POLYTROPIC GAS FLOW IN A CONSTANT-AREA DUCT UNDER THE SIMULTANEOUS EFFECTS OF FRICTION AND HEAT TRANSFER

by 45

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Diploma, Taiwan Provincial Taipei Institute of Technology, 1964

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

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1969

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INTRODUCTION

There are a number of studies published for steady, one-dimensional flow of a compressible fluid. Shapiro and Hawthorne (7) presented the equations of steady flow showing the combined effects of external heat exchange, friction, area change, drag of internal bodies, chemical reaction, change of phase, injection of gases and change in molecular weight and specific heat. Recently, Chen (3) obtained equations for changes in some fluid properties along the passage which include the thermodynamical behaviors of the characteristics of n ($\frac{dp}{p} + n\frac{dv}{v} = 0$) of the pressure-volume relation of a perfect gas for three special cases of subsonic heating: constant-heat flux, constant-wall temperature and exponential longitudinal fluid temperature distribution.

The main purpose of this report is to investigate the physical properties of a perfect gas flowing steadily, and which undergoes a polytropic process in a constant-area duct under the simultaneous effects of friction and heat transfer. It is assumed in this report that: (a) the gas is flowing in a duct according to the relation pv^n = constant, where n is constant, (b) the molecular weight and the specific heats of the gas are constants, and (c) the Reynolds Analogy is valid, the recovery factor is unity and the ratio of specific heats is 1.4.

A so-called "Critical Mach Number", which represents a limit for a continuous polytropic gas flow, is used in this report. This critical point means that no flow can occur such that the velocity of the gas passes through the critical Mach Number.

In addition to the investigations of fluid properties, equations and figures are developed for several cases for the working ranges of Mach Number of a polytropic gas flow in ducts. The working ranges were studied through an examination of the related factors, such as the impulse function, the stagnation temperature and the wall temperature ratio. From these investigations of the working ranges of Mach Number, it is shown that the isopiestic flow ($n = 0$) of a perfect gas is impossible in a constant-area passage with friction and heat transfer.

Based on the unpublished notes of Professor Wilson Tripp, Department of Mechanical Engineering, Kansas State University, and on reference 4, page 178 and reference 10, page 23, three particular cases were surveyed as a verification to those formulas derived in this report. Those three cases are: the isentropic flow, the isothermal flow, and the constant-Mach Number flow.

Finally, a numerical example is presented to show the details of the calculations required to obtain the variations of the physical properties of air flowing in a constant-area passage, according to the relation pv^n = constant, where n is equal to 1.2.

NOMENCLATURE

A	:	cross-sectional flow area, sq. ft.
A_w	:	wetted area, sq. ft.
c	:	velocity of sound, $\sqrt{kRT}$, ft/sec
c_p	:	specific heat at constant pressure, Btu/slug °R
c_v	:	specific heat at constant volume, Btu/slug °R
D	:	equivalent hydraulic diameter, $4Ax/A_w$
e_e	:	expansion efficiency, defined by Equation B-2
F	:	impulse function, lb
F_f	:	mechanical energy converted to thermal energy by friction, ft-lb/slug
f	:	local friction coefficient
$\bar{f}$	:	mean value of friction coefficient, defined as $\frac{1}{L}\int_o^L f\,dx$
G	:	mass velocity, w/A, slug/sec sq. ft.
H	:	coefficient of convective heat transfer, Btu/sec sq. ft. °R
J	:	mechanical equivalent of heat (778 ft-lb per Btu)
k	:	ratio of specific heats, c_p/c_v
L	:	total length of flow passage, ft.
M	:	Mach Number, $V/\sqrt{kRT}$
M_c	:	Critical Mach Number
n	:	constant of pv^n = constant
p	:	static pressure, lb/sq. ft. abs.
p_o	:	stagnation pressure, lb/sq. ft. abs.
Q	:	heat flow per unit mass, Btu/slug
R	:	gas constant, ft-lb/slug °R
r	:	recovery factor, $(T_{aw}-T)/(T_o-T)$

s : entropy per unit mass, Btu/°R slug

T_{aw} : adiabatic wall temperature, °R

T_o : stagnation temperature, °R

T_w : flow passage wall temperature, °R

u : internal energy per unit mass, Btu/slug

V : axial velocity in flow passage, ft/sec

v : specific volume, cu. ft./slug

W : work per unit mass, ft-lb/slug

w : rate of mass flow, slug/sec

x : axial distance through flow passage, ft.

Greek Letters

ρ : density, slug/cu. ft.

ϕ : a parameter, e_e pdv/J dQ

Subscripts

1 : signifies properties at initial section of flow passage

2 : signifies properties at final section of flow passage

F : properties evaluated at Fanno Line

R : properties evaluated at Rayleigh Line

c : properties evaluated at the Critical Mach Number

o : signifies stagnation state

w : signifies conditions of the wall of the duct

Superscripts

$*$: signifies properties of isentropic flow at Mach Number unity

$*^t$: signifies properties of isothermal flow at Mach Number $\frac{1}{\sqrt{k}}$

FUNDAMENTAL GOVERNING EQUATIONS

The characteristics of a polytropic gas flow in a constant-area passage under the combined effects of friction and heat transfer are analyzed in the following pages. All problems investigated here are based on the following basic hypotheses:

(1) The flow is steady and one-dimensional, that is to say all properties are uniform over each cross section.

(2) Changes are continuous in stream properties.

(3) The fluid is a perfect gas with $k = 1.4$.

(4) The specific heats, ratio of specific heats and molecular weight are all assumed to be constant.

(5) The fluid flow is polytropic according to the relation pv^n = constant, where n is constant.

(6) Heat is transferred instantaneously, completely and transversely throughout the cross section.

Assume a perfect gas is flowing into a duct at the inlet section named 1 and along the duct to an arbitrary section. Then

$$\left(\frac{T}{T_1}\right) = \left(\frac{\rho}{\rho_1}\right)^{n-1} \quad ; \quad \left(\frac{T}{T_1}\right) = \left(\frac{p}{p_1}\right)^{\frac{n-1}{n}} \tag{1}$$

By logarithmic differentiation the first part of Equation (1) gives

$$\frac{dT}{T} = (n-1)\,\frac{d\rho}{\rho} \tag{2}$$

From the expression for a perfect gas, the Mach Number is

$$M^2 = V^2/KRT$$

By logarithmic differentiation,

$$\frac{dM^2}{M^2} = 2\,\frac{dV}{V} - \frac{dT}{T} \tag{3}$$

The continuity equation is

$$G = W/A = \rho V = \text{constant}$$

or

$$\frac{d\rho}{\rho} + \frac{dV}{V} = 0 \tag{4}$$

Eliminating the terms $\frac{d\rho}{\rho}$ and $\frac{dV}{V}$ from Equations (2), (3), and (4) yields

$$\frac{dT}{T} = -\,\frac{n-1}{n+1}\,\frac{dM^2}{M^2} \tag{5}$$

The stagnation temperature is

$$T_o = T\left(1 + \frac{k-1}{2}\,M^2\right)$$

or in differential form,

$$\frac{dT_o}{T_o} = \frac{dT}{T} + \frac{\frac{k-1}{2}M^2}{1 + \frac{k-1}{2}M^2}\,\frac{dM^2}{M^2} \tag{6}$$

Substituting Equation (5) into (6) for $\frac{dT}{T}$,

$$\frac{dT_o}{T_o} = \frac{(k-1)M^2 - (n-1)}{(1+n)(1 + \frac{k-1}{2}M^2)} \frac{dM^2}{M^2} \qquad (7)$$

From Equations (2) and (5),

$$\frac{d\rho}{\rho} = \frac{-1}{n+1} \frac{dM^2}{M^2} \qquad (8)$$

Eliminating $\frac{d\rho}{\rho}$ from equations (4) and (8) gives

$$\frac{dV}{V} = \frac{1}{n+1} \frac{dM^2}{M^2} \qquad (9)$$

By logarithmic differentiation of the second part of Equation (1), and using the relation of Equation (5), one obtains

$$\frac{dp}{p} = - \frac{n}{n+1} \frac{dM^2}{M^2} \qquad (10)$$

From the definition of the stagnation pressure,

$$p_o = p(1 + \frac{k-1}{2}M^2)^{\frac{k}{k-1}}$$

After differentiation,

$$\frac{dp_o}{p_o} = \frac{dp}{p} + \frac{\frac{kM^2}{2}}{(1 + \frac{k-1}{2}M^2)} \frac{dM^2}{M^2} \qquad (11)$$

Combining Equations (10) and (11) gives

$$\frac{dp_o}{p_o} = \frac{(n+k)\frac{M^2}{2} - n}{(n+1)(1 + \frac{k-1}{2}M^2)} \frac{dM^2}{M^2} \qquad (12)$$

The definition of the impulse function is

$$F = pA + \rho AV^2 = pA(1 + kM^2)$$

In differential form

$$\frac{dF}{F} = \frac{dp}{p} + \frac{kM^2}{1 + kM^2} \frac{dM^2}{M^2}$$

From this expression, with the help of Equation (10) to eliminate the term $\frac{dp}{p}$, there results

$$\frac{dF}{F} = \frac{kM^2 - n}{(n+1)(1+kM^2)} \frac{dM^2}{M^2} \tag{13}$$

The change of entropy in differential form is

$$ds = c_p \frac{dT}{T} - \frac{R}{J} \frac{dp}{p} \tag{14}$$

Also

$$c_p = \frac{k}{k-1} \frac{R}{J} \tag{15}$$

Introducing Equations (5), (10), and (15) into (14) gives

$$ds = \frac{R}{J} \frac{(k-n)}{(n+1)(k-1)} \frac{dM^2}{M^2} \tag{16}$$

From Equation 8.40, reference 4, page 230, the change of Mach Number under the influence of both friction and heat transfer, for the conditions of this analysis, is expressed as

$$\frac{dM^2}{M^2} = \frac{(1+kM^2)(1 + \frac{k-1}{2}M^2)}{(1-M^2)} \frac{dT_o}{T_o} + \frac{kM^2(1 + \frac{k-1}{2}M^2)}{(1-M^2)} 4f \frac{dx}{D} \qquad (17)$$

Substituting Equation (6) into (17) and simplifying gives

$$\frac{dM^2}{M^2} = \frac{(n+1)\ kM^2\ (1 + \frac{k-1}{2}\ M^2)}{2n - M^2[(n+2k-kn) + k(k-1)M^2]} 4f \frac{dx}{D} \qquad (18)$$

QUALITATIVE EFFECTS OF FRICTION

Since the differential term $4f\frac{dx}{D}$ is always positive, it is selected here as an independent variable in order to investigate the characteristics of fluid flow in the duct. The remaining variables, such as dT/T, dT_o/T_o etc., may consequently be found in terms of $4f\frac{dx}{D}$ with the aid of Equations (5), (7), (8), (9), (10), (12), (13), (16), and (18). The final results are:

$$\frac{dT}{T} = \frac{k(1-n)M^2(1 + \frac{k-1}{2}M^2)}{2n - M^2[(n+2k-kn) + k(k-1)M^2]} 4f \frac{dx}{D} \tag{19}$$

$$\frac{dT_o}{T_o} = \frac{kM^2[(k-1)M^2 + (1-n)]}{2n - M^2[(n+2k-kn) + k(k-1)M^2]} 4f \frac{dx}{D} \tag{20}$$

$$\frac{d\rho}{\rho} = \frac{-kM^2(1 + \frac{k-1}{2}M^2)}{2n - M^2[(n+2k-kn) + k(k-1)M^2]} 4f \frac{dx}{D} \tag{21}$$

$$\frac{dV}{V} = \frac{kM^2(1 + \frac{k-1}{2}M^2)}{2n - M^2[(n+2k-kn) + k(k-1)M^2]} 4f \frac{dx}{D} \tag{22}$$

$$\frac{dp}{p} = \frac{-nkM^2(1 + \frac{k-1}{2}M^2)}{2n - M^2[(n+2k-kn) + k(k-1)M^2]} 4f \frac{dx}{D} \tag{23}$$

$$\frac{dp_o}{p_o} = \frac{kM^2[(n+k)M^2 - 2n]/2}{2n - M^2[(n+2k-kn) + k(k-1)M^2]} 4f \frac{dx}{D} \tag{24}$$

$$\frac{dF}{F} = \frac{kM^2(kM^2-n)(1+\frac{k-1}{2}M^2)}{2n - M^2[(n+2k-kn) + k(k-1)M^2]} \, 4f\frac{dx}{D} \tag{25}$$

$$ds = \frac{R}{(k-1)J} \, \frac{k(k-n)M^2(1+\frac{k-1}{2}M^2)}{\{2n - M^2[(n+2k-kn) + k(k-1)M^2]\}} \, 4f\frac{dx}{D} \tag{26}$$

From these equations it is seen that the direction of change depends not on whether the flow is subsonic or supersonic, but whether the denominator is greater or less than zero. Noting that $4f\frac{dx}{D}$ is always positive, the equations from (19) to (26) become infinite if the denominator is zero. This means that when the Mach Number is of such value as to cause the denominator to be zero then this Mach Number is a limiting one for the gas flow. This point is called the Critical Mach Number and is represented by M_c.

