A STUDY OF PULSE CODE MODULATION

by

P. R. Hariharan

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Electrical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1963

Approved by:

Major Professor

TABLE OF CONTENTS

INTRODUCTION

In amplitude modulation, phase modulation, and frequency modulation information is transmitted continuously in time domain, whereas in pulse modulation systems the information is transmitted intermittently. The carrier is a set of discrete pulses. These pulses are characterized by the rise time, decay time, (Proc. I.R.E., 1955) average pulse repetition rate which is given by the average number of pulses per unit time duration of one pulse and the amplitude of these pulses. Any of the quantities, repetition rate, duration of pulse, or amplitude of the pulse can be made to vary in accordance with the amplitude of the modulating wave. In pulse duration modulation the value of each instantaneous sample of the signal wave is made to vary the duration of a particular pulse. In pulse position modulation the value of each instantaneous sample of the signal wave varies the time of occurrence of a pulse relative to its unmodulated position.

In pulse code modulation the samples of the modulating wave are allowed to take only certain discrete values. These amplitudes are then assigned a code, where each such code is uniquely related to the magnitude of the sample.

The operation that is common to all these four systems of pulse modulation is the operation of obtaining the signal magnitude at prespecified intervals of time. This operation is known as sampling.

THE SAMPLING PRINCIPLE (Shannon, 1948)

A signal which contains no frequencies greater than B cycles/sec. cannot assume an infinite number of independent values per second. It can in fact assume 2B independent values per second and the amplitude at any set of points spaced τ seconds apart where $\tau = \frac{1}{2B}$, specifies the signal completely. Hence to transmit a bandlimited signal of duration T it is not necessary to send the entire continuous function of time. It suffices to send the finite number of 2BT independent values obtained by sampling of the signal at a regular rate of 2B samples per second.

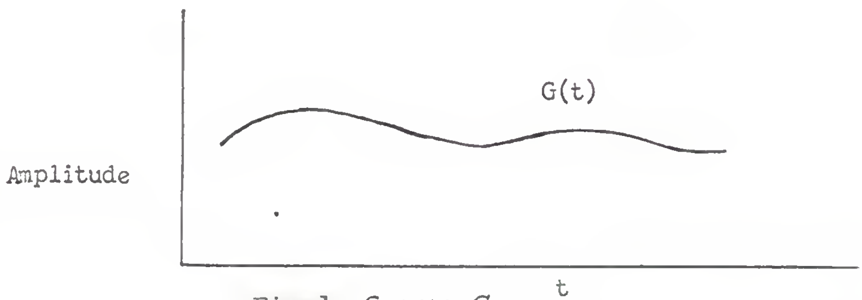


Fig-1. SIGNAL SAMPLED

Let G(t) be the signal that is periodically sampled. Let $\Omega(w)$ be the Fourier transform of G(t). Then G(t) is given by the relation

$$G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Omega(w)e^{jwt}dw \qquad (1)$$

Since $\Omega(w)$ does not exist outside the band B the above can be written as

$$G(t) = \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B}\Omega(\omega)e^{jwt}dw \qquad (2)$$

Let t = (n/2B). The above then reduces to

$$G(n/2B) = \frac{1}{2\pi}\int_{-2\pi B}^{2\pi B}\Omega(w)e^{jw(n/2B)}dw \qquad (3)$$

The set of values of G(n/2B) for all positive and negative integral values of n determines all the coefficients in the Fourier expansion of $\Omega(w)$. Consequently they determine $\Omega(w)$ itself in the range

$-2\pi B < w < 2\pi B$. Since $\Omega(w)$ is assumed to be zero outside of this range, the set of values of $G(n/2B)$ completely specify $\Omega(w)$. There is thus one and only one function whose spectrum $\Omega(w)$ is limited to the frequency bandwidth B and which passes through a set of given values at sample points spaced 1/2B seconds apart.

The proof of the sampling theorem (Shannon,1948) for a time limited case is given below. Suppose a function $f(x)$ is defined in the interval $-T$ to T and satisfies the Dirichlet's conditions which are

1. $f(x)$ should be defined and bounded in (a,b) where $b = a + 2T$ and there should be a finite positive number A such that at every point in (a,b)
$$|f(x)| \leq A$$
2. $f(x)$ should be integrable in (a,b)
3. $f(x)$ should have only a finite number of discontinuities for every finite interval interior to (a,b) and at every point of discontinuity the value of the function equals
$$\frac{1}{2}\left\{f(x+0) = f(x-0)\right\}$$
which is the arithmetic mean of the right hand and left hand limits.

Now $f(x)$ can be expanded in a series of trignometric functions (Tolstov,1962) such that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nwx + \sum_{n=1}^{\infty} b_n \sin nwx \quad (4)$$

Where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad (6)$$

and
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx \quad (7)$$

This trignometric relationship can also be expressed in the complex form since

$$a_n \cos nx + b_n \sin nx = a_n \frac{e^{jnx} + e^{-jnx}}{2} + b_n \frac{e^{jnx} - e^{-jnx}}{2j}$$

$$= \frac{(a_n - jb_n)}{2} e^{jnx} - \frac{(a_n - jb_n)}{2} e^{-jnx}$$

On writing

$$C_n = (a_n - jb_n)/2 \quad \text{and}$$

$$C_{-n} = (a_n - jb_n)/2$$

f(x) can be expresse as

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{jnx} \tag{8}$$

where

$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-jnx} dx \tag{9}$$

If the interval of expansion is 0 to T, then

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{j\frac{2\pi nx}{T}} \tag{10}$$

and

$$C_n = \frac{1}{T} \int_{0}^{T} f(x) e^{-jnx} dx \tag{11}$$

and for a perfectly general case $T_1 < x < T_2$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{j2\pi nx}{T_2 - T_1}} \tag{12}$$

and

$$C_n = \frac{1}{T_2-T_1} \int_{T_1}^{T_2} f(x)e^{+j\frac{2\pi}{T_1-T_2}nx}dx \tag{13}$$

$\Omega(w)$, the frequency spectrum of $G(t)$, can be expanded in a Fourier Series in the interval $-2\pi B \leqslant w \leqslant 2\pi B$

$$\Omega(w) = \sum_{n=-\infty}^{\infty} \overline{C_n}\, e^{-j\frac{2\pi nw}{4\pi nB}} \tag{14}$$

$$= \sum_{n=-\infty}^{\infty} \overline{C_n}\, e^{-\frac{jnw}{2B}} \tag{15}$$

where

$$\overline{C} = \frac{1}{4\pi B} \int_{-2\pi B}^{2\pi B} \Omega(w)e^{\frac{jwn}{2B}}\, dw \tag{16}$$

$$= \frac{1}{2B}\quad G(n/2B)$$

Hence

$$\Omega(w) = \frac{1}{2B} \sum_{n=-\infty}^{\infty} G(n/2B)e^{-\frac{jnw}{2B}} \tag{17}$$

$$G(t) = \frac{1}{2\pi} \int_{-2\pi B}^{2\pi B} \frac{1}{2B} \sum_{n=-\infty}^{\infty} G(n/2B)\, e^{-\frac{jnw}{2B}}\, e^{jwt}\cdot dw \tag{18}$$

i.e.

$$G(t) = \frac{1}{4\pi B} \sum_{n=-\infty}^{\infty} G(n/2B) \int_{-2\pi B}^{2\pi B} e^{jw(t - \frac{n}{2B})}\, dw \tag{19}$$

Now

$$\int_{-2\pi B}^{2\pi B} e^{jw(t - \frac{n}{2B})}\, dw = 2 \, \frac{\text{Sin } 2\pi B(t - n/2B)}{(t - n/2B)}$$

Hence from equation (18) it is seen that

$$G(t) = \frac{1}{4\pi B} \sum_{n=-\infty}^{\infty} G\left(\frac{n}{2B}\right) \cdot \frac{2 \text{ Sin } (2\pi Bt - \pi n)}{(2\pi Bt - \pi n)}$$

and thus completely defined.

This shows that an arbitrary function of time whose spectrum is limited to the bandwidth B, can be completely reconstituted by its samples at intervals of $\tau = 1/2B$ secs. apart.

If the function is present only for an interval of time T then there are a total of 2TB sampling points within this interval and the value of the function at these points give rise to G(t), the original time function.

RECONSTRUCTION OF SAMPLED DATA

Interpolation

From the discrete set of sampled data available it is required to regenerate the continuous signal. This involves filling in the intermediate values between the present discrete values.

Let the discrete samples of G(t) be denoted by G_1, G_2, ...G_n. These values are separated in time by $1/f_s$ where f_s is the sampling frequency. In order to produce a continuously varying time function h(t) which passes through the sample points of G(t), another function u(t), has to be introduced such that

$$h(t) = \sum_{n=-\infty}^{\infty} g_n u\,(t - n/f_s) \qquad \text{(Stiltz, 1961)} \qquad (22)$$

and in order for h(t) to be the desired continuous function, u(t) has to satisfy the conditions that

$$u\,(o) = 1$$

$$u\,(nf_s) = o \qquad \text{for } -\infty < \underset{n \neq o}{n} < \infty \qquad (23)$$

This is known as a step function interpolator or a holding filter and its Laplace transform representation is

$$\frac{1 - e^{-TS}}{S} \quad . \qquad (24)$$

where T is the sampling period. The response of this filter to periodically applied impulse functions is shown in Figure 2. This is a shallow skirted low pass filter. Towards proving this the time domain response of this should be obtained and towards proving this replace S by jW and find the spectrum.

$$\left|T\,(jW)\right| = \left|\frac{1 - e^{-jWT}}{JW}\right| \qquad (25)$$

$$= T\,\frac{\text{Sin } WT/2}{WT/2}$$

$$= T\,\frac{\text{Sin}\,\theta}{\theta} \qquad \text{where} \quad \theta = \frac{WT}{2} \qquad (26)$$

This function has a shallow, low pass characteristic with cut off occurring at $\Theta = \pi$ i.e., f cut off = 1/T. The phase angle Arg T(jw) = WT/2 is linear giving forth a constant delay for all frequencies.

