a}-u_3)(Y_{1a}+Y_{2a}+Y_{5a}) \\
&\quad - h^{-3}(\mathbf{u})(Y_{4a}-u_4)(Y_{1a}+Y_{2a}+Y_{5a}) + O_p(a^{\frac{-3}{2}}).
\end{aligned}$$

We have that,

$$\sqrt{a}g_{H_a}(\underline{\mathbf{Y}}) = G_1(\underline{\mathbf{Y}}) + G_2(\underline{\mathbf{Y}}) + G_3(\underline{\mathbf{Y}}) + O_p(a^{-1})$$

where

$$G_1(\underline{\mathbf{Y}}) = \frac{\sqrt{a}}{h(\mathbf{u})}(Y_{1a}+Y_{2a}+Y_{5a})$$

$$G_2(\underline{\mathbf{Y}}) = \frac{-\sqrt{a}h^{-3}(\mathbf{u})}{a-1}(Y_{3a} - u_3)(Y_{1a} + Y_{2a} + Y_{5a}) = O_p(a^{-2})$$

and

$$G_3(\underline{\mathbf{Y}}) = -\sqrt{a}h^{-3}(\mathbf{u})(Y_{4a} - u_4)(Y_{1a} + Y_{2a} + Y_{5a}).$$

Next, we apply Taylor series expansion to $g^*(\underline{\mathbf{Y}})$ at $\mathbf{u}$ and obtain

$$\begin{aligned} g^*(\underline{\mathbf{Y}}) &= g^*(\mathbf{u}) + \frac{\partial g^*(\mathbf{u})}{\partial \mathbf{u}}(\underline{\mathbf{Y}} - \mathbf{u})' + O_p(\|\underline{\mathbf{Y}} - \mathbf{u}\|^2) \\ &= \frac{c_a}{h(\mathbf{u})} + G_4(\underline{\mathbf{Y}}) + O_p(a^{-1}) \end{aligned}$$

where

$$G_4(\underline{\mathbf{Y}}) = -c_a h^{-3}(\mathbf{u})(Y_{4a} - u_4).$$

Therefore, the test statistic $W_a(\underline{\mathbf{Y}})$ under the local alternatives is written as

$$W_a(\underline{\mathbf{Y}}) = G_1(\underline{\mathbf{Y}}) + G_3(\underline{\mathbf{Y}}) + G_4(\underline{\mathbf{Y}}) + \frac{c_a}{h(\mathbf{u})} + O_p(a^{-1}).$$

Now, we write $W_a(\underline{\mathbf{Y}})$ as

$$W_a(\underline{\mathbf{Y}}) = G(\underline{\mathbf{Y}}) + \frac{c_a}{h(\mathbf{u})} + O_p(a^{-1})$$

where

$$G(\underline{\mathbf{Y}}) = G_1(\underline{\mathbf{Y}}) + G_3(\underline{\mathbf{Y}}) + G_4(\underline{\mathbf{Y}}).$$

To obtain the distribution function of $W_a(\underline{\mathbf{Y}})$, we need the first four moments of $G(\underline{\mathbf{Y}})$ as follows: The first moment of $G(\underline{\mathbf{Y}})$ is given by

$$E[G(\underline{\mathbf{Y}})] = E[G_1(\underline{\mathbf{Y}})] + E[G_3(\underline{\mathbf{Y}})] + E[G_4(\underline{\mathbf{Y}})].$$

$$E[G_1(\underline{\mathbf{Y}})] = \frac{\sqrt{a}}{h(\mathbf{u})}E[Y_{1a} + Y_{2a} + Y_{5a}] = 0,$$

since

$$E[Y_{1a}] = E[Y_{2a}] = E[Y_{5a}] = 0.$$

$$\begin{aligned}
E[G_3(\underline{\mathbf{Y}})] &= -\sqrt{a}h^{-3}(\mathbf{u})E[(Y_{4a}-u_4)(Y_{1a}+Y_{2a}+Y_{5a})] \\
&= -\sqrt{a}h^{-3}(\mathbf{u})\left\{E[(Y_{4a}-u_4)Y_{1a}] + E[(Y_{4a}-u_4)Y_{2a}] + E[(Y_{4a}-u_4)Y_{5a}]\right\} \\
&= -h^{-3}(\mathbf{u})\sum_{i=1}^{a}\left[\frac{2}{a^{3/2}}\frac{\sigma_i^6(\gamma_i^2-2)}{n_i^2(n_i-1)^2} + \frac{4}{a^{1/2}(a-1)}\frac{\alpha_i\gamma_i\sigma_i^5}{n_i^2(n_i-1)}\right]
\end{aligned}$$

since

$$E[(Y_{4a}-u_4)Y_{1a}] = \frac{2}{a^2}\sum_{i=1}^{a}\frac{\sigma_i^6(\gamma_i^2-2)}{n_i^2(n_i-1)^2}\ ;\ E[(Y_{4a}-u_4)Y_{2a}] = 0 \text{ and}$$

$$E[(Y_{4a}-u_4)Y_{5a}] = \frac{4}{a(a-1)}\sum_{i=1}^{a}\frac{\alpha_i\gamma_i\sigma_i^5}{n_i^2(n_i-1)}.$$

$$E[G_4(\underline{\mathbf{Y}})] = -c_a h^{-3}(\mathbf{u})E(Y_{4a}-u_4) = 0.$$

Therefore, the first moment of $G(\underline{\mathbf{Y}})$ is

$$E[G(\underline{\mathbf{Y}})] = -h^{-3}(\mathbf{u})\sum_{i=1}^{a}\left[\frac{2}{a^{3/2}}\frac{\sigma_i^6(\gamma_i^2-2)}{n_i^2(n_i-1)^2} + \frac{4}{a^{1/2}(a-1)}\frac{\alpha_i\gamma_i\sigma_i^5}{n_i^2(n_i-1)}\right].$$

Next, we obtain the second moment of $G(\underline{\mathbf{Y}})$, which is given by

$$\begin{aligned}
E[G^2(\underline{\mathbf{Y}})] &= E[(G_1(\underline{\mathbf{Y}})+G_3(\underline{\mathbf{Y}})+G_4(\underline{\mathbf{Y}}))^2] \\
&= E[G_1^2(\underline{\mathbf{Y}})] + E[G_3^2(\underline{\mathbf{Y}})] + E[G_4^2(\underline{\mathbf{Y}})] + 2E[G_1(\underline{\mathbf{Y}})G_3(\underline{\mathbf{Y}})] + 2E[G_1(\underline{\mathbf{Y}})G_4(\underline{\mathbf{Y}})] \\
&+ 2E[G_3(\underline{\mathbf{Y}})G_4(\underline{\mathbf{Y}})].
\end{aligned}$$

$$E[G_1^2(\underline{\mathbf{Y}})] = \frac{\sqrt{a}}{h^2(\mathbf{u})}E[Y_{1a}^2 + 2Y_{1a}Y_{2a} + Y_{2a}^2 + 2Y_{1a}Y_{5a} + 2Y_{2a}Y_{5a} + Y_{5a}^2].$$

$$E[Y_{1a}^2] = E\left[\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right]^2 = \frac{2}{a^2}\sum_{i=1}^{a}\frac{\sigma_i^4}{n_i(n_i-1)}.$$

$$E[Y_{1a}Y_{2a}] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\overline{\epsilon}_{i}.\overline{\epsilon}_{i'}.\right)\right] = 0.$$

$$E[Y_{2a}^2] = E\left[\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\overline{\epsilon}_{i}.\overline{\epsilon}_{i'}.\right]^2 = O(a^{-2}).$$

$$E[Y_{1a}Y_{5a}] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\overline{\epsilon}_{i}.\right)\right] = 0.$$

$$E[Y_{2a}Y_{5a}] = E\left[\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\overline{\epsilon}_{i}.\overline{\epsilon}_{i'}.\right)\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\overline{\epsilon}_{i}.\right)\right] = 0.$$

$$E[Y_{2a}^2] = E\left[\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\overline{\epsilon}_{i}.\right]^2 = \frac{4}{(a-1)^2}\sum_{i=1}^{a}\frac{\alpha_i^2\sigma_i^2}{n_i}.$$

