DIFFERENCES IN ELECTRICAL IMPEDANCE MEASUREMENTS DUE TO GUARDED/UNGUARDED ELECTRODES IN HOMOGENEOUS/INHOMOGENEOUS REGIONS

by

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I. Introduction

Electrical impedance measurement is of great interest in medical diagnoses. It is a low-cost, low-risk technique which may determine certain meaningful properties of tissues. Use of guarded electrodes was found experimentally helpful in improving the regional sensitivity of the measurements. [1-5] The reason for improved regional sensitivity is explained as guarding provides a more confined measuring volume. [6] However, the calculation of the impedance is a complex fields problem, and the published quantitative evaluations of the guarded impedance measurement are minimal.

Here, the effect of guarding is studied in a volume conductor with simple rectangular geometry. Both homogeneous and inhomogeneous cases are considered. The electrical potential is first calculated, then the electrode impedance measurements are computed. Equipotential lines and current pathways are also constructed.

II. General Problem Description

The general problem for which a solution is desired is that of calculating a guarded electrode measurement for a media with a variation in geometries and a range for the conductivity function. However, the solution of this fields problem with relatively simple geometry usually turns out to be quite complex. Here, only a simplified model is studied.

The problem is illustrated graphically in Fig. 2.1. The parallelpiped is an inhomogeneous volume conductor whose conductivity σ is a function of position. The guard electrode G and center electrode C can be moved anywhere along the X-axis, and are maintained at a constant electrical potential. The reference electrode R is grounded. The impedance measured between the center electrode and the reference electrode, and a description of the current pathways, are to be determined.

Assuming that the Z-dimension is large compared to the X and Y dimensions, the solution of the three-dimensional problem can be obtained by solving a two-dimensional problem as shown by the geometry in Fig. 2.2.

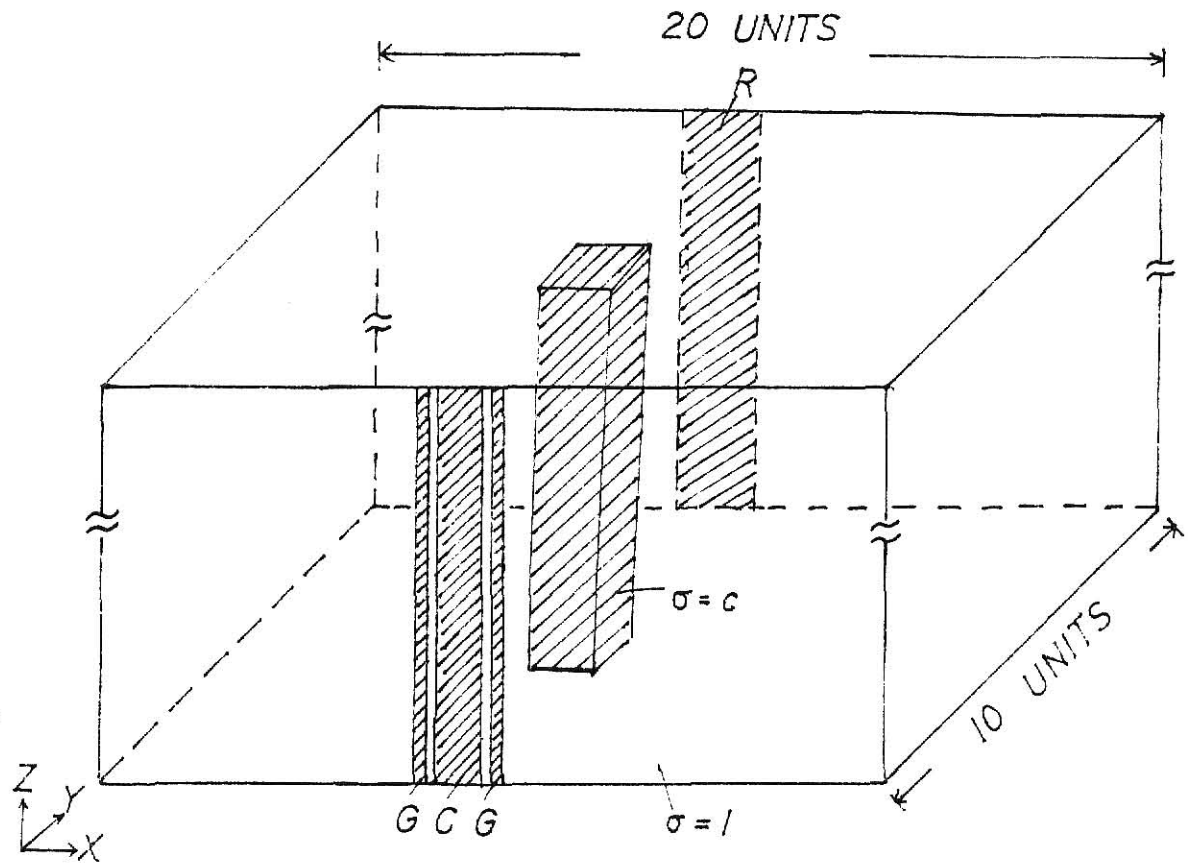
From basic electromagnetic theory the conduction current density and the electric field are related as follows:

$$\vec{J} = \sigma \vec{E} \tag{1}$$

For a time-independent problem

$$\vec{E} = - \nabla V \tag{2}$$

where V is the electric potential. From the continuity equation we



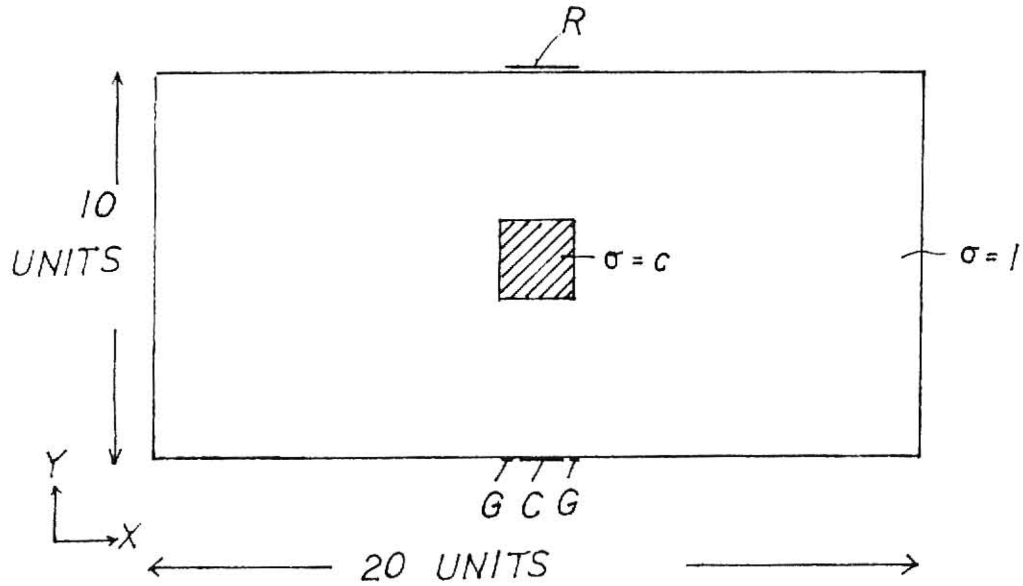
C: Center electrode (guarded electrode)

G: Guard electrode

R: Reference electrode

σ: Conductivity

Figure 2.1 Three Dimensional Model



C: Center electrode (guarded electrode)

G: Guard electrode

R: Reference electrode

σ: Conductivity

Figure 2.2 Two Dimensional Model

have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0 \qquad (3)$$

In steady state, $\frac{\partial \rho}{\partial t} = 0$; thus, Eqn. 3 becomes

$$\nabla \cdot \vec{J} = 0 \qquad (4)$$

Substituting Eqn. 1 into Eqn. 4 we have

$$\nabla \cdot (\sigma \vec{E}) = 0 \qquad (5)$$

Substituting Eqn. 2 into Eqn. 5 we have

$$\nabla \cdot (-\sigma \nabla V) = 0 \qquad (6)$$

For a two-dimensional system, Eqn. 6 can be expressed as

$$\frac{\partial}{\partial x} (\sigma \frac{\partial V}{\partial x}) + \frac{\partial}{\partial y} (\sigma \frac{\partial V}{\partial y}) = 0 \qquad (7)$$

We must determine the solution of Eqn. 7 with the following boundary conditions:

1) $V = V_o$ at points in guarded electrode G and center electrode C. In this study we use $V_o = 10V$.

2) $V = 0$ at points in the reference electrode.

3) $\sigma_+ E_{n+} = \sigma_- E_{n-}$ at boundary of discontinuity of conductivity.

The solutions are the electric potentials at each point in the domain as shown in Fig. 2.2. With the electric potentials known, one can find the equipotential lines, current pathways and the impedance.

III. Methods

3.1 Numerical Approximation of the Potential Equation

In the previous chapter, the problem was identified to be one of obtaining the solution for a two-dimensional potential equation with specified boundary conditions. To the present, only a limited number of special types of elliptic partial differential equations have been solved analytically and the usefulness of these solutions is further restricted to problems for which the boundary conditions are too difficult to satisfy. [7] In such cases, only approximated solutions can be obtained. Here, we use an iterative finite-difference method to approximate the solution of Eqn. 7.

3.1.1 Zoning

The first step of the finite difference method is that of subdividing the region into sets of smaller rectangles. The smaller the rectangles, the more accurate approximation one will have; but at the same time, more mesh points will require solving a larger system of equations. In this problem, for a better approximation of the guard electrodes, smaller zones are used for points occurring at the same column as the guard electrode. This is illustrated by Fig. 3.1.

With the zoning shown in Fig. 3.1, we define $V_{i,j}$ as the electrical potential at the i^{th} column and j^{th} row grid point. With the finite difference method, we find $V_{i,j}$ for each isolated grid point instead of developing a solution for every point in Fig. 2.2. The coordinates i,j do not have a linear relationship with x,y, the coordinates of each point since unequal zoning is used. Also notice that $V_{1,1}$ is the electrical potential at the origin (0,0).

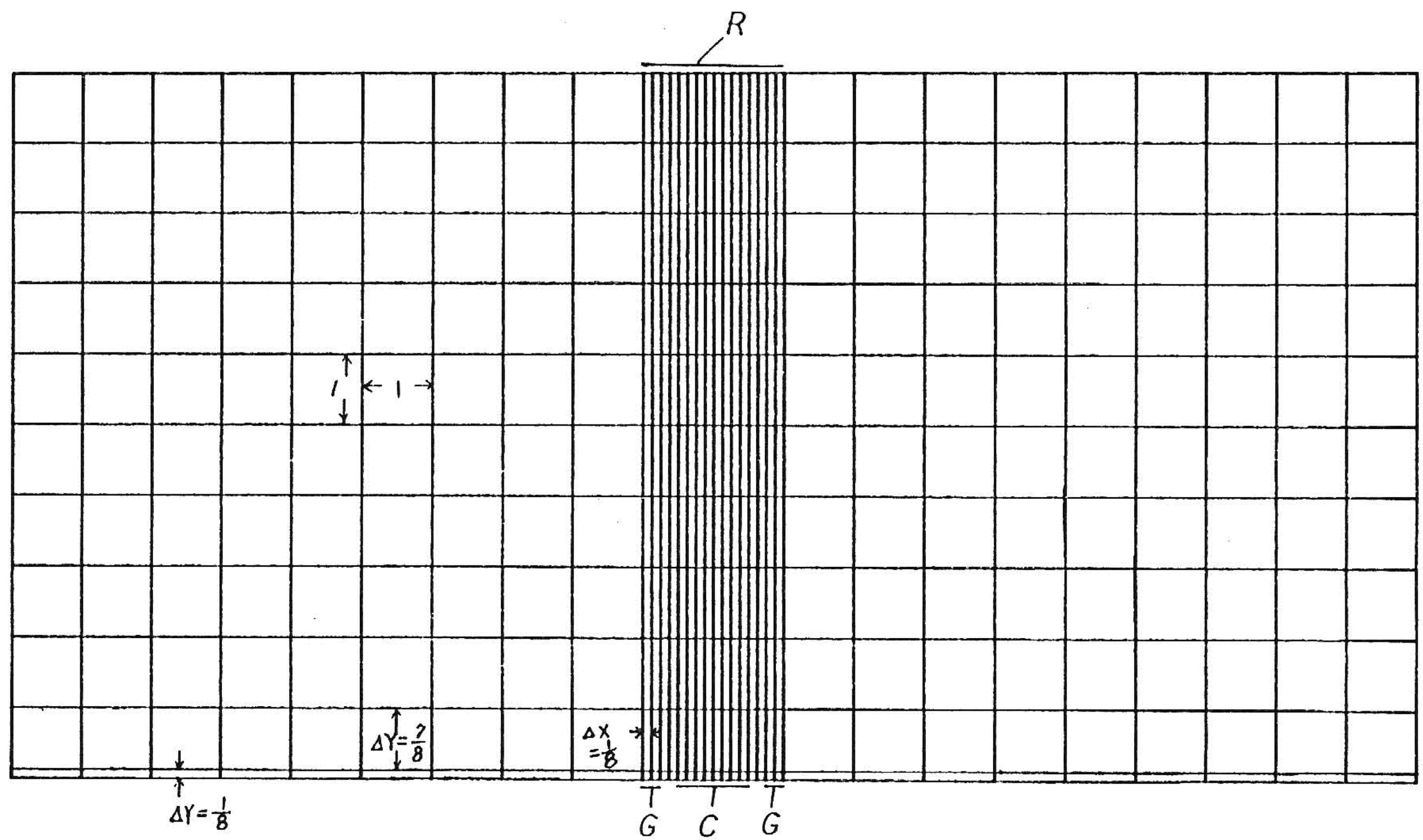


Figure 3.1 Zoning

3.1.2 Finite Difference Method

The finite difference method approximates the derivative by difference quotients over a small interval. It gives a satisfactory approximation if the difference is small enough.

Using the finite difference concept, $\frac{\partial V}{\partial x}$ is approximated as

$$\frac{\partial V(x,y)}{\partial x} = \frac{V(x+\Delta x/2,y) - V(x-\Delta x/2,y)}{\Delta x} \tag{8}$$

For an interior grid point (i,j) as shown in Fig. 3.2

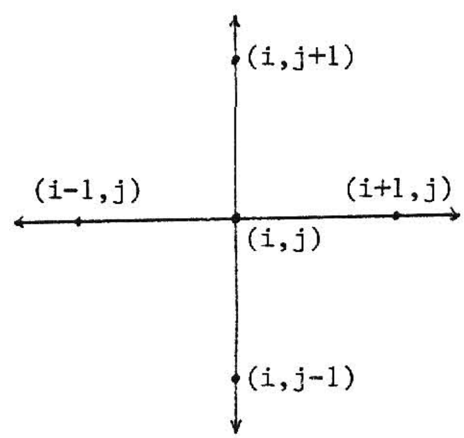


Figure 3.2 Interior Grid Point

$$\frac{\partial}{\partial x}\left(\sigma \frac{\partial V}{\partial x}\right)_{i,j} = \frac{1}{\Delta x_1}\left[\left(\sigma \frac{\partial V}{\partial x}\right)_{i+\frac{1}{2},j} - \left(\sigma \frac{\partial V}{\partial x}\right)_{i-\frac{1}{2},j}\right] \tag{9}$$

also

$$\left(\sigma \frac{\partial V}{\partial x}\right)_{i+\frac{1}{2},j} = \frac{1}{\Delta x_2}\left[\sigma_{i+\frac{1}{2},j}\left(V_{i+1,j} - V_{i,j}\right)\right] \tag{10}$$

$$\left(\sigma \frac{\partial V}{\partial x}\right)_{i-\frac{1}{2},j} = \frac{1}{\Delta x_3}\left[\sigma_{i-\frac{1}{2},j}\left(V_{i,j} - V_{i-1,j}\right)\right] \tag{11}$$

where $\Delta x_1 = \frac{1}{2}(\Delta x_2 + \Delta x_3)$

From Eqns. 9, 10 and 11 we have

$$\frac{\partial}{\partial x} (\sigma \frac{\partial V}{\partial x})_{i,j} = (\frac{1}{\Delta x_1}) [\frac{1}{\Delta x_2} \sigma_{i+\frac{1}{2},j} (V_{i+1,j} - V_{i,j}) -$$

$$\frac{1}{\Delta x_3} \sigma_{i-\frac{1}{2},j} (V_{i,j} - V_{i-1,j})] \tag{12}$$

Similarly

$$\frac{\partial}{\partial y} (\sigma \frac{\partial V}{\partial x})_{i,j} = (\frac{1}{\Delta y_1}) [\frac{1}{\Delta y_2} \sigma_{i,j+\frac{1}{2}} (V_{i,j+1} - V_{i,j}) -$$

$$\frac{1}{\Delta y_3} \sigma_{i,j-\frac{1}{2}} (V_{i,j} - V_{i,j-1})] \tag{13}$$

where $\Delta y_1 = \frac{1}{2}(\Delta y_2 + \Delta y_3)$

From Eqns. 12 and 13, Eqn. 7 is approximated by

$$(\frac{1}{\Delta x_1}) [\frac{1}{\Delta x_2} \sigma_{i+\frac{1}{2},j} (V_{i+1,j} - V_{i,j}) - \frac{1}{\Delta x_3} \sigma_{i-\frac{1}{2},j} (V_{i,j} - V_{i-1,j})] +$$

$$(\frac{1}{\Delta y_1}) [\frac{1}{\Delta y_2} \sigma_{i,j+\frac{1}{2}} (V_{i,j+1} - V_{i,j}) - \frac{1}{\Delta y_3} \sigma_{i,j-\frac{1}{2}} (V_{i,j} - V_{i,j-1})] = 0 \tag{14}$$

Eqn. 14 is used to derive the finite difference equations for grid points other than boundary points. The detailed derivations of all the finite difference equations are listed in Appendix A. As observed from Eqn. 14, grid points with different conductivity σ will have different finite difference equations.

3.1.3 Finite Difference Equation for Boundary Points

For points at the boundary of the discontinuity of conductivity, we require the continuity of the normal component of the current density.

$$J_{n+} = J_{n-} \tag{15}$$

i.e. $$\sigma_+ E_{n+} = \sigma_- E_{n-} \tag{16}$$

Since $\vec{E} = \overline{V}_{\nabla V}$ (2)

We obtain

$$E_{n+} = [(-\nabla V)_+ \cdot \hat{n}] \tag{17}$$

and $$E_{n-} = [(-\nabla V)_- \cdot \hat{n}] \tag{18}$$

For a boundary point at the discontinuty of conductivity as shown below we approximate $(\nabla V)_+$ by

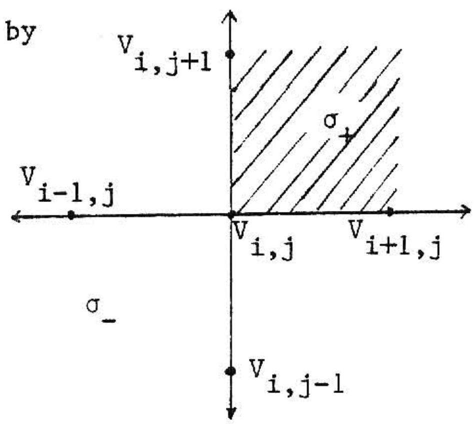


Fig. 3.3 A Boundary Point

$$(\nabla V)_+ = \frac{V_{i+1,j} - V_{i,j}}{\Delta x_1}\hat{x} + \frac{V_{i,j+1} - V_{i,j}}{\Delta y_1}\hat{y} \tag{19}$$

Approximating $(\nabla V)_-$ we have

$$(\nabla V)_- = \frac{V_{i,j} - V_{i-1,j}}{\Delta x_2}\hat{x} + \frac{V_{i,j} - V_{i,j-1}}{\Delta y_2}\hat{y} \tag{20}$$

With Eqns. 17-20, Eqn. 16 can be approximated by

$$\sigma_+ [[(\frac{V_{i+1,j} - V_{i,j}}{\Delta x_1})\hat{x} + (\frac{V_{i,j+1} - V_{i,j}}{\Delta y_1})\hat{y}] \cdot \hat{n}]$$

$$= \sigma_-[[(\frac{V_{i,j} - V_{i-1,j}}{\Delta x_2})\hat{x} + (\frac{V_{i,j} - V_{i,j-1}}{\Delta y_2})\hat{y}] \cdot \hat{n}] \tag{21}$$

Eqn. 21 is a finite difference equation for a boundary point at the discontinuity of conductivity. The boundary points may have different finite difference equations if they have different Δx, Δy, σ, or $\hat{n}$. Detailed derivation of all the finite difference equations for boundary points are also included in Appendix A.

From Eqns. 14 and 16 and

$$V = V_o \quad \text{for points at guard electrode}$$

$$V = 0 \quad \text{for points at reference electrode}$$

we get one finite difference equation for each grid point. Thus, the problem of solving a boundary value problem becomes a problem of solving a large system of linear algebraic equations. The numbering of the grid points is shown in Fig. 3.2. We use the same numbers for the system of finite difference equations.

3.1.4 Gauss-Siedel Iterative Method

We now have a set of linear algebraic equations to be solved:

$$A\ V = F \tag{22}$$

where A is the coefficient matrix depending on Δx, Δy, and the σ's. V is an unknown vector of all node potentials and F is a known column vector depending on the boundary conditions.

Using the Gauss-Siedel iteration method, we solve this set of linear algebraic equations by first assigning an initial approximation and then successively iterating the approximation according to the finite difference equations.

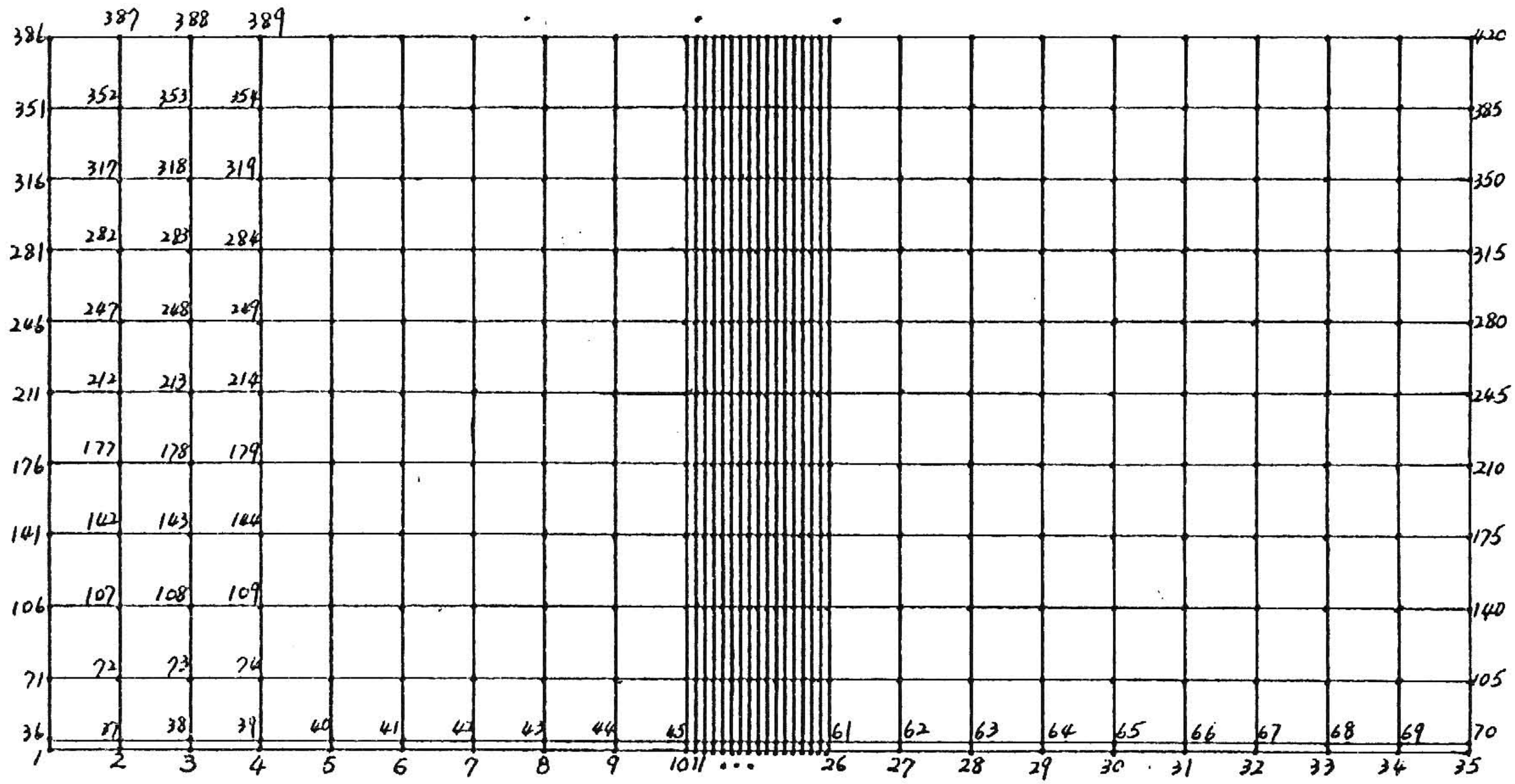


Figure 3.4 Numbering of Grid Points

The Gauss-Siedel iteration method can be expressed as

$$(D+L)V^{(k+1)} = F-UV^{(k)}$$

where k is the iteration number and L is the strictly lower triangular part of A, U is the strictly upper triangular part of A and D is the diagonal part of A. That is, for a nxn coefficient matrix A,

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & & & a_{2n} \\ \cdot & & & & & \cdot \\ & & & \cdot & & \\ \cdot & & & & & \cdot \\ & & & & \cdot & \\ \cdot & & & & & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nn} \end{bmatrix}$$

The matrices L, U and D are:

$$L = \begin{bmatrix} & & & & & \\ a_{21} & & & \bigcirc & & \\ & \cdot & & & & \\ \cdot & & \cdot & & & \\ \cdot & & & \cdot & & \\ \cdot & & & & & \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{n,n-1} & \end{bmatrix}$$

$$U = \begin{bmatrix} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ & & \cdot & & & \cdot \\ & & & \cdot & & \cdot \\ & \Large 0 & & & \cdot & \\ & & & & & a_{n-1,n} \\ & & & & & \end{bmatrix}$$

$$D = \begin{bmatrix} a_{11} & & & & \\ & a_{12} & & \Large 0 & \\ & & \cdot & & \\ & \Large 0 & & \cdot & \\ & & & & \cdot \\ & & & & a_{nn} \end{bmatrix}$$

3.1.5 Convergence

The coefficient matrix A in Eqn. 22 is a nxn matrix where n is the number of nodes.

Here we see that

$$\left|a_{ii}\right| = \sum_{j \neq i} \left|a_{ij}\right| \qquad \text{for node i not at the electrodes}$$

$$\left|a_{ii}\right| > \sum_{j \neq i} \left|a_{ij}\right| = 0 \qquad \text{for node i at the electrodes}$$

Therefore matrix A is diagonally dominant. In addition, for each pair (i,j) where i$\neq$j, there is a chain

$$a_{il_1} a_{l_1 l_2} \cdot \cdot \cdot a_{l_m j} \neq 0$$

The Gauss-Siedel iteration method will converge if the coefficient matrix A satisfies the conditions named above. [8]

3.1.6 Successive Over Relaxation Method

In real problems it is normal for iteration processes to be slowly convergent. The success of most iterations depends upon the application of a special technique called acceleration to hasten the convergence process. [9]

In this study, the successive over relaxation method is used to accelerate convergence. The successive over relaxation method can be expressed as

$$V^{(k)} = (1-w)\ V^{(k-1)} + w\ \bar{V}^{(k)}$$

where $\bar{V}^{(k)}$ is the components of the k^{th} Gauss-Siedel iteration. The factor w, called the acceleration parameter, generally lies in the range $1<w<2$. The choice of w determines the rapidity of the convergence, and the optimal w is related to the size of the matrix A. If $w=1$, this method is reduced to the Gauss-Siedel method. In this study we use 1.95 as the acceleration parameter.

