WIDE-FLANGE BEAM BUCKLING ANALYSIS BY

FINITE ELEMENT METHOD

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# CHAPTER I - INTRODUCTION

A problem of increasing practical importance is that of determining the elastic buckling strength of thin-walled structural members. Historically, these members have been treated as one-dimensional or line elements in determining their critical buckling stresses. Classical solutions of the governing equations are mathematically complex and are available for only a limited number of geometric shapes, support, and loading conditions (25)*. Valuable insight into the buckling phenomenon has been established in this way, however, the influence of cross-sectional distortion is not accounted for due to the simplifying assumption that cross sections do not distort during buckling. Further, the geometric idealization inherent in the line element analysis can only approximate members with variable depth and their assemblage into rectangular frameworks or gable frame structures.

A finite element procedure for the buckling analysis of beams (18) has been developed. In this application the prismatic wide-flange beam was idealized as an assemblage of two-dimensional finite elements as shown in Fig. 1a in which each element has both membrane and bending stiffnesses.

The buckling analysis of structural beam members having a plane of symmetry about the midsurface of the web may be simplified to a two-dimensional problem by replacing the flanges and stiffeners with one-dimensional elements while the web remains to be idealized with two-dimensional elements, as shown in Fig. 1b. The BASP** computer program, which is used in the report, applies this idealization for the wide-flange beam buckling analysis.

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*Numerals in parentheses refer to corresponding items in the References.

**This computer program, Buckling Analysis of Stiffened Plates BASP, was developed by H. U. Akay at the University of Texas at Austin ( 1).

The purpose of this report is:

1. To present a finite element procedure for wide-flange beam buckling analysis.

2. To use the BASP computer program to find the buckling load and the buckled shape of wide-flange beams.

3. To obtain a comparison with the theoretical results (15), the results (25) based on the idealization as shown in Fig. 1a and the results computed from the BASP computer program.

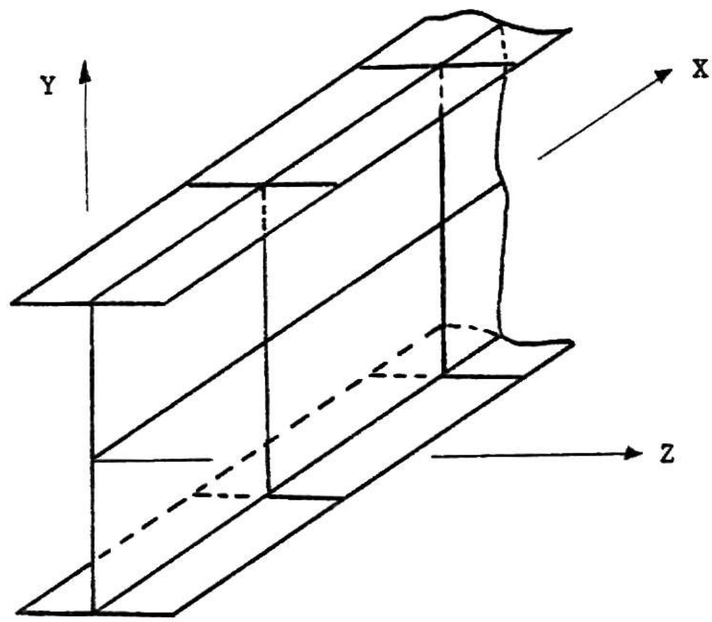



Fig. 1a - Idealization of Web and Flanges With Planar Elements

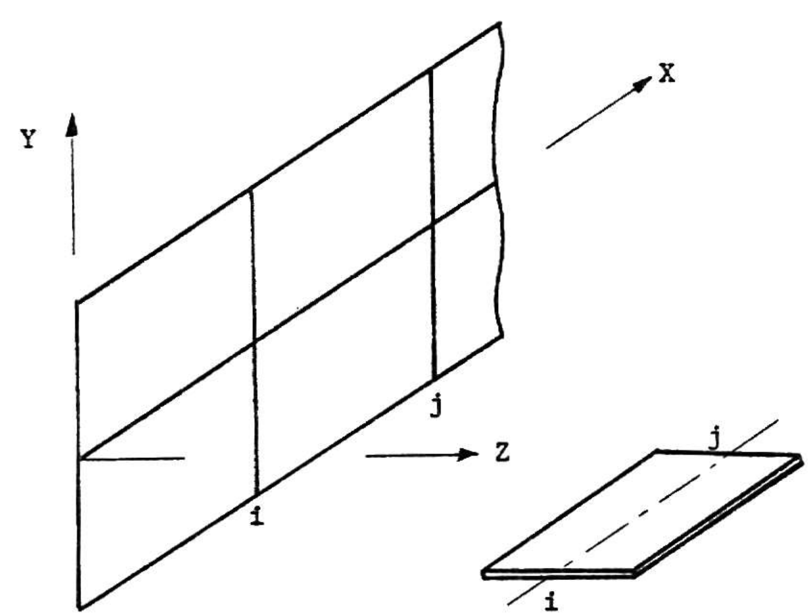



Fig. 1b - Idealization of Web With Planar Elements and Flanges With Beam Elements

# CHAPTER II - METHOD OF ANALYSIS

## 2-1 Introduction

The finite element method is presently recognized as an extremely powerful computer oriented method of analysis. It was introduced in the late fifties (26) and has subsequently been applied to a wide range of structural problems. With regard to stability, the method was first applied to truss and beam-column problems by use of one-dimensional elements (20). Subsequently, two-dimensional planar elements were used in the stability analysis of plates (19) and shells (8). The method has been applied to lateral-torsional beam buckling (10, 24) in which one-dimensional elements were used. A finite element procedure has also been developed for the buckling analysis of rigid frames (17) and beams (18) in which two-dimensional elements were used. Classical assumptions of the line element analysis (25) were maintained in that work.

In the previously mentioned stability analyses of plates (19), shells (8), rigid frames (17) and beams (18), in-plane or membrane element stresses were considered as initial stresses in computing the geometric stiffness. These in-plane stresses were considered to produce element geometric stiffnesses which were a function of the transverse displacements of the element. These geometric stiffnesses are referred to as the out-of-plane components. This treatment is applicable only to structures where buckling may be represented by the combination of the out-of-plane components of the individual elements of the assemblage.

The finite element procedure for the buckling analysis of beams consists of the following steps: 1) Generation of the conventional (elastic) structure stiffness, $K_c$, followed by a displacement analysis for the applied loading; 2) Generation of an additional stiffness matrix,

$K_g$ (herein referred to as the geometric stiffness), by using the stresses computed in Step 1; and 3) Computation of the critical buckling load and buckled shape by assuming that the geometric stiffness is linearly proportional to the applied loading. This may be stated mathematically (7) as

$$[K_c]\{r\} + \lambda\ [K_g]\{r\} = \{0\} \text{ ------------------------------ } (1)$$

in which vector $\{r\}$ represents the buckled shape, while constant $\lambda$ is a single parameter characterizing the applied loading and which determines the critical buckling load, i.e., the critical buckling load is equal to $\lambda$ times the applied load that was used to obtain the geometric stiffness. In general, Eq. 1 is only solved for the smallest $\lambda$ since it corresponds to the smallest buckling load. In the preceding formulation, the buckling problem is linearized in that the prebuckling displacements (as caused by the applied loading) are uncoupled from those caused by buckling. Buckling is therefore assumed to take place from the initial geometry. In this regard, both $K_c$ and $K_g$ are referenced from the initial geometry as well as the stresses that are used to construct $K_g$. This formulation is "equivalent" to the classical linearized buckling theory in which displacements are assumed to be "small"; therefore, the buckled mode shape may be determined while the actual magnitude of the buckled shape remains undefined.

## 2-2 The Finite Element Idealization

The basic concept of the finite element method is the idealization of the actual structure by an assemblage of structural elements in which the stiffness properties of each individual element are derived from assumed displacement functions. The accuracy of this method is directly related to the proper choice for the displacement function. For two-dimensional structural stability analysis there are a variety of elements available. Some of them are discussed in this chapter such as triangle, rectangular, quadrilateral and beam element.

Criteria for element selection should include elements whose geometric form closely approximates the actual structure, as well as exhibiting stiffness properties that accurately predict the response of the actual structure.

## 2-3 Basis for Derivation of Conventional and Geometric Stiffnesses

The derivation of both the conventional and geometric stiffnesses may be performed from an expression of the total strain energy of the individual element. The total strain energy of an element subjected to initial stresses, which subsequently undergoes a set of virtual displacements, can be taken as:

$$U = \tfrac{1}{2} \int_v \Delta\tau_{ij} \cdot \Delta\varepsilon_{ij}\, dv + \int_v \tau_{ij} \cdot \Delta\eta_{ij}\, dv \qquad \begin{matrix} i = 1,2,3 \\ j = 1,2,3 \end{matrix} \quad \text{-- (2)}$$

where

$U$ is the total strain energy

$\Delta\tau_{ij}$ are the conventional stresses due to the virtual displacements

$\Delta\varepsilon_{ij}$ are the linear components of strain due to the virtual displacements

$\tau_{ij}$ are the element stresses from the applied loading

$\Delta\eta_{ij}$ are the nonlinear components of strain induced by the virtual displacements.

In order to derive the element stiffnesses the total strain energy, U, must be expressed in terms of displacements, $\bar{u}$. This is shown symbolically as follows where a comma between integer subscripts denotes partial differentiation:

$$2\Delta\varepsilon_{ij} = \bar{u}_{i,j} + \bar{u}_{j,i} \qquad i = 1,2,3$$

$$2\Delta\eta_{ij} = \bar{u}_{k,i} \cdot \bar{u}_{k,j} \qquad j = 1,2,3 \quad k = 1,2,3 \qquad \text{------------------ (3)}$$

where

$\bar{u} = \bar{u}(x,y,z)_i$ ......., $i = 1,2,3$ represents the displacements in the x-, y-, and z-directions.

The total strain energy for each individual element is subsequently expressed in terms of element nodal point displacements by utilizing assumed displacement functions. The derivation of the element stiffness is then most conveniently accomplished by using Castigliano's first theorem. This procedure has been carried out for all elements described below in Section 2-4 and Section 2-5.

## 2-4 Conventional Element Stiffnesses

Three types of conventional elements are considered in this report. Each of these elements have both in-plane (membrane) and out-of-plane (bending) stiffnesses. These elements are described in the literature, thus only their basic characteristics are reviewed below.

a) Type 1C - QUADRILATERAL ELEMENT

The quadrilateral element (14) consists of four triangles, as shown in Fig. 2. The membrane stiffness for each triangle is derived from quadratic displacement functions in which the in-plane displacement of the external sides (1-2, 2-3, 3-4, 4-1), Fig. 2, are constrained to vary linearly. The quadratic membrane displacement functions for Triangle 1 together with its degrees-of-freedom (DOF)* system is shown in Fig. 4b. This element has twelve DOF (two at each corner node and at each midside

*The term, degrees of freedom, is herein abbreviated as DOF.

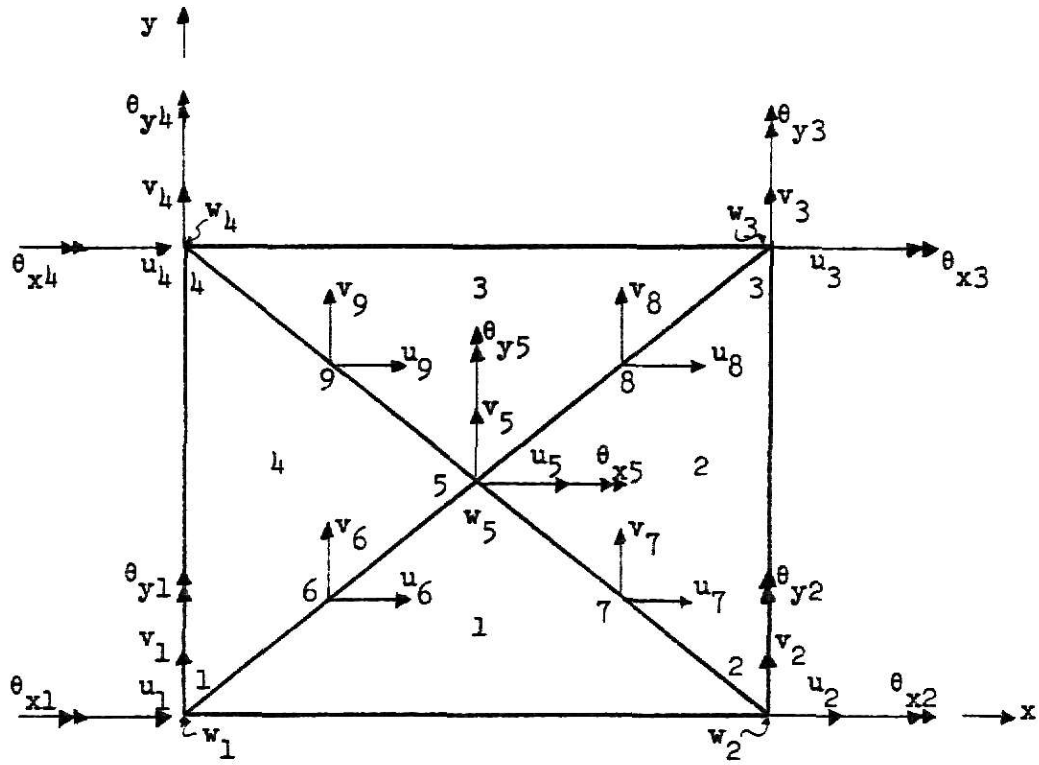



Fig. 2 - Planar Quadrilateral With 33 DOF

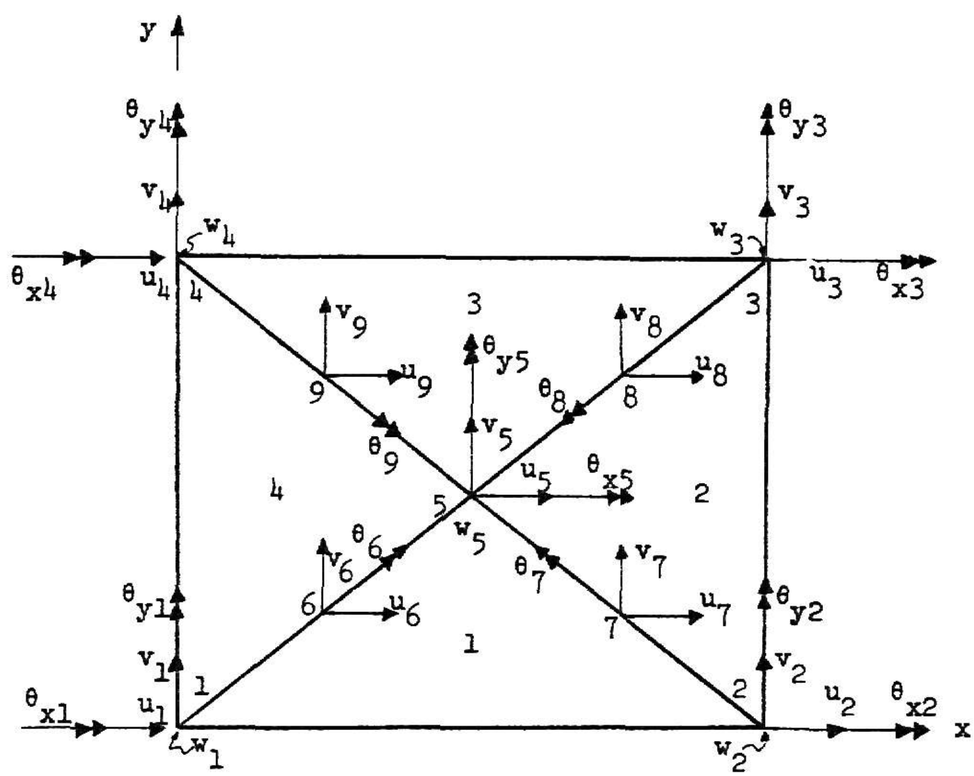



Fig. 3 - Planar Quadrilateral With 37 DOF

$$w^{(3)} = \{1, x_3, y_3, x_3^2, x_3y_3, y_3^2, x_3^3, x_3y_3^2, y_3^3\}\begin{Bmatrix}\bar{a}_1 \\ \vdots \\ \bar{a}_9\end{Bmatrix}$$

$$w^{(1)} = \{1, x_1, y_1, x_1^2, x_1y_1, y_1^2, x_1^3, x_1y_1^2, y_1^3\}\begin{Bmatrix}\bar{a}_{10} \\ \vdots \\ \bar{a}_{18}\end{Bmatrix}$$

$$w^{(2)} = \{1, x_2, y_2, x_2^2, x_2y_2, y_2^2, x_2^3, x_2y_2^2, y_2^3\}\begin{Bmatrix}\bar{a}_{19} \\ \vdots \\ \bar{a}_{27}\end{Bmatrix}$$

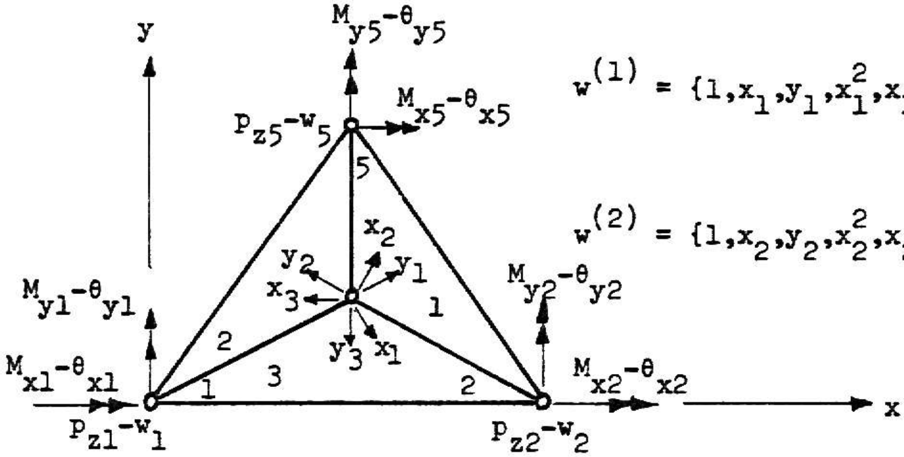



Fig. 4a - Displacement Functions and Nodal Point Systems for Triangular Plate Elements

$$u(x,y) = \{1, x, y, x^2, xy, y^2\}\begin{Bmatrix}\bar{\bar{a}}_1 \\ \vdots \\ \bar{\bar{a}}_6\end{Bmatrix}$$

$$v(x,y) = \{1, x, y, x^2, xy, y^2\}\begin{Bmatrix}\bar{\bar{a}}_7 \\ \vdots \\ \bar{\bar{a}}_{12}\end{Bmatrix}$$

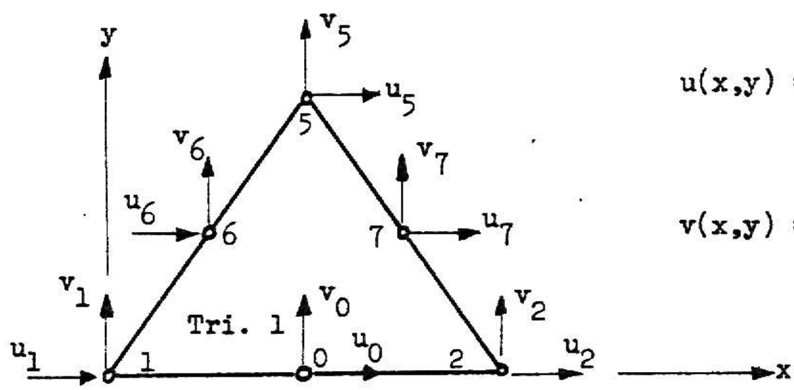



Fig. 4b - Linear Strain Triangle With 12 DOF

$$u_0 = (u_1 + u_2)/2$$

$$v_0 = (v_1 + v_2)/2$$

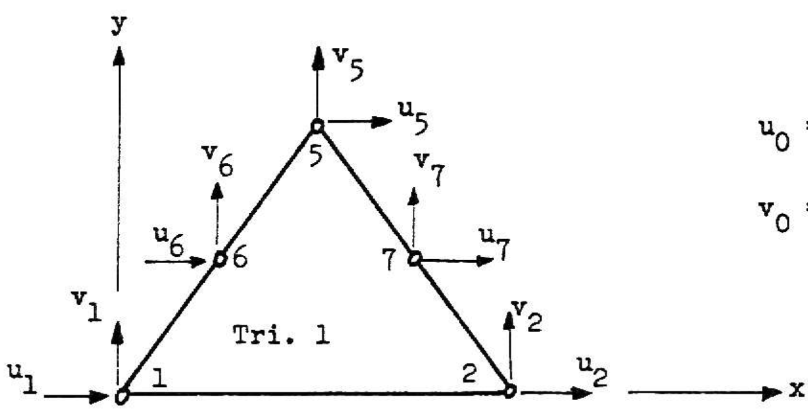



Fig. 4c - Constrained Linear Strain Triangle With 10 DOF node) and is called the "Linear Strain Triangle" (7). The displacement functions for this triangular element, Fig. 4c, are derived from the Linear Strain Triangle by setting $u_0 = (u_1 + u_2)/2$ and $v_0 = (v_1 + v_2)/2$, as shown in Fig. 4c. This eliminates the DOF at the external midside nodes, (i.e., Point 0 in Fig. 4b); and the resulting triangle has 10 DOF, as shown in Fig. 4c.