We have,

$$E[G_1^2(\underline{\mathbf{Y}})] = \frac{1}{h^2(\mathbf{u})}\sum_{i=1}^{a}\left[\frac{2}{a}\frac{\sigma_i^4}{n_i(n_i-1)} + \frac{4a}{(a-1)^2}\frac{\alpha_i\sigma_i^2}{n_i}\right] + O(a^{-1}).$$

$$\begin{aligned}
E[G_3^2(\underline{\mathbf{Y}})] &= ah^{-6}(\mathbf{u})E[\{(Y_{4a}-u_4)(Y_{1a}+Y_{2a}+Y_{5a})\}^2] \\
&= ah^{-6}(\mathbf{u})E[(Y_{4a}-u_4)^2(Y_{1a}^2+2Y_{1a}Y_{2a}+Y_{2a}^2+2Y_{1a}Y_{5a}+2Y_{2a}Y_{5a}+Y_{5a}^2)].
\end{aligned}$$

$$\begin{aligned}
E[(Y_{4a}-u_4)^2Y_{1a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right] \\
&= O(a^{-2}).
\end{aligned}$$

$$\begin{aligned}
E[(Y_{4a}-u_4)^2Y_{1a}Y_{2a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\overline{\epsilon}_{i}.\overline{\epsilon}_{i'}.\right)\right] \\
&= O(a^{-2}).
\end{aligned}$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2Y_{2a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}\overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot}\right)^2\right] \\
&= O(a^{-3}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2Y_{1a}Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right. \\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\overline{\epsilon}_{i\cdot}\right)\right] \\
&= O(a^{-9/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2Y_{2a}Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot}\right)\right. \\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\overline{\epsilon}_{i\cdot}\right)\right] \\
&= O(a^{-9/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\overline{\epsilon}_{i\cdot}\right)^2\right] \\
&= O(a^{-5/2}) \text{ by condition (2.7.2).}
\end{aligned}
$$

We obtain

$$
E[G_3^2(\underline{\mathbf{Y}})] = O(a^{-1}).
$$

$$
\begin{aligned}
E[G_4^2(\underline{\mathbf{Y}})] &= c_a^2h^{-6}(\mathbf{u})E[(Y_{4a}-u_4)^2] \\
&= c_a^2h^{-6}(\mathbf{u})E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2\right] \\
&= O(a^{-1}).
\end{aligned}
$$

$$
\begin{aligned}
E[G_1(\underline{\mathbf{Y}})G_3(\underline{\mathbf{Y}})] &= ah^{-4}(\mathbf{u})E[\{(Y_{4a}-u_4)(Y_{1a}+Y_{2a}+Y_{5a})^2\}] \\
&= ah^{-4}(\mathbf{u})E[(Y_{4a}-u_4)(Y_{1a}^2+2Y_{1a}Y_{2a}+Y_{2a}^2+2Y_{1a}Y_{5a}+2Y_{2a}Y_{5a}+Y_{5a}^2)].
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{1a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right] \\
&= O(a^{-2}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{1a}Y_{2a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)*\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right. \\
&\quad * \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right] \\
&= O(a^{-2}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{2a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^2\right] \\
&= O(a^{-3}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{1a}Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)*\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right. \\
&\quad * \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right] \\
&= O(a^{-9/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{2a}Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)*\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right. \\
&\quad * \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right] \\
&= O(a^{-9/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^2\right] \\
&= O(a^{-5/2}) \text{ by condition (2.7.2).}
\end{aligned}
$$

We end up with

$$E[G_1(\underline{\mathbf{Y}})G_3(\underline{\mathbf{Y}})] = O(a^{-1}).$$

$$\begin{aligned} E[G_1(\underline{\mathbf{Y}})G_4(\underline{\mathbf{Y}})] &= -\sqrt{a}c_a h^{-4}(\mathbf{u})E[(Y_{4a}-u_4)(Y_{1a}+Y_{2a}+Y_{5a})] \\ &= c_a h^{-1}(\mathbf{u})E[G_3(\underline{\mathbf{Y}})] \\ &= -c_a h^{-4}(\mathbf{u})\sum_{i=1}^{a}\left[\frac{2}{a^{3/2}}\frac{\sigma_i^6(\gamma_i^2-2)}{n_i^2(n_i-1)^2}+\frac{4}{a^{1/2}(a-1)}\frac{\alpha_i\gamma_i\sigma_i^5}{n_i^2(n_i-1)}\right]. \end{aligned}$$

$$\begin{aligned} E[G_3(\underline{\mathbf{Y}})G_4(\underline{\mathbf{Y}})] &= \sqrt{a}c_a h^{-6}(\mathbf{u})E[(Y_{4a}-u_4)^2(Y_{1a}+Y_{2a}+Y_{5a})] \\ &= \sqrt{a}c_a h^{-6}(\mathbf{u})\left\{E[(Y_{4a}-u_4)^2Y_{1a}]+E[(Y_{4a}-u_4)^2Y_{2a}]+E[(Y_{4a}-u_4)^2Y_{5a}]\right\}. \end{aligned}$$

$$\begin{aligned} E[(Y_{4a}-u_4)^2Y_{1a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right] \\ &= O(a^{-2}). \end{aligned}$$

$$\begin{aligned} E[(Y_{4a}-u_4)^2Y_{2a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right] \\ &= O(a^{-3}). \end{aligned}$$

$$\begin{aligned} E[(Y_{4a}-u_4)^2Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right] \\ &= O(a^{-9/4}) \text{ by condition (2.7.2).} \end{aligned}$$

We obtain

$$E[G_3(\underline{\mathbf{Y}})G_4(\underline{\mathbf{Y}})] = O(a^{-3/2}).$$

Thus, the second moment of $E[G^2(\underline{\mathbf{Y}})]$ is given by

$$\begin{aligned} E[G^2(\underline{\mathbf{Y}})] &= \frac{1}{h^2(\mathbf{u})}\sum_{i=1}^{a}\left[\frac{2}{a}\frac{\sigma_i^4}{n_i(n_i-1)}+\frac{4a}{(a-1)^2}\frac{\alpha_i\sigma_i^2}{n_i}\right] \\ &- 2c_a h^{-4}(\mathbf{u})\sum_{i=1}^{a}\left[\frac{2}{a^{3/2}}\frac{\sigma_i^6(\gamma_i^2-2)}{n_i^2(n_i-1)^2}+\frac{4}{a^{1/2}(a-1)}\frac{\alpha_i\gamma_i\sigma_i^5}{n_i^2(n_i-1)}\right]+O(a^{-1}). \end{aligned}$$

We now proceed to derive the third moment of $G(\underline{\mathbf{Y}})$.

$$\begin{aligned}
E[G^3(\underline{\mathbf{Y}})] &= E[(G_1(\underline{\mathbf{Y}}) + G_3(\underline{\mathbf{Y}}) + G_4(\underline{\mathbf{Y}}))^3] \\
&= E[G_1^3(\underline{\mathbf{Y}})] + E[G_3^3(\underline{\mathbf{Y}})] + E[G_4^3(\underline{\mathbf{Y}})] + 3E[G_1^2(\underline{\mathbf{Y}})G_3(\underline{\mathbf{Y}})] + 3E[G_1(\underline{\mathbf{Y}})G_3^2(\underline{\mathbf{Y}})] \\
&+ 3E[G_1^2(\underline{\mathbf{Y}})G_4(\underline{\mathbf{Y}})] + 3E[G_1(\underline{\mathbf{Y}})G_4^2(\underline{\mathbf{Y}})] + 3E[G_3^2(\underline{\mathbf{Y}})G_4(\underline{\mathbf{Y}})] + 3E[G_3(\underline{\mathbf{Y}})G_4^2(\underline{\mathbf{Y}})] \\
&+ 6E[G_1(\underline{\mathbf{Y}})G_3(\underline{\mathbf{Y}})G_4(\underline{\mathbf{Y}})].
\end{aligned}$$

$$\begin{aligned}
E[G_1^3(\underline{\mathbf{Y}})] &= \frac{a^{3/2}}{h^3(\mathbf{u})} E[(Y_{1a} + Y_{2a} + Y_{5a})^3] \\
&= \frac{a^{3/2}}{h^3(\mathbf{u})} E[Y_{1a}^3 + 3Y_{1a}^2 Y_{2a} + 3Y_{1a}Y_{2a}^2 + Y_{2a}^3 + 3Y_{1a}^2 Y_{5a} + 6Y_{1a}Y_{2a}Y_{5a} + 3Y_{2a}^2 Y_{5a} \\
&+ 3Y_{1a}Y_{5a}^2 + 3Y_{2a}Y_{5a}^2 + Y_{5a}^3].
\end{aligned}$$

$$E[Y_{1a}^3] = E\left[\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'} \frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right]^3 \;=\; \frac{4}{a^3}\sum_{i=1}^{a} \frac{\sigma_i^6[\gamma_i^2 + 2(n_i-2)]}{n_i^2(n_i-1)^2}.$$

$$E[Y_{1a}^2 Y_{2a}] \;=\; E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i} \frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2 \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a} \overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot}\right)\right] = 0.$$

$$E[Y_{1a} Y_{2a}^2] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i} \frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right) \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a} \overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot}\right)^2\right] = O(a^{-3}).$$

$$E[Y_{2a}^3] = E\left[\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a} \overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot}\right]^3 = O(a^{-3}).$$

$$\begin{aligned}
E[Y_{1a}^2 Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i} \frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2 \left(\frac{1}{a-1}\sum_{i=1}^{a} \alpha_i\overline{\epsilon}_{i\cdot}\right)\right] \\
&= \frac{16}{a^2(a-1)}\sum_{i=1}^{a} \frac{\alpha_i\gamma_i\sigma_i^5}{n_i^2(n_i-1)}.
\end{aligned}$$

$$E[Y_{1a}Y_{2a}Y_{5a}] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i} \frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a} \overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot}\right)\left(\frac{1}{a-1}\sum_{i=1}^{a} \alpha_i\overline{\epsilon}_{i\cdot}\right)\right] = 0.$$

$$\begin{aligned} E[Y_{2a}^2 Y_{5a}] &= E\left[\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^2\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right] \\ &= O(a^{-13/4}) \text{ by condition (2.7.2).} \end{aligned}$$

$$\begin{aligned} E[Y_{1a} Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^2\right] \\ &= O(a^{-5/2}) \text{ by condition (2.7.2).} \end{aligned}$$

$$\begin{aligned} E[Y_{2a} Y_{5a}^2] &= E\left[\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^2\right] \\ &= O(a^{-5/2}) \text{ by condition (2.7.2).} \end{aligned}$$

$$\begin{aligned} E[Y_{5a}^3] &= E\left[\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right]^3 \\ &= O(a^{-11/4}) \text{ by condition (2.7.2).} \end{aligned}$$