Let

$$2n - M_c^2[(n+2k-kn) + k(k-1)M_c^2] = 0$$

or

$$k(k-1)M_c^4 + (n+2k-kn)M_c^2 - 2n = 0$$

Solving this equation for M_c^2 gives

$$M_c^2 = \frac{-(n+2k-kn) + \sqrt{(n+2k-kn)^2 + 8kn(k-1)}}{2k(k-1)}$$

or

$$M_c = \sqrt{\frac{-(n+2k-kn) + \sqrt{(n+2k-kn)^2 + 8kn(k-1)}}{2k(k-1)}} \tag{27}$$

The positive signs before the square roots are chosen, since M_c and M_c^2 are always positive. From Equation (27) it is clearly shown that no real Critical Mach Number exists in a polytropic gas flow for any negative value of n.

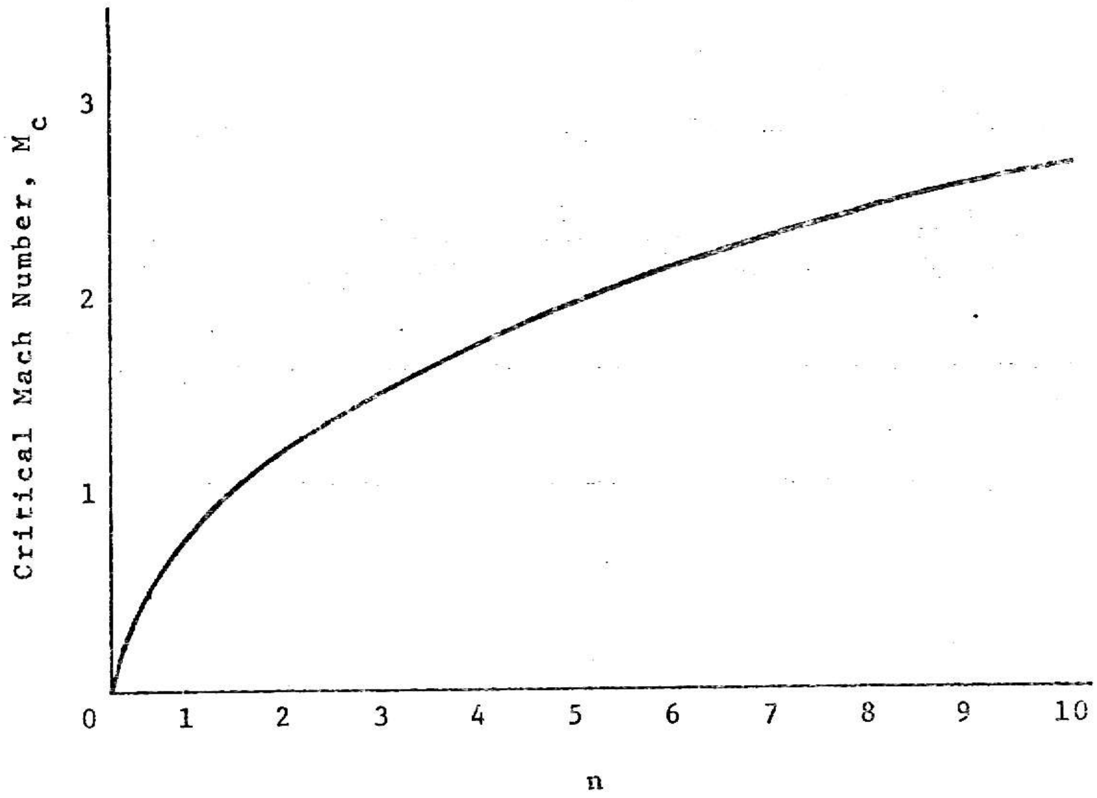

Figure 1. Critical Mach Number of a Perfect Gas as a Function of n, k = 1.4.

For polytropic flow of a perfect gas in a duct, the Critical Mach Number for twenty-two different values of positive n are calculated by assuming k = 1.4 and are shown on Table I. For any additional values of positive n, the Critical Mach Number of flow can be obtained from Figure 1.

Table 1. Critical Mach Number of a Perfect Gas, k = 1.4

n	M_c	n	M_c
0.0	0.000	5.0	1.890
0.5	0.598	5.5	1.982
1.0	0.845	6.0	2.070
1.4	1.000	6.5	2.155
1.5	1.035	7.0	2.236
2.0	1.195	7.5	2.315
2.5	1.336	8.0	2.391
3.0	1.464	8.5	2.464
3.5	1.581	9.0	2.536
4.0	1.690	9.5	2.605
4.5	1.793	10.0	2.673

According to Equations (19) through (26), the directions of changes in the stream properties can be easily examined. As an example, Table II summarizes the case of a perfect gas in which n = 1.2 and k = 1.4. Here it is seen that the Mach Number always tends toward the critical Mach Number 0.924. Continuous transitions either from M less than M_c to M greater than M_c, or from M greater than M_c to M less than M_c are consequently impossible.

For any gas flow with other values of n and k, the directions of changes of stream properties can also be found through examining the differential equations from (19) to (26).

Table II. Variations of Stream Properties of a Perfect Gas with $n = 1.2$ and $k = 1.4$; $M_c = 0.924$

Properties	$M<M_c$ (Subsonic)	$M>M_c$ (Subsonic or Supersonic)
Mach Number	increases	decreases
Temperature	decreases	increases
Stagnation Temperature	increases for $M>0.707$ decreases for $M<0.707$	decreases
Density	decreases	increases
Velocity	increases	decreases
Pressure	decreases	increases
Stagnation Pressure	decreases	increases for $M<0.961$ decreases for $M>0.961$
Impulse Function	decreases	increases for $M<0.936$ decreases for $M>0.936$
Entropy	increases	decreases

WORKING FORMULAS

In order to obtain formulas suitable for practical computation, the integration of the previously given differential equations is needed. The Mach Number will be selected as the independent variable for this purpose. Equation (18) is rearranged to read

$$\int_0^L 4f\,\frac{dx}{D} = \int_{M^2}^{M_c^2} \frac{2n - M^2[(n+2k-kn) + k(k-1)M^2]}{(n+1)kM^2(1 + \frac{k-1}{2}M^2)} \cdot \frac{dM^2}{M^2}$$

where the limits of integration are taken as (I) the section where the Mach Number is M, and where x is arbitrarily set equal to zero, and (II) the section where the Mach Number is M_c, and x is the length L of duct measured at $M = M_c$.

The integration of the above-mentioned equation is carried out in Appendix B. The final result is

$$4\bar{f}\,\frac{L}{D} = \frac{2}{n+1}\,\ell n\left(\frac{M}{M_c}\right)^2 + \frac{2n}{k(n+1)}\left(\frac{1}{M^2} - \frac{1}{M_c^2}\right) \tag{28}$$

where $\bar{f}$ is the mean friction coefficient with respect to length, defined by

$$\bar{f} = \frac{1}{L}\int_0^L f\,dx$$

By similar methods of integration other stream properties are found in terms of the local Mach Number. These integrated relations are:

$$\frac{T}{T_c} = \left(\frac{p}{p_c}\right)^{\frac{n-1}{n}} = \left(\frac{\rho}{\rho_c}\right)^{n-1} = \left(\frac{V_c}{V}\right)^{n-1} = \left(\frac{M_c}{M}\right)^{\frac{2(n-1)}{(n+1)}} \quad (29)$$

$$\frac{T_o}{T_{oc}} = \left(\frac{M_c}{M}\right)^{\frac{2(n-1)}{(n+1)}} \left(\frac{1 + \frac{k-1}{2}M^2}{1 + \frac{k-1}{2}M_c^2}\right) \quad (30)$$

$$\frac{p_o}{p_{oc}} = \left(\frac{M_c}{M}\right)^{\frac{2n}{n+1}} \left(\frac{1 + \frac{k-1}{2}M^2}{1 + \frac{k-1}{2}M_c^2}\right)^{\frac{k}{k-1}} \quad (31)$$

$$\frac{F}{F_c} = \left(\frac{M_c}{M}\right)^{\frac{2n}{n+1}} \left(\frac{1 + kM^2}{1 + kM_c^2}\right) \quad (32)$$

$$s - s_c = \frac{R}{J} \cdot \frac{(n-k)}{(n+1)(k-1)} \ln\left(\frac{M_c^2}{M^2}\right) \quad (33)$$

The quantities marked with a subscript c in these expressions, such as T_c, P_c, etc., represent the values of the stream properties at the section in the duct where $M = M_c$.

The next step is to find out the wall temperature of the constant-area passage for polytropic gas flow. Consider an infinitesimal length of duct dx. The rate of heat transferred from the wall to the fluid is equal to the rate absorbed by the fluid. From reference 4, pp. 213 and 243, this can be written as,

$$\omega dQ = \rho AV\, c_p\, d\, T_o = H\,(T_w - T_{aw})\, dA_w \quad (34)$$

The recovery factor is taken to be unity, i.e., $T_{aw} = T_o$. Thus, Equation (34) becomes

$$\frac{dT_o}{T_w - T_o} = \frac{H}{\rho V c_p} \cdot \frac{4}{D} dx \tag{35}$$

Furthermore, Reynolds Analogy, which relates friction factor and coefficient of heat transfer, is assumed to be valid. This is, from reference 4, pp 243,

$$\frac{H}{\rho V c_p} = \frac{f}{2} \tag{36}$$

Introducing Equation (36) into (35) gives

$$\frac{dT_o}{T_w - T_o} = \frac{2f}{D} dx$$

After rearrangement it becomes

$$2 \frac{dT_o}{T_o} = \left(\frac{T_w}{T_o} - 1\right) 4f \frac{dx}{D} \tag{37}$$

Equation (37) shows the relation between the change in stagnation temperature and the length of flow passage. Substituting Equations (7) and (18) for $\frac{dT_o}{T_o}$ and $4f\frac{dx}{D}$, into Equation (37), gives

$$\frac{2kM^2[(1-n) + (k-1)M^2]}{2n - M^2[(n+2k-kn) + k(k-1)M^2]} = \frac{T_w}{T_o} - 1$$

Simplifying this expression for T_w/T_o yields

$$\frac{T_w}{T_o} = \frac{2n - M^2[n(1+k) + k(1-k)M^2]}{2n - M^2[(n+2k-kn) + k(k-1)M^2]} \qquad (38)$$

This formula represents a ratio of the local wall temperature to the local stagnation temperature of the fluid flow. Since the absolute temperature of T_w or T_o is always greater than zero, the signs of the numerator and denominator of the right-hand side of Equation (38) must agree with each other. This characteristic defines a working range of Mach Number for any kind of polytropic gas flow in a constant-area duct under the influences of friction and heat transfer.

SPECIAL CASES

Based on the notes of Professor Wilson Tripp, Department of Mechanical Engineering, Kansas State University, on reference 4, page 178 and reference 10, page 23, three special cases of polytropic gas flow in a duct are to be used here as a verification of those formulas derived and investigated in this report.