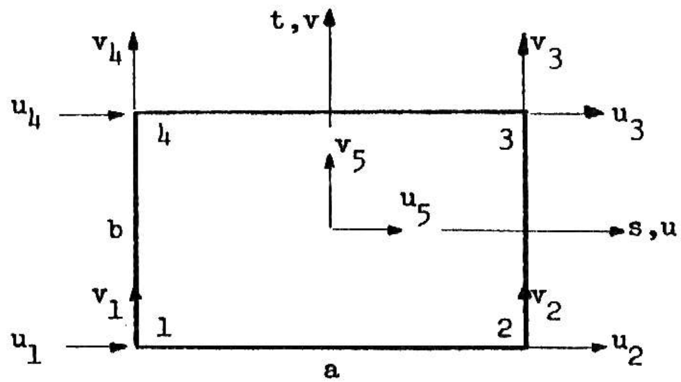
Each of the four triangles are fully compatible plate-bending elements (HCT), which is named after Hsieh, Clough, and Tocher (3). This element has 3 DOF at each corner (two rotations and a normal translation). Cubic displacement functions are assigned to each subelement of the triangle as shown in Fig. 4a. This results in 27 generalized coordinates for each triangle. The reduction of 9 DOF is accomplished by applying internal compatibility constraints between the subelements.

The stiffness of this quadrilateral element (Fig. 2) is then obtained by superposition of the bending elements (Fig. 4a) and the membrane elements (Fig. 4c). The interior points 5-9 in Fig. 2 are eliminated by static condensation (Appendix II); and the resulting condensed element has 20 DOF (five at each exterior corner node).

b) <u>Type 2C</u> - RECTANGULAR ELEMENT

The membrane and bending stiffnesses of the rectangle are derived from the displacement functions, as shown in Figs. 5 and 6. Displacements u and v are contained in the element's plane while the displacement w is normal to the element's plane. Constants $a_0 - a_4$, $b_0 - b_4$, and $\alpha_1 - \alpha_{12}$ are generalized coordinates which are subsequently related to the element displacements at its four corners in obtaining its stiffness properties.

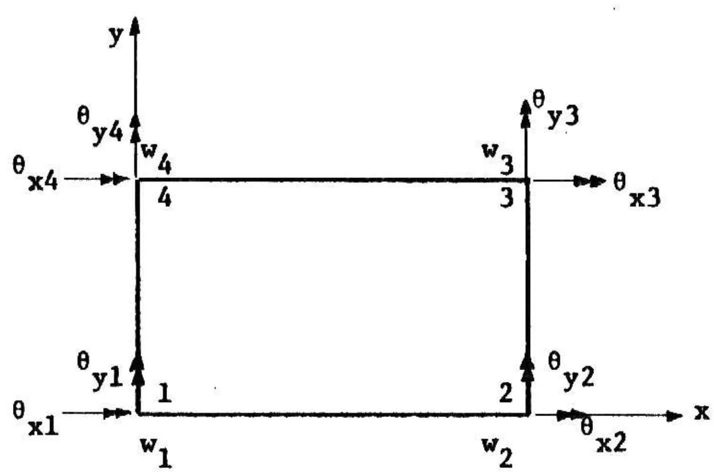
The membrane element of Fig. 5 was first considered in Ref. ( 6 ) and was termed the QM5. The in-plane displacements, u and v, are bilinear




$$u = a_0 + a_1 s + a_2 t + a_3 st + a_4 (1-s^2)(1-t^2)$$

$$v = b_0 + b_1 s + b_2 t + b_3 st + b_4 (1-s^2)(1-t^2)$$

Fig. 5 - Refined Membrane Element -- QM5




$$w = \{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, x^3y, xy^3\} \begin{Bmatrix} \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{12} \end{Bmatrix}$$

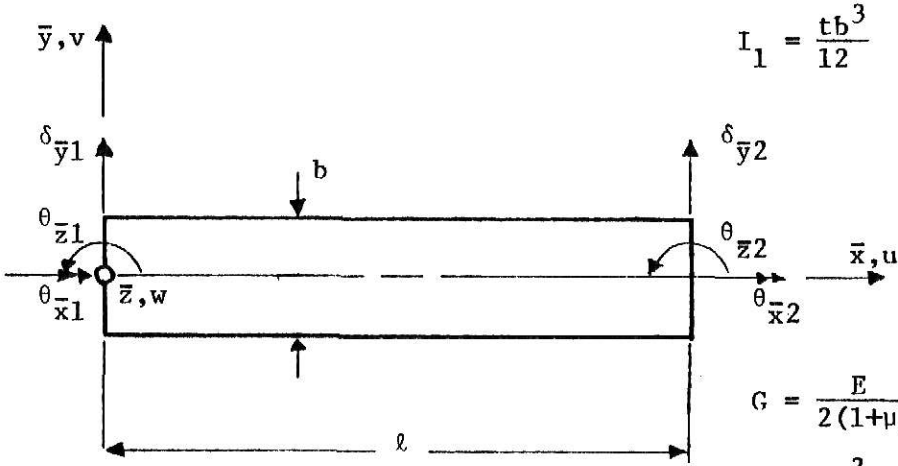
Fig. 6 - Rectangular Bending Element -- ACM

polynomials with one additional higher order term in each expansion, as shown in Fig. 5. These expansions give a linear variation of both in-plane components along the element interfaces, thus establishing membrane displacement compatibility requirements. Since each expansion has five terms, a total of 10 in-plane DOF are necessary. They consist of two DOF (translational) at each corner and the central interior point. The two interior DOF are eliminated by static condensation, thus the condensed element has only eight DOF (two at each corner node). The essential feature, which gives rise to an improved membrane stiffness as compared with elements with the same DOF at the corner nodes, is that of constraining the shear strain to be constant in the derivation of its stiffness.

The bending stiffness is derived from a 12-term polynomial and thus requires 12 DOF (three degrees-of-freedom at each corner), as shown in Fig. 6. Thus, each corner has three DOF consisting of two in-plane rotations and one out-of-plane translation. This element, which is called the ACM element, has been shown to give good results for plate bending problems (3), even though it is not completely compatible due to the fact that certain slope components are not fully matched along adjacent element interfaces.

c) <u>Type 3C</u> - QUADRILATERAL AND BEAM ELEMENTS

These elements are used when the web is idealized by quadrilateral elements while the flanges are idealized by beam elements, Fig. 1b. This quadrilateral is again constructed from four triangles, as shown in Fig. 3. Its membrane stiffness is identical to Type 1C. Its bending stiffness is similar to the HCT element except that there are four additional rotational DOF at the interior midside nodes, Fig. 3. This element was presented in




$$I_1 = \frac{tb^3}{12}$$

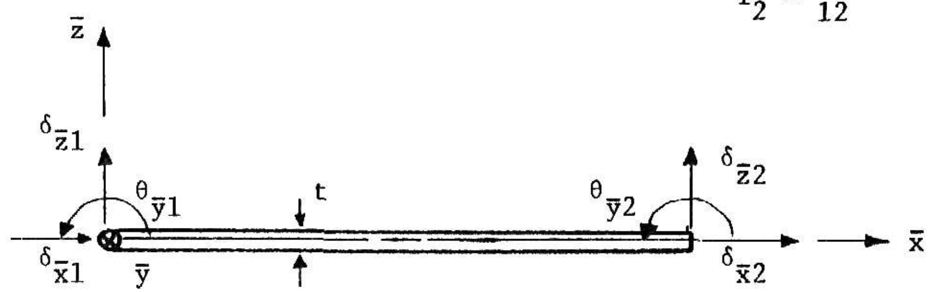
$$G = \frac{E}{2(1+\mu)}$$

$$K = \frac{bt^3}{3}\left(1 - .630\,\frac{t}{b}\right)$$

Fig. 7 - Beam Element and Stiffness for Buckling Normal to the Web

$$\begin{Bmatrix} P_{\bar{y}1} \\ M_{\bar{z}1} \\ P_{\bar{y}2} \\ M_{\bar{z}2} \end{Bmatrix} = \frac{EI_1}{\ell^3} \begin{bmatrix} 12 & 6\ell & -12 & 6\ell \\ 6\ell & 4\ell^2 & -6\ell & 2\ell^2 \\ -12 & -6\ell & 12 & -6\ell \\ 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix} \begin{Bmatrix} \delta_{\bar{y}1} \\ \theta_{\bar{z}1} \\ \delta_{\bar{y}2} \\ \theta_{\bar{z}2} \end{Bmatrix}$$

$$\begin{Bmatrix} M_{\bar{x}1} \\ M_{\bar{x}2} \end{Bmatrix} = \frac{GK}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_{\bar{x}1} \\ \theta_{\bar{x}2} \end{Bmatrix}$$




$$I_2 = \frac{bt^3}{12}$$

Fig. 8 - Beam Element and Stiffness for Buckling in the Plane of the Web

$$\begin{Bmatrix} P_{\bar{z}1} \\ M_{\bar{y}1} \\ P_{\bar{z}2} \\ M_{\bar{y}2} \end{Bmatrix} = \frac{EI_2}{\ell^3} \begin{bmatrix} 12 & 6\ell & -12 & 6\ell \\ 6\ell & 4\ell^2 & -6\ell & 2\ell^2 \\ -12 & -6\ell & 12 & -6\ell \\ 6\ell & 2\ell^2 & -6\ell & 4\ell^2 \end{bmatrix} \begin{Bmatrix} \delta_{\bar{z}1} \\ \theta_{\bar{y}1} \\ \delta_{\bar{z}2} \\ \theta_{\bar{y}2} \end{Bmatrix}$$

$$\begin{Bmatrix} P_{\bar{x}1} \\ P_{\bar{x}2} \end{Bmatrix} = \frac{btE}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \delta_{\bar{x}1} \\ \delta_{\bar{x}2} \end{Bmatrix}$$

Ref. (5). As in the case of the quadrilateral, Type 1C, the interior DOF are eliminated by static condensation and the resulting quadrilateral as 20 DOF (five at each exterior corner node).

A typical beam element for idealizing the flanges is shown in Figs. 7 and 8. When the structure is idealized in this manner, it is reduced to a two-dimensional structure which requires a plane of structural symmetry in the mid-surface of the web. The beam stiffness for buckling normal to the web is shown in Fig. 7. This stiffness is used in combination with the quadrilateral bending stiffness of the web for out-of-plane analysis. The beam stiffness for buckling in the plane of the web is shown in Fig. 8. It is used in combination with the quadrilateral membrane stiffness of the web for the in-plane analysis.

## 2-5 Geometric Element Stiffness

Two approaches are available for deriving the geometric stiffness. The consistent approach utilizes the same displacement assumptions that are used in evaluating the conventional stiffness. The second approach permits the use of simpler (lower order polynomial) displacement assumptions for the geometric stiffness than those used in forming the conventional stiffness. This approach has been shown to yield sufficient accuracy (5) for plate buckling. It requires less computational effort and simplifies the formulation of the problem. This approach is used in this report.

Since only the membrane stresses are considered in evaluating the element geometric stiffnesses, the second half of Eq. 2 has the following simplified form:

$$Ug = \int_v \tau_{ij} \cdot \Delta\eta_{ij} \, dv \doteq \int_v [\tau_{11} \cdot \eta_{11} + 2\, \tau_{12}\, \eta_{12} + \tau_{22}\, \eta_{22}] \, dv \quad \text{-- (4)}$$

where

$\tau_{11}$, $\tau_{12} \equiv \tau_{21}$, $\tau_{22}$ are in-plane stresses, as shown in Fig. 9.

The strains in Eq. 4 may be written as

$$\eta_{11} = \tfrac{1}{2}\,(\bar{u}^2_{1,1} + \bar{u}^2_{2,1} + \bar{u}^2_{3,1}) \equiv \tfrac{1}{2}\,(u^2_{,x} + v^2_{,x} + w^2_{,x})$$

$$\eta_{22} = \tfrac{1}{2}\,(\bar{u}^2_{1,2} + \bar{u}^2_{2,2} + \bar{u}^2_{3,2}) \equiv \tfrac{1}{2}\,(u^2_{,y} + v^2_{,y} + w^2_{,y})$$

$$\eta_{12} = \tfrac{1}{2}\,(\bar{u}_{1,1}\,\bar{u}_{1,2} + \bar{u}_{2,1}\,\bar{u}_{2,2} + \bar{u}_{3,1}\,\bar{u}_{3,2})$$

$$\equiv \tfrac{1}{2}\,(u_{,x}\,u_{,y} + v_{,x}\,v_{,y} + w_{,x}\,w_{,y}) \quad \text{------------------} \quad (5)$$

where $\bar{u}_{i,j} = \dfrac{\partial \bar{u}_i}{\partial x_j}$ , $\bar{u}_1 = u$ and $\bar{u}_2 = v$ are displacements in the plane of the element, while $\bar{u}_3 = w$ is the displacement normal to the element, and $x_1 = x$, $x_2 = y$ are as shown in Fig. 9.

In order to evaluate Eq. 4, it is convenient to express the displacement $\bar{u}_i$ in terms of nodal point displacements which are shown symbolically as:

$$u = \Phi u_i;\; v = \Phi v_i;\; w = \Phi w_i \quad \text{------------------------------} \quad (6)$$

where $u_i$, $v_i$, and $w_i$ are nodal point displacements and $\Phi$ is the appropriate interpolation function. The derivatives implied in Eq. 5 may now be expressed as

$$\left.\begin{aligned} u_{,x} &= \Phi_{,x} u_i;\; v_{,x} = \Phi_{,x} v_i;\; w_{,x} = \Phi_{,x} w_i \\ u_{,y} &= \Phi_{,y} u_i;\; v_{,y} = \Phi_{,y} v_i;\; w_{,y} = \Phi_{,y} w_i \end{aligned}\right. \quad \text{------------------} \quad (7)$$

By using Eq. 5-7, Eq. 4 may be expressed as:

$$Ug = \int_v \Big(\frac{\tau_{11}}{2}\,[u_i^T\,\Phi^T_{,x}\,\Phi_{,x}\,u_i + v_i^T\,\Phi^T_{,x}\,\Phi_{,x}\,v_i + w_i^T\,\Phi^T_{,x}\,\Phi_{,x}\,w_i] +$$

$$\frac{\tau_{22}}{2}\,[u_i^T\,\Phi^T_{,y}\,\Phi_{,y}\,u_i + v_i^T\,\Phi^T_{,y}\,\Phi_{,y}\,v_i + w_i^T\,\Phi^T_{,y}\,\Phi_{,y}\,w_i] +$$

$$\tau_{12}\,[u_i^T\,\Phi^T_{,x}\,\Phi_{,y}\,u_i + v_i^T\,\Phi^T_{,x}\,\Phi_{,y}\,v_i + w_i^T\,\Phi^T_{,x}\,\Phi_{,y}\,w_i]\Big)dv$$

---- (8)

Then taking the partial derivatives of $\bar{U}_g$ with respect to $u_i$, $v_i$, $w_i$, (i.e., using Castigliano's first theorem) we obtain the following form for the geometric stiffness for a two-dimensional element:

$$\begin{Bmatrix} P_{GX} \\ P_{GY} \\ P_{GZ} \end{Bmatrix} = \begin{bmatrix} \bar{K}_g & \cdot & \cdot \\ \cdot & \bar{K}_g & \cdot \\ \cdot & \cdot & \bar{K}_g \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} \qquad \text{------------------------} \quad (9)$$

where

$$\bar{K}_g = \int_v \{\Phi_{,x}\ \Phi_{,y}\} \begin{bmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{bmatrix} \begin{Bmatrix} \Phi_{,x} \\ \Phi_{,y} \end{Bmatrix} dv \qquad \text{----------------} \quad (10)$$

Eq. 9 may now be specialized for particular elements by the use of the appropriate nodal displacements ($u_i$, $v_i$, $w_i$) and the interpolation function, $\Phi$. Equation 9 is specialized for rectangular, triangular, quadrilateral and beam elements as described below.

a) Type 1G -- RECTANGULAR ELEMENT

For the rectangle, bi-linear displacement functions as shown in Fig. 13 are assumed for u, v, and w in Eq. 6 such that the interpolation function, $\Phi$, is

$$\Phi = \{(1-\xi)(1-\eta),\ \xi(1-\eta),\ \xi\eta,\ (1-\xi)\eta\} \qquad \text{-----------------} \quad (11)$$

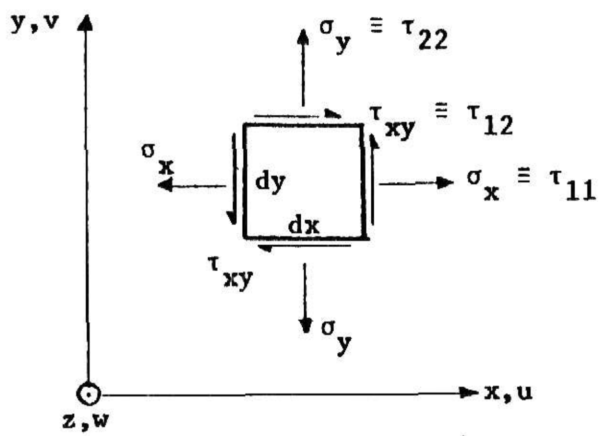



Fig. 9 - In-Plane Stresses

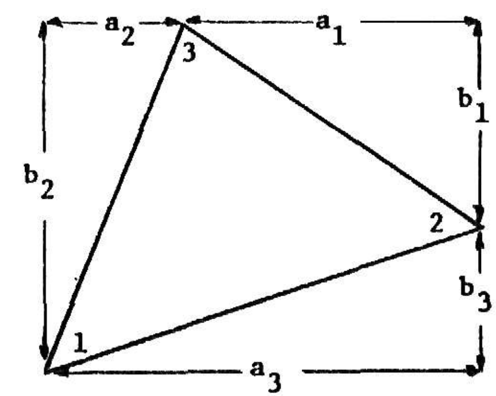



Fig. 10 - Geometry of Triangle

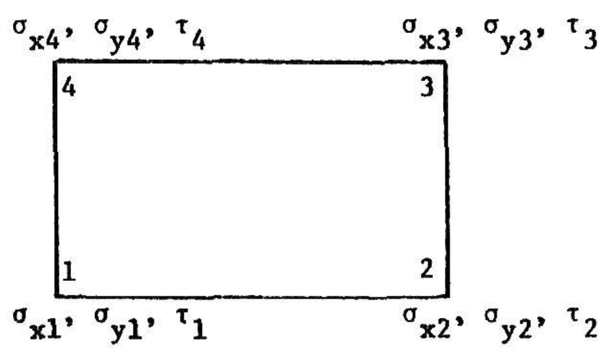



Fig. 11 - Nodal Point Stresses for Rectangle

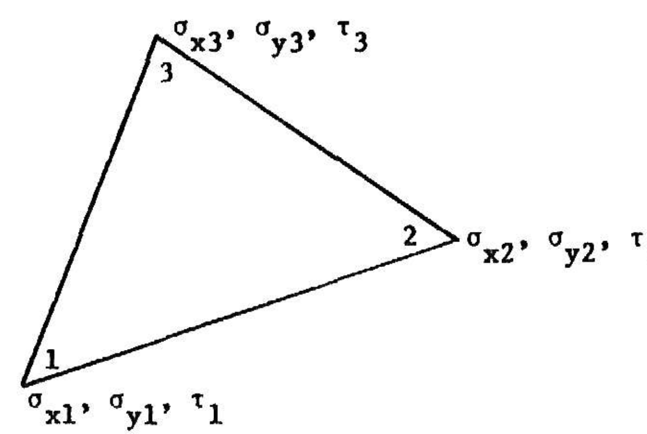



Fig. 12 - Nodal Point Stresses for Triangle

The derivatives of $\Phi$ are obtained by noting that

$$\frac{\partial \Phi}{\partial x} \equiv \Phi_{,x} = \frac{1}{a}\frac{\partial \Phi}{\partial \xi} \text{ and } \frac{\partial \Phi}{\partial y} \equiv \Phi_{,y} = \frac{1}{b}\frac{\partial \Phi}{\partial \eta} \quad \text{------------------} \quad (12)$$

In addition, a linear distribution of stress is assumed over the element which may be expressed in terms of element nodal point stresses shown in Fig. 11 as

$$\begin{Bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} \sigma_{x1} & \bar{\sigma}_{x2} & \bar{\sigma}_{x4} \\ \sigma_{y1} & \bar{\sigma}_{y2} & \bar{\sigma}_{y4} \\ \tau_1 & \bar{\tau}_2 & \bar{\tau}_4 \end{bmatrix} \begin{Bmatrix} 1 \\ \xi \\ \eta \end{Bmatrix} \quad \text{------------------------} \quad (13)$$

where

$$\bar{\sigma}_{x2} = \sigma_{x2} - \sigma_{x1},\ \bar{\sigma}_{y2} = \sigma_{y2} - \sigma_{y1},\ \bar{\tau}_2 = \tau_2 - \tau_1$$

$$\bar{\sigma}_{x4} = \sigma_{x4} - \sigma_{x1},\ \bar{\sigma}_{y4} = \sigma_{y4} - \sigma_{y1},\ \bar{\tau}_4 = \tau_4 - \tau_1$$

Using Eqs. 12 and 13, $\bar{K}_g$ in Eq. 10 may be evaluated by performing the required integration.