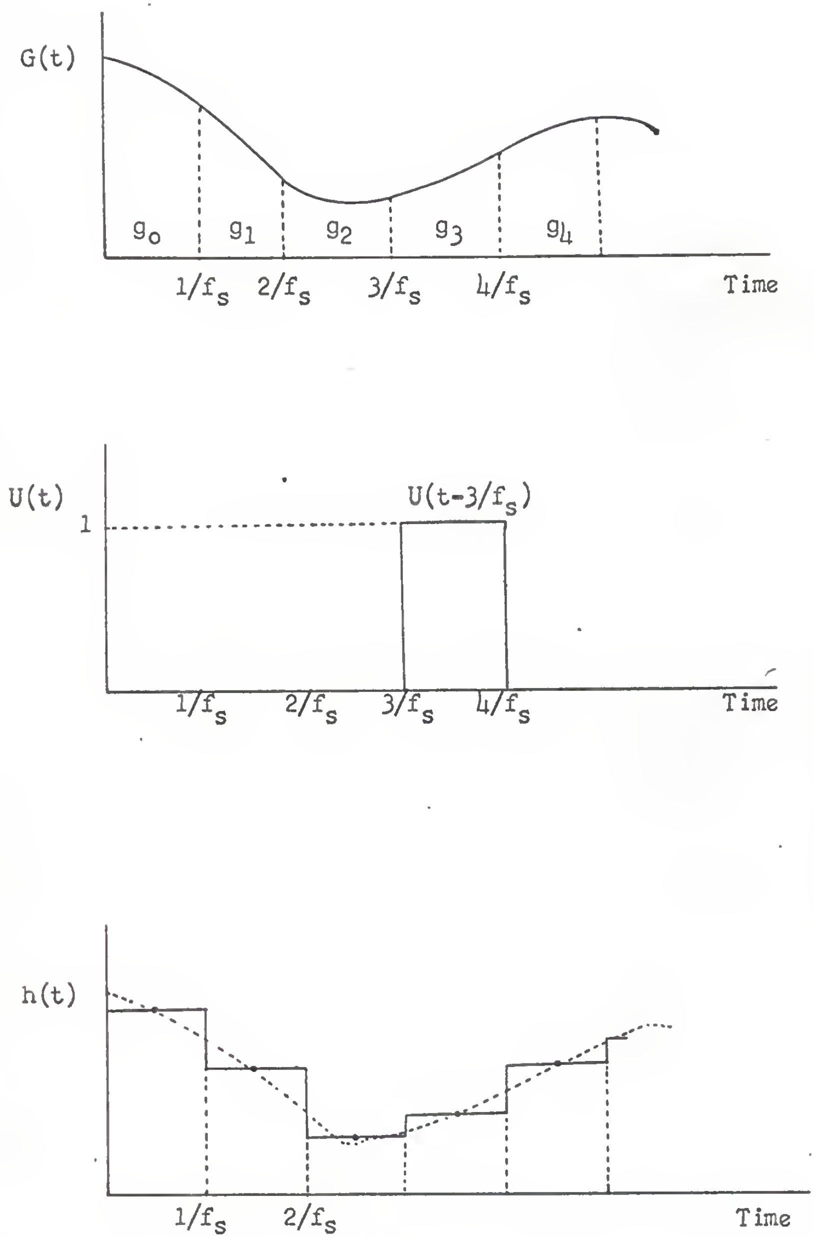


Figure 2. Response of interpolating filter to periodically applied impulse functions. (H. Stiltz, 1961)

There are various methods of realizing this. One method, which is quite complex is by means of making use of a delay time and integrators as shown in Figure 3 and this represents the ideal case.

In the Laplace domain integration is denoted by 1/S. This can be expressed in terms of an infinite series as below.

$$\frac{1}{1-Z} = 1 + Z + Z^2 + \ldots\ldots\ldots\ldots + Z^n + \ldots\ldots\ldots\ldots \quad (27)$$

Replacing Z by $1/1+S$ it is seen that the equation (27) can be written as

$$\frac{1}{1-\frac{1}{1+S}} = 1 + \frac{1}{1+S} + \frac{1}{1+S^2} + \ldots\ldots\ldots \quad (28)$$

$$\frac{1+S}{S} = \sum_{K=0}^{\infty} \left(\frac{1}{1+S}\right)^K$$

i.e.

$$\frac{1}{S} = \sum_{K=1}^{\infty} \left(\frac{1}{1+S}\right)^K \quad (29)$$

The summation is terminated after a finite number of terms to obtain an approximate finite integrator. In equation (29) $1/(1 + S)$ can be realized by an RC network, thus the 1/S can be realized to any desired degree of accuracy by means of cascading these networks and summing their output. Some of these circuits are shown in Figure 4.

OPTIMUM PHYSICALLY REALIZABLE TIME INVARIANT LINEAR SMOOTHING FILTER

The train of sample pulses h(t) can be represented by

$$h(t) = G(t) \sum_{n=-\infty}^{\infty} \delta(t - n/fs) \quad (30)$$

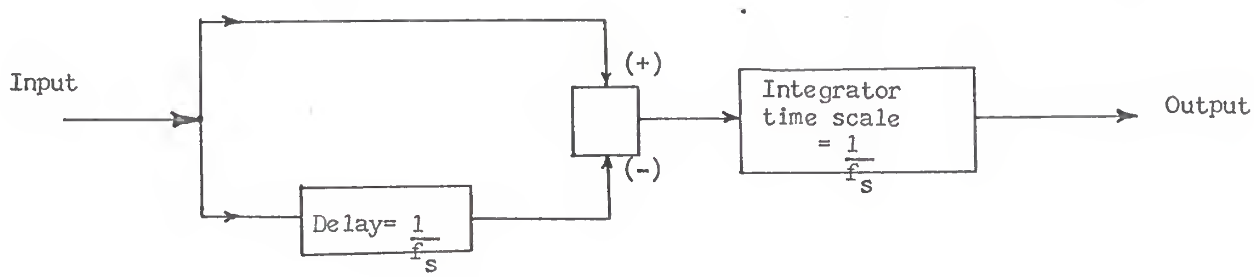


Figure 3. Realization of interpolating filter. (H. Stiltz, 1961)

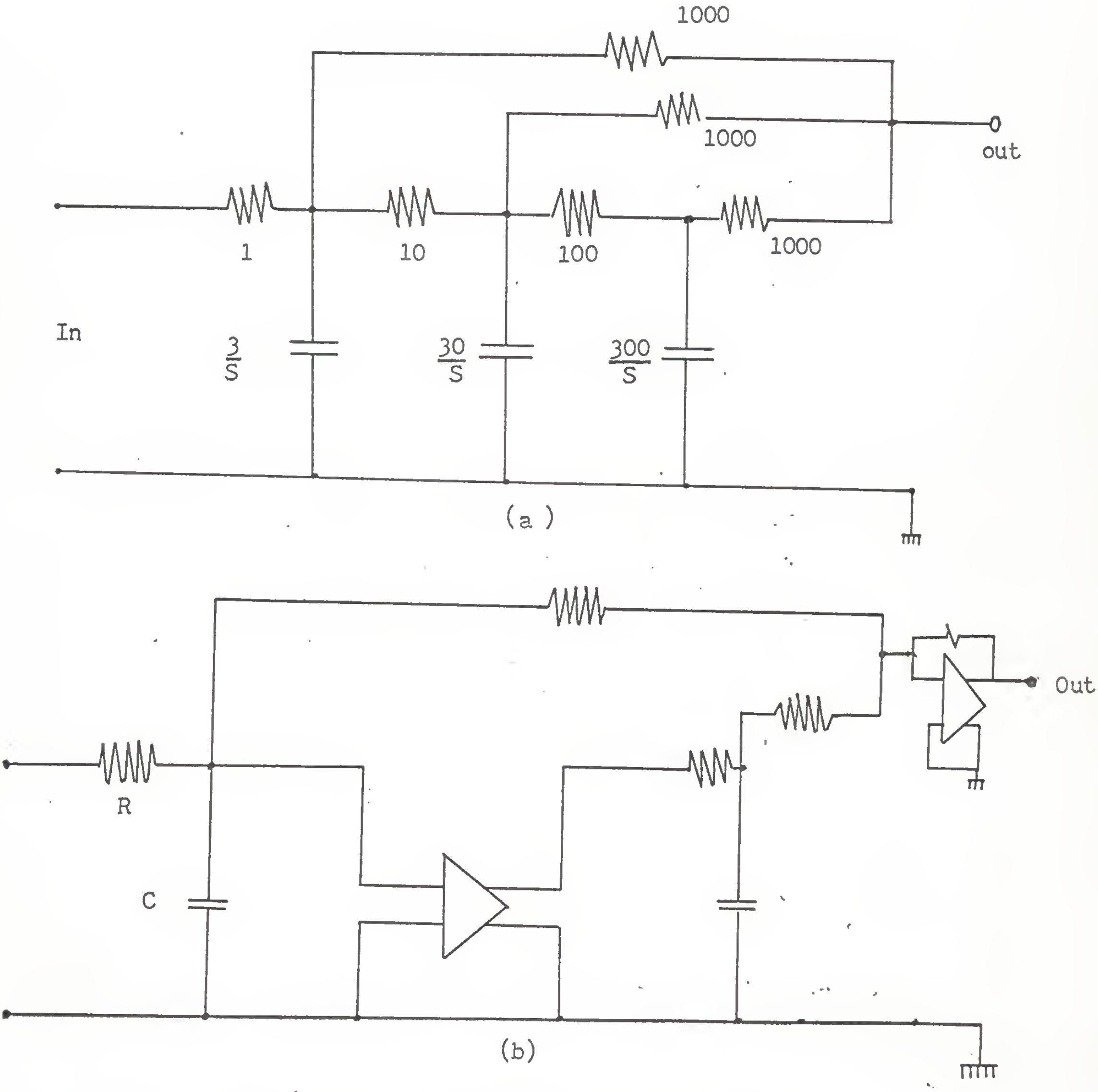


Figure 4.(a).Approximate third order finite integrator.

(b).A finite integrator making use of operational amplifiers.

From Fourier Series the trignometric approximation to a series of functions can be obtained as

$$\delta(t) = \frac{1}{T} + \frac{2}{T} \sum_{n=1}^{\infty} \cos \frac{2\pi nt}{T} \tag{31}$$

where T is the sampling period. This approximation to δ functions can be reduced as (Stewart, 1956)

$$\delta(t) = f_s \sum_{n=-\infty}^{\infty} e^{jn2\pi f_s t} \tag{32}$$

where $f_s = 1/T$

and hence

$$h(t) = G(t)\, f_s \sum_{n=-\infty}^{\infty} e^{j2\pi n f_s t} \tag{33}$$

Considering a finite section of both G(t) and h(t) extending from -T to +T and Fourier transforming both sides it is seen that

$$H_T(f) = f_s \sum_{n=-\infty}^{\infty} G_T(f - nf_s) \tag{34}$$

$$g_T(f) = YH_T(f) = f_s Y \left\{ G_T(f) + \sum_{n=1}^{\infty} G_T(f-nf_s) + G_T(f+nf_s) \right\} \tag{35}$$

Error in recovered message E_T is given by

$$E_T = g_T - G_T = (f_s Y - 1) G_T(f) + f_s Y \sum_{n=1}^{\infty} G_T(f-nf_s) + G_T(f+nf_s) \tag{36}$$

By an application of Parsevall's theorem it can be seen that the spectral density of the error is

$$\Phi_\epsilon(f) = \lim_{T\to\infty}\left(\frac{1}{T}\ \left|E_T\right|^2\right) = \lim_{T\to\infty}\left(\frac{1}{T}\cdot E_T E_T^{*}\right) \tag{37}$$

using equations (36 and (37) $\Phi_\epsilon(f)$ can be computed as

$$\Phi_\epsilon(f) = \left|f_s Y - 1\right|^2 \Phi_m(f) + f_s y^2 \sum_{n=1}^{\infty} \Phi_m(f - nf_s) + \Phi_m(f + nf_s)$$

$$+ \text{ cross spectral density terms.} \tag{38}$$

A typical cross spectral density term is

$$\Phi_c(f) = \lim_{T\to\infty} \frac{1}{T}\left(G_T(f-nf_s) + G_T(f+nf_s)\right)\left(G_T(f-mf_s) + G_T(f+mf_s)\right) \tag{39}$$