Therefore,

$$E[G_1^3(\underline{\mathbf{Y}})] = \frac{4}{h^3(\mathbf{u})}\sum_{i=1}^{a}\left[\frac{1}{a^{3/2}}\frac{\sigma_i^6[\gamma_i^2+2(n_i-2)]}{n_i^2(n_i-1)^2} + \frac{12}{a^{1/2}(a-1)}\frac{\alpha_i\gamma_i\sigma_i^5}{n_i^2(n_i-1)}\right] + O(a^{-1}).$$

$$\begin{aligned} E[G_3^3(\underline{\mathbf{Y}})] &= -a^{3/2}h^{-9}(\mathbf{u})E[\{(Y_{4a}-u_4)(Y_{1a}+Y_{2a}+Y_{5a})\}^3] \\ &= -a^{3/2}h^{-9}(\mathbf{u})E[(Y_{4a}-u_4)^3(Y_{1a}^3+3Y_{1a}^2Y_{2a}+3Y_{1a}Y_{2a}^2+Y_{2a}^3+3Y_{1a}^2Y_{5a} \\ &+ 6Y_{1a}Y_{2a}Y_{5a}+3Y_{2a}^2Y_{5a}+3Y_{1a}Y_{5a}^2+3Y_{2a}Y_{5a}^2+Y_{5a}^3)]. \end{aligned}$$

$$\begin{aligned} E[(Y_{4a}-u_4)^3 Y_{1a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^3\right] \\ &= O(a^{-3}). \end{aligned}$$

$$\begin{aligned} E[(Y_{4a}-u_4)^3 Y_{1a}^2 Y_{2a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right. \\ &\quad \left. * \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right] \\ &= O(a^{-4}). \end{aligned}$$

$$\begin{aligned}
E[(Y_{4a}-u_4)^3 Y_{1a} Y_{2a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right.\\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)^2\right]\\
&= O(a^{-4}).
\end{aligned}$$

$$\begin{aligned}
E[(Y_{4a}-u_4)^3 Y_{1a} Y_{2a} Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right.\\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right) * \left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)\right]\\
&= O(a^{-17/4}) \text{ by condition (2.7.2).}
\end{aligned}$$

$$\begin{aligned}
E[(Y_{4a}-u_4)^3 Y_{2a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)^3\right]\\
&= O(a^{-5}).
\end{aligned}$$

$$\begin{aligned}
E[(Y_{4a}-u_4)^3 Y_{1a}^2 Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right.\\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)\right]\\
&= O(a^{-13/4}) \text{ by condition (2.7.2).}
\end{aligned}$$

$$\begin{aligned}
E[(Y_{4a}-u_4)^3 Y_{1a} Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right.\\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)^2\right]\\
&= O(a^{-7/2}) \text{ by condition (2.7.2).}
\end{aligned}$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3Y_{2a}^2Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^2\right.\\
&\quad * \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right]\\
&= O(a^{-17/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3Y_{2a}Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right.\\
&\quad * \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^2\right]\\
&= O(a^{-9/2}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3Y_{5a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^3\right]\\
&= O(a^{-15/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

Therefore we obtain

$$
E[G_3^3(\underline{\mathbf{Y}})] = O(a^{-3/2}).
$$

$$
\begin{aligned}
E[G_4^3(\underline{\mathbf{Y}})] &= c_a^3h^{-9}(\mathbf{u})E[(Y_{4a}-u_4)^3]\\
&= c_a^3h^{-9}(\mathbf{u})E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3\right]\\
&= O(a^{-2}).
\end{aligned}
$$

$$
\begin{aligned}
E[G_1^2(\underline{\mathbf{Y}})G_4(\underline{\mathbf{Y}})] &= -ac_ah^{-5}(\mathbf{u})E[(Y_{4a}-u_4)(Y_{1a}+Y_{2a}+Y_{5a})^2]\\
&= c_ah^{-1}(\mathbf{u})E[G_1(\underline{\mathbf{Y}})G_3(\underline{\mathbf{Y}})]\\
&= O(a^{-1}).
\end{aligned}
$$

$$
\begin{aligned}
E[G_1(\underline{\mathbf{Y}})G_4^2(\underline{\mathbf{Y}})] &= \sqrt{a}c_a^2h^{-7}(\mathbf{u})E[(Y_{4a}-u_4)^2(Y_{1a}+Y_{2a}+Y_{5a})] \\
&= ch^{-1}(\mathbf{u})E[G_3(\underline{\mathbf{Y}})G_4(\underline{\mathbf{Y}})] \\
&= O(a^{-3/2}).
\end{aligned}
$$

$$
\begin{aligned}
E[G_1^2(\underline{\mathbf{Y}})G_3(\underline{\mathbf{Y}})] &= -ah^{-5}(\mathbf{u})E[(Y_{4a}-u_4)(Y_{1a}+Y_{2a}+Y_{5a})^3] \\
&= -ah^{-1}(\mathbf{u})E[(Y_{4a}-u_4)(Y_{1a}^3+3Y_{1a}^2Y_{2a}+3Y_{1a}Y_{2a}^2+Y_{2a}^3+3Y_{1a}^2Y_{5a} \\
&+ 6Y_{1a}Y_{2a}Y_{5a}+3Y_{2a}^2Y_{5a}+3Y_{1a}Y_{5a}^2+3Y_{2a}Y_{5a}^2+Y_{5a}^3)].
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{1a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^3\right] \\
&= O(a^{-2}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{1a}^2Y_{2a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)*\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right] \\
&= O(a^{-3}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{1a}Y_{2a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)*\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^2\right] \\
&= O(a^{-3}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{1a}Y_{2a}Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)*\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)*\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right] \\
&= O(a^{-13/4}) \text{ by condition (2.7.2)}.
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{2a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^3\right] \\
&= O(a^{-4}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{1a}^2Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)*\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right. \\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right] \\
&= O(a^{-9/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{1a}Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)*\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right. \\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^2\right] \\
&= O(a^{-5/2}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{2a}^2Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)*\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^2\right. \\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right] \\
&= O(a^{-13/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{2a}Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)*\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right. \\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^2\right] \\
&= O(a^{-7/2}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{5a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^3\right] \\
&= O(a^{-11/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

Therefore,

$$E[G_1^2(\underline{\mathbf{Y}})G_3(\underline{\mathbf{Y}})] = O(a^{-1}).$$

$$\begin{aligned}
E[G_1(\underline{\mathbf{Y}})G_3^2(\underline{\mathbf{Y}})] &= a^{3/2}h^{-7}(\mathbf{u})E[(Y_{4a}-u_4)^2(Y_{1a}+Y_{2a}+Y_{5a})^3] \\
&= a^{3/2}h^{-7}(\mathbf{u})E[(Y_{4a}-u_4)^2(Y_{1a}^3+3Y_{1a}^2Y_{2a}+3Y_{1a}Y_{2a}^2+Y_{2a}^3+3Y_{1a}^2Y_{5a} \\
&+ 6Y_{1a}Y_{2a}Y_{5a}+3Y_{2a}^2Y_{5a}+3Y_{1a}Y_{5a}^2+3Y_{2a}Y_{5a}^2+Y_{5a}^3)].
\end{aligned}$$

$$\begin{aligned}
E[(Y_{4a}-u_4)^2Y_{1a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^3\right] \\
&= O(a^{-3}).
\end{aligned}$$

$$\begin{aligned}
E[(Y_{4a}-u_4)^2Y_{1a}^2Y_{2a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right] \\
&= O(a^{-3}).
\end{aligned}$$

$$\begin{aligned}
E[(Y_{4a}-u_4)^2Y_{1a}Y_{2a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^2\right] \\
&= O(a^{-3}).
\end{aligned}$$

$$\begin{aligned}
E[(Y_{4a}-u_4)^2Y_{1a}Y_{2a}Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right) * \left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right] \\
&= O(a^{-13/4}) \text{ by condition (2.7.2)}.
\end{aligned}$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2 Y_{2a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^3\right] \\
&= O(a^{-4}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2 Y_{1a}^2 Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right. \\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right] \\
&= O(a^{-13/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2 Y_{1a} Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right. \\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^2\right] \\
&= O(a^{-7/2}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2 Y_{2a}^2 Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^2\right. \\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right] \\
&= O(a^{-13/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2 Y_{2a} Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right. \\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^2\right] \\
&= O(a^{-7/2}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2Y_{5a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^3\right] \\
&= O(a^{-15/4}) \text{ by condition (2.7.2)}
\end{aligned}
$$