b) Type 2G -- TRIANGULAR AND QUADRILATERAL ELEMENTS

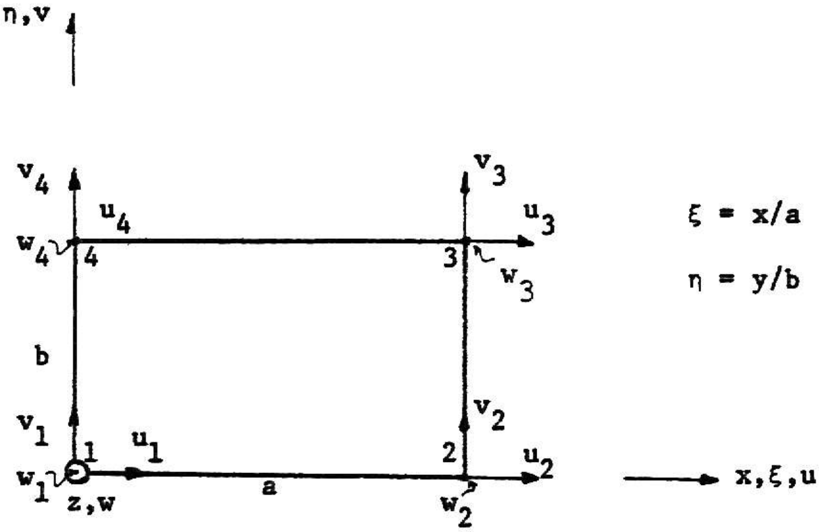
As shown in Fig. 14 for the triangle element, linear displacement functions are assumed for u, v, and w in Eq. 6 such that the interpolation function, $\Phi$, is

$$\Phi = \{\zeta_1, \zeta_2, \zeta_3\} \quad \text{-----------------------------------} \quad (14)$$

where $\zeta_1$, $\zeta_2$, $\zeta_3$ are triangular coordinates. The derivatives of $\Phi$ are taken by noting the following relationship:

$$\frac{\partial \zeta_i}{\partial x} = \frac{b_i}{2A} \text{ and } \frac{\partial \zeta_i}{\partial y} = \frac{a_i}{2A} \qquad i = 1, 2, 3 \quad \text{-----------------} \quad (15)$$

where $b_i$ and $a_i$ are element dimensions as shown in Fig. 10 and A is the area of the triangle. A linear stress distribution is assumed over the

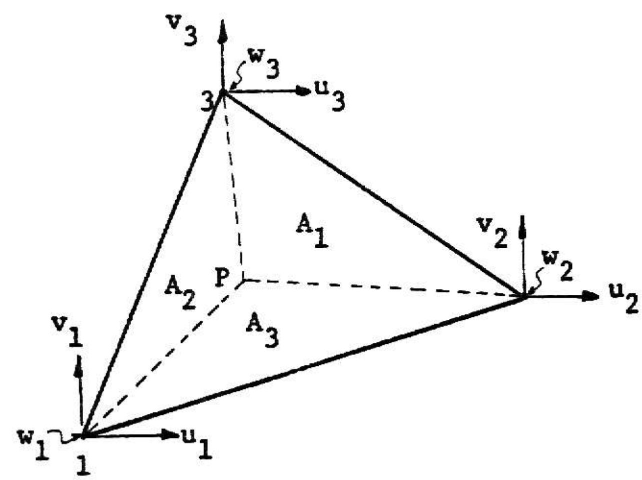



$\xi = x/a$

$\eta = y/b$

$$\{u,v,w\} = \{(1-\xi)(1-\eta),\ \xi(1-\eta),\ \xi\eta,\ (1-\xi)\eta\} \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \\ u_4 & v_4 & w_4 \end{bmatrix}$$

Fig. 13 - Displacement Functions for Rectangle -- Geometric Stiffness




$\zeta_1 = A_1/A$

$\zeta_2 = A_2/A$

$\zeta_3 = A_3/A$

$$\{u,v,w\} = \{\zeta_1,\ \zeta_2,\ \zeta_3\} \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix}$$

Fig. 14 - Displacement Functions for Triangle -- Geometric Stiffness

triangle which may be expressed in terms of element nodal point stresses shown in Fig. 12 as

$$\begin{Bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} \sigma_{x1} & \sigma_{x2} & \sigma_{x3} \\ \sigma_{y1} & \sigma_{y2} & \sigma_{y3} \\ \tau_1 & \tau_2 & \tau_3 \end{bmatrix} \begin{Bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{Bmatrix} \quad \text{-------------------------} \quad (16)$$

By noting that $\Phi_{,x}$ and $\Phi_{,y}$ are constant, the evaluation of $\bar{K}_g$, Eq. 10, is greatly simplified. We need only to integrate Eq. 16 over the volume of the triangle which yields

$$\int_v \tau_{11}\, dv = \frac{A \cdot t}{3} (\sigma_{x1} + \sigma_{x2} + \sigma_{x3}) = A \cdot t \cdot \sigma_{xo}$$

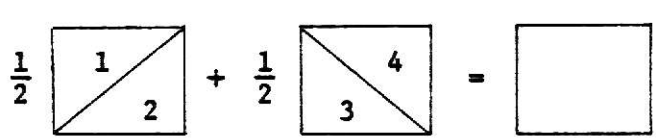
$$\int_v \tau_{22}\, dv = \frac{A \cdot t}{3} (\sigma_{y1} + \sigma_{y2} + \sigma_{y3}) = A \cdot t \cdot \sigma_{yo} \quad \text{-----------} \quad (17)$$

$$\int_v \tau_{12}\, dv = \frac{A \cdot t}{3} (\tau_1 + \tau_2 + \tau_3) = A \cdot t \cdot \tau_o$$

where $\sigma_{xo}$, $\sigma_{yo}$ and $\tau_o$ are simply the average of the corner element nodal point stresses and t is the element thickness. Thus, $\bar{K}_g$ may be expressed in matrix form as

$$\bar{K}_g = \frac{t}{4A} \begin{bmatrix} b_1 & a_1 \\ b_2 & a_2 \\ b_3 & a_3 \end{bmatrix} \begin{bmatrix} \sigma_{xo} & \tau_o \\ \tau_o & \sigma_{yo} \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{bmatrix} \quad \text{--------} \quad (18)$$

A quadrilateral geometric stiffness may be constructed from pairs of triangles. To obtain a more accurate geometric stiffness, two quadrilateral geometric stiffnesses may be constructed and added as shown symbolically in the following figure:



Triangles 1 and 2 are used to generate the first quadrilateral stiffness while triangles 3 and 4 are used to generate the second quadrilateral stiffness. The final quadrilateral geometric stiffness is simply taken as the average of the first and second stiffnesses.

c) Type 3G -- BEAM ELEMENT

For buckling normal to the web, only the geometric stiffness associated with the beam element of Fig. 7 needs to be considered. The displacements u, v, and w, are taken in the direction of $\bar{x}$, $\bar{y}$, and $\bar{z}$, respectively, as shown in Fig. 7. Linear distributions of the displacement v and the rotation $\theta_{\bar{x}}$ are assumed as

$$v = \{(1-\xi),\ \xi\} \begin{Bmatrix} \delta_{\bar{y}1} \\ \delta_{\bar{y}2} \end{Bmatrix} \qquad (19)$$

$$\theta_{\bar{x}} = \{(1-\xi),\ \xi\} \begin{Bmatrix} \theta_{\bar{x}1} \\ \theta_{\bar{x}2} \end{Bmatrix} \qquad (20)$$

By considering only the in-plane stress in the $\bar{x}$-direction and recognizing that u = 0, Eq. 4 has the following simplified form:

$$Ug = \int_v \tau_{11}\, \eta_{11}\, dv = \int_v \frac{\sigma_{\bar{x}}}{2} (v^2_{,\bar{x}} + w^2_{,\bar{x}})\, dv \qquad (21)$$

From Eq. 19 we have

$$v_{,\bar{x}} = \frac{1}{\ell} \{-1,\ 1\} \begin{Bmatrix} \delta_{\bar{y}1} \\ \delta_{\bar{y}2} \end{Bmatrix} \qquad (22)$$

and by noting that $w = \bar{y} \cdot \theta_{\bar{x}}$, Eq. 20 becomes

$$w = \bar{y} \cdot \theta_{\bar{x}} = \bar{y} \cdot \{(1-\xi),\ \xi\} \begin{Bmatrix} \theta_{\bar{x}1} \\ \theta_{\bar{x}2} \end{Bmatrix} \qquad (23)$$

and

$$w_{,\bar{x}} = \frac{\bar{y}}{\ell} \{-1, 1\} \begin{Bmatrix} \theta_{\bar{x}1} \\ \theta_{\bar{x}2} \end{Bmatrix} \quad \text{------------------------------------} \quad (24)$$

By substituting Eqs. 22 and 24 into Eq. 21, we obtain

$$Ug = \tfrac{1}{2} \int_v \frac{\sigma_{\bar{x}}}{\ell^2} \left\{ \{\delta_{\bar{y}1}, \delta_{\bar{y}2}\} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \{-1\ 1\} \begin{Bmatrix} \delta_{\bar{y}1} \\ \delta_{\bar{y}2} \end{Bmatrix} + \right.$$

$$\left. \bar{y}^2 \{\theta_{\bar{x}1}, \theta_{\bar{x}2}\} \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} \{-1\ 1\} \begin{Bmatrix} \theta_{\bar{x}1} \\ \theta_{\bar{x}2} \end{Bmatrix} \right\} dv \quad \text{--------------} \quad (25)$$

Then by assuming $\sigma_{\bar{x}}$ to be constant over the flange width with a linear variation in the $\bar{x}$-direction, we can write

$$\sigma_{\bar{x}} = \{(1-\xi), \xi\} \begin{Bmatrix} \sigma_{\bar{x}1} \\ \sigma_{\bar{x}2} \end{Bmatrix} \quad \text{------------------------------------} \quad (26)$$

where $\sigma_{\bar{x}1}$ and $\sigma_{\bar{x}2}$ are nodal stresses at $\bar{x} = 0$, $\bar{x} = \ell$, respectively. Substituting Eq. 26 into Eq. 25, performing the required intergration and taking partial derivatives with respect to the nodal point displacements and rotations we obtain the following geometric stiffnesses.

$$\begin{Bmatrix} P_{G\bar{y}1} \\ P_{G\bar{y}2} \end{Bmatrix} = G_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \delta_{\bar{y}1} \\ \delta_{\bar{y}2} \end{Bmatrix} \quad \text{-------------------------} \quad (27)$$

$$\begin{Bmatrix} M_{G\bar{x}1} \\ M_{G\bar{x}2} \end{Bmatrix} = G_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \theta_{\bar{x}1} \\ \theta_{\bar{x}2} \end{Bmatrix} \quad \text{-------------------------} \quad (28)$$

where

$$G_1 = \left(\frac{\sigma_{\bar{x}1} + \sigma_{\bar{x}2}}{2}\right) \frac{t \cdot b}{\ell}$$

$$G_2 = \left(\frac{\sigma_{\bar{x}1} + \sigma_{\bar{x}2}}{2}\right) \frac{t \cdot b^3}{12 \cdot \ell}$$

In the lateral buckling analysis of wide flange beam, only the geometric stiffness of Eq. 27 is considered. On the other hand, Eq. 28 should be included in predicting local (torsional) buckling of the flange.

## 2-6 The Element Assemblage

The stiffness matrix of the complete assemblage is obtained by the direct stiffness procedure (APPENDIX I). It consists of first deriving the individual element stiffnesses in terms of element coordinates, as previously described, followed by a coordinate transformation and the subsequent superposition of each element stiffness to form the stiffness of the complete assemblage. This applies to both the conventional and the geometric stiffnesses. The coordinate transformation is performed by rotating the element coordinates, so that the translational and rotational degrees of freedom of all elements which share a common nodal point are expressed in the same coordinates. The superposition of each transformed stiffness is accomplished by adding its individual terms into the complete stiffness matrix according to the element nodal point numbers.

The element assemblage of two-dimensional elements may consist of Type 1C combined with Type 1G (16), or Type 2C also combined with Type 1G (18) for wide-flange beam buckling analysis. The element assemblage of two-dimensional web elements and one-dimensional flange elements, which is used in the BASP computer program, consist of the combined use of Type 3C with Type 2G and Type 3G.

## 2-7 Determination of Buckling Load

The conventional and geometric stiffnesses of Eq. 1 are symmetrical and banded. These properties are recognized in the solution. Even with the relative economy offered by these properties, the solution of Eq. 1 poses a computational problem of considerable magnitude. Although Eq. 1 represents the standard eigenvalue problem, the order of this matrix prohibits the use of well known algorithms based on direct methods.

Inverse iteration was used for the solution of Eq. 1 (7, 24). It generally converges to the smallest load which would cause the structure to buckle. It should be kept in mind that a positive $\lambda$ means the buckling load is in the same direction of the applied loading, whereas a negative $\lambda$ implies a complete reversal in sign of the applied loading. If the applied loading cannot be reversed, a negative $\lambda$ would represent a fictious loading which should be ignored. Unfortunately a negative $\lambda$ may be obtained if it is smaller in absolute value than the smallest positive $\lambda$. This was avoided as described herein.

Eq. 1 may be made to converge to the smallest positive $\lambda$ by shifting the origin of the eigenvalue calculation from zero to a positive value. The shifting procedure is applied to Eq. 1 by letting $\lambda = \bar{\lambda} + s$ in which $s$ = a shift; and $\bar{\lambda}$ = the shifted eigenvalue. Then we have

$$[K_c]\{r\} + (\bar{\lambda} + s)[K_g]\{r\} = 0 \qquad (29)$$

or by rearranging

$$[\bar{K}]\{r\} = -\lambda[K_g]\{r\} \qquad (30)$$

in which $[\bar{K}] = [K_c] + s[K_g]$. Eq. 30 is solved for $\bar{\lambda}$ and thus $\lambda$ by the inverse iteration procedure as well as the buckled mode shape. Upon convergence, the buckling load is computed by multiplying $\lambda$ by the applied loading.

# CHAPTER III - COMPUTER PROGRAM

## 3-1 Introduction

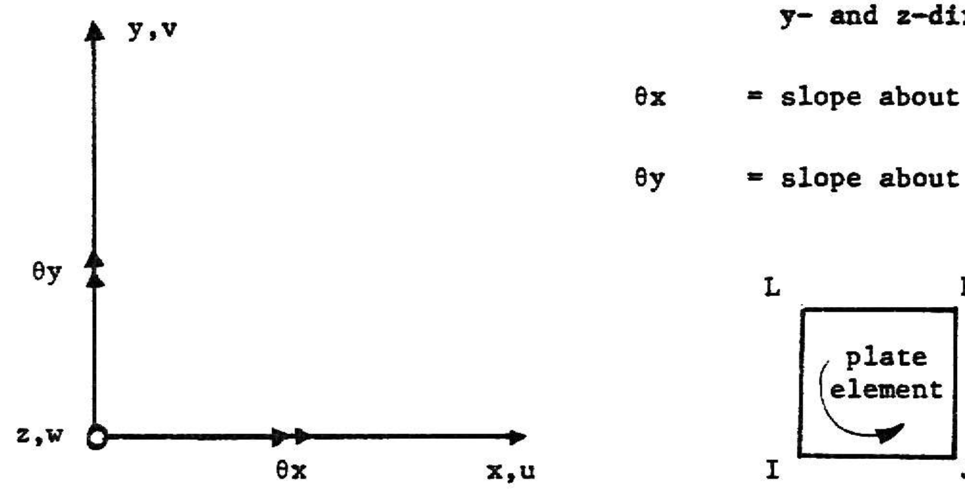
The computer program, Buckling Analysis of Stiffened Plates--BASP ( 1 ), provides a general capability for the buckling analysis of wide flange beams having stiffener elements placed symmetrically about the beam. The beam web is idealized by two-dimensional finite elements while the flanges are idealized by conventional one-dimensional elements. Arbitrary planar geometry, loadings, boundary conditions and elastic restraints may be accurately represented. This procedure accounts for the cross-section distortion as well as local buckling of the web and torsional buckling of the flanges.

The program consists of two types of analysis, the first one is the stress analysis under the applied loadings and is called in-plane stress analysis or membrane analysis. The second one is the buckling analysis, which uses the stresses calculated under the applied loading, and it is sometimes called the out-of-plane buckling analysis.

The problem is treated as a linear-elastic buckling problem. Inverse iteration is used to solve the resulting eigenvalue problem, yielding the smallest load causing buckling. This program is restricted to the determination of buckling normal to plane of web. In the linearized problem, buckling in the plane of the web is uncoupled from that normal to the web and hence may be ignored.

## 3-2 Use of the Program

1. A global Cartesian coordinate system, x, y, and z, as shown in Fig. 15 must be chosen for the structure to be analyzed. The midsurface of the plate (web) is contained in the x-y plane.

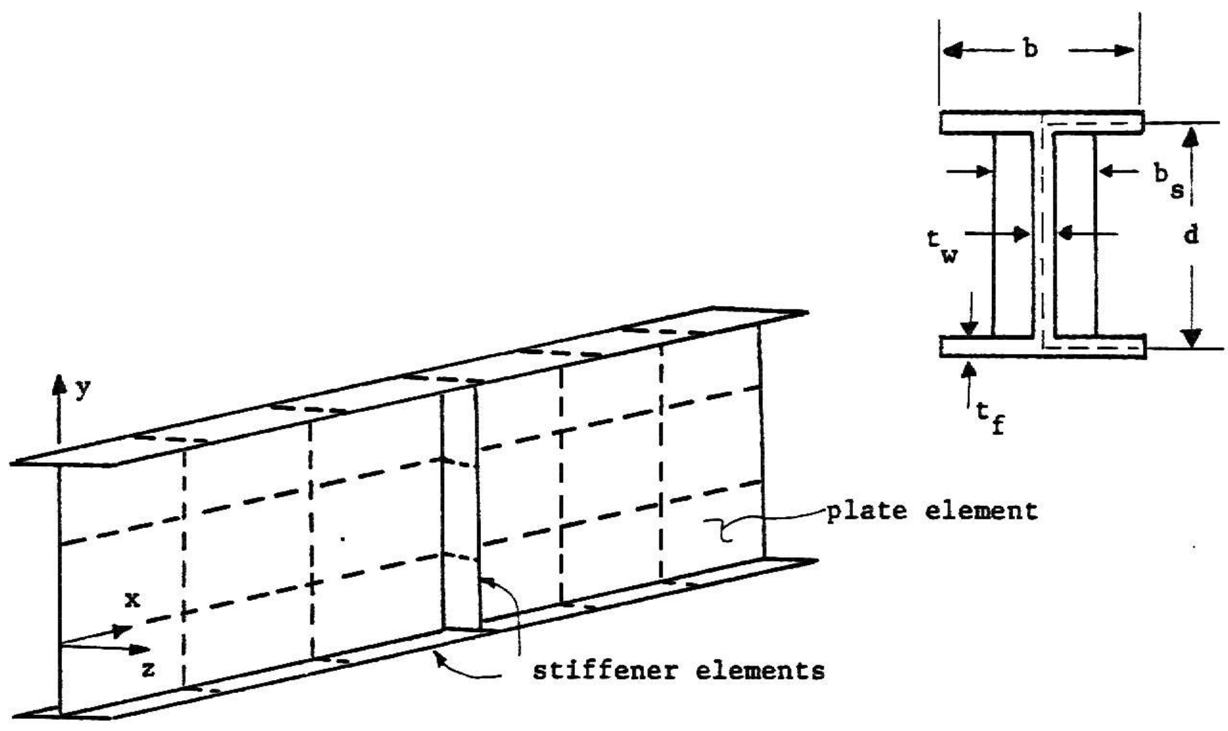



x,v,w = displacements in x-, y- and z-directions

θx = slope about x = $(\frac{\partial w}{\partial y})$

θy = slope about y = $(\frac{-\partial w}{\partial x})$

Fig. 15 - The Sign Convention for the Displacements and Rotations at each Nodal Point




b = width of flange stiffener element

$b_s$ = width of web stiffener element

Fig. 16 - The Idealization of Wide-Flange Beam

2. The structure is idealized as an assemblage of quadrilateral plate elements and uniform stiffener elements, as shown in Fig. 16. Use of only one type of element, like only plates or stiffeners, is permitted.

3. Nodal points are numbered sequentially. Plate elements and stiffener elements are numbered separately. All numbering should begin with the interger 1. The nodal point numbering should run in the direction of the smallest number of division in order to minimize the nodal point half band width of the assemblage, which is governed by the maximum element nodal difference. Quadrilateral plate elements have four nodal points, I, J, K, and L. Element nodal point numbers are input in a counterclockwise sequence in either direction for input.

4. In-plane analysis involves two degrees of freedom at each nodal point, which are displacements u and v in x- and y-directions. The out-of-plane analysis requires three degrees of freedom at each nodal point and these are displacement w in z-direction and rotations $\theta_x$ and $\theta_y$. The sign convention shown in Fig. 15 is used for the displacements and rotations at each nodal point.

5. The first part of the output consists of the nodal point displacements u and v, which are in x- and y-directions respectively, and the computed stresses under the applied loading. The second part of the output consists of the eigenvalue, $\lambda$, and the corresponding normalized eigenvector. Where $\lambda$ = critical load/applied load, and the eigenvector corresponds to the buckled mode shape. The lateral displacements and two rotations about x- and y-axes are printed for each nodal point.

6. This program is dimensioned for a maximum of 90 nodes, 60 plate elements and 65 stiffener elements. The maximum allowable number of plate elements in the short direction is 4.