The expression above represents the cross spectral density between two real functions

$$\begin{aligned} &2\ G\ (t) \cos 2\ nf_s t \\ \text{and}\quad &2\ G\ (t) \cos 2\ mf_s t \end{aligned} \tag{40}$$

and must equal the cross correlation Fourier transform of these two functions

$$\Phi_c(f) = 2\int_{-\infty}^{\infty} \phi_c(\tau)\, e^{-j2\pi f\tau}\, d\tau \tag{41}$$

where

$$\phi_c(\tau) = \overline{2G(t) \cos 2\pi nf_s t}\quad \overline{2G(t+\tau) \cos 2\pi mf_s(t+\tau)} \tag{42}$$

$$= [\overline{4\, G(t)G(t+\tau)}]\ [\overline{\cos 2\pi n f_s t \cos 2\pi m f_s (t+\tau)}] \tag{43}$$

The bar denotes the time average. Taking the average of the product of the two bracketed terms as equal to the product of the average values it is seen that

$$\Phi_c(\tau) = 4\ \overline{G(t)G(t+\tau)}\ [\overline{\cos 2\pi n f_s t \cos 2\pi m f_s (t+\tau)}] \tag{44}$$

If $m \neq n$ the average value of the quantity inside the bracket is zero. If $m = n$, equation (44) reduces to

$$\Phi_c(\tau) = 2\ \Phi_m(\tau) \cos 2\pi f_s n \tau \tag{45}$$

Hence by a direct transformation, from (45) the relationship

$$\Phi_n(f) = [\Phi_m(f - nf_s) + \Phi_m(f + nf_s)] \tag{46}$$

is obtained. The sideband spectrum $\Phi_s(f)$ is

$$\Phi_s(f) = \sum_{n=1}^{\infty} \Phi_m(f - nf_s) + \Phi_m(f + nf_s) \tag{47}$$

This spectrum is shown in Figure 5. The expression for $\Phi_\varepsilon(f)$ is identical in form with the error spectral density for continuous smoothing of message plus noise if the noise were independent of the message and had a spectral density equal to the sideband spectrum.

Weiner's Method can be applied to give the optimum linear filter function $f_s y$ for any given sampling rate.

$\Phi_\varepsilon(f)$ is the spectral density of the error function and the mean square error can then be written as (Bendat, 1958)

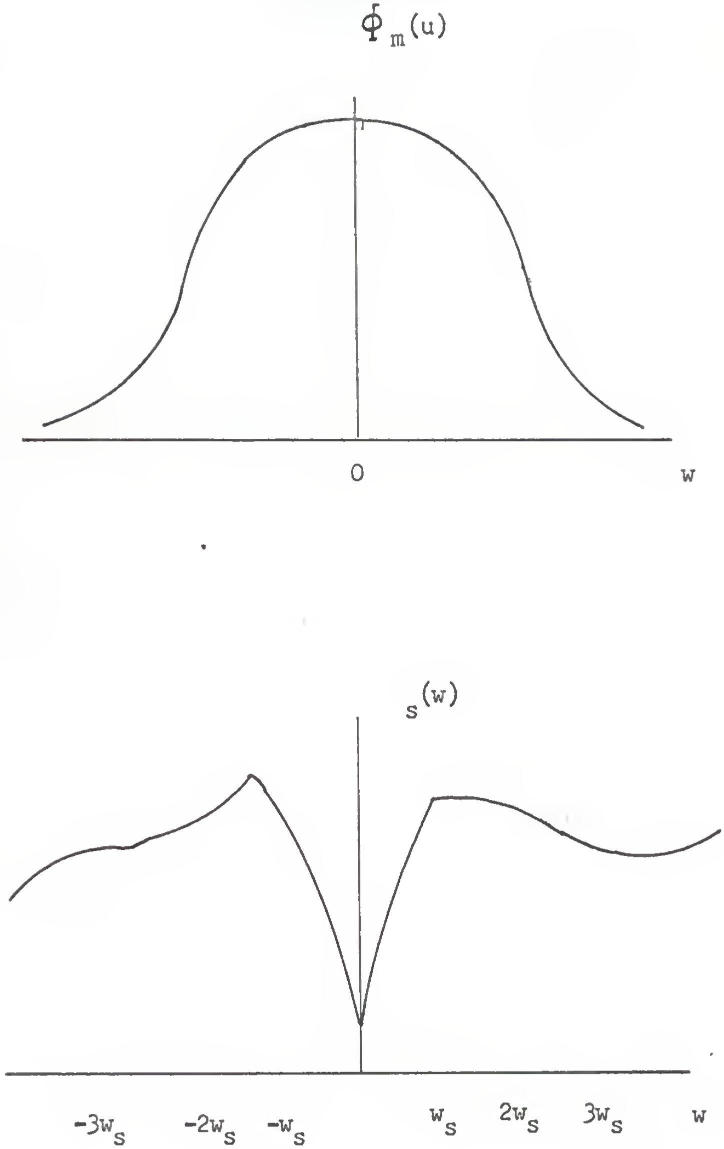


Figure 5. Spectra of typical original message and of sidebands of effective noise. (Stewart, 1956)

$$\sigma_{\varepsilon}^{2} = \int_{0}^{\infty} \Phi_{\varepsilon}(f)\, df \tag{48}$$

If the sampling rate is high compared to the effective bandwidth of G(t), then all except the first sideband to the right can be neglected giving for $\Phi_s(f)$

$$\Phi_s(f) \doteq \Phi_m(f - f_s) \tag{49}$$

ALIASING

When the sampling frequency is less than twice the highest frequency contained in the signal, recovery of a signal identical to the original to the originally sampled signal is not possible. In this case a downward transposition of the spectra of the signal occurs. This particular phenomenon is known as aliasing.

It is necessary to reduce aliasing errors to arbitrarily small proportions and use a sampling frequency not very much in excess of twice the highest significant frequency contained in the signal. For mathematical convenience the power spectral density of the signal is expressed as (Stiltz, 1961)

$$G(f) = \frac{A}{1 + (f/f_o)^{2m}} \tag{50}$$

where f_o is the 3 db point as denoted in Figure 6. A is the low frequency power spectral density, $f_s/2$ is the Nyquist frequency and m is the rate of spectrum cut off. One half of the minimum sampling frequency is referred to as the Nyquist frequency.

This mathematical definition is consistent with the physical systems when the signal is passed through a Butterworth's low pass filter. If it

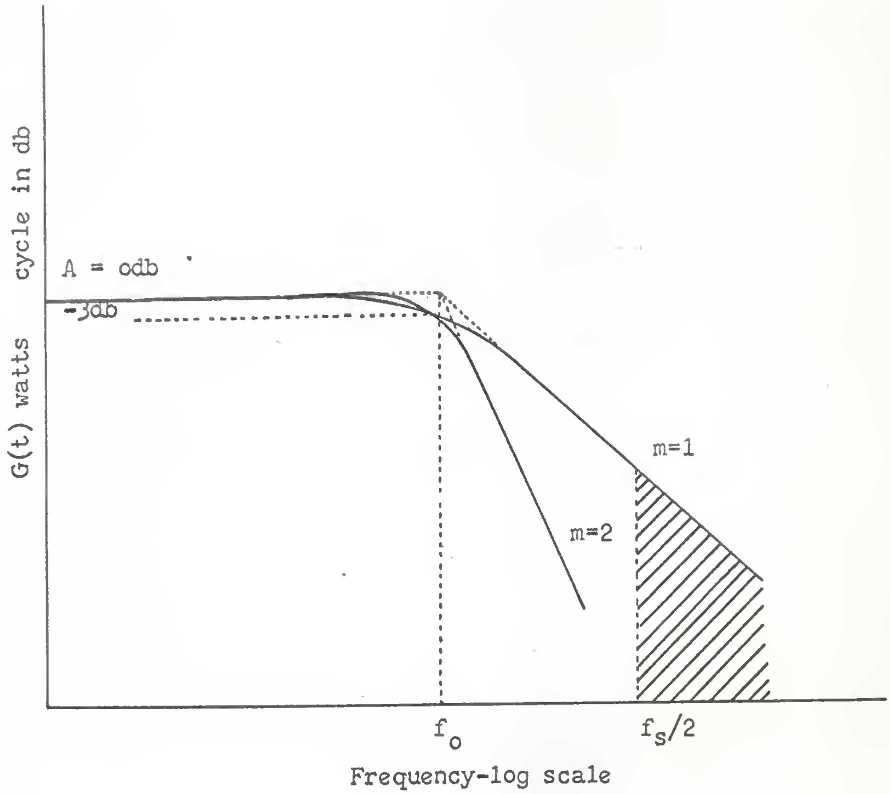


Figure 6. Maximally flat response. (Stiltz, 1961)

is assumed that $f_o << f_s$ then

$$G(f) \doteq A\,(f_o/f)^{2m} \tag{51}$$

Making use of the relation that

$$P_{AV}\text{ (all frequencies)} = \int_0^{\infty} G(f)\,df \tag{52}$$

and

$$P_{AV}\text{ (frequency band } f_r \text{ to } f_s) = \int_{f_r}^{f_s} G(f)\,df \tag{53}$$

the aliasing error power which is equal to the power in frequencies from $f_s/2$ to ∞ is obtained as

$$Va^2 = P_{AV} = \int_{fs/2}^{\infty} G(f)df = \int_{fs/2}^{\infty} A\left(\frac{f_o}{f}\right)^{2m} df \tag{54}$$

i.e.

$$Va^2 = Afo^{2m} \int_{f_s/2}^{\infty} f^{-2m}df$$

$$= \frac{Afo^{2m}}{2m-1} \cdot \frac{1}{f^{2m-1}} \Bigg|_{\infty}^{f_s/2}$$

$$= \frac{2^{2m-1}\ Afo}{(2m-1)} \cdot \left(\frac{fo}{fx}\right)^{2m-1} \tag{55}$$

The power contained in the signal is given by

$$V_s^{\,2} = P = \int_0^{\infty} \frac{A}{1 + (f/fo)^{2m}} \cdot df$$

$$= \frac{Af_o}{2m} \text{ Cosec } (\pi/2m) \tag{56}$$

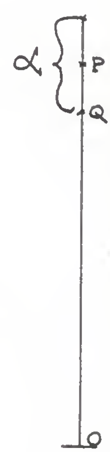
The relative error due to aliasing is given by

$$V_E = V_a^2/V_s^2 = 2\pi\left[\frac{m}{(2m-1)} (f_o/f)^{2m-1} \text{ Sin } (\pi/2m)\right]^{1/2} \tag{57}$$

This analysis shows that sampling frequencies much higher than the nominal bandwidth of the signal should be used if the low-pass filter used does not have a sharp cut-off characteristic in order to maintain the aliasing error within tolerable limits. It is also to be noted that the filters normally used cuts off at the rate of 60 db/octave and so a ratio of sampling frequency to nominal highest frequency in signal of three is sufficient for keeping the error to less than 1 per cent.