Therefore we obtain,

$$
E[G_1(\underline{\mathbf{Y}})G_3^2(\underline{\mathbf{Y}})] = O(a^{-3/2}).
$$

$$
\begin{aligned}
E[G_3^2(\underline{\mathbf{Y}})G_4(\underline{\mathbf{Y}})] &= -ac_a h^{-9}(\mathbf{u})E[(Y_{4a}-u_4)^3(Y_{1a}+Y_{2a}+Y_{5a})^2] \\
&= -ac_a h^{-9}(\mathbf{u})E[(Y_{4a}-u_4)^3(Y_{1a}^2+2Y_{1a}Y_{2a}+Y_{2a}^2+2Y_{1a}Y_{5a}+2Y_{2a}Y_{5a}+Y_{5a}^2)].
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3Y_{1a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right] \\
&= O(a^{-3}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3Y_{1a}Y_{2a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right] \\
&= O(a^{-3}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3Y_{2a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^2\right] \\
&= O(a^{-4}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3Y_{1a}Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right. \\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right] \\
&= O(a^{-13/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3 Y_{2a} Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)\right.\\
&\quad\left. * \left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)\right]\\
&= O(a^{-13/4}) \text{ by condition (2.7.2)}.
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3 Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 \left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)^2\right]\\
&= O(a^{-7/2}) \text{ by condition (2.7.2)}.
\end{aligned}
$$

Therefore, we obtain

$$
E[G_3^2(\underline{\mathbf{Y}})G_4(\underline{\mathbf{Y}})] = O(a^{-2}).
$$

$$
\begin{aligned}
E[G_3(\underline{\mathbf{Y}})G_4^2(\underline{\mathbf{Y}})] &= -\sqrt{a}c_a^2 h^{-9}(\mathbf{u})E[(Y_{4a}-u_4)^3(Y_{1a}+Y_{2a}+Y_{5a})]\\
&= -\sqrt{a}c_a^2 h^{-9}(\mathbf{u})\left\{E[(Y_{4a}-u_4)^3 Y_{1a}]+E[(Y_{4a}-u_4)^3 Y_{2a}]+E[(Y_{4a}-u_4)^3 Y_{5a}]\right\}.
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3 Y_{1a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right]\\
&= O(a^{-2}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3 Y_{2a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)\right]\\
&= O(a^{-3}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3 Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 \left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)\right]\\
&= O(a^{-9/4}) \text{ by condition (2.7.2)}
\end{aligned}
$$

We obtain

$$E[G_3(\underline{\mathbf{Y}})G_4^2(\underline{\mathbf{Y}})] = O(a^{-3/2}).$$

$$\begin{aligned}
E[G_1(\underline{\mathbf{Y}})G_3(\underline{\mathbf{Y}})G_4(\underline{\mathbf{Y}})] &= ac_a h^{-7}(\mathbf{u})E[(Y_{4a}-u_4)^2(Y_{1a}+Y_{2a}+Y_{5a})^2] \\
&= ac_a h^{-7}(\mathbf{u})E[G_3^2(\underline{\mathbf{Y}})] \\
&= O(a^{-1}).
\end{aligned}$$

Thus, the third moment of $G(\underline{\mathbf{Y}})$, $E[G^3(\underline{\mathbf{Y}})]$ is given by

$$E[G^3(\underline{\mathbf{Y}})] = \frac{4}{h^3(\mathbf{u})}\sum_{i=1}^{a}\left[\frac{1}{a^{3/2}}\frac{\sigma_i^6[\gamma_i^2+2(n_i-2)]}{n_i^2(n_i-1)^2} + \frac{12}{a^{1/2}(a-1)}\frac{\alpha_i\gamma_i\sigma_i^5}{n_i^2(n_i-1)}\right] + O(a^{-1}).$$

Next, we derive the fourth moment of $G(\underline{\mathbf{Y}})$.

$$\begin{aligned}
E[G^4(\underline{\mathbf{Y}})] &= E[(G_1(\underline{\mathbf{Y}})+G_3(\underline{\mathbf{Y}})+G_4(\underline{\mathbf{Y}}))^4] \\
&= E[G_1^4(\underline{\mathbf{Y}})+4G_1^3(\underline{\mathbf{Y}})G_4(\underline{\mathbf{Y}})+6G_1^2(\underline{\mathbf{Y}})G_4^2(\underline{\mathbf{Y}})+4G_1(\underline{\mathbf{Y}})G_4^3(\underline{\mathbf{Y}})+G_4^4(\underline{\mathbf{Y}})] \\
&+ 4E[G_1^3(\underline{\mathbf{Y}})G_3(\underline{\mathbf{Y}})+3G_1^2(\underline{\mathbf{Y}})G_3(\underline{\mathbf{Y}})G_4(\underline{\mathbf{Y}})+3G_1(\underline{\mathbf{Y}})G_3(\underline{\mathbf{Y}})G_4^2(\underline{\mathbf{Y}})+G_3(\underline{\mathbf{Y}})G_4^3(\underline{\mathbf{Y}})] \\
&+ 6E[G_1^2(\underline{\mathbf{Y}})G_3^2(\underline{\mathbf{Y}})+2G_1(\underline{\mathbf{Y}})G_3^2(\underline{\mathbf{Y}})G_4(\underline{\mathbf{Y}})+G_4^2(\underline{\mathbf{Y}})G_4^2(\underline{\mathbf{Y}})] \\
&+ 4E[G_1(\underline{\mathbf{Y}})G_3^3(\underline{\mathbf{Y}})+G_3^3(\underline{\mathbf{Y}})G_4(\underline{\mathbf{Y}})]+E[G_3(\underline{\mathbf{Y}})^4].
\end{aligned}$$

$$\begin{aligned}
E[G_1^4(\underline{\mathbf{Y}})] &= \frac{a^2}{h^4(\mathbf{u})}E[(Y_{1a}+Y_{2a}+Y_{5a})^4] \\
&= \frac{a^2}{h^4(\mathbf{u})}E[Y_{1a}^4+4Y_{1a}^3Y_{2a}+6Y_{1a}^2Y_{2a}^2+4Y_{1a}Y_{2a}^3+Y_{2a}^4+4Y_{1a}^3Y_{5a}+12Y_{1a}^2Y_{2a}Y_{5a} \\
&+ 12Y_{1a}Y_{2a}^2Y_{5a}+4Y_{2a}^3Y_{5a}+6Y_{1a}^2Y_{5a}^2+12Y_{1a}Y_{2a}Y_{5a}^2+6Y_{2a}^2Y_{5a}^2+4Y_{1a}Y_{5a}^3 \\
&+ 4Y_{2a}Y_{5a}^3+Y_{5a}^4].
\end{aligned}$$

$$E[Y_{1a}^4] = E\left[\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right]^4 = \frac{12}{a^4}\sum_{i\neq i'}^{a}\frac{\sigma_i^4\sigma_{i'}^4}{n_i(n_i-1)n_{i'}(n_{i'}-1)} + O(a^{-3}).$$

$$E[Y_{1a}^3Y_{2a}] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^3\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right] = O(a^{-3}).$$

$$E[Y_{1a}^2 Y_{2a}^2] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)^2\right] = O(a^{-3}).$$

$$E[Y_{1a} Y_{2a}^3] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)^3\right] = O(a^{-4}).$$

$$E[Y_{2a}^4] = E\left[\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right]^4 = O(a^{-4}).$$

$$\begin{aligned} E[Y_{1a}^3 Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^3\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)\right] \\ &= O(a^{-13/4}) \text{ by condition (2.7.2).} \end{aligned}$$

$$\begin{aligned} E[Y_{1a}^2 Y_{2a} Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)\right] \\ &= O(a^{-13/4}) \text{ by condition (2.7.2).} \end{aligned}$$

$$\begin{aligned} E[Y_{1a} Y_{2a}^2 Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)^2\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)\right] \\ &= O(a^{-13/4}) \text{ by condition (2.7.2).} \end{aligned}$$

$$\begin{aligned} E[Y_{2a}^3 Y_{5a}] &= E\left[\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)^3\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)\right] \\ &= O(a^{-17/4}) \text{ by condition (2.7.2).} \end{aligned}$$

$$\begin{aligned} E[Y_{1a}^2 Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)^2\right] \\ &= \frac{8}{a^2(a-1)^2}\sum_{i\neq i'}^{a}\frac{\alpha_{i'}^2\sigma_i^4\sigma_{i'}^2}{n_i(n_i-1)n_{i'}} + O(a^{-7/2}) \text{ by condition (2.7.2).} \end{aligned}$$

$$
\begin{aligned}
E[Y_{1a}Y_{2a}Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^2\right] \\
&= O(a^{-7/2}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[Y_{2a}^2Y_{5a}^2] &= E\left[\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^2\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^2\right] \\
&= O(a^{-7/2}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[Y_{1a}Y_{5a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^3\right] \\
&= O(a^{-15/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[Y_{2a}Y_{5a}^3] &= E\left[\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^3\right] \\
&= O(a^{-15/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
E[Y_{5a}^4] = E\left[\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right]^4 = O(a^{-3}) \text{ by condition (2.7.2).}
$$