## 3-3 Data Input Requirements

Abbreviations: A = Alphanumeric Field: the max., unit Alphanumeric Field in this program is 4.

I = Integer value; must be packed to the right of the field, and is always I5.

F = Floating point number, must be punched with a decimal, and is always F10.0.

1. Identification of Run (A) - Two alphanumeric cards are needed for each run. These cards may contain any information related to the run.

2. Identification of Problem (One card)

   Cols. 1-5 (I) Problem number; program stops when problem number is zero or blank.

   11-80 (A) Alphanumeric information for the descriptions of the problem.

3. Control Data (One card) (See pg. 34 for some options)

   Cols. 1-5 (I) NUMEL: Number of plate elements

   6-10 (I) NUMBM: Number of beam elements (stiffeners)

   11-15 (I) NUPTS: Number of nodal points

   16-20 (I) NLPTS: Number of loaded nodal points for in-plane analysis

   21-25 (I) NMBPTS: Number of nodal points with displacement boundary conditions for in-plane analysis

   26-30 (I) NSPRM: Number of nodal points with elastic springs for in-plane analysis

   31-35 (I) NBBPTS: Number of nodal points with displacement boundary conditions for out-of-plane buckling analysis

   36-40 (I) NSPRB: Number of nodal points with elastic springs for out-of-plane buckling analysis

4. Material Properties (One card) (See pg. 34 for some options)

   Cols. 1-10 (F) Modulus of Elasticity

   11-20 (F) Poisson's Ratio

5. Nodal Coordinate Generation (Straight lines)

| | | | |
|---|---|---|---|
| Cols. | 1-5 | (I) | Node I of straight line |
| | 6-10 | (I) | Node J of straight line ($J \geq I$) |
| | 11-15 | (I) | Nodal point increment (INCRN) |
| | 21-30 | (F) | Global x coordinate of point I; (X1) |
| | 31-40 | (F) | Global y coordinate of point I; (Y1) |
| | 41-50 | (F) | Global x coordinate of point J; (X2) |
| | 51-60 | (F) | Global y coordinate of point J; (Y2) |

The straight line is divided into (J-I)/INCRN equal parts and the intermediate global nodal coordinates are computed.

If only a single point is input, J, INCRN, X2 and Y2 may be left blank.

Note: Generation stops if I or J is equal to NUPTS. Therefore, card must be taken to generate the nodal coordinate of the last node number in the last card.

6. Nodal Point Number System Generation for Quadrilateral Plate Elements (No card is needed if NUMEL = 0)

| | | | |
|---|---|---|---|
| Cols. | 1-5 | (I) | From element number (N1) |
| | 6-10 | (I) | To element number (N2) |
| | 11-15 | (I) | Nodal point number increment (INCRN) |
| | 16-20 | (I) | Element number increment (INCRE) |
| | 21-25 | (I) | Node number I of element N1 |
| | 26-30 | (I) | Node number J of element N1 |
| | 31-35 | (I) | Node number K of element N1 |
| | 36-40 | (I) | Node number L of element N1 |

Nodal points are generated for quadrilaterals N1, N1 + INCRE, N1 + 2*INCRE,....N2 with nodal point increments of INCRN. If an individual element is input, N2, INCRN, and INCRE may be left blank.

Note: Generation stops when N1 or N2 is equal to NUMEL.

7. Nodal Point Number System Generation for Beam Elements (Stiffeners)

(No card is needed if NUMBM = 0)

| | | | |
|---|---|---|---|
| Cols. | 1-5 | (I) | From element number (N1) |
| | 6-10 | (I) | To element number (N2) |
| | 11-15 | (I) | Nodal point number increment (INCRN) |
| | 16-20 | (I) | Element number increment (INCRE) |
| | 21-25 | (I) | Node number I of element N1 |
| | 26-30 | (I) | Node number J of element N1 |

Nodal points are generated for beams N1, N1 + INCRE, N1 + 2*INCRE,.... N2 with nodal point increment of INCRN. If an individual element is input, then N2, INCRN and INCRE may be left blank. Stiffeners should have their own independent numbering system starting from 1.

Note: Generation stops when N1 or N2 is equal to NUMBM.

8. Generation of the Properties of Quadrilateral Plate Elements

(No card is needed if NUMEL = 0)

| | | | |
|---|---|---|---|
| Cols. | 1-5 | (I) | From element number (N1) |
| | 6-10 | (I) | To element number (N2) |
| | 11-15 | (I) | Element number increment (INCRE) |
| | 21-30 | (F) | Thickness of plate element |

If an individual element is input N2 and INCRE may be left blank.

Note: Generation stops when N1 or N2 is equal to NUMEL.

9. Generation of the Properties of Stiffener Elements

(No card is needed if NUMBM = 0)

| | | | |
|---|---|---|---|
| Cols. | 1-5 | (I) | From element number (N1) |
| | 6-10 | (I) | To element number (N2) |
| | 11-15 | (I) | Element number increment (INCRE) |

21-30 (F) Thickness of stiffener

31-40 (F) Width of stiffener

If an individual element is input, N2 and INCRE may be left blank.

Note: Generation stops when N1 or N2 is equal to NUMBM.

10. Generation of Nodal Point Loads

(No card is needed if NLPTS = 0)

Cols. 1-5 (I) From node (I)

6-10 (I) To Node (J)

11-15 (I) Node increments (INCRN)

21-30 (F) Load in x-direction

31-40 (F) Load in y-direction

11. Generation of Specified Boundary Conditions for In-Plane Analysis

(No card is needed if NMBPTS = 0)

Cols. 1-5 (I) From node (I)

6-10 (I) To node (J)

11-15 (I) Node increments (INCRN)

16-20 (I) Displacement in x-direction (u)
If = 1; x-displacement is specified
If = 0 or blank; free to displace

21-25 (I) Displacement in y-direction (v)
If = 1; y-displacement is specified
If = 0 or blank; free to displace

31-40 (F) Specified value of x-displacement

41-50 (F) Specified value of y-displacement

12. Generation of Elastic Spring for In-Plane Analysis

(No card is needed if NSPRM = 0)

Cols. 1-5 (I) From node (I)

6-10 (I) To node (J)

11-15 (I) Node increments (INCRN)

21-30 (F) x-spring constant

31-40 (F) y-spring constant

13. Generation of Boundary Conditions for Out-of-Plane Buckling Analysis (No card is needed if NBBPTS = 0)

Cols. 1-5 (I) From node (I)

6-10 (I) To node (J)

11-15 (I) Node increments (INCRN)

16-20 (I) Displacement in z-direction (w)
If = 1, z-displacement is zero
If = 0 or blank, free to displace

21-25 (I) Slope about x-direction ($\theta_x$)
If = 1; slope is zero
If = 0 or blank; free to rotate

26-30 (I) Slope about y-direction ($\theta_y$)
If = 1; slope is zero
If = 0 or blank; free to rotate

14. Generation of Elastic Springs for Out-of-Plane Buckling Analysis (No card is needed if NSPRB = 0)

Cols. 1-5 (I) From node (I)

6-10 (I) To node (J)

11-15 (I) Node increments (INCRN)

21-30 (F) z-spring constant (lateral)

31-40 (F) Rotational spring constant about x

41-50 (F) Rotational spring constant about y

15. Initial Mode Shape Generation (Eigenvector)

Cols. 1-5 (I) From nodal point I

6-10 (I) To nodal point J

11-15 (I) Nodal point increment (INCRN)

21-30 (F) Lateral deflection

31-40 (F) Rotation about x ($\theta_x$)

41-50 (F) Rotation about y ($\theta_y$)

The iterative process used to find the eigenvalue usually closes faster if the approximate buckling shape is input. In general, however, all displacements and rotations are assigned a value of unity.

Note: Generation stops if I or J is equal to NUPTS.

16. Iteration Control Data Card (One card)

| | | | |
|---|---|---|---|
| Cols. | 1-5 | (I) | Maximum number of shifts to be made (JMAX) |
| | 6-10 | (I) | Maximum number of cycles before a new shift is made (IMAX) |
| | 11-20 | (F) | A factor ($\leq$1.0), which is to be multiplied by the current eigenvalue each time JMAX is increased, and is used as a shift for the next IMAX number of cycles |
| | 21-30 | (F) | Estimated eigenvalue--shift |

Iterations close much faster if the approximate eigenvalue (buckling stress) is known, otherwise it is safer to input zero as a shift. Over-estimating the shift value may lead to the calculation of an eigenvalue and thus a buckling stress higher than the lowest one. After the first IMAX number of iterations, the current calculated eigenvalue is multiplied by FAC, and used as a shift for the next IMAX number of iterations. This continues until either the convergence criteria is satisfied or when JMAX*IMAX iterations are completed. There is no general rule for an optimum JMAX, IMAX, and FAC combination, but experience has shown that JMAX = 3, IMAX = 10, FAC = .8 is a fairly safe set of values to use.

If the eigenvalue obtained is negative, then the solution corresponds to the case where the input loading is reversed. This will be a valid solution only if the beam is perfectly symmetric about the longitudinal (x) axis. If the solution is not valid then a shift other than zero should be used. A good rule is to use a shift equal to the absolute value of the negative eigenvalue.

Any number of problems may be solved in a single run. Each problem requires a separate data deck beginning with a Problem Identification Card and ending with an Iteration Control Data Card. Put a blank card at the end if no other problem is required.

## OPTIONS - BASP

A. On the "Control Data" Card

Column 45: Output option*

If = 0; long output is printed

If = 1; medium output is printed

If = 2; short output is printed

Column 50: Torsional buckling option for beam effects

If = 0; no torsional buckling of beam elements is considered

If = 1; torsional buckling of beam elements is considered

B. On the "Material Properties" Card

Column 35: Mode shape plot option for I-beam and narrow rectangular beams

If = 0; no mode shape is plotted

If = n; mode shape is plotted, where n = no. web plate elements over the depth of the beam

---

*The short output prints the input data, the final eigenvalue and the buckled shape.

The medium output, gives in addition to the above, the coordinate system, the in-plane stresses, and the eigenvalue at each iteration.

The long output gives all of the above plus the buckled shape at each iteration.

# CHAPTER IV - RESULTS OF ANALYSIS

Problems for which classical solutions are available were selected as examples so that the accuracy of the finite element program could be checked. The results computed by the BASP computer program are compared to the analytical solutions of Timoshenko (25) and, when available, to results determined by another computer program (18).

## 4-1 Numerical Examples

The following numerical examples, which were worked using the BASP computer program, are presented in this report:

Problem 1: Simply Supported Wide-Flange Beam Under Concentrated Load at the Midspan.

case (a): applied load on top flange. (Fig. III-1a)

case (b): applied load on centroid of beam. (Fig. III-1b)

case (c): applied load on bottom flange. (Fig. III-1c)

Problem 2: Simply Supported Wide-Flange Beam With Uniform Load.

case (a): applied load on top flange. (Fig. III-2a)

case (b): applied load on centroid of beam. (Fig. III-2b)

case (c): applied load on bottom flange. (Fig. III-2c)

Problem 3: Cantilever Wide-Flange Beam Under Concentrated Load at Centroid at Free End.

case (a): use 150 inch length. (Fig. III-3a)

case (b): use 300 inch length. (Fig. III-3c)

Problem 4: Simply Supported Wide-Flange Beam With Lateral Constraint at the Ends.

case (a): applied concentrated load at centroid at midspan. (Fig. III-4a) case (b): applied uniform load on centroid of beam. (Fig. III-4b)

Problem 5: Simply Supported Wide-Flange Beam With Intermediate Lateral Support.

case (a): applied a concentrated load at centroid at midspan. (Fig. III-5a)

case (b): applied uniform load on centroid of beam. (Fig. III-5b)

Problem 6: Simply Supported Wide-Flange Beam Under Constant Moment. (Fig. III-6)

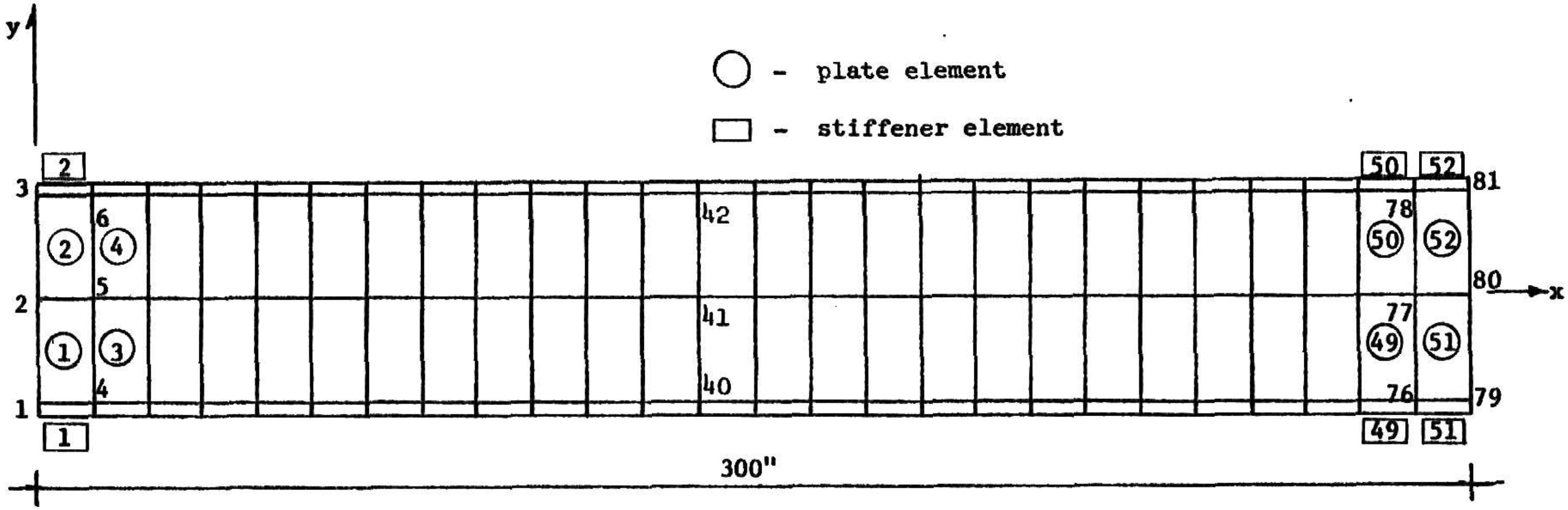
The beam cross-section properties and the discretization used in the above examples are shown in Fig. 17. The necessary values for solving Timoshenko's equations are also shown in Fig. 17, in which: 1) $E$ = Young's modulus; 2) $\mu$ = Poisson's ratio; 3) $G$ = shear modulus = $E/2(1+\mu)$; 4) $J$ = torsion constant = $t^3(2b+h)/3$; 5) $C_w$ = warping constant = $th^2b^3/24$; 6) $C = CJ$ = torsional rigidity; 7) $C_1 = EC_w$ = warping rigidity; 8) $I_y$ = the moment of inertia of the cross section about y axis.

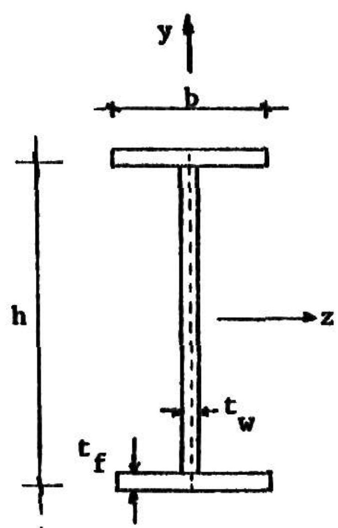
Table 1 gives the input boundary conditions used in the examples studied using the BASP computer program. The notations u, v, w, $\theta_x$, and $\theta_y$ are the displacements in the x-, y-, and z-directions and the rotations about the x- and y-axes, respectively.

## 4-2 Specification of the Load for Input in the BASP Computer Program

(a) Simply Supported Beam Under Constant Moment.

Consider a beam subjected to constant moment, as shown in Fig. 18. The end moments can be replaced by a couple of magnitude $P = M/d$ where d is the distance between the centroid of the flanges (12).




$h = 21$ in

$b = 10$ in

$t_w = t_f = t = 1$ in

$E = 30000\ \mathrm{K/in^2}$

$\mu = 0.3$

$G = 11538\ \mathrm{K/in^2}$

$J = 13.667\ \mathrm{in^4}$

$C_w = 18375\ \mathrm{in^6}$

$C = GJ = 157692\ \mathrm{K/in^2}$

$C_1 = EC_w = 551250000\ \mathrm{K/in^4}$

$I_y = 168.42\ \mathrm{in^4}$

**Fig. 17 - Elevation Showing Mesh Used in Finite Element Solution and Cross-Sectional Properties of Beam**

Table 1 - Input Boundary Conditions Used in BASP Computer Program

| Problem | In-plane Boundary Conditions | Out-of-plane Boundary Conditions |
|---|---|---|
| 1 | $u_1 = v_1 = 0$<br>$v_{79} = 0$ | $w_{1,2,3} = 0;\ w_{79,80,81} = 0$<br>$\theta_{x_{1,2,3}} = 0;\ \theta_{x_{79,80,81}} = 0$ |
| 2 | $u_1 = v_1 = 0$<br>$v_{79} = 0$ | $w_{1,2,3} = 0;\ w_{79,80,81} = 0$<br>$\theta_{x_{1,2,3}} = 0;\ \theta_{x_{79,80,81}} = 0$ |
| 3 | $u_{79,80,81} = 0$<br>$v_{79,80,81} = 0$ | $w_{79,80,81} = 0;\ \theta_{y_{79,80,81}} = 0$<br>$\theta_{x_{79,80,81}} = 0$ |
| 4 | $u_1 = v_1 = 0$<br>$v_{79} = 0$ | $w_{1,2,3} = 0;\ w_{79,80,81} = 0;\ \theta_{x_{1,2,3}} = 0$<br>$\theta_{x_{79,80,81}} = 0;\ \theta_{y_{1,2,3}} = 0;\ \theta_{y_{79,80,81}} = 0$ |
| 5 | $u_1 = v_1 = 0$<br>$v_{79} = 0$ | $w_{1,2,3} = 0;\ w_{79,80,81} = 0$<br>$w_{40,41,42} = 0;\ \theta_{x_{1,2,3}} = 0$<br>$\theta_{x_{40,41,42}} = 0;\ \theta_{x_{79,80,81}} = 0$ |
| 6 | $u_1 = v_1 = 0$<br>$v_{79} = 0$ | $w_{1,2,3} = 0;\ w_{79,80,81} = 0$<br>$\theta_{x_{1,2,3}} = 0;\ \theta_{x_{79,80,81}} = 0$ |

(b) Beam Subjected to Uniform Load.

A uniform load is idealized by concentrated loads $P = w\ell/n$ at internal nodal points and $P = w\ell/2n$ applied at the end nodal points, as shown in Fig. 19, where w is the load per unit length, $\ell$ is the length and n is the number of horizontal divisions.

## 4-3 The Critical Load

The results obtained from the BASP computer program and the theoretical results computed by equations given in Timoshenko (25) are shown in Table 2. As can be seen in this table, the results of the BASP computer program compare favorably with the solutions given in Timoshenko.

Three of these solutions are available in Johnson's paper (18). For problems 1a and 1b, Johnson, using 117 nodal points and 96 elements for a half beam analysis, found the critical loads of 146 kips and 198 kips as compared to the BASP computer program results of 147 kips and 196 kips, respectively. For problem 3a, Johnson used the same mesh as above, but considered the whole beam, and found the critical load of 308 kips as compared to the BASP computer program result of 315 kips. Although both results are close to the theoretical results computed from Timoshenko, the BASP computer program simplifies the coding of the required data and the computation time for a given problem is reduced.

