THE APPLICATION OF THE DISCRETE MAXIMUM PRINCIPLE TO TRANSPORTATION PROBLEMS WITH LINEAR COST FUNCTIONS

by

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INTRODUCTION

Transportation problems involving optimizing objective cost function of the linear form can be regarded as a generalization of the assignment problems and thus can be solved by the Simplex Method of Linear Programming. But the special Transportation Methods like Northwest Corner Method, Unit Penalty Method, or Vogel's Approximation Method are developed, which are easy to apply and are less tedious methods than the Simplex Method (2, 7, 9).

Transportation problems involving optimizing objective cost function of non-linear type are solved by methodology of Dynamic Programming (1). Recently attempts were successfully made to solve these transportation problems, involving non-linear cost function, with the Discrete Maximum Principle (5). The Maximum Principle for continuous processes was originally developed by Pontryagin (8), and the Discrete version of this Maximum Principle was independently proposed by Chang (3) and Katz (6) and was developed further by Fan and Wang (5).

The aim of this report is to find and to show a method of solving the transportation problems involving optimizing objective function of the linear form, with the application of The Discrete Maximum Principle. To show the application of this Discrete Maximum Principle to the transportation problems involving linear cost function, two specific examples are solved and explained in detail. In the first problem, the optimal solution is derived by the application of this method for the problem with two origins (say factories) where the resource is located, and five destinations (say warehouses) where demand for this resource exists. In the second problem, consideration is given to the problem with three origins and four destinations. In either case of these problems, all the feasible solutions are derived in order to represent the application of this technique. Examples 3 and 4 are solved, showing the method of attacking and solving this type of problem in order to achieve optimal solution easily and directly by following the general rules mentioned on pages 22 to 25.

THE DISCRETE MAXIMUM PRINCIPLE

The following is an outline of the general algorithm of The Discrete Maximum Principle given by Fan and Wang (5).

Figure 1 represents a multistage decision process consisting of N stages in sequence. The state of the process stream denoted by an s-dimensional vector, X, is transformed at each stage according to the decision made on the control actions denoted by t-dimentional vector, θ. The transformation of the process stream thus resulted at the n^{th} stage is given by the transformation operator,

$$X_i^n = T_i^n \,(X_k^{n-1};\ \theta^n), \quad \text{- -} \quad (1)^1$$

$$n = 1, 2, \ldots, N; \; i = 1, 2, \ldots, s.$$

The optimization problem is to find the sequence of θ^n, subject to condition, $\alpha^n \leq \theta^n \leq \beta^n$, n = 1, 2,...., N, which will minimize X_j^N, with X_i^0 preassigned, i = 1, 2,...., s.

The procedure for finding the optimal sequence of θ^n which minimizes X_j^N, is to introduce an s-dimensional covariant vector, Z^n, and a Hamiltonian function H^n satisfying

$$H^n = \sum_{i=1}^{s} Z_i^n \, T_i^n \,(X_k^{n-1};\ \theta^n) \quad \text{- - - - - - - - - - - - - - - - - - -} \quad (2)$$

$$n = 1, 2, \ldots, N; \; i = 1, 2, \ldots, s$$

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[1] The superscript, n, indicates the stage number, and the exponents will be written in parenthesis like $(X^n)^2$.

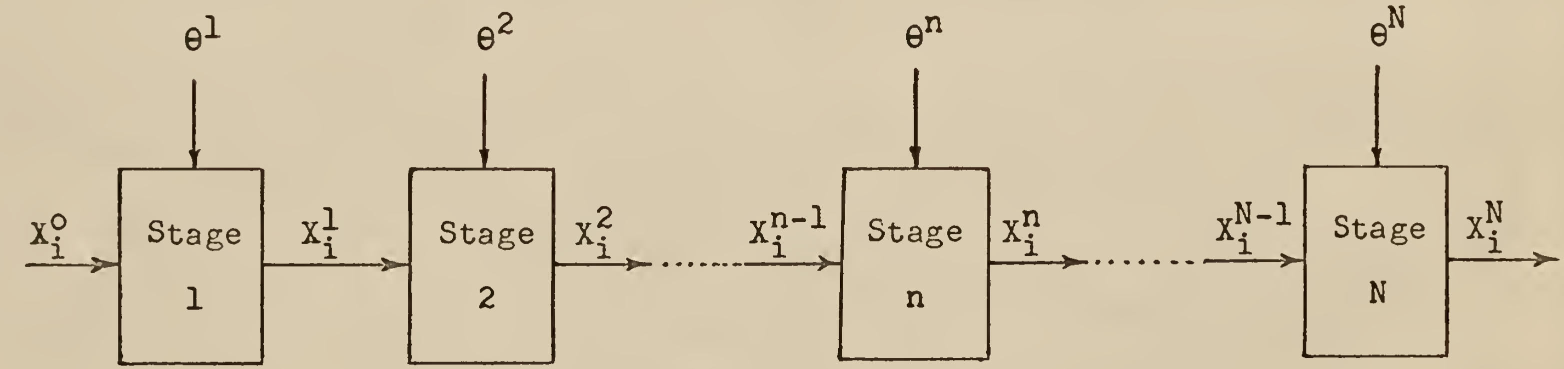

Fig. 1. Multistage decision process.

$$X_i^n = \frac{\partial H^n}{\partial Z_i^n}, \quad n = 1, 2, \ldots, N; \; i = 1, 2, \ldots, s \text{ - - - - } (3)$$

$$Z_i^{n-1} = \frac{\partial H^n}{\partial X_i^{n-1}}, \; n = 1, 2, \ldots, N; \; i = 1, 2, \ldots, s \text{ - - - } (4)$$

$$Z_i^N = \delta_{ij} \; ; \quad i = 1, 2, \ldots, s \text{ - - - - - - - - - - - - - - - } (5)$$

and to determine the optimal sequence of control actions, θ^n, such that H^n = minimum, $n = 1, 2, \ldots, N$ - - - - - - - - - - - - (6)

where δ_{ij} is the kronecker delta ($\delta_{ij} = 1$, if $i = j$ and $\delta_{ij} = 0$, if $i \neq j$).

For the problem in which the final values of state variables, say X_a^N and X_b^N are to be kept constant equal to given values W_a and W_b, respectively, that is

$$X_a^N = W_a \text{ and } X_b^N = W_b \text{ - } (7)$$

then the general algorithm is still applicable with the modification in equation (5), as follows:

$$Z_i^N = \delta_{ij} \begin{cases} = 1, \text{ for } i = j \\ = 0, \text{ for } i \neq a, b, j \\ = C_i, \text{ some constant when } i = a, b. \end{cases} \text{ - - - - - - - - - - - - - - } (8)$$

The missing conditions for $i = a$ and $i = b$ are made up by equation (7).

If the formulation of the problem is such as to maximize X_j^N, the determination of the optimal sequence of the control actions θ^n, (see equation (6)) must be such that

$$H^n = \text{maximum}, \; n = 1, 2, \ldots, N \text{ - - - - - - - - - - - - - - } (9)$$

STATEMENT OF THE TRANSPORTATION PROBLEM

The problem is schematically shown in Fig. 2 on page 7, where $i = 1, 2, \ldots, s$, are the number of origins (factories) where the resource is located, and $n = 1, 2, \ldots, N$, are the number of destinations (warehouses), where the demand for this resource exists.

θ_i^n = number of units of the resource supplied from the i^{th} origin to the n^{th} destination.

C_i^n = the cost incurred in supplying one unit of this resource from the i^{th} origin to the n^{th} destination.

Then $C_{sN} = \sum_{i=1}^{s} \sum_{n=1}^{N} C_i^n \theta_i^n$ represents the objective cost function which is to be minimized and is subjected to the following conditions:

(i) $\theta_i^n \geq 0$

(ii) $\sum_{i=1}^{s} \theta_i^n = D^n$, number of units of the resource required by the n^{th} destination, $n = 1, 2, \ldots, N$.

(iii) $\sum_{n=1}^{N} \theta_i^n = W_i$, number of units of the resource available at the i^{th} origin, $i = 1, 2, \ldots, s$.

	Origins						Demand
	i n	1	2	.... i	k	s	D^n
	1	C_1^1 θ_1^1					D^1
	2		C_2^2 θ_2^2				D^2
	3						D^3
Destinations	⋮						
	n			C_i^n θ_i^n	C_k^n θ_k^n	C_s^n θ_s^n	D^n
	⋮						
	N			C_i^N θ_i^N	C_k^N θ_k^N	C_s^N θ_s^N	D^N
Supply	W_i	W_1	W_2	W_i	W_k	W_s	

Fig. 2. Schematical diagram of transportation problem, showing transportation costs and requirements.

FORMULATION OF TRANSPORTATION PROBLEM IN TERMS OF THE DISCRETE MAXIMUM PRINCIPLE

The following is the formulation of the transportation problem given by Fan and Wang (5):

In the case of transportation problem as shown by Fig. 2 on page 7, we can write the transformation operator as follows:

$$X_i^n = X_i^{n-1} + \theta_i^n; \quad X_i^0 = 0; \quad X_i^N = W_i \text{ - - - - - - - - - - - - - } (10)$$

$$i = 1, 2, \ldots, s-1; \; n = 1, 2, \ldots, N.$$

where X_i^n, for i = 1, 2,...., s-1, are the number of state variables representing the total number of units of resource supplied from the i^{th} origin to the first n-destinations. It must be noted that though there are "s" origins in the problem, there are only (s-1) state variables. This is due to the fact that the demand by each destination is preassigned; hence the number of units supplied from the s^{th} origin to the n^{th} destination can be obtained by subtracting the sum of the units supplied to the n^{th} destination by (1) to (s-1) origins from the total number of units required by the n^{th} destination. That is to say that

$$\theta_s^n = D^n - \sum_{i=1}^{s-1} \theta_i^n$$

As our objective is to minimize the total cost of transportation, we state this objective in terms of the s^{th} state variable and we rewrite the same in terms of the transformation operator as follows:

$$X_s^n = X_s^{n-1} + \sum_{i=1}^{s} C_i^n \, \theta_i^n; \quad X_s^o = 0 \text{ - - - - - - - - - - - - - - } (11)$$

$n = 1, 2, \ldots, N.$

Hence, from the above equation, it is noticed that the problem of minimizing the total cost of transportation has turned to the problem of minimizing the s^{th} state variable X_s^N by choosing the values of θ_i^n, $i = 1, 2, \ldots, s-1$, and $n = 1, 2, \ldots, N$, for the process described by the equations (10) and (11).