3.2 Calculation of Impedance

The resistance measured by the electrodes can be calculated by

$$Z = \frac{V}{I} \tag{23}$$

where V is the voltage of the center electrode with respect to the reference and I is the current per unit length going out from the center electrode. In two dimensional representation, the current per unit length is calculated by

$$I = \int \vec{J} \cdot d\vec{l} \tag{24}$$

where $\vec{J}$ is the current density.

From Eqns. 1 and 2,

$$\vec{J} = \sigma \vec{E} \qquad (1)$$

and

$$\vec{E} = -\nabla V \qquad (2)$$

and knowing the electrical potential at each grid point, the current density $\vec{J}$ can be obtained. From Eqns. 23 and 24, the impedance value can also be calculated.

3.3 Computer Implementation

A computer program "ITERATION" is written in BASIC to first find the electrical potential at each grid point and then calculate the impedance measured by the electrodes. A listing of the computer program and the flowchart are included in Appendices B and C.

The computer program "PLOT" is written in FORTRAN to plot the equipotential lines. A list of the program "PLOT" and its flowchart are included in Appendices D and E.

3.3.1 Computer Program "ITERATION"

The program "ITERATION" is written in BASIC to run on the HP9826 desktop minicomputer. The inputs for the program are listed here:

1. location of electrode
2. size of disturbance (inhomogeneous rectangle)
3. location of disturbance
4. conductivity of disturbance
5. acceleration factor w
6. error criterion

With the inputs listed above, the "ITERATION" program sets up an operator matrix IOPMAT, where IOPMAT(I,J) is the number of the finite difference equation for grid point (I,J). Then, it uses the corresponding equation to iteratively calculate the electrical potential at each grid point (I,J) until the preset error criterion is satisfied. In this study, 10^{-5} is used as the error criterion. Therefore, the program stops its iteration process when the difference of the electrical potential at the $k+1^{st}$ iteration and that of the k^{th} iteration is no more than 10^{-5} at any grid point, and the electrical potential at each grid point is obtained. With the electrical potential values known, the "ITERATION" program thus calculates the impedance as illustrated in Sec. 3.2.

The outputs of the program are listed here:

1. electrode location
2. disturbance location
3. operator matrix IOPMAT
4. disturbance conductivity
5. iteration number
6. error criterion
7. acceleration factor
8. electrical potential at each grid point
9. impedance-unit length

3.3.2 Computer Program "PLOT"

A computer program "PLOT" is written to plot out the equipotential lines. This program uses the existing plotter software routines at the Kansas State University Computing Center, and the plots are made by a Calcomp Model 1051 digital incremental drum plotter.

The inputs for the program "PLOT" are:

1. electrode locations
2. electrical potential at every grid point

The program "PLOT" takes the electrical potential at each grid point and plots the contours. The electrode locations are also shown in the plot.

IV. Results

The effect of guarding is studied for several different electrode placements. As shown in Fig. 4.1, an inhomogeneous rectangle with size 4 x 4 units and conductivity 0.01 is centrally located in the system. The reference electrode is located as shown and the various locations of the guarded electrode are labeled as A,B,C,D,E. Both guarded and unguarded cases are simulated.

With the electrical potential at each grid point known from the simulation result, the equipotential lines are plotted by the Calcomp plotter. Then the current pathways are plotted by drawing lines perpendicular to the equipotential lines. For guarded cases, only the center electrode current pathways are plotted since the center electrode, not the guard electrode, is the measuring electrode.

The equipotential lines and the current pathways for only cases A, B and C are plotted since cases D and E are mirror images of cases B and A respectively. Plots of the equipotential lines are shown in Figs. 4.2-4.7 and the current pathway plots are shown in Figs. 4.8-4.13. The impedance versus electrode locations plots for guarded and unguarded cases are shown in Figs. 4.14 and 4.15.