Therefore, we end up with

$$
E[G_1^4(\underline{\mathbf{Y}})] = \frac{12}{h^4(\mathbf{u})}\sum_{i\neq i'}^{a}\left[\frac{1}{a^2}\frac{\sigma_i^4\sigma_{i'}^4}{n_i(n_i-1)n_{i'}(n_{i'}-1)} + \frac{4}{(a-1)^2}\frac{\alpha_{i'}^2\sigma_i^4\sigma_{i'}^2}{n_i(n_i-1)n_{i'}}\right] + O(a^{-1}).
$$

$$
\begin{aligned}
E[G_1^3(\underline{\mathbf{Y}})G_4(\underline{\mathbf{Y}})] &= -a^{3/2}c_a h^{-6}(\mathbf{u})E[(Y_{4a}-u_4)(Y_{1a}+Y_{2a}+Y_{5a})^3] \\
&= \sqrt{a}c_a h^{-1}(\mathbf{u})E[G_1^2(\underline{\mathbf{Y}})G_3(\underline{\mathbf{Y}})] \\
&= -12\sqrt{a}c_a h^{-6}(\mathbf{u})\sum_{i\neq i'}^{a}\left[\frac{1}{a^{5/2}}\frac{\sigma_i^6(\gamma_i^2-1)\sigma_{i'}^4}{n_i^2(n_i-1)^2 n_{i'}(n_{i'}-1)}\right] \\
&- 12\sqrt{a}c_a h^{-6}(\mathbf{u})\sum_{i\neq i'}^{a}\left[\frac{2}{a^{3/2}(a-1)}\frac{\gamma_i\sigma_i^5\sigma_{i'}^4\alpha_i}{n_i^2(n_i-1)n_{i'}(n_{i'}-1)}\right] \\
&+ O(a^{-1}).
\end{aligned}
$$

$$
\begin{aligned}
E[G_1^2(\underline{\mathbf{Y}})G_4^2(\underline{\mathbf{Y}})] &= ac_a h^{-8}(\mathbf{u})E[(Y_{4a}-u_4)^2(Y_{1a}+Y_{2a}+Y_{5a})^2] \\
&= c_a h^{-1}(\mathbf{u})E[G_3^2(\underline{\mathbf{Y}})] \\
&= O(a^{-1}).
\end{aligned}
$$

$$
\begin{aligned}
E[G_1(\underline{\mathbf{Y}})G_4^3(\underline{\mathbf{Y}})] &= -\sqrt{a}c_a^3 h^{-10}(\mathbf{u})E[(Y_{4a}-u_4)^3(Y_{1a}+Y_{2a}+Y_{5a})] \\
&= c_a h^{-1}(\mathbf{u})E[G_3(\underline{\mathbf{Y}})G_4^2(\underline{\mathbf{Y}})] \\
&= O(a^{-3/2}).
\end{aligned}
$$

$$
\begin{aligned}
E[G_4^4(\underline{\mathbf{Y}})] &= c_a^4 h^{-12}(\mathbf{u})E[(Y_{4a}-u_4)^4] \\
&= c_a^4 h^{-12}(\mathbf{u})E\left[\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right]^4 \\
&= O(a^{-2}).
\end{aligned}
$$

$$
\begin{aligned}
E[G_1^3(\underline{\mathbf{Y}})G_3(\underline{\mathbf{Y}})] &= -a^2 h^{-6}(\mathbf{u})E[(Y_{4a}-u_4)(Y_{1a}+Y_{2a}+Y_{5a})^4] \\
&= -a^2 h^{-6}(\mathbf{u})E[(Y_{4a}-u_4)(Y_{1a}^4+4Y_{1a}^3Y_{2a}+6Y_{1a}^2Y_{2a}^2+4Y_{1a}Y_{2a}^3 \\
&+ Y_{2a}^4+4Y_{1a}^3Y_{5a}+12Y_{1a}^2Y_{2a}Y_{5a}+12Y_{1a}Y_{2a}^2Y_{5a} \\
&+ 4Y_{2a}^3Y_{5a}+6Y_{1a}^2Y_{5a}^2+12Y_{1a}Y_{2a}Y_{5a}^2+6Y_{2a}^2Y_{5a}^2+4Y_{1a}Y_{5a}^3 \\
&+ 4Y_{2a}Y_{5a}^3+Y_{5a}^4)].
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{1a}^4] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^4\right] \\
&= O(a^{-3}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{1a}^3Y_{2a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right) * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^3\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot}\right)\right] \\
&= O(a^{-3}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{1a}^2Y_{2a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)*\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right.\\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^2\right]\\
&= O(a^{-4}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{1a}Y_{2a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)*\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right.\\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^3\right]\\
&= O(a^{-4}).
\end{aligned}
$$

$$
E[(Y_{4a}-u_4)Y_{2a}^4] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^4\right] = O(a^{-5}).
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{1a}^3Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)*\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^3\right.\\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right]\\
&= O(a^{-13/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{1a}^2Y_{2a}Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)*\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right.\\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)*\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right]\\
&= O(a^{-13/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{1a}Y_{2a}^2Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)*\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right.\\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^2\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right]\\
&= O(a^{-17/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{2a}^3Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)*\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^3\right.\\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right]\\
&= O(a^{-17/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{1a}^2Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)*\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right.\\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^2\right]\\
&= O(a^{-7/2}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{1a}Y_{2a}Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)*\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right.\\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)*\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^2\right]\\
&= O(a^{-9/2}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{2a}^2Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)*\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^2\right.\\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^2\right]\\
&= O(a^{-9/2}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{1a}Y_{5a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right) * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right. \\
&\quad * \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^3\right] \\
&= O(a^{-15/4}) \text{ by condition (2.7.2)}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{2a}Y_{5a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right) * \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right. \\
&\quad * \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^3\right] \\
&= O(a^{-15/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)Y_{5a}^4] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^4\right] \\
&= O(a^{-4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

Therefore, we end up with

$$E[G_1^3(\underline{\mathbf{Y}})G_3(\underline{\mathbf{Y}})] = O(a^{-1}).$$

$$
\begin{aligned}
E[G_1^2(\underline{\mathbf{Y}})G_3(\underline{\mathbf{Y}})G_4(\underline{\mathbf{Y}})] &= a^{3/2}c_a h^{-8}(\mathbf{u})E[(Y_{4a}-U_4)^2(Y_{1a}+Y_{2a}+Y_{5a})^3] \\
&= c_a h^{-1}(\mathbf{u})E[G_1(\underline{\mathbf{Y}})G_3^2(\underline{\mathbf{Y}})] \\
&= O(a^{-3/2}).
\end{aligned}
$$

$$
\begin{aligned}
E[G_1(\underline{\mathbf{Y}})G_3(\underline{\mathbf{Y}})G_4^2(\underline{\mathbf{Y}})] &= a c_a^2 h^{-10}(\mathbf{u})E[(Y_{4a}-U_4)^3(Y_{1a}+Y_{2a}+Y_{5a})^2] \\
&= c_a h^{-1}(\mathbf{u})E[G_3^2(\underline{\mathbf{Y}})G_4(\underline{\mathbf{Y}})] \\
&= O(a^{-2}).
\end{aligned}
$$

$$
\begin{aligned}
E[G_3(\underline{\mathbf{Y}})G_4^3(\underline{\mathbf{Y}})] &= \sqrt{a}c_a^3 h^{-12}(\mathbf{u})E[(Y_{4a}-u_4)^4(Y_{1a}+Y_{2a}+Y_{5a})] \\
&= \sqrt{a}c_a^3 h^{-12}(\mathbf{u})[E(Y_{4a}-u_4)^4Y_{1a}+E(Y_{4a}-u_4)^4Y_{2a}+E(Y_{4a}-u_4)^4Y_{5a})].
\end{aligned}
$$

$$E[(Y_{4a}-u_4)^4Y_{1a}] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right] = O(a^{-3}).$$

$$E[(Y_{4a}-u_4)^4Y_{2a}] = E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right] = O(a^{-3}).$$

$$\begin{aligned} E[(Y_{4a}-u_4)^4Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right] \\ &= O(a^{-13/4}) \text{ by condition (2.7.2).} \end{aligned}$$

