FREE VIBRATIONS OF CYLINDRICALLY AEOLOTROPIC CIRCULAR PLATES

by

HUA-KAWI WOO

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Approved by

Chi-lung Huang

Major Professor

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NOMENCLATURE

a	radius of the plate
$a_{ij}(i,j=1,2,..,6)$	elastic contants
b	radius of the isotropic core
D_r	$= E_r h^3/12$
$E_r, E_\theta, E_{r\theta}, G$	material constants
$\bar{E}$	error vector
$e_r, e_\theta, \gamma_{r\theta}$	radial, tangential and shearing strains
$\bar{F}, \bar{Y}, \bar{k}, \delta\bar{k}$	vector functions
h	thickness of the plate
(J)	jacobian matrix
$k_i(i=1,2,...,n)$	unknown values
$M_r, M_\theta, M_{r\theta}$	radial, tangential and twisting moments per unit length
Q_r, Q_θ	shearing forces per unit length of element
q	lateral load per unit area
r, θ, z	cylindrical coordinates
t	time variable
u, v	radial, tangential displacements
V_o	element of volume
w	lateral displacement, function of space and time
W	lateral displacement, function of space only
$y_i(i=1,2,...,5)$	subsidiary dependent variables
$(y_i)_l, (y_i)_r$	value of dependent variable at meeting point obtained by integrating from left, from right, respectively
α	$= E_{r\theta}/E_r$
β	$= E_\theta/E_r$

γ	$= G/E_r$
λ^4	linear eigenvalue
λ^2	$= \frac{\rho h}{D_r} a^2 \omega$, frequency parameter
ρ	mass density of the plate
$\sigma_r, \sigma_\theta, \tau_{r\theta}$	radial, tangential and shearing stresses
ω	circular frequency, rad./sec.
ξ	non-dimensional space variable
$( )_,$	partial derivative of ( ) with respect to the subscripts following the comma
$( )'$	$= d( )/d\xi$
$\{ \}$	column vector

INTRODUCTION

Due to the fast development of material science, anisotropic materials play an important role in modern technology. The experimental studies of a material such as plywood show great difference in elastic modulus and flexural rigidities between the principal directions.

If a circular plate with cylindrical aeolotropy is at the same time also orthotropic, this kind of plate is called the cylindrically orthotropic circular plate. That means the principal directions of aeolotropy at a point are in radial and tangential directions. Physically it is obvious that cylindrical orthotropy can not exist at the center of the circular plate since the property of material should be isotropic there. Therefore, there must be a region, including the origin, where there is a transition of properties. That area of the core having properties other than those ascribed to be the general element of the material, will, of course, depend upon the fabrication of the material. If it is formed in the manner previously suggested, the aeolotropic theory would be applied only in that region containing the material so formed, and the solution would be meaningless at the origin in any event. If, however, the property of material is continuous through the origin, correction must be made to the results in this neighborhood, since the mathematical theory does not take into account this singularity. Such aeolotropy may occur in nature as in some cross sections of wood, or may be manufactured, at least approximately, by reinforced materials or plates with stiffeners attached.

The literature contains many analyses of transverse vibrations of cylindrically orthotropic circular plates from the standpoint of small-deflection, thin plate theory. The governing differential equation is derived in terms of the lateral deflections of their middle surface. Past researchers [1-4] dealt with the frequency equation by expressing the function of lateral deflection as an infinite series and considering the regular conditions at the center point and the boundary conditions along the edge. However, the results shown in those papers are not in agreement.

In the present analysis, the shooting method is used to convert the boundary-value problem into an initial-value problem. With a suitable choice of initial values the integration of differential equations are carried out numerically. The frequency parameters are thus found for clamped and simply supported plates with various material constants. Finally a brief conclusion is presented.

DERIVATION OF THE GOVERNING EQUATION

Consider the thin solid plate shown in Fig. 1 and located in its initially undeformed configuration by cylindrical coordinates r, θ and z.

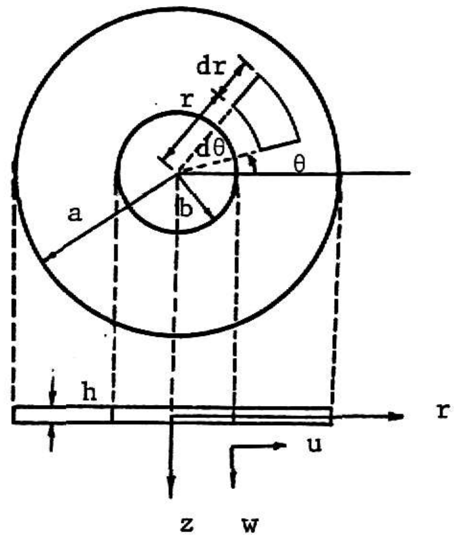

Fig. 1

The plate, except an isotropic core of small radius (r=b) occupying the central portion, is composed of an elastic, homogenous and cylindrically orthotropic medium bounded by the planes $z=\pm h(r)/2$ and cylinders r=b and r=a. From the following, the governing differential equation for free flexural vibration of the plate is derived.

First, it is assumed that the circular plates analyzed in the present investigation are governed by the small-deflection and thin plate theory. Then the bending theory of the plate can be developed by making the following assumptions:

1. The normals of the middle plane before bending are deformed into the normals of the middle plane after bending.

2. The middle plane remains unstrained after bending.

3. The normal stress σ_z is small compared with the other stress components and can be neglected in the stress-strain relations.

Based on these assumptions, the strain-stress relations for a case of plane stress in the cylindrical coordinate system are:

$$\begin{aligned} \sigma_r &= E_r e_r + E_{r\theta} e_\theta \\ \sigma_\theta &= E_{r\theta} e_r + E_\theta e_\theta \\ \tau_{r\theta} &= G\gamma_{r\theta} \end{aligned} \tag{1}$$

where σ_r and σ_θ are normal stresses in the radial and tangential directions respectively, $\tau_{r\theta}$ is the shearing stress, E_r, E_θ, $E_{r\theta}$ and G are the material constants, and e_r and e_θ are the normal strains in the radial and tangential directions respectively, $\tau_{r\theta}$ is the shearing strain.

The strains are defined as

$$\begin{aligned} e_r &= u,_r \\ e_\theta &= u/r + v,_\theta/r \\ \gamma_{r\theta} &= u,_\theta/r + v,_r - v/r \end{aligned} \tag{2}$$

where u,v are radial and tangential displacements, respectively. The comma in the above equations denotes partial differentiation with respect to the coordinate indicated by the subscript following the comma.