Case 1 Isentropic Flow

When n = k, the polytropic gas flow is isentropic and the governing physical differential equations (5), (7), (8), (9), (10), (12), (13), and (16) become

$$\frac{dT}{T} = (k-1)\frac{d\rho}{\rho} = -(k-1)\frac{dV}{V} = \frac{(k-1)}{k}\frac{dp}{p} = -\frac{(k-1)}{(k+1)}\frac{dM^2}{M^2} \quad (39)$$

$$\frac{dT_o}{T_o} = \frac{(k-1)(M^2-1)}{(k+1)(1+\frac{k-1}{2}M^2)}\frac{dM^2}{M^2} \quad (40)$$

$$\frac{dp_o}{p_o} = \frac{k(M^2-1)}{(k+1)(1+\frac{k-1}{2}M^2)}\frac{dM^2}{M^2} \quad (41)$$

$$\frac{dF}{F} = \frac{k(M^2-1)}{(k+1)(1+kM^2)}\frac{dM^2}{M^2} \quad (42)$$

$$ds = \frac{R}{J}\left[-\frac{k}{k+1} + \frac{k}{k+1}\right]\frac{dM^2}{M^2} = 0$$

Through steps of simplifying Equation (18) for the case $n = k$, the frictional term of isentropic flow is

$$4f\,\frac{dx}{D} = \frac{2k - M^2[k(3-k) + k(k-1)\,M^2]}{k(k+1)M^2(1 + \frac{k-1}{2}\,M^2)}\,\frac{dM^2}{M^2}$$

$$= \frac{1 - M^2[(\frac{3-k}{2}) + (\frac{k-1}{2})\,M^2]}{M^2(1 + \frac{k-1}{2}\,M^2)}\,(\frac{2}{k+1})\,\frac{dM^2}{M^2}$$

$$= (\frac{2}{k+1})\,\frac{(1-M^2)(1 + \frac{k-1}{2}\,M^2)}{M^2(1 + \frac{k-1}{2}\,M^2)}\,\frac{dM^2}{M^2}$$

$$= \frac{2}{(k+1)}\,\frac{(1-M^2)}{M^2}\,\frac{dM^2}{M^2} \tag{43}$$

or, in an altered form,

$$\frac{dM^2}{M^2} = \frac{(k+1)}{2}\,\frac{M^2}{(1-M^2)}\,4f\,\frac{dx}{D} \tag{44}$$

Substituting Equation (44) into (40) yields

$$\frac{dT_o}{T_o} = -\,\frac{(\frac{k-1}{2})\,M^2}{(1 + \frac{k-1}{2}\,M^2)}\,4f\,\frac{dx}{D} \tag{44A}$$

From this equation it is clear that $\frac{dT_o}{T_o}$ is negative, since $4f\frac{dx}{D}$ is always greater than zero. This says that only a cooling process can occur for isentropic flow in ducts with friction and heat transfer.

From Equation (38), the wall temperature ratio for the case $n = k$ is simplified as

$$\frac{T_w}{T_o} = \frac{2k - M^2[k(k+1) + k(1-k)M^2]}{2k - M^2[k(3-k) + k(k-1)M^2]}$$

$$= \frac{1 - M^2[\frac{k+1}{2} - \frac{k-1}{2} M^2]}{1 - M^2[1 - \frac{k-1}{2} + \frac{k-1}{2} M^2]}$$

$$= \frac{(1 - \frac{k-1}{2} M^2)(1-M^2)}{(1 + \frac{k-1}{2} M^2)(1-M^2)}$$

$$= \frac{(1 - \frac{k-1}{2} M^2)}{(1 + \frac{k-1}{2} M^2)} \qquad (45)$$

Noting that the absolute stagnation temperature is always greater than zero, the wall temperature ratio of the present case is never negative. That is to say

$$(1 - \frac{k-1}{2} M^2) \geq 0$$

or

$$M \leq \sqrt{\frac{2}{k-1}} \qquad (45A)$$

Equations (44A) and (45A) show that isentropic gas flow in a constant-area duct is possible only for the cooling process and only when the Mach Number is less than or equal to $\sqrt{\frac{2}{k-1}}$.
For air, $M \leq \sqrt{\frac{2}{k-1}} = 2.237$.

The working formulas for isentropic flow can be obtained either by integrating the governing physical differential equations from (39) to (43), or by substituting k for n in the integrated working formulas of polytropic flow, on page . The results are

$$\frac{T}{T^*} = \left(\frac{p}{p^*}\right)^{\frac{k-1}{k}} = \left(\frac{\rho}{\rho^*}\right)^{k-1} = \left(\frac{V^*}{V}\right)^{k-1} = \left(\frac{1}{M^2}\right)^{\frac{k-1}{k+1}} \tag{46}$$

$$\frac{T_o}{T_o^*} = \left(\frac{1}{M^2}\right)^{\frac{k-1}{k+1}} \left(\frac{2}{k+1}\right)\left(1 + \frac{k-1}{2} M^2\right) \tag{47}$$

$$\frac{p_o}{p_o^*} = \left(\frac{1}{M^2}\right)^{\frac{k}{k+1}} \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}} \left(1 + \frac{k-1}{2} M^2\right)^{\frac{k}{k-1}} \tag{48}$$

$$4f \frac{L_{max}}{D} = \frac{2}{k+1} \left(\frac{1}{M^2} - 1 + \ell_n M^2\right) \tag{49}$$

The quantities marked with an asterisk in these expressions, such as p*, V*, etc., represent the values of the stream properties at the section in the duct where the Mach Number is unity. The Critical Mach Number for isentropic flow is equal to unity.

The formulas and conclusions presented here for isentropic flow agree with those given in reference (9).

Case II Isothermal Flow

When n is unity the fluid flow is isothermal and the governing differential equations become

$$\frac{dp}{p} = \frac{d\rho}{\rho} = -\frac{dV}{V} = -\frac{1}{2}\frac{dM^2}{M^2} \tag{50}$$

$$\frac{dT_o}{T_o} = \frac{\frac{k-1}{2}M^2}{1+\frac{k-1}{2}M^2} \cdot \frac{dM^2}{M^2} \tag{51}$$

$$\frac{dp_o}{p_o} = \frac{(1+k)M^2 - 2}{4(1+\frac{k-1}{2}M^2)} \cdot \frac{dM^2}{M^2} \tag{52}$$

$$\frac{dF}{F} = \frac{kM^2 - 1}{2(1+kM^2)} \frac{dM^2}{M^2} \tag{53}$$

$$ds = \frac{1}{2}\frac{R}{J}\frac{dM^2}{M^2} \tag{54}$$

From Equation (18), the friction term for the case n = 1 is

$$4f\frac{dx}{D} = \frac{2 - M^2[(1+k) + k(k-1)M^2]}{kM^2[1+\frac{k-1}{2}M^2]} \frac{1}{2} \frac{dM^2}{M^2}$$

or, rearranging,

$$\frac{dM^2}{2M^2} = \frac{kM^2(1+\frac{k-1}{2}M^2)}{2 - M^2[(1+k) + k(k-1)M^2]} 4f\frac{dx}{D}$$

$$= \frac{(kM^2)(1+\frac{k-1}{2}M^2)}{2(1-kM^2)(1+\frac{k-1}{2}M^2)} 4f\frac{dx}{D}$$

$$= \frac{kM^2}{2(1-kM^2)} 4f\frac{dx}{D} \tag{55}$$

Substituting this relation into Equations (51) and (52) gives

$$\frac{dT_o}{T_o} = \frac{k(k-1)M^4}{2(1-kM^2)(1 + \frac{k-1}{2} M^2)} 4f \frac{dx}{D} \tag{56}$$

$$\frac{dp_o}{p_o} = \frac{kM^2(1 - \frac{k+1}{2} M^2)}{2(kM^2-1)(1 + \frac{k-1}{2} M^2)} 4f \frac{dx}{D} \tag{57}$$

Equations (50), (51), (55), (56) and (57) are the same as those presented on page 179, reference 4. To obtain the working formulas for isothermal flow, the usual integration methods are used. For example, Equation (55) may be rearranged to give

$$\int_0^{L_{max}} 4f\frac{dx}{D} = \int_{M^2}^{\frac{1}{k}} \frac{1-kM^2}{kM^4} \, dM^2$$

Carrying out the integration yields

$$4\bar{f} \frac{L_{max}}{D} = \frac{1-kM^2}{kM^2} + \ell_n kM^2 \tag{58}$$

Using similar methods the formulas which follow are obtained

$$\frac{p}{p^{*t}} = \frac{\rho}{\rho^{*t}} = \frac{V^{*t}}{V} = \frac{1}{\sqrt{k}M} \tag{59}$$

$$\frac{p_o}{p_o^{*t}} = \frac{1}{\sqrt{k}M} \left(\frac{2k}{3k-1}\right)^{\frac{k}{k-1}} \left(1 + \frac{k-1}{2} M^2\right)^{\frac{k}{k-1}} \tag{60}$$

$$\frac{T_o}{T_o^{*t}} = \frac{2k}{(3k-1)} \left(1 + \frac{k-1}{2} M^2\right) \tag{61}$$

$$\frac{F}{F^{*t}} = \frac{1+kM^2}{2\sqrt{k}M} \tag{62}$$

$$s - s^{*t} = \frac{R}{J} \ell_n M \tag{63}$$

Such symbols as p^{*t}, V^{*t}, etc., in the relations from Equations (59) to (63), denote the stream properties at a Mach Number equal to the Critical Mach Number (i.e. $\frac{1}{\sqrt{k}}$). The wall temperature ratio for the present case is

$$\frac{T_w}{T_o} = \frac{2 - M^2[(1+k) + k(1-k) M^2]}{2 - M^2[(1+k) + k(k-1) M^2]}$$

$$= \frac{1 - M^2[(1+k)/2 + k(1-k)M^2/2]}{(1-kM^2)(1 + \frac{k-1}{2} M^2)} \tag{64}$$

An examination of Equation (56) shows that heat is added to the stream when M is less than $\frac{1}{\sqrt{k}}$, and heat is rejected from the stream when M exceeds $\frac{1}{\sqrt{k}}$. But from Equation (64), investigations give the results that the heating process is from Mach Number equal to zero to the value of $\frac{1}{\sqrt{k}}$, and the cooling process is confined in a working range of Mach Number, decreasing from 1.774 to 1.063. This behavior of working range and the term T_w/T_o was not presented in reference 4.

These working ranges of Mach Number for isothermal flow in a duct are:

$$M < \frac{1}{\sqrt{k}} \quad ; \quad \text{heating process}$$

$$M_c = \frac{1}{\sqrt{k}} \quad ; \quad \frac{T_w}{T_o} = \infty$$

$$1.063 \leq M \leq 1.774 \quad ; \quad \text{cooling process.}$$

Case III Constant-Mach Number Flow

When $n = -1$, constant-Mach Number flow exists in the constant-area passage, as shown in Appendix C. The governing physical differential equations are obtained by eliminating $\frac{dM^2}{M^2}$ from those simultaneous equations (5), (7), (8), (9), (10), (12), (13) and (16). For example, the term $\frac{dM^2}{M^2}$ may be cancelled when Equations (5) and (13) are divided one into the other, yielding

$$\frac{dT}{T} = \frac{(1-n)(1 + kM^2)}{(kM^2 - n)} \frac{dF}{F}$$

When $n = -1$,

$$\frac{dT}{T} = 2 \frac{dF}{F}$$

Integrating this relation from state 1 to any arbitrary state gives

$$\frac{T}{T_1} = (\frac{F}{F_1})^2$$

Using similar methods the remaining stream properties are

$$\frac{T}{T_1} = \frac{T_o}{T_{01}} = \left(\frac{V}{V_1}\right)^2 = \left(\frac{\rho_1}{\rho}\right)^2 = \left(\frac{p}{p_1}\right)^2 = \left(\frac{p_o}{p_{01}}\right)^2 = \left(\frac{F}{F_1}\right)^2 \qquad (65)$$

From Equations (5) and (16), the entropy change is

$$ds = \frac{R}{J}\,\frac{(k+1)}{2(k-1)}\,\frac{dT}{T}$$

Integrating

$$s - s_1 = \frac{R}{J}\,\frac{(k+1)}{2(k-1)}\,\ln\frac{T}{T_1} \qquad (66)$$

Equation (17) can be simplified by setting $dM = 0$ (for constant Mach Number flow) giving

$$\frac{dT_o}{T_o} = -\frac{kM^2}{(1+kM^2)}\,4f\,\frac{dx}{D} \qquad (67)$$

The wall temperature ratio is

$$\frac{T_w}{T_o} = \frac{-2 + M^2[(1+k) + k(k-1)M^2]}{-2 - M^2[(3k-1) + k(k-1)M^2]}$$

$$= \frac{1 - M^2[(k+1)/2 + k(k-1)M^2/2]}{1 + M^2[(3k-1)/2 + k(k-1)M^2/2]}$$

$$= \frac{(1-kM^2)(1 + (k-1)M^2/2)}{(1+kM^2)(1 + (k-1)M^2/2)}$$

$$= \frac{(1-kM^2)}{(1+kM^2)} \qquad (68)$$

Equation (67) shows that the stagnation temperature of constant-Mach Number flow always decreases. Thus only a cooling process can occur. Equation (68) shows that this cooling process must be in the range that the Mach Number is equal to or less than $\frac{1}{\sqrt{k}}$. That is $M \leq \frac{1}{\sqrt{k}}$.