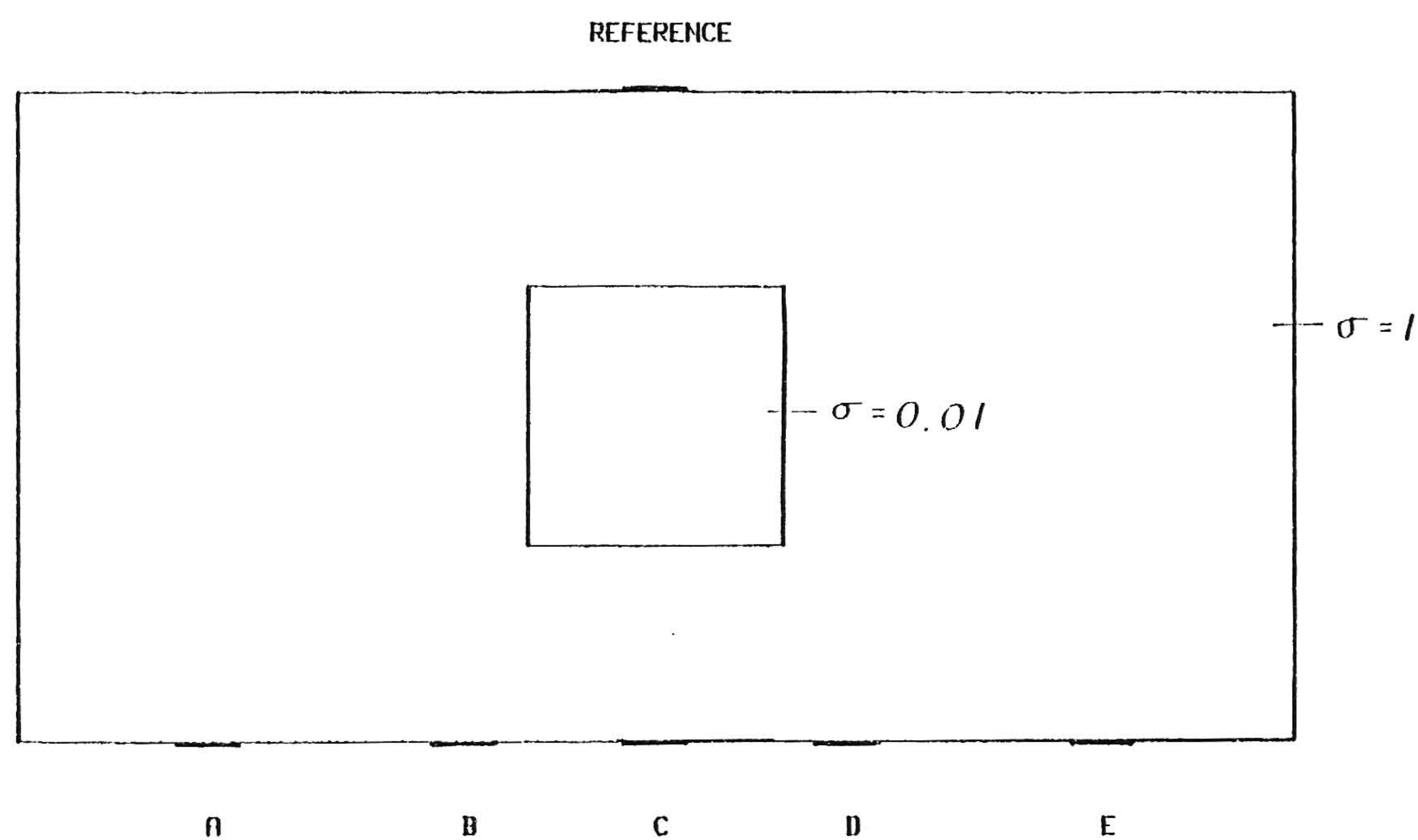


Figure 4.1 Five Different Electrode Placements and a Fixed Reference Electrode

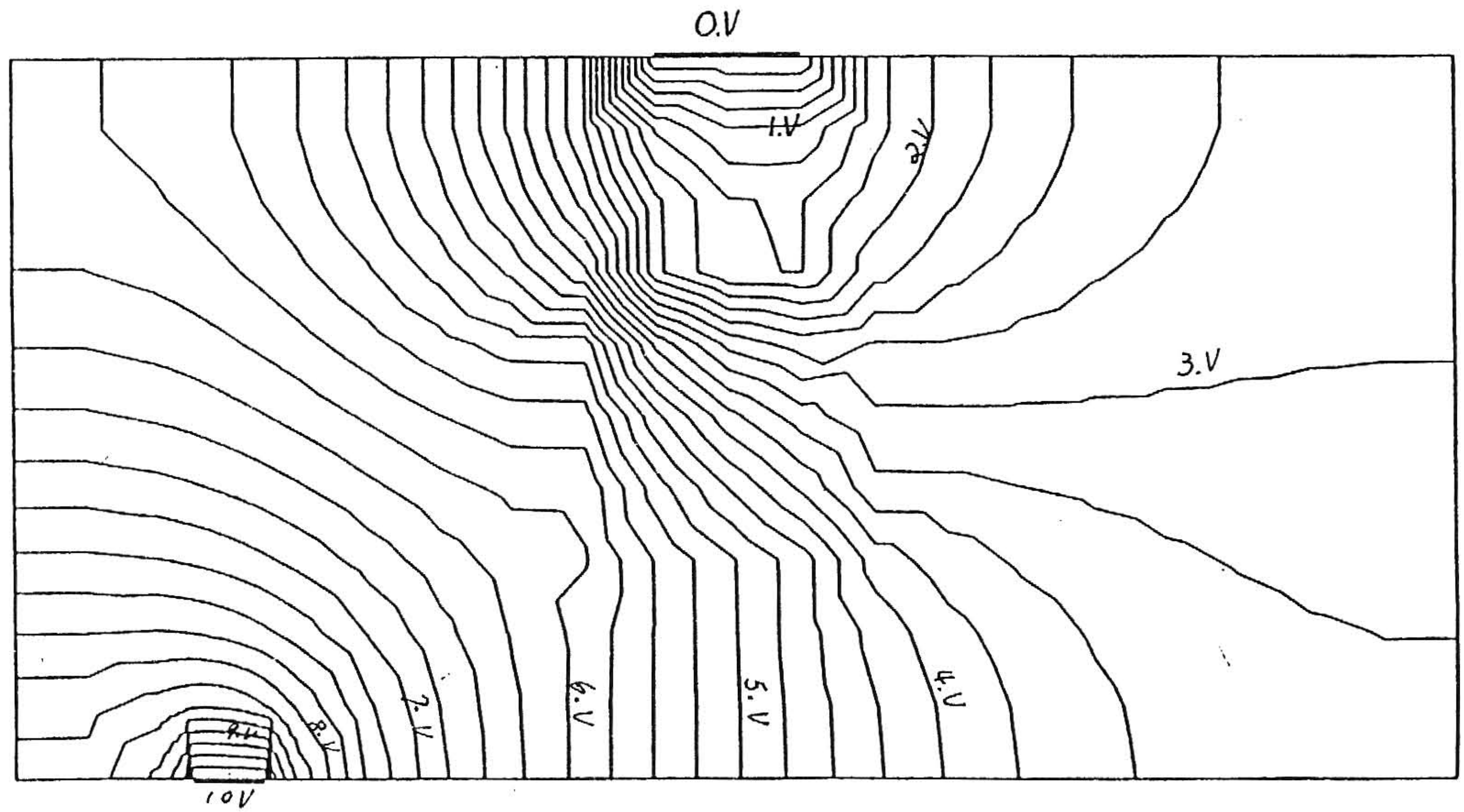


Figure 4.2 Equipotential Lines for Unguarded Electrode

at Location A

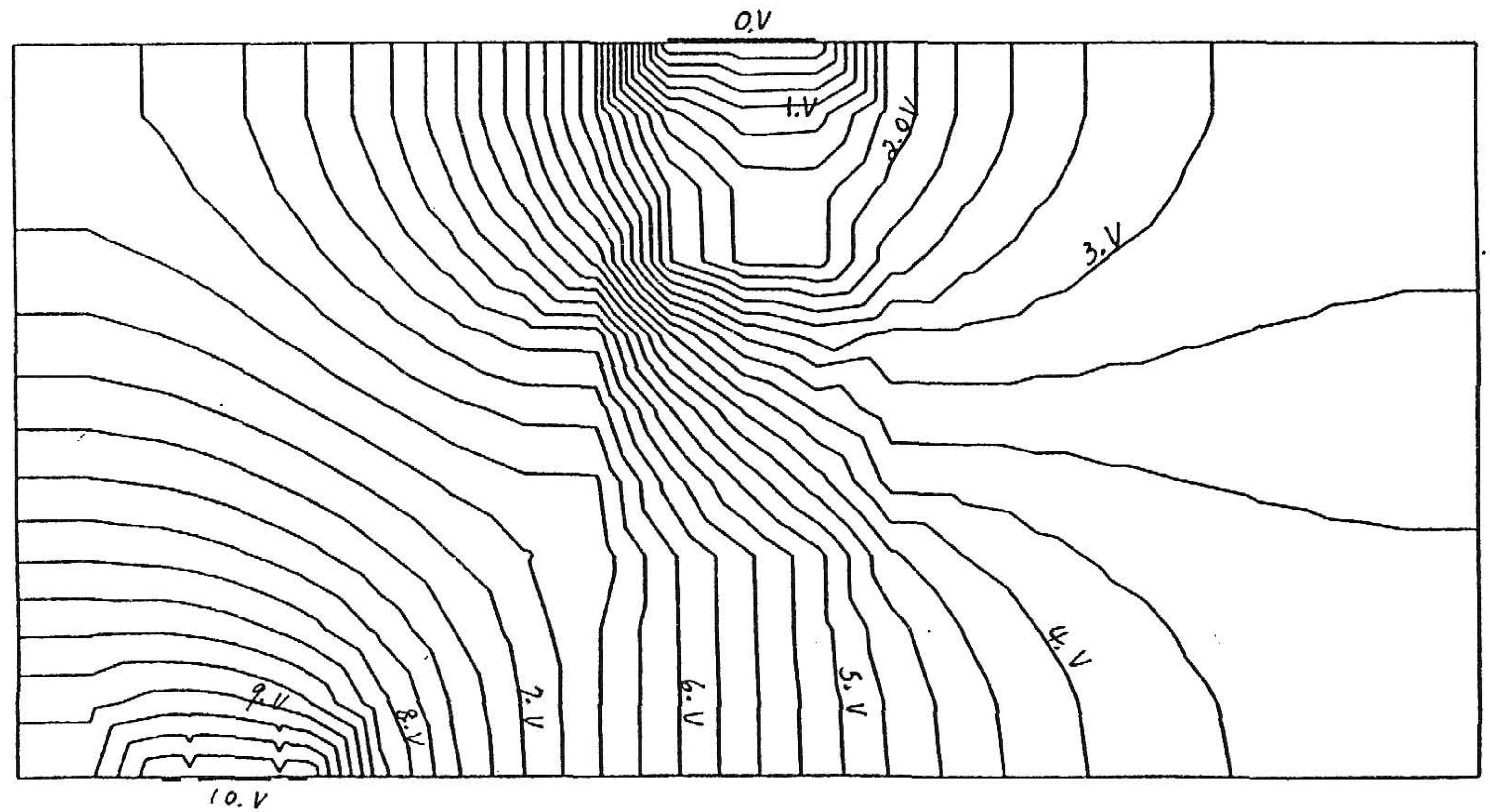


Figure 4.3 Equipotential Lines for Guarded Electrode

at Location A

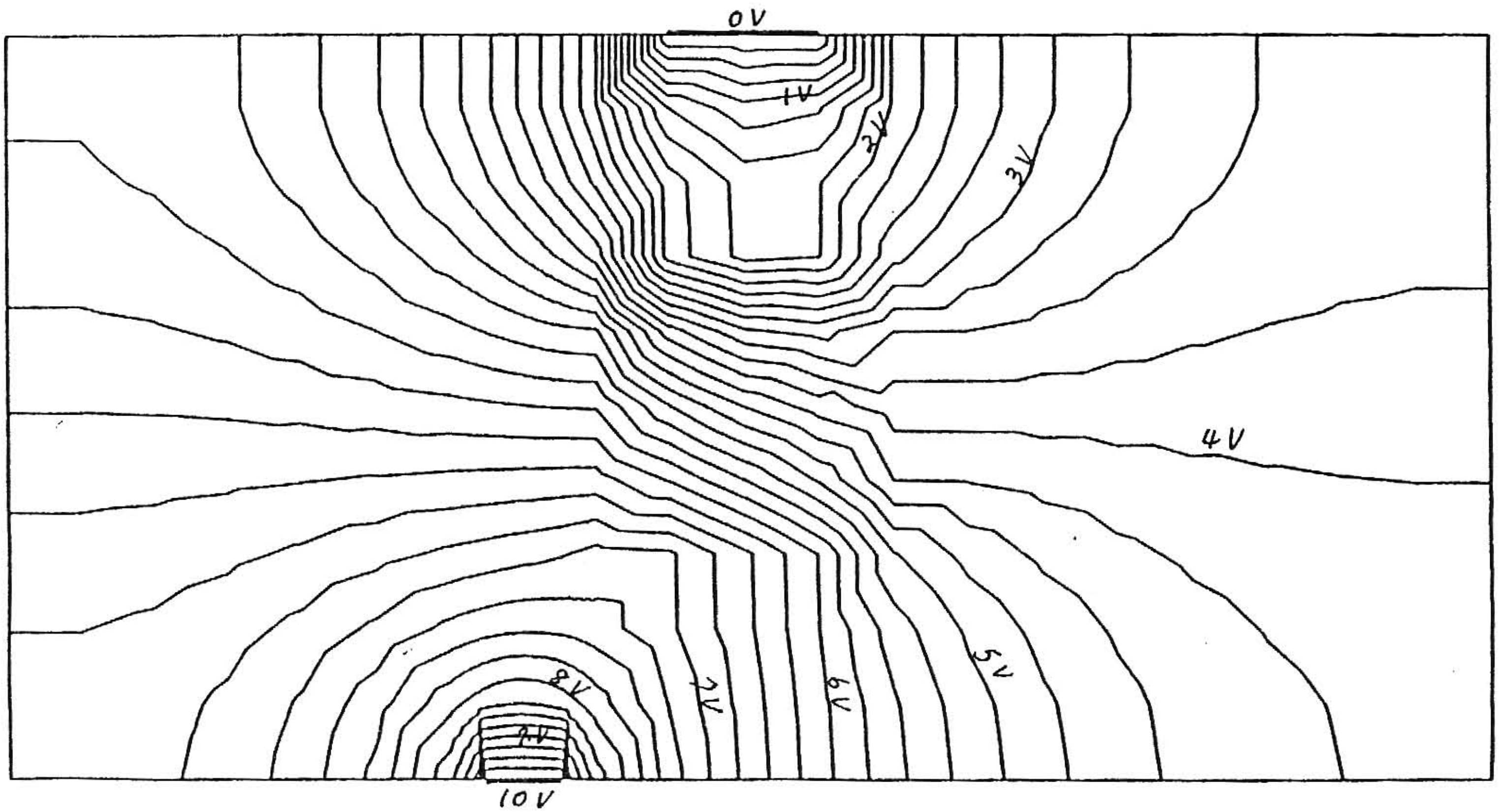


Figure 4.4 Equipotential Lines for Unguarded Electrode at Location B

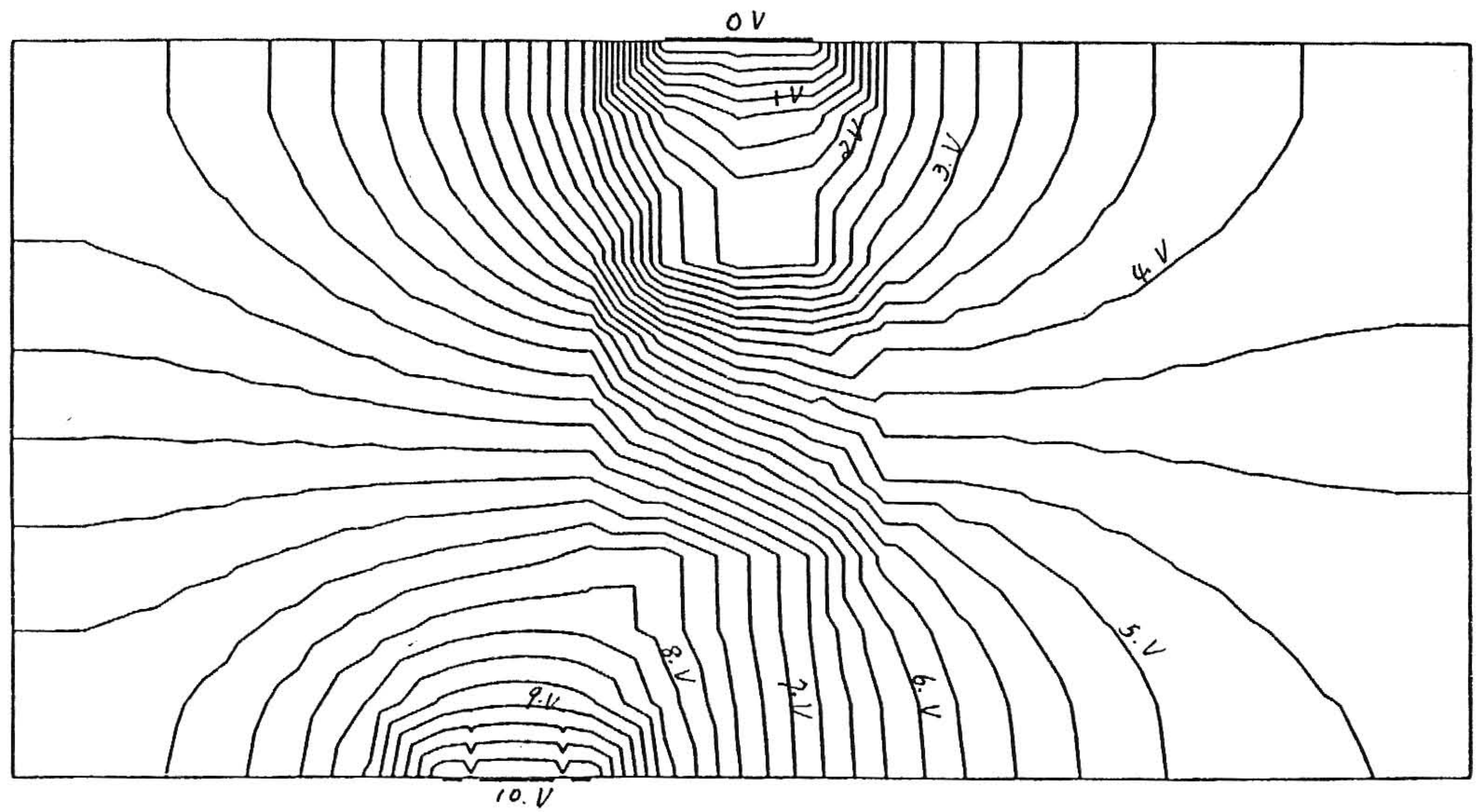


Figure 4.5 Equipotential Lines for Guarded Electrode at Location B

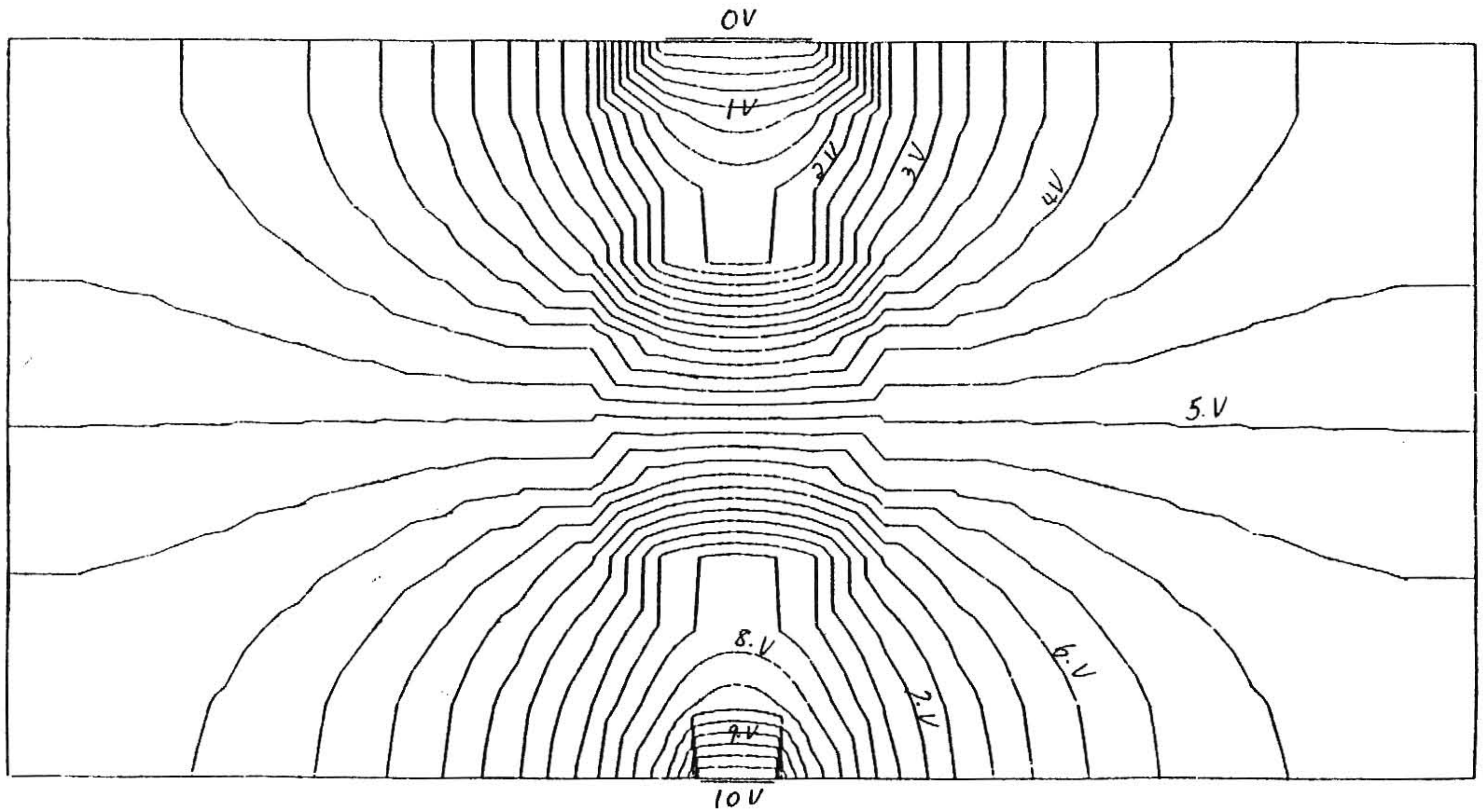


Figure 4.6 Equipotential Lines for Unguarded Electrode

at Location C

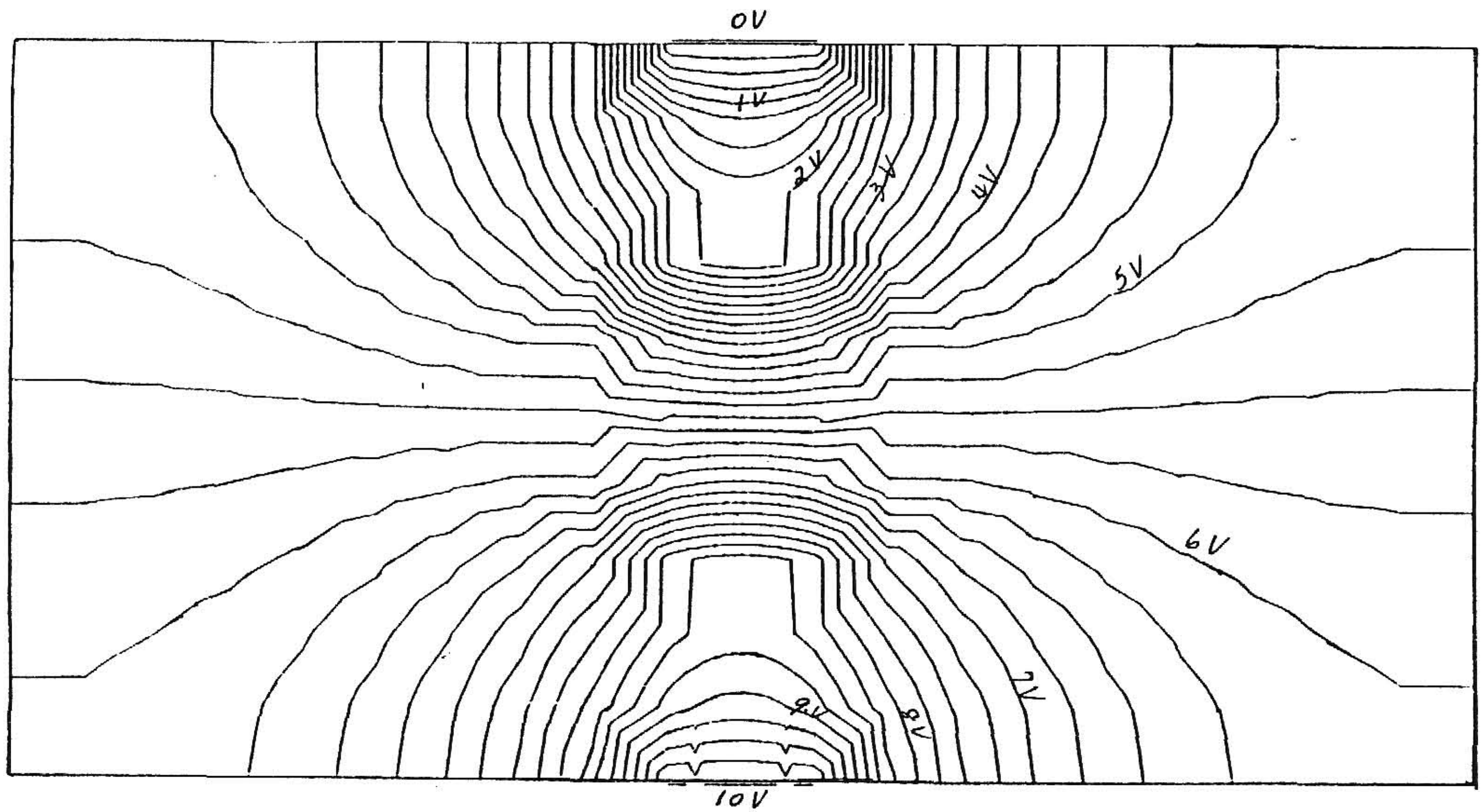


Figure 4.7 Equipotential Lines for Guarded Electrode at Location C

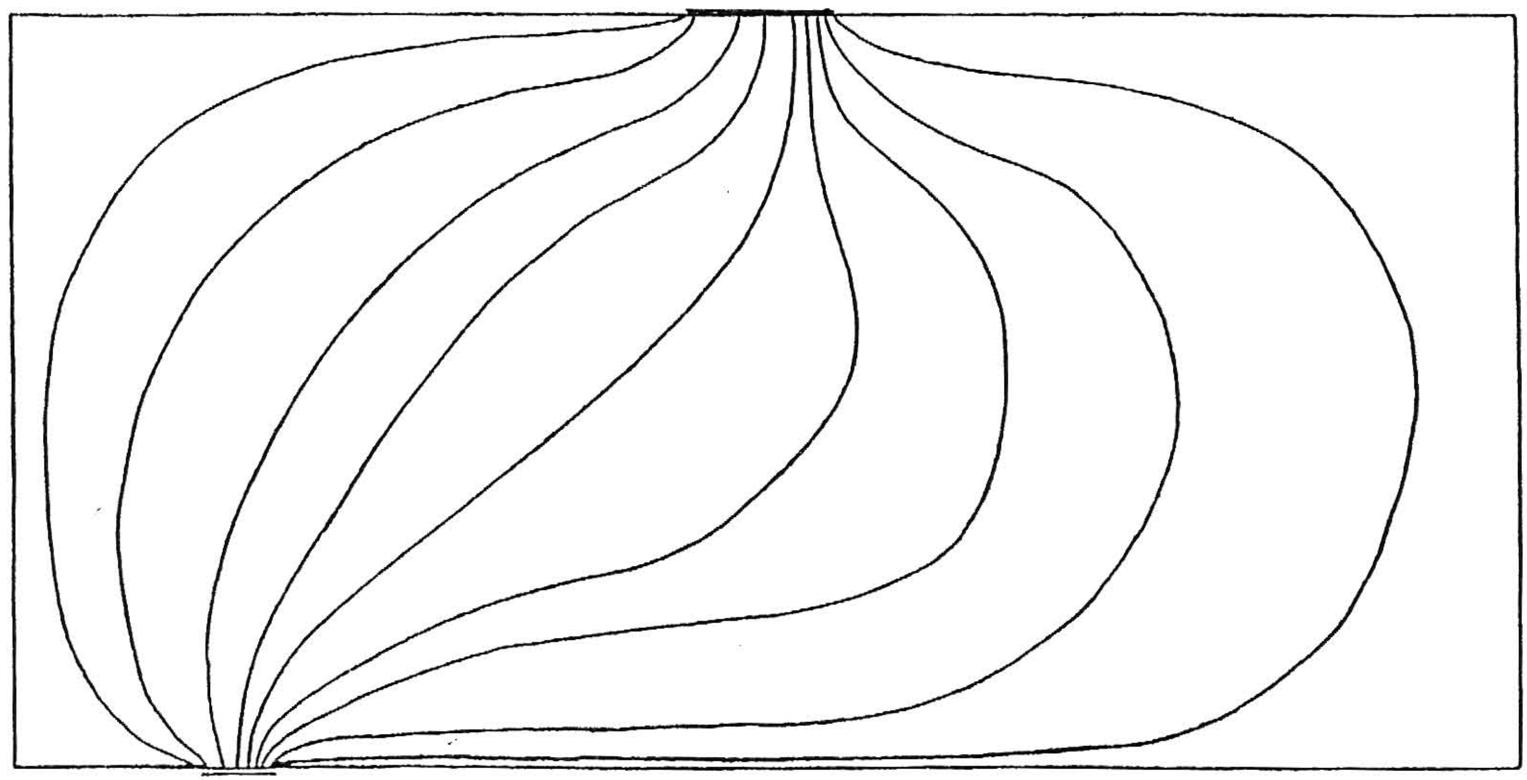


Figure 4.8 Current Pathways for Unguarded Electrode at Location A

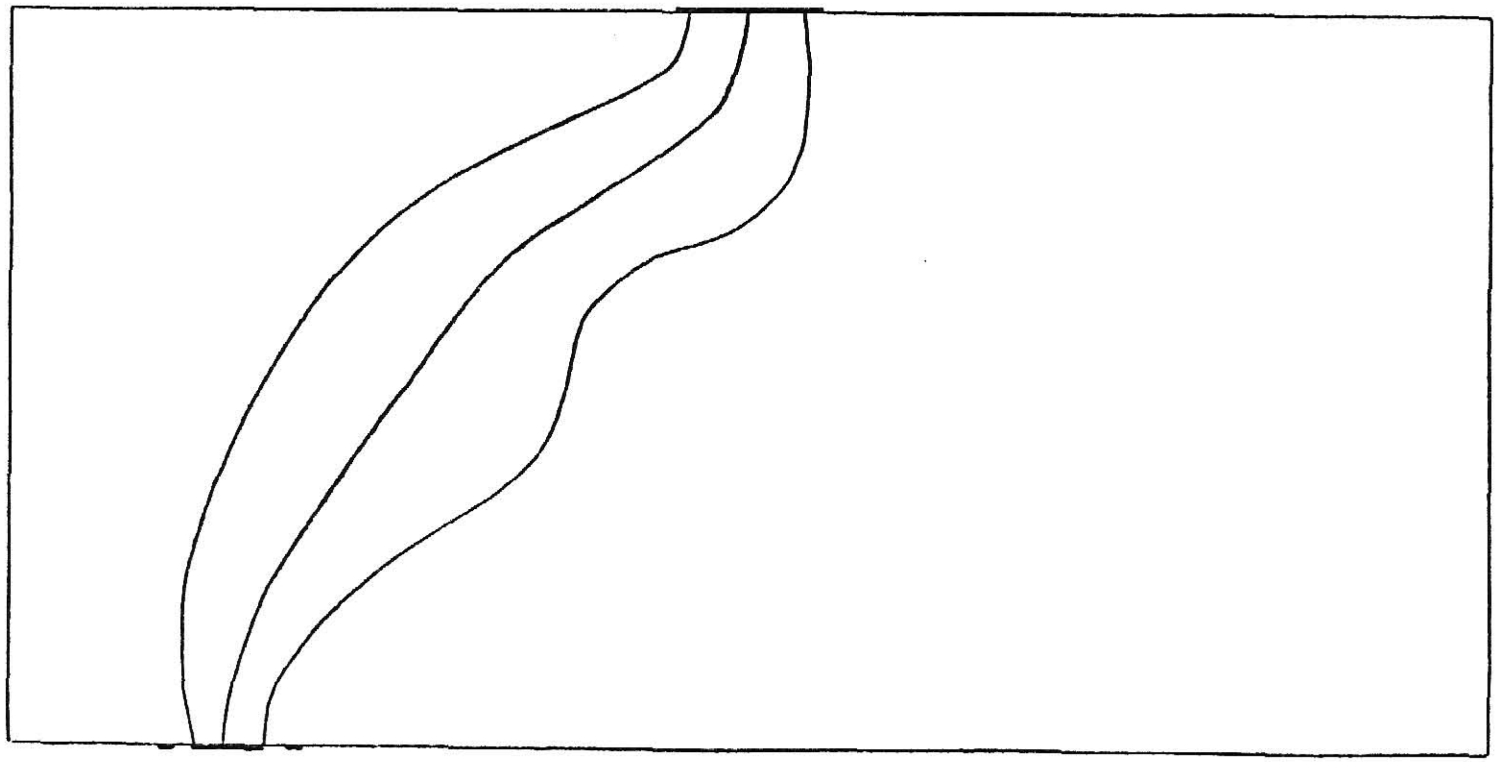


Figure 4.9 Current Pathways for Guarded Electrode at Location A

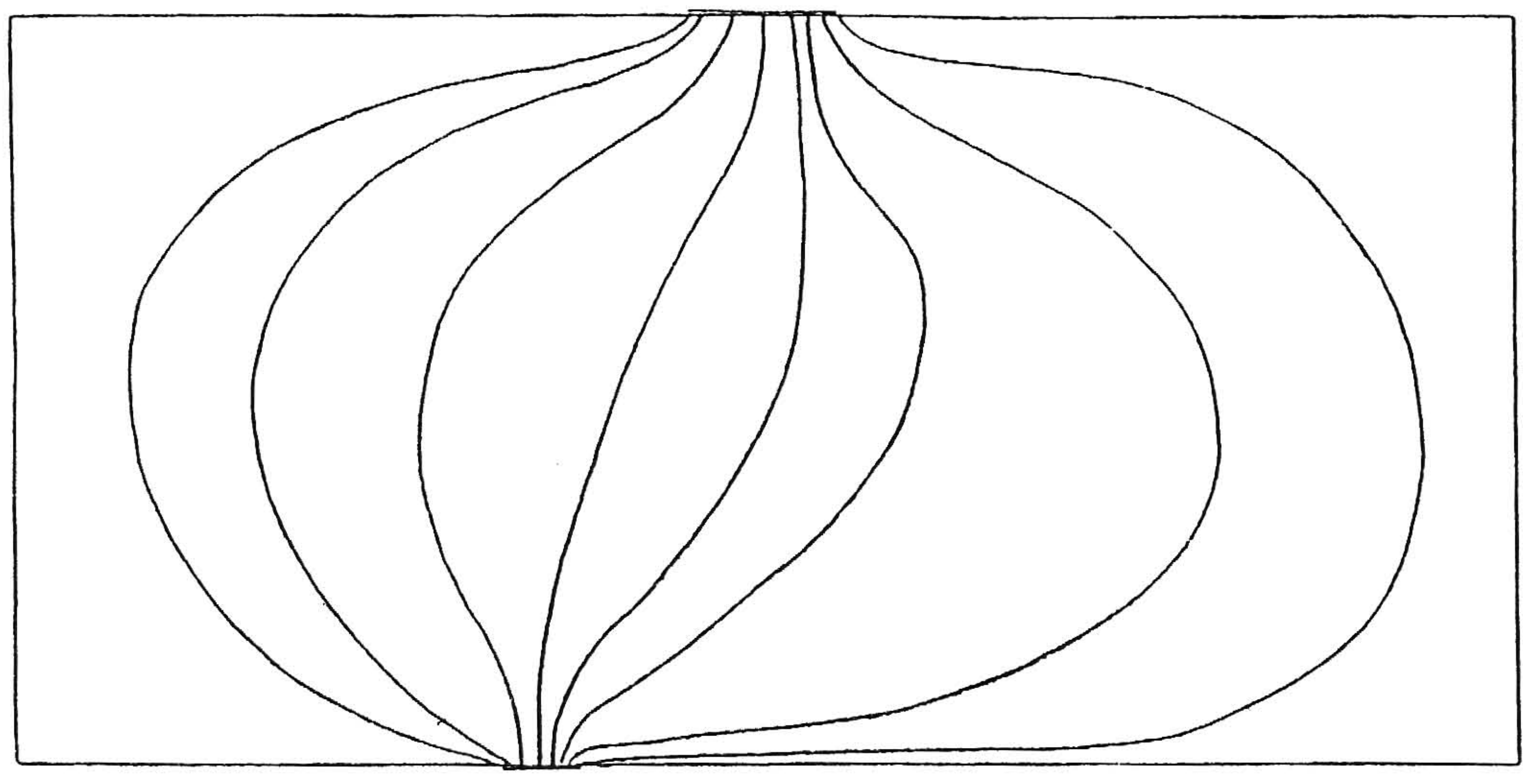


Figure 4.10 Current Pathways for Unguarded Electrode at Location B

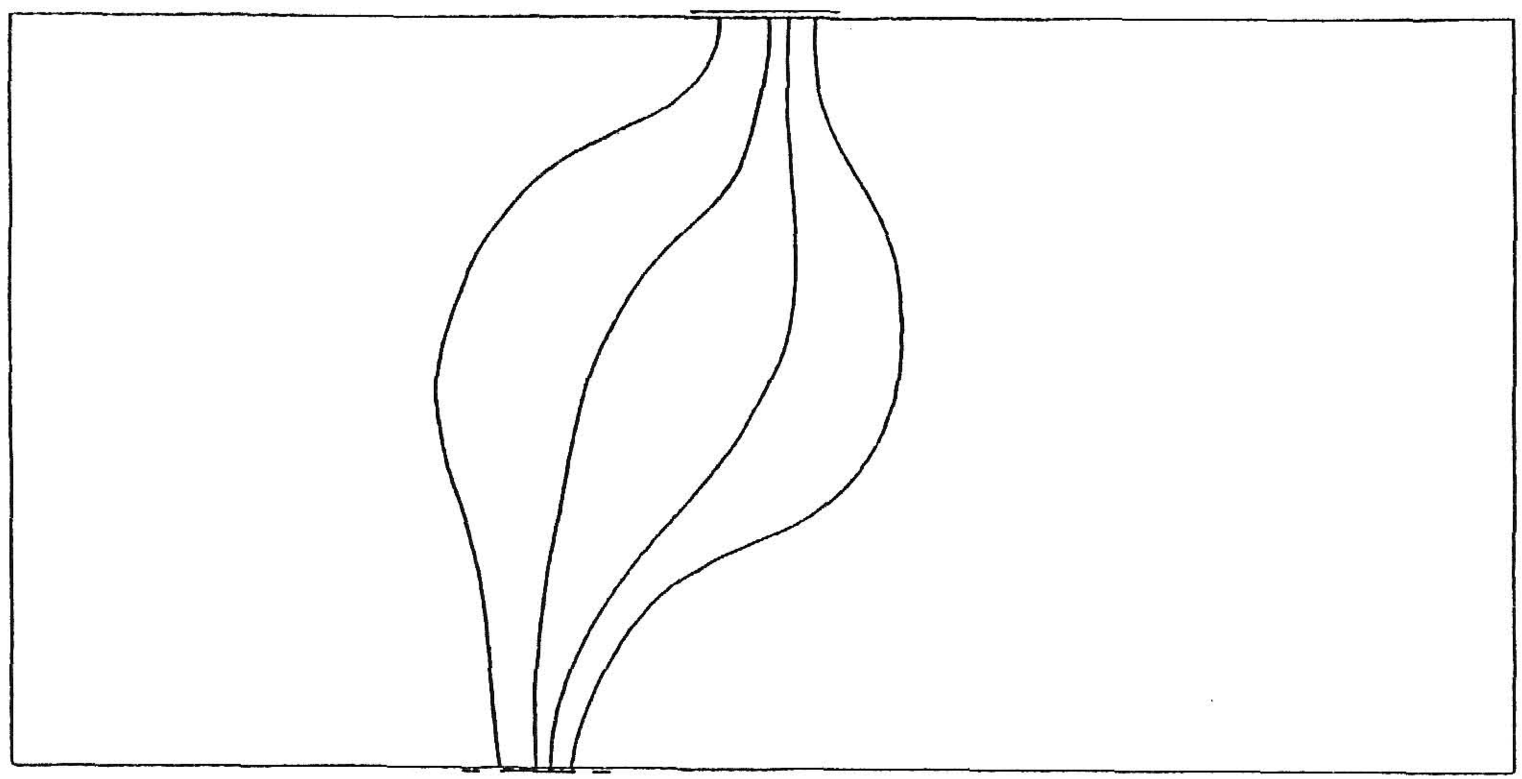


Figure 4.11 *Current Pathways for Guarded Electrode at Location B*

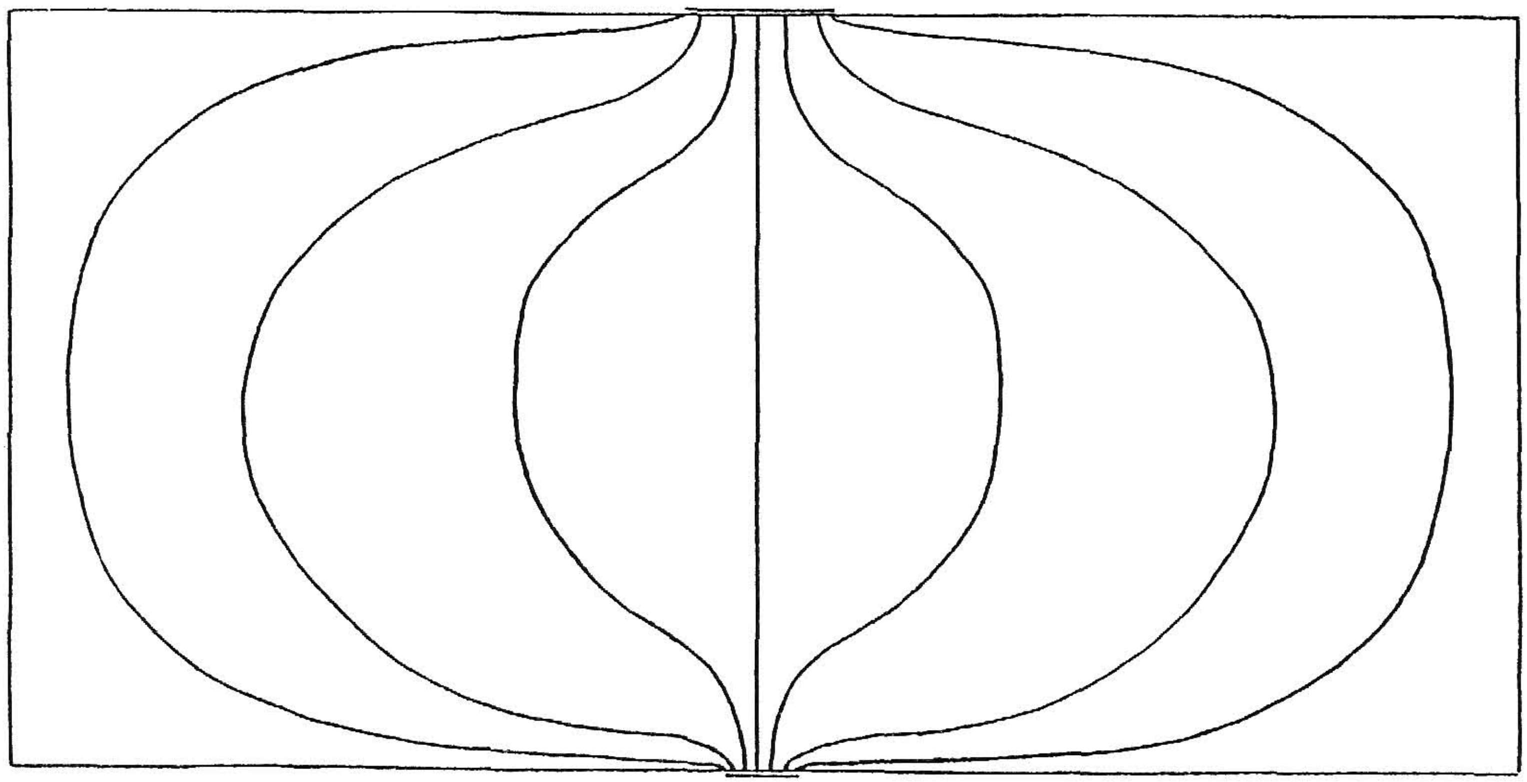


Figure 4.12 Current Pathways for Unguarded Electrode at Location C

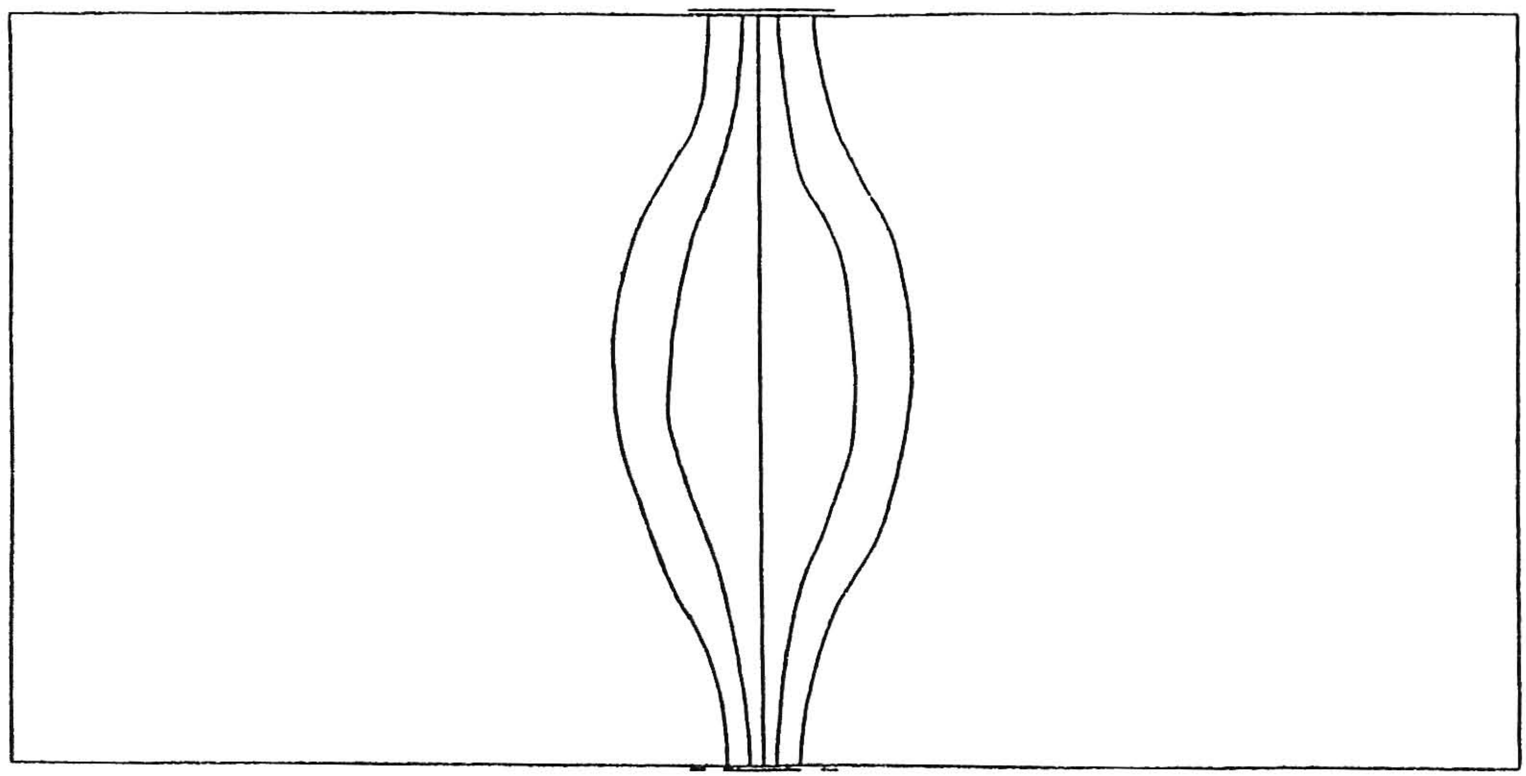


Figure 4.13 Current Pathways for Guarded Electrode at Location C

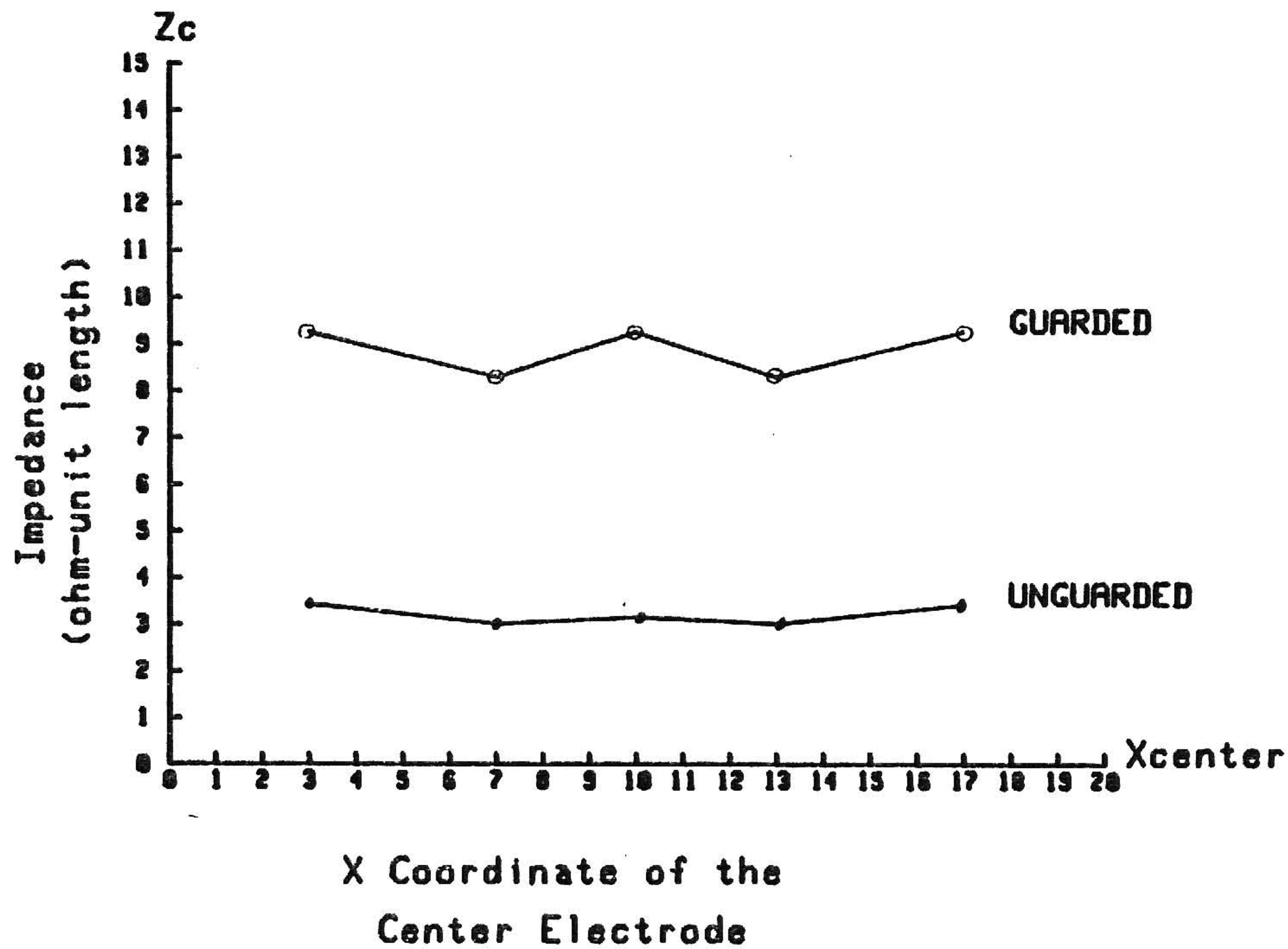


Figure 4.14 Plot of Impedance-Length versus Center Electrode Location

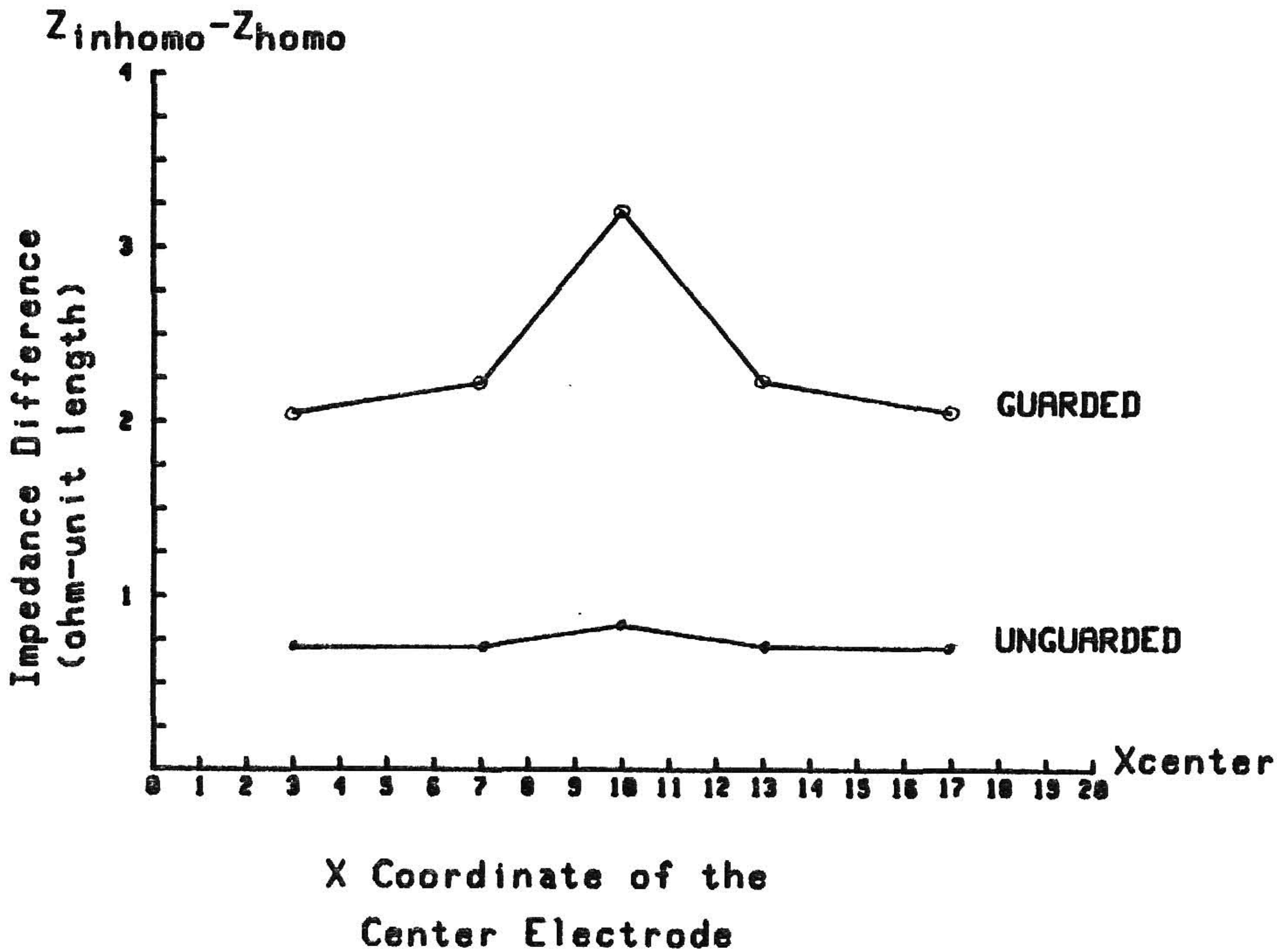


Figure 4.15 Impedance Change versus
Center Electrode Location

V. Discussion and Recommendations

With the equipotential line plots and the current pathway plots we can compare the guarded cases with the unguarded cases. We see that the current pathways spread out more for unguarded cases (Figs. 4.8, 4.10, and 4.12) when the measurement field is the entire bounded region. For guarded cases (Figs. 4.9, 4.11, and 4.13) the current pathways emanating from the measuring electrode are more confined, thus giving a more confined measurement field. A confined measurement field increases the regional sensitivity of the measurement; therefore, we obtain more information near the electrodes.

With the inhomogeneous region in the middle of the system and the reference electrode location fixed as shown in Fig. 4.1 impedance values are simulated for measuring electrodes at different locations. The relationship between impedance and the electrode location is shown by an impedance versus electrode location plot (Fig. 4.14). We see from this plot that unguarded measurements do not give any indication as to where the inhomogeneous region might be located. For guarded cases, we see that location C is close to the inhomogeneous region and the impedance measurement at location C is large compared with the impedance at locations B and D. Locations A and E are far away from the inhomogeneous region, but the impedances at locations A and E are also large compared with impedances at locations B and D. This can be explained as follows.

The resistance of a piece of material can be expressed as

$$R = \frac{L}{\sigma A} \tag{25}$$

where L is the length of the material
A is the cross sectional area
σ is the conductivity

where σ and A are constants. If σ and A vary along L, the resistance R can be expressed as

$$R = \int \frac{1}{\sigma A} \, dL \tag{26}$$

From Figs. 4.8–4.13 we see that for cases A and E, L is large compared with the other cases. These give a large resistance R. For case C, an inhomogeneous region which has smaller conductivity σ, the resistance R is large. Therefore, a larger impedance measurement does not necessarily mean the existence of an inhomogeneous region with smaller conductivity in the neighborhood.

Fig. 4.15 shows the impedance change for different locations of the measuring electrode. The value of the impedance change is obtained by subtracting the impedance for the homogeneous case (no inhomogeneous region in the middle) from the impedance for the inhomogeneous case. The justification for this approach is to try to cancel out the effect that a different length L and cross-sectional area A might have on the impedance. The impedance difference will then show mainly the effect of the existence of the inhomogeneous region. As shown in Fig. 4.15, we get the largest impedance difference from the guarded electrode at location C. We have a considerably smaller impedance change in the other cases. This indicates the location of an inhomogeneous region with smaller conductivity close to location C. We also notice that for the unguarded cases the impedance change is not pronounced.

From the discussion above, we have the following conclusions:

1. Compared with the unguarded electrode, the guarded electrode gives a more confined measurement field and a better measure of regional sensitivity.

2. There are several factors that affect the impedance reading. Differential conductivity, cross-sectional area, and length of the measurement field will all contribute to a change of impedance. Therefore, a correct interpretation of the impedance measurement is important.
3. With the impedance of the homogeneous case (system with no inhomogeneous region) as the reference, the change of impedance is a good indication of a change of conductivity close to the electrodes.

In this study, we see the interesting effect that guarding has on the measurement field and impedance changes. However, the practical application of detecting an inhomogeneous region will require further study. A study of a three-dimensional model with more complicated geometry will be necessary.

VI. Bibliography

[1] M. Graham, "Guard ring use in physiological measurement," IEEE Trans. Biomed. Eng., vol. BME-15, pp. 197-198, 1965.

[2] W. L. Cooley, and R. L. Longini, "A new design for an impedance pneumograph," J. Appl. Physiol., vol. 25, pp. 429-432, 1968.

[3] J. L. Schmalzel and R. R. Gallagher, "What are the limitations of the guarded-electrode technique?", IEEE Frontiers of Engineering in Health Care, pp. 20-23, 1981.

[4] J. L. Schmalzel, "The guarded electrode: Studies using computer simulation and physical modeling," Ph.D. Thesis, Kansas State University, Manhattan, KS 1980.

[5] J. M. Barquest, "The use of a guarded electrode impedance pneumograph for the assessment of regional pulmonary ventilation in the canine," Ph.D. Thesis, Kansas State University, Manhattan, KS, 1979.

[6] R. P. Henderson, and J. G. Webster, "An impedance camera for spatially specific measurement of the thorax," IEEE Trans. Biomed. Eng., vol BME-25, pp. 250-254, 1978.

[7] G. D. Smith, Numerical Solutions of Partial Differential Equations, pp. 2-3, 1978.

[8] J. M. Ortega, Numerical Analysis, pp. 120-121, 1972.

[9] W. F. Ames, Numerical Methods for Partial Differential Equations, pp. 119-125, 1977.

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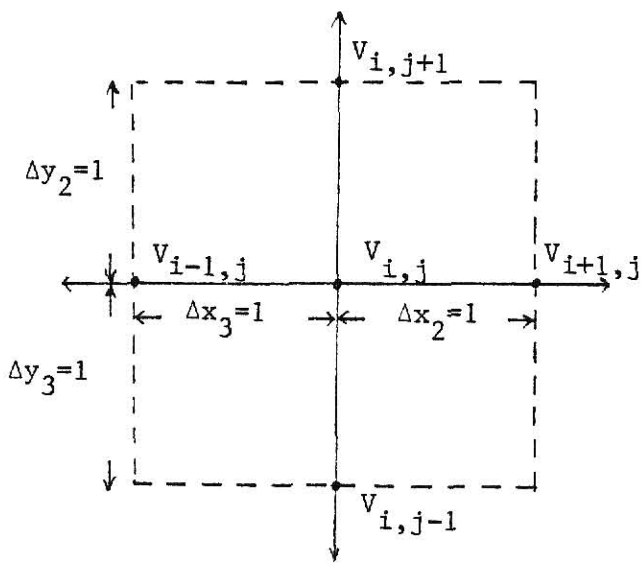
My family, who helped me in numerous ways through the years deserve my deepest thanks. And to my husband, Michael, who provided me with needed spiritual support, goes my eternal thankfulness. I would also like to thank Michael's grandparents for just being there when we needed them.