Since the middle surface remains undisplaced horizontally, the values of u and v at any point z are given by

$$\begin{aligned} u &= -z\, W_{,r} \\ v &= -(z/r) W_{,\theta} \end{aligned} \tag{3}$$

where W is the lateral displacement and z is measured from the undeformed middle plane.

The relations between strain and displacement can be obtained by substituting Equation (3) into (2)

$$
\begin{aligned}
e_r &= -z\,W_{,rr} \\
e_\theta &= -z(W_{,r}/r + W_{,\theta\theta}/r^2) \qquad (4) \\
\gamma_{r\theta} &= -2z(W_{,\theta}/r)_{,r}
\end{aligned}
$$

The bending moments per unit length can be expressed as

$$
\begin{aligned}
M_r &= \int_{-h/2}^{h/2} \sigma_r\, zdz = -D_r[W_{,rr} + \alpha(W_{,r}/r + W_{,\theta\theta}/r^2)] \\
M_\theta &= \int_{-h/2}^{h/2} \sigma_\theta\, zdz = -D_r[\alpha W_{,rr} + \beta(W_{,r}/r + W_{,\theta\theta}/r^2)] \qquad (5) \\
M_{r\theta} &= \int_{-h/2}^{h/2} \tau_{r\theta} zdz = 2\gamma D_r(W_{,\theta}/r)_{,r}
\end{aligned}
$$

where M_r, M_θ and $M_{r\theta}$ are the radial, tangential, and twisting moments per unit length respectively, and h is the thickness of the plate, $D_r = E_r h^3/12$, $\alpha = E_{r\theta}/E_r$, $\beta = E_\theta/E_r$ and $\gamma = G/E_r$.

The equations of equilibrium of the volumetric element (dr)(rdθ)h has shown in Fig. 1 are

$$
\begin{aligned}
(M_r)_{,r} - (M_{r\theta})_{,\theta}/r + (M_r - M_\theta)/r - Q &= 0 \\
(M_\theta)_{,\theta}/r - (M_{r\theta})_{,r} - 2M_{r\theta}/r - Q_r &= 0 \qquad (6) \\
(rQ_r)_{,r} + (Q_\theta)_{,\theta} + rq &= 0
\end{aligned}
$$

where q is the intensity of the lateral load acting on the plate, and Q_r and Q_θ are the shearing forces per unit length parallel to the z axis and perpendicular to r and θ axes respectively.

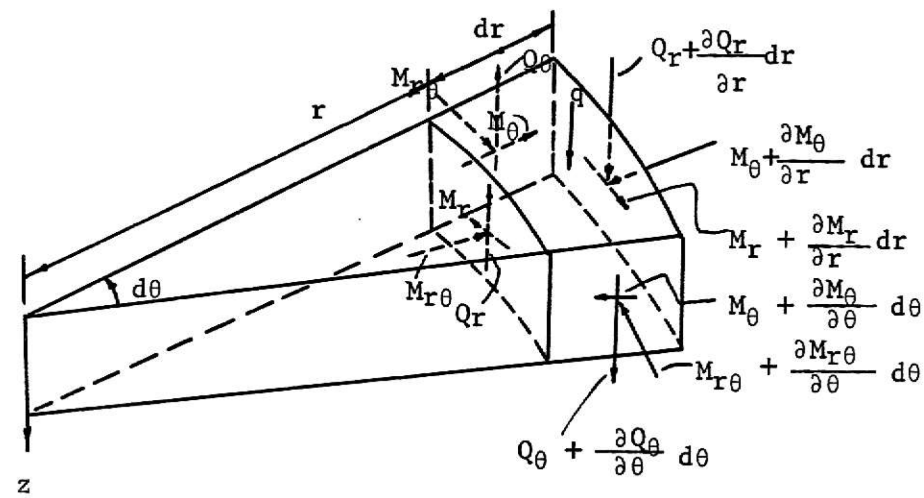

Fig. 2 Sectorial Element Under General Loading

The set of equations in (6) may be written as one equation

$$(rM_r)_{,rr}/r - 2(rM_{r\theta})_{,r\theta}/r^2 + M_{\theta,\theta\theta}/r^2 - M_{\theta,r}/r = -q \tag{7}$$

Since the present problem is concerned with the free vibration of the plate, the loading intensity q becomes

$$q = -\rho h\, W_{,tt} \tag{8}$$

where ρ is the mass density of the plate, and t is the time.

Substitution of Equations (6) and (8) into (7) yields the governing equation for the free vibration of the plate.

$$D_r \; W_{,rrrr} + 2\,\frac{W_{,rrr}}{r} + 2\alpha\left(\frac{W_{,rr\theta\theta}}{r^2} - \frac{W_{,r\theta\theta}}{r^3} + \frac{W_{,\theta\theta}}{r^2}\right)$$

$$+ \beta\left(-\frac{W_{,rr}}{r^2} + \frac{W_{,r}}{r^3} + 2\,\frac{W_{,\theta\theta}}{r^4} + \frac{W_{,\theta\theta\theta\theta}}{r^4}\right)$$

$$+ 4\gamma\left(\frac{W_{,rr\theta\theta}}{r^2} - \frac{W_{,r\theta\theta}}{r^3} + \frac{W_{,\theta\theta}}{r^4}\right)$$

$$- D_{r,r} \quad 2W_{,rrr} - 2\,\frac{W_{,rr}}{r} - \alpha\left(\frac{W_{,rr}}{r} + 2\,\frac{W_{,r\theta\theta}}{r^2} - \frac{W_{,\theta\theta}}{r^3}\right)$$

$$- \beta\left(\frac{W_{,r}}{r^2} - \frac{W_{,\theta\theta}}{r^3}\right) + 4\gamma\left(\frac{W_{,r\theta\theta}}{r^2} - \frac{W_{,\theta\theta}}{r^3}\right)$$

$$+ D_{r,rr} \quad W_{,rr} + \alpha\left(\frac{W_{,r}}{r} - \frac{W_{,\theta\theta}}{r^2}\right)$$

$$+ D_{r,\theta} \quad 2\alpha\,\frac{W_{,rr\theta}}{r^2} + 2\beta\left(\frac{W_{,r\theta}}{r^3} - \frac{W_{,\theta\theta\theta}}{r^4}\right)$$

$$+ 4\gamma\left(\frac{W_{,rr\theta}}{r^2} - \frac{W_{,r\theta}}{r^3} - \frac{W_{,\theta}}{r^4}\right)$$

$$+ D_{r,\theta\theta} \quad \alpha\,\frac{W_{,rr}}{r^2} + \beta\left(\frac{W_{,r}}{r^3} + \frac{W_{,\theta\theta}}{r^4}\right)$$

$$+ D_{r,r\theta} \quad 4\gamma\left(\frac{W_{,r\theta}}{r^2} - \frac{W_{,\theta}}{r^3}\right) \quad = -\rho h W_{,tt} \tag{9}$$