It is important to note that the problem has to be rerun to obtain higher eigenvalues if the initial solution gives a negative eigenvalue, except for beams which are perfectly symmetric about the longitudinal (x) axis. This is accomplished by using the absolute value of the negative eigenvalue as input data (instead of zero) in the shift position on input card number sixteen. (page 33)

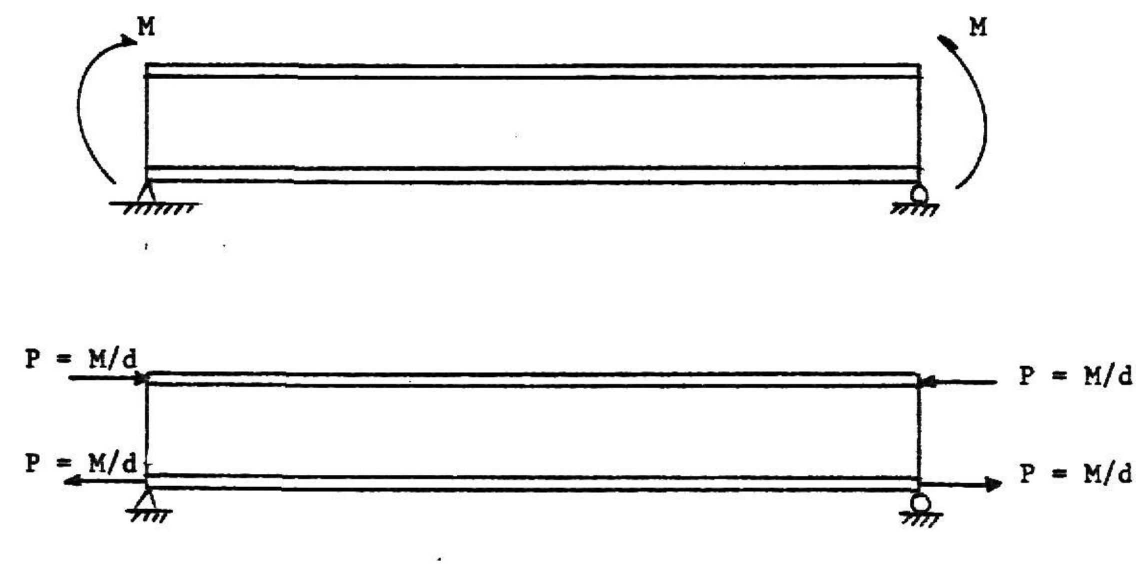



Fig. 18 - Simply Supported Beam Under Constant Moment

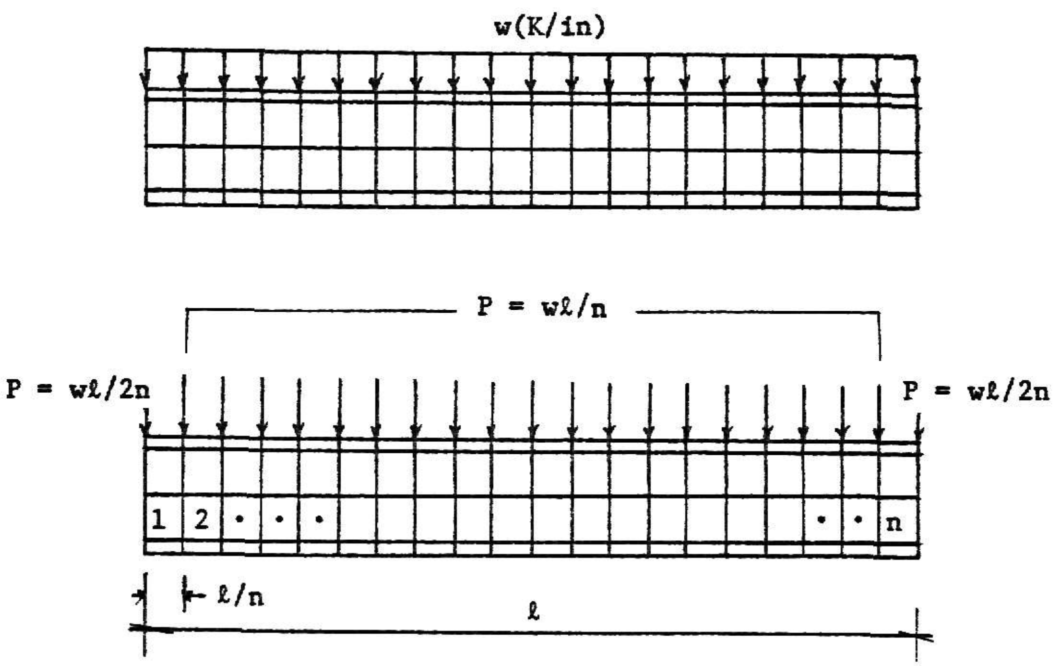



Fig. 19 - Beam Subjected to Uniform Load

Table 2 – The Comparison Between Timoshenko's Solutions and the Results of BASP Computer Program

| Problem | Eigenvalue | | Critical Load | | Error (%) |
|---|---|---|---|---|---|
| | $\lambda_1$ | $\lambda_2$* | Timoshenko's | BASP | |
| 1a | 1.4678 | ------ | 149 Kips | 147 Kips | -1.34 |
| 1b | -1.9564 | 1.9574 | 200 Kips | 196 Kips | -2.00 |
| 1c | -1.4588 | 2.6583 | 267 Kips | 266 Kips | -0.38 |
| 2a | 0.86219 | ------ | 0.876 K/in | 0.862 K/in | -1.60 |
| 2b | -1.0955 | 1.1014 | 1.12 K/in | 1.10 K/in | -1.78 |
| 2c | -0.85254 | 1.3972 | 1.42 K/in | 1.40 K/in | -1.40 |
| 3a | -3.0188 | ------ | 339 Kips | 302 Kips | -10.91 |
| 3b | -0.60331 | ------ | 61 Kips | 60.3 Kips | -1.14 |
| 4a | -4.1298 | 4.1328 | 420 Kips | 413 Kips | -1.67 |
| 4b | -2.5773 | 2.5766 | 2.578 K/in | 2.577 K/in | -0.30 |
| 5a | -7.3195 | 7.3341 | 724 Kips | 733 Kips | +1.24 |
| 5b | -3.5387 | 3.5516 | 3.574 K/in | 3.55 K/in | -0.67 |
| 6 | -5.1857 | 5.1982 | $11000^{k-in}$ | $10920^{k-in}$ | -0.70 |

*Shift = the absolute value of eigenvalue $\lambda_1$.

A subroutine in the BASP computer program uses the Calcomp Plotter to graph the buckled mode shape of the beams. These graphs show the normalized lateral displacements of the top and bottom flanges and the centroidal axis. The buckled mode shapes of the thirteen examples are presented in Appendix III.

An IBM 370/158 computer system was used to solve all the numerical examples presented in this report. Table 3 is included to give an indication of the running time and costs involved in working these problems.

Three different output options may be selected when running the BASP computer program. The medium and short output for the beam subjected to constant moment (problem 6) are given in Appendices IV and V.

Table 3 - Running Time and Costs for Numerical Examples

| Problem | Running Time (Seconds) | | Cost (Dollars) | |
|---|---|---|---|---|
| | A | B* | A | B* |
| 1a | 52.76 | ----- | 14.57 | ----- |
| 1b | 57.63 | 45.79 | 15.91 | 13.18 |
| 1c | 51.10 | 51.61 | 14.66 | 14.80 |
| 2a | 55.94 | ----- | 15.88 | ----- |
| 2b | 58.33 | 45.72 | 16.08 | 13.16 |
| 2c | 52.33 | 48.02 | 14.96 | 13.83 |
| 3a | 54.85 | ----- | 15.11 | ----- |
| 3b | 62.16 | ----- | 17.60 | ----- |
| 4a | 53.00 | 46.60 | 15.19 | 13.40 |
| 4b | 53.39 | 45.14 | 15.29 | 13.03 |
| 5a | 55.31 | 47.00 | 15.72 | 13.48 |
| 5b | 52.85 | 45.57 | 15.01 | 13.15 |
| 6 | 58.06 | 58.94 | 16.01 | 17.22 |

*Shift = the absolute value of eigenvalue $\lambda_1$.

# CHAPTER V - CONCLUSIONS

1. The finite element procedure and a computer program for beam buckling have been presented. Comparison of the results of the BASP computer program to the results computed from Timoshenko's equations show that excellent accuracy may be obtained.
2. The BASP computer program may be applied to simply supported wide-flange beams with stiffeners, lateral restraints, and various rotational restraints against lateral buckling.
3. The point of application of the load over the depth of beam, as it influences buckling, may be determined as evidenced by problems 1 and 2. It is interesting to note that the critical load is less when applied on the compression flange than when applied on the tension flange.
4. Applying a lateral constraint on a beam will give higher buckling load than a beam without lateral constraint. This result is shown in problems 4 and 5.
5. The BASP computer program is limited to linear elastic buckling. Lateral torsional buckling was evident in all examples presented.
6. The results of the BASP computer program compare favorably with the results of another computer program (18) where the beam is idealized as an assemblage of two-dimensional finite elements as shown in Fig. 1a. The advantages of the BASP computer program is that it uses a one-dimensional beam element for the flanges which simplifies the coding required and the computation time for a given problem is reduced.

# REFERENCES


1. Akay, H., "Buckling and Post-Buckling Analysis of Stiffened Plates by Finite Element Method," Ph.D. Dissertation, The University of Texas at Austin, 1972.

2. Clough, R. W., "The Finite Element Method in Structural Mechanics," Stress Analysis, O. C. Zienkiewicz and G. S. Holister, Eds., John Wiley and Sons, 1965.

3. Clough, R. W. and Tocher, J. L., "Finite Element Stiffness Matrices for the Analysis of Plate Bending," Proceedings, Conference on Matrix Methods in Structural Mechanics, Air Force Institute of Technology, Wright-Patterson Air Force Base, Ohio, Oct., 1965 (AFFDL-TR-66-60., Nov., 1966).

4. Clough, R. W. and Johnson, C. P., "A Finite Element Approximation for the Analysis of Thin Shells," International Journal of Solids Structures, Vol. 4, Pergamon Press, Inc., London, England, 1968.

5. Clough, R. W. and Felippa, Carlso, A., "A Refined Quadrilateral Element for Analysis of Plate Bending," Proceedings, Second Conference on Matrix Method in Structural Mechanics, Air Force Institute of Technology, Wright-Patterson Air Force Base, Ohio, 1968 (AFFDL-TR-68-150., Dec., 1969).

6. Doherty, W. P., Wilson, E. L., and Taylor, R. L., "Stress Analysis of Axisymmetric Solids Utilizing Higher-Order Quadrilateral Elements," Structural Engineering Laboratory Report No. 69-3, University of California, Berkeley, Jan., 1969.

7. Felippa, C. A., "Refined Finite Element Analysis of Linear and Non-linear Two-Dimensional Structures," Structural Engineering Laboratory Report No. 66-22, University of California, Berkeley, California, 1966.

8. Gallagher, R. H., et al., "A Discrete Element Procedure for Thin Shell Instability Analysis," AIAA Journal, American Institute for Aeronautics and Astronautics, Vol. 5, No. 1, Jan., 1967.

9. Gallagher, R. H., "Analysis of Plate and Shell Structures," Proceedings, Symposium on Application of Finite Element Methods in Civil Engineering, Vanderbilt University, Nashville, Tenn., 1969.

10. Gallagher, R. H. and Barsoum, R. S., "Finite Element Analysis of Torsional and Torsional-Flexural Stability Problems," International Journal of Numerical Methods in Engineering, Vol. 2, 1970.

11. Gallagher, R. H., Finite Element Analysis--Fundamentals, Prentice-Hall Book Company, Englewood Cliffs, New Jersey, 1975.

12. Javier Horvilleur, "Stability of Beams and Columns", Master's Report, University of Texas at Austin.

13. Johnson, C. P., "The Analysis of Thin Shells by a Finite Element Procedure," Structural Engineering Laboratory Report No. 67-22, University of California, Berkeley, Calif., 1967.

14. Johnson, C. P., "The Analysis of Thin Shells by a Finite Element Procedure," Structural Engineering Laboratory Report No. 67-22, University of California, Berkeley, California, 1967.

15. Johnson, C. P. and Smith, P. G., "A Computer Program for the Analysis of Thin Shells," Structural Engineering Laboratory Report No. 69-5, University of California, Berkeley, California, 1969.

16. Johnson, C. P., "Lateral Buckling of Compression Flanges in Rigid Frame Structures by a Finite Element Method," Report to University of Arkansas and Varco-Pruden, May, 1969.

17. Johnson, C. P., "Lateral Buckling of Rigid Frames by Finite Element Procedure," CESM Report No. 72-1, Department of Civil Engineering, The University of Texas, Austin, Tex., Jan., 1972.

18. Johnson, C. P. and Will, K. M., "Beam Buckling by Finite Element Procedure," Journal of Structural Division, ASCE, Vol. 100, No. ST3, March, 1974.

19. Kapur, K. K. and Hartz, B. J., "Stability of Plates Using the Finite Element Method," Journal of the Engineering Mechanics Division, ASCE, Vol. 92, No. EM2, Proc. Paper 4790, Apr., 1966, pp. 177-196.

20. Martin, H. C., "On the Derivation of Stiffness Matrices for the Analysis of Large Deflection and Stability Problems," Proceedings, First Conference of Matrix Methods in Structural Mechanics, Air Force Institute of Technology, Wright-Patterson Air Force Base, Ohio, Oct., 1965 (AFFDL-TR-66-80, Nov., 1966).

21. Martin, H. C., Introduction to Matrix Methods of Structural Analysis, McGraw-Hill Book Company, 1966.

22. Oral, Oktay, Finite Element Method--Basic Concepts and Applications, Intext Educational Publishers, New York, 1973.

23. Plotter Guide, Computing Center, Kansas State University, 1976.

24. Powell, G. and Klingner, R., "Elastic Lateral Buckling of Steel Beams," Journal of the Structural Division, ASCE, Vol. 96, No. ST9, Proc. Paper 7555, Sept., 1970, pp. 1919-1932.

25. Timoshenko, S. P. and Gere, J. M., Theory of Elastic Stability, McGraw-Hill Book Company, New York, 1961.

26. Turner, M. J., et al., "Stiffness and Deflection Analysis of Complex Structures," Journal of Aeronautical Science, Vol. 23, No. 9, 1956.

27. Will, K. M., "Elastic Instability Analysis of Thin-Walled Structures by a Refined Finite Element Procedure, CESM Report No. 71-2, Department of Civil Engineering, The University of Texas at Austin, 1971.

28. Winter, G., "Lateral Bracing of Beams and Columns," Trans. ASCE, Vol. 125, 1960, p. 807.

29. Yura, J. A. and Oral, A., "Web Behavior at Points of Concentrated Loads in Steel Beams," AISI Project No. 157, Report No. 4, Department of Civil Engineering, The University of Texas, Austin, Tex., Sept., 1972.

30. Zienkiewicz, O. C., The Finite Element Method in Engineering Science, McGraw-Hill Book Co., Inc., London, England, 1971.

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# APPENDIX I - THE DIRECT STIFFNESS METHOD

(a) Derivation of element force-displacement equation

For each element, relationships between forces and displacements at the node points can be derived in matrix form referred to the local coordinates:

$$\{f\} = [\bar{K}]\{\delta\}$$

where $\{f\}$ and $\{\delta\}$ are column matrices of forces and displacements respectively at the nodes, and $[\bar{K}]$ is a square, element stiffness matrix referred to the local coordinate axes.

(b) Transformation of the element stiffness matrix

Each element stiffness matrix is then transformed from the local coordinate system to the global coordinate system of the complete structure (21) by the transformation equation.

$$[K] = [T]^T[K][T] \qquad \text{(I-1)}$$

where $[K]$ is the element stiffness matrix referred to the global coordinates, $[T]$ is the transformation matrix relating the local coordinates and the global coordinates, and $[T]^T$ is the transpose of $[T]$.

(c) Assembly of the total stiffness matrix

The element stiffness matrix can be expressed as

$$[K^1] = \begin{bmatrix} K^1_{ii} & K^1_{ij} \\ K^1_{ji} & K^1_{jj} \end{bmatrix} \qquad \text{(I-2)}$$

for each element. The superscripts refer to the element number and the subscripts are referred to the joint name as shown in Fig. I-1. For the entire structure, the total stiffness matrix is obtained by superposition of each of the element stiffness matrices which gives

$$K = \begin{bmatrix} K^1_{ii} & K^1_{ji} & 0 & 0 & 0 \\ K^1_{ji} & K^1_{jj} + K^2_{jj} + K^3_{jj} + K^4_{jj} & K^2_{jk} & K^3_{jl} & K^4_{jm} \\ 0 & K^2_{kj} & K^2_{kk} & 0 & 0 \\ 0 & K^3_{lj} & 0 & K^3_{ll} & 0 \\ 0 & K^4_{mj} & 0 & 0 & K^4_{mm} \end{bmatrix} \tag{I-3}$$

It is easy to visualize that member 2 connecting joints j and k will only influence the equilibrium equations of these two joints, and therefore will only affect the elements $K_{jj}$, $K_{jk}$, $K_{kj}$, and $K_{kk}$ of the complete structure stiffness matrix.

The force-displacement relation for the entire structure is then

$$\{f\} = [K]\{U\} + \{F_o\} \tag{I-4}$$

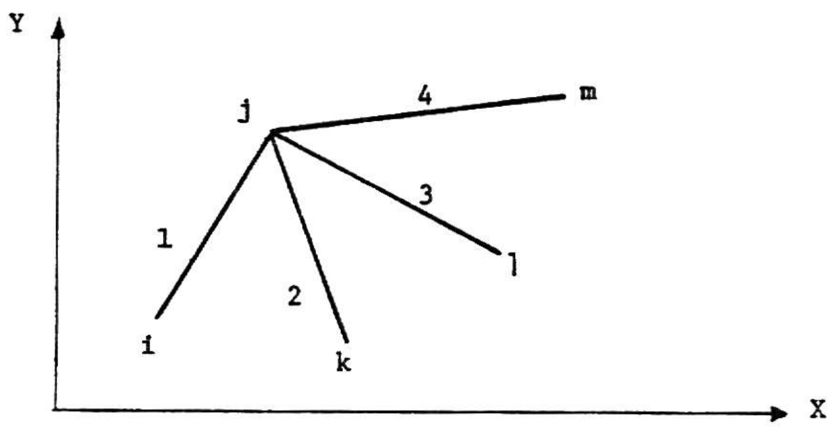


Fig. I-1 Element Assembly

where $\{F\}$ is the column matrix of applied external loads including reactions, $F_o$ is the column matrix of nodal forces introduced to maintain the initial structural shape in the presence of thermal gradients or equivalent effects, $[K]$ is the total stiffness matrix, and $\{U\}$ is the column matrix of actual nodal displacement of whole structure.

(d) Determination of nodal displacements

The complete structural stiffness matrix $[K]$ is singular and cannot be inverted. However, the set of Eqs. (I-4) can be arranged and partitioned in the form

$$\left\{\begin{array}{c} F' \\ \hline R \end{array}\right\} = \left[\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right] \left\{\begin{array}{c} U' \\ \hline C \end{array}\right\} \qquad \text{(I-5)}$$

where $\{F'\}$ are the specified loads applied to the structure, $\{R\}$ are the reactions at points of support or constrained deflections, $\{U'\}$ are the unknown displacements, and $\{C\}$ are the specified displacements (zero for undeformed support points). For the case where thermal gradients are present, the column matrix of left side of Eqs. (I-4) includes $F_o$, i.e., $F' = F - F_o$. The unknown displacements may then be found as

$$\{U'\} = [K_{11}]^{-1} (\{F'\} - [K_{12}]\{C\}) \qquad \text{(I-6)}$$

The reactions can be obtained in terms of the nodal displacements as,

$$\{R\} = [K_{21}]\{U'\} + [K_{22}]\{C\} \qquad \text{(I-7)}$$

(e) Determination of stresses

In each element, stresses (or internal forces) may be found directly from the nodal displacements (21):

$$\{\sigma\} = [S]\{\delta\} \qquad \text{(I-8)}$$

where $[S]$ is the stress matrix for the elements.

# APPENDIX II - STATIC CONDENSATION PROCEDURE

The term condensation refers here to the contraction in size of a system of equations by elimination of certain DOF. The condensed equations are to be expressed in terms of preselected DOF $\{\Delta_c\}$ that are to be retained, together with the surplus DOF $\{\Delta_b\}$, to comprise the total original DOF set. The original equations of the form

$$\left[\begin{array}{c|c} K_{bb} & K_{bc} \\ \hline K_{cb} & K_{cc} \end{array}\right] \left\{\begin{array}{c} \Delta_b \\ \hline \Delta_c \end{array}\right\} = \left\{\begin{array}{c} F_b \\ \hline F_c \end{array}\right\} \tag{II-1}$$

are to be condensed to the form

$$\left[\hat{K}_{cc}\right]\{\Delta_c\} = \{\hat{F}_c\} \tag{II-2}$$

An approach to condensation is adopted that employs the concept of a coordinate transformation. Thus, the objective is to construct the relationships

$$\left\{\begin{array}{c} \Delta_b \\ \Delta_c \end{array}\right\} = [T_o]\{\Delta_c\} \tag{II-3}$$

where $[T_o]$ is the desired transformation matrix. To do so we first solve the upper partition of Eq. (II-1):

$$\{\Delta_b\} = - [K_{bb}]^{-1} [K_{bc}]\{\Delta_c\} + [K_{bb}]^{-1} \{F_b\} \tag{II-4}$$

Since the second term on the right-hand side is constant for given loads, the stiffness relationship between the DOF ($\{\Delta_c\}$ and $\{\Delta_b\}$) is given by $-[K_{bb}]^{-1} [K_{bc}]$. Noting also that $\{\Delta_c\} = [I]\{\Delta_c\}$, the following transformation of coordinates can be written:

$$\left\{\begin{array}{c} \Delta_b \\ \Delta_c \end{array}\right\} = \left[\begin{array}{c} -[K_{bb}]^{-1} [K_{bc}] \\ \hline I \end{array}\right] \{\Delta_c\} = [T_o]\{\Delta_c\}$$

Applying this to Eq. (II-1) in the manner of a conventional coordinate transformation, $\hat{K}_{cc}$ and $\hat{F}_c$ of Eq. II-2 becomes

$$[\hat{K}_{cc}] = \Big[[K_{cc}] - [K_{cb}][K_{bb}]^{-1}[K_{bc}]\Big] \qquad \text{(II-6)}$$

$$\{\hat{F}_c\} = [T_o]^T \begin{Bmatrix} F_b \\ F_c \end{Bmatrix} = \{F_c\} - [K_{cb}]\,[K_{bb}]^{-1}\,\{F_b\} \qquad \text{(II-7)}$$

Note that this transformation, constructed on the basis of the relationship between the DOF alone, serves also to transform the right-hand side vector.