WORKING RANGES OF MACH NUMBER

From the last three particular cases it is shown that there always exist some restrictions either for the heating or for the cooling process for polytropic gas flow in a duct. These restrictions will be further investigated here in order to obtain a general expression for any value of n.

The working range of Mach Number is defined by:

(1) The differential governing equation of stagnation temperature, i.e., Equation (20).

(2) The friction term, $4f\frac{dx}{D}$.

(3) The wall temperature ratio (T_w/T_o), which is greater than zero, i.e., $0 \leq T_w/T_o < \infty$

(4) The Critical Mach Number

(5) The impulse function.

Through a study of these factors, the working range of Mach Number will be obtained. For isentropic gas flow (i.e., $n = k$), the working range of Mach Number is determined as follows. (1) From Equations (44A) and (45A) it is seen that only cooling processes can occur and the Mach Number must be less than or equal to $\sqrt{\frac{2}{k-1}}$. Beyond this value no isentropic gas flow can exist in a duct. (2) The directions of flow can be determined by the differential form of the impulse function and the value of the Critical Mach Number.

Substituting Equation (44) into (42) gives

$$\frac{dF}{F} = - \frac{kM^2}{1 + kM^2} 4f\frac{dx}{D}$$

This shows that the impulse function always decreases because of the friction. From Equation (42), the relation of impulse function to Mach Number is

$$\frac{dF}{F} = \frac{k(M^2-1)}{(k+1)(1+kM^2)} \frac{dM^2}{M^2} \tag{42}$$

According to this expression, when the isentropic gas flow is subsonic (M<1), the Mach Number increases to unity ($dM^2/M^2>0$) because the change of impulse function is negative, and when the flow is supersonic the Mach Number decreases to the sonic velocity.

Figure (2) shows the working ranges of Mach Number for isentropic gas flow in a constant-area duct.

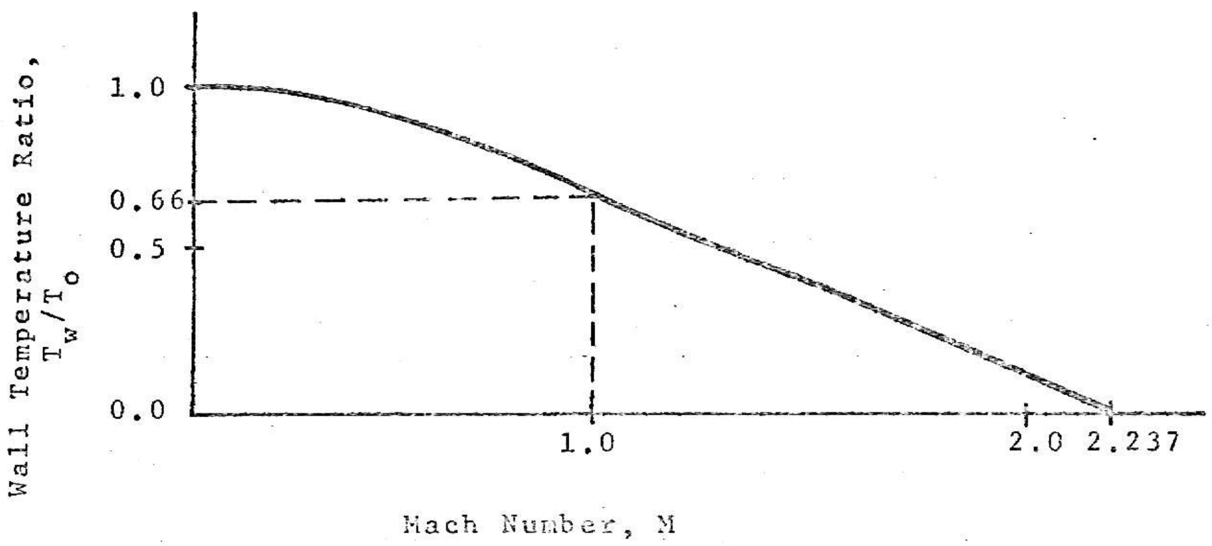

Fig. 3 Working Range of Mach Number for Isentropic Gas Flow in a Constant-Area Duct; k = 1.4

Using similar procedures, the working ranges of Mach Number can be obtained for other cases of n, such as 2, 1, 0.5, -0.5, -1 and -2. These special cases are shown on Figures 7 to 9 in Appendix D.

Figure 3 shows the combined working ranges of Mach Number for polytropic gas flow in constant-area ducts.

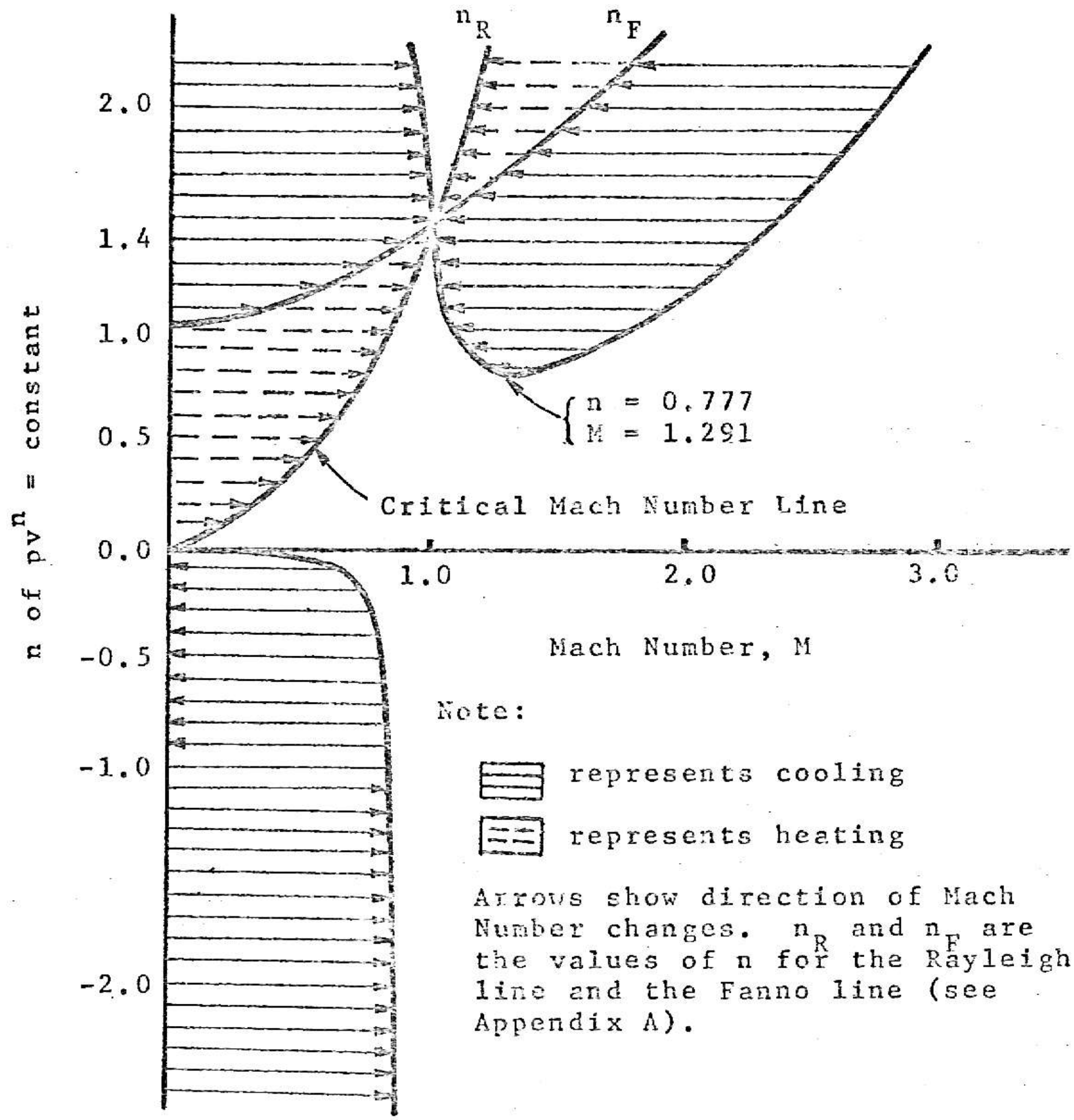

Fig. 3 Working Ranges of Mach Number for Polytropic Gas Flow in Constant-Area Ducts, k = 1.4.

From Figure (3) it is seen that: (1) a gap exists between the heating and cooling processes as n becomes greater than 0.777, (2) only cooling processes can occur when n is less than zero, and (3) when the gas flow in duct is isopiestic (i.e., n = 0) neither heating nor cooling can occur in a constant area passage under the combined effects of friction and heat transfer.

A proof that the isopiestic flow is impossible in a duct is demonstrated in the following paragraphs. The impulse function is

$$F = pA + \rho AV^2 = pA(1+kM^2)$$

After differentiating, and noting that both the pressure and area of impulse function are constant, there results

$$\frac{dF}{F} = \frac{k}{(1+kM^2)} dM^2$$

Dividing both sides by dx,

$$\frac{1}{F}\frac{dF}{dx} = \frac{k}{(1+kM^2)}\frac{dM^2}{dx} \tag{A-1}$$

where $\frac{dM^2}{dx}$ must be negative since the impulse function decreases in the direction of flow.

From Equation (7), the stagnation temperature for n = 0, is

$$\frac{dT_o}{T_o} = \frac{1 + (k-1)M^2}{1 + \frac{(k-1)}{2}M^2}\frac{dM^2}{M^2} \tag{A-2}$$

Substituting Equation (A-2) into (17), and solving for the Mach Number, yields

$$\frac{dM^2}{M^2} = - \frac{1 + \frac{k-1}{2} M^2}{2 + (k-1) M^2} 4f\frac{dx}{D} = -2f\frac{dx}{D}$$

Dividing both sides by dx,

$$\frac{dM^2}{dx} = -2f\frac{M^2}{D} \tag{A-3}$$

Replacing $\frac{dT_o}{T_o}$ of Equation (37) by (A-2), and solving for $\frac{T_w}{T_o}$, gives

$$\frac{T_w}{T_o} = 1 + \frac{D}{2f} \frac{1 + (k-1) M^2}{M^2(1 + \frac{k-1}{2} M^2)} \frac{dM^2}{dx} \tag{A-4}$$

Substituting Equation (A-3) into (A-4),

$$\frac{T_w}{T_o} = 1 + \frac{D}{2f} \frac{1 + (k-1)M^2}{M^2(1 + \frac{k-1}{2} M^2)} (-2f \frac{M^2}{D})$$

$$= - \frac{(k-1) M^2}{2 + (k-1) M^2} \tag{A-5}$$

From thermodynamics the value of k is known always to be greater than unity. Therefore the ratio $\frac{T_w}{T_o}$ is negative. This is impossible because the absolute temperature of T_w and T_o are always greater than zero.