For the axisymmetric case, Equation (9) reduces to

$$W_{,rrrr} + 2W_{,rrr}/r - \beta\; W_{,rr}/r^2 + \beta\; W_{,r}/r^3 = -\,\frac{\rho h}{D_r}\,W_{,tt} \tag{10}$$

The solution to Equation (10) must satisfy the boundary conditions which depend upon the manner in which the edge of the plate is supported. For a clamped plate, the boundary conditions are

$$W = 0 \quad \text{at} \quad r = a$$
$$W_{,r} = 0 \quad \text{at} \quad r = a \tag{11}$$

where a is the radius of the plate.

Also the boundary conditions for a simply supported plate are

$$W = 0 \qquad \text{at} \quad r = a$$
$$W_{,rr} + \alpha\, W_{,r}/r = 0 \qquad \text{at} \quad r = a \tag{12}$$

For axisymmetric vibrations the shearing forces in the plate at radius r must balance the inertia forces as follows:

$$2\pi r Q_r = \int_0^r \rho h\; 2\pi r W_{,tt} dr \tag{13}$$

where Q_r is the intensity of shearing force in the plate at radius r.

By combining Equations (7) and (8) and substituting the result into Equation (13), the following expression is obtained

$$rQ_r = (rM_r)_{,r} - M_\theta \tag{14}$$

Equation (14) may also be expressed in terms of the displacement of W as

$$Q_r = -D_r(rW_{,rrr} + W_{,rr} - \beta\, W_{,r}/r^2) \tag{15}$$

The left hand side of Equation (15) becomes zero at the center (r = 0). As for the right hand side of the Equation (15), it is evident that at r = 0

$$D_r(rW_{,rrr} + W_{,rr} - \beta\, W_{,r}/r) = 0 \quad .$$

Also at r = 0
$$W_{,r} = 0 \tag{16}$$

since the slope of the plate must be zero at the center.

For simplicity, it is convenient to normalize the radius of the plate by introducing the non-dimensional variable $\xi = r/a$. Then Equations (10), (11) and (12) become

$$W_{,\xi\xi\xi\xi} + 2\,W_{,\xi\xi\xi}/\xi - \beta\,W_{,\xi\xi}/\xi^2 + \beta\,W_{,\xi}/\xi^3 = -\frac{\rho h a^4}{D_r}\,W_{,tt} \tag{17}$$

$$\begin{aligned} W &= 0 \qquad \text{at} \qquad \xi = 1 \\ W_{,\xi} &= 0 \qquad \text{at} \qquad \xi = 1 \end{aligned} \tag{18}$$

$$\begin{aligned} W &= 0 \qquad \text{at} \qquad \xi = 1 \\ W_{,\xi\xi} + \alpha\,W_{,\xi}/\xi &= 0 \qquad \text{at} \qquad \xi = 1 \end{aligned} \tag{19}$$

Equation (17) can be solved by the Method of separation of variables. Let

$$W(\xi,t) = \overline{W}(\xi)e^{i\omega t} \tag{20}$$

where $\overline{W}$ is a function of ξ only, $i = \sqrt{-1}$, and ω is the circular frequency.

By substituting Equation (20) into (17), the following expression is obtained

$$\overline{W}'''' + 2\,\overline{W}'''/\xi - \beta\,\overline{W}''/\xi^2 + \beta\,\overline{W}'/\xi^3 = \lambda^4\overline{W} \tag{21}$$

where $\lambda^4 = \dfrac{\rho h a^4 \omega^2}{D_r}$, and a prime over a symbol denotes differentiation with respect to ξ.

LIMITATIONS OF THE THEORY

The generalized Hooke's law for the case of cylindrical aeolotropy can be written in the form

$$\begin{aligned} e_\theta &= a_{11}\sigma_\theta + a_{12}\sigma_r \\ e_r &= a_{12}\sigma_\theta + a_{22}\sigma_r \\ e_z &= a_{13}\sigma_\theta + a_{23}\sigma_r \\ \gamma_{r\theta} &= a_{66}\tau_{r\theta} \end{aligned} \tag{22}$$

where a_{ij} are elastic constants.

After considering the limitations on the values of the elastic constants, under which this theory is applicable to aeolotropic plates, Carrier (5) observes that we must restrict the values of the ratios a_{12}/a_{11} and a_{12}/a_{22}.

Unfortunately very little experimental data have been published on the values of elastic constants of aeolotropic materials, so that the arguments used here must be based on the properties observed in isotropic materials, and hence these arguments are hypothetical in nature.

It seems logical to assume that, in general, the strain in a direction normal to an applied tensile stress will be negative and that the change in the volume of an element subjected to a tensile stress will be positive. Hence, a_{12}, a_{13} and a_{23} will be negative and such that the change in an element of volume V_o, under tensile stress, will obey the inequality

$$V_o(e_r + e_\theta + e_z) > 0 \; . \tag{23}$$

When there is only a radial tensile stress σ_r the elemental volume change becomes

$$V_o\sigma_r(a_{12} + a_{22} + a_{23}) > 0 \tag{24a}$$

and for a tangential tensile stress σ_θ

$$V_o\sigma_\theta(a_{11} + a_{12} + a_{13}) > 0 \tag{24b}$$

Equations (24.a) and (24.b) can be rewritten as follows:

$$\begin{aligned} &- \frac{a_{12}}{a_{22}} - \frac{a_{23}}{a_{22}} < 1 \\ &- \frac{a_{12}}{a_{11}} - \frac{a_{13}}{a_{11}} < 1 \end{aligned} \tag{25}$$

where, under the former of the two assumptions just made, each of the left-hand terms is positive. This restricts $-a_{12}/a_{22}$ and $-a_{12}/a_{11}$ to be between 0 and 1; i.e.,

$$\begin{aligned} &0 < - \frac{a_{12}}{a_{22}} < 1 \\ &0 < - \frac{a_{12}}{a_{11}} < 1 \end{aligned} \tag{26}$$