Hence, changing the problem in terms of Hamiltonian function equation (2) we have

$$H^n = \sum_{i=1}^{s-1} Z_i^n \, (X_i^{n-1} + \theta_i^n) + Z_s^n \, (X_s^{n-1} + \sum_{i=1}^{s} C_i^n \, \theta_i^n) \text{ - - - } (12)$$

$n = 1, 2, \ldots, N$

and applying equation (4) to the above equation (12), we have

$$Z_i^{n-1} = \frac{\partial H^n}{\partial X_i^{n-1}}$$

$$= Z_i^n \text{ - } (13)$$

$i = 1, 2, \ldots, s-1; \quad n = 1, 2, \ldots, N$

and $$Z_s^{n-1} = \frac{\partial H^n}{\partial X_s^{n-1}}$$

$$= Z_s^n \text{ - } (14)$$

But from equation (8) page 5,

$$\left.\begin{aligned} & Z_s^N = 1, \text{ as } i = s \\ \text{and } & Z_i^N = C_i, \text{ for } i = 1, 2, \ldots, s-1; \text{ (as } W_i \text{ are prescribed)} \end{aligned}\right\} (15)$$

Hence, from equations (14) and (15) we have

$$\left.\begin{aligned} &Z_s^n = 1, \text{ for } i = s \text{ and } n = 1, 2, \ldots, N \\ \text{and } &Z_i^n = C_i, \text{ for } i = 1, 2, \ldots, s-1; \text{ and } n = 1, 2, \ldots, N \end{aligned}\right) \quad (16)$$

Therefore, the Hamiltonian equation (12) can be written as

$$H^n = \sum_{i=1}^{s-1} Z_i^n (X_i^{n-1} + \theta_i^n) + X_s^{n-1} + \sum_{i=1}^{s} C_i^n \theta_i^n \text{ - - - - - - - } (17)$$

$n = 1, 2, \ldots, N.$

Since the values of X_i^N, $i = 1, 2, \ldots, s-1$, are prescribed by W_i, $i = 1, 2, \ldots, s-1$, the values of Z_i^N, $i = 1, 2, \ldots, s-1$, are undetermined at the beginning of the calculation. The values of θ_i^n, $i = 1, 2, \ldots, s-1$, are determined in such a way that H^n is the absolute minimum, by selecting the values of Z_i^n in a particular chosen limit, so as to make the computed values of X_i^N, $i = 1, 2, \ldots, s-1$, equal to the given values of W_i, $i = 1, 2, \ldots, s-1$.

But the values of Z_i^n and X_i^{n-1} for $n = 1, 2, \ldots, N$ are each considered as constants at each step in the minimization of the Hamiltonian equation (17); hence, it is possible for us to define and minimize only the variable part of this Hamiltonian equation, which is as follows:

$$H_v^n = \sum_{i=1}^{s-1} Z_i^n \theta_i^n + \sum_{i=1}^{s} C_i^n \theta_i^n \text{ - - - - - - - - - - - - - - - - - } (18)$$

$n = 1, 2, \ldots, N.$

EXAMPLE (1) WITH TWO ORIGINS AND FIVE DESTINATIONS

The problem is represented by Table 1. Values of C_i^n (in \$), D^n and W_i are shown in this table, and total number of units required by n-destinations is equal to the total number of units supplied from the i-origins, i.e.,

$$\sum_{n=1}^{N} D^n = \sum_{i=1}^{s} W_i .$$

It is required to allocate the number of units of resource in such a way as to minimize the total cost of transportation.

Table 1. Transportation cost and requirements.

$n \backslash i$	1	2	D^n
1	9	4	7
2	7	9	19
3	1	4	13
4	8	2	5
5	6	10	6
W_i	22	28	

Solution:

Considering the variable part of the Hamiltonian equation (18), page 10, we get the general form of the variable part of the Hamiltonian equation for this problem as follows:

$$H_v^n = \sum_{i=1}^{s-1} Z_i^n \Theta_i^n + \sum_{i=1}^{s} C_i^n \Theta_i^n, \quad n = 1, 2, \ldots, N$$

$$\therefore \quad H_v^n = Z_1^n \Theta_1^n + C_1^n \Theta_1^n + C_2^n \Theta_2^n$$

$$\text{but } \Theta_2^n = D^n - \Theta_1^n,$$

$$\therefore \quad H_v^n = (Z_1^n + C_1^n - C_2^n)\,\Theta_1^n + C_2^n D^n; \; n = 1, 2, \ldots, N \; - - - \; (19)$$

Substituting n = 1 in the above equation (19), we have the variable part of the Hamiltonian equation for the first origin as,

$$H_v^1 = (Z_1^1 + C_1^1 - C_2^1)\,\Theta_1^1 + C_2^1 D^1$$

$$= (Z_1^1 + 5)\,\Theta_1^1 + 28 \; - - - - - - - - - - - - - - - - - - - \; (20)$$

From this equation (20), we see that

(a) H_v^1 is minimum, when $\Theta_1^1 = 0$ and if $Z_1^1 > -5$

(b) H_v^1 is minimum, when $\Theta_1^1 = 7$ and if $Z_1^1 < -5$

The conditions (a) and (b) are shown in Fig. 3, page 13.

In a similar manner, for the rest of the stages, n = 2, 3, 4, and 5. The values of Z_1^n and Θ_1^n are determined which makes H_v^n, a minimum. These values of Z_1^n and Θ_1^n are shown in Table 2, page 14.

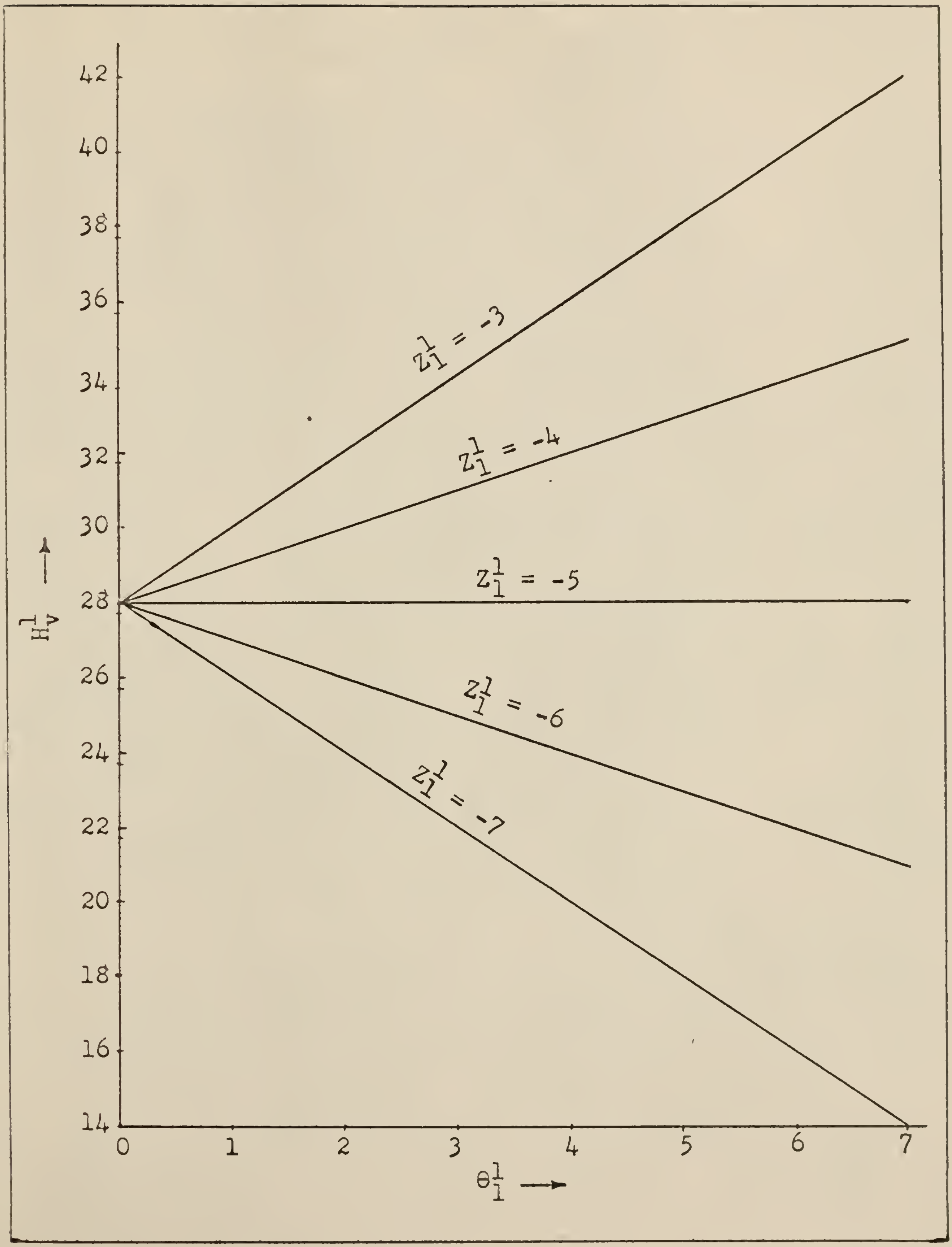

Fig. 3. Covariant vector Z_1^1, showing selection of θ_1^1.