Therefore, we have

$$E[G_3(\underline{\mathbf{Y}})G_4^3(\underline{\mathbf{Y}})] = O(a^{-5/2}).$$

$$\begin{aligned} E[G_1^2(\underline{\mathbf{Y}})G_3^2(\underline{\mathbf{Y}})] &= a^2h^{-8}(\mathbf{u})E[(Y_{4a}-u_4)^2(Y_{1a}+Y_{2a}+Y_{5a})^4] \\ &= -a^2h^{-8}(\mathbf{u})E[(Y_{4a}-u_4)^2(Y_{1a}^4+4Y_{1a}^3Y_{2a}+6Y_{1a}^2Y_{2a}^2+4Y_{1a}Y_{2a}^3 \\ &+ Y_{2a}^4+4Y_{1a}^3Y_{5a}+12Y_{1a}^2Y_{2a}Y_{5a}+12Y_{1a}Y_{2a}^2Y_{5a} \\ &+ 4Y_{2a}^3Y_{5a}+6Y_{1a}^2Y_{5a}^2+12Y_{1a}Y_{2a}Y_{5a}^2+6Y_{2a}^2Y_{5a}^2+4Y_{1a}Y_{5a}^3 \\ &+ 4Y_{2a}Y_{5a}^3+Y_{5a}^4)]. \end{aligned}$$

$$\begin{aligned} E[(Y_{4a}-u_4)^2Y_{1a}^4] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^4\right] \\ &= O(a^{-3}). \end{aligned}$$

$$\begin{aligned} E[(Y_{4a}-u_4)^2Y_{1a}^3Y_{2a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^3\right. \\ &* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right] \\ &= O(a^{-4}). \end{aligned}$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2 Y_{1a}^2 Y_{2a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot}\right)^2\right] \\
&= O(a^{-4}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2 Y_{1a} Y_{2a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot}\right)^3\right] \\
&= O(a^{-5}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2 Y_{2a}^4] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot}\right)^4\right] \\
&= O(a^{-5}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2 Y_{1a}^3 Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^3\right. \\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\overline{\epsilon}_{i\cdot}\right)\right] \\
&= O(a^{-13/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2 Y_{1a}^2 Y_{2a} Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot}\right) * \left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\overline{\epsilon}_{i\cdot}\right)\right] \\
&= O(a^{-17/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2 Y_{1a} Y_{2a}^2 Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right.\\
&\quad * \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)^2\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)\right]\\
&= O(a^{-17/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2 Y_{2a}^3 Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)^3\right.\\
&\quad * \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)\right]\\
&= O(a^{-21/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2 Y_{1a}^2 Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right.\\
&\quad * \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)^2\right]\\
&= O(a^{-7/2}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2 Y_{1a} Y_{2a} Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right.\\
&\quad * \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right) * \left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)^2\right]\\
&= O(a^{-9/2}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2 Y_{2a}^2 Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)^2\right.\\
&\quad * \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)^2\right]\\
&= O(a^{-9/2}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2 Y_{1a} Y_{5a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right. \\
&\quad \left. * \left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^3\right] \\
&= O(a^{-15/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2 Y_{2a} Y_{5a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 * \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right. \\
&\quad \left. * \left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^3\right] \\
&= O(a^{-19/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^2 Y_{5a}^4] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^2 \left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^4\right] \\
&= O(a^{-4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

Therefore, we end up with

$$
E[G_1^2(\underline{\mathbf{Y}})G_3^2(\underline{\mathbf{Y}})] = O(a^{-1}).
$$

$$
\begin{aligned}
E[G_1(\underline{\mathbf{Y}})G_3^2(\underline{\mathbf{Y}})G_4(\underline{\mathbf{Y}})] &= -a^{3/2}c_a h^{-10}(\mathbf{u})E[(Y_{4a}-u_4)^3(Y_{1a}+Y_{2a}+Y_{5a})^3] \\
&= c_a h^{-1}(\mathbf{u})E[G_3^3(Y)] \\
&= O(a^{-3/2}).
\end{aligned}
$$

$$
\begin{aligned}
E[G_3^2(\underline{\mathbf{Y}})G_4^2(\underline{\mathbf{Y}})] &= ac_a^2 h^{-12}(\mathbf{u})E[(Y_{4a}-u_4)^4(Y_{1a}+Y_{2a}+Y_{5a})^2] \\
&= ac_a^2 h^{-12}(\mathbf{u})E[(Y_{4a}-u_4)^4(Y_{1a}^2+2Y_{1a}Y_{2a}+Y_{2a}^2+2Y_{1a}Y_{5a}+2Y_{2a}Y_{5a}+Y_{5a}^2)].
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4 Y_{1a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right] \\
&= O(a^{-3}).
\end{aligned}
$$

$$\begin{aligned}E[(Y_{4a}-u_4)^4Y_{1a}Y_{2a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right.\\ &\quad * \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right]\\ &= O(a^{-4}).\end{aligned}$$

$$\begin{aligned}E[(Y_{4a}-u_4)^4Y_{2a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^2\right]\\ &= O(a^{-4}).\end{aligned}$$

$$\begin{aligned}E[(Y_{4a}-u_4)^4Y_{1a}Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right.\\ &\quad * \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right]\\ &= O(a^{-13/4}) \text{ by condition (2.7.2)}.\end{aligned}$$

$$\begin{aligned}E[(Y_{4a}-u_4)^4Y_{2a}Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right.\\ &\quad * \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right]\\ &= O(a^{-17/4}) \text{ by condition (2.7.2)}.\end{aligned}$$

$$\begin{aligned}E[(Y_{4a}-u_4)^4Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^2\right]\\ &= O(a^{-7/2}) \text{ by condition (2.7.2)}.\end{aligned}$$

We obtain

$$E[G_3^2(\underline{\mathbf{Y}})G_4^2(\underline{\mathbf{Y}})] = O(a^{-2}).$$

$$
\begin{aligned}
E[G_3^3(\underline{\mathbf{Y}})G_4(\underline{\mathbf{Y}})] &= a^{3/2}c_a h^{-12}(\mathbf{u})E[(Y_{4a}-u_4)^4(Y_{1a}+Y_{2a}+Y_{5a})^3] \\
&= a^{3/2}c_a h^{-12}(\mathbf{u})E[(Y_{4a}-u_4)^4(Y_{1a}^3+3Y_{1a}^2Y_{2a}+3Y_{1a}Y_{2a}^2+Y_{2a}^3+3Y_{1a}^2Y_{5a} \\
&+ 6Y_{1a}Y_{2a}Y_{5a}+3Y_{2a}^2Y_{5a}+3Y_{1a}Y_{5a}^2+3Y_{2a}Y_{5a}^2+Y_{5a}^3)].
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4Y_{1a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^3\right] \\
&= O(a^{-4}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4Y_{1a}^2Y_{2a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right] \\
&= O(a^{-4}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4Y_{1a}Y_{2a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^2\right] \\
&= O(a^{-5}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4Y_{1a}Y_{2a}Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right) * \left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right] \\
&= O(a^{-17/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4Y_{2a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^3\right] \\
&= O(a^{-5}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4 Y_{1a}^2 Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right.\\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right]\\
&= O(a^{-17/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4 Y_{1a} Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right.\\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^2\right]\\
&= O(a^{-9/2}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4 Y_{2a}^2 Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^2\right.\\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right]\\
&= O(a^{-21/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4 Y_{2a} Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right.\\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^2\right]\\
&= O(a^{-11/2}) \text{ by condition (2.7.2)}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4 Y_{5a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 \left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^3\right]\\
&= O(a^{-19/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