The relation between elastic constants and material constants of cylindrically aeolotropic material can be established by comparing Equation (22) with (1)

$$\begin{aligned} &- \frac{a_{12}}{a_{11}} = \frac{E_{r\theta}}{E_r} = \alpha \\ &- \frac{a_{12}}{a_{22}} = \frac{E_{r\theta}}{E_\theta} = \frac{E_{r\theta}}{E_r}\,\frac{E_r}{E_\theta} = \frac{\alpha}{\beta} \end{aligned} \tag{27}$$

The combination of Equations (26) and (27) yields the following restrictions on the values of α and β

$$\begin{aligned} &0 < \alpha < 1 \\ &\alpha < \beta \end{aligned} \tag{28}$$

ANALYSIS OF THE SHOOTING METHOD (6)

I. General Formulation

Any n^{th}-order differential equation, linear or non-linear, may be reduced to n simultaneous first-order equations

$$\frac{dy_1}{dr} = f_1(y_1, y_2, \cdots, y_n, r)$$

$$\frac{dy_2}{dr} = f_2(y_1, y_2, \cdots, y_n, r) \qquad (29a)$$

- - - - - - - - - - - - - - - -

$$\frac{dy_n}{dr} = f_n(y_1, y_2, \cdots, y_n, r)$$

which can be conveniently written in the form

$$\frac{d\bar{Y}}{dr} = \bar{F}(\bar{Y}, r) \qquad (29b)$$

where $y_i (i=1,2,\ldots,n)$ are dependent variables and r is an independent variable.

To specify a unique solution of Equation (29.b) the n boundary conditions must be provided at the two points $r=r_0$, $r=r_1$ $(>r_0)$. In case that n_1 condiations are given at $r=r_0$, then n_0 $(=n-n_1)$ conditions are given at $r=r_1$.

If the unknown boundary values at one boundary $r=r_0$ say, are assigned arbitrary values, step-by-step integration of Equation (29) from $r=r_0$ to $r=r_1$ would then be possible. Apparently the values of y at $r=r_1$ thus obtained would not in general satisfy the given conditions there. The problem is therefore to determine changes of the assumed unknown values at $r=r_0$ so that the solution can satisfy the given conditions at $r=r_1$.

In general it is more convenient and often far more practical to estimate the boundary values at both boundaries, and then integrate inwards to a meeting point. Changes can be made in all the unknown boundary values to make two integration curves satisfy the continuity conditions at the meeting point.

Let $k_1, k_2, \ldots, k_n$ denote the n unknown boundary values of the y_i at the two boundaries. The n_o unknowns $k_1, k_2, \ldots, k_{n_o}$ are at $r=r_0$ and n_1 unknowns $k_{n_o+1}, \ldots, k_n$ are at $r=r_1$. Once an estimate of the k_i has been made, integration can be performed from both boundaries to the meeting point. If the values of y_i at this point found by integration from the left are $(y_i)_l$ and those found by integration from the right are $(y_i)_r$ then

$$g_i = (y_i)_\ell - (y_i)_r \tag{30}$$

A solution of the boundary value problem is obtained if $g_i=0$. For arbitrarily chosen k_i, the g_i will not be zero. Nevertheless, it is possible to calculate the changes required in k_i to make the g_i arbitrarily small.

The functions g_i are not explicit functions of k_i. However they do depend solely upon the k_i through the integration of Equation (29). If $g_i(k_1, k_2, \ldots, k_n)=0$, then, by Newton's Method, a better approximation of $k_i+\delta k_i$ can be found by solving the linear algebraic equations

$$\sum_{j=1}^{n} \frac{\partial g_i}{\partial k_j} \delta k_j = -g_i(k_1, k_2, \ldots, k_n) \tag{31}$$

$$\text{i.e.} \quad (J)\delta\bar{k} = -\bar{g}(k_i)$$

The n^2 elements of the Jacobian matrix, $\frac{\partial g_i}{\partial k_j}$ $(i,j=1,2,\ldots,n)$, may be determined as follows. From the definitions of the g_i, it can easily be seen that

$$\frac{\partial g_i}{\partial k_j} = \begin{cases} (\partial y_i/\partial k_j)_\ell & \text{if} \quad j = 1,2,\ldots,n_o \\ -(\partial y_i/\partial k_j)_r & \text{if} \quad j = n_o+1,\ldots,n \end{cases} \tag{32}$$

Differential equations for the variables $\partial y_i/\partial k_j$ can be found by formally differentiating in Equation (29.b) with respect to k_j. Thus

$$\frac{\partial}{\partial k_j}\left(\frac{d\bar{Y}}{dr}\right) = \sum_{i=1}^{n} \frac{\partial \bar{F}}{\partial y_i}\left(\frac{\partial y_i}{\partial k_j}\right) + \frac{\partial F}{\partial k_j}\,, \qquad (j = 1,2,\ldots,n_o) \qquad (33.a)$$

where the second term on the right hand side of Equation (33.a) is zero if $\bar{F}$ does not depend explicitly upon the k_j. The order of differentiation on the left-hand side can be interchanged to give

$$\frac{d}{dr}\left(\frac{\partial \bar{Y}}{\partial k_j}\right) = \sum_{i=1}^{n} \frac{\partial F}{\partial y_i}\left(\frac{\partial y_i}{\partial k_j}\right) + \frac{\partial \bar{F}}{\partial k_j}\,, \qquad (j = 1,2,\ldots,n_o) \qquad (33.b)$$

This is a set of first-order differential equations for the variables $\partial y_i/\partial k_j$. The coefficients $\partial \bar{F}/\partial y_j$ can easily be found by differentiating the right-hand side of Equation (29.b) with respect to y_i have been renamed k_j, the boundary values of $\partial y_i/\partial k_j$ at $r = r_0$, which are needed for the integration of Equation (33.b), is unity if k_j is the renamed y_i, and otherwise zero.

II. Treatment of Eigenvalue Problems

If the differential equation contains an eigenvalue λ the shooting method can still be applied. The eigenvalue is considered as another variable, say y_{n+1}, and the equation $dy_{n+1}/dr=0$ is added to the set of Equation (29.a). Thus an n^{th}-order differential equation which contains an eigenvalue is considered as a standard $(n+1)^{th}$-order equation with two-point boundary conditions.

To solve this new set of n+1 equations, n+1 boundary conditions are needed. There are n conditions provided with the original equation, and the remaining condition is obtained from a normalizing condition which is used to fix the value of the eigenfunction or one of its derivatives at one of the boundaries.