Table 2. Conditions necessary for H_v^n to be minimum.

n	Minimum of H_v^n, occurs at	
	θ_1^n	Z_1
1	0	> -5
	7	< -5
2	0	> 2
	19	< 2
3	0	> 3
	13	< 3
4	0	> -6
	5	< -6
5	0	> 4
	6	< 4

But by equation (16) page 10, $Z_i^n = C_i$ for $n = 1, 2, \ldots, N$, hence $Z_1^1 = Z_1^2 = Z_1^3 = Z_1^4 = Z_1^5$, are all equal to some constant value. Thus, the values of Z_1^n for all $n = 1, 2, \ldots, N$, represents the same value Z_1 which is equal to some constant value. For these above values of Z_1, we have Fig. 4 on page 15, showing limiting values of covariant vectors Z_1 representing the positions of the stages $n = 1, 2, \ldots, 5$ with respect to one another. From this Fig. 4, we notice that there are four possible limiting values for Z_1, within which Z_1 will be equal to all values of Z_1^n, for $n = 1, 2, \ldots, 5$. These possible limiting values for Z_1 are as follows:

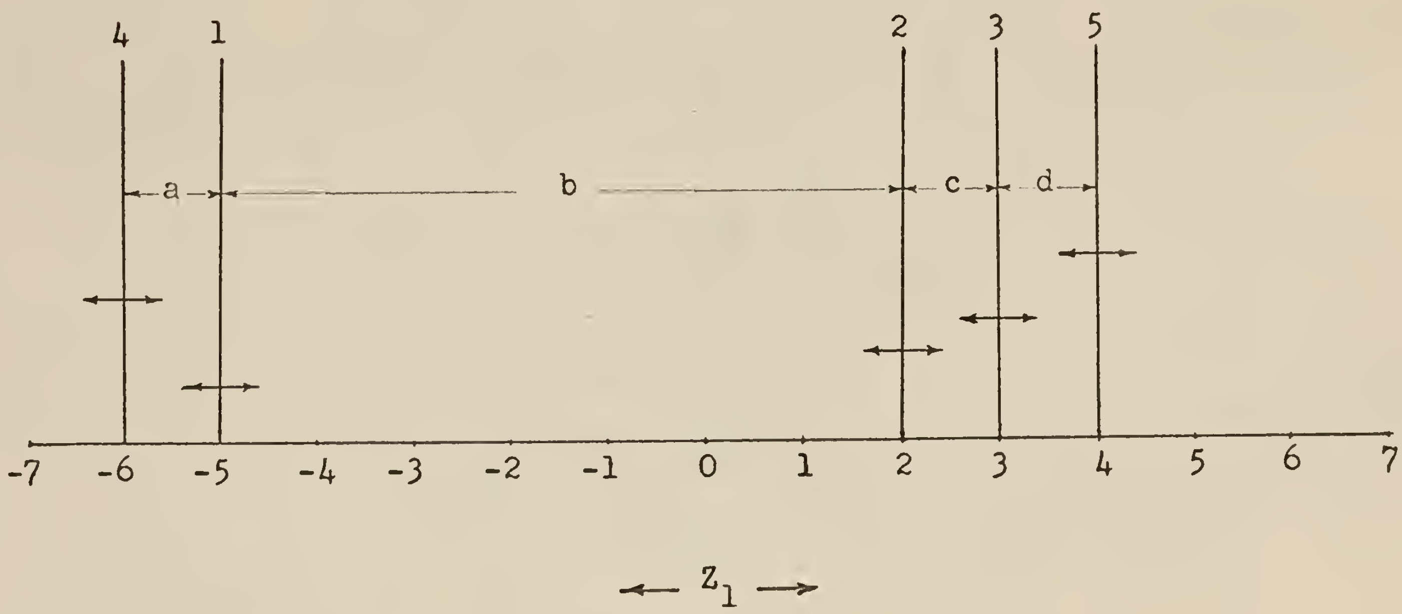

Fig. 4. Limiting values of covariant vectors Z_1.

(a) $-6 < Z_1 < -5$

(b) $-5 < Z_1 < 2$

(c) $2 < Z_1 < 3$

(d) $3 < Z_1 < 4$

Now considering these four possible limiting values of Z_1, we will derive all the feasible solutions as follows:

Solution from limiting value, $-6 < Z_1 < -5$:

Now comparing the values of Z_1^n, of all the stages n = 1, 2,...., 5, as shown in Table 2, with the above limits of Z_1, we assign values to θ_1^n and thus derive the following solution:

Table 3. Step 1 of solution for limiting value $-6 < Z_1 < -5$.

n \ i	1	2	D^n
1	7	0	7
2	19	0	19
3	13	0	13
4	0	5	5
5	6	0	6

This above solution does not satisfy our end-point conditions of $W_1 = 22$ and $W_2 = 28$, as in the above case $W_1 = 45$ and $W_2 = 5$. Hence, to get the feasible solution with these end-point conditions, we refer to Fig. 4, page 15. From this figure we notice that the limiting values of Z_1 shown in Table 3 were given by the stages (4) and (1). Hence, it is possible for us to adjust the values of θ_1^4 and θ_2^4 or θ_1^1 and θ_2^1. In this case it is only possible to adjust the values of θ_1^1 and θ_2^1; therefore, the value of $\theta_1^1 = 7$ is changed to $\theta_1^1 = 0$, and that of $\theta_2^1 = 0$ to $\theta_2^1 = 7$. Hence we have Step 2.

Table 4. Step 2 of solution for limiting value $-6 < Z_1 < -5$.

n \ i	1	2	D^n
1	0	7	7
2	19	0	19
3	13	0	13
4	0	5	5
5	6	0	6

Still, the above does not satisfy our end-point conditions of $W_1 = 22$ and $W_2 = 28$. Hence, we again refer to Fig. 4 and notice that Stage (2) is the next stage to Stage (1); hence, by changing the values of θ_1^2 and θ_2^2 to $\theta_1^2 = 3$ and $\theta_2^2 = 16$, satisfying end-point conditions of $W_1 = 22$, and $W_2 = 28$; we have the feasible solution as follows:

Table 5. Feasible solution for limiting value $-6 < Z_1 < -5$.

n \ i	1	2	D^n
1	0	7	7
2	3	16	19
3	13	0	13
4	0	5	5
5	6	0	6

with the total cost $= \sum_{i=1}^{2} \sum_{n=1}^{5} C_i^n \theta_i^n$

$= \$252.00.$

In a similar way considering the remaining possible limiting values for Z_1, given by b, c, and d, we determine the feasible solutions as shown on the following pages.

The solution from limiting value $-5 < Z_1 < 2$, is shown in Table 6, page 19. This solution in Table 6 is not a feasible one as end-point conditions $W_1 = 22$ and $W_2 = 28$ are not satisfied. Hence, to make this solution a feasible one, we again refer to Fig. 4 and notice that this time the limiting values of Z_1 are given by stages (1) and (2). Hence, adjusting the value of $\theta_1^2 = 3$ and $\theta_2^2 = 16$, we have the same feasible solution as shown by Table 5 above, with cost of \$252.00.

Solution from limiting value, $-5 < Z_1 < 2$:

Table 6. Step 1 of solution for limiting value $-5 < Z_1 < 2$.

n \ i	1	2	D^n
1	0	7	7
2	19	0	19
3	13	0	13
4	0	5	5
5	6	0	6

Solution from limiting value $2 < Z_1 < 3$:

Table 7. Step 1 of solution for limiting value $2 < Z_1 < 3$:

n \ i	1	2	D^n
1	0	7	7
2	0	19	19
3	13	0	13
4	0	5	5
5	6	0	6

With the similar reasoning as in cases (a) and (b), we adjust the values to $\theta_1^2 = 3$ and $\theta_2^2 = 16$, and have the same feasible solution as shown by Table 5, with cost of $252.00.

Solution from limiting value $3 < Z_1 < 4$:

Table 8. Step 1 of solution for limiting value $3 < Z_1 < 4$.

n \ i	1	2	D^n
1	0	7	7
2	0	19	19
3	0	13	13
4	0	5	5
5	6	0	6

In the above solution $W_1 \neq 22$ and $W_2 \neq 28$; hence, we refer to Fig. 4 and notice that the limiting values of Z_1 are given by stage (3) and stage (5), but it is only possible to change the values of θ_1^3 and θ_2^3 to $\theta_1^3 = 13$ and $\theta_2^3 = 0$. Still this does not satisfy the end-point conditions of $W_1 = 22$ and $W_2 = 28$. So, again referring to Fig. 4, we notice that stage (2) is the nearest stage to stage (3), and hence, changing the values of θ_1^2 and θ_2^2 to $\theta_1^2 = 3$ and $\theta_2^2 = 16$, we satisfy the end-point conditions, $W_1 = 22$ and $W_2 = 28$. Thus we have the same feasible solution as shown by Table 5, with cost of $252.00.

Thus by selecting any limiting value of Z_1, we get the optimal solution (as shown by Table 5) with cost of \$252.00. It must be noted that solving the above problem by regular transportation methods we get the same optimal solution.

GENERAL RULES AND STEPS TO BE CARRIED OUT FOR SOLVING THE PROBLEMS WITH THREE OR MORE ORIGINS

(1) Enumerate all "Fixed States."

Fixed State is the state (origin) in which values of θ's are fixed. This occurs when a state has (N-1) number of stages with $\theta = 0$; and hence, the remaining one stage must have the value of $\theta = W$.

(2) Enumerate all "Fixed Stages."

Fixed Stages are those stages in which values of θ's are fixed. This occurs when (s-1) number of states have $\theta = 0$; and hence, the remaining one state must have the value of $\theta = D^n$.

(3) "Common Stages."