NUMERICAL EXAMPLE

Problem: One pound of air flows steadily in a constant-area duct with friction and heat transfer, according to the relation pv^n = constant, where n = 1.2. At one section in the duct, air is at M = 1, p = 100 psia and T = 1000°R. It is desired to determine all the physical properties of the air flow from p = 50 psia to p = 150 psia, and show the final results on graphs. (Assume k = 1.4, R = 53.3 ft-lb/lb_m °R, c_p = 0.24 Btu/lb_m °R, c_v = 0.17 Btu/lb_m °R)

Solution: For polytropic air flow with n = 1.2, the Critical Mach Number is equal to 0.924 (see Fig. 1). The mass flow per unit area can be obtained as follows,

$$V = MC = 1\ (49.02\sqrt{1000} = 1550 \text{ ft/sec}$$

$$\rho = P/RT = 100 \times 144/53.3 \times 1000 = 0.270 \text{ lb}_m/\text{ft}^3$$

$$w/A = \rho V = 0.270 \times 1550 = 418.2 \text{ lb}_m/\text{ft}^2\text{sec}$$

The physical properties of air at the critical Mach Number may be obtained by using these working formulas for n = 1.2.

$$\frac{T}{T_c} = \left(\frac{M_c}{M}\right)^{2\left(\frac{n-1}{n+1}\right)} = (0.924)^{0.182} = 0.985$$

$$T_c = \frac{T}{0.985} = \frac{1000}{0.985} = 1014.45\ °R$$

$$\frac{p}{p_c} = \left(\frac{M_c}{M}\right)^{\frac{2n}{n+1}} = (0.924)^{1.091} = 0.917$$

$$p_c = \frac{p}{0.917} = \frac{100}{0.917} = 108.98 \text{ psia}$$

$$\frac{\rho}{\rho_c} = \left(\frac{M_c}{M}\right)^{\frac{2}{n+1}} = (0.924)^{0.909} = 0.931$$

$$\rho_c = \frac{\rho}{0.931} = \frac{0.2701}{0.9310} = 0.290 \ lb_m/ft^3$$

$$\frac{V}{V_c} = \left(\frac{M}{M_c}\right)^{\frac{2}{n+1}} = \left(\frac{1}{0.924}\right)^{0.909} = 1.0742$$

$$V_c = \frac{V}{1.0742} = \frac{1550}{1.0742} = 1443 \text{ ft/sec}$$

$$\frac{T_o}{T_{oc}} = \left(\frac{M_c}{M}\right)^{2\left(\frac{n-1}{n+1}\right)} \left(\frac{1 + \frac{k-1}{2}M^2}{1 + \frac{k-1}{2}M^2}\right) = 0.985 \left(\frac{1.2}{1.1706}\right) = 1.010$$

$$T_{oc} = \frac{T_o}{1.010} = T\left(1 + \frac{k-1}{2} M^2\right)/1.010 = 1200/1.010 = 1188 \ °R$$

$$\frac{p_o}{p_{oc}} = \left(\frac{M_c}{M}\right)^{\frac{2n}{n+1}} \left(\frac{1 + \frac{k-1}{2} M^2}{1 + \frac{k-1}{2} M^2}\right)^{\frac{k}{k-1}} = 0.917 \left(\frac{1.738}{1.892}\right) = 1.001$$

$$p_{oc} = \frac{p_o}{1.001} = p\left(1 + \frac{k-1}{2} M^2\right)^{\frac{k}{k-1}}/1.001 = 100(1.892)/1.001 =$$

$$183.8 \text{ psia}$$

$$\frac{F}{F_c} = \left(\frac{M_c}{M}\right)^{\frac{2n}{n+1}} \left(\frac{1 + kM^2}{1 + kM_c^2}\right) = 1.002$$

$$F_c/A = \frac{F/A}{1.002} = p(1+kM^2)/1.002 = 100(2.1942)/1.002 =$$

$$219\ lb_f/ft^2$$

$$\frac{(s-s_c)}{R/J} = \frac{1}{2.2}\, \ell_n \left(\frac{M}{M_c}\right) = 0.0358$$

$$\frac{T_w}{T_o} = \frac{2.4 - M^2[2.88 - 0.56M^2]}{2.4 - M^2[2.32 + 0.56M^2]} = -0.1666$$

$$4f\,\frac{x}{D} = 1.818\ \ell_n\left(\frac{M}{M_c}\right) + 0.78\ \left(\frac{1}{M^2} - \frac{1}{M_c^2}\right) = 0.0173$$

Using similar methods, the stream properties at other pressure could be obtained if the Mach Number of fluid flow is known. For example, suppose that the pressure of air is 150 psia, then

$$\frac{T}{T_c} = \left(\frac{p}{p_c}\right)^{\frac{n-1}{n}} = \left(\frac{150}{109}\right)^{0.1665} = 1.0546$$

$$T = T_c(1.0546) = (1014.45)(1.0546) = 1070\ {}^\circ R$$

$$C = 49.02\sqrt{T} = 49.02\sqrt{1070} = 1602\ ft/sec$$

Since the flow is steady,

$$\frac{w}{A} = \rho V = \frac{p}{RT}\quad V = \left(\frac{w}{A}\right)_{M=1}$$

and

$$V = \frac{(w/A)_{M=1}}{(p/RT)} = \frac{418.2}{150 \times 144/53.3 \times 1070} = 1103 \text{ ft/sec}$$

$$M = V/c = 1103/1602 = 0.688$$

The corresponding stream properties are as follows,

$$\frac{T}{T_c} = \left(\frac{M_c}{M}\right)^{2\left(\frac{n-1}{n+1}\right)} = \left(\frac{0.924}{0.688}\right)^{0.182} = 1.0546$$

$$\frac{p}{p_c} = \left(\frac{M_c}{M}\right)^{\frac{2n}{n+1}} = \left(\frac{0.924}{0.688}\right)^{1.091} = 1.3783$$

$$\frac{\rho}{\rho_c} = \left(\frac{M_c}{M}\right)^{\frac{2}{n+1}} = \left(\frac{0.924}{0.688}\right)^{0.908} = 1.306$$

$$\frac{V}{V_c} = \left(\frac{M}{M_c}\right)^{\frac{2}{n+1}} = \left(\frac{0.688}{0.924}\right)^{0.908} = 0.766$$

$$\frac{T_o}{T_{oc}} = \left(\frac{M_c}{M}\right)^{\frac{2(n-1)}{(n+1)}} \left(\frac{1 + \frac{k-1}{2}M^2}{1 + \frac{k-1}{2}M^2}\right) = 0.986$$

$$\frac{p_o}{p_{oc}} = \left(\frac{M_c}{M}\right)^{\frac{2n}{n+1}} \left(\frac{1 + \frac{k-1}{2}M^2}{1 + \frac{k-1}{2}M^2}\right)^{\frac{k}{k-1}} = 1.088$$

$$\frac{F}{F_c} = \left(\frac{M_c}{M}\right)^{\frac{2n}{n+1}} \left(\frac{1 + kM^2}{1 + kM_c^2}\right) = 1.05$$

$$4f\frac{x}{D} = 1.818\ \ell_n \frac{M}{M_c} + 0.78\left(\frac{1}{M^2} - \frac{1}{M_c^2}\right) = 0.1978$$

$$\frac{T_w}{T_o} = \frac{2.4 - M^2[2.88 - 0.56M^2]}{2.4 - M^2[2.32 + 0.56M^2]} = 0.992$$

$$\frac{s-s_c}{R/J} = \frac{1}{2.2}\ \ell_n \frac{M}{M_c} = -0.1338$$

For other values of pressure, the physical properties are tabulated in Table III for air flowing in the constant-area duct.

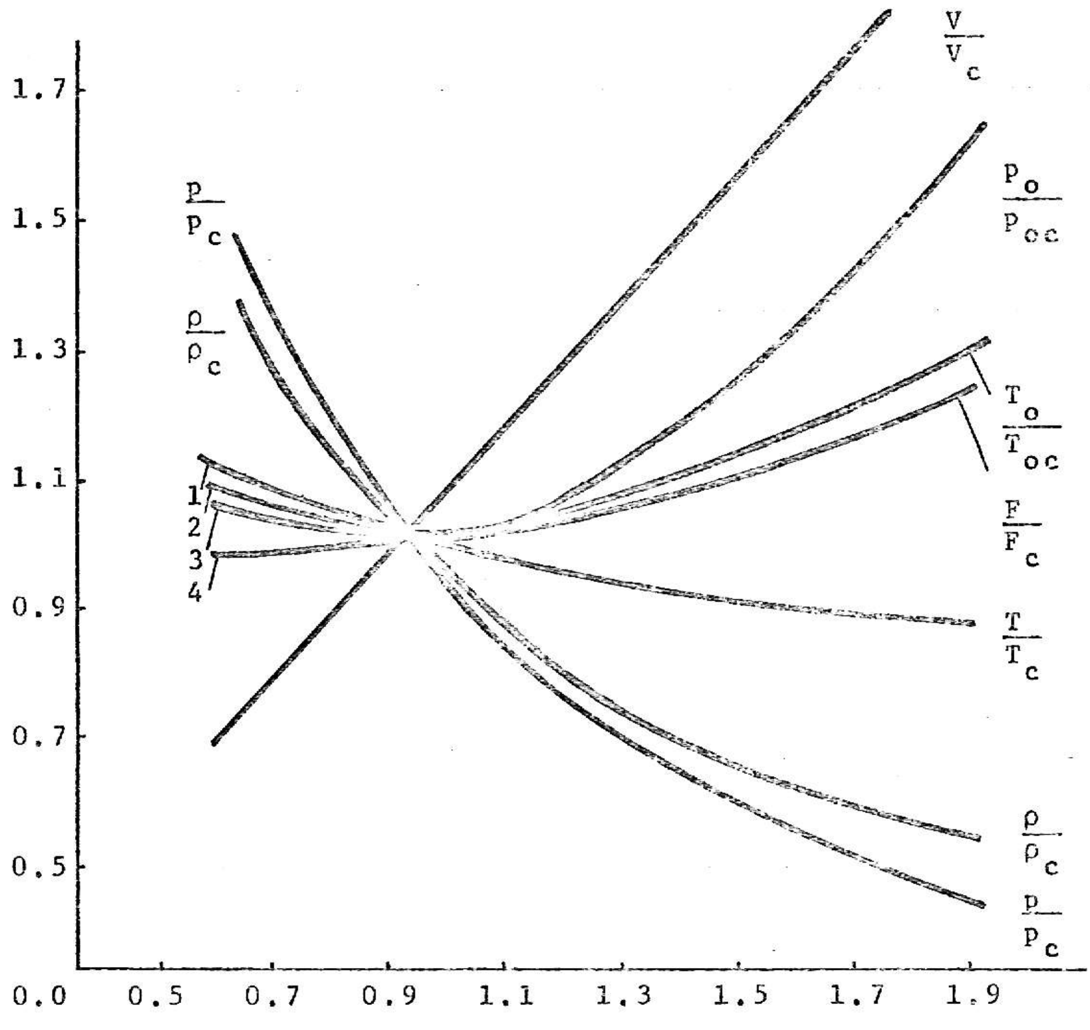

Fig. 4 Stream Properties of Polytropic Air Flow, $n = 1.2$; $k = 1.4$. Note that the Symbols: $1 = p_o/p_{oc}$, $2 = T/T_c$, $3 = F/F_c$, and $4 = T_o/T_{oc}$.

Table III Stream Properties of Polytropic Air Flow, n = 1.2; k = 1.4

p psia	150	140	130	120	110	109	100	90	80	70	60	50
M	0.688	0.734	0.784	0.845	0.917	0.924	1	1.098	1.225	1.386	1.598	1.885
T/T_c	1.0546	1.0423	1.0296	1.016	1.0015	1	0.985	0.9687	0.9498	0.9288	0.9055	0.878
p/p_c	1.3783	1.2843	1.1958	1.118	1.0076	1	0.917	0.8288	0.7343	0.6424	0.5498	0.459
ρ/ρ_c	1.306	1.2318	1.1603	1.084	1.0064	1	0.931	0.8555	0.774	0.6922	0.608	0.5235
v/v_c	0.766	0.812	0.862	0.922	0.994	1	1.0742	1.169	1.292	1.446	1.644	1.91
T_o/T_{oc}	0.986	0.986	0.989	0.992	0.999	1	1.010	1.051	1.055	1.099	1.169	1.283
p_o/p_{oc}	1.088	1.058	1.032	1.028	1.001	1	1.001	1.016	1.06	1.155	1.340	1.73
F/F_c	1.05	1.028	1.014	1.0018	0.998	1	1.002	1.015	1.038	1.078	1.146	1.25
$4fx/D$	0.1978	0.1178	0.0548	0.0173	0.0022	0	0.0108	0.0478	0.1195	0.2298	0.3887	0.6026
$\frac{s-s_c}{R/J}$	-0.1338	-0.1042	-0.0745	-0.040	-0.0032	0	0.0358	0.0783	0.1282	0.1842	0.249	0.324
T_w/T_o	0.992	1.026	1.105	1.363	2.78	27.02	-0.166	0.2106	0.282	0.2575	0.182	0.0597

Figures (4) and (5) represent the continuous variations of stream properties of polytropic air flow in a duct. Figure (6) shows the working ranges of Mach Number for heating and cooling of the present problem. This is to say that the range of Mach Number from 0.0 to 0.707 is for cooling, from 0.707 to 0.924 is for heating, and from 1.02 to 2.022 is cooling again.