QUANTIZATION

Speech has a continuous range of amplitudes, and hence the sampled wave also has a continuous variation in the amplitude scale. Human ear cannot detect minute variations in intensity. For example consider one sample and offer a corresponding sound pulse to the ear. It will judge different samples like OP to be equal, even though P lies within a certain range of amplitudes. By taking advantage og this phenomenon it is permissible to transmit all amplitude levels in this range by the one discrete amplitude level OQ. It is also seen that deviation from fidelity can be kept within tolerable limits by using a large number of steps.



Speech transmission can therefore be effectively achieved by transmission of a finite number of discrete amplitude levels.

The signal that is recovered at the receiver will not be identical with the transmitted signal because of quantization. The maximum error should not in any case exceed one quantum step. In order to keep this deviation from the original signal within limits, a sufficiently large number of quantum steps are required. The actual number used depends upon the fidelity required.

Consider one particular amplitude level OQ = F. A possible measure of fidelity with respect to this particular amplitude level is the mean square of the distance, d, between OP and OQ

$$d^2 = \left[(OP)^2 - (OQ)^2 \right]^2$$

Making the assumption that the point P takes on values in the range of α with equal density, one obtains (Bennett, 1941)

$$d^2 = \frac{1}{\alpha} \int_{-\alpha/2}^{\alpha/2} x^2 \, dx = \frac{\alpha^2}{12} \qquad (58)$$

$$\overline{OP^2} = \overline{OQ^2} + d^2$$

$$= F^2 + \alpha^2/12$$

The power of the signal amplitude in the range α without quantization is $\overline{OP}^2$ and $\widetilde{F}^2$ is the contribution of the same after quantization. Hence the quantizing error which is known as quantization noise, being the difference between these two is $\alpha^2/12$.

Physical Interpretation of Quantization Noise

Let $G(t)$ denote the signal function before quantizing and P the power associated with it. Let $G_Q(t)$ be the signal function after quantization, which is received by the receiver, and P_Q the power associated with it. $G(t)$ and $G_Q(t)$ are different because of quantization. Also, P_Q, the power received due to transmission of the quantized samples differs from P, the power associated with the unquantized sample, by the quantizing noise power.

i.e. $$P = P_Q + N_Q$$

where N_Q is the quantizing noise power.

This can also be viewed as the quantizer splitting up the power P into the signal power P_Q and noise power N_Q which hinders the signal detection. Even if the transmission channel is noiseless the quantization noise is present at the receiver. Let A be the unquantized amplitude of $G(t)$ and it be divided into n equal units. The size of every step is

$$\alpha = A/n$$

If A_Q denotes the range of the quantized sample the relationship

$$A = A_Q + \alpha$$

is always satisfied. From this it is seen that the relationship between the number of steps n, size of every step α and the quantum range A_Q is given by

$$n = 1 + \frac{A_Q}{\alpha}$$

The signal to noise ratio is given by (Mayer, 1957)

$$\frac{P_Q}{N_Q} = (n^2 - 1)$$

The following table gives the number of steps versus the signal to noise ratio:

n	2	4	8	16	32	64	128
n^2-1	3	15	63	255	1023	4095	16383
P_Q/N_Q in db	4.77	11.76	17.99	24.08	30.1	36.13	42.13

All experiments conducted so far for the determination of the number of steps required for generation of good quality speech with good intelligibility are subjective in nature. It is generally agreed that 64 steps regenerates the original signal with a very high degree of accuracy. (Mayer, 1957)

COMPANDORS IN QUANTIZER

A compandor is used to achieve noise reduction. By compression is meant that the effective gain which is applied to the signal is varied as a function of its magnitude such that the gain is greater for small rather than for large signals.

The weak signals are most susceptible to degradation by noise and other unwanted interference. These weak signals are highly amplified by the compressor and are carried at a relatively high amplitude level in the presence of noise.

The compandor provides a means for making the noise susceptibility a function of the magnitude of the signal. The noise susceptibility is made less during one portion and greater than that of a linear system during some other portion of the input.

Analysis

The block diagram of a system employing a compressor is as shown in Figure 7. The input signal is filtered by a low pass filter LPF-1 with cut off frequency B. Its signal output occupies all frequencies in the band B. This signal is sampled at the rate of 2B samples per second and thus the conversion of the signal to PAM pulses is achieved. According to the sampling theorem the signal can be reconstructed from the samples. These PAM pulses make up the input to the compressor.

At the receiver an expandor is used to compensate for the effects of the compandor. The input versus the output characteristics of this expandor are exactly opposite to that of the compressor used.

A compressor is called instantaneous if its bandwidth is wide enough so that it can accomplish the change in the magnitude of each pulse without increasing its duration. Theoretically the bandwidth required before and after compression for transmission of the signal are the same. Also, since the compression is performed in accordance with some known law, the inverse operation can be performed in the receiver for an accurate recovery of the signal information.

Let the pulse impressed on the input of the expandor have a magnitude of $V_1 + v_1$ where V_1 is the signal amplitude and v_1 the noise amplitude. The maximum values of the input and output are kept equal. A typical expandor characteristic is shown in the Figure 8. Here V_1 is the value of the input pulse when there is no noise and E_1 is the corresponding magnitude of the output pulse. When noise is present the magnitude of the output is $E_1 + \Delta E_1$. These pulses serve as the input to the low pass filter F_3.

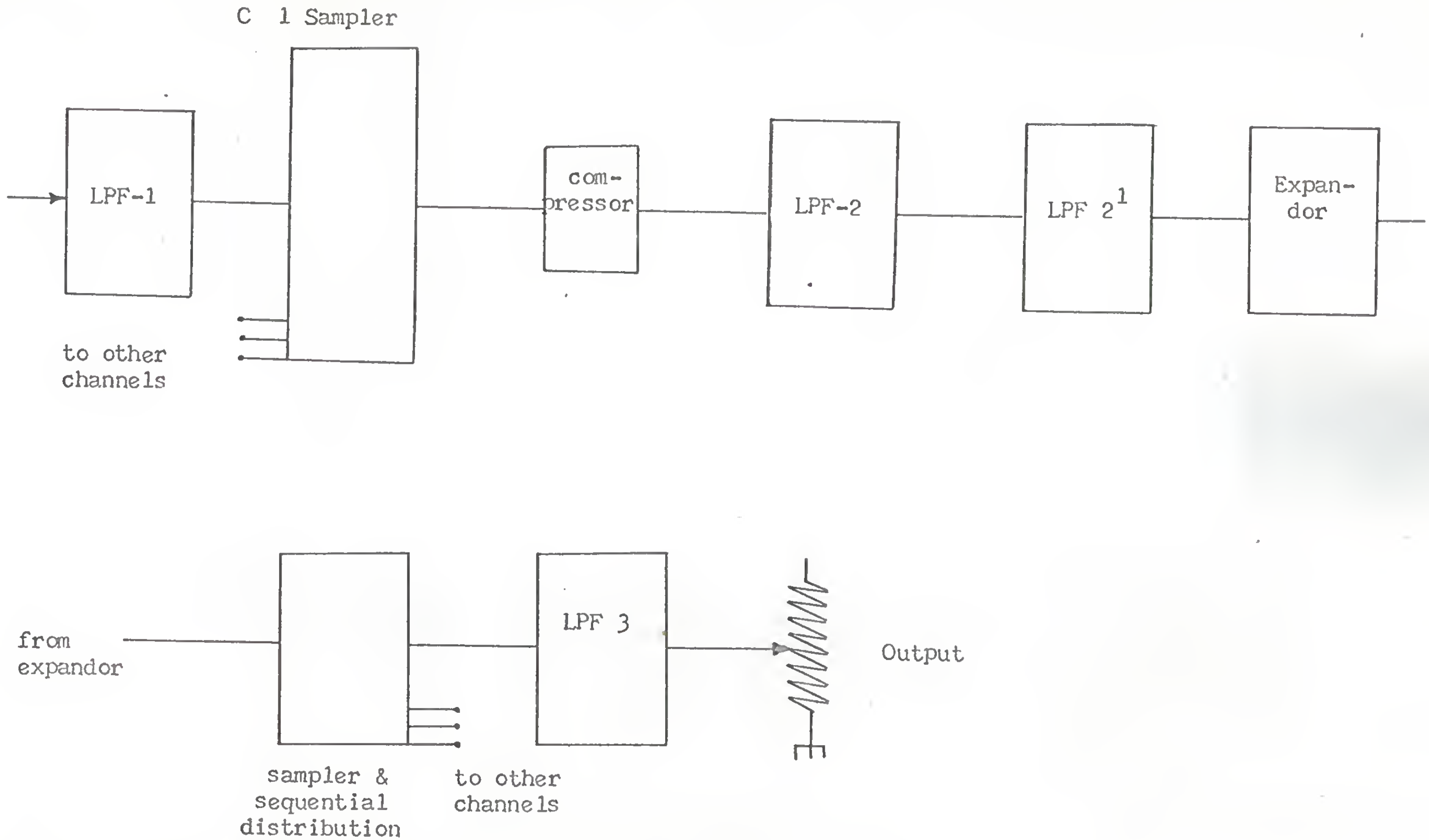


Figure 7. A system using the compressor. (Mallinkrodt, 1951)

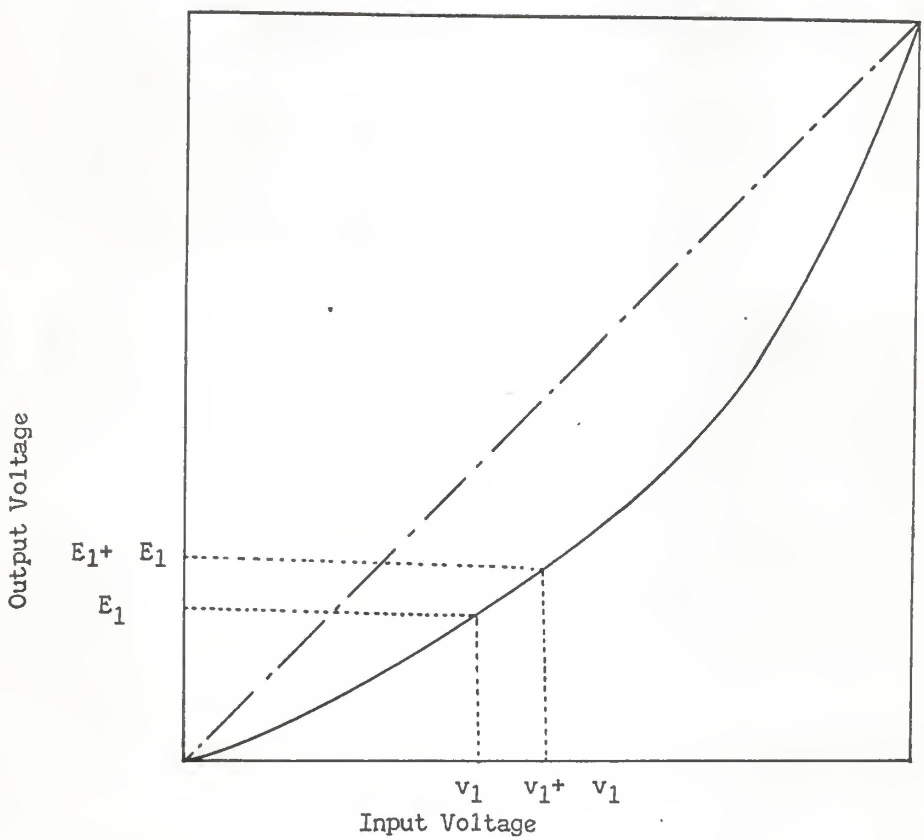


Figure 8. Expandor-input voltage versus output voltage characteristic. (Mallinkrodt, 1951)

Let the instantaneous noise voltages at the output of this low pass filter be denoted by S_1 and N_1.

$$S_1 = kE_1 \tag{59a}$$

$$N = kv_1 \frac{\Delta E_1}{E_1} \tag{59b}$$

where k is a constant which depends upon the design of the system.