Common Stages are those stages which are common in forming the limiting values of Z_1, Z_2,...., of the particular combination (of limiting values of Z_1, Z_2,....) which is selected for solving the problem. It must be noted that there may be more than one common stage in a combination.

(4) Selection of stages for assigning the values of θ's must be carried out in the following manner:

(a) As mentioned in (1) above, in the case of Fixed State, we have one stage with the value of $\theta = W$. First preference must be given to these stages.

(b) Second preference must be given to the stages which have maximum number of θ's equal to zero. That is, the number of θ's equal to zero is one less than the number of θ's equal to zero in the Fixed Stage.

(c) Third preference must be given to the stages which have number of θ's (equal to zero), which is less by one than in the case of (b).

(d) This preference of the stages is carried on until the group of stages which do not contain a single value of θ equal to zero. Thus, last preference is given to the stages which do not have a single θ equal to zero.

(5) Preferences in selecting the stages in any of the above cases, b, c, and d of (4), must be given in the following manner:

(a) First preference must be given to the common stages. If there is more than one common stage, then any combination of selection in preference of these stages will give the optimal solution.

(b) Second preference must be given one by one to any one of the stages which are forming the limiting values of Z_1, Z_2,...., of that particular combination (of limiting values of Z_1, Z_2,....) which is taken into consideration for solving the problem. Then any one of these combinations of these stages will give the optimal solution.

(6) The method of assigning the values to θ's.

(a) In the case of Fixed Stage, the values of θ's of the first preferred stage must be assigned in such a way as to satisfy the end-point condition (i.e., the value of W_i) of the state, which corresponds to the value of θ, equal to the requirement of the stage (destination) which is D^n. If this is not possible, then assign all possible values to θ's of this first preferred stage and then satisfy the previously mentioned W_i by assigning the values to θ's of the subsequent preferred stage. (Hence, all the alternatives must be carried out.) In case this does not satisfy the above-mentioned end-point (W_i), then above steps must be repeated with the subsequent preferred stages until the end-point is satisfied. (Generally this does not happen, and even if this occurs, then all the alternatives must be carried out.)

(b) In the case where there is no Fixed State, then assign all possible values to θ's of the first preferred stage. This will give more than one alternative. Then assign the values to the θ's of the subsequent preferred stage in such a way as to satisfy W_i, which corresponds to the assigned values of θ's of the first preferred stage. In case this does not satisfy the above-mentioned end-point (W_i), then above steps must be repeated, with the subsequent preferred stages, until the end-point is satisfied. (Generally this will not occur, and even if this occurs, then all alternatives must be carried out.)

(c) Whenever the values of W_i are satisfied as mentioned in the above cases (a) and (b), then all the possible values must be assigned to θ's of subsequent stage. This will give more than one alternative. Then the process must be repeated by assigning the values to the θ's of the subsequent preferred stages in such a way as to satisfy W_i, which corresponds to the assigned values of θ's of the above-mentioned stage. If W_i is not satisfied, then steps must be repeated as mentioned in (b).

(7) Best approach of selecting the limiting values of Z_1, Z_2,......

It will always be preferable to try to achieve optimal solution by selecting the limiting values of Z_1, Z_2,.... in such a way as the limits of these Z's are positive and negative, i.e., $-p < Z_i < +q$, where p and q are some constants.

EXAMPLE (2) WITH THREE ORIGINS AND FOUR DESTINATIONS

The problem is represented by the following Table 9. Values of C_i^n, (in \$), D^n and W_i are given; and the total number of units required by n-destinations is equal to the total number of units supplied from i-origins (i.e., $\sum_{n=1}^{N} D^n = \sum_{i=1}^{s} W_i$). It is required to allocate the number of units of resource in such a way as to minimize the total cost of transportation.

Table 9. Transportation costs and requirements.

i / n	1	2	3	D^n
1	2	8	5	21
2	4	2	1	19
3	7	3	5	13
4	2	6	10	17
W_i	15	20	35	70

Considering the variable part of the Hamiltonian equation (18), page 10, we have

$$H_v^n = \sum_{i=1}^{s-1} Z_i^n \theta_i^n + \sum_{i=1}^{s} C_i^n \theta_i^n, \quad n = 1, 2, \ldots, N$$

but i = 1, 2, and 3

$$\therefore \quad H_v^n = Z_1^n \theta_1^n + Z_2^n \theta_2^n + C_1^n \theta_1^n + C_2^n \theta_2^n + C_3^n \theta_3^n$$

but $\theta_3^n = D^n - \theta_1^n - \theta_2^n$

$$\therefore \quad H_v^n = (Z_1^n + C_1^n - C_3^n)\,\theta_1^n + (Z_2^n + C_2^n - C_3^n)\,\theta_2^n + C_3^n D^n \quad - - -(21)$$

Substituting n = 1 in the foregoing equation, we have the variable part of the Hamiltonian equation for the first destination as

$$H_v^1 = (Z_1^1 + C_1^1 - C_3^1)\ \theta_1^1 + (Z_2^1 + C_2^1 - C_3^1)\ \theta_2^1 + C_3^1\ D^1$$

$$= (Z_1^1 - 3)\ \theta_1^1 + (Z_2^1 + 3)\ \theta_2^1 + 105$$

From this we see that

(a) H_v^1 is minimum, when $\theta_1^1 = 0$ and $\theta_2^1 = 0$ and if $Z_1^1 > 3$ and $Z_2^1 > -3$.

(b) H_v^1 is minimum, when $0 \leq \theta_1^1 \leq 21$ and $\theta_2^1 = 0$ and if $Z_1^1 < 3$ and $Z_2^1 > -3$.

(c) H_v^1 is minimum, when $\theta_1^1 = 0$ and $0 \leq \theta_2^1 \leq 21$ and if $Z_1^1 > 3$ and $Z_2^1 < -3$.

(d) H_v^1 is minimum, when $0 \leq \theta_1^1 \leq 21$ and $0 \leq \theta_2^1 \leq 21$ and if $Z_1^1 < 3$ and $Z_2^1 < -3$.

In a similar manner by substituting the remaining three values of n = 2, 3 and 4, we get the similar types of results as shown by the above conditions, a, b, c, and d, at which the minima occurs. These conditions are shown in Table 10 on the next page.

As shown before by equation 16, page 10, Z_1^1, Z_1^2, Z_1^3, and Z_1^4 are all equal and constant (C_1), and similarly Z_2^1, Z_2^2, Z_2^3, and Z_2^4 are all equal and constant = C_2. From the values of Z_1 and Z_2

Table 10. Conditions necessary for H_v^n to be minimum.

Stages n	Solutions	Minimum of H_v^n occurs at:			
		θ_1^n	θ_2^n	Z_1	Z_2
1	a	0	0	> 3	> -3
	b	$0 \le \theta_1^1 \le 21$	0	< 3	> -3
	c	0	$0 \le \theta_2^1 \le 21$	> 3	< -3
	d	$0 \le \theta_1^1 \le 21$	$0 \le \theta_2^1 \le 21$	< 3	< -3
2	a	0	0	> -3	> -1
	b	$0 \le \theta_1^2 \le 19$	0	< -3	> -1
	c	0	$0 \le \theta_2^2 \le 19$	> -3	< -1
	d	$0 \le \theta_1^2 \le 19$	$0 \le \theta_2^2 \le 19$	< -3	< -1
3	a	0	0	> -2	> 2
	b	$0 \le \theta_1^3 \le 13$	0	< -2	> 2
	c	0	$0 \le \theta_2^3 \le 13$	> -2	< 2
	d	$0 \le \theta_1^3 \le 13$	$0 \le \theta_2^3 \le 13$	< -2	< 2
4	a	0	0	> 8	> 4
	b	$0 \le \theta_1^4 \le 17$	0	< 8	> 4
	c	0	$0 \le \theta_2^4 \le 17$	> 8	< 4
	d	$0 \le \theta_1^4 \le 17$	$0 \le \theta_2^4 \le 17$	< 8	< 4

(of Table 10), for n = 1, 2, 3, and 4, we draw the Figs. 5 and 6, showing limiting values of covariant vectors Z_1 and Z_2, as shown on page 30. From Fig. 5, we see that there are three possible limiting values for Z_1, within which Z_1 will be equal to all the values of Z_1^n for stages n = 1, 2, 3, and 4. These possible limiting values for Z_1 are as follows:

(a_1) $-3 < Z_1 < -2$

(b_1) $-2 < Z_1 < 3$

(c_1) $3 < Z_1 < 8$

Similarly from Fig. 6, we see that there are three possible limiting values of Z_2, within which Z_2 will be equal to all the values of Z_2^n for stages n = 1, 2, 3, and 4. These limiting values for Z_2 are as follows:

(a_2) $-3 < Z_2 < -1$

(b_2) $-1 < Z_2 < 2$

(c_2) $2 < Z_2 < 4$

Now considering the above limiting values of Z_1 and Z_2, we see that there are nine possible combinations for the solution of this problem, and these are