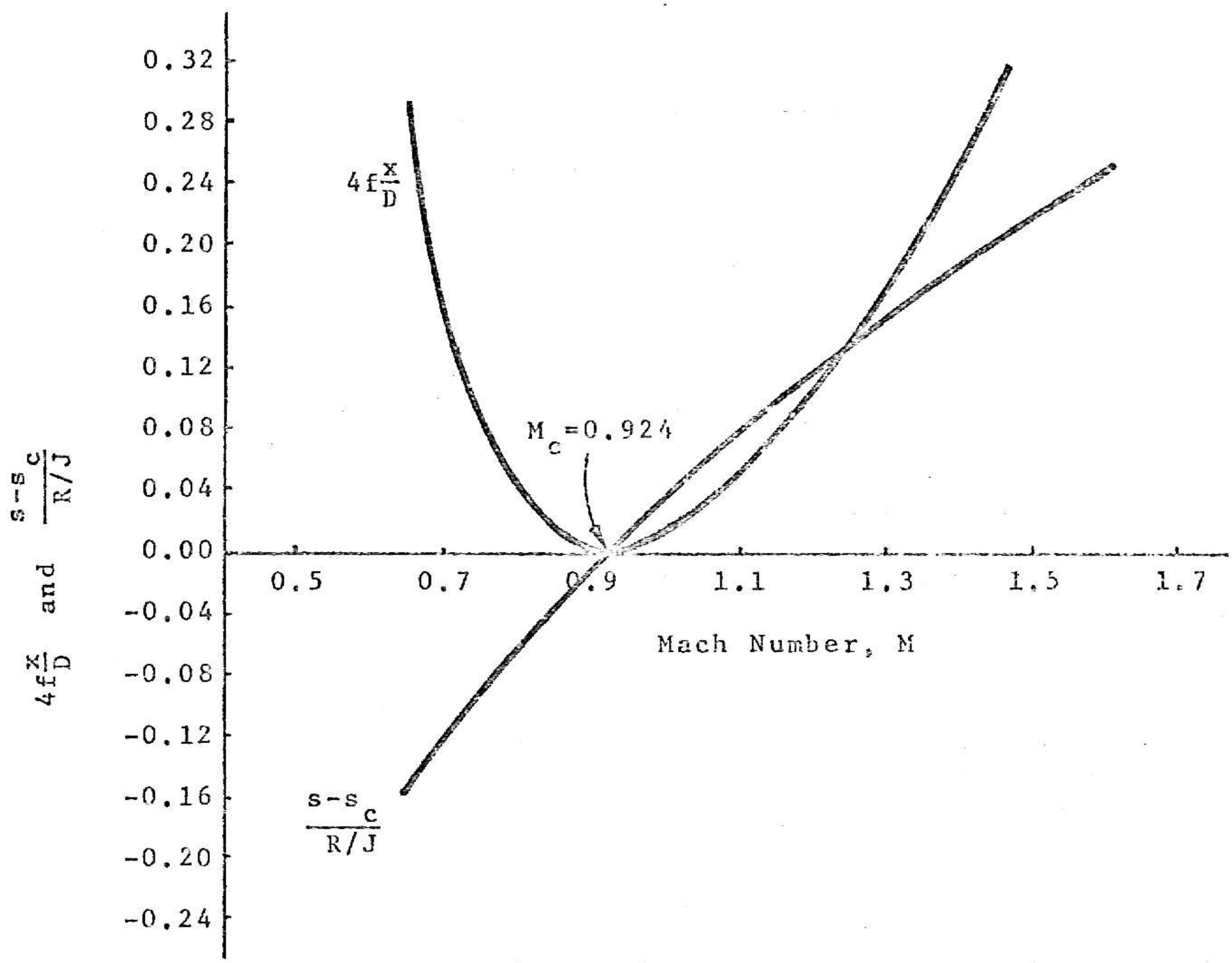

Fig. 5 $4f\frac{x}{D}$ and $\frac{(s-s_c)}{R/J}$ for Polytropic Air Flow, n = 1.2; k = 1.4

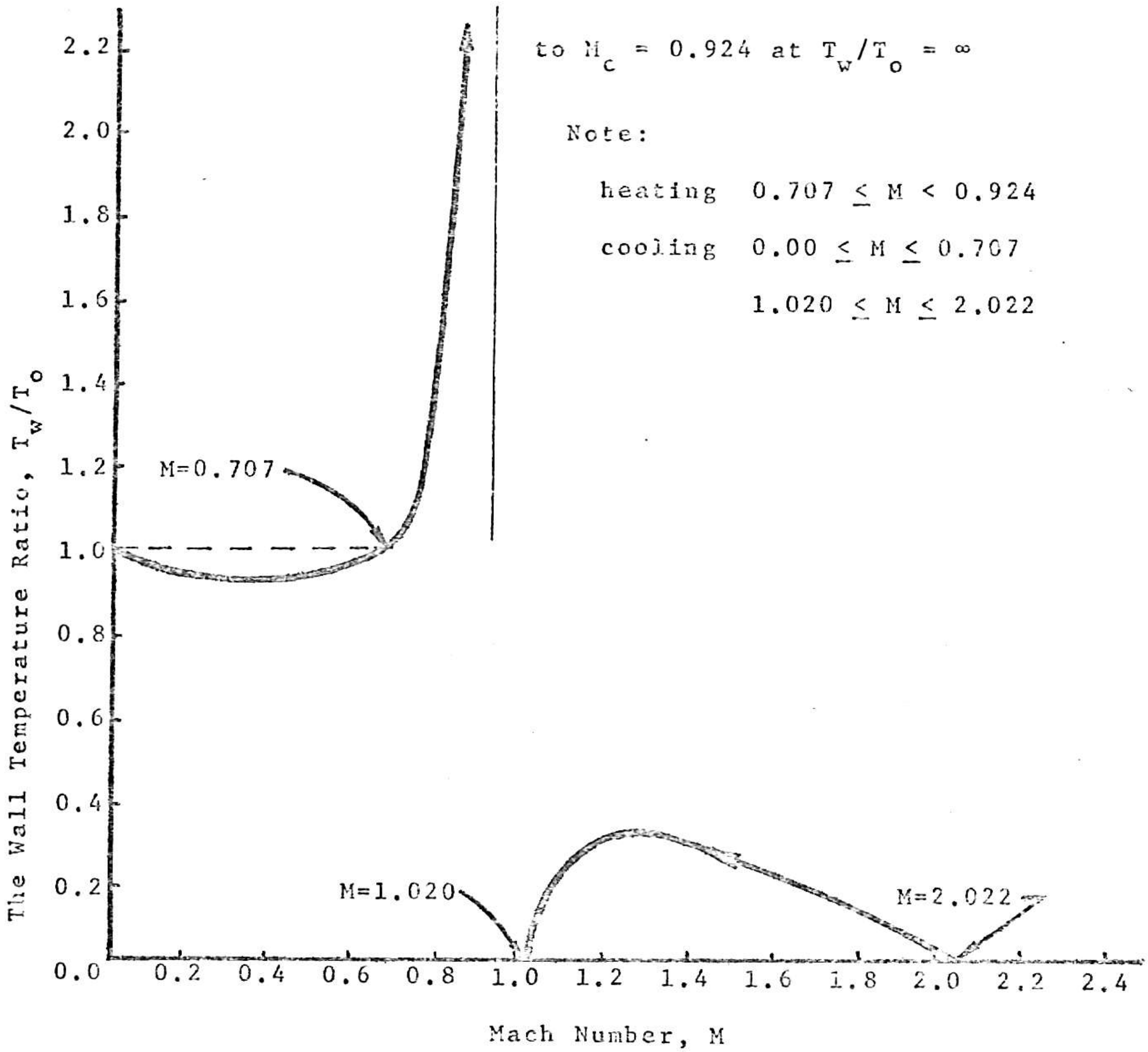

Fig. 6 Working Ranges of Mach Number for Heating and Cooling of Polytropic Air Flow with n = 1.2; k = 1.4

CONCLUSIONS

The steady, one-dimensional polytropic gas flow in constant-area ducts under the simultaneous effects of friction and heat transfer was investigated. It is assumed in this report that the area of duct, specific heats, molecular weight, and the value of n of the pressure-volume relation of a perfect gas, are all constant. In addition the Reynolds Analogy is assumed and the recovery factor is taken as unity.

The general working formulas of polytropic gas flow were obtained in terms of n, k and M, and an important ratio of the local wall temperature to the local stagnation temperature was also found. Three particular cases, (the isentropic flow, the isothermal flow, and the constant-Mach Number flow) were used to verify the formulas derived in this report.

A so-called "Critical Mach Number", which represents a limit for continuous polytropic gas flow in constant-area ducts, was defined and investigated. For the Critical Mach Number, there are four final conclusions: (a) for every positive value of n there is always a definite and unique value of the Critical Mach Number, (b) there is a parabolic relation between the Critical Mach Number and the positive value of n, (c) the Critical Mach Number is zero when n is zero, and (d) no Critical Mach Number exists for negative values of n.

Besides the investigation of the stream properties, the working ranges of Mach Number for polytropic gas flow were investigated. As to the characteristics of the working ranges of

Mach Number, four conclusions were made: (1) the isopiestic flow can never occur in a constant-area passage under the combined effects of friction and heat transfer, (2) for positive values of n the flow either for heating or for cooling is in the direction of approaching the Critical Mach Number, (3) when the value of n is negative the flow is always subsonic and no heating can occur for a polytropic gas flow; only the cooling process can occur and the range of Mach Number for this cooling varies from zero to a value approaching unity as n becomes more negative, (4) the direction of flow for the cooling process is to decrease the Mach Number to zero when n is less than zero but greater than -1; when n is less than the value -1, the flow is in the direction to increase the Mach Number from zero to some limiting value defined by $\frac{T_w}{T_o}$.

Only the heating process exists for the case of any positive value of n from zero to 0.777, and the Mach Number for range of heating increases as n increase. When n is greater than 0.777, but less than unity, there always exist both heating and cooling processes for gas flowing polytropically in a duct. Whenever the value of n is beyond the value of unity, there exist one Mach Number range for heating and two Mach Number ranges for cooling.

When n equals k, (isentropic flow) it is noted that only the cooling process exists, and its range of Mach Number is between zero and $\sqrt{\frac{2}{K-1}}$. When n is equal to -1 the flow of gas is the constant-Mach Number case.

For any value of n there exists either one or two definite ranges of Mach Numbers for which neither heating nor cooling can occur. This is due to the assumption of the Reynolds Analogy and to the assumption that the value of the recovery factor being equal to unity, which results in a negative value of the wall-temperature ratio, T_w/T_o, which is an impossibility.

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APPENDIX A

Apolytropic Flow

From the first law of thermodynamics,

$$dQ = du + dW \tag{B-1}$$

If an irreversible diabatic expansion process of a perfect gas is considered, Equation (B-1) becomes

$$dQ = c_v\ dT + e_e\ p\ dv/J \tag{B-2}$$

Now, let

$$\phi = \frac{e_e\ p\ dv/J}{dQ}$$

Arranging this relation for dQ, gives

$$dQ = \frac{e_e\ p\ dv/J}{\phi} \tag{B-3}$$

Substituting Equation (B-3) into (B-2), gives

$$\frac{e_e\ p\ dv/J}{\phi} = \frac{RdT}{J(k-1)} + \frac{e_e\ p\ dv}{J} \tag{B-4}$$

From the equation of state, $pv = RT$,

$$R\ dT = p\ dv + v\ dp \tag{B-5}$$

Substituting (B-5) into (B-4) gives

$$\frac{e_e\ p\ dv}{\phi} = \frac{p\ dv + v\ dp}{(k-1)} + e_e\ p\ dv \tag{B-6}$$

After combining and rearranging,

$$[1 + e_e(k-1)(1-1/\phi)]p\ dv + vdp = 0 \qquad (B-7)$$

Let

$$1 + e_e(k-1)(1-1/\phi) = n \qquad (B-8)$$

then Equation (B-7) becomes,

$$n\ p\ dv + v\ dp = 0 \qquad (B-9)$$

If n is a constant, integrating Equation (B-9) yields

$$pv^n = \text{constant}$$

This is a polytropic flow, in that n is a constant. If n is not a constant apolytropic flow will occur. The Fanno Line or the Rayleigh Line processes are examples of apolytropic flow in which n in Equation (B-9) is a function of k and M.