To start the calculation of the eigensolutions, values of unknown k_j must be supplied. Any physical information which is available will be useful in this context. When several independent eigensolutions have been computed, good estimate for the k_j of other eigensolutions can be found by extrapolation.

SOLUTION FOR THE FREE VIBRATION OF CYLINDRICALLY AEOLOTROPIC CIRCULAR PLATES

In an attempt to solve the governing Equation (21), the differential equation can be transformed by introducing

$$
\begin{aligned}
y_1 &= \bar{W} \\
y_2 &= \bar{W}' \\
y_3 &= \bar{W}'' \\
y_4 &= \bar{W}''' \\
y_5 &= \lambda^4
\end{aligned}
\tag{34}
$$

and their derivatives

$$
\begin{aligned}
y_1' &= y_2 \\
y_2' &= y_3 \\
y_3' &= y_4 \\
y_4' &= \bar{W}'''' \\
y_5' &= 0
\end{aligned}
\tag{35}
$$

into a set of five simultaneous first-order differential equations

$$y_1' = y_2$$

$$y_2' = y_3$$

$$y_3' = y_4 \tag{36}$$

$$y_4' = y_1 y_5 - \frac{\beta}{\xi^3} y_2 + \frac{\beta}{\xi^2} y_3 - \frac{2}{\xi} y_4$$

$$y_5' = 0$$

which can be conveniently written in the form

$$\frac{d\bar{Y}}{d\xi} = \bar{F}(\bar{Y},\xi) \qquad (0 < \xi \leq 1) \tag{37}$$

where

$$\bar{Y}(\xi) = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{Bmatrix}, \qquad \bar{F}(\bar{Y},\xi) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ y_5 & -\beta/\xi^3 & \beta/\xi^2 & -2/\xi & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Examination of Equation (36) shows that there exists a singular point at the origin (ξ=0). This singular point causes unboundedness unless some special treatment is added. Since it is known that in the case of isotropic circular plate $\bar{F}$ is an analytic function in the neighborhood of the origin, MacLaurin Series Expansion is then permissibly employed to remove this singular point.

$$y_1 = y_1^o + \frac{y_2^o}{1!}\xi + \frac{y_3^o}{2!}\xi^2 + \frac{y_4^o}{3!}\xi^3 + \frac{(y_4^o)'}{4!}\xi^4 + \ldots$$

$$y_2 = y_2^o + \frac{y_3^o}{1!}\xi + \frac{y_4^o}{2!}\xi^2 + \frac{(y_4^o)'}{3!}\xi^3 + \ldots$$

$$y_3 = y_3^o \quad \frac{y_4^o}{1!}\xi + \frac{(y_4^o)'}{2!}\xi^2 + \ldots$$

$$y_4 = y_4^o + \frac{(y_4^o)'}{1!}\xi + \ldots$$

Substituting these expansions into Equation (36) and letting β be equal to 1 yields the derivative $\dot{y}_4$ as follows:

$$(y_4^o)' = y_1^o y_5^o - \frac{1}{\xi^3} y_2^o - \frac{3}{2}\frac{y_4^o}{\xi} - \frac{5}{3}(y_4^o)' \tag{38}$$

Boundedness and continuity at the origin result in

$$y_2^o = 0, \qquad y_4^o = 0 \tag{39}$$

With the above conditions, Equation (36) reduces

$$\begin{aligned} (y_1^o)' &= 0 \\ (y_2^o)' &= y_3^o \\ (y_3^o)' &= 0 \\ (y_4^o)' &= 3/8\ y_1^o y_5^o \\ (y_5^o)' &= 0 \end{aligned} \tag{40}$$

Since the differential equations and the boundary conditions are homogeneous, the solution can be normalized so that y_1=1 at ξ=0. The rest of two known boundary conditions will be found at $\xi=1$. Thus, for the clamped plate, the starting values for the integration are

$$\bar{Y}(0) = \begin{Bmatrix} 1 \\ 0 \\ k_1 \\ 0 \\ k_2 \end{Bmatrix} \qquad \bar{Y}(1) = \begin{Bmatrix} 0 \\ 0 \\ k_3 \\ k_4 \\ k_5 \end{Bmatrix} \tag{41.a}$$

while for the simply supported plate the initial values become

$$\bar{Y}(0) = \begin{Bmatrix} 1 \\ 0 \\ k_1 \\ 0 \\ k_2 \end{Bmatrix} \qquad \bar{Y}(1) = \begin{Bmatrix} 0 \\ k_3 \\ -\alpha k_3 \\ k_4 \\ k_5 \end{Bmatrix} \tag{41.b}$$

where $k_i (i=1,2,...,5)$ are the unknown boundary values.

To start the integration for the set of Equation (37) the unknown values k_1,k_2 at $\xi=0$ and k_3,k_4 and k_5 at $\xi=1$ must be guessed. It should be noted that since k_2 and k_5 both represent λ^4, they must have same value throughout the calculation. Consequently, they must be made equal initially and then the process automatically ensures that they remain equal. Therefore, instead of solving that boundary-value problem we simply work on the initial-value problem. The numerical integration can be obtained by the usual Runge-Kutta-Gill integration process. The meeting point for two directional integrations is chosen to be $\xi=b/a$. At that point the difference between the nondimensional quantities of each w, $w,_r$, M_r, Q_r, λ^4 obtained from the integration is indicated by E_i.

$$E_1 = (y_1)_\ell - (y_1)_r$$

$$E_2 = (y_2)_\ell - (y_2)_r$$

$$E_3 = (y_3)_\ell - (y_3)_r + \frac{a}{b}\alpha[(y_2)_\ell - (y_2)_r] \tag{42}$$

$$E_4 = (y_4)_\ell - (y_4)_r + \frac{a}{b}[(y_3)_\ell - (y_3)_r] - \frac{a}{b}[(y_2)_\ell - (y_2)_r]$$

$$E_5 = (y_5)_\ell - (y_5)_r$$

If $(y_i)_1$ are really close to $(y_i)_r$ at $\xi=b/a$, the linear approximations yield

$$(J)\begin{Bmatrix}\delta k_1\\ \delta k_2\\ \delta k_3\\ \delta k_4\\ \delta k_5\end{Bmatrix} = -\begin{Bmatrix}E_1\\ E_2\\ E_3\\ E_4\\ E_5\end{Bmatrix} \tag{43}$$