$\Delta E_1/v_1$ is a function of the slope of the expandor characteristics and is called the noise susceptibility s of the system. From the above two equations (59a) and (59b) it can be written that

$$\frac{S}{N} = \frac{E}{vs} \tag{60}$$

S/N is the ratio of the instantaneous signal to instantaneous noise at the output of the low pass filter F_3 and E/v is the corresponding ratio in the absence of the compandor. If the ratio of signal to noise is high, it can be written that

$$s = dE/dv \tag{61}$$

When s is unity, then the noise susceptibility equals that of a linear system. s varies as a function of the signal input. Companding makes s vary as a certain predetermined function of the magnitude of the input signal. The input-output characteristic of the compressor should be a single valued function as otherwise it could create ambiguities at the receiver.

CHOOSING OF THE EXPANDOR CHARACTERISTIC

A compandor is said to be logarithamic when the output voltage of the compressor is a logarithamic function of its input voltage. The output-input characteristics of such an expandor are exponential and is expressed by the equation

$$E = ae^{bV} \tag{62a}$$

where a and b are arbitrary constants, v is the expandor input boltage and E is the output voltage. The characteristics should not follow an exponential law at very low values of input voltage, since if the relationship is expotential, E is not zero when the input v is zero. This difficulty of the system producing an output without an input signal is avoided by using a characteristic which is linear for input voltages below a given value and exponential for input voltages above this value. The transition point is defined as the point where the input-output characteristics changes from the linear to the exponential relation. The characteristics and its first derivative are continuous at this point. Over the exponential portion of the characteristics the relationship can be written as

$$E = e^{(V-1)/V_t} \tag{62b}$$

indicating that $E = 1$ when $v = 1$ and $de/dv = E_t/V_t$ where the voltages at the transition points are given as E_t and V_t, so that

$$E_t = e^{(V_t-1)/V_t} \tag{63}$$

The expansion ratio is defined as the ratio of Em/E_t to Vm/V_t where Em and Vm are the maximum values of the expandor output and input voltages respectively.

Signal to Noise Ratio

On differentiating equation (62a) with respect to V and substituting equation (61) one obtains

$$s = \frac{e^{(V-1)/V_t}}{V_t} \qquad (64)$$

This equation expresses the relationship between noise susceptibility and the compressor output voltage v. From equation (60) it is seen that

$$\frac{S}{N} = \frac{E}{vs}$$

and from equation (64) it can be written

$$s = \frac{e^{(V-1)/V_t}}{V_t}$$

and from these two equations (60) and (64) the relationship that

$$\frac{S}{N} = \frac{V_t}{v} \qquad (65)$$

is obtained. The above relationship brings out the fact that when the input signal is such as to operate the expandor in the exponential portion of its characteristics the ratio of the instantaneous signal to instantaneous noise is independent of the magnitude of the signal

Noise Advantage Achieved

The permissible noise increase at the output of the system when a compandor is used has to be obtained for the comparison to be justifiable, the noise at the output of the system during intervals when the signal voltage is zero must be the same for the two conditions, when the system is equipped with an instantaneous compandor and when linear networks having linear characteristics are used. When the compandor is used S/N must be x db where x db is the improvement achieved by use of the compandor.

Let v_x^1 represent the r.m.s value of the noise voltage at the output of the transmitting medium upon using the compandor and v_x be the corresponding value when the compandor is not used. The noise at the output of the system during intervals of zero signal input will be the same for the two conditions when the relationship

$$v_x = v_x^1 \times \frac{1}{k} \qquad (66)$$

where 1/k is the compression ratio is satisfied. Also, usually an optimum value of 22 db compression is used giving sin = 22 db and

$$12.59 = V_t/Vx^1$$

The quality of the two systems will be the same when both the equations (66) and (67) are simultaneously satisfied.

Use of Logarthamic Compandor

In work connected with speech a logarthamic compandor is of help in reducing the quantization distortion to an acceptable level for weak signals with an acceptable level of impairment for strong signals. The standard symbol for the compression ratio is μ. Certain considerations encourage high values of μ, certain other considerations discourage high values of μ and the actual value of μ selected is a compromise between the two.

The considerations that encourage high values of μ are:

(a). Obtaining a large companding improvement for weak signals.

(b). Reduction of idle circuit noise and interchannel cross talk due to the irregular excitation of weak steps.

(c). Prevention of clipping of the signal at its maximum level. For this a high system overload value relative to the weak signal level has to be maintained.

Considerations against using a high value of μ are:

(a). The difficulty of achieving sufficient stability in system net loss for high level signals.

(b). The difficulty of achieving and maintaining satisfactory "tracking" between compressor and expandor.

(c). Obtaining sufficient bandwidths in the compandor networks.

(d). The difficulty of holding the d.c. value of the multiplexed signals to a low enough d.c. value for full exploitation of high μ.

QUANTIZATION ERROR - NON UNIFORM SAMPLING OF LEVELS

Let the signal be symmetrically disposed on either side of the zero level in the range -A to +A. The levels are as denoted in the Figure 19. The signal value is transmitted as X_k provided it satisfies the condition

$$X_{(k - \frac{1}{2})} < X < X_{(k + \frac{1}{2})} \qquad (68)$$

Let $(X-X_k)$ denote the error of the transmitted signal and P(x) the probability density of the signal, assumed to have a normal distribution. The mean square distortion voltage is given as (Bendat, 1958)

$$\sigma_k = \int_{X_{(k - \frac{1}{2})}}^{X_{(k + \frac{1}{2})}} (X-Xk)^2 \, P(x) \, dx \qquad (69)$$

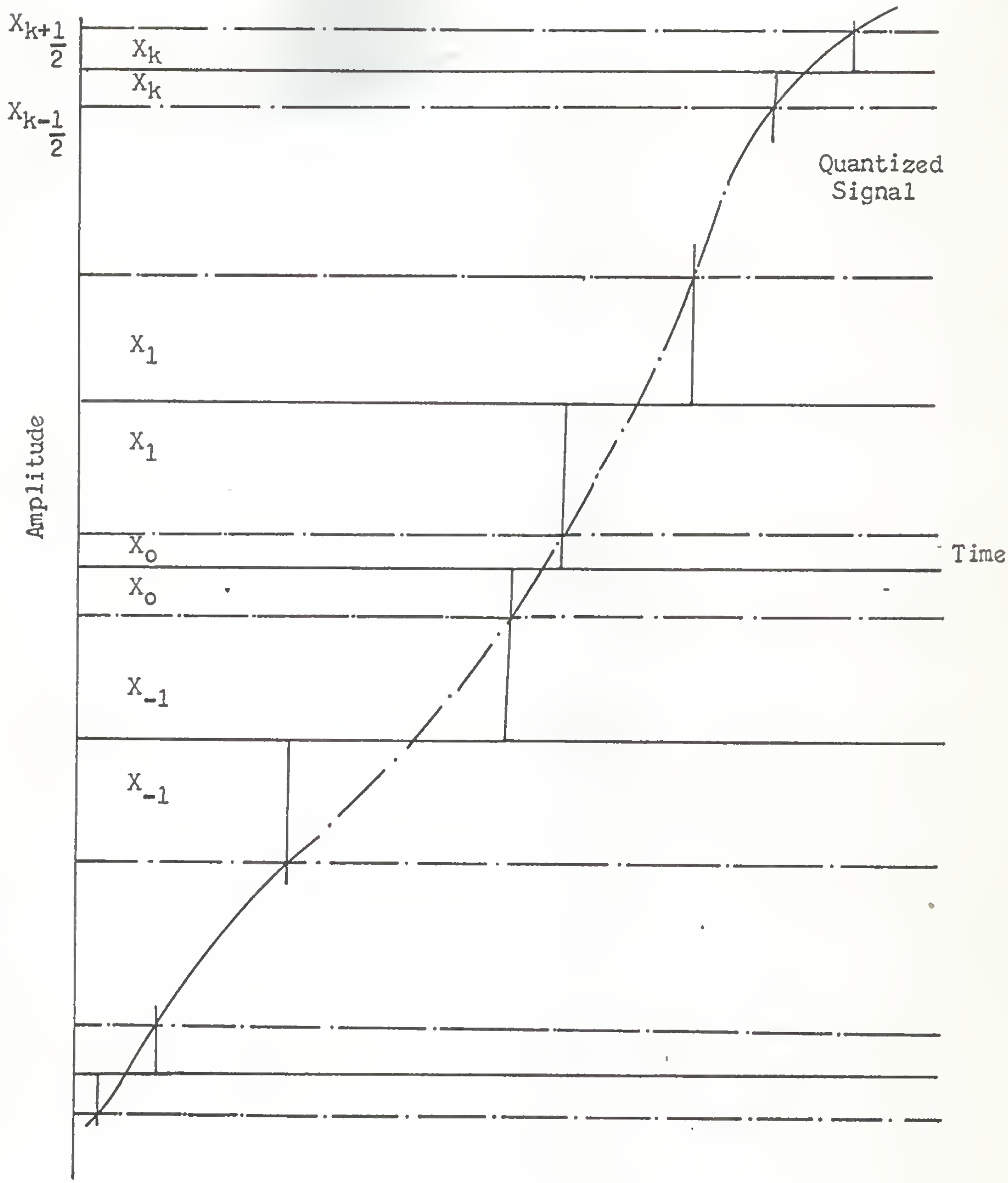


Figure 9. Non-uniform quantization. (Panter and Dite, 1951)

In case a large number of steps are used then the assumption can be made that P(x) can be considered to be constant over the region of integration and equal to $P(X_{AV})$ where X_{AV} is given by

$$X = \frac{X_{(k-\frac{1}{2})} + X_{(k+\frac{1}{2})}}{2} \tag{70}$$

In this case σ_k^2 reduces to

$$\sigma_k^2 = P(X_{AV}) \int_{X(k-\frac{1}{2})}^{X(k+\frac{1}{2})} (X - Xk)^2 dx$$

$$= P(X_{AV}) \frac{(X_{(k+\frac{1}{2})} - X_k)^3 - (X_{(k-\frac{1}{2})} - X_k)^3}{3} \tag{71}$$