<u>Rayleigh Line</u>

This is a reversible diabatic process for which $e_e = 1$. Thus, Equation (B-7) reduces to

$$n_R = 1 + (k-1)(1-1/\phi) \qquad (B-10)$$

Solving Equation (B-6) for ϕ,

$$\phi = \frac{(k-1)}{vdp/pdv + k} \qquad (B-11)$$

From page 196, Equations 7.18 and 7.17, reference 4,

$$\frac{p}{p^*} = \frac{(1+k)}{(1+kM^2)} \tag{B-12}$$

$$\frac{V}{V^*} = \frac{(1+k)M^2}{(1+kM^2)} \tag{B-13}$$

or in differential form,

$$\frac{dp}{p} = \frac{-kdM^2}{(1+kM^2)} \tag{B-14}$$

$$\frac{dv}{v} = \frac{dM^2}{M^2(1+kM^2)} \tag{B-15}$$

Dividing Equation (B-14) by (B-15), yields

$$vdp/pdv = -kM^2 \tag{B-16}$$

Substituting Equation (B-16) into (B-11),

$$\phi = \frac{(k-1)}{k(1-M^2)} \tag{B-17}$$

Combining Equation (B-17) with (B-10), gives

$$n_R = kM^2 \tag{B-18}$$

<u>Fanno Line</u>

This is an irreversible adiabatic process for which $\phi = \infty$. Thus Equation (B-8) reduces to

$$n_F = 1 + e_e(k-1) \tag{B-19}$$

Using the relations $c_v = c_p - R/J$ and $c_p/c_v = k$ Equation (B-2) becomes, with $dQ = 0$,

$$\frac{k}{k-1} RdT - (1-e_e)pdv - vdp = 0 \tag{B-20}$$

Now, Bernoulli's Equation gives

$$-v\ dp = v\ dV + dF_f \tag{B-21}$$

where dF_f is defined as

$$dF_f = (1-e_e)pdv \tag{B-22}$$

Combining Equations (B-20), (B-21) and (B-22), gives

$$\frac{k}{(k-1)} RdT + VdV = 0 \tag{B-23}$$

Substituting Equation (B-5) into (B-23) for RdT,

$$VdV = -\frac{k}{(k-1)} (pdv + vdp) \tag{B-24}$$

Eliminating VdV, from Equations (B-21) and (B-24), gives

$$dF_f = \frac{k\ pdv + vdp}{(k-1)} \tag{B-25}$$

By Equation (B-22), e_e becomes

$$e_e = -\frac{1 + vdp/pdv}{(k-1)} \tag{B-26}$$

From p. 168, Equations 6.22 and 6.23, reference 4,

$$\frac{p}{p^*} = \frac{1}{M}\left[\frac{(k+1)}{2(1 + \frac{k-1}{2}M^2)}\right]^{1/2}$$

$$\frac{v}{v} = M\left[\frac{(k+1)}{2(1 + \frac{k-1}{2}M^2)}\right]^{1/2}$$

or, in differential form,

$$\frac{dp}{p} = -\frac{1 + (k-1)M^2}{2M^2(1 + \frac{k-1}{2}M^2)}\, dM^2 \qquad \text{(B-27)}$$

$$\frac{dv}{v} = \frac{1}{2M^2(1 + \frac{k-1}{2}M^2)}\, dM^2 \qquad \text{(B-28)}$$

Dividing Equation (B-27) by (B-28), yielding

$$vdp/pdv = -[1 + (k-1)M^2] \qquad \text{(B-29)}$$

Substituting Equation (B-29) into (B-26)

$$e_e = M^2$$

and, by Equation (B-19)

$$n_F = 1 + (k-1)M^2 \qquad \text{(B-30)}$$

APPENDIX B

Integration of Equation (28)

From Equation (18), $4f\frac{dx}{D}$ in terms of the Mach Number $\frac{dM^2}{M^2}$ can be represented as

$$4f\ \frac{dx}{D} = \frac{2n - M^2[(n+2k-kn) + k(k-1)M^2]}{(n+1)kM^2(1 + \frac{k-1}{2}M^2)}\ \frac{dM^2}{M^2} \qquad \text{(C-1)}$$

or, in integral form,

$$\int_0^L 4f\ \frac{dx}{D} = \frac{2n}{(n+1)k}\int_{M^2}^{M_c^2} \frac{dM^2}{(1 + \frac{k-1}{2}M^2)M^4} -$$

$$\frac{n+2k-kn}{(n+1)k}\int_{M^2}^{M_c^2} \frac{dM^2}{(1 + \frac{k-1}{2}M^2)M^2} - \frac{(k-1)}{(n+1)}\int_{M^2}^{M_c^2} \frac{dM^2}{(1 + \frac{k-1}{2}M^2)}$$

(C-2)

Since the right-hand side of Equation (C-2) is complicated, a separate mathematical operation should be used for each of the three integrals. From the first term, the integral is

$$\int_{M^2}^{M_c^2} \frac{dM^2}{(1 + \frac{k-1}{2}M^2)M^4} \qquad \text{(C-3)}$$

Let

$$Z = \frac{1}{M^2} + \frac{(k-1)}{2} \qquad \text{(C-4)}$$

then

$$M^2 = \frac{1}{Z - (k-1)/2} \quad , \quad dM^2 = \frac{-dZ}{(Z - (k-1)/2)^2} \tag{C-5}$$

and

$$(1 + \frac{k-1}{2} M^2) = \frac{Z}{Z - (k-1)/2} \tag{C-6}$$

Substituting Equations (C-4), (C-5) and (C-6) into (C-3), gives

$$\int_{M^2}^{M_c^2} \frac{dM^2}{(1 + \frac{k-1}{2} M^2) M^4} = \int_Z^{Z_c} (\frac{k-1}{2} \frac{dZ}{Z} - dZ) = (\frac{k-1}{2} \ell_n Z - Z)_Z^{Z_c}$$

$$= \frac{k-1}{2} \ell_n \frac{1/M_c^2 + (k-1)/2}{1/M^2 + (k-1)/2} + \frac{1}{M^2} - \frac{1}{M_c^2}$$

$$= \frac{k-1}{2} \ell_n \frac{(1 + \frac{k-1}{2} M_c^2)}{(1 + \frac{k-1}{2} M^2)} \frac{M^2}{M_c^2} + (\frac{1}{M^2} - \frac{1}{M_c^2}) \tag{C-7}$$

Using similar assumptions and method, the integral of the second term becomes

$$\int_{M^2}^{M_c^2} \frac{dM^2}{(1 + \frac{k-1}{2} M^2) M^2} = \int_Z^{Z_c} - \frac{dZ}{Z} = [-\ell_n Z]_Z^{Z_c} = \ell_n \frac{Z}{Z_c}$$

$$= \ell_n \frac{(1 + \frac{k-1}{2} M^2)}{(1 + \frac{k-1}{2} M^2)} \cdot \frac{M_c^2}{M^2} \tag{C-8}$$

The integration of the third term gives

$$\int_{M^2}^{M_c^2} \frac{dM^2}{(1 + \frac{k-1}{2} M^2)} = [\frac{2}{k-1} \ell_n (1 + \frac{k-1}{2} M^2)]_{M^2}^{M_c^2} =$$

$$\frac{2}{(k-1)} \ell_n \frac{(1 + \frac{k-1}{2} M_c^2)}{(1 + \frac{k-1}{2} M^2)} \tag{C-9}$$

Substituting Equations (C-7), (C-8) and (C-9) into (C-2),

$$\int_0^L 4f \frac{dx}{D} = \frac{2n}{(n+1)k} [\frac{k-1}{2} \ell_n \frac{(1 + \frac{k-1}{2} M_c^2)}{(1 + \frac{k-1}{2} M^2)} \cdot \frac{M^2}{M_c^2} + \frac{1}{M^2} - \frac{1}{M_c^2}]$$

$$- \frac{n+2k-kn}{(n+1)k} [\ell_n \frac{(1 + \frac{k-1}{2} M^2)}{(1 + \frac{k-1}{2} M_c^2)} \cdot \frac{M_c^2}{M^2}] - \frac{2}{(n+1)} \ell_n \frac{(1 + \frac{k-1}{2} M_c^2)}{(1 + \frac{k-1}{2} M^2)}$$

(C-10)

After simplifying the right-hand side, and noting that a constant, average friction coefficient $\overline{f}$ is used for the integration, Equation (C-10) becomes

$$4\overline{f} \frac{L}{D} = (\frac{2}{n+1}) \ell_n \frac{M^2}{M_c^2} + \frac{2n}{(n+1)k} (\frac{1}{M^2} - \frac{1}{M_c^2}) \tag{C-11}$$

APPENDIX C

Constant-Mach Number Flow

It is the purpose here to prove that the polytropic gas flow is constant-Mach Number flow when $n = -1$ in the relation pv^n = constant.

The relation between pressure and specific volume for a polytropic flow $n = -1$ of a perfect gas is

$$\frac{p}{v} = \frac{p_1}{v_1} \qquad \text{or,} \qquad \frac{p}{p_1} = \frac{v}{v_1} \tag{D-1}$$

For a constant-area duct the continuity equation is

$$\frac{\rho}{\rho_1} = \frac{V_1}{V} \tag{D-2}$$

Therefore, from Equations (D-1) and (D-2),

$$\frac{p}{p_1} = \frac{v}{v_1} = \frac{\rho_1}{\rho} = \frac{V}{V_1} \tag{D-3}$$

From the equation of state of a perfect gas,

$$p = \rho RT \qquad \text{and,} \qquad p_1 = \rho_1 RT_1$$

$$\frac{p}{p_1} = \frac{\rho}{\rho_1} \cdot \frac{T}{T_1} \tag{D-4}$$

or

$$\frac{T}{T_1} = \frac{p}{p_1} \cdot \frac{\rho_1}{\rho} \tag{D-5}$$

Substituting Equation (D-3) into (D-5) for $\frac{\rho_1}{\rho}$, gives

$$\frac{T}{T_1} = \left(\frac{p}{p_1}\right)^2 = \left(\frac{V}{V_1}\right)^2$$

From the definition of Mach Number,

$$V^2 = M^2 kRT \qquad \text{and,} \qquad V_1^2 = M_1^2 kRT_1 \tag{D-7}$$

$$\left(\frac{V}{V_1}\right)^2 = \left(\frac{M}{M_1}\right)^2 \left(\frac{T}{T_1}\right) \tag{D-8}$$

From Equations (D-6) and (D-8),

$$\left(\frac{M}{M_1}\right)^2 = 1$$

This shows the flow is constant-Mach Number as $n = -1$ of the polytropic gas flow.