These results would, of course, have followed directly by substitution of Eq. (II-4) into the lower partition of Eq. (II-1). However, the notion of condensation by means of a transformation of the DOF ($[T_0]$) proves extremely valuable in dynamic and elastic stability analyses and may even prove convenient from a programming standpoint in a linear static analysis.

# APPENDIX III

## THE NORMALIZED BUCKLED SHAPE OF PROBLEMS

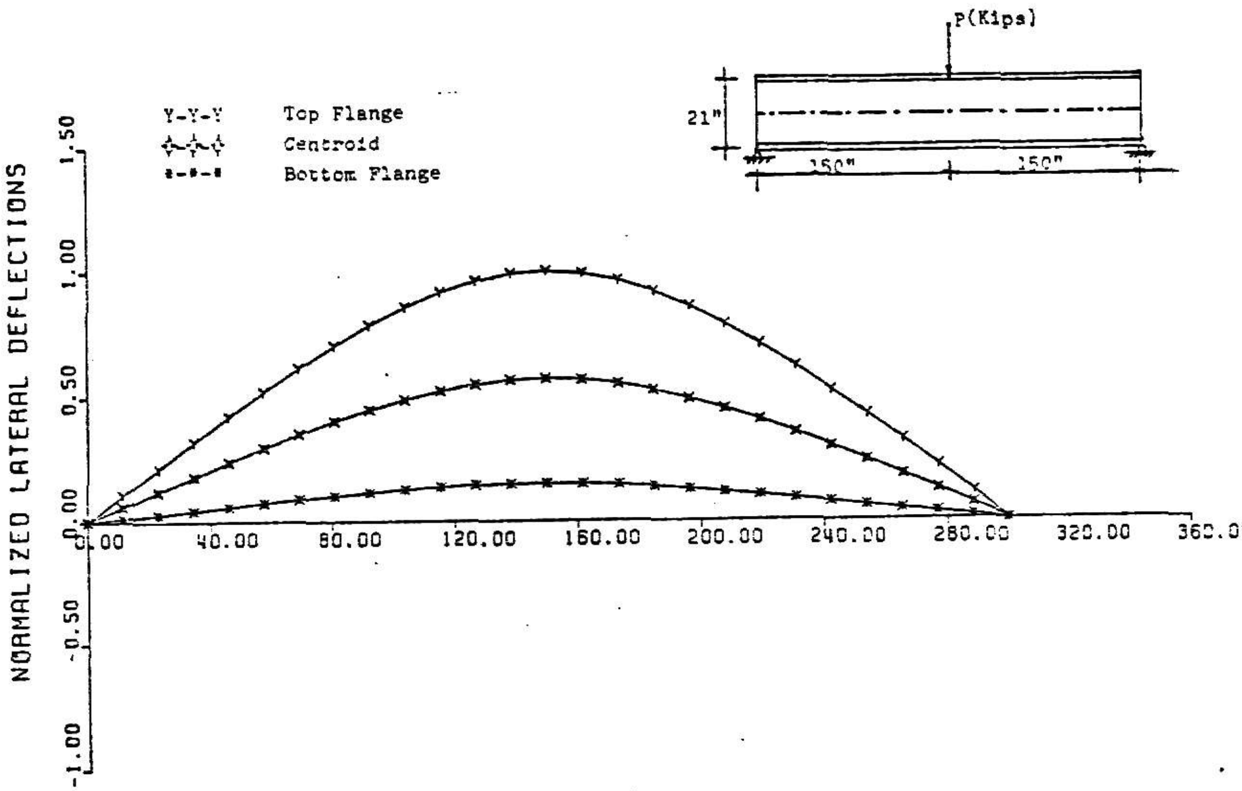


Fig. III-1a - The Normalized Buckled Shape of Problem 1a

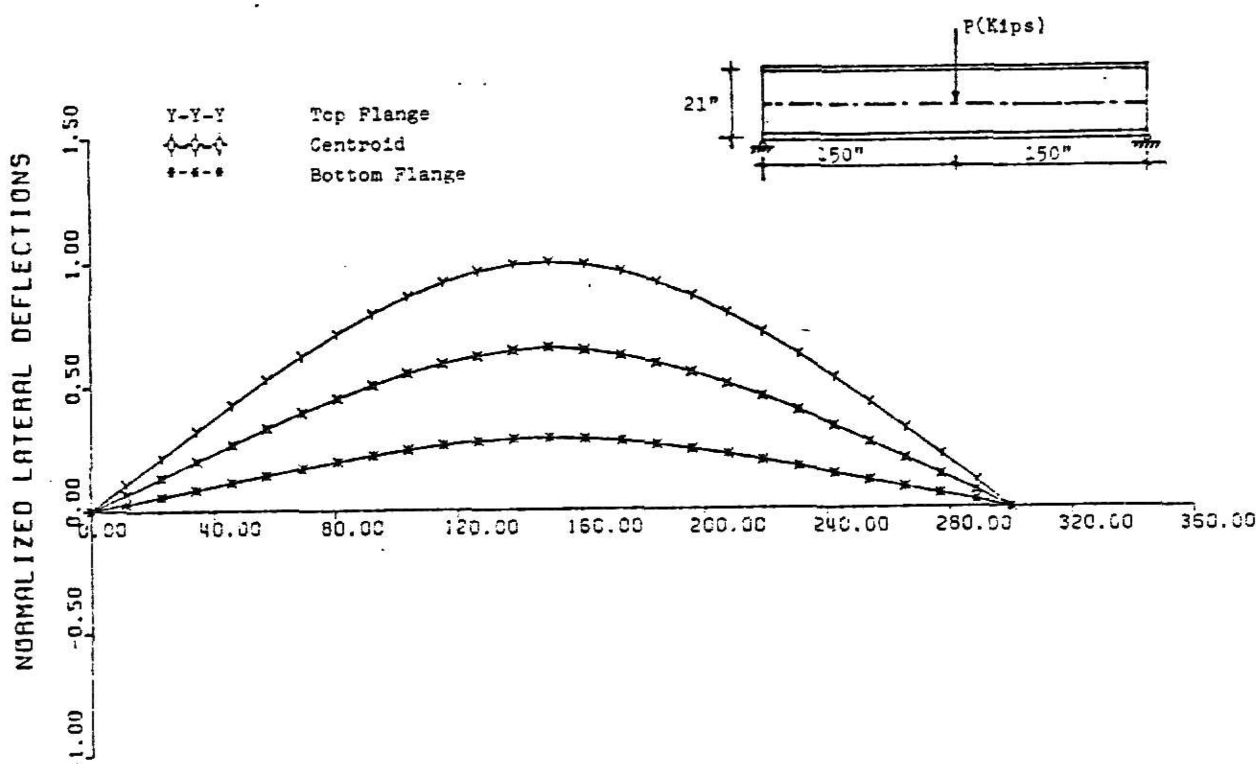


Fig. III-1b - The Normalized Buckled Shape of Problem 1b

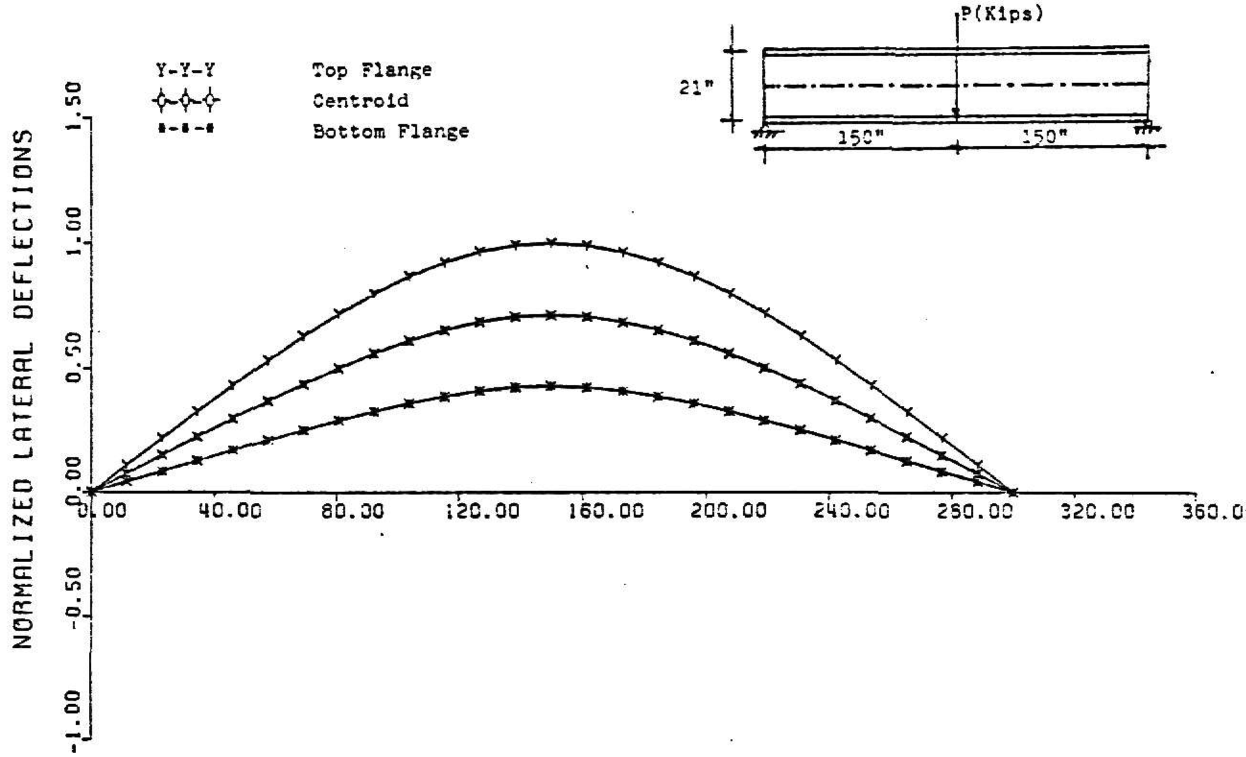


Fig. III-1c - The Normalized Buckled Shape of Problem 1c

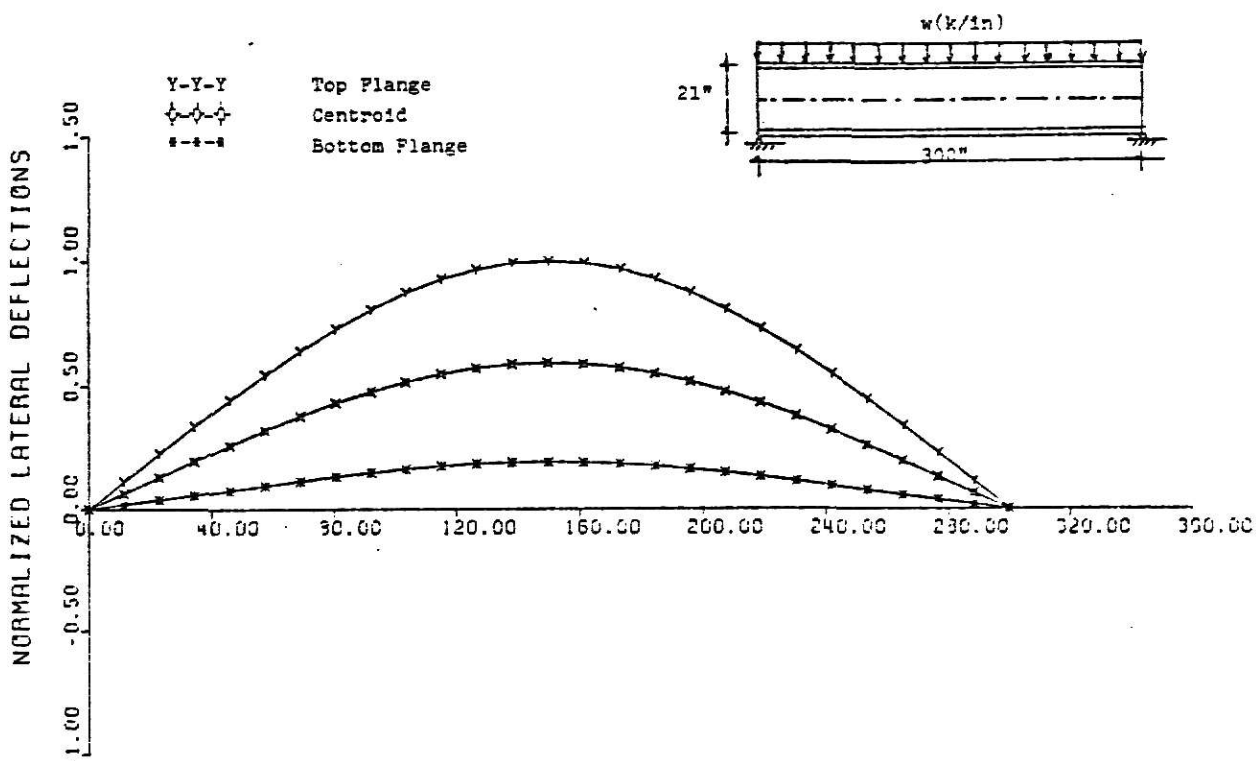


Fig. III-2a - The Normalized Buckled Shape of Problem 2a

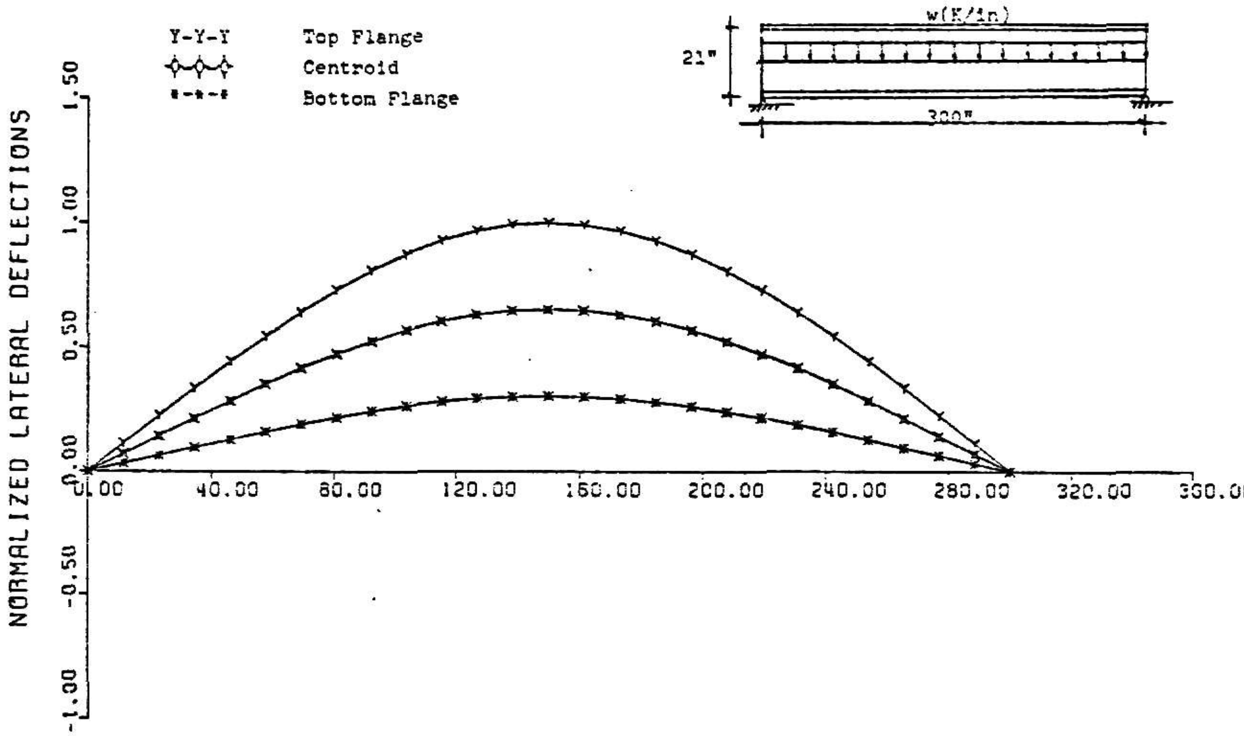


Fig. III-2b - The Normalized Buckled Shape of Problem 2b

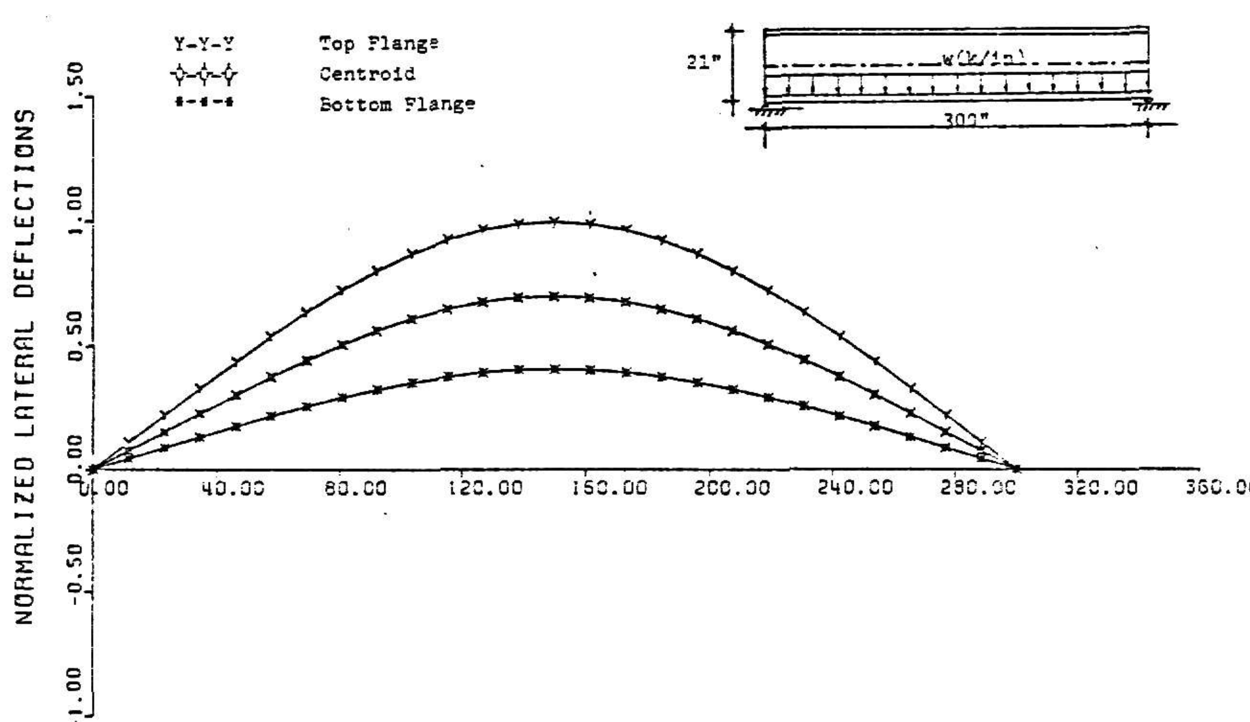


Fig. III-2c - The Normalized Buckled Shape of Problem 2c

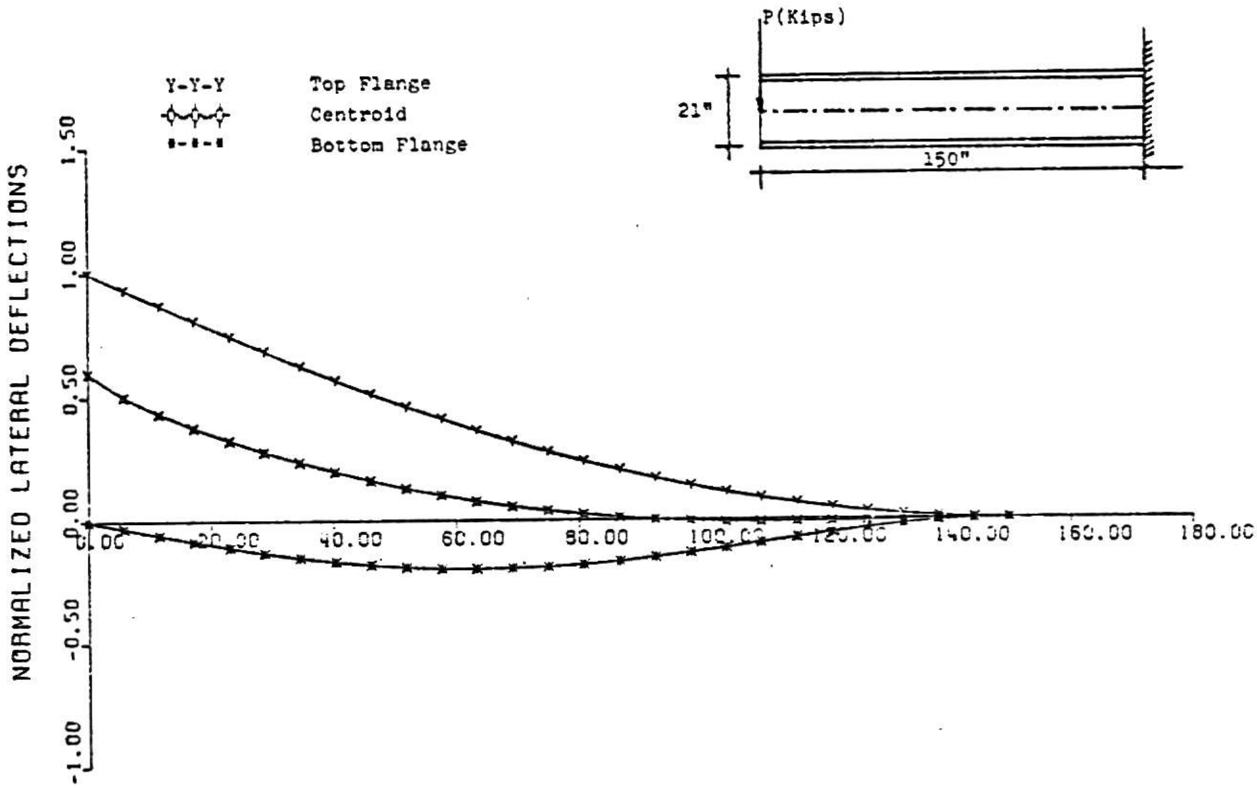


Fig. III-3a - The Normalized Buckled Shape of Problem 3a

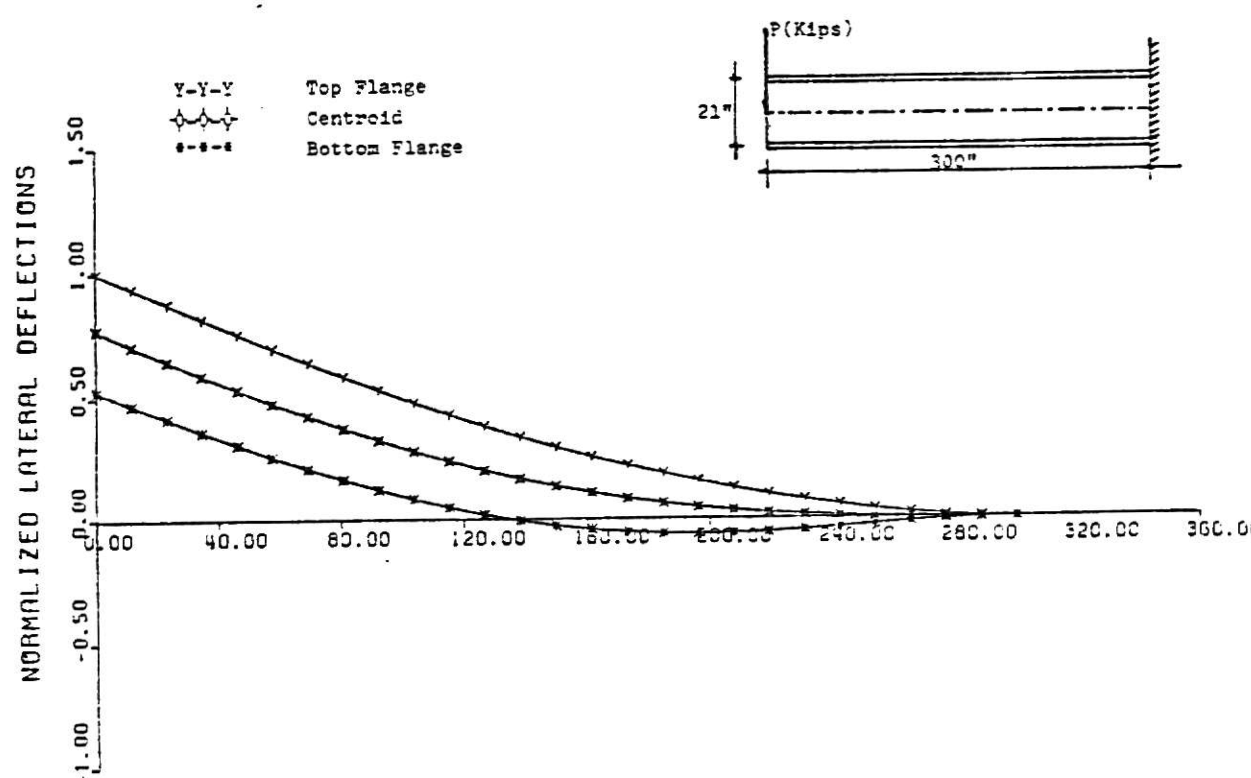


Fig. III-3b - The Normalized Buckled Shape of Problem 3b

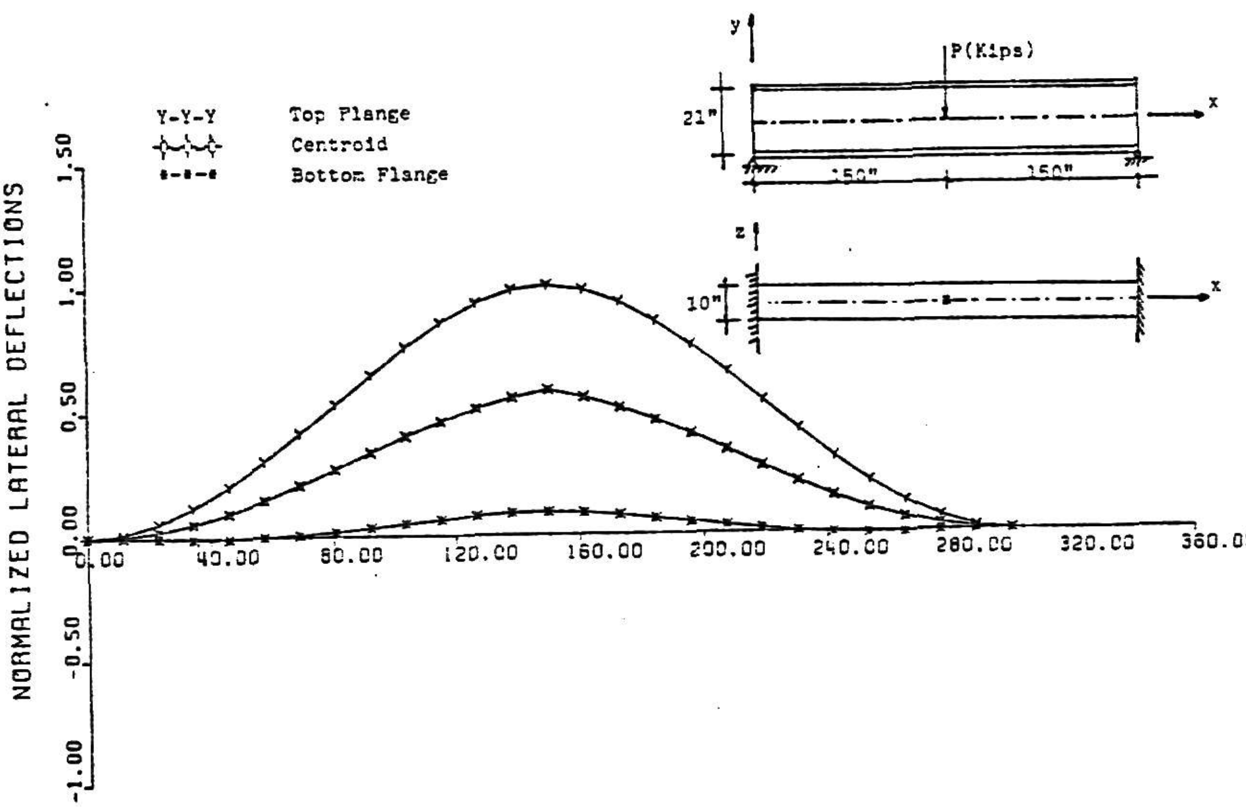


Fig. III-4a - The Normalized Buckled Shape of Problem 4a

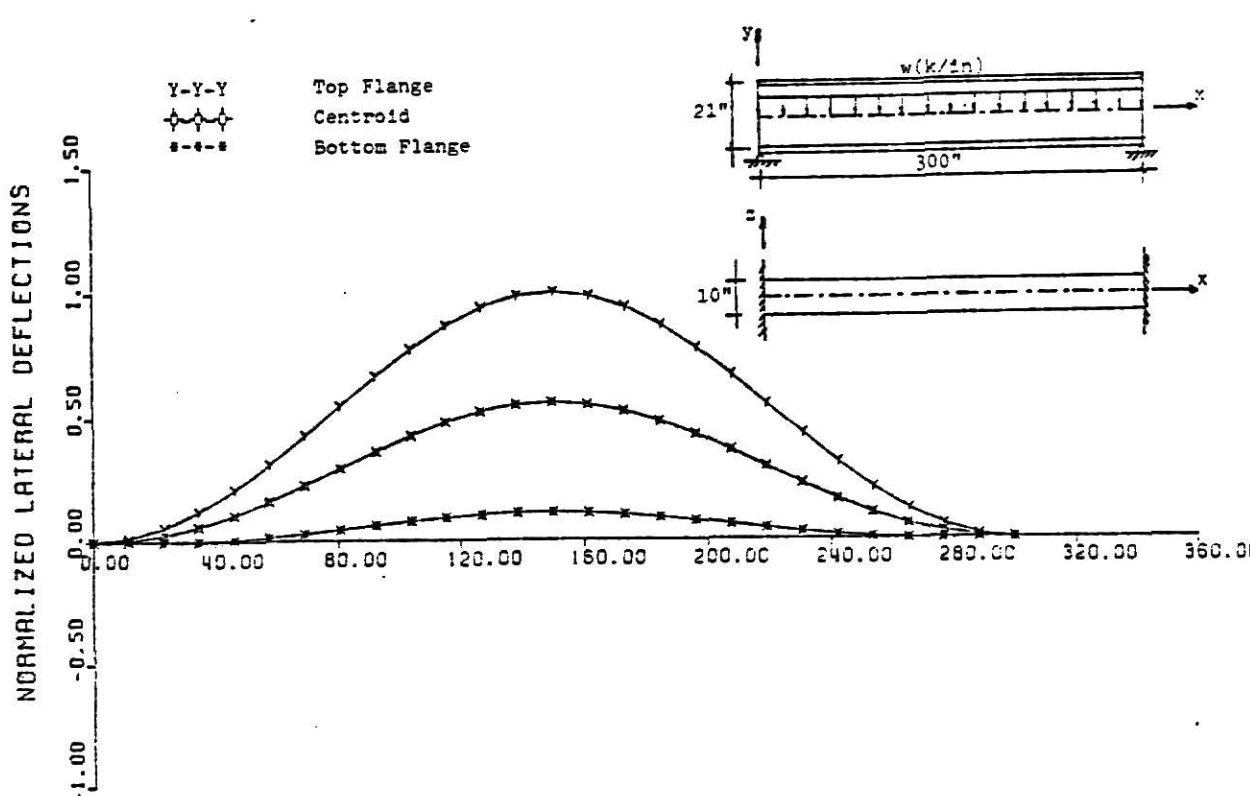


Fig. III-4b - The Normalized Buckled Shape of Problem 4b