APPENDIX A. Finite Difference Equations

Case 1: An interior grid point in a homogeneous region



From Eqn. 14

$$(\frac{1}{\Delta x_1})\ [\frac{1}{\Delta x_2}\ \sigma_{i+\frac{1}{2},j}\ (V_{i+1,j} - V_{i,j}) - \frac{1}{\Delta x_3}\ \sigma_{i-\frac{1}{2},j}\ (V_{i,j} - V_{i-1,j})] +$$

$$(\frac{1}{\Delta y_1})\ [\frac{1}{\Delta y_2}\ \sigma_{i,j+\frac{1}{2}}\ (V_{i,j+1} - V_{i,j}) - \frac{1}{\Delta y_3}\ \sigma_{i,j-\frac{1}{2}}\ (V_{i,j} - V_{i,j-1})] = 0$$

where $\Delta x_1 = \frac{1}{2}(\Delta x_2 + \Delta x_3)$, $\Delta y_1 = \frac{1}{2}\ (\Delta y_2 + \Delta y_3)$

with

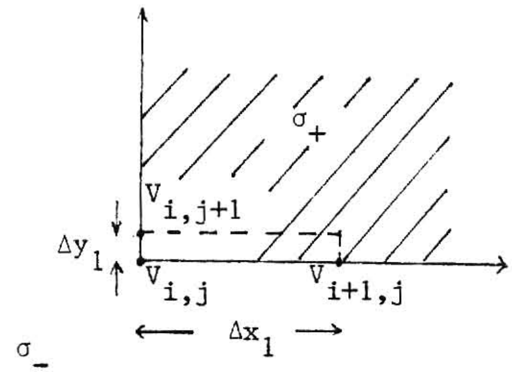
$$\Delta x_2 = \Delta y_2 = \Delta x_3 = \Delta y_3 = 1$$

$$\sigma_{i+\frac{1}{2},j} = \sigma_{i-\frac{1}{2},j} = \sigma_{i,j+\frac{1}{2}} = \sigma_{i,j-\frac{1}{2}} = 1 .$$

We get the finite difference equation as

$$V_{i,j} = 1/4\ (V_{i-1,j} + V_{i+1,j} + V_{i,j-1} + V_{i,j+1}) .$$

Case 2: A boundary point



From Eqn. 21

$$\sigma_+ \;[[(\frac{V_{i+1,j}-V_{i,j}}{\Delta x_1})\;\hat{X} + (\frac{V_{i,j+1}-V_{i,j}}{\Delta y_1})\;\hat{y})]\cdot\hat{n}]$$

$$= \;\sigma_- \;[[(\frac{V_{i,j}-V_{i-1,j}}{\Delta x_2})\hat{n} + (\frac{V_{i,j}-V_{i,j-1}}{\Delta y_2})\hat{y}]\cdot\hat{n}]$$

and with

$$\sigma_+ = 1, \quad \sigma_- = 0$$

$$\text{and} \quad \Delta x_1 = 1, \quad \Delta y_1 = \frac{1}{8}$$

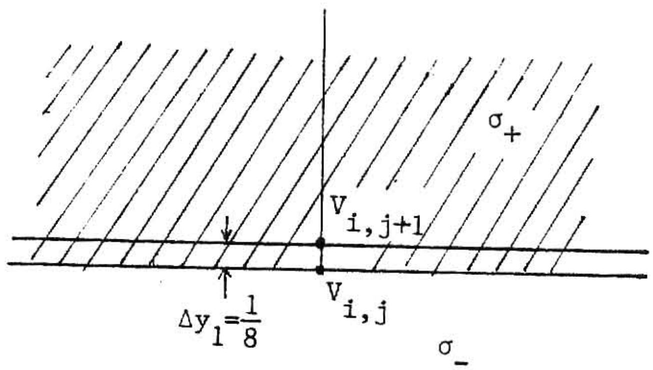
$$\hat{n} = -\frac{1}{\sqrt{2}}\,\hat{x} - \frac{1}{\sqrt{2}}\,\hat{y}$$

we have $$[(\frac{V_{i+1,j} - V_{i,j}}{1})\;\hat{x} + (\frac{V_{i,j+1}-V_{i,j}}{1/8})\;\hat{y}]\;\cdot\;(-\frac{1}{\sqrt{2}}\,\hat{x} - \frac{1}{\sqrt{2}}\,\hat{y}) = 0\;,$$

i.e.

$$V_{i,j} = (V_{i+1,j} + 8\;V_{i,j+1})/9\;.$$

Case 3: A boundary point



From Eqn. 16.

$$\sigma_+ E_{n+} = \sigma_- E_{n-}$$

with $\sigma_+ = 1$, $\sigma_- = 0$, we get $\sigma_+ E_{n+} = 0$.

Approximating E_{n+} by

$$E_{n+} = [-\nabla V)_+ \cdot \hat{n}]$$

where $\nabla V_+ = \dfrac{V_{i+1,j} - V_{i,j}}{\Delta x_1} \hat{x} + \dfrac{V_{i,j+1} - V_{i,j}}{\Delta y_1} \hat{y}$

and $\hat{n} = -\hat{y}$, $\Delta y_1 = \frac{1}{8}$.

The finite difference equation is obtained as

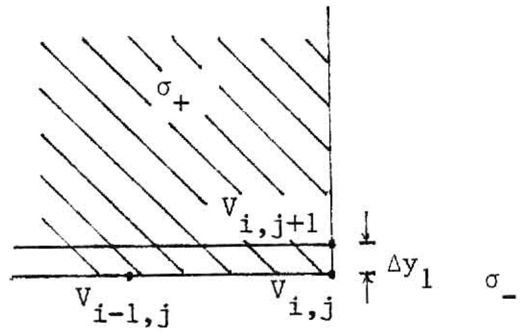
$$V_{i,j} = V_{i+1,j} \ .$$

Case 4: A boundary point at the measuring electrode

For point at the measuring electrode, the electrical potential is maintained at 10V. Therefore, we have the following equation:

$$V_{i,j} = 10$$

Case 5: A boundary point



From Eqn. 16

$$\sigma_+ E_{n+} = \sigma_- E_{n-}$$

with $\sigma_+ = 1$, $\sigma_- = 0$, we get $\sigma_+ E_{n+} = 0$.

Approximating E_{n+} by

$$E_{n+} = [(-\nabla V)_+ \cdot \hat{n}]$$

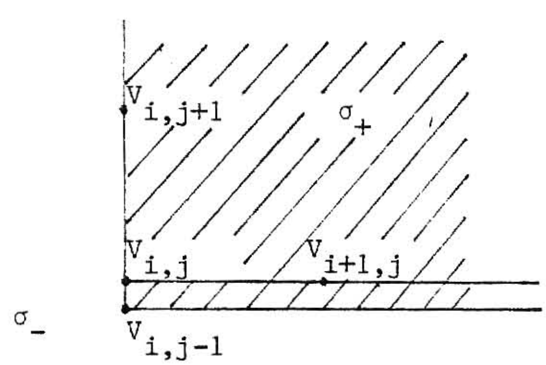
$$\text{where } \nabla V_+ = \frac{V_{i,j+1} - V_{i,j}}{\Delta y_1} \hat{y} + \frac{V_{i,j} - V_{i-1,j}}{\Delta x_1} \hat{x}$$

$$\text{and } \hat{n} = \frac{1}{\sqrt{2}} \hat{x} - \frac{1}{\sqrt{2}} \hat{y}$$

the finite difference equation is obtained as

$$V_{i,j} = (V_{i-1,j} + 8\, V_{i,j+1})/9 .$$

Case 6: A boundary point



From Eqn. 16. $\quad \sigma_+ E_{n+} = \sigma_- E_{n-}$

and here $\sigma_- = 0$, we get $\sigma_+ E_{n+} = 0$.

Approximating E_{n+} by

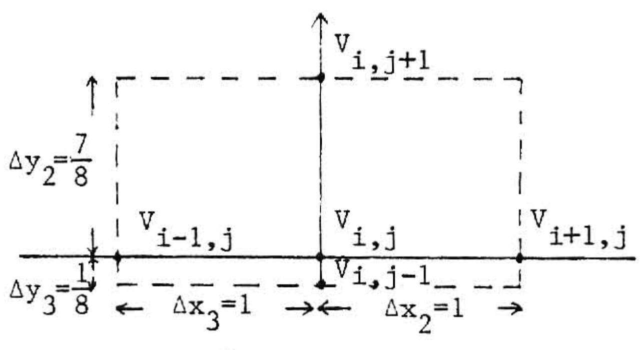
$$E_{n+} = [(-\nabla V)_+ \cdot \hat{n}]$$

$$\text{where } \nabla V_+ = \left(\frac{V_{i+1,j}-V_{i,j}}{\Delta x}\hat{x} + \frac{V_{i,j+1}-V_{i,j}}{\Delta y}\hat{y}\right)$$

and $\hat{n} = -\hat{x}$

we get $V_{i,j} = V_{i+1,j}$.

Case 7: An interior grid point in a homogeneous region



From Eqn. 14

$$\left(\frac{1}{\Delta x_1}\right)\left[\frac{1}{\Delta x_2}\sigma_{i+\frac{1}{2},j}(V_{i+1,j}-V_{i,j}) - \frac{1}{\Delta x_3}\sigma_{i-\frac{1}{2},j}(V_{i,j}-V_{i-1,j})\right] +$$

$$\left(\frac{1}{\Delta y_1}\right)\left[\frac{1}{\Delta y_2}\sigma_{i,j+\frac{1}{2}}(V_{i,j+1}-V_{i,j}) - \frac{1}{\Delta y_3}\sigma_{i,j-\frac{1}{2}}(V_{i,j}-V_{i,j-1})\right] = 0$$

where $\Delta x_1 = \frac{1}{2}(\Delta x_2+\Delta x_3)$, $\Delta y_1 = \frac{1}{2}(\Delta y_2+\Delta y_3)$

and

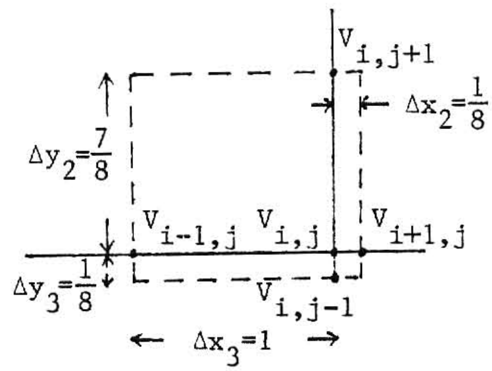
$\Delta x_2 = \Delta x_3 = 1$, $\Delta y_2 = 7/8$, $\Delta y_3 = 1/8$,

$\sigma_{i,j+\frac{1}{2}} = \sigma_{i,j-\frac{1}{2}} = \sigma_{i+\frac{1}{2},j} = \sigma_{i-\frac{1}{2},j} = 1$

we obtain

$$V_{i,j} = (7V_{i+1,j} + 7V_{i-1,j} + 16V_{i,j+1} + 112V_{i,j-1})/142 .$$

Case 8: An interior grid point in a homogeneous region



From Eqn. 14

$$\left(\frac{1}{\Delta x_1}\right)\left[\frac{1}{\Delta x_2}\sigma_{i+\frac{1}{2},j}(V_{i+1,j}-V_{i,j}) - \frac{1}{\Delta x_3}\sigma_{i-\frac{1}{2},j}(V_{i,j}-V_{i-1,j})\right] +$$

$$\left(\frac{1}{\Delta y_1}\right)\left[\frac{1}{\Delta y_2}\sigma_{i,j+\frac{1}{2}}(V_{i,j+1}-V_{i,j}) - \frac{1}{\Delta y_3}\sigma_{i,j-\frac{1}{2}}(V_{i,j}-V_{i,j-1})\right] = 0$$

where $\Delta x_1 = \frac{1}{2}(\Delta x_2+\Delta x_3)$, $\Delta y_1 = \frac{1}{2}(\Delta y_2+\Delta y_3)$,

and

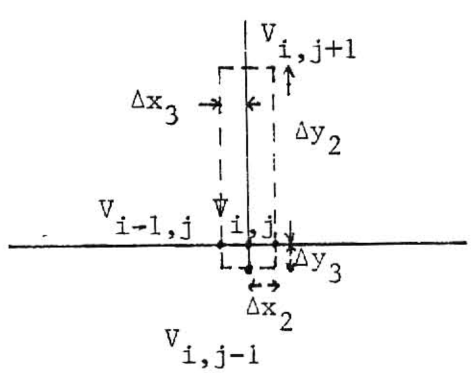
$\Delta x_2 = \frac{1}{8}$, $\Delta x_3 = 1$, $\Delta y_2 = \frac{7}{8}$, $\Delta y_3 = \frac{1}{8}$,

$\sigma_{i+\frac{1}{2},j} = \sigma_{i-\frac{1}{2},j} = \sigma_{i,j+\frac{1}{2}} = \sigma_{i,j-\frac{1}{2}} = 1$,

we obtain

$$V_{i,j} = (56V_{i+1,j} + 7V_{i-1,j} + 9V_{i,j+1} + 63V_{i,j-1})/135 .$$

Case 9: An interior grid point in a homogeneous region



From Eqn. 14

$$\left(\frac{1}{\Delta x_1}\right)\left[\frac{1}{\Delta x_2}\sigma_{i+\frac{1}{2},j}\,(V_{i+1,j}-V_{i,j}) - \frac{1}{\Delta x_3}\sigma_{i-\frac{1}{2},j}\,(V_{i,j}-V_{i-1,j})\right] +$$

$$\left(\frac{1}{\Delta y_1}\right)\left[\frac{1}{\Delta y_2}\sigma_{i,j+\frac{1}{2}}\,(V_{i,j+1}-V_{i,j}) - \frac{1}{\Delta y_3}\sigma_{i,j-\frac{1}{2}}\,(V_{i,j}-V_{i,j-1})\right] = 0$$

where $\Delta x_1 = \frac{1}{2}(\Delta x_2+\Delta x_3)$, $\Delta y_1 = \frac{1}{2}(\Delta y_2+\Delta y_3)$,

and

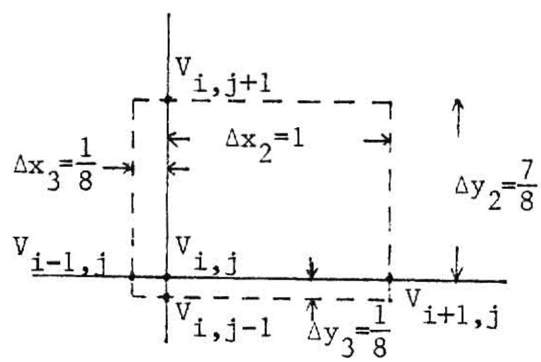
$$\Delta x_2 = \Delta y_3 = \Delta x_3 = \frac{1}{8}\ ,\ \Delta y_2 = \frac{7}{8}\ ,$$

$$\sigma_{i+\frac{1}{2},j} = \sigma_{i-\frac{1}{2},j} = \sigma_{i,j+\frac{1}{2}} = \sigma_{i,j-\frac{1}{2}} = 1\ ,$$

the finite difference equation is obtained as

$$V_{i,j} = [28V_{i+1,j} + 28V_{i-1,j} + V_{i,j+1} + 7V_{i,j-1})/64\ .$$

Case 10: An interior grid point in a homogeneous region



From Eqn. 14

$$\left(\frac{1}{\Delta x_1}\right)\left[\frac{1}{\Delta x_2}\sigma_{i+\frac{1}{2},j}\,(V_{i+1,j}-V_{i,j}) - \frac{1}{\Delta x_3}\sigma_{i-\frac{1}{2},j}\,(V_{i,j}-V_{i-1,j})\right] +$$

$$\left(\frac{1}{\Delta y_1}\right)\left[\frac{1}{\Delta y_2}\sigma_{i,j+\frac{1}{2}}\,(V_{i,j+1}-V_{i,j}) - \frac{1}{\Delta y_3}\sigma_{i,j-\frac{1}{2}}\,(V_{i,j}-V_{i,j-1})\right] = 0$$

where $\Delta x_1 = \frac{1}{2}(\Delta x_2+\Delta x_3)$, $\Delta y_1 = \frac{1}{2}(\Delta y_2+\Delta y_3)$

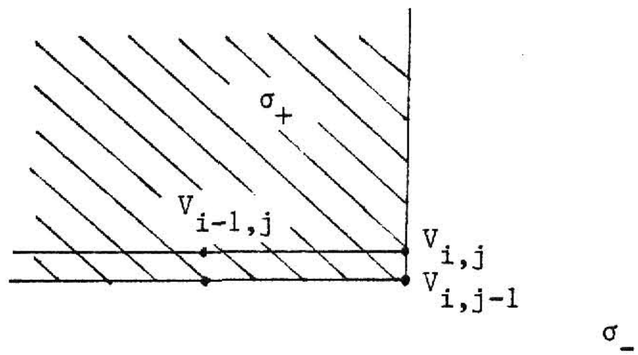
and

$$\Delta x_2 = 1\ ,\ \Delta x_3 = \frac{1}{8}\ ,\ \Delta y_2 = \frac{7}{8}\ ,\ \Delta y_3 = \frac{1}{8}\ ,$$

the finite difference equation is obtained as

$$V_{i,j} = (7V_{i+1,j} + 56V_{i-1,j} + 9V_{i,j+1} + 63V_{i,j-1})/135 \ .$$

<u>Case 11</u>: A boundary point



From Eqn. 16 , $\sigma_+ E_{n+} = \sigma_- E_{n-}$

and with $\sigma_- = 0$, we get $E_{n+} = 0$.