I . . . . . $-3 < Z_1 < -2$ and $-3 < Z_2 < -1$

II . . . . . $-3 < Z_1 < -2$ and $-1 < Z_2 < 2$

III . . . . . $-3 < Z_1 < -2$ and $2 < Z_2 < 4$

IV . . . . . $-2 < Z_1 < 3$ and $-3 < Z_2 < -1$

V . . . . . $-2 < Z_1 < 3$ and $-1 < Z_2 < 2$

VI . . . . . $-2 < Z_1 < 3$ and $2 < Z_2 < 4$

VII . . . . . $3 < Z_1 < 8$ and $-3 < Z_2 < -1$

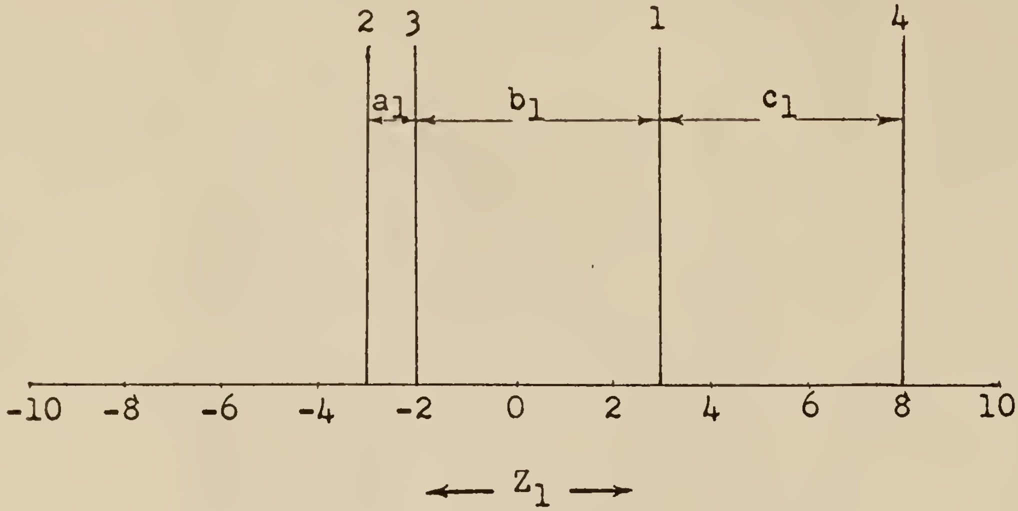

Fig. 5. Limiting values of covariant vectors Z_1.

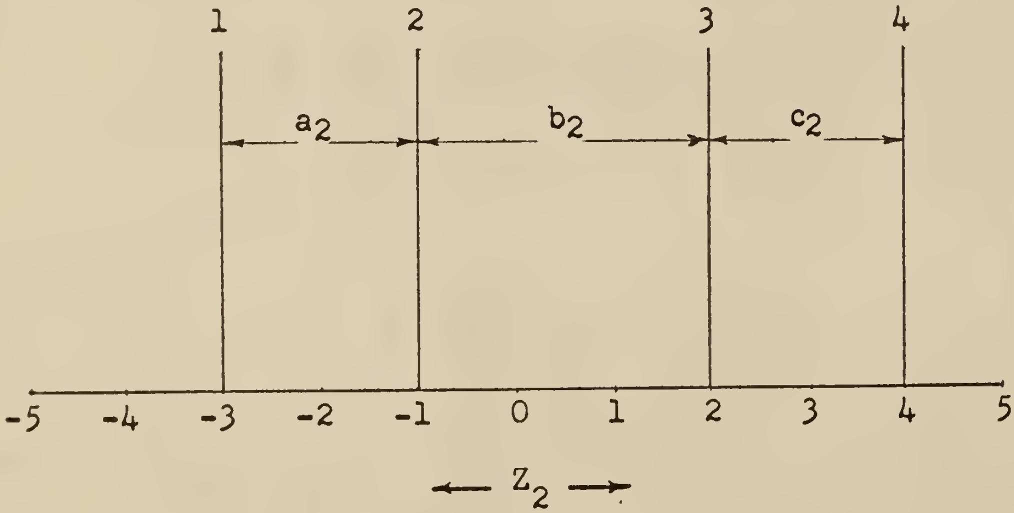

Fig. 6. Limiting values of covariant vectors Z_2.

VIII . . . . . $3 < Z_1 < 8$ and $-1 < Z_2 < 2$

IX . . . . . $3 < Z_1 < 8$ and $2 < Z_2 < 4$

From these nine combinations of Z_1 and Z_2, we will derive all feasible solutions as follows:

Solution from combination I, $-3 < Z_1 < -2$ and $-3 < Z_2 < -1$:

Comparing the values of Z_1 and Z_2 for n = 1, 2, 3, and 4, (as shown in Table 10) with the above limits of Z_1 and Z_2, we determine values of θ_1^n and θ_2^n, which makes H_v^n (for n = 1, 2, 3, and 4) minimum. These values are shown in Table 11, page 32.

From Table 11, we make the following Table 12, showing limiting values of control actions θ's.

Table 12. Limiting values of θ's.

n \ i	1	2	3	D^n
1	$0 \leq \theta_1^1 \leq 21$	$\oplus$*		21
2	$\oplus$	$0 \leq \theta_2^2 \leq 19$		19
3	$0 \leq \theta_1^3 \leq 13$	$0 \leq \theta_2^3 \leq 13$		13
4	$0 \leq \theta_1^4 \leq 17$	$0 \leq \theta_2^4 \leq 17$		17
W_i	15	20	35	70

*$\oplus$ shows that $\theta_i^n = 0$, which is fixed by values of Z_i^n.

Table 11. Conditions necessary for H_v^n's to be minimum.

Stage 1	(a)	$\theta_1^1 \neq 0$	$\theta_2^1 = 0$
	(b)	$0 \leq \theta_1^1 \leq 21$	$\theta_2^1 = 0$
	(c)	$\theta_1^1 \neq 0$	$0 \nleq \theta_2^1 \nleq 21$
	(d)	$0 \leq \theta_1^1 \leq 21$	$0 \nleq \theta_2^1 \nleq 21$
	$\therefore$	$0 \leq \theta_1^1 \leq 21$ and	$\theta_2^1 = 0$
Stage 2	(a)	$\theta_1^2 = 0$	$\theta_2^2 \neq 0$
	(b)	$0 \nleq \theta_1^2 \nleq 19$	$\theta_2^2 \neq 0$
	(c)	$\theta_1^2 = 0$	$0 \leq \theta_2^2 \leq 19$
	(d)	$0 \nleq \theta_1^2 \nleq 19$	$0 \leq \theta_2^2 \leq 19$
	$\therefore$	$\theta_1^2 = 0$ and	$0 \leq \theta_2^2 \leq 19$
Stage 3	(a)	$\theta_1^3 \neq 0$	$\theta_2^3 \neq 0$
	(b)	$0 \leq \theta_1^3 \leq 13$	$\theta_2^3 \neq 0$
	(c)	$\theta_1^3 \neq 0$	$0 \leq \theta_2^3 \leq 13$
	(d)	$0 \leq \theta_1^3 \leq 13$	$0 \leq \theta_2^3 \leq 13$
	$\therefore$	$0 \leq \theta_1^3 \leq 13$ and	$0 \leq \theta_2^3 \leq 13$
Stage 4	(a)	$\theta_1^4 \neq 0$	$\theta_2^4 \neq 0$
	(b)	$0 \leq \theta_1^4 \leq 17$	$\theta_2^4 \neq 0$
	(c)	$\theta_1^4 \neq 0$	$0 \leq \theta_2^4 \leq 17$
	(d)	$0 \leq \theta_1^4 \leq 17$	$0 \leq \theta_2^4 \leq 17$
	$\therefore$	$0 \leq \theta_1^4 \leq 17$ and	$0 \leq \theta_2^4 \leq 17$

Now following the general rules and steps shown before on pages 22-25, we derive the optimal solution for Table 12 as follows:

First preference will be from stages (1) and (2), but stage (2) is the common stage (see Figs. 5 and 6); hence, first preference is given to stage (2), second preference to stage (1), third preference to stage (3), and last to the remaining stage (4). Thus we have the following solutions:

```
            Ia                                  Ib

15    : θ : 6     :2nd         :5     : θ : 16    : 2nd
   /0:   :    /21: choice      :  /15 :   :    /6: choice
     :   :        :            :      :   :       :
  θ   :19 :  0    :1st choice  :  θ   : 0 :    19 : 1st choice
```

In case I_a, we notice that it is not possible to satisfy the end-point condition (W_2) which we are supposed to fulfill, as the assignment of $\theta_2^2 = 19$ was made by the first preferred stage (2). In case of I_b, we are able to satisfy end-point condition of $W_3 = 35$; hence, we continue with this one alternative which satisfies $W_3 = 35$ and neglect the other. Similarly we neglect I_a as one of the alternatives of I_b satisfies end-point condition $W_3 = 35$. Thus we continue with this alternative giving 3rd preference to stage (3):

```
                 Ib

:      5 :   θ    : 16 :
:        :        :    :
:      θ :   0    : 19 :
:        :        :    :
: 10     : 3      :  0 : 3rd choice
:     /0 :    /13 :    :
:        :        :    :
:        :        :  0 :
```

$$I_{b1} = \begin{array}{ccc} 5 & \theta & 16 \\ \theta & 0 & 19 \\ 10 & 3 & 0 \\ 0 & 17 & 0 \end{array} \qquad I_{b2} = \begin{array}{ccc} 5 & \theta & 16 \\ \theta & 0 & 19 \\ 0 & 13 & 0 \\ 10 & 7 & 0 \end{array}$$

Note that while considering 3rd choice of stage (3), we go for both the alternatives as shown by I_{b1} and I_{b2} above, as per our general rules, 6 (c), page 24. Total cost ($\sum_{i=1}^{3} \sum_{n=1}^{4} (C_i^n \, \theta_i^n)$) given by I_{b1} is \$290.00, while that of I_{b2} is \$210.00. Thus we have I_{b2} as an optimal solution.

In a similar way (as I above) we will derive the optimal solution from the remaining eight combinations of Z_1 and Z_2 one by one as follows:

Solution from combination II, $-3 < Z_1 < -2$ and $-1 < Z_2 < 2$:

Table 13. Limiting values of θ's.

n \ i	1	2	3	D^n
1	$0 \le \theta_1^1 \le 21$	θ		21
2	θ	θ	19	19
3	$0 \le \theta_1^3 \le 13$	$0 \le \theta_2^3 \le 13$		13
4	$0 \le \theta_1^4 \le 13$	$0 \le \theta_2^4 \le 17$		17
W_i	15	20	35	

II_a

15	θ	6	1st choice
θ	θ	19	

II_b

5	θ	16	
θ	θ	19	
10	3	0	2nd choice
/0	/13		
		0	

II_{b1}

5	θ	16
θ	θ	19
10	3	0
0	17	0

II_{b2}

5	θ	16
θ	θ	19
0	13	0
10	7	0

Note that we do not continue II_a, as II_b satisfies $W_3 = 35$. This combination II has given us the same solutions as I.