APPENDIX D

(GRAPHS)

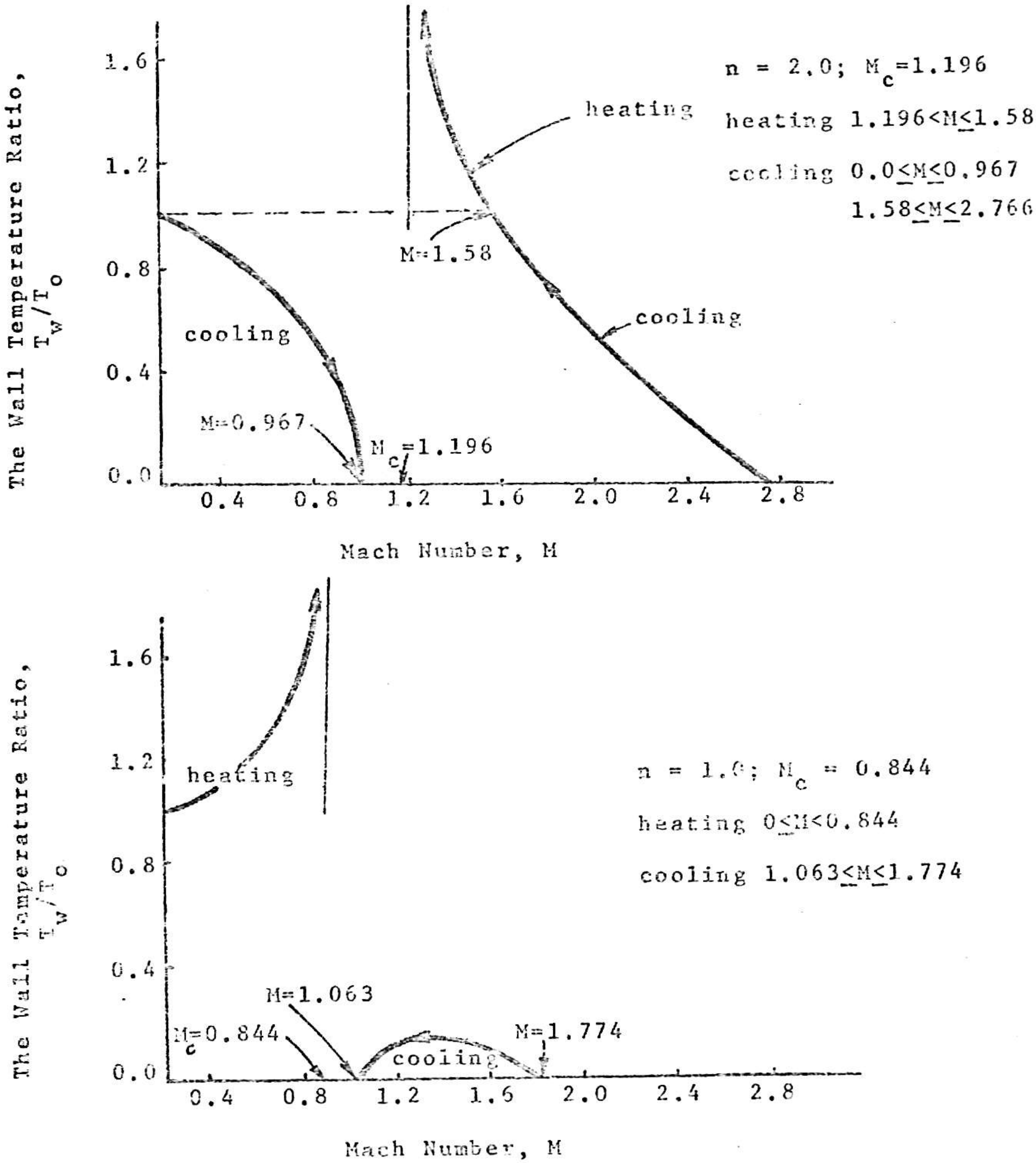

Fig. 7 Working Range of Mach Number for Heating and Cooling of Polytropic Air Flow with n = 2.0 and with n = 1.0; k = 1.4

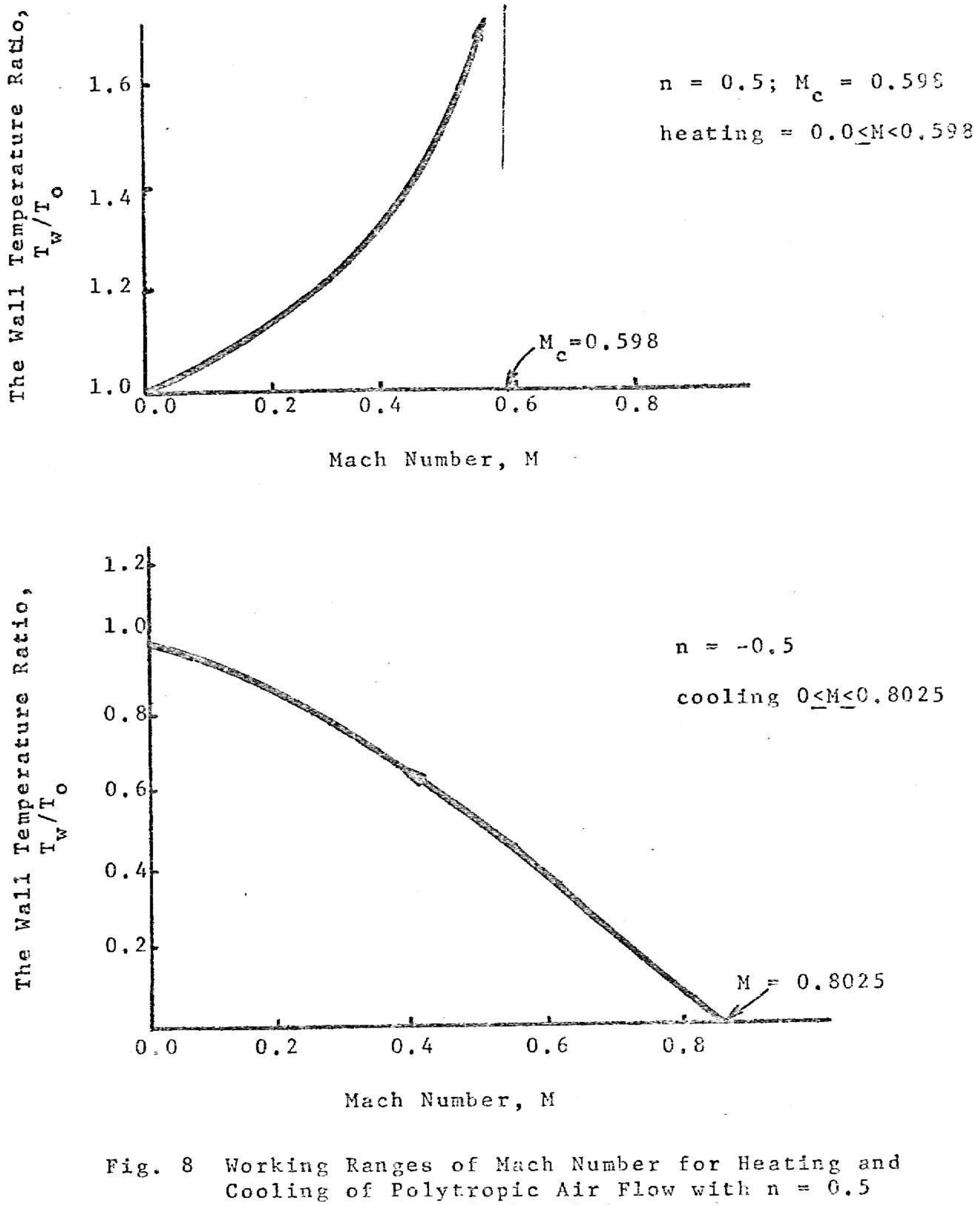

Fig. 8 Working Ranges of Mach Number for Heating and Cooling of Polytropic Air Flow with n = 0.5 and with n = -0.5; k = 1.4

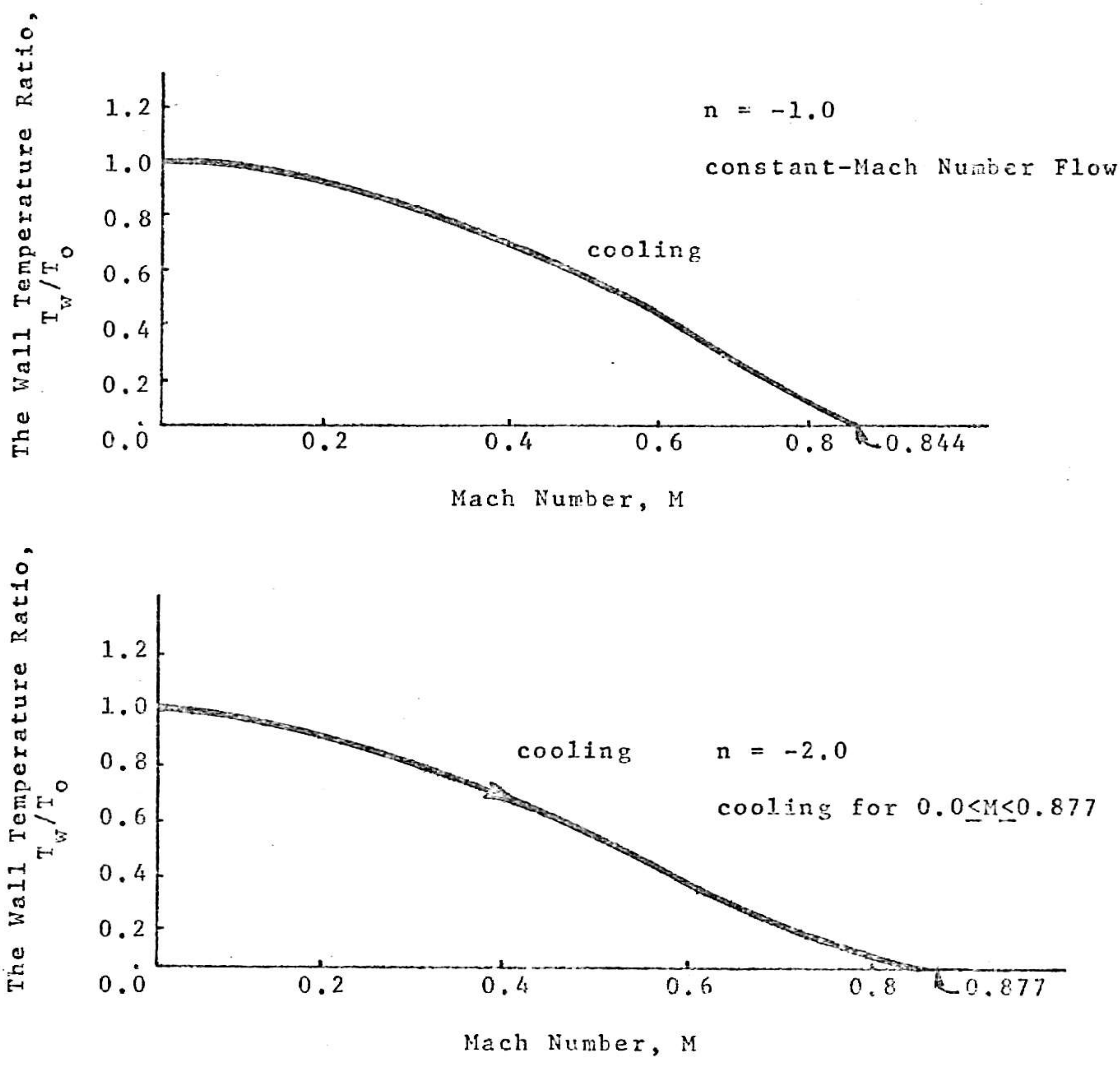

Fig. 9 Working Ranges of Mach Number for Heating and Cooling of Polytropic Air Flow with n = -1.0 and n = -2.0; k = 1.4

ACKNOWLEDGEMENT

The writer wishes to express his sincere gratitude to Professor Wilson Tripp of the Department of Mechanical Engineering, Kansas State University, for his guidance, counsel, and help in preparing this report. Thanks are given also to Professors Robert L. Gorton and Robert E. Crank of the Department of Mechanical Engineering, and Professor Thomas S. Parker of the Department of Mathematics, for their advice and encouragement.

POLYTROPIC GAS FLOW IN A CONSTANT-AREA DUCT UNDER THE SIMULTANEOUS EFFECTS OF FRICTION AND HEAT TRANSFER

by

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Diploma, Taiwan Provincial Taipei Institute of Technology, 1964

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1969

The main purpose of the work in this report is to investigate the physical properties of a steady polytropic flow of a perfect gas in a constant-area duct under the simultaneous effects of friction and heat transfer. The assumptions made in the analysis of this report are that the exponent "n" of the relation pv^n = constant, the molecular weight and the specific heats of the perfect gas are all constant, and that the Reynolds Analogy is valid, the recovery factor is unity and the ratio of specific heats is 1.4.

The governing physical equations and the working formulas for the polytropic gas flow in ducts were derived in terms of n, k and M, and the ratio of local wall temperature to local stagnation temperature was also found.

A newly named "Critical Mach Number", which represents a limit for continuous polytropic gas flow, was investigated and found that no Critical Mach Number existed for any negative value of n.

Particular treatment was given as a verification to those formulas derived in this report for the cases of the isentropic flow, the isothermal flow, and the constant-Mach Number flow. It is found that the final results of the cases are entirely the same as that shown on references (4), (9) and (10).

The working ranges of Mach Number for heating and cooling processes of a polytropic gas flow were also studied. The final important findings are that the isopiestic flow of a perfect gas is impossible in a constant-area duct with friction and heat transfer, and that if the value of n is negative only a cooling process can occur.