Approximating E_{n+} by

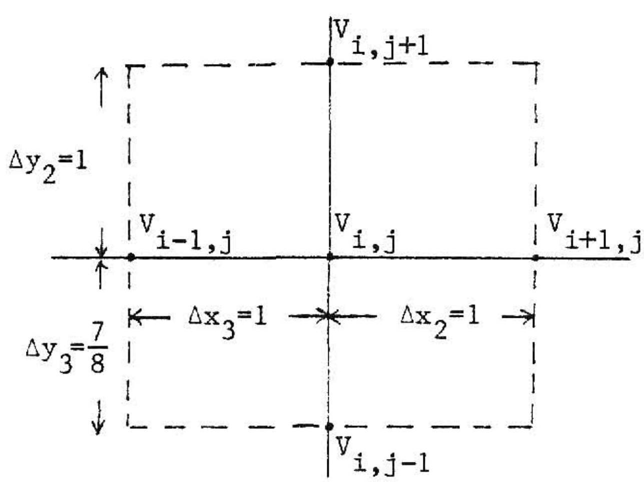
$$E_{n+} = [(-\nabla V)_+ \cdot \hat{n}]$$

$$\text{where } \nabla V_+ = \frac{V_{i,j}-V_{i-1,j}}{\Delta x}\hat{x} + \frac{V_{i,j}-V_{i,j-1}}{\Delta y}\hat{y}$$

and $\hat{n} = \hat{x}$

we get $V_{i,j} = V_{i-1,j}$.

<u>Case 12</u>: An interior grid point in a homogeneous region



From Eqn. 14

$$\left(\frac{1}{\Delta x_1}\right)\left[\frac{1}{\Delta x_2}\sigma_{i+\frac{1}{2},j}\left(V_{i+1,j}-V_{i,j}\right)-\frac{1}{\Delta x_3}\sigma_{i-\frac{1}{2},j}\left(V_{i,j}-V_{i-1,j}\right)\right]+$$

$$\left(\frac{1}{\Delta y_1}\right)\left[\frac{1}{\Delta y_2}\sigma_{i,j+\frac{1}{2}}\left(V_{i,j+1}-V_{i,j}\right)-\frac{1}{\Delta y_3}\sigma_{i,j-\frac{1}{2}}\left(V_{i,j}-V_{i,j-1}\right)\right]=0$$

where $\Delta x_1 = \frac{1}{2}(\Delta x_2+\Delta x_3)$, $\Delta y_1 = \frac{1}{2}(\Delta y_2+\Delta y_3)$,

and

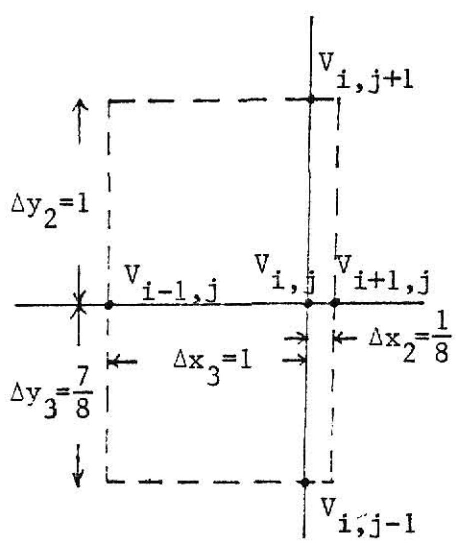
$$\Delta x_2 = \Delta x_3 = \Delta y_2 = 1 \;,\; \Delta y_3 = \frac{7}{8} \;,$$

$$\sigma_{i+\frac{1}{2},j} = \sigma_{i-\frac{1}{2},j} = \sigma_{i,j+\frac{1}{2}} = \sigma_{i,j-\frac{1}{2}} \;,$$

the finite difference equation is obtained as

$$V_{i,j} = (105V_{i+1,j} + 105V_{i-1,j} + 112V_{i,j+1} + 128V_{i,j-1})/450 \;.$$

Case 13: An interior grid point in a homogeneous region



From Eqn. 14

$$\left(\frac{1}{\Delta x_1}\right)\left[\frac{1}{\Delta x_2}\sigma_{i+\frac{1}{2},j}\left(V_{i+1,j}-V_{i,j}\right)-\frac{1}{\Delta x_3}\sigma_{i-\frac{1}{2},j}\left(V_{i,j}-V_{i-1,j}\right)\right]+$$

$$\left(\frac{1}{\Delta y_1}\right)\left[\frac{1}{\Delta y_2}\sigma_{i,j+\frac{1}{2}}\left(V_{i,j+1}-V_{i,j}\right)-\frac{1}{\Delta y_3}\sigma_{i,j-\frac{1}{2}}\left(V_{i,j}-V_{i,j-1}\right)\right]=0$$

where $\Delta x_1 = \frac{1}{2}(\Delta x_2+\Delta x_3)$, $\Delta y_1 = \frac{1}{2}(\Delta y_2+\Delta y_3)$

and

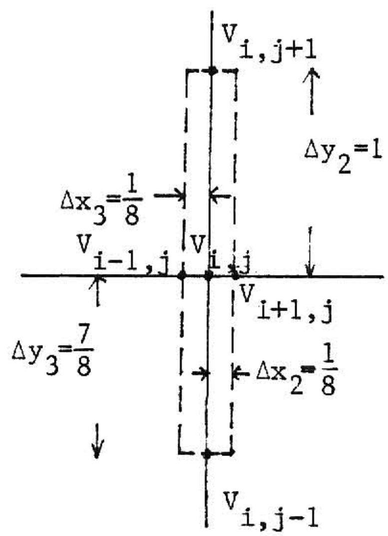
$$\Delta x_2 = \frac{1}{8}\, , \Delta x_3 = 1\, , \;\; \Delta y_2 = 1\, , \Delta y_3 = \frac{7}{8}$$

$$\sigma_{i+\frac{1}{2},j} = \sigma_{i-\frac{1}{2},j} = \sigma_{i,j+\frac{1}{2}} = \sigma_{i,j-\frac{1}{2}}\, ,$$

the finite difference equation is obtained as

$$V_{i,j} = (280V_{i+1,j} + 35V_{i-1,j} + 21V_{i,j+1} + 26V_{i,j-1})/360\, .$$

<u>Case 14</u>: An interior grid point in a homogeneous region



From Eqn. 14

$$(\frac{1}{\Delta x_1})\ [\frac{1}{\Delta x_2}\ \sigma_{i+\frac{1}{2},j}\ (V_{i+1,j}-V_{i,j}) - \frac{1}{\Delta x_3}\ \sigma_{i-\frac{1}{2},j}\ (V_{i,j}-V_{i-1,j})] +$$

$$(\frac{1}{\Delta y_1})\ [\frac{1}{\Delta y_2}\ \sigma_{i,j+\frac{1}{2}}\ (V_{i,j+1}-V_{i,j}) - \frac{1}{\Delta y_3}\ \sigma_{i,j-\frac{1}{2}}\ (V_{i,j}-V_{i,j-1})] = 0$$

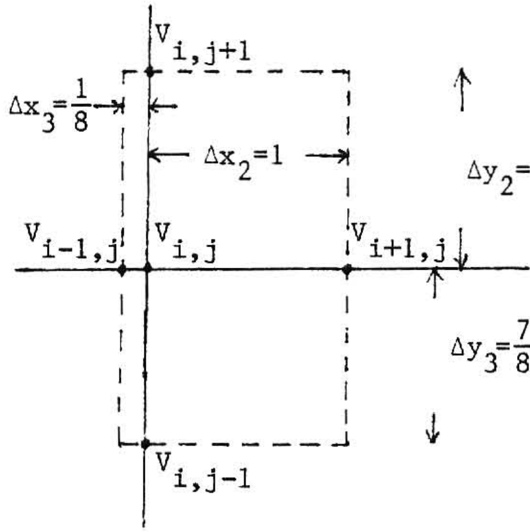
where $\Delta x_1 = \frac{1}{2}(\Delta x_2+\Delta x_3)$, $\Delta y_1 = \frac{1}{2}(\Delta y_2+\Delta y_3)$

$$\sigma_{i+\frac{1}{2},j} = \sigma_{i-\frac{1}{2},j} = \sigma_{i,j+\frac{1}{2}} = \sigma_{i,j-\frac{1}{2}}\ ,$$

the finite difference equation is obtained as

$$V_{i,j} = (420V_{i+1,j} + 420V_{i-1,j} + 7V_{i,j+1} + 8V_{i,j-1})/855\, .$$

<u>Case 15</u>: An interior grid point in a homogeneous region



From Eqn. 14

$$\left(\frac{1}{\Delta x_1}\right)\left[\frac{1}{\Delta x_2}\sigma_{i+\frac{1}{2},j}\left(V_{i+1,j}-V_{i,j}\right)-\frac{1}{\Delta x_3}\sigma_{i-\frac{1}{2},j}\left(V_{i,j}-V_{i-1,j}\right)\right]+$$

$$\left(\frac{1}{\Delta y_1}\right)\left[\frac{1}{\Delta y_2}\sigma_{i,j+\frac{1}{2}}\left(V_{i,j+1}-V_{i,j}\right)-\frac{1}{\Delta y_3}\sigma_{i,j-\frac{1}{2}}\left(V_{i,j}-V_{i,j-1}\right)\right]=0$$

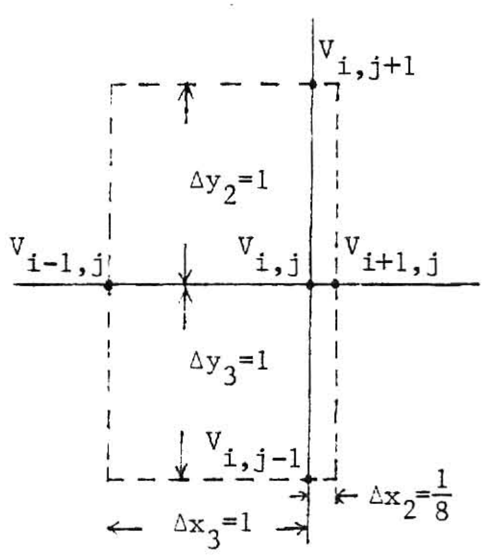
where $\Delta x_1 = \frac{1}{2}(\Delta x_2+\Delta x_3)$, $\Delta y_1 = \frac{1}{2}(\Delta y_2+\Delta y_3)$

$\sigma_{i+\frac{1}{2},j} = \sigma_{i-\frac{1}{2},j} = \sigma_{i,j+\frac{1}{2}} = \sigma_{i,j-\frac{1}{2}}$,

the finite difference equation obtained is

$$V_{i,j} = (35V_{i+1,j} + 280V_{i-1,j} + 21V_{i,j+1} + 24V_{i,j-1})/360 .$$

<u>Case 16</u>: An interior grid point in a homogeneous region



From Eqn. 14

$$\left(\frac{1}{\Delta x_1}\right)\left[\frac{1}{\Delta x_2}\sigma_{i+\frac{1}{2},j}\left(V_{i+1,j}-V_{i,j}\right)-\frac{1}{\Delta x_3}\sigma_{i-\frac{1}{2},j}\left(V_{i,j}-V_{i-1,j}\right)\right]+$$

$$\left(\frac{1}{\Delta y_1}\right)\left[\frac{1}{\Delta y_2}\sigma_{i,j+\frac{1}{2}}\left(V_{i,j+1}-V_{i,j}\right)-\frac{1}{\Delta y_3}\sigma_{i,j-\frac{1}{2}}\left(V_{i,j}-V_{i,j-1}\right)\right]=0$$

where $\Delta x_1 = \frac{1}{2}(\Delta x_2+\Delta x_3)$, $\Delta y_1 = \frac{1}{2}(\Delta y_2+\Delta y_3)$

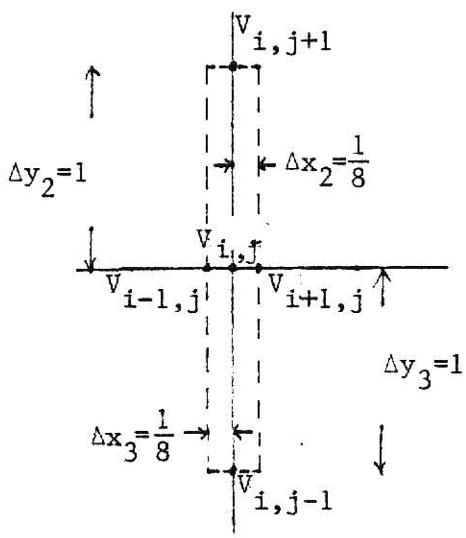
$\sigma_{i+\frac{1}{2},j} = \sigma_{i-\frac{1}{2},j} = \sigma_{i,j+\frac{1}{2}} = \sigma_{i,j-\frac{1}{2}}$,

and $\Delta x_2 = \frac{1}{8}$, $\Delta x_3 = 1$, $\Delta y_2 = 1$, $\Delta y_3 = 1$,

the finite difference equation is obtained as

$$V_{i,j} = (128V_{i+1,j} + 16V_{i-1,j} + 9V_{i,j+1} + 9V_{i,j-1})/162 .$$

<u>Case 17</u>: An interior grid point in a homogeneous region



From Eqn. 14

$$\left(\frac{1}{\Delta x_1}\right)\left[\frac{1}{\Delta x_2}\sigma_{i+\frac{1}{2},j}\left(V_{i+1,j}-V_{i,j}\right)-\frac{1}{\Delta x_3}\sigma_{i-\frac{1}{2},j}\left(V_{i,j}-V_{i-1,j}\right)\right]+$$

$$\left(\frac{1}{\Delta y_1}\right)\left[\frac{1}{\Delta y_2}\sigma_{i,j+\frac{1}{2}}\left(V_{i,j+1}-V_{i,j}\right)-\frac{1}{\Delta y_3}\sigma_{i,j-\frac{1}{2}}\left(V_{i,j}-V_{i,j-1}\right)\right]=0$$

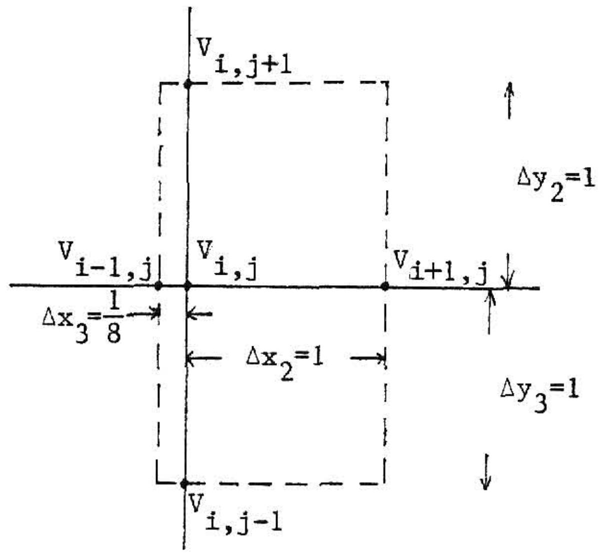
where $\Delta x_1 = \frac{1}{2}(\Delta x_2+\Delta x_3)$, $\Delta y_1 = \frac{1}{2}(\Delta y_2+\Delta y_3)$

$\sigma_{i+\frac{1}{2},j} = \sigma_{i-\frac{1}{2},j} = \sigma_{i,j+\frac{1}{2}} = \sigma_{i,j-\frac{1}{2}}$

and $\Delta x_2 = \Delta x_3 = \frac{1}{8}$, $\Delta y_2 = \Delta y_3 = 1$, the finite difference equation is obtained as

$$V_{i,j} = (64V_{i+1,j} + 64V_{i-1,j} + V_{i,j+1} + V_{i,j-1})/130.$$

Case 18: An interior grid point in a homogeneous region



From Eqn. 14

$$\left(\frac{1}{\Delta x_1}\right)\left[\frac{1}{\Delta x_2}\sigma_{i+\frac{1}{2},j}(V_{i+1,j}-V_{i,j}) - \frac{1}{\Delta x_3}\sigma_{i-\frac{1}{2},j}(V_{i,j}-V_{i-1,j})\right] +$$

$$\left(\frac{1}{\Delta y_1}\right)\left[\frac{1}{\Delta y_2}\sigma_{i,j+\frac{1}{2}}(V_{i,j+1}-V_{i,j}) - \frac{1}{\Delta y_3}\sigma_{i,j-\frac{1}{2}}(V_{i,j}-V_{i,j-1})\right] = 0$$

where $\Delta x_1 = \frac{1}{2}(\Delta x_2+\Delta x_3)$, $\Delta y_1 = \frac{1}{2}(\Delta y_2+\Delta y_3)$,

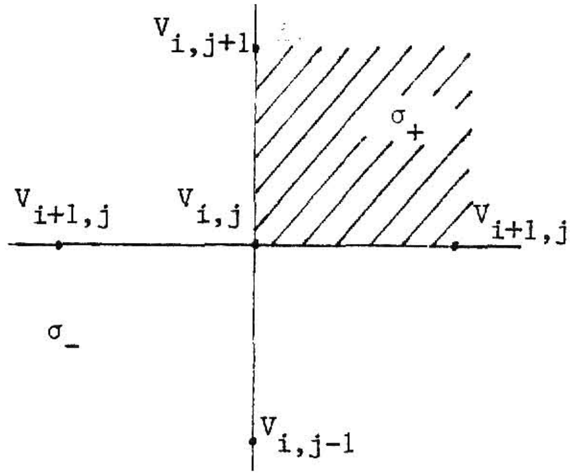
$$\sigma_{i+\frac{1}{2},j} = \sigma_{i-\frac{1}{2},j} = \sigma_{i,j+\frac{1}{2}} = \sigma_{i,j-\frac{1}{2}} ,$$

and $\Delta x_2 = 1$, $\Delta x_3 = \frac{1}{8}$, $\Delta y_2 = \Delta y_3 = 1$,

the finite difference equation obtained is

$$V_{i,j} = (16\,V_{i+1,j} + 128V_{i-1,j} + 9V_{i,j+1} + 9V_{i,j-1})/162 .$$

Case 19: An interior grid point at discontinuity of conductivity



From Eqns. 16, 17 and 18 we have

$$\sigma_+ \left[(-\nabla V)_+ \cdot \hat{n}\right] = \sigma_- \left[(-\nabla V)_- \cdot \hat{n}\right] .$$

Approximating ∇V_+ by

$$\nabla V_+ = \frac{V_{i+1,j} - V_{i,j}}{\Delta x_1} \hat{x} + \frac{V_{i,j+1} - V_{i,j}}{\Delta y_1} \hat{y} ,$$

approximating ∇V_- by

$$\nabla V_- = \frac{V_{i,j} - V_{i-1,j}}{\Delta x_2} \hat{x} + \frac{V_{i,j} - V_{i,j-1}}{\Delta y_2} \hat{y} ,$$

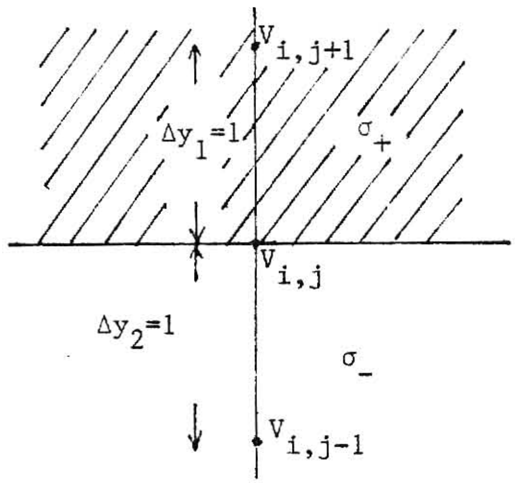
and with $\sigma_+ = C$, $\sigma_- = 1$, $\Delta x_1 = \Delta x_2 = \Delta y_1 = \Delta y_2 = 1$

and $\hat{n} = -\frac{1}{\sqrt{2}} \hat{x} - \frac{1}{\sqrt{2}} \hat{y}$

the finite difference equation obtained is

$$V_{i,j} = (C\, V_{i+1,j} + V_{i-1,j} + C\, V_{i,j+1} + V_{i,j-1})/(2C+2) .$$

Case 20: An interior grid point at discontinuity of conductivity



From Eqns. 16, 17, and 18 we have

$$\sigma_+ \; [(-\nabla V)_+ \cdot \hat{n}] = \sigma_- \; [(-\nabla V)_- \cdot \hat{n}] \; .$$

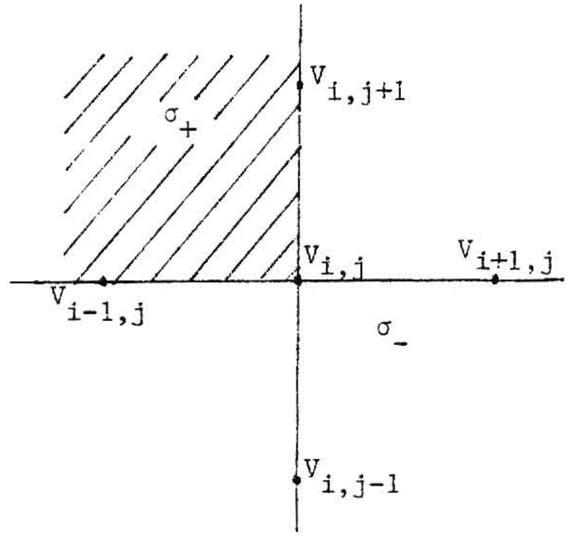
Here, with $\sigma_+ = C$, $\sigma_- = 1$ and $\hat{n} = -\hat{y}$, the above equation can be approxmiated by

$$C \cdot [\frac{V_{i,j+1} - V_{i,j}}{\Delta y_1} = [\frac{V_{i,j} - V_{i,j-1}}{\Delta y_2}] \; ,$$

for $\Delta y_1 = \Delta y_2 = 1$, the finite difference equation is obtained as

$$V_{i,j} = (C \; V_{i,j+1} + V_{i,j-1})/(C+1) \; .$$

Case 21: An interior grid point at the discontinuity of conductivity.



From Eqns. 16, 17, and 18, we have

$$\sigma_+ \left[(-\nabla V)_+ \cdot \hat{n}\right] = \sigma_- \left[(-\nabla V)_- \cdot \hat{n}\right] .$$

Approximating ∇V_+ by

$$\nabla V_+ = \frac{V_{i,j} - V_{i-1,j}}{\Delta x_1} \hat{x} + \frac{V_{i,j+1} - V_{i,j}}{\Delta y_1} \hat{y} ,$$

approximating ∇V_- by

$$\nabla V_- = \frac{V_{i+1,j} - V_{i,j}}{\Delta x_2} \hat{x} + \frac{V_{i,j} - V_{i,j-1}}{\Delta y_2} \hat{y} .$$

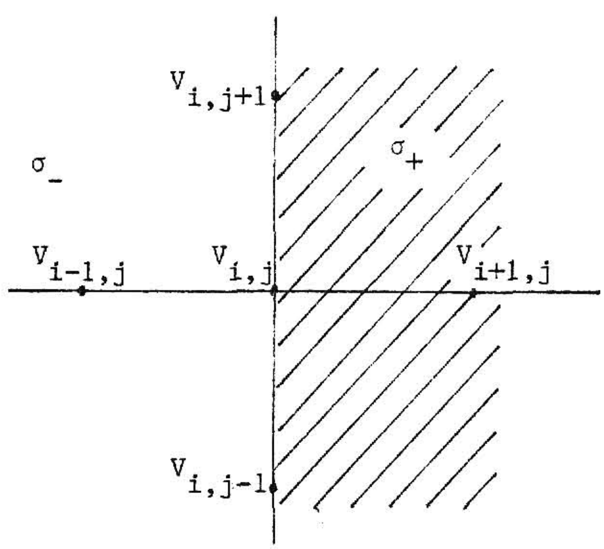
and with $\sigma_+ = C$, $\sigma_- = 1$, $\Delta x_1 = \Delta x_2 = \Delta y_1 = \Delta y_2 = 1$,

and $\hat{n} = \frac{1}{\sqrt{2}} \hat{x} - \frac{1}{\sqrt{2}} \hat{y}$,

the finite difference equation obtained is

$$V_{i,j} = (V_{i+1,j} + C\, V_{i-1,j} + V_{i,j-1} + C\, V_{i,j+1})/(2C+2) .$$

Case 22: An interior grid point at the discontinuity of conductivity



From Eqns. 16, 17, and 18, we have

$$\sigma_+ \, [(-\nabla V)_+ \cdot \hat{n}] = \sigma_- \, [(-\nabla V)_- \cdot \hat{n}] \, .$$

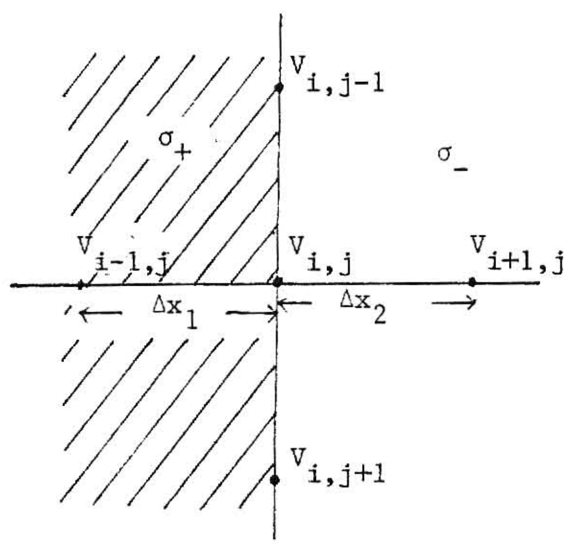
Here, with $\sigma_+ = C$, $\sigma_- = 1$ and $\hat{n} = -\hat{x}$, the above equation can be approximated by

$$C \, [\frac{V_{i+1,j} - V_{i,j}}{\Delta x_1}] = [\frac{V_{i,j} - V_{i-1,j}}{\Delta x_2}] \, .$$

For $\Delta x_1 = \Delta x_2 = 1$, the finite difference equation is obtained as

$$V_{i,j} = (C \, V_{i+1,j} + V_{i-1,j})/(C+1) \, .$$

Case 23: An interior grid point at the discontinuity of conductivity.



From Eqns. 16, 17, and 18, we have

$$\sigma_+ \, [(-\nabla V)_+ \cdot \hat{n}] = \sigma_- \, [(-\nabla V)_- \cdot \hat{n}] \, ,$$

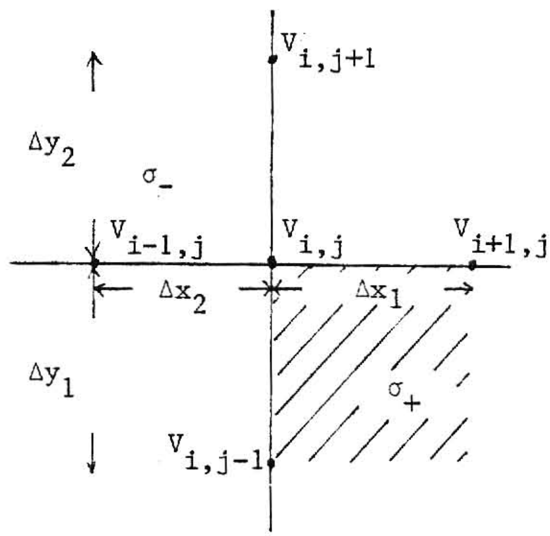
with $\sigma_+ = C$, $\sigma_- = 1$ and $\hat{n} = \hat{x}$, the above equation can be approximated by

$$C \, [\frac{V_{i,j} - V_{i-1,j}}{\Delta x_1}] = [\frac{V_{i+1,j} - V_{i,j}}{\Delta x_2}] \, .$$

Here, $\Delta x_1 = \Delta x_2 = 1$, the finite difference equation is obtained as

$$V_{i,j} = (C\, V_{i-1,j} + V_{i+1,j})/(C+1) .$$

Case 24: An interior grid point at the discontinuity of conductivity.