Solution from combination III, $-3 < Z_1 < -2$ and $2 < Z_2 < 4$:

Table 14. Limiting values of θ's.

n \ i	1	2	3	D^n
1	$0 \le \theta_1^1 \le 21$	θ		21
2	θ	θ	19	19
3	$0 \le \theta_1^3 \le 13$	θ		13
4	$0 \le \theta_1^4 \le 17$	$0 \le \theta_2^4 \le 17$		17
W_i	15	20	35	

Here we notice that stage (2) is the fixed stage and also notice that state (2) is also a fixed state, and we are not able to satisfy our end-point condition of $W_2 = 20$, as D^4 is only 17. Hence, it is not possible to have a feasible solution by this combination.

Solution from combination IV, $-2 < Z_1 < 3$ and $-3 < Z_2 < -1$:

Table 15. Limiting values of Θ's.

n \ i	1	2	3	D^n
1	$0 \leq \Theta_1^1 \leq 21$	Θ		21
2	Θ	$0 \leq \Theta_2^2 \leq 19$		19
3	Θ	$0 \leq \Theta_2^3 \leq 13$		13
4	$0 \leq \Theta_1^4 \leq 17$	$0 \leq \Theta_2^4 \leq 17$		17
W_i	15	20	35	

Here we have the following alternatives for solutions A and B.

Solution A:

1st choice for stage (1)

2nd choice for stage (2)

3rd choice for stage (3)

4th choice for stage (4)

Solution B:

1st choice for stage (1)

2nd choice for stage (3)

3rd choice for stage (2)

4th choice for stage (4)

Solution for A

A_1

15	θ	6	1st choice
θ	0 /19	19 /0	2nd choice
θ			

A_2

0	θ	21
θ	5	14
θ	13	0
15	2	0

Note that the optimal solution (A_2) with the optimal cost of $210.00 is different from the optimal solution obtained in I and II. Similarly we go for the second alternative of B.

Solution for B

B_a

15	θ	6	1st choice
θ			
θ	0 /13	13 /0	2nd choice
0			

B_b

0	θ	21
θ		
θ	0 /13	13 /0

B_{a1}

15	θ	6	
θ	3	16	3rd choice
θ	0	13	
0	17	0	

B_{a2}

15	θ	6
θ	0	19
θ	13	0
0		

B_{b1}

0	θ	21	
θ	18	1	3rd choice
θ	0	13	
15	2	0	

B_{b2}

0	θ	21
θ	5	14
θ	13	0
15	2	0

Note that here we have three feasible solutions, B_{a1}, B_{b1}, and B_{b2}, with cost of \$249.00, \$249.00, and \$210.00, respectively. Hence, with B also we obtained the same optimal solution as A_2.

Solution for combination V, $-2 < Z_1 < 3$ and $-1 < Z_2 < 2$:

Table 16. Limiting values of θ's.

n \ i	1	2	3	D^n
1	$0 \leq \theta_1^1 \leq 21$	θ		21
2	θ	θ	19	19
3	θ	$0 \leq \theta_2^3 \leq 13$		13
4	$0 \leq \theta_1^4 \leq 17$	$0 \leq \theta_2^4 \leq 17$		17
W_i	15	20	35	

	V_a				V_b	
5	θ	16	2nd choice	19	θ	3
θ	θ	19		θ	θ	19
0	13	0	1st choice	0	0	13
10	7	0				

Here we have optimal solution given by V_a with the cost of \$210.00. In case of V_b we are not able to satisfy $W_1 = 15$ and hence, $W_2 = 20$. Therefore, the solution is not feasible.

Solution for combination VI, $-2 < Z_1 < 3$ and $2 < Z_2 < 4$:

Table 17. Limiting values of θ's.

n \ i	1	2	3	D^n
1	$0 \leq \theta_1^1 \leq 21$	θ		21
2	θ	θ	19	19
3	θ	θ	13	13
4	$0 \leq \theta_1^4 \leq 17$	$0 \leq \theta_1^4 \leq 17$		17
W_i	15	20	35	

The foregoing solution is not feasible, as State (2) is the Fixed State and we are not able to satisfy end-point condition of $W_2 = 20$.

Solution from combination VII, $3 < Z_1 < 8$ and $-3 < Z_2 < -1$:

Table 18. Limiting values of θ's.

n \ i	1	2	3	D^n
1	θ	θ	21	21
2	θ	$0 \leq \theta_2^2 \leq 19$		19
3	θ	$0 \leq \theta_2^3 \leq 13$		13
4	15	$0 \leq \theta_2^4 \leq 17$		17
W_i	15	20	35	

VII_a				VII_b		
θ	θ	21		θ	θ	21
θ	7	12	2nd choice	θ	5	14
θ	13	0		θ	13	0
15	0	2	1st choice	15	2	0

In the above case we have Fixed State (1) and Fixed Stage (1), so first we fulfill the end-point conditions of $W_1 = 15$ and $D^1 = 21$. Then first preference must be given to Stage (4) (see general rule 4 (a), page 22). Here cost incurred by solution VII_a is \$220.00 and that by VII_b is \$210.00. We have optimal solution given by VII_b.

Solution from combination VIII, $3 < Z_1 < 8$ and $2 < Z_2 < 4$:

Table 19. Limiting values of θ's.

n \ i	1	2	3	D^n
1	θ	θ	21	21
2	θ	θ	19	19
3	θ	$0 \leq \theta_2^3 \leq 13$		13
4	15	$0 \leq \theta_2^4 \leq 17$		17
W_i	15	20	35	

The above solution is not feasible as W_3 is already greater than 35.

Solution from combination IX, $3 < Z_1 < 8$ and $2 < Z_2 < 4$:

Table 20. Limiting values of θ's.

n \ i	1	2	3	D^n
1	θ	θ	21	21
2	θ	θ	19	19
3	θ	θ	13	13
4	$0 \leq \theta_1^4 \leq 17$	$0 \leq \theta_2^4 \leq 17$		17
W_i	15	20	35	

In the above case we cannot have feasible solution as we cannot satisfy the values of W_i.

Thus we have two optimal solutions with cost of \$210.00 as follows:

Table 21. Optimal solution A.

n \ i	1	2	3	D^n
1	5	0	16	21
2	0	0	19	19
3	0	13	0	13
4	10	7	0	17
W_i	15	20	35	70

Table 22. Optimal solution B.

n \ i	1	2	3	D^n
1	0	0	21	21
2	0	5	14	19
3	0	13	0	13
4	15	2	0	17
W_i	15	20	35	70

Thus we see that any combination of Z_1 and Z_2, which has a feasible solution, must give us the optimal solution directly by following the general rules and steps given on pages 22-25. It must also be noted that solving this above problem, with regular transportation methods, we get the same optimal solution.

EXAMPLE (3) WITH THREE ORIGINS AND FOUR DESTINATIONS

The problem is represented by Table 23 shown below, and it is required to find the optimal solution with minimum of cost.

Table 23. Transportation costs and requirements.

n \ i	1	2	3	D^n
1	1	5	3	19
2	5	3	7	29
3	2	7	5	23
4	4	9	7	19
W_i	27	24	39	90

We have equation (21) on page 26:

$$H_v^n = (Z_1^n + C_1^n - C_3^n)\ \theta_1^n + (Z_2^n + C_2^n - C_3^n)\ \theta_2^n + C_3^n\ (D^n)$$

From which we get

$$H_v^1 = (Z_1^1 - 2)\ \theta_1^1 + (Z_2^1 + 2)\ \theta_2^1 + 3(19)$$

$$H_v^2 = (Z_1^2 - 2)\ \theta_1^2 + (Z_2^2 - 4)\ \theta_2^2 + 7(29)$$

$$H_v^3 = (Z_1^3 - 3)\ \theta_1^3 + (Z_2^3 + 2)\ \theta_2^3 + 5(23)$$

$$H_v^4 = (Z_1^4 - 3)\ \theta_1^4 + (Z_2^3 + 2)\ \theta_2^3 + 7(18)$$

From the above equations of H_v^n, for n = 1, 2, 3, and 4, we determine conditions necessary for H_v^n's to be minimum, as shown in Table 24.

Table 24. Conditions necessary for H_v^n's to be minimum.

n	Minimum of H_v^n occurs at:			
	θ_1^n	θ_2^n	Z_1	Z_2
1	0	0	> 2	> -2
	$0 \leq \theta_1^1 \leq 19$	$0 \leq \theta_2^1 \leq 19$	< 2	< -2
2	0	0	> 2	> 4
	$0 \leq \theta_1^2 \leq 29$	$0 \leq \theta_2^2 \leq 29$	< 2	< 4
3	0	0	> 3	> -2
	$0 \leq \theta_1^3 \leq 23$	$0 \leq \theta_2^3 \leq 23$	< 3	< -2
4	0	0	> 3	> -2
	$0 \leq \theta_1^4 \leq 19$	$0 \leq \theta_2^4 \leq 19$	< 3	< -2

Hence, we have Figs. 7 and 8 for limiting values of Z's, as shown on page 45.

From Figs. 7 and 8, we note that though there are four stages, we have only one possible combination of Z_1 and Z_2, which must give us our desired optimal solution. This is $2 < Z_1 < 3$ and $-2 < Z_2 < 4$.

For this combination we derive Table 25, on page 46, showing limiting values of θ's.