Therefore we obtain

$$E[G_3^3(\underline{\mathbf{Y}})G_4(\underline{\mathbf{Y}})] = O(a^{-5/2}).$$

$$\begin{aligned}
E[G_1(\underline{\mathbf{Y}})G_3^3(\underline{\mathbf{Y}})] &= -a^2h^{-10}(\mathbf{u})E[(Y_{4a}-u_4)^3(Y_{1a}+Y_{2a}+Y_{5a})^4] \\
&= -a^2h^{-10}(\mathbf{u})E[(Y_{4a}-u_4)^3(Y_{1a}^4+4Y_{1a}^3Y_{2a}+6Y_{1a}^2Y_{2a}^2+4Y_{1a}Y_{2a}^3 \\
&+ Y_{2a}^4+4Y_{1a}^3Y_{5a}+12Y_{1a}^2Y_{2a}Y_{5a}+12Y_{1a}Y_{2a}^2Y_{5a} \\
&+ 4Y_{2a}^3Y_{5a}+6Y_{1a}^2Y_{5a}^2+12Y_{1a}Y_{2a}Y_{5a}^2+6Y_{2a}^2Y_{5a}^2+4Y_{1a}Y_{5a}^3 \\
&+ 4Y_{2a}Y_{5a}^3+Y_{5a}^4)].
\end{aligned}$$

$$\begin{aligned}
E[(Y_{4a}-u_4)^3Y_{1a}^4] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3\left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^4\right] \\
&= O(a^{-4}).
\end{aligned}$$

$$\begin{aligned}
E[(Y_{4a}-u_4)^3Y_{1a}^3Y_{2a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^3\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot}\right)\right] \\
&= O(a^{-4}).
\end{aligned}$$

$$\begin{aligned}
E[(Y_{4a}-u_4)^3Y_{1a}^2Y_{2a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}\overline{\epsilon}_{i\cdot}\overline{\epsilon}_{i'\cdot}\right)^2\right] \\
&= O(a^{-5}).
\end{aligned}$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3 Y_{1a} Y_{2a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right.\\
&\quad * \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^3\right]\\
&= O(a^{-5}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3 Y_{2a}^4] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^4\right]\\
&= O(a^{-5}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3 Y_{1a}^3 Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^3\right.\\
&\quad * \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right]\\
&= O(a^{-17/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3 Y_{1a}^2 Y_{2a} Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right.\\
&\quad * \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right) * \left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right]\\
&= O(a^{-17/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3 Y_{1a} Y_{2a}^2 Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right.\\
&\quad * \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^2 \left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right]\\
&= O(a^{-21/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3Y_{2a}^3Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)^3\right.\\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)\right]\\
&= O(a^{-21/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3Y_{1a}^2Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right.\\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)^2\right]\\
&= O(a^{-9/2}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3Y_{1a}Y_{2a}Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right.\\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right) * \left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)^2\right]\\
&= O(a^{-9/2}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3Y_{2a}^2Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)^2\right.\\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)^2\right]\\
&= O(a^{-11/2}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3Y_{1a}Y_{5a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right.\\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)^3\right]\\
&= O(a^{-19/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3 Y_{2a} Y_{5a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 * \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\overline{\epsilon}_{i}.\overline{\epsilon}_{i'}.\right)\right. \\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\overline{\epsilon}_{i}.\right)^3\right] \\
&= O(a^{-19/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^3 Y_{5a}^4] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^3 \left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\overline{\epsilon}_{i}.\right)^4\right] \\
&= O(a^{-5}) \text{ by condition (2.7.2).}
\end{aligned}
$$

Therefore, we end up with

$$
E[G_1(\underline{\mathbf{Y}})G_3^3(\underline{\mathbf{Y}})] = O(a^{-2})
$$

$$
\begin{aligned}
E[G_3^4(\underline{\mathbf{Y}})] &= a^2 h^{-12}(\mathbf{u})E[(Y_{4a}-u_4)^4(Y_{1a}+Y_{2a}+Y_{5a})^4] \\
&= a^2 h^{-12}(\mathbf{u})E[(Y_{4a}-u_4)^4(Y_{1a}^4+4Y_{1a}^3Y_{2a}+6Y_{1a}^2Y_{2a}^2+4Y_{1a}Y_{2a}^3 \\
&+ Y_{2a}^4+4Y_{1a}^3Y_{5a}+12Y_{1a}^2Y_{2a}Y_{5a}+12Y_{1a}Y_{2a}^2Y_{5a} \\
&+ 4Y_{2a}^3Y_{5a}+6Y_{1a}^2Y_{5a}^2+12Y_{1a}Y_{2a}Y_{5a}^2+6Y_{2a}^2Y_{5a}^2+4Y_{1a}Y_{5a}^3 \\
&+ 4Y_{2a}Y_{5a}^3+Y_{5a}^4)].
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4 Y_{1a}^4] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^4\right] \\
&= O(a^{-4}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4 Y_{1a}^3 Y_{2a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^3\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\overline{\epsilon}_{i}.\overline{\epsilon}_{i'}.\right)\right] \\
&= O(a^{-5}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4 Y_{1a}^2 Y_{2a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^2\right] \\
&= O(a^{-5}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4 Y_{1a} Y_{2a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^3\right] \\
&= O(a^{-6}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4 Y_{2a}^4] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)^4\right] \\
&= O(a^{-6}).
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4 Y_{1a}^3 Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^3\right. \\
&* \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right] \\
&= O(a^{-17/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4 Y_{1a}^2 Y_{2a} Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right. \\
&* \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right) * \left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)\right] \\
&= O(a^{-21/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4 Y_{1a} Y_{2a}^2 Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right.\\
&\quad * \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)^2\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)\right]\\
&= O(a^{-21/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4 Y_{2a}^3 Y_{5a}] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)^3\right.\\
&\quad * \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)\right]\\
&= O(a^{-25/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4 Y_{1a}^2 Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)^2\right.\\
&\quad * \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)^2\right]\\
&= O(a^{-9/2}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4 Y_{1a} Y_{2a} Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right.\\
&\quad * \left.\left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right) * \left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)^2\right]\\
&= O(a^{-11/2}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4 Y_{2a}^2 Y_{5a}^2] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i}.\bar{\epsilon}_{i'}.\right)^2\right.\\
&\quad * \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i}.\right)^2\right]\\
&= O(a^{-11/2}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4 Y_{1a} Y_{5a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{1}{a}\sum_{i=1}^{a}\sum_{j\neq j'}^{n_i}\frac{\epsilon_{ij}\epsilon_{ij'}}{n_i(n_i-1)}\right)\right.\\
&\quad * \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^3\right]\\
&= O(a^{-19/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4 Y_{2a} Y_{5a}^3] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 * \left(\frac{-1}{a(a-1)}\sum_{i\neq i'}^{a}\bar{\epsilon}_{i\cdot}\bar{\epsilon}_{i'\cdot}\right)\right.\\
&\quad * \left.\left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^3\right]\\
&= O(a^{-23/4}) \text{ by condition (2.7.2).}
\end{aligned}
$$

$$
\begin{aligned}
E[(Y_{4a}-u_4)^4 Y_{5a}^4] &= E\left[\left(\frac{1}{a}\sum_{i=1}^{a}\frac{\left(\widehat{\sigma_i^4}-E\left(\widehat{\sigma_i^4}\right)\right)}{n_i(n_i-1)}\right)^4 \left(\frac{1}{a-1}\sum_{i=1}^{a}\alpha_i\bar{\epsilon}_{i\cdot}\right)^4\right]\\
&= O(a^{-5}) \text{ by condition (2.7.2).}
\end{aligned}
$$

Therefore, we end up with

$$
E[G_3^4(\underline{\mathbf{Y}})] = O(a^{-2})
$$

Thus, the fourth moment of $G(\underline{\mathbf{Y}})$, $E[G^4(\underline{\mathbf{Y}})]$ is given by

$$
\begin{aligned}
E[G^4(\underline{\mathbf{Y}})] &= \frac{12}{h^4(\mathbf{u})}\sum_{i\neq i'}^{a}\left[\frac{1}{a^2}\frac{\sigma_i^4\sigma_{i'}^4}{n_i(n_i-1)n_{i'}(n_{i'}-1)} + \frac{4}{(a-1)^2}\frac{\alpha_{i'}^2\sigma_i^4\sigma_{i'}^2}{n_i(n_i-1)n_{i'}}\right]\\
&- 48\sqrt{a}ch^{-6}(U)\sum_{i\neq i'}^{a}\left[\frac{1}{a^{5/2}}\frac{\sigma_i^6(\gamma_i^2-1)\sigma_{i'}^4}{n_i^2(n_i-1)^2 n_{i'}(n_{i'}-1)} + \frac{2}{a^{3/2}(a-1)}\frac{\gamma_i\sigma_i^5\sigma_{i'}^4\alpha_i}{n_i^2(n_i-1)n_{i'}(n_{i'}-1)}\right]\\
&+ O(a^{-1}).
\end{aligned}
$$

Let κ_{1a}^G, κ_{2a}^G, κ_{3a}^G and κ_{4a}^G be the first four cumulants of $G(Y)$. Then using the first four moments, we obtain the cumulants as follows:

$$
\begin{aligned}
\kappa_{1a}^G &= E[G(\underline{\mathbf{Y}})]\\
&= \frac{1}{\sqrt{a}}[\kappa_{11}^g + \kappa_{11}^{g_1}]
\end{aligned}
$$

where κ_{11}^{g} and $\kappa_{11}^{g_1}$ are defined in equations (2.3.2) and (2.7.5) respectively.