From Eqns. 16, 17, and 18, we have

$$\sigma_+ \,[(-\nabla V)_+ \cdot \hat{n}] = \sigma_- \,[-(\nabla V)_- \cdot \hat{n}]$$

Approximating ∇V_+ by

$$\nabla V_+ = \frac{V_{i+1,j} - V_{i,j}}{\Delta x_1}\,\hat{x} + \frac{V_{i,j} - V_{i,j-1}}{\Delta y_1}\,\hat{y} ,$$

approximating ∇V_- by

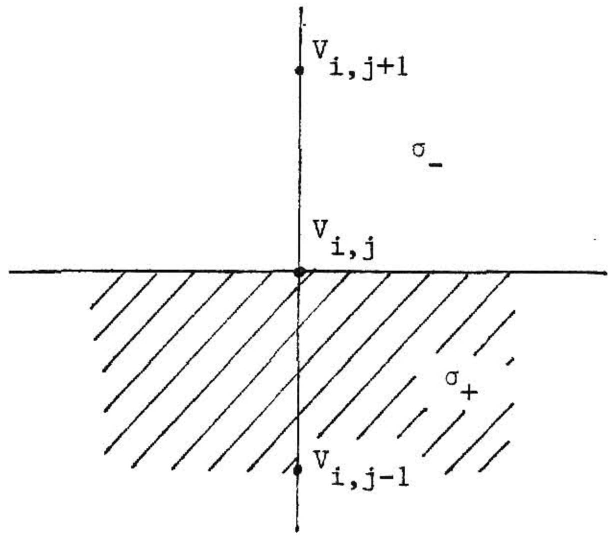
$$\nabla V_- = \frac{V_{i,j} - V_{i-1,j}}{\Delta x_2}\,\hat{x} + \frac{V_{i,j+1} - V_{i,j}}{\Delta y_2}\,\hat{y} ,$$

and with $\sigma_+ = C$, $\sigma_- = 1$, $\Delta x_1 = \Delta x_2 = \Delta y_1 = \Delta y_2 = 1$,

and $\hat{n} = -\frac{1}{\sqrt{2}}\,\hat{x} + \frac{1}{\sqrt{2}}\,\hat{y}$, the finite difference equation obtained is

$$V_{i,j} = (V_{i-1,j} + V_{i,j+1} + C\,V_{i+1,j} + C\,V_{i,j-1})/(2C+2)\ .$$

<u>Case 25</u>: An interior grid point at the discontinuity of conductivity.



From Eqns. 16, 17, and 18, we have

$$\sigma_+\ [(-\nabla V)_+ \cdot \hat{n}] = \sigma_-\ [(-\nabla V)_- \cdot \hat{n}]\ ,$$

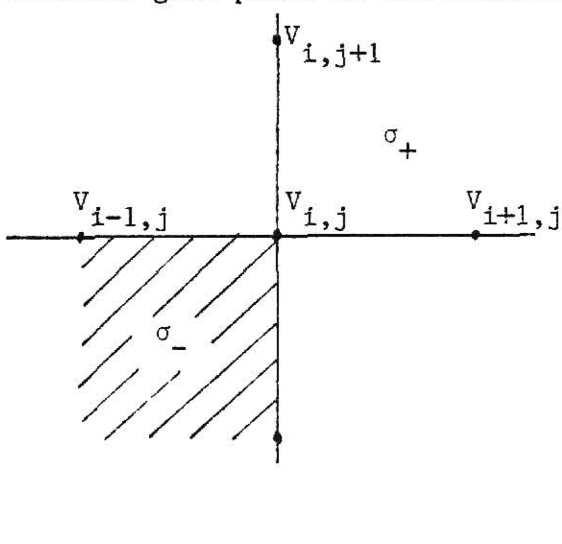
with $\sigma_+ = C$, $\sigma_- = 1$, and $\hat{n} = \hat{y}$, the above equation can be approximated by

$$C\ [\frac{V_{i,j} - V_{i,j-1}}{\Delta y_1}] = [\frac{V_{i,j+1} - V_{i,j}}{\Delta y_2}]\ .$$

Here $\Delta y_1 = \Delta y_2 = 1$, the finite difference equation is obtained as

$$V_{i,j} = (V_{i,j+1} + C\,V_{i,j-1})/(C+1)\ .$$

<u>Case 26</u>: An interior grid point at the dicsontinuity of conductivity.



From Eqn. 16, 17, and 18, we have

$$\sigma_+ \, [(-\nabla V)_+ \cdot \hat{n}] = \sigma_- \, [(-\nabla V)_- \cdot \hat{n}] \, .$$

Approximating ∇V_+ by

$$\nabla V_+ = \frac{V_{i+1,j} - V_{i,j}}{\Delta x_1} \, \hat{x} + \frac{V_{i,j+1} - V_{i,j}}{\Delta y_1} \, \hat{y} \, ,$$

approximating ∇V_- by

$$\nabla V_- = \frac{V_{i,j} - V_{i-1,j}}{\Delta x_2} \, \hat{x} + \frac{V_{i,j} - V_{i,j-1}}{\Delta y_2} \, \hat{y}$$

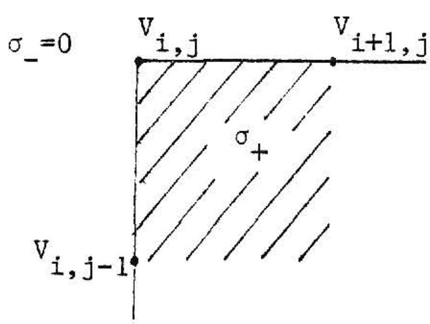
and with $\sigma_+ = 1$, $\sigma_- = C$, $\Delta x_1 = \Delta x_2 = \Delta y_1 = \Delta y_2 = 1$

and $\hat{n} = \frac{1}{\sqrt{2}} \hat{x} + \frac{1}{\sqrt{2}} \hat{y}$

the finite difference equation is obtained as

$$V_{i,j} = (C \, V_{i-1,j} + V_{i+1,j} + V_{i,j+1} + C \, V_{i,j-1})/(2C+2) \, .$$

Case 27: A boundary point.



From Eqns. 16, 17, and 18, we have

$$\sigma_+ \, [(-\nabla V)_+ \cdot \hat{n}] = \sigma_- \, [(-\nabla V)_- \cdot \hat{n}] \, .$$

Here $\sigma_+ = 1$ and $\sigma_- = 0$; thus we get

$$\sigma_+ \, [(-\nabla V)_+ \cdot \hat{n}] = 0 \, .$$

Approximating ∇V_+ by

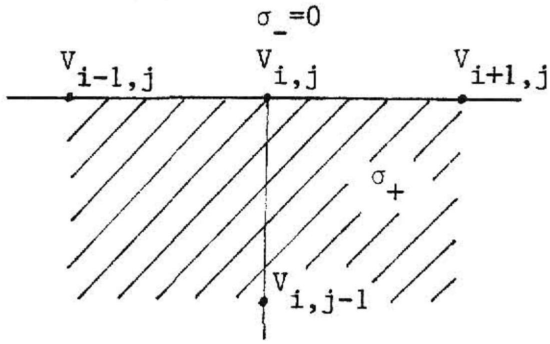
$$\nabla V_+ = \left(\frac{V_{i+1,j} - V_{i,j}}{\Delta x}\right) \hat{x} + \left(\frac{V_{i,j} - V_{i,j-1}}{\Delta y}\right) \hat{y} \, ,$$

and with $\Delta x = \Delta y = 1$, $\hat{n} = -\frac{1}{\sqrt{2}}\hat{x} + \frac{1}{\sqrt{2}}\hat{y}$,

the finite difference equation is obtained as

$$V_{i,j} = (V_{i+1,j} + V_{i,j-1})/2 .$$

Case 28: A boundary point.



From Eqns. 16, 17, and 18, we have

$$\sigma_+ [(-\nabla V)_+ \cdot \hat{n}] = \sigma_- [(-\nabla V)_- \cdot \hat{n}] ,$$

with $\sigma_- = 0$, we have

$$[(-\nabla V)_+ \cdot \hat{n}] = 0 .$$

Approximating ∇V_+ by

$$\nabla V_+ = [\frac{V_{i+1,j} - V_{i,j}}{\Delta x}\hat{x} + \frac{V_{i,j} - V_{i,j-1}}{\Delta y}\hat{y}] ,$$

here $\hat{n} = \hat{y}$, the finite difference equation is obtained as

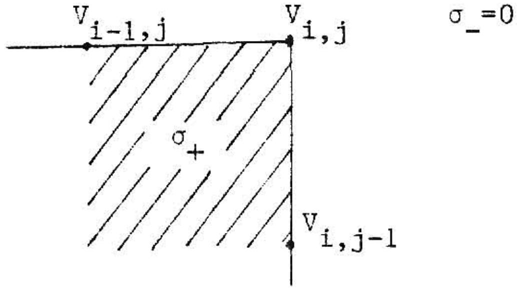
$$V_{i,j} = V_{i,j-1} ,$$

Case 29: A point at the reference electrode.

For point at the reference electrode, the finite different equatin is:

$$V_{i,j} = 0$$

Case 30: A boundary point.



From Eqns. 16, 17, and 18, we have

$$\sigma_+ \, [(-\nabla V)_+ \cdot \hat{n}] = \sigma_- \, [(-\nabla V)_- \cdot \hat{n}]$$

with $\sigma_- = 0$, we get

$$\sigma_+ \, [(-\nabla V_+) \cdot \hat{n}] = 0$$

Approximating ∇V_+ by

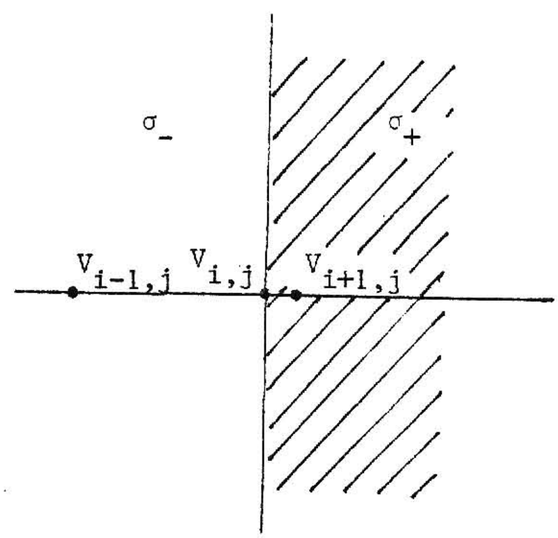
$$\nabla V_+ = [\frac{V_{i,j} - V_{i-1,j}}{\Delta x} \hat{x} + \frac{V_{i,j} - V_{i,j-1}}{\Delta y} \hat{y}]$$

and with $\Delta x = \Delta y = 1$, $\hat{n} = \frac{1}{\sqrt{2}} \hat{x} + \frac{1}{\sqrt{2}} \hat{y}$,

the finite difference equation is obtained as

$$V_{i,j} = (V_{i-1,j} + V_{i,j-1})/2 \; .$$

Case 31: An interior grid point at the discontinuty of conductivity.



From Eqns. 16, 17, and 18, we have

$$\sigma_+ \, [(-\nabla V)_+ \cdot \hat{n}] = \sigma_- \, [(-\nabla V)_- \cdot \hat{n}] \; ,$$

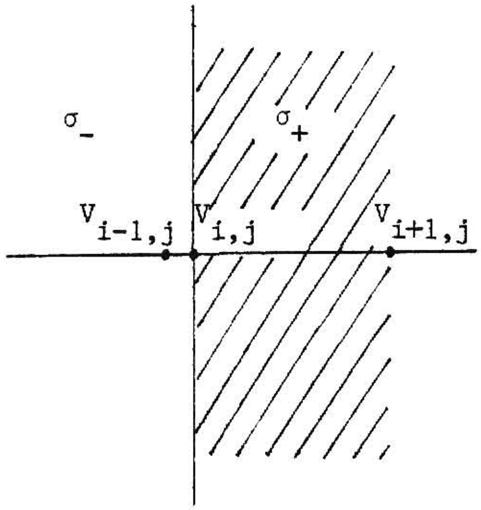
with $\sigma_+ = C$, $\sigma_- = 1$ and $\hat{n} = -\hat{x}$, the above equation can be approximated by

$$C \, [\frac{V_{i+1,j} - V_{i,j}}{\Delta x_1}] = [\frac{V_{i,j} - V_{i-1,j}}{\Delta x_2}] \; .$$

Here $\Delta x_1 = \frac{1}{8}$, $\Delta x_2 = 1$. The finite difference equation is obtained as

$$V_{i,j} = (V_{i-1,j} + 8C\, V_{i+1,j})/(1+8C) .$$

<u>Case 32</u>: An interior grid point at the discontinuity of conductivity.



From Eqns. 16, 17, and 18, we have

$$\sigma_+ \, [(-\nabla V)_+ \cdot \hat{n}] = \sigma_- \, [(-\nabla V)_- \cdot \hat{n}] ,$$

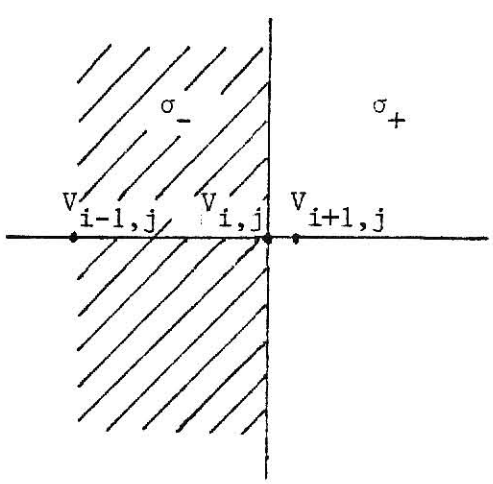
with $\sigma_+ = C$, $\sigma_- = 1$, and $\hat{n} = -\hat{x}$, the above equation can be approximated by

$$C \left[\frac{V_{i+1,j} - V_{i,j}}{\Delta x_1}\right] = \left[\frac{V_{i,j} - V_{i-1,j}}{\Delta x_2}\right] .$$

Here $\Delta x_1 = 1$, $\Delta x_2 = \frac{1}{8}$. The finite difference equation is obtained as

$$V_{i,j} = (C\, V_{i+1,j} + 8\, V_{i-1,j})/(8+C) .$$

<u>Case 33</u>: An interior grid point at the discontinuity of conductivity.



From Eqns. 16, 17, and 18, we have

$$\sigma_+ \, [(-\nabla V)_+ \cdot \hat{n}] = \sigma_- \, [(-\nabla V)_- \cdot \hat{n}] \, ,$$

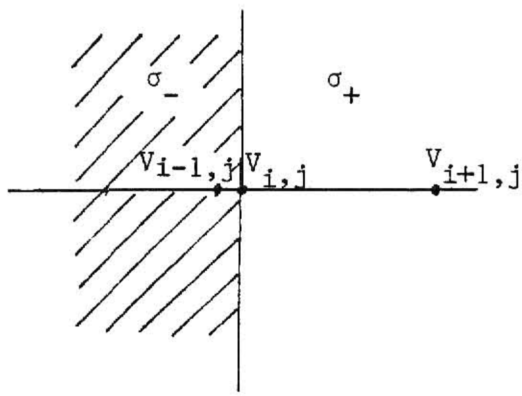
with $\sigma_+ = 1$ and $\sigma_- = C$ and $\hat{n} = \hat{x}$, the above equation can be approximated by

$$[\frac{V_{i+1,j} - V_{i,j}}{\Delta x_1}] = C \, [\frac{V_{i,j} - V_{i-1,j}}{\Delta x_2}] \, .$$

Here $\Delta x_1 = \frac{1}{8}$, $\Delta x_2 = 1$. The finite difference equation is obtained as

$$V_{i,j} = (8 \, V_{i+1,j} + C \, V_{i-1,j})/(8+C) \, .$$

Case 34: An interior grid point at the discontinuity of conductivity



From Eqns. 16, 17, and 18, we have

$$\sigma_+ \, [(-\nabla V)_+ \cdot \hat{n}] = \sigma_- \, [(-\nabla V)_- \cdot \hat{n}] \, .$$

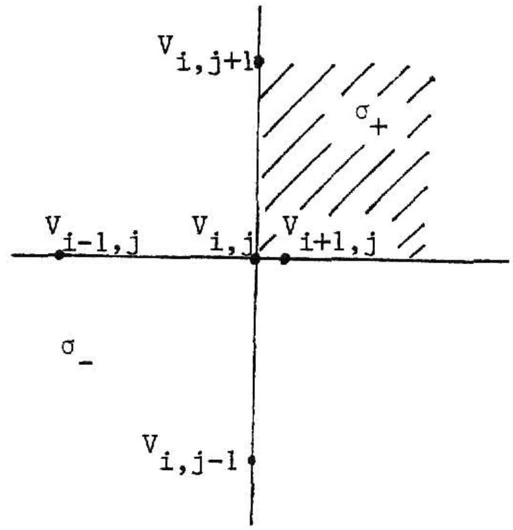
With $\sigma_+ = 1$, $\sigma_- = C$ and $\hat{n} = \hat{x}$, the above equation can be approximated by

$$[\frac{V_{i+1,j} - V_{i,j}}{\Delta x_1}] = C \, [\frac{V_{i,j} - V_{i-1,j}}{\Delta x_2}] \, .$$

Here $\Delta x_1 = 1$, $\Delta x_2 = \frac{1}{8}$. The finite difference equation is obtained as

$$V_{i,j} = (V_{i+1,j} + 8C \, V_{i-1,j})/(1+8C) \, .$$

Case 35: An interior grid point at the discontinuity of conductivity.



From Eqns. 16, 17, and 18, we have

$$\sigma_+ \,[(-\nabla V)_+ \cdot \hat{n}] = \sigma_- \,[(-\nabla V)_- \cdot \hat{n}] \;.$$

Approximating ∇V_+ by

$$\nabla V_+ = \frac{V_{i+1,j} - V_{i,j}}{\Delta x_1}\,\hat{x} + \frac{V_{i,j+1} - V_{i,j}}{\Delta y_1}\,\hat{y} \;,$$

approximating ∇V_- by

$$\nabla V_- = \frac{V_{i,j} - V_{i-1,j}}{\Delta x_2}\,\hat{x} + \frac{V_{i,j} - V_{i,j-1}}{\Delta y_2}\,\hat{y} \;,$$

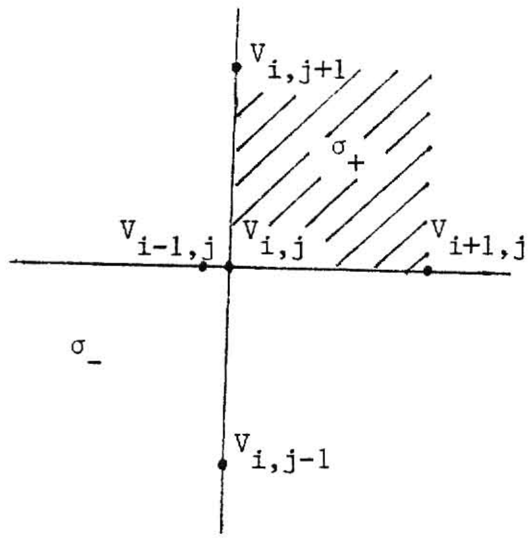
and with $\Delta x_1 = \frac{1}{8}$, $\Delta y_1 = 1$, $\Delta x_2 = 1$, $\Delta y_2 = 1$,

$$\sigma_+ = C, \; \sigma_- = 1 \text{ and } \hat{n} = -\frac{1}{\sqrt{2}}\,\hat{x} - \frac{1}{\sqrt{2}}\,\hat{y} \;,$$

the finite difference equation is obtained as

$$V_{i,j} = (8C\, V_{i+1,j} + C\, V_{i,j+1} + V_{i-1,j} + V_{i,j-1})/(9C+2) \;.$$

Case 36: An interior grid point at the discontinuity of conductivity.



From Eqns. 16, 17, and 18, we have

$$\sigma_+ \, [(-\nabla V)_+ \cdot \hat{n}] = \sigma_- \, [(-\nabla V)_- \cdot \hat{n}]$$

Approximating ∇V_+ by

$$\nabla V_+ = \frac{V_{i+1,j} - V_{i,j}}{\Delta x_1}\,\hat{x} + \frac{V_{i,j+1} - V_{i,j}}{\Delta y_1}\,\hat{y} \quad ,$$

approximating ∇V_- by

$$\nabla V_- = \frac{V_{i,j} - V_{i-1,j}}{\Delta x_2}\,\hat{x} + \frac{V_{i,j} - V_{i,j-1}}{\Delta y_2}\,\hat{y} \quad ,$$

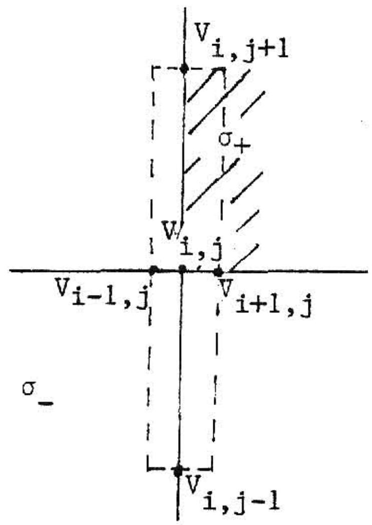
and with $\sigma_+ = C$, $\sigma_- = 1$, $\Delta x_1 = 1$, $\Delta x_2 = \frac{1}{8}$, $\Delta y_1 = \Delta y_2 = 1$,

and $\hat{n} = -\frac{1}{\sqrt{2}}\,\hat{x} - \frac{1}{\sqrt{2}}\,\hat{y}$

the finite difference equation obtained is

$$V_{i,j} = (C\,V_{i+1,j} + C\,V_{i,j+1} + 8\,V_{i-1,j} + V_{i,j-1})/(2C+9) \ .$$

Case 37: An interior grid point at the discontinuity of conductivity.



From Eqns. 16, 17, and 18, we have

$$\sigma_+ \left[(-\nabla V)_+ \cdot \hat{n}\right] = \sigma_- \left[(-\nabla V)_- \cdot \hat{n}\right] .$$

Approximating ∇V_+ by

$$\nabla V_+ = \frac{V_{i+1,j} - V_{i,j}}{\Delta x_1} \hat{x} + \frac{V_{i,j+1} - V_{i,j}}{\Delta y_1} \hat{y} ,$$

approximating ∇V_- by

$$\nabla V_- = \frac{V_{i,j} - V_{i-1,j}}{\Delta x_2} \hat{x} + \frac{V_{i,j} - V_{i,j-1}}{\Delta y_2} \hat{y} ,$$

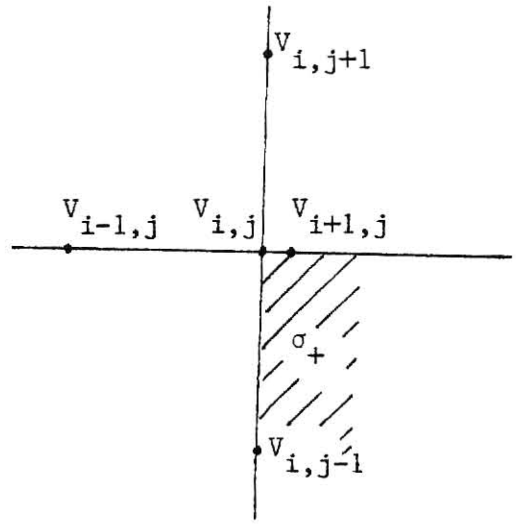
and with $\sigma_+ = C$, $\sigma_- = 1$, $\Delta x_1 = \Delta x_2 = \frac{1}{8}$, $\Delta y_1 = \Delta y_2 = 1$,

and $\hat{n} = -\frac{1}{\sqrt{2}} \hat{x} - \frac{1}{\sqrt{2}} \hat{y}$

the finite difference equation obtained is

$$V_{i,j} = (8C\, V_{i+1,j} + C\, V_{i,j+1} + 8\, V_{i-1,j} + V_{i,j-1})/(9C+9) .$$

<u>Case 38</u>: An interior grid point at the discontinuity of conductivity.



From Eqns. 16, 17, and 18, we have

$$\sigma_+ \, [(-\nabla V)_+ \cdot \hat{n}] = \sigma_- \, [(-\nabla V)_- \cdot \hat{n}] \ .$$

Approximating ∇V_+ by

$$\nabla V_+ = \frac{V_{i+1,j} - V_{i,j}}{\Delta x_1} \hat{x} + \frac{V_{i,j} - V_{i,j-1}}{\Delta y_1} \hat{y} \ ,$$

approximating ∇V_- by

$$\nabla V_- = \frac{V_{i,j} - V_{i-1,j}}{\Delta x_2} \hat{x} + \frac{V_{i,j+1} - V_{i,j}}{\Delta y_2} \hat{y} \ ;$$

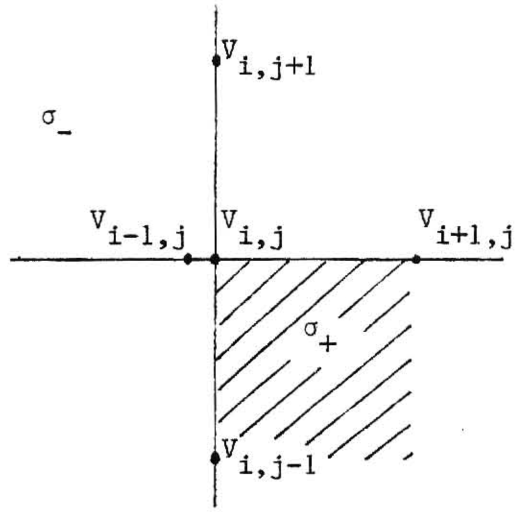
and with $\sigma_+ = C$, $\sigma_- = 1$, $\Delta x_1 = \frac{1}{8}$, $\Delta y_1 = \Delta x_2 = \Delta y_2 = 1$,

and $\hat{n} = -\frac{1}{\sqrt{2}} \hat{x} + \frac{1}{\sqrt{2}} \hat{y}$

the finite difference equation obtained is

$$V_{i,j} = (V_{i-1,j} + V_{i,j+1} + 8C \, V_{i+1,j} + C \, V_{i,j-1})/(9C+2) \ .$$

Case 39: An interior grid point at the discontinuity of conductivity.



From Eqns. 16, 17, and 18, we have

$$\sigma_+ \; [(-\nabla V)_+ \cdot \hat{n}] = \sigma_- \; [(-\nabla V)_- \cdot \hat{n}]$$

Approximating ∇V_+ by

$$\nabla V_+ = \frac{V_{i+1,j} - V_{i,j}}{\Delta x_1} \hat{x} + \frac{V_{i,j} - V_{i,j-1}}{\Delta y_1} \hat{y} ,$$

approximating ∇V_- by

$$\nabla V_- = \frac{V_{i,j} - V_{i-1,j}}{\Delta x_2} \hat{x} + \frac{V_{i,j+1} - V_{i,j}}{\Delta y_2} \hat{y} ,$$

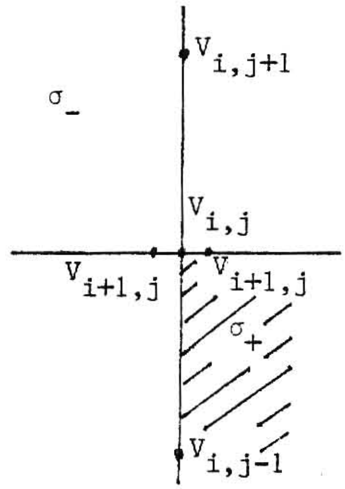
and with $\sigma_+ = C$, $\sigma_- = 1$, $\Delta x_1 = 1$, $\Delta x_2 = \frac{1}{8}$, $\Delta y_1 = \Delta y_2 = 1$,

and $\hat{n} = -\frac{1}{\sqrt{2}} \hat{x} + \frac{1}{\sqrt{2}} \hat{y}$

the finite difference equation obtained is

$$V_{i,j} = (8\, V_{i-1,j} + V_{i,j+1} + C\, V_{i+1,j} + C\, V_{i,j-1})/(2C+9) .$$

Case 40: An interior grid point at the discontinuity of conductivity.