We note from said Table 25 that State (2) is the Fixed State with $\theta_2^2 = 24$ (i.e., D^2), and Stage (1) is the Fixed Stage with $\theta_3^1 = 19$. Then to satisfy $D^2 = 29$, we have to assign $\theta_3^2 = 5$. Now we have only two stages, (3) and (4), left for selection of preference; but we notice that both these stages are common

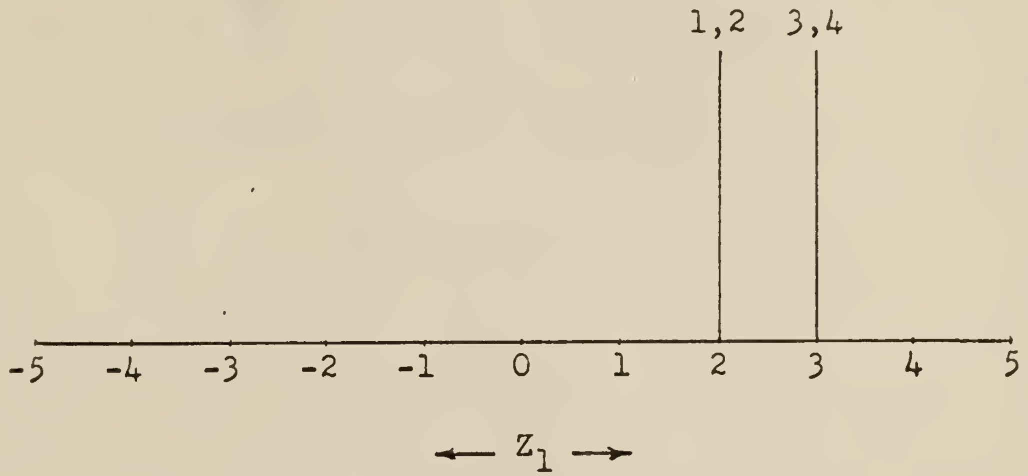

Fig. 7. Limiting values of covariant vectors Z_1.

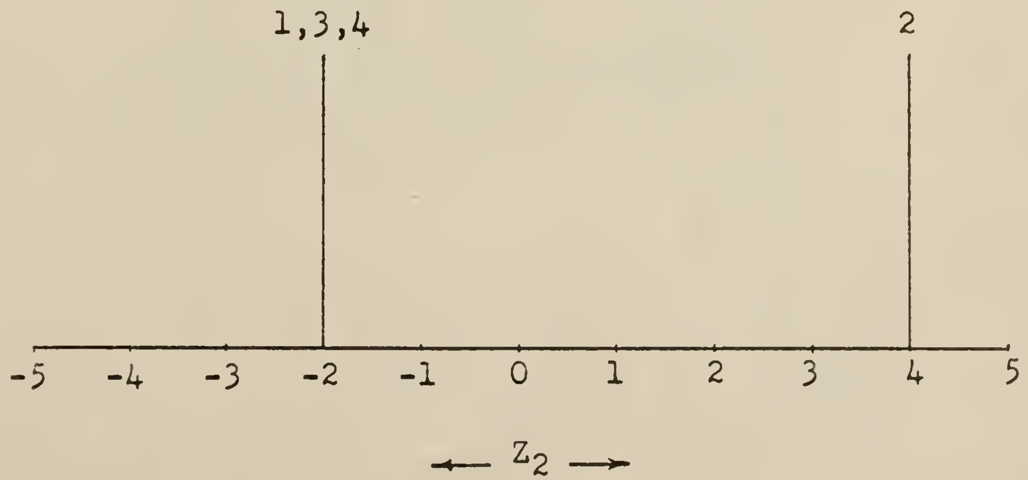

Fig. 8. Limiting values of covariant vectors Z_2.

Table 25. Limiting value of θ's.

n \ i	1	2	3	D^n
1	θ	θ	19	19
2	θ	24	5	29
3	$0 \leq \theta_1^3 \leq 23$	θ		23
4	$0 \leq \theta_1^4 \leq 19$	θ		19
W_i	27	24	39	90

stages. Then giving them first preference, one by one, we get the following feasible solutions:

Solutions

A

θ	θ	19	fixed stage
θ	24	5	fixed stage
8	θ	15	1st choice
19	θ	0	

B

θ	θ	19	fixed stage
θ	24	5	fixed stage
23	θ	0	
4	θ	15	1st choice

Hence we have the above two feasible solutions, which are both optimal solutions with total transportation cost of $331.00.

The same optimal solutions are derived by the regular transportation methods.

EXAMPLE (4) WITH FOUR ORIGINS AND SIX DESTINATIONS[1]

The problem is represented by Table 26, as shown below, and it is required to find the optimal solution with minimum transportation cost.

Table 26. Transportation costs and requirements.

n \ i	1	2	3	4	D^n
1	9	7	6	6	4
2	12	3	5	8	4
3	9	7	9	11	6
4	6	7	11	2	2
5	9	5	3	2	4
6	10	5	11	10	2
W_i	5	6	2	9	22

From Equation (18) on page 10, we have

$$H_v^n = (Z_1^n + C_1^n - C_4^n)\,\theta_1^n + (Z_2^n + C_2^n - C_4^n)\,\theta_2^n + (Z_3^n + C_3^n - C_4^n)\,\theta_3^n + C_4^n\,D^n.$$

From the above equation we determine the conditions necessary for H_v^n for n = 1, 2,...., 6 must be minimum. These are shown in Table 27, page 48.

[1]This example is solved by "Unit Penalty Cost Method" on pages 196 to 218 of Reference (9).

Table 27. Conditions necessary for H_v^n's to be minimum.

n	Minimum of H_v^n occurs at:					
	θ_1^n	θ_2^n	θ_3^n	Z_1	Z_2	Z_3
1	0	0	0	> -3	> -1	> 0
	$0 \leq \theta_1^1 \leq 4$	$0 \leq \theta_2^1 \leq 4$	$0 \leq \theta_3^1 \leq 4$	< -3	< -1	< 0
2	0	0	0	> -4	> 5	> 3
	$0 \leq \theta_1^2 \leq 4$	$0 \leq \theta_2^2 \leq 4$	$0 \leq \theta_3^2 \leq 4$	< -4	< 5	< 3
3	0	0	0	> 2	> 4	> 2
	$0 \leq \theta_1^3 \leq 6$	$0 \leq \theta_2^3 \leq 6$	$0 \leq \theta_3^3 \leq 6$	< 2	< 4	< 2
4	0	0	0	> -4	> -5	> -9
	$0 \leq \theta_1^4 = 2$	$0 \leq \theta_2^4 \leq 2$	$0 \leq \theta_3^4 \leq 2$	< -4	< -5	< -9
5	0	0	0	> -7	> -3	> -1
	$0 \leq \theta_1^5 \leq 4$	$0 \leq \theta_2^5 \leq 4$	$0 \leq \theta_3^5 \leq 4$	< -7	< -3	< -1
6	0	0	0	> 0	> 5	> -1
	$0 \leq \theta_1^6 \leq 2$	$0 \leq \theta_2^6 \leq 2$	$0 \leq \theta_3^6 \leq 2$	< 0	< 5	< -1

From the above we have Figs. 9, 10, and 11 for limiting values of Z's, as shown on page 49.

Here there is a total of 64 possible combinations of Z_1, Z_2, and Z_3 for getting our feasible solutions, but as mentioned on page 25, Item (7), we will take one combination as follows:

$$0 < Z_1 < 2, \quad -1 < Z_2 < 4 \quad \text{and} \quad -1 < Z_3 < 0$$

and we derive the following Table 28.

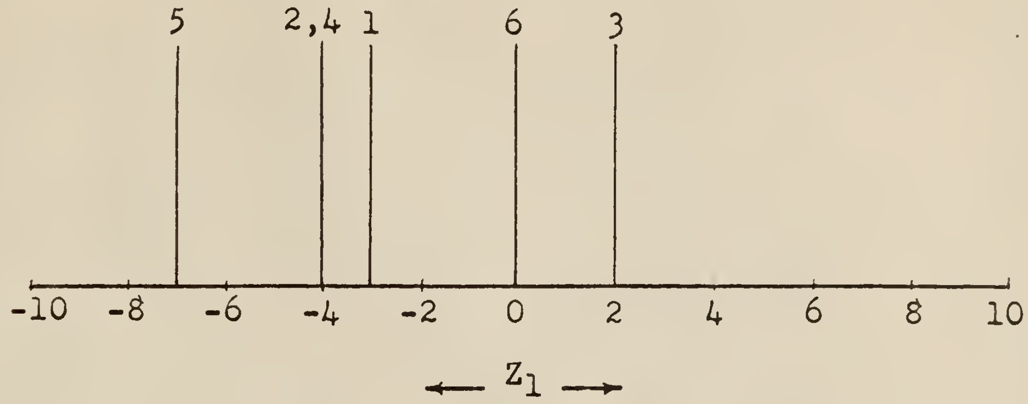

Fig. 9. Limiting values of covariant vectors Z_1.

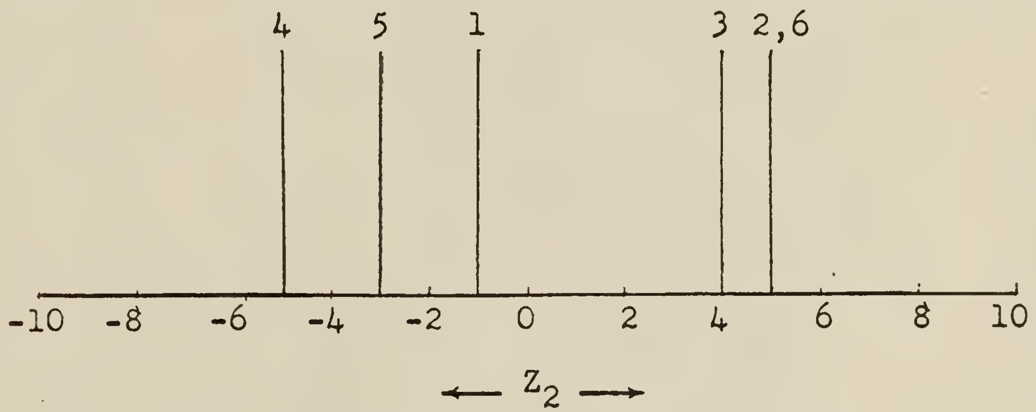

Fig. 10. Limiting values of covariant vectors Z_2.