The relationship between X_k and X_{AV} is obtained by differentiation of above with respect to Xk and setting it equal to zero.

$$\frac{d\sigma_k^2}{dX_k} = P(X_{AV}) \; (X_{(k-\frac{1}{2})} - X_k)^2 - (X_{(k+\frac{1}{2})} - X_k)^2 \tag{72}$$

i.e. $$X = \frac{X_{(k+\frac{1}{2})} + X_{(k-\frac{1}{2})}}{2} \tag{73}$$

which by definition is equal to X_{AV}. Thus it can be seen that the condition for making σ_k^2 a minimum is that X_k should be equal to X_{AV}. If it is made such that

$$\begin{aligned} X_{(k+\frac{1}{2})} &= X_k + \Delta X_k \\ X_{(k-\frac{1}{2})} &= X_k - \Delta X_k \end{aligned} \tag{74}$$

where X_k is arbitrarily small. Then σ_k^2 is seen to be equal to

$$\sigma_k^2 = P(X_k) \quad \frac{2\,\Delta X_k{}^3}{3}$$

Under the assumption that the distortion voltage is the same for all steps, the total mean square distortion voltage is obtained by summing it up for all the steps.

$$\sigma_{total}^2 = \sum_{-n}^{n} P(X_k) \cdot \frac{2}{3} \Delta X_k{}^3$$

The definition of the integral gives

$$\sigma_{total}^2 = \sum_{-n}^{n} P(X_k) \frac{2\,\Delta X_k{}^3}{3} = \frac{2}{3}\left[\int_{-V}^{V} P(x)^{1/3} dx\right]^3$$

The above is a constant, K, and is a function of its limits. If μ_k represents $P(X_k)^{1/3} \Delta X_k$ then

$$\sigma_{total}^2 = \frac{2}{3} \sum_{-n}^{n} \mu^3 \qquad (75)$$

and

$$K = \sum_{-n}^{n} \mu_k \qquad (76)$$

σ_{total} will be a minimum if the sum of the cubes is a minimum, at the same time satisfying the condition that equation (76) is a constant, i.e., the sum of the variables is a constant. In order to achieve this let (Kaplan, 1949)

$$\sigma^2(\mu_k) = \frac{2}{3} \sum_{-n}^{n} \mu^3 \qquad (77)$$

$$g\,(\mu_k) = K - \sum_{-n}^{n} \mu_k = 0 \tag{78}$$

From equations (77) and (78)

$$\frac{\partial\sigma^2}{\partial\mu_k} + \lambda_1 \frac{\partial g}{\partial\mu_k} = 0 \tag{79}$$

$$K - \sum_{-n}^{n} \mu_k = 0$$

It is seen that there are (2n+2) equations and (2n+2) unknowns.

From the first of the above two equations in (79) it is seen that

$$\lambda_1 = 2\,\mu_k^2\ . \tag{80}$$

All μ_k are equal

$$(2n \neq 1)\,\mu_k = K \tag{81}$$

$$\mu_k = \frac{K}{2n+1} \tag{82}$$

Hence

$$P^{1/3}(X_k)\Delta X_k = \frac{K}{(2n+1)} \tag{83}$$

Minimum mean square distortion voltage is

$$\sigma_m^2 = \frac{2}{3}\ \frac{K^3}{(2n+1)^3} \tag{84}$$

But $\int_{-V}^{V} P^{1/3}(x)dx = 2K$ giving K as

$$K = \frac{1}{2} \int_{-V}^{V} P^{1/3}(x)dx \tag{85}$$

Hence

$$\sigma_m^2 = \frac{1}{12(2n+1)^3} \left[\int_{-V}^{V} P^{1/3}(x)dx \right]^3 \tag{86}$$

Since P(x) is an even function it is seen that

$$\sigma_m^2 = \frac{2}{3(2n+1)^3} \left[\int_{0}^{V} P^{1/3}(x)dx \right]^3 \tag{87}$$

The ratio of mean square distortion voltage to the mean square signal voltage is given by

$$\frac{\sigma_m^2}{\sigma^2} = \frac{2}{3(2n+1)^3} \frac{\left[\int_{0}^{V} P^{1/3}(x)dx \right]^3}{\int_{0}^{V} x^2 P(x)dx} \tag{88}$$

The above equation gives the minimum distortion resulting from optimum level spacing.

WEIGHTED PCM

As has been shown previously the mean square quantization noise for quantization step size α is $\alpha^2/12$. The quantized samples are transmitted as binary code pulses. Usually a pulse of amplitude +1 unit is used to denote the digit 1 while a pulse of -1 in amplitude is used to represent the digit 0. In the receiver, when the received pulse amplitude is greater than a+1/2, then the transmitted digit is a+1, whereas if the received pulse amplitude is less than $-\frac{1}{2}$, then the transmitted pulse is taken to be zero. Perfect reception is achieved if the instantaneous peak to peak amplitude of the noise is less than 1. Under these conditions the only noise present at the output is the quantization noise.

If the assumption is made that the noise is essentially Gaussian then, it has a finite probability of its instantaneous amplitude exceeding the value required to produce an error in the received pulse. Also in pulse code modulation it is the position of the received pulse and not its amplitude that determines the amplitude of the signal transmitted. Under these circumstances an error involved in the identification of a certain pulse in the pulse sequence is greater than the error involved in the identification of certain other pulses. On the other hand since all the pulse amplitudes in the transmitted pulse sequence are equal the probability of correct identification of each of these pulses is the same. In order to overcome this difficulty, Bedrosian (1958) has suggested a weighting of the pulse amplitudes so that the higher the power of 2 represented by the pulse the higher the transmitted pulse amplitude, which, gives it a higher probability of being identified correctly.

The assumptions made for optimum weighting are as discussed below. For representation of any one sampled amplitude there are n pulses within the code group of transmitted pulses and they are labelled 1 through n. The i^{th} pulse with an amplitude of a_i represents the $(i-1)^{St}$ power of 2. This amplitude is negative if the i^{th} digit is a zero and is $+^{ve}$ if the i^{th} digit is a 1. Also by employment of compression or expansion (discussed previously) the probabilities for all the 2^n pulse groups are made equal. Because of this the probability of occurrence of either a zero or one in any particular pulse position are equal. Hence even though the pulse amplitudes are weighted the mean value of the ensemble of pulse sequences is zero.

The noise added to the pulse sequence is assumed to be Gaussian. The power spectral density of white noise is a constant and is expressed by the relation

$$G_{xx}(f) = A \tag{89}$$

where A is the constant. The relation between the auto correlation function and power spectral density is given by

$$\Gamma_{xx}(\tau) = \frac{1}{2} \int_{-\infty}^{\infty} G_{xx}(f) \; e^{j2\pi f\tau} \; df \tag{90}$$

From equations (89) and (90) it is seen that the auto correlation function of the white noise is (Bendat, 1958; Hayre, 1963)

$$\Gamma_{xx}(\tau) = \frac{A}{2} \cdot \delta(\tau) \tag{91}$$

where $\delta(\tau)$ equals the unit impulse function. The probability density function of the white noise is given by the formulae

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} \tag{92}$$

where its mean value is zero and σ^2 is the variance. If the variance is taken to be unity then

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{93}$$

The probability of making an error in identifying the i^{th} pulse equals the probability of the pulse being positive times the probability of noise being more negative than the pulse amplitude plus the probability of the pulse being negative times the probability of noise being more positive. If P_i equals the probability of error in identifying the i^{th} pulse

$$P_i = \frac{1}{2}\int_{a_i}^{\infty} p(x)dx + \frac{1}{2}\int_{-\infty}^{a_i} p(x)dx \qquad (94)$$

where 1/2 is the probability of the pulse being positive in one case and negative in the other case. Making the substitution for p(x) from equation (93), it is seen that

$$P_i = \frac{1}{2}\int_{a_i}^{\infty} e^{-x^2/2}dx \qquad (95)$$

The average signal power is merely the mean square pulse amplitude and hence is given by

$$S = \frac{K^2}{n}\sum_{i=1}^{n} a_i^{2} \qquad (96)$$

where K is the duty cycle.

For an operating system to be useful the errors have to be kept a minimum and hence the assumption $P_i \ll 1$ can be made. And because of this the occurrence of more than one error in a given pulse code group may be disregarded. The probability of a single error occurring in the i^{th} position of the pulse group equals the product of the probabilities of each pulse other than the i^{th} pulse being identified correctly and the i^{th}

being in error (Bedrosian, 1958).

$$P_i\ (1) = Pi \left(\prod_{\substack{j=1 \\ i \neq j}}^{n} (1-Pj) \right) \tag{97}$$

According to the assumptions previously made, $P_i \ll 1$ and hence the quantity within brackets is close to unity, which reduces the above expression to

$$P_i\ (1) \doteq Pi \tag{98}$$

Similarly the probability of making two errors in the i^{th} and j^{th} position of the pulse group is given by

$$P_{ij}\ (2) = P_i\ P_j \prod_{j \neq i \neq k}^{n} (1-P_k) \tag{99}$$

which gives

$$P_{ij}\ (2) = P_i\ P_j \tag{100}$$

Thus the probability of two errors in one group is much less than the probability of one error in the group. Hence the assumption that there is at the most one error in a given code group is valid.

Let the error in the output pulse due to the incorrect identification of the i^{th} pulse be Ei. The errors are equally likely due to the addition or subtraction of noise, to (from) the pulses and since the pulses denote binary digits

$$E_i = \pm \alpha \cdot 2^{(i-1)} \tag{101}$$

depending upon whether the pulse was positive or negative. Here α is the size of the quantum step.

Let Ne denote the mean square noise in the output due to the errors in the identification. If there is a single error in the identification of the code group Ne is given by the relation

$$Ne = K^2 \sum_{i=1}^{n} P_i\ (1)\ E_i^2 \qquad (103)$$

and since $P_i\ (1) \doteq P_i$, the above reduces to

$$Ne = K^2 \sum_{i=1}^{n} P_i\ Ei^2 \qquad (104)$$

Making the substitution for Ei from equation (102), equation (104) can be written as

$$Ne = K^2 \alpha^2 \sum_{i=1}^{n} P_i\ 4^{i-1} \qquad (105)$$

For simplification K and α can be taken to be unity and the simplified expressions written as

$$S = \frac{1}{n} \sum_{i=1}^{n} a_i^2 \qquad (106)$$

$$Ne = \sum_{i=1}^{n} P_i\ 4^{i-1} \qquad (107)$$

Both S and Ne are functions of the a_i's. The find the weighting function of a_i the above expression is minimized subject to the condition that S = constant. The system of equations are

$$f\ (a_1, a_2, \ldots\ldots\ldots, a_n) = \sum_{i=1}^{n} P_i\ 4^{i-1} \qquad (108)$$

$$g\,(a_1, a_2, \ldots\ldots, a_n) = S - \frac{1}{n}\sum_{i=1}^{n} ai^2 = 0 \qquad (109)$$

To solve for the a_i's to make (S/N) a maximum, the method of Lagrange multipliers is used and the Lagrangian Function is $F = f + \lambda g$ where is the Lagrange multiplier.