From Eqns. 16, 17, and 18, we have

$$\sigma_+ \, [(-\nabla V)_+ \cdot \hat{n}] = \sigma_- \, [(-\nabla V)_- \cdot \hat{n}] \, .$$

approximating ∇V_+ by

$$\nabla V_+ = \frac{V_{i+1,j} - V_{i,j}}{\Delta x_1} \hat{x} + \frac{V_{i,j} - V_{i,j-1}}{\Delta y_1} \hat{y} \, ,$$

Approximating ∇V_- by

$$\nabla V_- = \frac{V_{i,j} - V_{i-1,j}}{\Delta x_2} \hat{x} + \frac{V_{i,j+1} - V_{i,j}}{\Delta y_2} \hat{y} \, ,$$

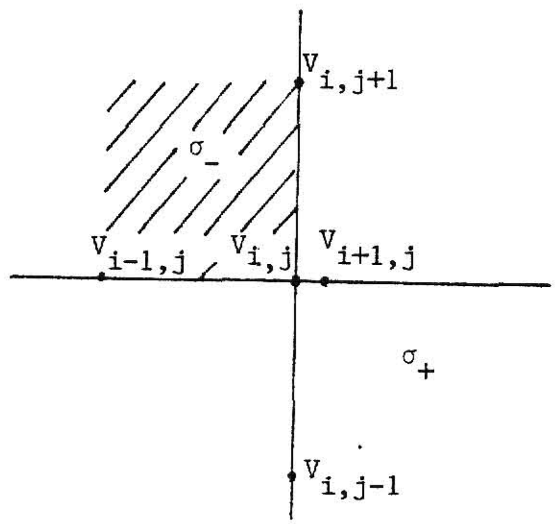
and with $\sigma_+ = C$, $\sigma_- = 1$, $\Delta x_1 = \Delta x_2 = \frac{1}{8}$, $\Delta y_1 = \Delta y_2 = 1$,

and $\hat{n} = -\frac{1}{\sqrt{2}} \hat{x} + \frac{1}{\sqrt{2}} \hat{y}$

the finite difference equation is obtained as

$$V_{i,j} = (8\, V_{i-1,j} + V_{i,j+1} + 8C\, V_{i+1,j} + C\, V_{i,j-1})/(9C+9) \, .$$

Case 41: An interior grid point at the discontinuity of conductivity.



From Eqns. 16, 17, and 18, we have

$$\sigma_+ \, [(-\nabla V)_+ \cdot \hat{n}] = \sigma_- \, [(-\nabla V)_- \cdot \hat{n}]$$

Approximating ∇V_+ by

$$\nabla V_+ = \frac{V_{i+1,j} - V_{i,j}}{\Delta x_1} \hat{x} + \frac{V_{i,j} - V_{i,j-1}}{\Delta y_1} \hat{y} \ ,$$

approximating ∇V_- by

$$\nabla V_- = \frac{V_{i,j} - V_{i-1,j}}{\Delta x_2} \hat{x} + \frac{V_{i,j+1} - V_{i,j}}{\Delta y_2} \hat{y} \ ,$$

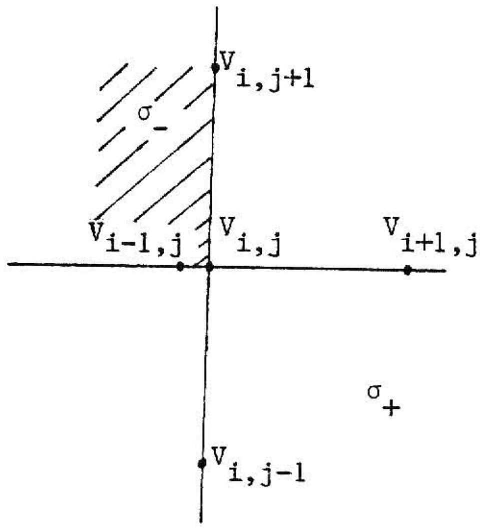
and with $\sigma_+ = 1$, $\sigma_- = C$, $\Delta x_1 = \frac{1}{8}$, $\Delta y_1 = \Delta x_2 = \Delta y_2 = 1$,

$$\hat{n} = \frac{1}{\sqrt{2}} \hat{x} - \frac{1}{\sqrt{2}} \hat{y} \ ,$$

the finite difference equation obtained is

$$V_{i,j} = (C \, V_{i-1,j} + C \, V_{i,j+1} + 8 \, V_{i+1,j} + V_{i,j-1})/(2C+9) \ .$$

Case 42: An interior grid point at the discontinuity of conductivity.



From Eqns. 16, 17, and 18, we have

$$\sigma_+ \left[(-\nabla V)_+ \cdot \hat{n}\right] = \sigma_- \left[(-\nabla V)_- \cdot \hat{n}\right] .$$

Approximating ∇V_+ by

$$\nabla V_+ = \frac{V_{i+1,j} - V_{i,j}}{\Delta x_1} \hat{x} + \frac{V_{i,j} - V_{i,j-1}}{\Delta y_1} \hat{y} ,$$

approximating ∇V_- by

$$\nabla V_- = \frac{V_{i,j} - V_{i-1,j}}{\Delta x_2} \hat{x} + \frac{V_{i,j+1} - V_{i,j}}{\Delta y_2} \hat{y} ,$$

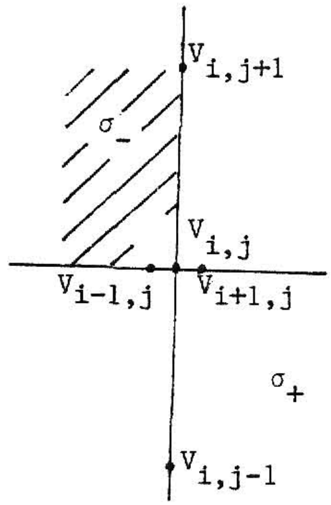
and with $\sigma_+ = 1$, $\sigma_- = C$, $\Delta x_1 = 1$, $\Delta x_2 = \frac{1}{8}$, $\Delta y_1 = \Delta y_2 = 1$,

and $\hat{n} = \frac{1}{\sqrt{2}} \hat{x} - \frac{1}{\sqrt{2}} \hat{y}$,

the finite difference equation is obtained as

$$V_{i,j} = (8C\, V_{i-1,j} + C\, V_{i,j+1} + V_{i+1,j} + V_{i,j-1})/(9C+2) .$$

Case 43: An interior grid point at the discontinuity of conductivity.



From Eqns. 16, 17, and 18, we have

$$\sigma_+ \, [(-\nabla V)_+ \cdot \hat{n}] = \sigma_- \, [(-\nabla V)_- \cdot \hat{n}] \, .$$

Approximating ∇V_+ by

$$\nabla V_+ = \frac{V_{i+1,j} - V_{i,j}}{\Delta x_1} \hat{x} + \frac{V_{i,j} - V_{i,j-1}}{\Delta y_1} \hat{y} \, ,$$

approximating ∇V_- by

$$\nabla V_- = \frac{V_{i,j} - V_{i-1,j}}{\Delta x_2} \hat{x} + \frac{V_{i,j+1} - V_{i,j}}{\Delta y_2} \hat{y} \, ,$$

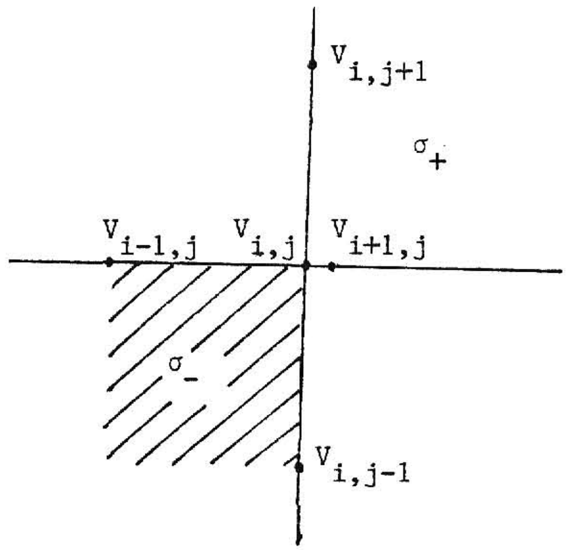
and with $\sigma_+ = 1$, $\sigma_- = C$, $\Delta x_1 = \Delta x_2 = \frac{1}{8}$, $\Delta y_1 = \Delta y_2 = 1$,

and $\hat{n} = \frac{1}{\sqrt{2}} \hat{x} - \frac{1}{\sqrt{2}} \hat{y}$,

the finite difference equation is obtained as

$$V_{i,j} = (8C \, V_{i-1,j} + C \, V_{i,j+1} + 8 \, V_{i+1,j} + V_{i,j-1})/(9C+9) \, .$$

Case 44: An interior grid point at the discontinuity of conductivity.



From Eqns. 16, 17, and 18, we have

$$\sigma_+ \left[(-\nabla V)_+ \cdot \hat{n}\right] = \sigma_- \left[(-\nabla V)_- \cdot \hat{n}\right] .$$

Approxiamting ∇V_+ by

$$\nabla V_+ = \frac{V_{i+1,j} - V_{i,j}}{\Delta x_1} \hat{x} + \frac{V_{i,j+1} - V_{i,j}}{\Delta y_1} \hat{y} ,$$

approximating ∇V_- by

$$\nabla V_- = \frac{V_{i,j} - V_{i-1,j}}{\Delta x_2} \hat{x} + \frac{V_{i,j} - V_{i,j-1}}{\Delta y_2} \hat{y} ,$$

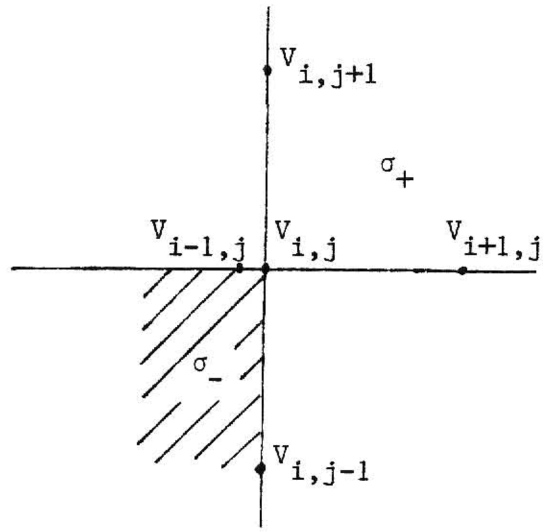
and with $\sigma_+ = 1$, $\sigma_- = C$, $\Delta x_1 = \frac{1}{8}$, $\Delta y_1 = \Delta x_2 = \Delta y_2 = 1$,

and $\hat{n} = \frac{1}{\sqrt{2}} \hat{x} + \frac{1}{\sqrt{2}} \hat{y}$,

the finite difference equation is obtained as

$$V_{i,j} = (8\, V_{i+1,j} + V_{i,j+1} + C\, V_{i-1,j} + C\, V_{i,j-1})/(2C+9) .$$

<u>Case 45</u>: An interior grid point at the discontinuity of conductivity.



From Eqns. 16, 17, and 18, we have

$$\sigma_+ \, [(-\nabla V)_+ \cdot \hat{n}] = \sigma_- \, [(-\nabla V)_- \cdot \hat{n}] \, .$$

Approximating ∇V_+ by

$$\nabla V_+ = \frac{V_{i+1,j} - V_{i,j}}{\Delta x_1} \, \hat{x} + \frac{V_{i,j+1} - V_{i,j}}{\Delta y_1} \, \hat{y} \, ,$$

approximating ∇V_- by

$$\nabla V_- = \frac{V_{i,j} - V_{i-1,j}}{\Delta x_2} \, \hat{x} + \frac{V_{i,j} - V_{i,j-1}}{\Delta y_2} \, \hat{y} \, ,$$

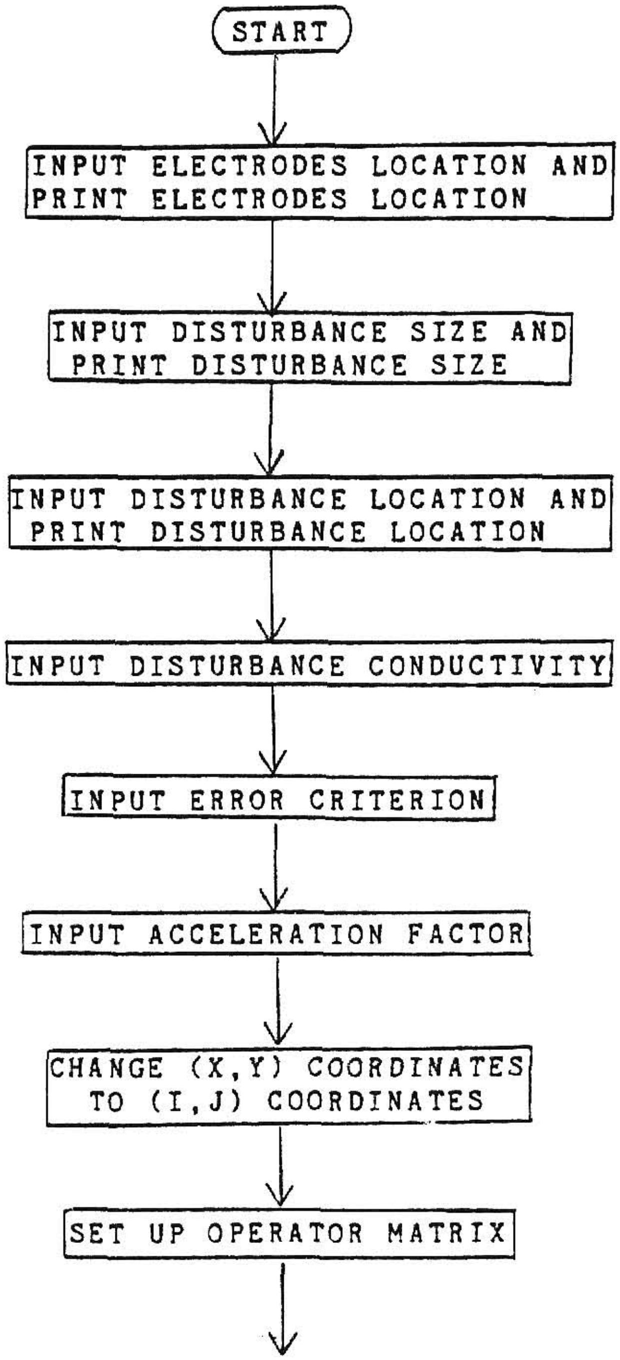
and with $\sigma_+ = 1$, $\sigma_- = C$, $\Delta x_1 = 1$, $\Delta x_2 = \frac{1}{8}$, $\Delta y_1 = \Delta y_2 = 1$,

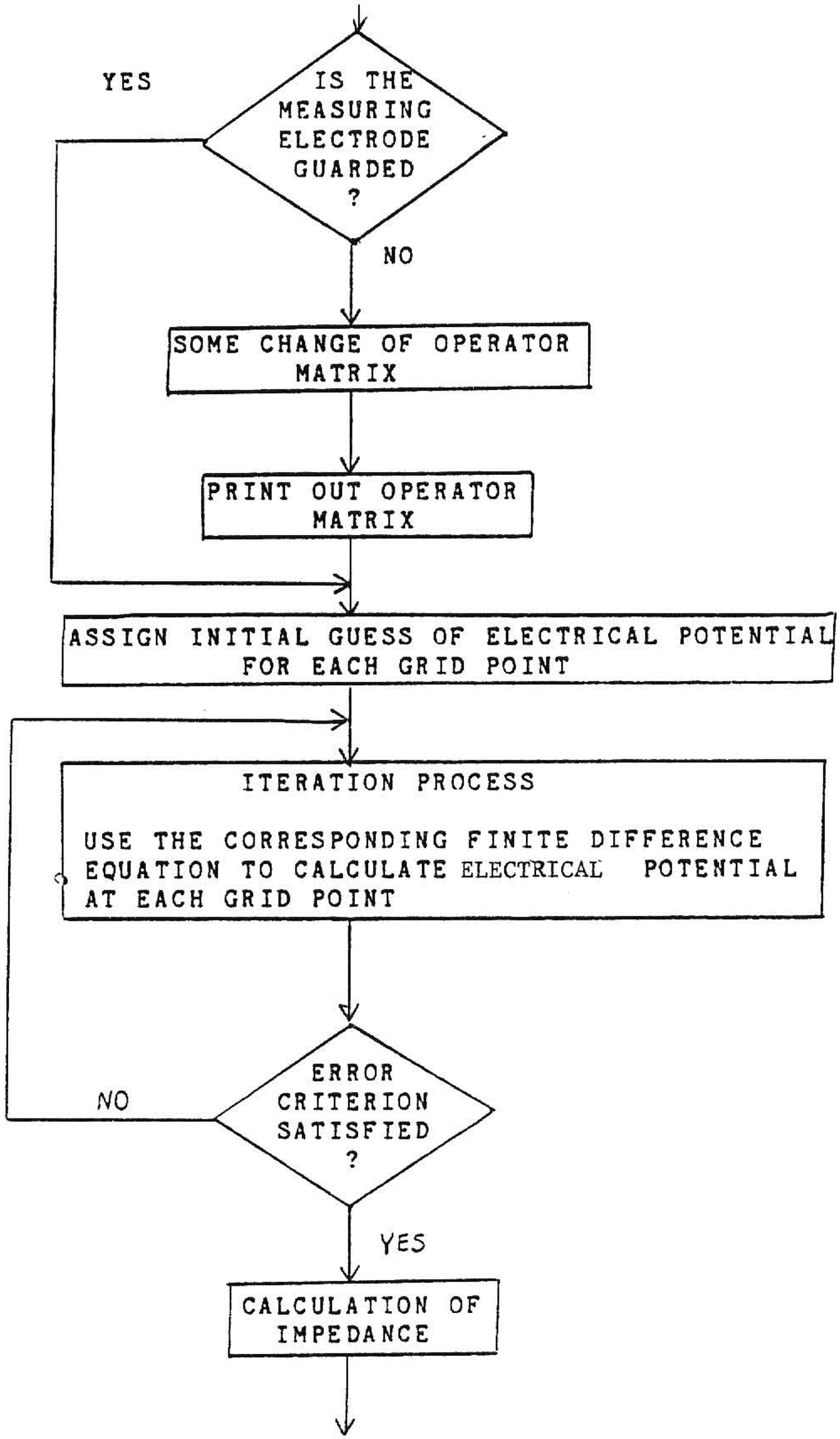
and $\hat{n} = \frac{1}{\sqrt{2}} \, \hat{x} + \frac{1}{\sqrt{2}} \, \hat{y}$,

the finite difference equation obtained is

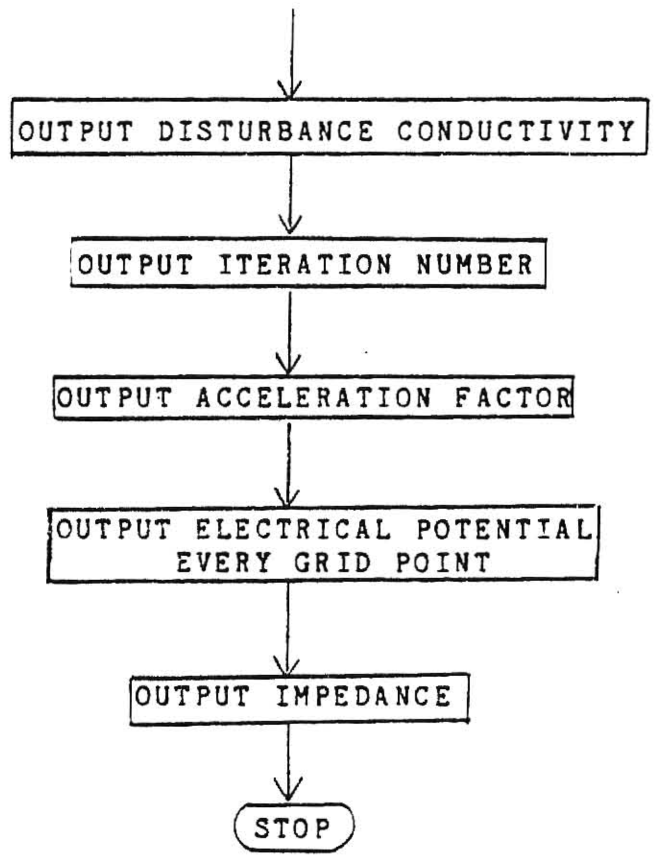
$$V_{i,j} = (V_{i+1,j} + V_{i,j+1} + 8C \, V_{i-1,j} + C \, V_{i,j-1})/(9C+2) \, .$$

Flowchart of Program "ITERATION"





YES
IS THE MEASURING ELECTRODE GUARDED ?
NO
SOME CHANGE OF OPERATOR MATRIX
PRINT OUT OPERATOR MATRIX
ASSIGN INITIAL GUESS OF ELECTRICAL POTENTIAL FOR EACH GRID POINT
ITERATION PROCESS
USE THE CORRESPONDING FINITE DIFFERENCE EQUATION TO CALCULATE ELECTRICAL POTENTIAL AT EACH GRID POINT
NO
ERROR CRITERION SATISFIED ?
YES
CALCULATION OF IMPEDANCE