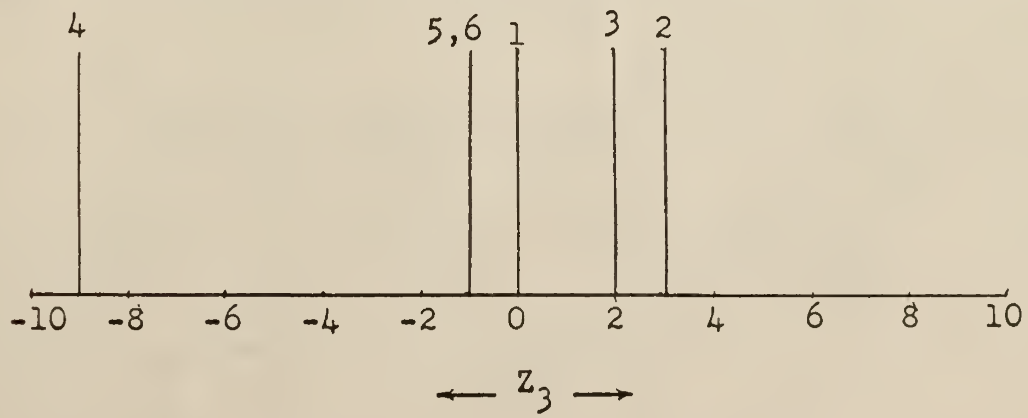

Fig. 11. Limiting values of covariant vectors Z_3.

Table 28. Limiting values of θ's.

n \ i	1	2	3	4	D^n
1	θ	θ	$0 \le \theta_3^1 \le 4$		4
2	θ	$0 \le \theta_2^2 \le 4$	$0 \le \theta_3^2 \le 4$		4
3	$0 \le \theta_1^3 \le 6$	$0 \le \theta_2^3 \le 6$	$0 \le \theta_3^3 \le 6$		6
4	θ	θ	θ	2	2
5	θ	θ	θ	4	4
6	θ	$0 \le \theta_2^6 \le 2$	θ		2
W_i	5	6	2	9	22

In the above case we notice that state (1) is the Fixed State; hence $\theta_1^3 = 5$, and so we must give first choice to stage (3) while selecting the stages for their preference. Stages (4) and (5) are also Fixed Stages; hence we have $\theta_4^4 = 2$ and $\theta_4^5 = 4$. Now there are two possibilities in selecting the stages for their preference. These possibilities are shown below, giving solutions A and B.

Solution A

First choice of stage 3

Second choice of stage 1

Third choice of stage 6

Fourth choice of stage 2

Solution B

First choice of stage 3

Second choice of stage 6

Third choice of stage 1

Fourth choice of stage 2

Solution A

A_a					A_b				A_c			
θ	θ	1	3	2nd choice	θ	θ	1	3	θ	θ	2	2
θ	4	0	0		θ	3	1	0	θ	4	0	0
5	0	1	0	1st choice	5	1	0	0	5	0	0	1
θ	θ	θ	2	fixed	θ	θ	θ	2	θ	θ	θ	2
θ	θ	θ	4	fixed	θ	θ	θ	4	θ	θ	θ	4
θ	2	θ	0		θ	2	θ	0	θ	2	θ	0

Solutions A_a and A_b are optimal solutions with total cost of \$112.00, while solution A_c has cost of \$114.00.

Solution B

B_a					B_b				B_c			
θ	θ				θ	θ			θ	θ		
θ					θ				θ			
5	1	0	0	1st choice	5	0	1	0	5	0	0	1
θ	θ	θ	2	fixed	θ	θ	θ	2	θ	θ	θ	2
θ	θ	θ	4	fixed	θ	θ	θ	4	θ	θ	θ	4
θ	2 /0	θ	0 /2	2nd choice	θ	2 /0	θ	0 /2	θ	2 /0	θ	0 /2

$B_{a,1}$					$B_{a,2}$			
θ	θ	1	3	3rd choice	θ	θ	3	1
θ	3	1	0		θ			0
5	1	0	0		5	1	0	0
θ	θ	θ	2		θ	θ	θ	2
θ	θ	θ	4		θ	θ	θ	4
θ	2	θ	0		θ	0	θ	2

Solution from B_b

$B_{b,1}$

θ	θ	1	3	3rd choice
θ	4	0	0	
5	0	1	0	
θ	θ	θ	2	
θ	θ	θ	4	
θ	2	θ	0	

$B_{b,2}$

θ	θ	3	1
θ			0
5	0	1	0
θ	θ	θ	2
θ	θ	θ	4
θ	0	θ	2

Solution from B_c

$B_{c,1}$

θ	θ	2	2	3rd choice
θ	4	0	0	
5	0	0	1	
θ	θ	θ	2	
θ	θ	θ	4	
θ	2	θ	0	

$B_{c,2}$

θ	θ	4	0
θ			0
5	0	0	1
θ	θ	θ	2
θ	θ	θ	4
θ	0	θ	2

Solution $B_{a,2}$ is not feasible as $W_2 = 2$;

Solution $B_{b,2}$ is not feasible as $W_2 = 2$, and

Solution $B_{c,2}$ is not feasible as $W_2 = 2$.

From the above we again have the same two optimal solutions with cost of $112.00. These are represented by solutions $B_{a,1}$ and $B_{b,1}$, while feasible solution $B_{c,1}$ is similar to A_c with cost of $114.00.

It must be noted that the same optimal solution (similar to A_a and $B_{b,1}$) is secured on page 217, Reference (9), by solving this problem with "unit penalty cost" method.

SUMMARY AND CONCLUSION

The "Discrete Maximum Principle" method reduces to a standard routine form for solving transportation problems involving linear cost functions, with the sum of the demands (requirements) of all the destinations equal to the sum of the supplies available at the origins.

From problem (2), it is noticed that any possible combination of the covariant vectors, Z_1, Z_2,....., which gives a feasible solution, will directly give an optimal solution with the proper use of the general rules and steps shown on pages 22 to 25. But failure in following these rules will not give the optimal solution. It is also noticed that any transportation problem involving any number of origins and any number of destinations and without slack variables is easily solved by this method of the "Discrete Maximum Principle" and it is easy to get optimal solution directly.

In the case of transportation problems involving slack variables and having two origins and any number of destinations, the same above method is applicable with slight modification. By assigning a unit value of cost per unit of resource (i.e., taking values of C_i^n for slack variable as unity instead of zero) for the slack variables and increasing other given cost (i.e., C_i^n) by unity and proceeding in the same way as above, we directly achieve the optimal solution for this problem. However, in the case of problems involving slack variables and having more than two origins, this above method does not seem to be an applicable one. It may be possible to solve this type of problem with modified approach of this technique. Hence, the further work may be carried on, and it is quite possible to obtain some modified approach to solve this type of problem.

Lastly, it must be noted that transportation problems of the above type (i.e., without slack variables), involving set-up cost associated with each origin and destination, are not possible to solve by the method of Linear Programming, but can easily be solved by the above method of the "Discrete Maximum Principle."

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THE APPLICATION OF THE DISCRETE MAXIMUM PRINCIPLE TO TRANSPORTATION PROBLEMS WITH LINEAR COST FUNCTIONS

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The aim of this report is to develop and illustrate a method of solving transportation problems involving an optimizing objective function of linear form, through the application of the "Discrete Maximum Principle." In order to do so, first an outline of general algorithm of the Discrete Maximum Principle, the statement of the transportation problem, and the formulation of the transportation problem in terms of the Discrete Maximum Principle are given in brief.

To show the application of the Discrete Maximum Principle to transportation problems with linear cost functions, a simple problem (Example 1) with two origins and five destinations is solved and explained in detail. For solving transportation problems with three or more origins and any number of destinations, the general rules and steps were found and are given under a title "General Rules and Steps to be Carried out for Solving the Problems With Three or More Origins." By the application of the above general rules and steps, Example 2, with three origins and four destinations, is solved and explained in detail. In a similar way Example 3, with three origins and four destinations, and Example 4, with four origins and six destinations, are solved by applying the same general rules and steps. This is done to show the method of attacking and solving this type of problem in order to achieve an optimal solution easily and directly.

The Discrete Maximum Principle reduces to a standard form for solving transportation problems which do not involve slack variables. It should be noticed that any possible combinations of covariant vectors Z_1, Z_2,...., which give a feasible solution, will directly give an optimal solution with the proper use of the general rules and steps. If, instead of the costs of transportation per unit of resource (i.e., C_i^n), profits of transportation per unit of resource are given, then the above method is applicable; but instead of finding the values of Z_i^n by minimizing the Hamiltonian equations, we have to find the same by maximizing these Hamiltonian equations; and proceeding in a similar manner, an optimal solution giving maximum profit is directly achieved. For the problems involving slack variables and having two origins and any number of destinations, the same above method is applicable, if a unit value of cost per unit of resource is assigned to the slack variables and other given values of C_i^n are increased by unity. However, this method does not seem to be applicable to problems involving slack variables and having more than two origins. It may be possible to solve this type of problem with modified approach of this technique.

Lastly, it should be noted that the transportation problems of the above type (i.e., without slack variables), involving setup cost associated with each origin and destination, are not possible to solve by the method of Linear Programming, but can easily be solved by the above method of the Discrete Maximum Principle.