There are (n+1) equations and the equations to be solved for a_i's are (Kaplan, 1949)

$$\begin{aligned} &\frac{\partial F}{\partial a_1} + \lambda_1 \frac{\partial g}{\partial a_1} = 0 \\ &\quad\vdots \qquad\qquad \vdots \\ &\frac{\partial F}{\partial a_n} + \lambda_1 \frac{\partial g}{\partial a_n} = 0 \\ &\qquad\qquad g = 0 \end{aligned} \qquad (110)$$

According to Bedrosian an approximate solution to this set of equations is given by

$$a_i^2 = S + \frac{S}{1+S}\left(i - \frac{n+1}{2}\right) \ln 16 \qquad (111)$$

This expression brings out the fact that a_i depends only upon S, the mean square value of the signal and n the number of digits per code group. a_i increases as i increases and the variation in a_i becomes less as S increases.

To get the average output power So, there are 2^n steps and each quantum step has already been assumed to be of unit amplitude.

$$So = \frac{1}{2n} \cdot 2 \cdot \sum_{i=1}^{2^{n-1}} \frac{(2i-1)^2}{4} \qquad (112)$$

$$= \frac{1}{2^{n-1}} \cdot \frac{1}{4} \sum_{i=1}^{2^{n-1}} 4i^2 - 4i + 1 \qquad (113)$$

Since

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \quad \text{and} \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

it can be seen that

$$S_\delta = \frac{1}{4} \cdot \frac{1}{2^{n-1}} \left\{ \frac{2}{3} \cdot 2^{n-1}(2^{n-1}+1)(2.2^{n-1}+1) - 2.2^{n-1}(2^{n-1}+1) + 2^{n-1} \right\}$$

$$= \frac{1}{4} \left\{ \frac{2}{3} (2^{n-1}+1)(2.2^{n-1}+1) - 2(2^{n-1}+1) + 1 \right\}$$

$$= \frac{1}{4} \left\{ \frac{2}{3} (2.2^{2n-2} + 3.2^{n-1}+1) - 2.2^{n-1} - 1 \right\}$$

$$= \frac{1}{4} \left\{ \frac{4}{3} \cdot 2^{2n-2} - \frac{1}{3} \right\}$$

$$= \frac{1}{4} \frac{4^n - 1}{3} \qquad (114)$$

Hence average power at the output is

$$S_\delta = \frac{4^n - 1}{12}$$

As has been already shown the quantization noise power is given by

$$N_Q = \alpha^2/12$$

and if $\alpha = 1$, then

$$N_Q = 1/12 \qquad (115)$$

The total noise power at the output is $(Ne + N_Q)$ and as such

$$(S/N)_{out} = So/(Ne + N_Q) \tag{116}$$

In a conventional PCM system all pulses are of equal amplitude and the P_i's are all equal. For this case, Ne^1, the noise power in the output due to an error is

$$Ne^1 = P \sum_{i=1}^{n} 4^{i-1} \tag{117}$$

In this case the signal to noise ratio at the output is

$$\left(\frac{S}{N}\right)^1_{out} = \frac{So}{Ne^1 + N} \tag{118}$$

Figure 10 shows the improvement in signal to noise ratio obtained by the method of optimum weighting. If the "knee" of output versus input signal to noise ratio is defined as the point where output (S/N) is 3 db down from the quantizing noise level, then the knee occurs at lower Sin/N in for the unweighted PCM case by approximately 1.5 db for n=5, 7, 9 and 11.

Information Rate

According to Shannon, the ideal rate of information transmission is given by

$$C = B \log \left(1 + \frac{S}{N}\right) \text{ bits/sec} \tag{119}$$

where C is channel capacity, B is channel bandwidth, S is average signal power and N is noise power, both these being referred to input.

In this case the information capacity of a system is thought of as

Top curves are for weighted PCM.

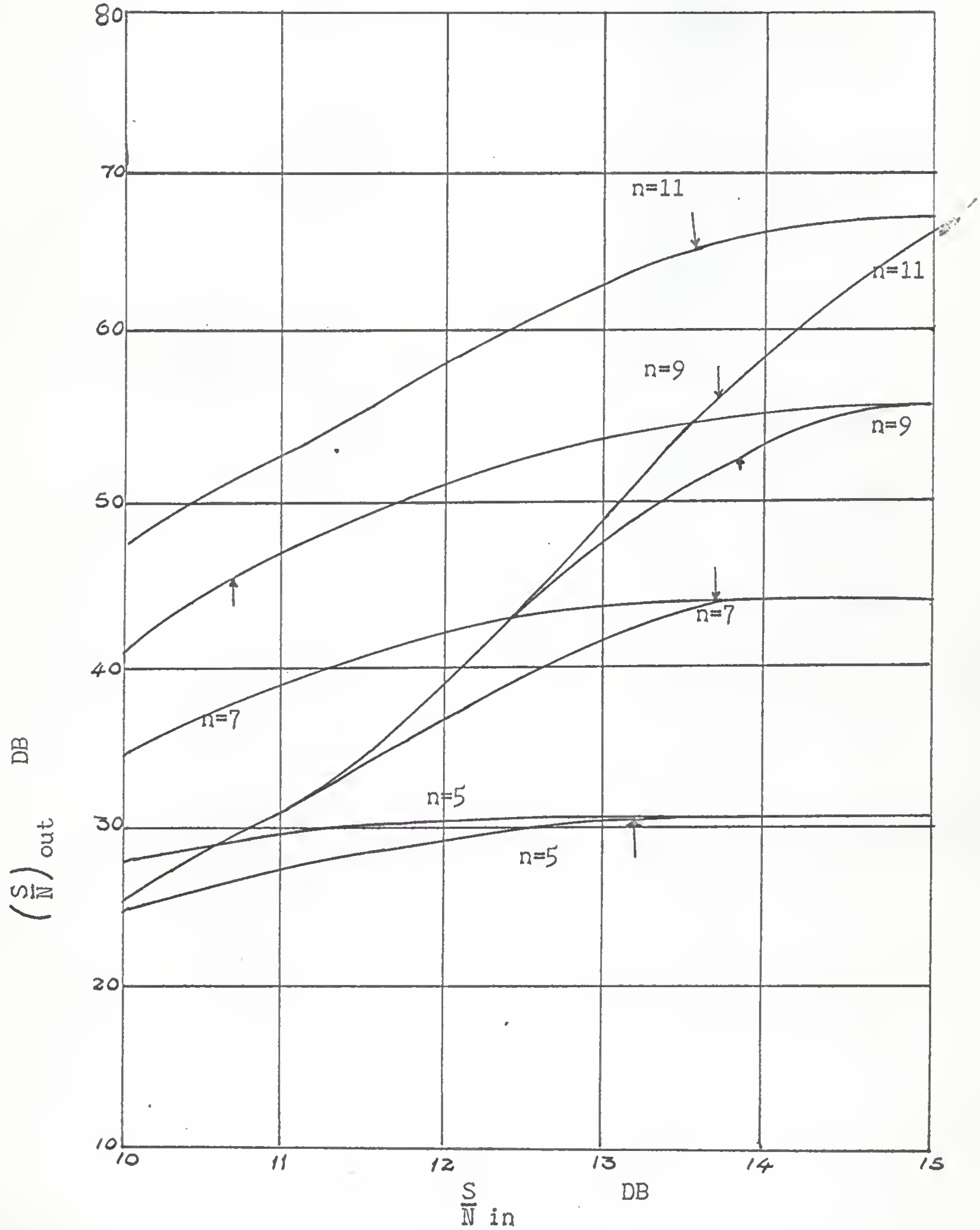


Figure 10. Comparison of weighted and unweighted PCM. (Bedrosian, 1958)

the number of independent symbols or characters which can be transmitted without an error in unit time.

In theorem 23 in the article "Mathematical Theory of Communication," Shannon states an inequality that the actual rate of information transmission, R, should satisfy. It is given by the expression

$$B \log_2 \frac{Q_1}{N} \leqq R \leqq B \log_2 \frac{Q}{N} \tag{120}$$

where B is source bandwidth, Q is average source power, Q_1 its entropy power and N the allowed mean square error. For the present case $N = Ne + N_Q$.

Shannon defines the entropy power of a source as the average power of a normally distributed source having the same entropy as the original source. Since in the source all values were assumed to be equally probable

$$p(x) = \begin{cases} \frac{1}{2}a & |x| \leqq a \\ o & |x| \gtreqqless a \end{cases} \tag{121}$$

where x denotes a typical signal limited in amplitude between -a and a.

The mean square value of x gives the mean square power, Q of the source. It is given by

$$\begin{aligned} Q &= \int_{-\infty}^{\infty} x^2 p(x)\, dx \\ &= \int_{-a}^{a} x^2 \cdot \frac{1}{2a} \cdot dx \\ &= \frac{a^2}{3} \end{aligned} \tag{122}$$

The entropy of the continuous distribution with density distribution function p(x) is (Shannon, 1949)

$$= - \int_{-\infty}^{\infty} p(x) \log p(x)\, dx$$

$$= -\log \frac{1}{2a} \quad = \log 2a \qquad (123)$$

From equation (122) it is seen that $3Q = a^2$, and hence from equation (123)

$$H = \log 2\sqrt{3Q} \qquad (124)$$

The source is assumed to be normally distributed and since Q_1 is its average power

$$p(y) = \frac{1}{\sqrt{2\pi Q_1}} e^{-y^2/2Q_1}$$

$$H_1 = - \int_{-\infty}^{\infty} p(y) \log p(y) dy$$

$$= - \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi Q_1}} e^{-y^2/2Q_1} \log \frac{e^{-y^2/2Q_1}}{\sqrt{2\pi Q_1}}\, dy$$

$$= \frac{1}{\sqrt{2\pi Q_1}} \int_{-\infty}^{\infty} e^{-y^2/2Q_1} \left\{ \log\sqrt{\pi 2 Q_1} - \log e^{-y^2/2Q_1} \right\} dy$$

$$= \log\sqrt{2\pi Q_1} \int_{-\infty}^{\infty} \frac{1}{2\pi Q_1} e^{-y^2/2Q_1}\, dy + \frac{\log e}{2Q_1} \int_{-\infty}^{\infty} \frac{1}{2\pi Q_1} y^2 e^{-y^2/2Q_1} \cdot dy$$

The integrand in the first case is the probability density function which upon integration between $-\infty$ to $+\infty$ gives unity. The integrand in the second case upon integration gives the variance which in this case is Q_1.

Hence after integration H_1 reduces to

$$H_1 = \log\sqrt{2\pi Q_1} + \tfrac{1}{2} \log e$$

$$= \log\sqrt{2\pi e Q_1} \qquad (125)$$

The relation between Q and Q_1 is obtained by equating the entropies and from equations (124) and (125) it be seen to be

$$Q_1 = \frac{6}{\pi e} Q \tag{126}$$

In the source the highest frequency is B and it has to be sampled at least at the rate of 2B samples per second. Since n digits are required in order to transmit one sample, at least 2nB samples have to be transmitted per second.