$$\begin{aligned}\kappa_{2a}^{G} &= E[G^2(\underline{\mathbf{Y}})] - \{E[G(\underline{\mathbf{Y}})]\}^2 \\ &= 1 + \frac{1}{\sqrt{a}}[\kappa_{22}^{g_2} + 2c_a h^{-1}(\mathbf{u})(\kappa_{11}^{g} + \kappa_{11}^{g_1})] + O(a^{-1}).\end{aligned}$$

where $\kappa_{22}^{g_2}$ is defined in equation (2.7.5).

$$\begin{aligned}\kappa_{3a}^{G} &= E[G^3(\underline{\mathbf{Y}})] - 3E[G^2(\underline{\mathbf{Y}})]E[G(\underline{\mathbf{Y}})] + \{E[G(\underline{\mathbf{Y}})]\}^3 \\ &= \frac{1}{\sqrt{a}}[\kappa_{33}^{g} + \kappa_{33}^{g_3}] + O(a^{-1})\end{aligned}$$

where κ_{33}^{g} and $\kappa_{33}^{g_3}$ are defined in equations (2.3.2) and (2.7.5) respectively.

$$\begin{aligned}\kappa_{4a}^{G} &= E[G^4(\underline{\mathbf{Y}})] - 4E[G^3(\underline{\mathbf{Y}})]E[G(\underline{\mathbf{Y}})] - 3\left\{E[G^2(\underline{\mathbf{Y}})]\right\}^2 + 12E[G^2(\underline{\mathbf{Y}})]\left\{E[G(\underline{\mathbf{Y}})]\right\}^2 \\ &- 6\left\{E[G(\underline{\mathbf{Y}})]\right\}^4 \\ &= O(a^{-1}).\end{aligned}$$

Using the cumulants, we now proceed to obtain the characteristic function of $G(\underline{\mathbf{Y}})$. Let χ_G be the characteristic function of $G(\underline{\mathbf{Y}})$. Then,

$$\begin{aligned}\chi_G(t) &= exp\left\{\kappa_{1a}^{G}(it) + \kappa_{2a}^{G}\frac{(it)^2}{2} + \kappa_{3a}^{G}\frac{(it)^3}{6} + \kappa_{4a}^{G}\frac{(it)^4}{24}\right\} \\ &= exp[\frac{1}{\sqrt{a}}\left\{\kappa_{11}^{g} + \kappa_{11}^{g_1}\right\}(it) + \left\{1 + \frac{1}{\sqrt{a}}(\kappa_{22}^{g_2} + 2ch^{-1}(\mathbf{u})(\kappa_{11}^{g} + \kappa_{11}^{g_1}))\right\}\frac{(it)^2}{2} \\ &+ \frac{1}{\sqrt{a}}\left\{\kappa_{33}^{g} + \kappa_{33}^{g_3}\right\}\frac{(it)^3}{6} + O(a^{-1})] \\ &= exp(-\frac{t^2}{2})exp[\frac{1}{\sqrt{a}}\left\{\kappa_{11}^{g} + \kappa_{11}^{g_1}\right\}(it) + \frac{1}{\sqrt{a}}\left\{\kappa_{22}^{g_2} + 2ch^{-1}(\mathbf{u})(\kappa_{11}^{g} + \kappa_{11}^{g_1})\right\}\frac{(it)^2}{2} \\ &+ \frac{1}{\sqrt{a}}\left\{\kappa_{33}^{g} + \kappa_{33}^{g_3}\right\}\frac{(it)^3}{6} + O(a^{-1})].\end{aligned}$$

By Taylor series expansion, we obtain

$$\begin{aligned}\chi_G(t) &= exp(-\frac{t^2}{2})exp[1 + \frac{1}{\sqrt{a}}[(\kappa_{11}^{g} + \kappa_{11}^{g_1})(it) + (\kappa_{22}^{g_2} + 2ch^{-1}(\mathbf{u})(\kappa_{11}^{g} + \kappa_{11}^{g_1}))\frac{(it)^2}{2} \\ &+ (\kappa_{33}^{g} + \kappa_{33}^{g_3})\frac{(it)^3}{6}] + O(a^{-1})].\end{aligned}$$

Applying the inverse fourier transform, we obtain the pdf of G as,

$$
\begin{aligned}
f_G(x) &= \int_{-\infty}^{\infty} e^{-itx}\chi_G(t)dt \\
&= \int_{-\infty}^{\infty} e^{-itx} exp(-\frac{t^2}{2}) exp[1 + \frac{1}{\sqrt{a}}[(\kappa_{11}^{g} + \kappa_{11}^{g_1})(it) + (\kappa_{22}^{g_2} + 2ch^{-1}(\mathbf{u})(\kappa_{11}^{g} + \kappa_{11}^{g_1}))\frac{(it)^2}{2} \\
&+ (\kappa_{33}^{g} + \kappa_{33}^{g_3})\frac{(it)^3}{6}] + O(a^{-1})]dt \\
&= \phi(x) + \frac{1}{\sqrt{a}}[(\kappa_{11}^{g} + \kappa_{11}^{g_1})H_1(x) + (\kappa_{22}^{g_2} + 2ch^{-1}(U)(\kappa_{11}^{g} + \kappa_{11}^{g_1}))\frac{H_2(x)}{2} \\
&+ (\kappa_{33}^{g} + \kappa_{33}^{g_3})\frac{H_3(x)}{6}]\phi(x) + O(a^{-1}).
\end{aligned}
$$

where $H_0(x)$, $H_1(x)$, $H_2(x)$, and $H_3(x)$ are Hermite polynomials. We now obtain the cdf of G as;

$$
\begin{aligned}
F_G(x) &= \int_{-\infty}^{x} f_G(u)du \\
&= \Phi(x) - \frac{1}{\sqrt{a}}[(\kappa_{11}^{g} + \kappa_{11}^{g_1})H_0(x) + (\kappa_{22}^{g_2} + 2ch^{-1}(\mathbf{u})(\kappa_{11}^{g} + \kappa_{11}^{g_1}))\frac{H_1(x)}{2} \\
&+ (\kappa_{33}^{g} + \kappa_{33}^{g_3})\frac{H_2(x)}{6}]\phi(x) + O(a^{-1}) \\
&= \Phi(x) + \frac{1}{\sqrt{a}}Q_1^{alt}(x)\phi(x) + O(a^{-1}).
\end{aligned}
$$

where

$$
Q_1^{alt}(x) = -[(\kappa_{11}^{g} + \kappa_{11}^{g_1}) + \frac{1}{2}(\kappa_{22}^{g_2} + 2ch^{-1}(\mathbf{u})(\kappa_{11}^{g} + \kappa_{11}^{g_1}))x + \frac{1}{6}(\kappa_{33}^{g} + \kappa_{33}^{g_3})(x^2 - 1)].
$$

Chapter 3

Summary and Contributions

3.1 Summary

In summary, we have described a test statistic suitable for bootstrap inference in ANOVA with a large number of treatment levels and small replications. The first part of the dissertation establishes the bootstrap inference in both one- and two-way ANOVA for a large number of treatment levels with small replications for skewed and heteroscedastic variances. Theoretical results show that the bootstrap inference based on asymptotically pivotal statistic has the same type I error accuracy as bootstrap inference based on non-pivotal statistic and the test based on limiting distribution of the test statistic. In the second part of the dissertation, we proposed an asymptotically pivotal statistic and a new test based on asymptotic expansions for one-way ANOVA with a large number of factor levels and small replications under heteroscedastic variances and skewed data. Theoretical results demonstrate that the type I error-rate of our asymptotic expansion of pivotal statistic has a better accuracy up to order $O(a^{-1})$. The connection between the test based on our asymptotic expansions and the bootstrap test has been demonstrated.

Numerical results show that the test based on our asymptotic expansions outperforms the bootstrap test and the test based on limiting normal distribution for both heteroscedastic and homoscedastic data in terms of type I error-rate and power. While the bootstrap test based on asymptotically pivotal statistic has better type I error accuracy in the classical large sample sizes but small number of treatment levels, it is not the case in our current setting of a large treatment levels with small replications. Moreover the bootstrap test requires more time due to the its resampling nature. Another limitation of the bootstrap test is the use of biased estimates in computing population parameters such as the population skewness. For one-way ANOVA with a large number of factor levels and small replications, we recommend to use our test based on asymptotic expansions.

3.2 Contributions of the dissertation

- Established bootstrap inference in ANOVA for a large number of treatment levels with small replications for skewed data under heteroscedastic variances.
- Demonstrated a criteria to determine a suitable test statistic for bootstrap test in high dimensional ANOVA.
- Derived the theoretical type I error accuracy of Akritas and Papadatos (2004) and Wang and Akritas (2006) in one- and two-way ANOVA, respectively.
- Proposed a test statistic which is asymptotically pivotal.
- Improved the order of approximation by deriving the Edgeworth expansion of the test statistic up to order $O(a^{-1})$.
- Proposed a new rejection region through Cornish-Fisher expansion of quantiles.
- Derived the type I error-rate of the proposed new test.

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