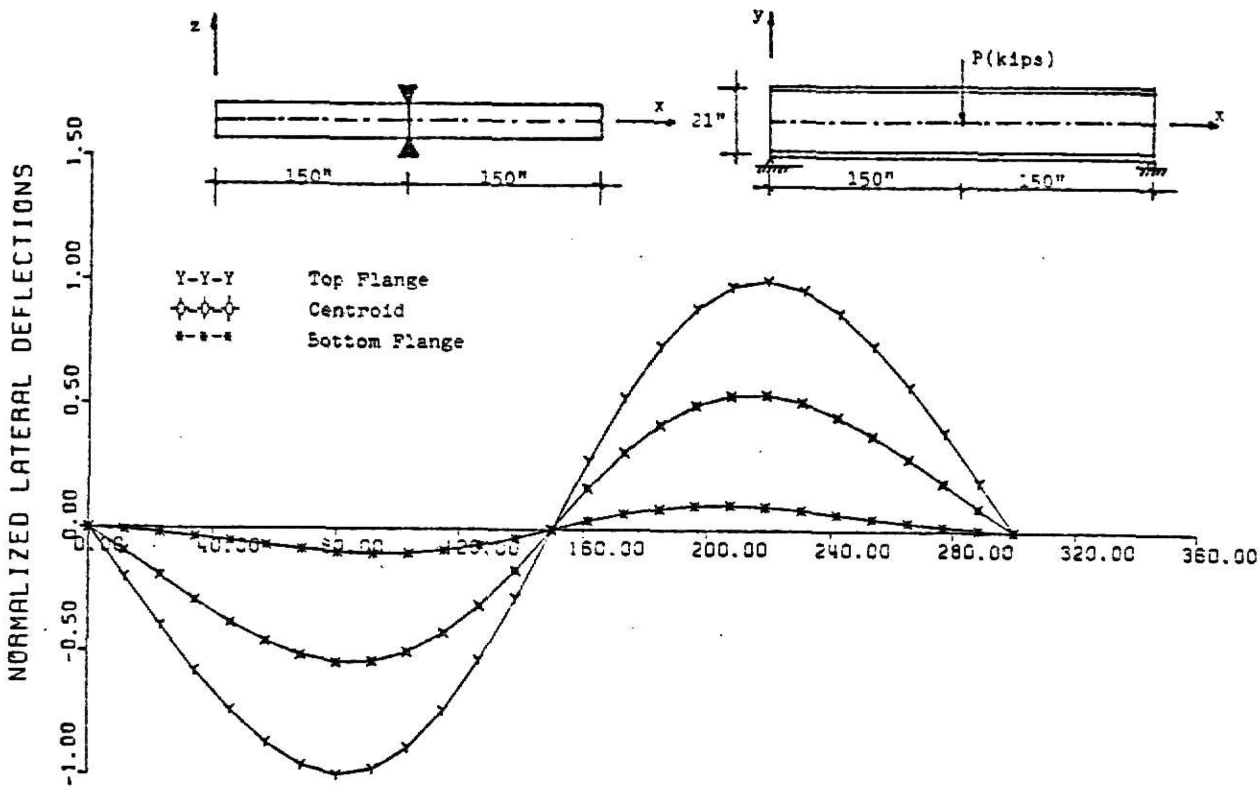


Fig. III-5a - The Normalized Buckled Shape of Problem 5a

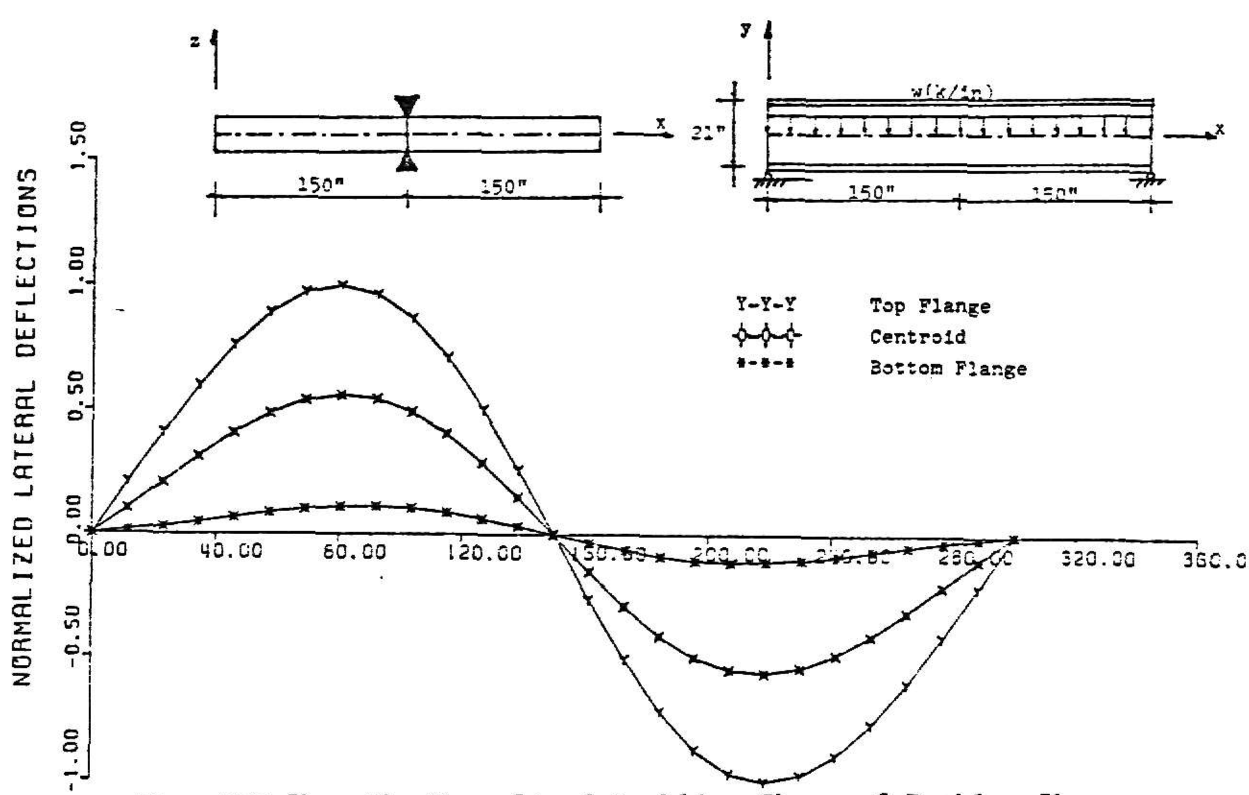


Fig. III-5b - The Normalized Buckling Shape of Problem 5b

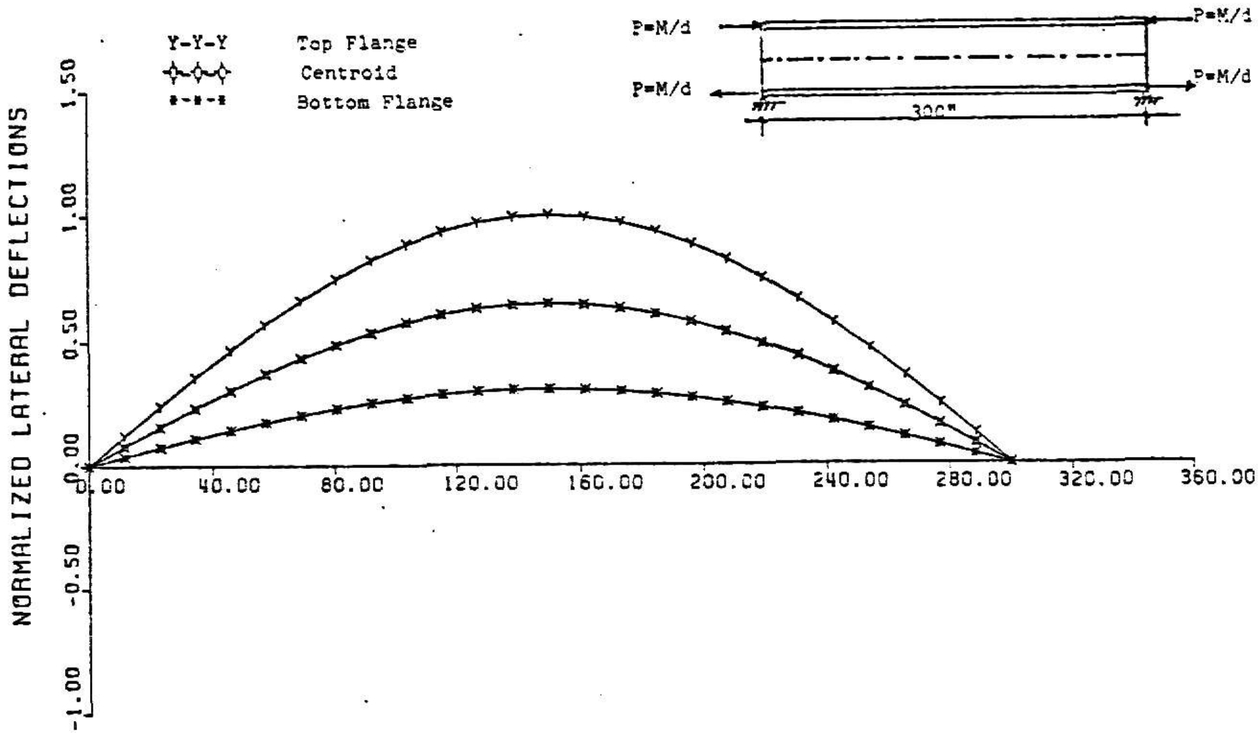


Fig. III-6 - The Normalized Buckled Shape of Problem 6

# APPENDIX IV

## INPUT DATA AND COMPUTER OUTPUT OF PROBLEM 6 (Medium Output)

FIELD IDENTIFICATION

| | 1–10 | 11–20 | 21–30 | 31–40 | 41–50 | 51–60 | 61–70 | 71–80 |
|---|---|---|---|---|---|---|---|---|
| 1 | BASP .... | | | | | | | |
| 2 | EXAMPLE PR | OBLEM.... | CHE | CKED BY | Y.H.HWANG | | | |
| 3 | 113 | SS WF BEAM | UNDER UNI | FORM MOMEN | T (WHOLE B | EAM) 2X26 | DIVISIONS | |
| 4 | 52 52 | 81 4 | 2 0 | 6 0 | 1 | | | |
| 5 | 30000.000 | 0.300 | 0 | 2 | | | | |
| 6 | 1 79 | 3 | 0. | -10.500 | 300. | -10.500 | | |
| 7 | 2 80 | 3 | 0. | 10.500 | 300. | 0.000 | | |
| 8 | 3 81 | 3 | 0. | 10.500 | 300. | 10.500 | | |
| 9 | 1 51 | 3 2 | 1 4 | 5 2 | | | | |
| 10 | 2 52 | 3 2 | 2 5 | 6 3 | | | | |
| 11 | 1 51 | 3 2 | 1 4 | | | | | |
| 12 | 2 52 | 3 2 | 3 6 | | | | | |
| 13 | 1 52 | 1 | 1.0000 | | | | | |
| 14 | 1 52 | 1 | 1.0000 | 10.000 | | | | |
| 15 | 1 | | -100.100 | | | | | |
| 16 | 3 | | 100.100 | | | | | |
| 17 | 79 | | 100.100 | | | | | |
| 18 | 81 | | -100.100 | | | | | |
| 19 | 1 0 | 0 1 | 1 | 0.000 | 0.000 | | | |
| 20 | 79 0 | 0 0 | 1 | | 0.000 | | | |