where $\delta k_i (i=1,2,...,5)$ are the elements of the correction vector, and

$$(J) = \begin{bmatrix} \frac{\partial E_1}{\partial k_1} & \frac{\partial E_1}{\partial k_2} & \frac{\partial E_1}{\partial k_3} & \frac{\partial E_1}{\partial k_4} & \frac{\partial E_1}{\partial k_5} \\ \frac{\partial E_2}{\partial k_1} & \frac{\partial E_2}{\partial k_2} & \frac{\partial E_2}{\partial k_3} & \frac{\partial E_2}{\partial k_4} & \frac{\partial E_2}{\partial k_5} \\ - & - & - & - & - \\ \frac{\partial E_5}{\partial k_1} & \frac{\partial E_5}{\partial k_2} & \frac{\partial E_5}{\partial k_3} & \frac{\partial E_5}{\partial k_4} & \frac{\partial E_5}{\partial k_5} \end{bmatrix}$$

the correction vector as obtained from Equation (43) is

$$\begin{Bmatrix} \delta k_1 \\ \delta k_2 \\ \delta k_3 \\ \delta k_4 \\ \delta k_5 \end{Bmatrix} = -\ (J)^{-1} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{Bmatrix} \tag{44}$$

In an attempt to find a set of values of k_i, Newton's Method is applied thus

$$(k_i)_{n+1} = (k_i)_n + (\delta k_i)_n \qquad (i=1,2,\ldots,5) \tag{45}$$

where

$$\begin{Bmatrix} \delta k_1 \\ \delta k_2 \\ \delta k_3 \\ \delta k_4 \\ \delta k_5 \end{Bmatrix}_n = -\ (J)_n^{-1} \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \end{Bmatrix}_n$$

and $(J)_n$ and $(E)_n$ are formed at the n^{th} step. In obtaining the final values the successive corrections of k_i are performed until the error vector $\bar{\bar{E}}$ at $\xi=b/a$ satisfies a given norm 10^{-6}.

Differentiating Equation (36) with respect to k_j, the equations for the variable $\partial y_i/\partial k_j$ of Equation (33.b) can be written as follows $(j=1,2,\ldots,5)$:

$$\frac{d}{d\xi}\left(\frac{\partial y_1}{\partial k_j}\right) = \frac{\partial y_2}{\partial k_j}$$

$$\frac{d}{d\xi}\left(\frac{\partial y_2}{\partial k_j}\right) = \frac{\partial y_3}{\partial k_j}$$

$$\frac{d}{d\xi}\left(\frac{\partial y_3}{\partial k_j}\right) = \frac{\partial y_4}{\partial k_j} \tag{46}$$

$$\frac{d}{d\xi}\left(\frac{\partial y_4}{\partial k_j}\right) = y_5 \frac{\partial y_1}{\partial k_j} + y_1 \frac{\partial y_5}{\partial k_j} - \frac{2}{\xi}\frac{\partial y_4}{\partial k_j} + \frac{\beta}{\xi^2}\frac{\partial y_3}{\partial k_j} - \frac{\beta}{\xi^3}\frac{\partial y_2}{\partial k_j}$$

$$\frac{d}{d\xi}\left(\frac{\partial y_5}{\partial k_j}\right) = 0$$

and at $\xi=0$, the derivative of $\partial y_4/\partial k_j$ is

$$\frac{d}{d\xi}\left(\frac{\partial y_4}{\partial k_j}\right) = \frac{3}{8} \quad y_5 \frac{\partial y_1}{\partial k_j} + y_1 \frac{\partial y_5}{\partial k_j} \tag{47}$$

Starting values for the integration of Equation (46) are

$$\frac{\partial\bar{Y}(0)}{\partial k_1} = \begin{Bmatrix}0\\0\\1\\0\\0\end{Bmatrix}, \quad \frac{\partial\bar{Y}(0)}{\partial k_2} = \begin{Bmatrix}0\\0\\0\\0\\1\end{Bmatrix}, \quad \frac{\partial\bar{Y}(1)}{\partial k_3} = \begin{Bmatrix}0\\0\\1\\0\\0\end{Bmatrix}, \quad \frac{\partial\bar{Y}(1)}{\partial k_4} = \begin{Bmatrix}0\\0\\0\\1\\0\end{Bmatrix}, \quad \frac{\partial\bar{Y}(1)}{\partial k_5} = \begin{Bmatrix}0\\0\\0\\0\\1\end{Bmatrix}$$

(48.a)

and for the simply supported plate the starting values becomes

$$\frac{\partial\bar{Y}(0)}{\partial k_1} = \begin{Bmatrix}0\\0\\1\\0\\0\end{Bmatrix}, \quad \frac{\partial\bar{Y}(0)}{\partial k_2} = \begin{Bmatrix}0\\0\\0\\0\\1\end{Bmatrix}, \quad \frac{\partial\bar{Y}(1)}{\partial k_3} = \begin{Bmatrix}0\\0\\-\alpha\\0\\0\end{Bmatrix}, \quad \frac{\partial\bar{Y}(1)}{\partial k_4} = \begin{Bmatrix}0\\0\\0\\1\\0\end{Bmatrix}, \quad \frac{\partial\bar{Y}(1)}{\partial k_5} = \begin{Bmatrix}0\\0\\0\\0\\1\end{Bmatrix}$$

(48.b)

By integrating Equations (46) simultaneously with initial conditions, the values necessary to construct the Jacobian matrix (J) at every step point are obtained.

NUMERICAL RESULTS

In view of the limitation on the ratio of the numerical constants, α is taken to be 0.3 and β is chosen from 0.1 to 2.0. It should be noted that $\beta = 1$ is the particular case of isotropy and α is a number corresponding to the Poisson's ratio appearing in the isotropic theory. The computed frequency parameters $\lambda^2 = \sqrt{\frac{\rho h}{D_r}}\, a^2 \omega$, corresponding to the lowest frequencies of a circular plate with an isotropic core of radius b at the central portion, are presented in the following tables and graphs. All the computations were performed on an IBM 360/50 computer at the University's Computer Center.