OUTPUT DISTURBANCE CONDUCTIVITY
OUTPUT ITERATION NUMBER
OUTPUT ACCELERATION FACTOR
OUTPUT ELECTRICAL POTENTIAL EVERY GRID POINT
OUTPUT IMPEDANCE
STOP

```
10     ! PROGRAM NAME: ITERATION
20     DIM Iopmat(35,12),V(35,12)
30     Numx=35
40     Numy=12
50     INPUT "GUARD RING ELECTRODE LOCATION=?",Iguard
60     INPUT "REFERENCED ELECTGRODE LOCATUION=?",Iref
70     PRINT "ELECTRODES LOCATION"
80     PRINT "IGUARD=",Iguard,"IREF=",Iref
90     !IGUARD IS THE X COORDINATE OF THE CENTER
100    !POINT OF THE MEASURING ELECTRODE
110    !IREF IS THE X COORDINATE OF THE CENTER
120    !POINT OF THE REFERENCE ELECTRODE
130    PRINT "SIZE OF DISTURBANCE"
140    INPUT "IXDIS=",Ixdis
150    INPUT "IYDIS=",Iydis
160    PRINT "IXDIS=",Ixdis,"IYDIS=",Iydis
170    !
180    !IXDIS,IYDIS ARE SUBJECT TO CHANGE IF DISTURBNANCE IS
190    !LINED UP WITH GUARD ELECTRODE
200    !
210    !
220    PRINT "LOCATION OF DISTURBANCE"
230    INPUT "IX1ST=?",Ix1st
240    INPUT "IY1ST=?",Iy1st
250    PRINT "IX1ST=",Ix1st,"IY1ST=",Iy1st
260    INPUT "DISTURBANCE CONDUCTIVITY=?",C
270    INPUT "ERROR CRITERION=?",Xeps
280    INPUT "ACCELERATION FACTOR=?",W
290    Nguard=17
300    !NGUARD IS NO OF POINTS USED TO REPREENT GUARD RING ELECTRODE
310    Nref=3
320    !NREF IS NO OF POINTS USED TO REPRESENT REFERENCE ELECTRODE
330    !NREF IS SUBJECT TO CHANGE IF REFERENCE
340    !ELECTRODE IS LINED UP WITH GUARD ELECTRODE
350    !
360    !*******************************************
370    !                                          *
380    !  MORE POINTS ARE USED TO REPRESENT AREA  *
390    !  AROUND GUARD RING ELECTRODE             *
400    !                                          *
410    !*******************************************
```

```
420    !
430    ! REDEFINE NREF
440    !
450      IF Iref=Iguard THEN Nref=17
460      IF (ABS(Iref-Iguard)=1) THEN Nref=10
470    !
480    ! REDEFINE IY1ST
490    !
500      Iy1st=Iy1st+1
510    !
520    !ADD POINTS TO DISTURBANCE
530      IF Ix1st<=Iguard THEN Check2
540      Ntest=Ix1st-Iguard
550      IF Ntest=1 THEN Ixdis=Ixdis+7
560      GOTO Add_pt
570 Check2:Itest=Ix1st+Ixdis-Iguard
580         IF Itest=1 THEN Ixdis=Ixdis+7
590         IF Itest>=2 THEN Ixdis=Ixdis+14
600    !
610    !
620    !
630    !REDEFINE IREF SINCE MORE POINTS ARE USED
640    !
650 Add_pt: IF (Iref-Iguard)=1 THEN Iref=Iref+7
660      IF (Iref-Iguard)>1 THEN Iref=Iref+14
670    !
680    !
690    !REDEFINE IX1ST SINCE MORE POINTS ARE USED
700      IF Ix1st-Iguard>=2 THEN Ix1st=Ix1st+14
710      IF Ix1st-Iguard=1 THEN Ix1st=Ix1st+7
720    !
730    !
740    !
750    !*****************************************
760    !                                        *
770    !  SET UP OPERATOR MATRIX IOPMAT         *
780    !                                        *
790    !  IOPMAT(I,J) IS THE NO OF  DIFFERENCE  *
800    ! EQUATION CORRESPONDING TO POINT(I,J)   *
810    !                                        *
820    !                                        *
830    !*****************************************
```

```
840    !
850    !
860    FOR N1=1 TO Numx
870     FOR N2=1 TO Numy
880      Iopmat(N1,N2)=1
890     NEXT N2
900    NEXT N1
910    FOR N3=1 TO Numx
920      Iopmat(N3,1)=3
930      Iopmat(N3,Numy)=28
940      Iopmat(N3,2)=7
950      Iopmat(N3,3)=12
960    NEXT N3
970    FOR N4=1 TO Numy
980      Iopmat(1,N4)=6
990      Iopmat(Numx,N4)=11
1000   NEXT N4
1010   Igmp=Nguard-2
1020   FOR N5=1 TO Igmp
1030     Iopmat(Iguard+N5,2)=9
1040     Iopmat(Iguard+N5,3)=14
1050      FOR N6=4 TO (Numy-1)
1060       Iopmat(Iguard+N5,N6)=17
1070      NEXT N6
1080   NEXT N5
1090   Iopmat(Iguard,2)=8
1100   Iopmat(Iguard+Nguard-1,2)=10
1110   Iopmat(Iguard,3)=13
1120   Iopmat(Iguard+Nguard-1,3)=15
1130   FOR N7=4 TO 11
1140     Iopmat(Iguard,N7)=16
1150     Iopmat(Iguard+Nguard-1,N7)=18
1160   NEXT N7
1170   Ixmp=Ixdis-1
1180   FOR N8=1 TO Ixmp
1190     Iopmat(Ix1st+N8,Iy1st)=20
1200     Iopmat(Ix1st+N8,Iy1st+Iydis)=25
1210   NEXT N8
1220   ! DEFINE LTXDIS AS THE RIGHT MOST POINT OF DISTURBANCE
1230   Ltxdis=Ix1st+Ixdis
1240 Cnt:   FOR N9=1 TO Iydis
1250          Iopmat(Ix1st,Iy1st+N9)=22
```

```
1260          IF Ix1st=Iguard THEN Iopmat(Ix1st,Iy1st+N9)=31
1270          IF Ix1st=Iguard+16 THEN Iopmat(Ix1st,Iy1st+N9)=32
1280          Iopmat(Ltxdis,Iy1st+N9)=23
1290           IF Ltxdis=Iguard THEN Iopmat(Ltxdis,Iy1st+N9)=33
1300           IF Ltxdis=Iguard+16 THEN Iopmat(Ltxdis,Iy1st+N9)=34
1310        NEXT N9
1320  Iopmat(Ix1st,Iy1st)=19
1330   IF Ix1st=Iguard THEN Iopmat(Ix1st,Iy1st)=35
1340   IF Ix1st=Iguard+16 THEN Iopmat(Ix1st,Iy1st)=36
1350   IF (Ix1st>Iguard) AND (Ix1st<Iguard+16) THEN Iopmat(Ix1st,Iy1st)=37
1360  Iopmat(Ix1st,Iy1st+Iydis)=24
1370   IF Ix1st=Iguard THEN Iopmat(Ix1st,Iy1st+Iydis)=38
1380   IF Ix1st=Iguard+16 THEN Iopmat(Ix1st,Iy1st+Iydis)=39
1390   IF (Ix1st>Iguard) AND (Ix1st<Iguard+16) THEN Iopmat(Ix1st,Iy1st+Iydis)=40
1400  Iopmat(Ltxdis,Iy1st)=21
1410   IF Ltxdis=Iguard THEN Iopmat(Ltxdis,Iy1st)=41
1420   IF Ltxdis=Iguard+16 THEN Iopmat(Ltxdis,Iy1st)=42
1430   IF (Ltxdis>Iguard) AND (Ltxdis<Iguard+16) THEN Iopmat(Ltxdis,Iy1st)=43
1440  Iopmat(Ltxdis,Iy1st+Iydis)=26
1450   IF Ltxdis=Iguard THEN Iopmat(Ltxdis,Iy1st+Iydis)=44
1460   IF Ltxdis=Iguard+16 THEN Iopmat(Ltxdis,Iy1st+Iydis)=45
1470   IF (Ltxdis>Iguard) AND (Ltxdis<Iguard+16) THEN Iopmat(Ltxdis,Iy1st+Iydis)
=46
1480  Iopmat(1,1)=2
1490  Iopmat(Numx,1)=5
1500  Iopmat(1,Numy)=27
1510  Iopmat(Numx,Numy)=30
1520  !ASSIGN PARAMETERS TO ELECTRODES
1525   INPUT "IS ELECTRODE GUARDED? (Y/N)",G$
1526   IF G$="N" THEN Ungurd
1530  FOR N10=1 TO Nguard
1540    Iopmat(Iguard+N10-1,1)=4
1550   NEXT N10
1553 Ungurd:  FOR M10=4 TO 13
1554      Iopmat(Iguard+M10-1,1)=4
1555      NEXT M10
1560   FOR N11=1 TO Nref
1570    Iopmat(Iref+N11-1,Numy)=29
1580  NEXT N11
1590  Iopmat(Iguard+3,1)=3
```

```
1600  Iopmat(Iguard+13,1)=3
1610  !
1620  !****************************************
1630  !                                       *
1640  !  PRINT OUT OPERATOR MATRIX            *
1650  !                                       *
1660  !****************************************
1670  !
1680 Pt_mat:  PRINT "OUTPUT OF OPERATOR MATRIX"
1690    FOR J=1 TO Numx
1700     FOR I=1 TO Numy-1
1710      PRINT Iopmat(J,I);
1720      IF Iopmat(J,I)<10 THEN PRINT " ";
1730     NEXT I
1740     PRINT Iopmat(J,Numy)
1750     IF J<Iguard THEN PRINT " "
1760     IF J>(Iguard+15) THEN PRINT " "
1770    NEXT J
1780    PRINT USING "@,#"
1790  !
1800  !
1810  !****************************************
1820  !                                       *
1830  !                                       *
1840  !     NOW READY FOR CALCULATION         *
1850  !                                       *
1860  !                                       *
1870  !****************************************
1880  !
1890  !
1900  !ASSIGN INITIAL GUESS
1910  !
1920    FOR I=1 TO Numx
1930     FOR J=1 TO Numy
1940      V(I,J)=5.0
1950     NEXT J
1960    NEXT I
1970  !
1980  !
1990  Iterat=0
```

```
2000 Calc:  Convs=0
2010  FOR J=1 TO Numy
2020   FOR I=1 TO Numx
2030    Oldv=V(I,J)
2040    IF Iopmat(I,J)>40 THEN J4
2050    IF Iopmat(I,J)>30 THEN J3
2060    IF Iopmat(I,J)>20 THEN J2
2070    IF Iopmat(I,J)>10 THEN J1
2080    ON Iopmat(I,J) GOTO C1,C2,C3,C4,C5,C6,C7,C8,C9,C10
2090 J1: ON Iopmat(I,J)-10 GOTO C11,C12,C13,C14,C15,C16,C17,C18,C19,C20
2100 J2: ON Iopmat(I,J)-20 GOTO C21,C22,C23,C24,C25,C26,C27,C28,C29,C30
2110 J3: ON Iopmat(I,J)-30 GOTO C31,C32,C33,C34,C35,C36,C37,C38,C39,C40
2120 J4: ON Iopmat(I,J)-40 GOTO C41,C42,C43,C44,C45,C46
2130 C1: V(I,J)=.25*(V(I-1,J)+V(I+1,J)+V(I,J-1)+V(I,J+1))
2140     GOTO Tst
2150 C2: V(I,J)=(V(I+1,J)+8*V(I,J+1))/9
2160     GOTO Tst
2170 C3: V(I,J)=V(I,J+1)
2180     GOTO Tst
2190 C4: V(I,J)=10
2200     GOTO Tst
2210 C5: V(I,J)=(V(I-1,J)+8*V(I,J+1))/9
2220     GOTO Tst
2230 C6: V(I,J)=V(I+1,J)
2240     GOTO Tst
2250 C7: V(I,J)=(7*V(I+1,J)+7*V(I-1,J)+16*V(I,J+1)+112*V(I,J-1))/142
2260     GOTO Tst
2270 C8: V(I,J)=(56*V(I+1,J)+7*V(I-1,J)+9*V(I,J+1)+63*V(I,J-1))/135
2280     GOTO Tst
2290 C9: V(I,J)=(28*V(I+1,J)+28*V(I-1,J)+V(I,J+1)+7*V(I,J-1))/64
2300     GOTO Tst
2310 C10: V(I,J)=(7*V(I+1,J)+56*V(I-1,J)+9*V(I,J+1)+63*V(I,J-1))/135
2320     GOTO Tst
2330 C11: V(I,J)=V(I-1,J)
2340     GOTO Tst
2350 C12: V(I,J)=(105*V(I+1,J)+105*V(I-1,J)+112*V(I,J+1)+128*V(I,J-1))/450
2360     GOTO Tst
2370 C13: V(I,J)=(280*V(I+1,J)+35*V(I-1,J)+21*V(I,J+1)+24*V(I,J-1))/360
2380     GOTO Tst
2390 C14: V(I,J)=(420*V(I+1,J)+420*V(I-1,J)+7*V(I,J+1)+8*V(I,J-1))/855
```

```
2400      GOTO Tst
2410 C15: V(I,J)=(35*V(I+1,J)+280*V(I-1,J)+21*V(I,J+1)+24*V(I,J-1))/360
2420      GOTO Tst
2430 C16: V(I,J)=(128*V(I+1,J)+16*V(I-1,J)+9*V(I,J+1)+9*V(I,J-1))/162
2440      GOTO Tst
2450 C17: V(I,J)=(64*V(I+1,J)+64*V(I-1,J)+V(I,J+1)+V(I,J-1))/130
2460      GOTO Tst
2470 C18: V(I,J)=(16*V(I+1,J)+128*V(I-1,J)+9*V(I,J+1)+9*V(I,J-1))/162
2480      GOTO Tst
2490 C19: V(I,J)=(C*V(I+1,J)+V(I-1,J)+C*V(I,J+1)++V(I,J-1))/(2*C+2)
2500      GOTO Tst
2510 C20: V(I,J)=(C*V(I,J+1)+V(I,J-1))/(C+1)
2520      GOTO Tst
2530 C21: V(I,J)=(V(I+1,J)+C*V(I-1,J)+V(I,J-1)+C*V(I,J+1))/(2*C+2)
2540      GOTO Tst
2550 C22: V(I,J)=(C*V(I+1,J)+V(I-1,J))/(C+1)
2560      GOTO Tst
2570 C23: V(I,J)=(C*V(I-1,J)+V(I+1,J))/(C+1)
2580      GOTO Tst
2590 C24: V(I,J)=(V(I-1,J)+V(I,J+1)+C*V(I+1,J)+C*V(I,J-1))/(2*C+2)
2600      GOTO Tst
2610 C25: V(I,J)=(V(I,J+1)+C*V(I,J-1))/(C+1)
2620      GOTO Tst
2630 C26: V(I,J)=(C*V(I-1,J)+V(I+1,J)+V(I,J+1)+C*V(I,J-1))/(2*C+2)
2640      GOTO Tst
2650 C27: V(I,J)=(V(I+1,J)+V(I,J-1))/2
2660      GOTO Tst
2670 C28: V(I,J)=V(I,J-1)
2680      GOTO Tst
2690 C29: V(I,J)=0
2700      GOTO Tst
2710 C30: V(I,J)=(V(I-1,J)+V(I,J-1))/2
2720      GOTO Tst
2730 C31: V(I,J)=(V(I-1,J)+8*C*V(I+1,J))/(1+8*C)
2740      GOTO Tst
2750 C32: V(I,J)=(C*V(I+1,J)+8*V(I-1,J))/(8+C)
2760      GOTO Tst
2770 C33: V(I,J)=(8*V(I+1,J)+C*V(I-1,J))/(C+8)
2780      GOTO Tst
2790 C34: V(I,J)=(V(I+1,J)+8*C*V(I-1,J))/(1+8*C)
2800      GOTO Tst
```

```
2810 C35: V(I,J)=(C*8*V(I+1,J)+C*V(I,J+1)+V(I-1,J)+V(I,J-1))/(C*9+2)
2820      GOTO Tst
2830 C36: V(I,J)=(C*V(I+1,J)+C*V(I,J+1)+8*V(I-1,J)+V(I,J-1))/(2*C+9)
2840      GOTO Tst
2850 C37: V(I,J)=(8*C*V(I+1,J)+C*V(I,J+1)+8*V(I-1,J)+V(I,J-1))/(9*C+9)
2860      GOTO Tst
2870 C38: V(I,J)=(V(I-1,J)+V(I,J+1)+8*C*V(I+1,J)+C*V(I,J-1))/(9*C+2)
2880      GOTO Tst
2890 C39: V(I,J)=(8*V(I-1,J)+V(I,J+1)+C*V(I+1,J)+C*V(I,J-1))/(2*C+9)
2900      GOTO Tst
2910 C40: V(I,J)=(8*V(I-1,J)+V(I,J+1)+8*C*V(I+1,J)+C*V(I,J-1))/(9*C+9)
2920      GOTO Tst
2930 C41: V(I,J)=(C*V(I-1,J)+C*V(I,J+1)+8*V(I+1,J)+V(I,J-1))/(2*C+9)
2940      GOTO Tst
2950 C42: V(I,J)=(8*C*V(I-1,J)+C*V(I,J+1)+V(I+1,J)+V(I,J-1))/(9*C+2)
2960      GOTO Tst
2970 C43: V(I,J)=(C*8*V(I-1,J)+C*V(I,J+1)+8*V(I+1,J)+V(I,J-1))/(9*C+9)
2980      GOTO Tst
2990 C44: V(I,J)=(8*V(I+1,J)+V(I,J+1)+C*V(I-1,J)+C*V(I,J-1))/(2*C+9)
3000      GOTO Tst
3010 C45: V(I,J)=(V(I+1,J)+V(I,J+1)+C*8*V(I-1,J)+C*V(I,J-1))/(9*C+2)
3020      GOTO Tst
3030 C46: PRINT "NOT DEFINED"
3040      GOTO Quit
3050 Tst: V(I,J)=W*V(I,J)-(W-1)*Oldv
3060       IF (V(I,J)-Oldv)>Xeps THEN Convg=Convg-1
3070      NEXT I
3080     NEXT J
3090     Iterat=Iterat+1
3100     PRINTER IS 1
3110     PRINT Iterat
3120     IF Convg<0 THEN GOTO Calc
3130     !
3140     !
3150     !*************************************
3160     !                                    *
3170     !          END OF ITERATION          *
3180     !                                    *
3190     !*************************************
3200     !
3210     !
```

```
3220     !
3230     !****************************************
3240     !                                       *
3250     !                                       *
3260     !  CALCULATION OF IMPEDANCE MEASURED    *
3270     !         BY CENTRAL ELECTRODE          *
3280     !                                       *
3290     !                                       *
3300     !****************************************
3310     !
3320       Sum=0
3330       Sum=Sum+.5*(10-V(Iguard+4,2))
3340       FOR I=1 TO 7
3350        Sum=Sum+(10-V(Iguard+4+I,2))
3360       NEXT I
3370       Sum=Sum+.5*(10-V(Iguard+12,2))
3380       Zc=10/Sum
3390     !
3400     !
3410     !****************************************
3420     !                                       *
3430     !                                       *
3440     !  END OF CALCULATION,READY FOR OUTPUT  *
3450     !                                       *
3460     !                                       *
3470     !****************************************
3480     !
3490     !
3500      PRINTER IS 9
3560       PRINT "DISTURBANCE CONDUCTIVITY=",C
3570       PRINT "ITERATION NUMBER=",Iterat
3580       PRINT "ERROR CRITERION=",Xeps
3590       PRINT "ACCELERATION FACTOR= ",W
3600       PRINT " "
3610 PRINT "(I)";" V(I,1) ";" V(I,2) ";" V(I,3) ";" V(I,4) ";" V(I,5) ";" V(I,6)
"
3620       FOR I=1 TO Numx
3630         PRINT USING Format;I,V(I,1);V(I,2);V(I,3);V(I,4);V(I,5);V(I,6)
3640       NEXT I
3650       PRINT " "
```

```
3660       PRINT "(I)";" V(I,7) ";" V(I,8) ";" V(I,9) ";" V(I,10)";" V(I,11)";" V
(I,12)"
3670       FOR I=1 TO Numx
3680       PRINT USING Format;I;V(I,7);V(I,8);V(I,9);V(I,10);V(I,11);V(I,12)
3690       NEXT I
3700       PRINT " "
3701 Format:      IMAGE  DD,3D.4D,3D.4D,3D.4D,3D.4D,3D.4D,3D.4D
3750       PRINT " "
3760       PRINT " "
3770       PRINT "IMPEDANCE-UNIT LENGTH=",Zc
3780 Quit:END
```

Appendix D. Sample of Computer Printout

ELECTRODES LOCATION
IGUARD= 10 IREF= 10
SIZE OF DISTURBANCE
IXDIS= 4 IYDIS= 4
LOCATION OF DISTURBANCE
IX1ST= 9 IY1ST= 4
OUTPUT OF OPERATOR MATRIX

2	6	6	6	6	6	6	6	6	6	6	27
3	7	12	1	1	1	1	1	1	1	1	28
3	7	12	1	1	1	1	1	1	1	1	28
3	7	12	1	1	1	1	1	1	1	1	28
3	7	12	1	1	1	1	1	1	1	1	28
3	7	12	1	1	1	1	1	1	1	1	28
3	7	12	1	1	1	1	1	1	1	1	28
3	7	12	1	1	1	1	1	1	1	1	28
3	7	12	1	19	22	22	22	24	1	1	28
4	8	13	16	20	16	16	16	25	16	16	29
4	9	14	17	20	17	17	17	25	17	17	29
4	9	14	17	20	17	17	17	25	17	17	29
3	9	14	17	20	17	17	17	25	17	17	29
4	9	14	17	20	17	17	17	25	17	17	29
4	9	14	17	20	17	17	17	25	17	17	29
4	9	14	17	20	17	17	17	25	17	17	29
4	9	14	17	20	17	17	17	25	17	17	29
4	9	14	17	20	17	17	17	25	17	17	29
4	9	14	17	20	17	17	17	25	17	17	29
4	9	14	17	20	17	17	17	25	17	17	29
4	9	14	17	20	17	17	17	25	17	17	29
4	9	14	17	20	17	17	17	25	17	17	29
3	9	14	17	20	17	17	17	25	17	17	29
4	9	14	17	20	17	17	17	25	17	17	29
4	9	14	17	20	17	17	17	25	17	17	29
4	10	15	18	20	18	18	18	25	18	18	29
3	7	12	1	21	23	23	23	26	1	1	28
3	7	12	1	1	1	1	1	1	1	1	28
3	7	12	1	1	1	1	1	1	1	1	28
3	7	12	1	1	1	1	1	1	1	1	28
3	7	12	1	1	1	1	1	1	1	1	28
3	7	12	1	1	1	1	1	1	1	1	28
3	7	12	1	1	1	1	1	1	1	1	28
3	7	12	1	1	1	1	1	1	1	1	28
5	11	11	11	11	11	11	11	11	11	11	30

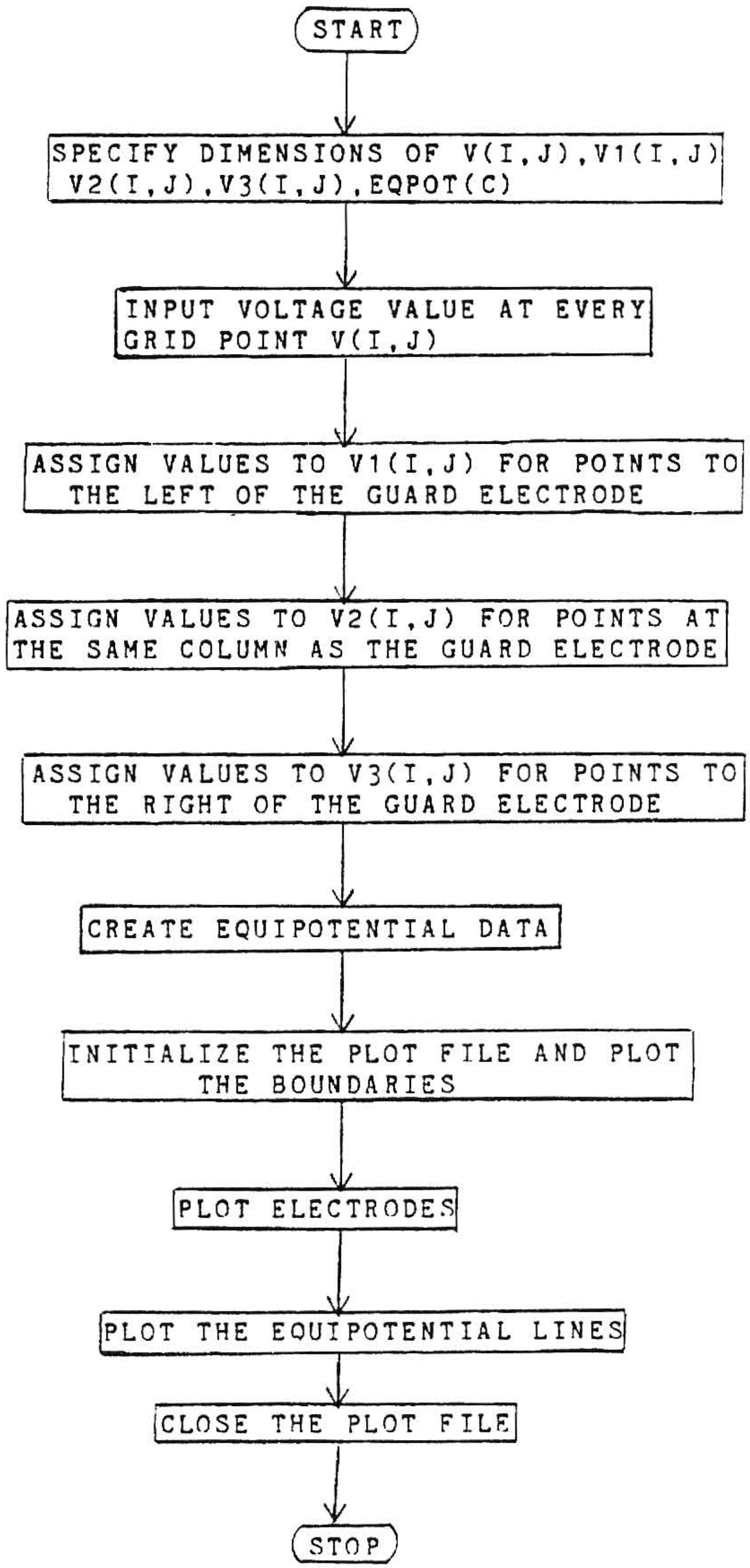
DISTURBANCE CONDUCTIVITY= .01
ITERATION NUMBER= 344
ERROR CRITERION= 1.E-5
ACCELERATION FACTOR= 1.95

(I)	V(I,1)	V(I,2)	V(I,3)	V(I,4)	V(I,5)	V(I,6)
1	6.0471	6.0471	6.0252	5.9552	5.8450	5.7064
2	6.0471	6.0471	6.0252	5.9552	5.8450	5.7064
3	6.0973	6.0973	6.0730	5.9955	5.8734	5.7199
4	6.2032	6.2032	6.1738	6.0804	5.9332	5.7483
5	6.3762	6.3762	6.3385	6.2189	6.0309	5.7946
6	6.6355	6.6355	6.5847	6.4259	6.1767	5.8626
7	7.0107	7.0107	6.9386	6.7232	6.3875	5.9540
8	7.5509	7.5509	7.4342	7.1409	6.6962	6.0530
9	8.3578	8.3578	8.1003	7.7100	7.2033	6.0591
10	10.0000	9.7533	8.8688	8.3956	8.3784	6.6628
11	10.0000	9.7923	8.9246	8.4492	8.4320	6.7053
12	10.0000	9.8104	8.9718	8.4957	8.4783	6.7410
13	9.8110	9.8110	9.0109	8.5350	8.5175	6.7705
14	10.0000	9.8401	9.0427	8.5672	8.5496	6.7941
15	10.0000	9.8579	9.0672	8.5921	8.5745	6.8122
16	10.0000	9.8683	9.0846	8.6100	8.5923	6.8249
17	10.0000	9.8737	9.0950	8.6207	8.6030	6.8325
18	10.0000	9.8754	9.0984	8.6242	8.6065	6.8350
19	10.0000	9.8737	9.0950	8.6207	8.6030	6.8325
20	10.0000	9.8683	9.0846	8.6100	8.5923	6.8249
21	10.0000	9.8579	9.0672	8.5921	8.5745	6.8122
22	10.0000	9.8401	9.0427	8.5672	8.5496	6.7941
23	9.8110	9.8110	9.0109	8.5350	8.5175	6.7705
24	10.0000	9.8104	8.9718	8.4957	8.4783	6.7410
25	10.0000	9.7923	8.9246	8.4492	8.4320	6.7053
26	10.0000	9.7533	8.8688	8.3956	8.3784	6.6628
27	8.3578	8.3578	8.1003	7.7100	7.2033	6.0591
28	7.5509	7.5509	7.4342	7.1409	6.6962	6.0530
29	7.0107	7.0107	6.9386	6.7232	6.3875	5.9540
30	6.6355	6.6355	6.5847	6.4259	6.1767	5.8626
31	6.3762	6.3762	6.3385	6.2189	6.0309	5.7946
32	6.2032	6.2032	6.1738	6.0804	5.9332	5.7483
33	6.0973	6.0973	6.0730	5.9955	5.8734	5.7199
34	6.0471	6.0471	6.0252	5.9552	5.8450	5.7064
35	6.0471	6.0471	6.0252	5.9552	5.8450	5.7064

(I)	V(I,7)	V(I,8)	V(I,9)	V(I,10)	V(I,11)	V(I,12)
1	5.5543	5.4051	5.2753	5.1793	5.1283	5.1283
2	5.5543	5.4051	5.2753	5.1793	5.1283	5.1283
3	5.5514	5.3859	5.2413	5.1343	5.0774	5.0774
4	5.5455	5.3456	5.1699	5.0392	4.9695	4.9695
5	5.5366	5.2812	5.0536	4.8831	4.7919	4.7919
6	5.5252	5.1888	4.8801	4.6477	4.5231	4.5231
7	5.5127	5.0689	4.6303	4.3045	4.1297	4.1297
8	5.5029	4.9438	4.2676	3.8102	3.5615	3.5615
9	5.5020	4.9356	3.6863	3.1073	2.7446	2.7446
10	5.4166	4.1180	2.2071	2.1880	1.5651	0.0000
11	5.4096	4.0589	2.1347	2.1155	1.4839	0.0000
12	5.4034	4.0087	2.0720	2.0526	1.4161	0.0000
13	5.3982	3.9670	2.0188	1.9993	1.3604	0.0000
14	5.3938	3.9333	1.9753	1.9558	1.3159	0.0000
15	5.3905	3.9075	1.9415	1.9219	1.2821	0.0000
16	5.3880	3.8891	1.9174	1.8977	1.2583	0.0000
17	5.3866	3.8782	1.9029	1.8832	1.2441	0.0000
18	5.3861	3.8746	1.8981	1.8783	1.2394	0.0000
19	5.3866	3.8782	1.9029	1.8832	1.2441	0.0000
20	5.3880	3.8891	1.9174	1.8977	1.2583	0.0000
21	5.3905	3.9075	1.9415	1.9219	1.2821	0.0000
22	5.3938	3.9333	1.9753	1.9558	1.3159	0.0000
23	5.3982	3.9670	2.0188	1.9993	1.3604	0.0000
24	5.4034	4.0087	2.0720	2.0526	1.4161	0.0000
25	5.4096	4.0589	2.1347	2.1155	1.4839	0.0000
26	5.4166	4.1180	2.2071	2.1880	1.5651	0.0000
27	5.5020	4.9356	3.6863	3.1073	2.7446	2.7446
28	5.5029	4.9438	4.2676	3.8102	3.5615	3.5615
29	5.5127	5.0689	4.6303	4.3045	4.1297	4.1297
30	5.5252	5.1888	4.8801	4.6477	4.5231	4.5231
31	5.5366	5.2812	5.0536	4.8831	4.7919	4.7919
32	5.5455	5.3456	5.1699	5.0392	4.9695	4.9695
33	5.5514	5.3859	5.2413	5.1343	5.0774	5.0774
34	5.5543	5.4051	5.2753	5.1793	5.1283	5.1283
35	5.5543	5.4051	5.2753	5.1793	5.1283	5.1283

IMPEDANCE-UNIT LENGTH= 9.21844426169

Flowchart for program "PLOT"



Appendix F. List of Program PLOT

```
      DIMENSION  V1(7,11),V2(17,11),V3(13,11),V(35,11),EQPOT(39)
C *   *   *   *   *   *   *   *   *   *   *   *
C *   READ ELECTRICAL POTENTIAL VALUES V(I,J) *
C *   *   *   *   *   *   *   *   *   *   *   *
      READ(5,30) ((V(I,J),I=1,35),J=1,11)
30    FORMAT(F7.4)
      DO 40 J=1,11
      DO 50 I=1,7
        V1(I,J)=V(I,J)
50    CONTINUE
40    CONTINUE
      DO 60 J=1,11
      DO 70 I=1,17
        V2(I,J)=V(I+6,J)
70    CONTINUE
60    CONTINUE
      DO 80 J=1,11
       DO 90 I=1,13
        V3(I,J)=V(I+22,J)
90    CONTINUE
80    CONTINUE
C *   *   *   *   *   *   *   *   *
C *   CREATE EQUAL POTENTIAL DATA *
C *   *   *   *   *   *   *   *   *
      READ(5,95) (EQPOT(K),K=1,39)
95      FORMAT(F5.2)
C *   *   *   *   *   *   *   *   *   *
C *   INITIALIZE THE PLOT FILE AND    *
C *   PLOT THE BOUNDARIES             *
C *   *   *   *   *   *   *   *   *   *
      CALL PLOTS
      CALL PLOT(2.,2.,23)
      CALL PLOT(10.,0.,2)
      CALL PLOT(10.,5.,2)
      CALL PLOT(0.,5.,2)
      CALL PLOT(0.,0.,2)
C *   *   *   *   *   *   *   *   *
C *   PLOT GUARD ELECTRODE AND    *
C *   REFERENCE ELECTRODE         *
C *   *   *   *   *   *   *   *   *
      CALL PLOT(3.25,-0.025,3)
      CALL PLOT(3.75,-0.025,2)
      CALL PLOT(4.5,5.025,3)
      CALL PLOT(5.5,5.025,2)
C *   *   *   *   *   *   *
C *   PLOT THE CONTOUR
C *   *   *   *   *   *   *
      CALL CONTUR(V1,7,11,EQPOT,39,3.,5.)
      CALL PLOT(3.0,0.,23)
      CALL CONTUR(V2,17,11,EQPOT,39,1.,5.)
```

```
      CALL PLOT(1.,0.,23)
      CALL CONTUR(V3,13,11,EQPOT,39,6.,5.)
C *   *   *   *   *   *   *   *
C *   CLOSE THE PLOT FILE      *
C *   *   *   *   *   *   *   *
      CALL PLOT(0.,0.,999)
      STOP
      END
```

DIFFERENCES IN ELECTRICAL IMPEDANCE MEASUREMENTS DUE TO GUARDED/UNGUARDED ELECTRODES IN HOMOGENEOUS/INHOMOGENEOUS REGIONS

by

Wen-Yin F. Reed

B.S., National Taiwan University, 1979

AN ABSTRACT OF A MASTER'S THESIS

submitted in partial fulfillment of the

requirement for the degree

MASTER OF SCIENCE

Department of Electrical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1983

Abstract

This research concerns the numerical simulation of the electrical impedance variations and improved regional sensitivity through the use of guarded electrodes.

The approach involves calculating measurements on guarded and unguarded electrodes, using homogeneous and inhomogeneous regions, in a media having two dimensional rectangular geometry.

Using a numerical approximation for the potential equation and the Gauss-Siedel iteration method, the electrical potentials at each point on a grid system are found. Then, impedance measurements are calculated, and equipotential lines and current pathways are plotted.

With the equipotential line plots and the current pathway plots the guarded cases are compared with the unguarded cases. The conclusions are expressed as follows. The guarded electrode gives a more confined field and a better measure of regional sensitivity when a proper reference is used. The unguarded electrode gives no indication of the existence and/or location of the inhomogeneous region.