From equation (120) it is seen that

$$\frac{1}{2n} \log \frac{6}{\pi e} \left(\frac{S}{N}\right)_{out} \leq \frac{R}{2Bn} \leq \frac{1}{2n} \log \left(\frac{S}{N}\right)_{out}$$

since

$$\frac{Q}{N} = \left(\frac{S}{N}\right)_{out}$$

This is compared with the channel capacity given by

$$C = B \log (1 + S/N)$$

and since for the present case bandwidth is nB

$$\frac{C}{2nB} = \frac{1}{2} \log (1 + S/N) \tag{127}$$

These calculations are plotted in Figure 11. It shows optimum weighting of PCM. It allows lowering (S/N) in by about 1.5 db while retaining the same R provided (S/N) in is below the knee of the unweighted PCM curve. The maximum possible lowering of (S/N) in for constant R is

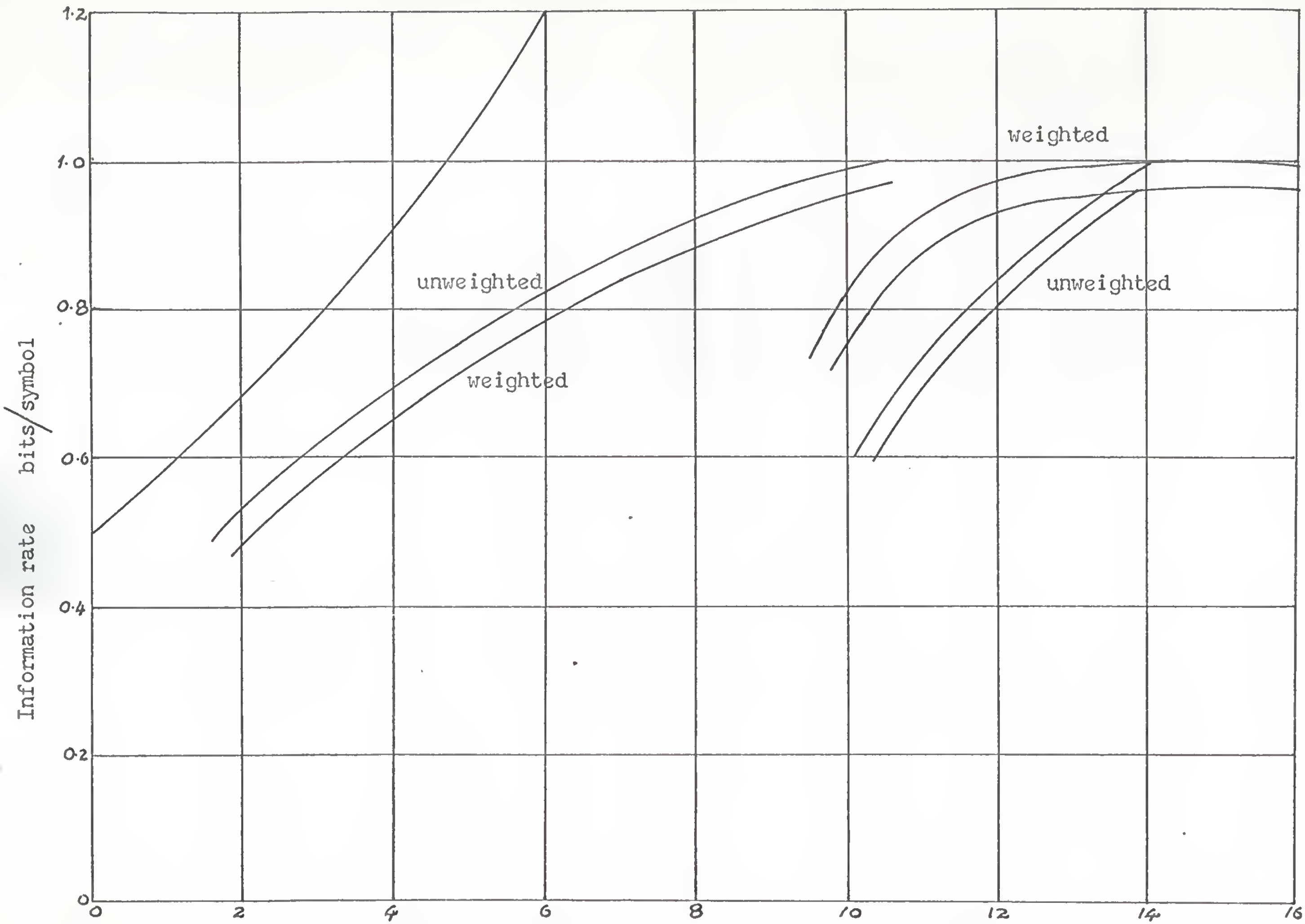


Figure 11. Channel signal to noise ratio in db. (Bedrosian, 1958)

9db, thus showing that weighted PCM has made a significant improvement.

The probability of error in the detection of a single pulse is a minimum when all pulses are of equal height. The rate of errors on a per symbol basis is higher for the case of weighted PCM compared to that for standard PCM. Equivocation which is defined as the uncertainty of the message sent given the message received is greater for weighted PCM since here also all pulses are of equal importance.

In conclusion, Bedrosian, mentions the fact that the improvement of about 1·5 db obtained by weighting is not very encouraging since a lot of complicated circuitry has to be employed to achieve this.

CONCLUSION

Pulse modulation systems enables the multiplexing of channels by time division as distinguished from frequency division. This is due to the discrete nature of the signal obtained after sampling. Multiplexing by time division simplifies the equipment necessary for the same since relatively simple gating circuits can replace the modulators, demodulators, bandpass filters, etc., necessary in the case of frequency division multiplexing.

Aliasing errors cannot be eliminated altogether. It can be minimized and brought to a negligible level by the use of sharp cut off filters to limit the highest frequency contained in the signal and then choosing the sampling frequency properly. Too high a sampling frequency should not be chosen since it might tend to limit the number of channels that can be used by time division multiplex. The actual sampling frequency chosen is a compromise between these two factors.

There are two types of noise introduced by a PCM system. One is the quantization noise. This is introduced at the transmitting end of the system and nowhere else. The other is the false pulse noise caused by the incorrect interpretation of the intended amplitude of a pulse by the receiver or by any repeater. This could arise anywhere along the system and is cumulative. This could be reduced by a proper weighting of the pulse amplitudes within a pulse code group as suggested by Bedrosian (1958). This noise decreases rapidly when the signal power is increased above a certain amplitude, so that in any practical system it can be made small by proper design without resorting to the complicated circuitry required for the case of the weighted PCM. As a result signal to noise ratio is set in a PCM system by the quantizing noise only.

The mean square quantization noise which is proportional to the square of the quantization step, can be reduced by a reduction of the size of a quantum step. This increases the number of steps required for a coverage of the peak signal amplitude and correspondingly the number of pulses within a given pulse code group. This tends to place severe constrictions on the pulse generating circuits or upon the number of channels that can be multiplexed. Hence in the choice of the size of a quantization step also a compromise solution between these various factors is necessary.

For PCM in the UHF range there are many more problems involved in both sampling and quantization. A survey of the literature on PCM so far published shows that it has not been considered so far and this present study could be used as the basic theory behind understanding the problem of PCM in the UHF range.

ACKNOWLEDGMENT

The author wishes to express his gratitude to his major professor, Dr. H. S. Hayre, for his help and guidance throughout this study.

BIBLIOGRAPHY

1. Bendat, J. S. Principles and Applications of Random Noise Theory. New York: John Wiley and Sons, Inc., 1958.

2. Black, H. S. Modulation Theory. New York: C. Van Nostrand, 1953.

3. Black, H. S., J. O. Edson. Pulse Code Modulation. Bell Telephone Systems Technical Publications Mono. B. 1491. June, 1947.
Sampling principle, quantization, number and size of steps required, as also synchronization and transmission performance have been discussed.

4. Bedrosian, E. Weighted PCM. I.R.E. Transactions on Information Theory. March, 1958. pp. 45-49.

5. Carbrey, R. L. Pulse Code Modulation, Terminal and Repeater Methods. Bell Telephone Systems Technical Publications Monograph 3703. May, 1960.
A discussion of sampling, time separation multiplex and binary code representation.

6. Goodal, W. M. Telephony by PCM. Bell Systems Technical Journal. Vol. 26. July, 1947. p395-403.

7. Hayre, H. S. Class Notes on Noise Theory. 1963.

8. Harry L. Stiltz (ed.). Aerospace Telemetry. Chapter IV. New Jersey: Prentice Hall, Inc., 1961.
Discusses sampling, interpolation, aliasing error in sampling, multiplexing, etc.

9. Kaplan. Advanced Calculus. Reading: Addison Wesley Publishing Company, Inc., 1959.

10. Mallinkrodt, C. O. Instantaneous Compandors. Bell Systems Technical Journal Monograph B. 1895. November, 1951.
Advantages of using a compressor in pulse modulation systems as also the theory behind the same and the improvement in signal to noise ratio obtained.

11. Mayer, H. F. Advances in Electronics III. New York: Academic Press, Inc., 1957.
Chapter on principles of pulse code modulation discusses fundamentals of PCM, quantization, quantization error, encoding and decoding including the methods involved for each of these.

12. Meacham, L. A. and E. A. Peterson. An Experimental Multichannel PCM System of Toll Quality. Bell System Technical Journal. Vol. 27. January, 1948. pp. 1-48.

13. Oliver, B.M., J.R.Pierce and C.E. Shannon. The Philosophy of PCM. Proc. I.R.E. Vol. 36. November, 1948. pp. 1324-1331.

14. Panter, P.F. and W.Dite. Quantization Distortion in Pulse Count Modulation with Non-uniform Sampling. Proc. I.R.E. Vol. 39. January, 1951. pp.44-48.

15. Shannon, C.E. A Mathematical Theory of Communication. Bell System Technical Journal, Vol. 27. July, 1948. pp. 379-423 and Vol. 27. October, 1948. pp. 623-656.

16. Shennum, R,H. and J.R. Gray. Performance limitations of a Practical PCM Terminal. Bell Systems Technical Journal. Vol. 41. January, 1962. pp. 143-171.

17. Stewart, R.M. Statistical Design and Evaluation of Filters for the restoration of Sampled data. Proc. I.R.E. Vol.44. February, 1956. p.243.

18. Tolstov, G.P. Fourier Series. New York: Prentice Hall, 1962.

A STUDY OF PULSE CODE MODULATION

by

P. R. Hariharan

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Electrical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1963

This report is a study of pulse code modulation. In the first section the theory of sampling, recovery of sampled information known as interpolation, and aliasing errors in sampling are discussed.

Quantization, which involves analog to digital conversion, quantization noise, error in quantization with unequal steps, the improvement in signal to noise ratio obtained by use of a compandor are included in the second section.

The last part of this report considers a modified form of pulse code modulation called "weighted PCM". In weighted pulse code modulation the amplitudes of the pulses within a pulse code group are suitably adjusted as to minimize the noise power in the reconstructed signal due to errors in transmission.