β	λ^2 for values of b/a			
	0.3	0.2	0.1	0.025
0.1	9.152	8.852	8.542	8.284
0.2	9.279	9.024	8.776	8.598
0.3	9.404	9.190	8.993	8.871
0.4	9.527	9.351	9.197	9.113
0.5	9.647	9.506	9.389	9.335
0.6	9.764	9.657	9.571	9.534
0.7	9.880	9.802	9.743	9.720
0.8	9.994	9.944	9.907	9.895
0.9	10.105	10.081	10.065	10.059
1.0	10.215	10.215	10.215	10.215
1.1	10.323	10.345	10.360	10.364
1.2	10.429	10.472	10.499	10.506
1.3	10.534	10.596	10.634	10.642
1.4	10.637	10.717	10.764	10.774
1.5	10.738	10.835	10.890	10.990
1.6	10.838	10.951	11.013	11.023
1.7	10.937	11.064	11.131	11.142
1.8	11.033	11.175	11.247	11.258
1.9	11.129	11.283	11.360	11.370
2.0	11.224	11.389	11.470	11.480

Table I. Axisymmetric Frequency Parameters $\lambda^2 = \sqrt{\frac{\rho h}{D_r}}\, a^2 \omega$ for a Clamped Circular Plate Having Polar Orthotropy and with an Isotropic Core in the Central Portion

β	λ^2 for values of b/a			
	0.3	0.2	0.1	0.025
0.1	3.606	3.403	3.192	3.009
0.2	3.781	3.616	3.457	3.343
0.3	3.946	3.814	3.695	3.622
0.4	4.104	4.000	3.911	3.865
0.5	4.256	4.175	4.110	4.081
0.6	4.401	4.341	4.296	4.277
0.7	4.542	4.499	4.469	4.458
0.8	4.677	4.651	4.626	4.627
0.9	4.808	4.796	4.787	4.785
1.0	4.935	5.935	5.935	4.935
1.1	5.058	5.069	5.076	5.077
1.2	5.178	4.199	4.211	4.213
1.3	5.295	5.324	5.341	5.344
1.4	5.409	5.446	5.446	5.469
1.5	5.519	5.564	5.587	5.590
1.6	5.628	5.679	5.704	5.707
1.7	5.734	5.790	5.818	5.821
1.8	5.837	5.899	5.928	5.931
1.9	5.939	6.005	6.035	6.038
2.0	6.039	6.109	6.140	6.143

Table II. Axisymmetric Frequency Parameters $\lambda^2 = \sqrt{\frac{\rho h}{D_r}}\, a^2 \omega$ for a simply Supported Circular Plate Having Polar Orthotropy and with an Isotropic Core in the Central Portion

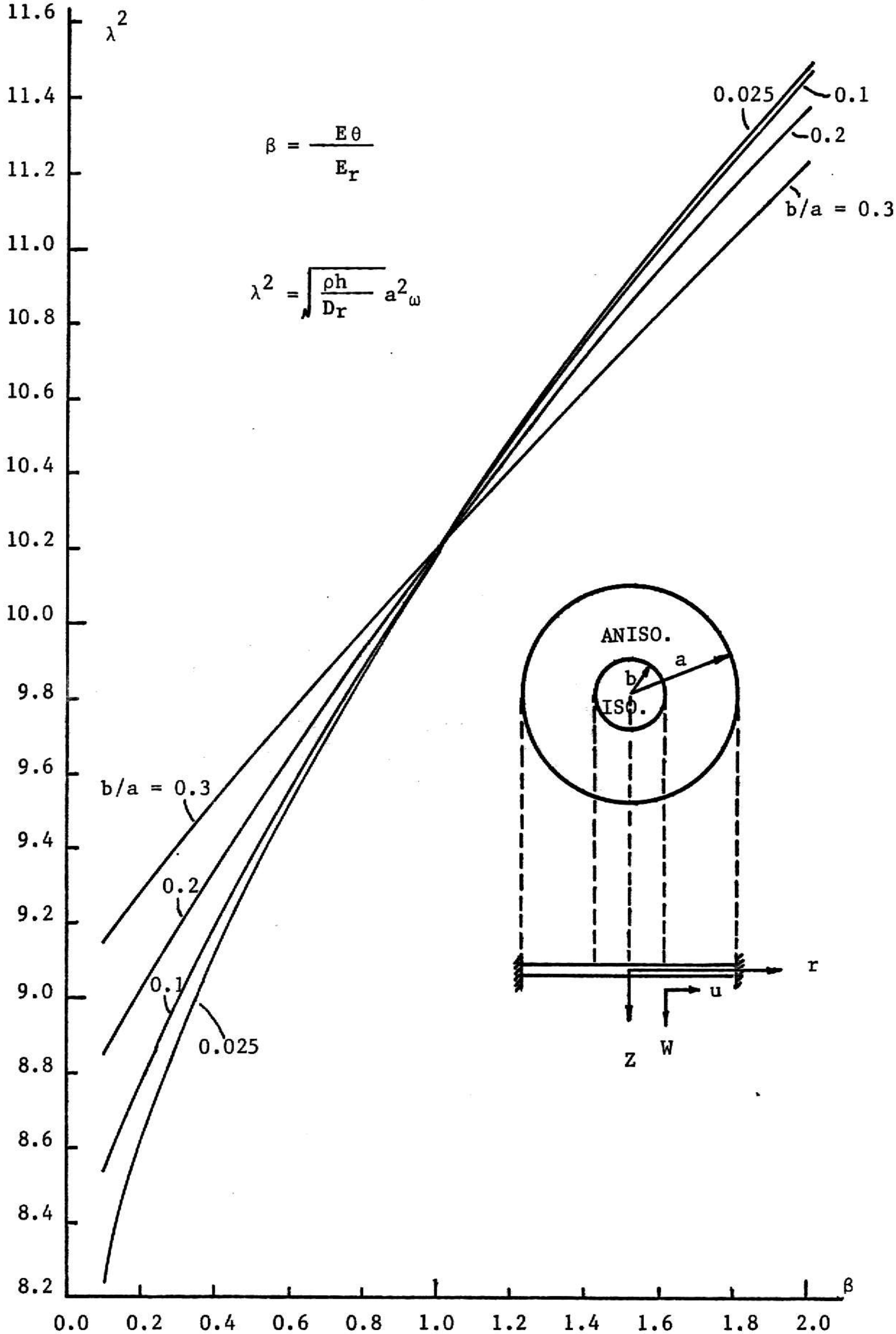

Fig. 3 Frequency Parameters for a Clamped Plate Having Polar Orthotropy and with an Isotropic Core in the Central Portion

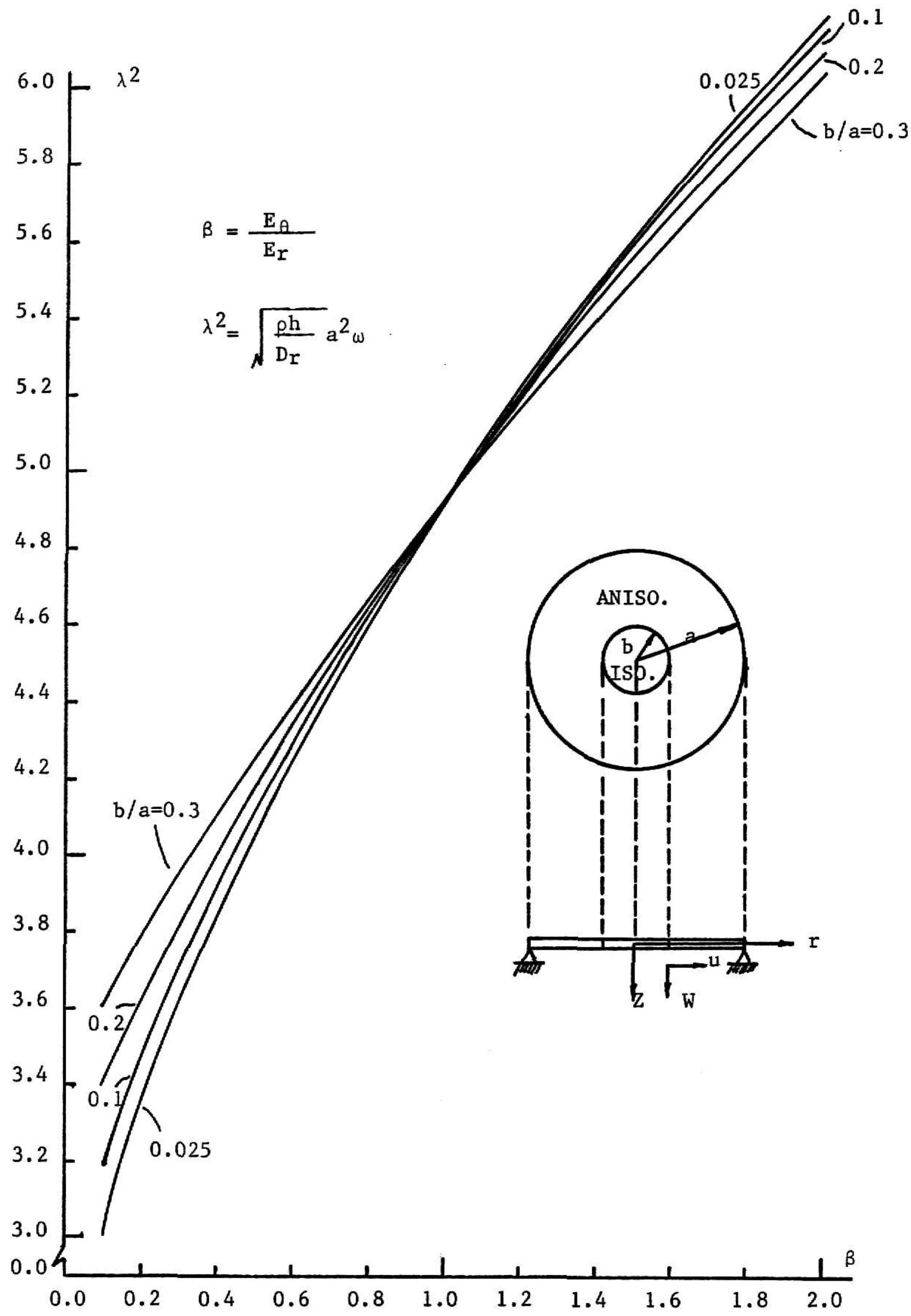

Fig. 4 Frequency parameters for a Simply Supported Circular Plate Having Polar Orthotropy and with a Isotropic Core in the Central Portion

CONCLUSION

The shooting method for determining the natural frequencies of the cylindrically aeolotropic circular plates has been presented. This method is very useful and has the property of the quadratic convergence if the initial estimates of the unknown boundary values are close enough to the solution for the boundary-value problem.

It has been noted that the cylindrical orthotropy cannot exist physically at the center of the plate and the center of the plate must be treated as an isotropic core of a small radius Δ. The "regularity" conditions at the center are then applied to the solution for the isotropic core; and for the solution of the problem the continuity of W, $W_{,r}$, M_r, Q_r are enforced at $r=\Delta$, the boundary of the isotropic core, and the orthotropic plate. Numerical results show the frequency parameters for the plate with an isotropic core of radius 0.025 are the upper bounds for the cases where $\Delta < 0.025$ and $\beta<1$; and are the lower bounds for the cases where $\Delta < 0.025$ and $\beta>1$. In addition, it is seen the frequency parameters λ^2 for an axisymmetric vibrations increase as the material constant β increases.

REFERENCES

[1] Akasaka, T., and Takagishi, T., "Vibration of Corrugated Diaphragm," Bull. JSME, Vol. 1, 3, pp. 215-221, 1958.

[2] Mossakowski, J., and Borsuk, K., "Buckling and Vibrations of Circular Plates with Cylindrical Orthotropy," Applied Mechanics, 10th Internation Applied Mechanics Conference, Italy, edited by Rolla and Koiter, Elsevier, pp. 266-269, 1960.

[3] Minkarah, I. A., and Hoppmann, W. H. II, "Flexural Vibrations of Cylindrically Aeolotropic Circular Plates," Jour. Acoust. Soc. Am., Vol. 36, 3, pp. 470-475, 1964.

[4] Pandalai, K. A. V., and Patel, S. A., "Natural Frequencies of Orthotropic Circular Plates," AIAA, Vol. 3, 4, pp. 780-781, 1965.

[5] Carrier, G. F., "The Bending of the Cylindrically Aeolotropic Plate," Journal of Applied Mechanics, Sept. 1944.

[6] Wadsworth, M., and Wilde, E., "Differential Eigenvalue Problem with Particular Reference to Rotor Blade Bending," The Aero. Quart. 19, pp. 192-204, 1968.

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FREE VIBRATIONS OF CYLINDRICALLY
AEOLOTROPIC CIRCULAR PLATES

by

HUA-KAWI WOO
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ABSTRACT

This report is concerned with the determination of the cylindrically aeolotropic circular plates having uniform thickness. The derivation of the differential equation governing the motion of the plate is based on the usual assumption of small-deflection and thin plate theory. The solution of the equation is obtained by an application of the shooting method. Numerical results of frequency parameters corresponding to various material constants are shown by tables and graphs for clamped and simply supported plates.