FIELD IDENTIFICATION

| | 1-10 | 11-20 | 21-30 | 31-40 | 41-50 | 51-60 | 61-70 | 71-80 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 3 | 1 1 | 1 0 | | | | | |
| 2 | 7.9 8.1 | 1 1 | 1 0 | | | | | |
| 3 | 1 8.1 | 1 | 1. | 1. | 1. | | | |
| 4 | 3 1.0 | 0. 8 | 0.10 | | | | | |
| 5 | | | | | | | | |
| 6 | 1.* | | | | | | | |
| 7 | | | | | | | | |
| 8 | | | | | | | | |
| 9 | | | | | | | | |
| 10 | | | | | | | | |
| 11 | | | | | | | | |
| 12 | | | | | | | | |
| 13 | | | | | | | | |
| 14 | | | | | | | | |
| 15 | | | | | | | | |
| 16 | | | | | | | | |
| 17 | | | | | | | | |
| 18 | | | | | | | | |
| 19 | | | | | | | | |
| 20 | | | | | | | | |

```
BASP  ....
EXAMPLE PROBLEM....           CHECKED  BY  Y.H.HWANG.


PRCB
  113     SS WF BEAM UNDER UNIFORM MOMENT (WHCLE BEAM) 2X26 DIVISICNS


NUM OF ELEMENTS                     52
NUM OF STIFF ELEMENTS               52
NUM OF POINTS                       81
NUM CF LCADED POINTS                 4
NUM OF MEMBRANE BCUNDARY PTS         2
NUM OF ELAST SPRINGS(IN-PLANE)       0
NUM OF BENDING BCUNDARY PTS          6
NUM OF ELAST SPRINGS(BUCKLING)       0


NOTE... SHCRT OUTPUT CPTICN IS USED ....


NOTE... GECMETRY PLCT CPTICN      =      0
    ... STRESS PLCT OPTION       =     0
    ... BUCKLED SHAPE PLCT OPTICN =      2


MOD OF ELASTICITY           0.300E 05
PCISSCNS RATIO              0.300E 00


 INPUT... NODAL POINT CCORDINATES ....
 FROM..TC..INCR...X1..Y1..X2..Y2..

   1   79   3    0.0        -0.105E 02    0.300E 03   -C.105E 02
   2   80   3    0.0         0.0          0.300E 03    C.C
   3   81   3    0.0         0.105E 02    0.300E 03    0.105E 02

 INPUT.. NODAL POINT NUMBER SYSTEM FOR QUADS ...
 FROM ELEM..TO ELEM..NODE INCR..ELEM INCR...I1,I2,I3,I4...

   1   51   3    2    1    4    5    2
   2   52   3    2    2    5    6    3

 INPUT... NODAL PCINT NUMBER SYSTEM FOR BEAMS....
 FROM ELEM..TC ELEM..NCDE INCR..ELEM INCR...I1,I2..
```

| | | | | | |
|---|---|---|---|---|---|
| 1 | 51 | 3 | 2 | 1 | 4 |
| 2 | 52 | 3 | 2 | 3 | 6 |

INPUT... PROPERTIES OF QUADS .....
FROM ELEM...TO ELEM...INCR....THICKNESS...

| | | | |
|---|---|---|---|
| 1 | 52 | 1 | 0.100E 01 |

INPUT...PROPERTIES OF BEAM ELEM....
FROM ELEM...TO ELEM...INCR....THICKNESS...WIDTH...

| | | | | |
|---|---|---|---|---|
| 1 | 52 | 1 | 0.100E 01 | 0.100E 02 |

| | X-COORDINATE | Y-COORDINATE |
|---|---|---|
| 1 | 0.0 | -0.105E 02 |
| 2 | 0.0 | 0.0 |
| 3 | 0.0 | 0.105E 02 |
| 4 | 0.115E 02 | -0.105E 02 |
| 5 | 0.115E 02 | 0.0 |
| 6 | 0.115E 02 | 0.105E 02 |
| 7 | 0.231E 02 | -0.105E 02 |
| 8 | 0.231E 02 | 0.0 |
| 9 | 0.231E 02 | 0.105E 02 |
| 10 | 0.346E 02 | -0.105E 02 |
| 11 | 0.346E 02 | 0.0 |
| 12 | 0.346E 02 | 0.105E 02 |
| 13 | 0.462E 02 | -0.105E 02 |
| 14 | 0.462E 02 | 0.0 |
| 15 | 0.462E 02 | 0.105E 02 |
| 16 | 0.577E 02 | -0.105E 02 |
| 17 | 0.577E 02 | 0.0 |
| 18 | 0.577E 02 | 0.105E 02 |
| 19 | 0.692E 02 | -0.105E 02 |
| 20 | 0.692E 02 | 0.0 |
| 21 | 0.692E 02 | 0.105E 02 |
| 22 | 0.808E 02 | -0.105E 02 |
| 23 | 0.808E 02 | 0.0 |
| 24 | 0.808E 02 | 0.105E 02 |
| 25 | 0.923E 02 | -0.105E 02 |
| 26 | 0.923E 02 | 0.0 |
| 27 | 0.923E 02 | 0.105E 02 |
| 28 | 0.104E 03 | -0.105E 02 |
| 29 | 0.104E 03 | 0.0 |
| 30 | 0.104E 03 | 0.105E 02 |
| 31 | 0.115E 03 | -0.105E 02 |
| 32 | 0.115E 03 | 0.0 |
| 33 | 0.115E 03 | 0.105E 02 |
| 34 | 0.127E 03 | -0.105E 02 |
| 35 | 0.127E 03 | 0.0 |
| 36 | 0.127E 03 | 0.105E 02 |
| 37 | 0.138E 03 | -0.105E 02 |
| 38 | 0.138E 03 | 0.0 |
| 39 | 0.138E 03 | 0.105E 02 |

| | | |
|---|---|---|
| 40 | 0.150E 03 | -0.105E 02 |
| 41 | 0.150E 03 | 0.0 |
| 42 | 0.150E 03 | 0.105E 02 |
| 43 | 0.162E 03 | -0.105E 02 |
| 44 | 0.162E 03 | 0.0 |
| 45 | 0.162E 03 | 0.105E 02 |
| 46 | 0.173E 03 | -0.105E 02 |
| 47 | 0.173E 03 | 0.0 |
| 48 | 0.173E 03 | 0.105E 02 |
| 49 | 0.185E 03 | -0.105E 02 |
| 50 | 0.185E 03 | 0.0 |
| 51 | 0.185E 03 | 0.105E 02 |
| 52 | 0.196E 03 | -0.105E 02 |
| 53 | 0.196E 03 | 0.0 |
| 54 | 0.196E 03 | 0.105E 02 |
| 55 | 0.208E 03 | -0.105E 02 |
| 56 | 0.208E 03 | 0.0 |
| 57 | 0.208E 03 | 0.105E 02 |
| 58 | 0.219E 03 | -0.105E 02 |
| 59 | 0.219E 03 | 0.0 |
| 60 | 0.219E 03 | 0.105E 02 |
| 61 | 0.231E 03 | -0.105E 02 |
| 62 | 0.231E 03 | 0.0 |
| 63 | 0.231E 03 | 0.105E 02 |
| 64 | 0.242E 03 | -0.105E 02 |
| 65 | 0.242E 03 | 0.0 |
| 66 | 0.242E 03 | 0.105E 02 |
| 67 | 0.254E 03 | -0.105E 02 |
| 68 | 0.254E 03 | 0.0 |
| 69 | 0.254E 03 | 0.105E 02 |
| 70 | 0.265E 03 | -0.105E 02 |
| 71 | 0.265E 03 | 0.0 |
| 72 | 0.265E 03 | 0.105E 02 |
| 73 | 0.277E 03 | -0.105E 02 |
| 74 | 0.277E 03 | 0.0 |
| 75 | 0.277E 03 | 0.105E 02 |
| 76 | 0.288E 03 | -0.105E 02 |
| 77 | 0.288E 03 | 0.0 |
| 78 | 0.288E 03 | 0.105E 02 |
| 79 | 0.300E 03 | -0.105E 02 |
| 80 | 0.300E 03 | 0.0 |
| 81 | 0.300E 03 | 0.105E 02 |

| ELEM | NODES (I,J,K,L) | | | | THICKNESS |
|---|---|---|---|---|---|
| 1 | 1 | 4 | 5 | 2 | 0.100E 01 |
| 2 | 2 | 5 | 6 | 3 | 0.100E 01 |
| 3 | 4 | 7 | 8 | 5 | 0.100E 01 |
| 4 | 5 | 8 | 9 | 6 | 0.100E 01 |
| 5 | 7 | 10 | 11 | 8 | 0.100E 01 |
| 6 | 8 | 11 | 12 | 9 | 0.100E 01 |
| 7 | 10 | 13 | 14 | 11 | 0.100E 01 |
| 8 | 11 | 14 | 15 | 12 | 0.100E 01 |
| 9 | 13 | 16 | 17 | 14 | 0.100E 01 |
| 10 | 14 | 17 | 18 | 15 | 0.100E 01 |
| 11 | 16 | 19 | 20 | 17 | 0.100E 01 |
| 12 | 17 | 20 | 21 | 18 | 0.100E 01 |

| | | | | | |
|---|---|---|---|---|---|
| 13 | 19 | 22 | 23 | 20 | C.100E 01 |
| 14 | 20 | 23 | 24 | 21 | C.100E 01 |
| 15 | 22 | 25 | 26 | 23 | C.100E 01 |
| 16 | 23 | 26 | 27 | 24 | C.100E 01 |
| 17 | 25 | 28 | 29 | 26 | C.100E 01 |
| 18 | 26 | 29 | 30 | 27 | C.100E 01 |
| 19 | 28 | 31 | 32 | 29 | C.100E 01 |
| 20 | 29 | 32 | 33 | 30 | 0.100E 01 |
| 21 | 31 | 34 | 35 | 32 | C.100E 01 |
| 22 | 32 | 35 | 36 | 33 | C.100E 01 |
| 23 | 34 | 37 | 38 | 35 | 0.100E 01 |
| 24 | 35 | 38 | 39 | 36 | C.100E 01 |
| 25 | 37 | 40 | 41 | 38 | 0.100E 01 |
| 26 | 38 | 41 | 42 | 39 | C.100E 01 |
| 27 | 40 | 43 | 44 | 41 | C.100E 01 |
| 28 | 41 | 44 | 45 | 42 | 0.100E 01 |
| 29 | 43 | 46 | 47 | 44 | C.100E 01 |
| 30 | 44 | 47 | 48 | 45 | C.100E 01 |
| 31 | 46 | 49 | 50 | 47 | C.100E 01 |
| 32 | 47 | 50 | 51 | 48 | C.100E 01 |
| 33 | 49 | 52 | 53 | 50 | 0.100E 01 |
| 34 | 50 | 53 | 54 | 51 | 0.100E 01 |
| 35 | 52 | 55 | 56 | 53 | C.100E 01 |
| 36 | 53 | 56 | 57 | 54 | C.100E 01 |
| 37 | 55 | 58 | 59 | 56 | C.100E C1 |
| 38 | 56 | 59 | 60 | 57 | C.100E 01 |
| 39 | 58 | 61 | 62 | 59 | 0.100E 01 |
| 40 | 59 | 62 | 63 | 60 | C.100E C1 |
| 41 | 61 | 64 | 65 | 62 | 0.100E 01 |
| 42 | 62 | 65 | 66 | 63 | C.100E 01 |
| 43 | 64 | 67 | 68 | 65 | C.100E 01 |
| 44 | 65 | 68 | 69 | 66 | 0.100E 01 |
| 45 | 67 | 70 | 71 | 68 | C.100E 01 |
| 46 | 68 | 71 | 72 | 69 | C.100E 01 |
| 47 | 70 | 73 | 74 | 71 | C.100E 01 |
| 48 | 71 | 74 | 75 | 72 | C.100E 01 |
| 49 | 73 | 76 | 77 | 74 | 0.100E 01 |
| 50 | 74 | 77 | 78 | 75 | C.100E 01 |
| 51 | 76 | 79 | 80 | 77 | C.100E 01 |
| 52 | 77 | 80 | 81 | 78 | 0.100E 01 |

PLATE STIFFENERS... NCDES (I,J)... THICKNESS... WIDTH.....

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 4 | 0.100E 01 | 0.100E 02 |
| 2 | 3 | 6 | 0.100E 01 | 0.100E 02 |
| 3 | 4 | 7 | 0.100E 01 | C.100E 02 |
| 4 | 6 | 9 | 0.100E 01 | 0.100E 02 |
| 5 | 7 | 10 | 0.100E 01 | 0.100E 02 |
| 6 | 9 | 12 | 0.100E 01 | 0.100E 02 |
| 7 | 10 | 13 | 0.100E 01 | 0.100E 02 |
| 8 | 12 | 15 | 0.100E 01 | 0.100E C2 |
| 9 | 13 | 16 | 0.100E 01 | 0.100E 02 |
| 10 | 15 | 18 | 0.100E 01 | 0.100E 02 |
| 11 | 16 | 19 | 0.100E C1 | C.100E C2 |
| 12 | 18 | 21 | 0.100E 01 | 0.100E 02 |
| 13 | 19 | 22 | 0.100E 01 | 0.100E 02 |
| 14 | 21 | 24 | 0.100E 01 | 0.100E C2 |
| 15 | 22 | 25 | 0.100E 01 | 0.100E 02 |

| | | | | |
|---|---|---|---|---|
| 16 | 24 | 27 | 0.100E 01 | 0.100E 02 |
| 17 | 25 | 28 | 0.100E 01 | 0.100E 02 |
| 18 | 27 | 30 | 0.100E 01 | 0.100E 02 |
| 19 | 28 | 31 | 0.100E 01 | 0.100E 02 |
| 20 | 30 | 33 | 0.100E 01 | 0.100E 02 |
| 21 | 31 | 34 | 0.100E 01 | 0.100E 02 |
| 22 | 33 | 36 | 0.100E 01 | 0.100E 02 |
| 23 | 34 | 37 | 0.100E 01 | 0.100E 02 |
| 24 | 36 | 39 | 0.100E 01 | 0.100E 02 |
| 25 | 37 | 40 | 0.100E 01 | 0.100E 02 |
| 26 | 39 | 42 | 0.100E 01 | 0.100E 02 |
| 27 | 40 | 43 | 0.100E 01 | 0.100E 02 |
| 28 | 42 | 45 | 0.100E 01 | 0.100E 02 |
| 29 | 43 | 46 | 0.100E 01 | 0.100E 02 |
| 30 | 45 | 48 | 0.100E 01 | 0.100E 02 |
| 31 | 46 | 49 | 0.100E 01 | 0.100E 02 |
| 32 | 48 | 51 | 0.100E 01 | 0.100E 02 |
| 33 | 49 | 52 | 0.100E 01 | 0.100E 02 |
| 34 | 51 | 54 | 0.100E 01 | 0.100E 02 |
| 35 | 52 | 55 | 0.100E 01 | 0.100E 02 |
| 36 | 54 | 57 | 0.100E 01 | 0.100E 02 |
| 37 | 55 | 58 | 0.100E 01 | 0.100E 02 |
| 38 | 57 | 60 | 0.100E 01 | 0.100E 02 |
| 39 | 58 | 61 | 0.100E 01 | 0.100E 02 |
| 40 | 60 | 63 | 0.100E 01 | 0.100E 02 |
| 41 | 61 | 64 | 0.100E 01 | 0.100E 02 |
| 42 | 63 | 66 | 0.100E 01 | 0.100E 02 |
| 43 | 64 | 67 | 0.100E 01 | 0.100E 02 |
| 44 | 66 | 69 | 0.100E 01 | 0.100E 02 |
| 45 | 67 | 70 | 0.100E 01 | 0.100E 02 |
| 46 | 69 | 72 | 0.100E 01 | 0.100E 02 |
| 47 | 70 | 73 | 0.100E 01 | 0.100E 02 |
| 48 | 72 | 75 | 0.100E 01 | 0.100E 02 |
| 49 | 73 | 76 | 0.100E 01 | 0.100E 02 |
| 50 | 75 | 78 | 0.100E 01 | 0.100E 02 |
| 51 | 76 | 79 | 0.100E 01 | 0.100E 02 |
| 52 | 78 | 81 | 0.100E 01 | 0.100E 02 |

INPUT...APPLIED LOADS...
FROM NODE...TO NODE...INCR...X-LOAD...Y-LOAD...

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 1 | -0.100E 03 | 0.0 |
| 3 | 3 | 1 | 0.100E 03 | 0.0 |
| 79 | 79 | 1 | 0.100E 03 | 0.0 |
| 81 | 81 | 1 | -0.100E 03 | 0.0 |

INPUT... IN-PLANE BOUNDARY CONDITIONS...
FROM NODE...TO NODE...INCR...DEFLX...DEFLY...VALUEX...VALUEY...

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 0.0 | 0.0 |
| 79 | 79 | 1 | 0 | 1 | 0.0 | 0.0 |

....ELAPSED TIME BEFORE IN-PLANE SOLUTION STARTS(SEC)= 1.36

....ELAPSED TIME FOR IN-PLANE SOLUTION(SEC)= 0.50

COMPUTED NODAL POINT DISPLACEMENTS,

| NODE | X-DIRECTION | Y-DIRECTION |
|---|---|---|
| 1 | 0.0 | 0.0 |
| 2 | 0.3673134E-01 | -0.3702280E-03 |
| 3 | 0.7347828E-01 | 0.2379896E-04 |
| 4 | 0.2839294E-02 | -0.3878140E-01 |
| 5 | 0.3673128E-01 | -0.3917290E-01 |
| 6 | 0.7063085E-01 | -0.3878500E-01 |
| 7 | 0.5676340E-02 | -0.7446116E-01 |
| 8 | 0.3673205E-01 | -0.7484770E-01 |
| 9 | 0.6778783E-01 | -0.7446051E-01 |
| 10 | 0.8510381E-02 | -0.1070191E 00 |
| 11 | 0.3672992E-01 | -0.1074055E 00 |
| 12 | 0.6494945E-01 | -0.1070178E 00 |
| 13 | 0.1134096E-01 | -0.1364629E 00 |
| 14 | 0.3672773E-01 | -0.1368487E 00 |
| 15 | 0.6211466E-01 | -0.1364617E 00 |
| 16 | 0.1416826E-01 | -0.1627963E 00 |
| 17 | 0.3672561E-01 | -0.1631817E 00 |
| 18 | 0.5928319E-01 | -0.1627951E 00 |
| 19 | 0.1699257E-01 | -0.1860231E 00 |
| 20 | 0.3672358E-01 | -0.1864081E 00 |
| 21 | 0.5645480E-01 | -0.1860220E 00 |
| 22 | 0.1981417E-01 | -0.2061464E 00 |
| 23 | 0.3672157E-01 | -0.2065310E 00 |
| 24 | 0.5362922E-01 | -0.2061452E 00 |
| 25 | 0.2263346E-01 | -0.2231691E 00 |
| 26 | 0.3671963E-01 | -0.2235537E 00 |
| 27 | 0.5080608E-01 | -0.2231681E 00 |
| 28 | 0.2545079E-01 | -0.2370939E 00 |
| 29 | 0.3671780E-01 | -0.2374782E 00 |
| 30 | 0.4798510E-01 | -0.2370930E 00 |
| 31 | 0.2826653E-01 | -0.2479229E 00 |
| 32 | 0.3671605E-01 | -0.2483069E 00 |
| 33 | 0.4516583E-01 | -0.2479221E 00 |
| 34 | 0.3108114E-01 | -0.2556576E 00 |
| 35 | 0.3671437E-01 | -0.2560415E 00 |
| 36 | 0.4234789E-01 | -0.2556567E 00 |
| 37 | 0.3389503E-01 | -0.2602990E 00 |
| 38 | 0.3671274E-01 | -0.2606830E 00 |
| 39 | 0.3953090E-01 | -0.2602983E 00 |
| 40 | 0.3670868E-01 | -0.2618482E 00 |
| 41 | 0.3671130E-01 | -0.2622321E 00 |
| 42 | 0.3671441E-01 | -0.2618476E 00 |
| 43 | 0.3952255E-01 | -0.2603053E 00 |
| 44 | 0.3671010E-01 | -0.2606894E 00 |

| | | |
|---|---|---|
| 45 | 0.3389802E-01 | -0.2603047E 00 |
| 46 | 0.4233710E-01 | -0.2556702E 00 |
| 47 | 0.3670894E-01 | -0.2560542E 00 |
| 48 | 0.3108120E-01 | -0.2556697E 00 |
| 49 | 0.4515276E-01 | -0.2479416E 00 |
| 50 | 0.3670788E-01 | -0.2483259E 00 |
| 51 | 0.2826349E-01 | -0.2479411E 00 |
| 52 | 0.4797008E-01 | -0.2371182E 00 |
| 53 | 0.3670705E-01 | -0.2375029E 00 |
| 54 | 0.2544444E-01 | -0.2371179E 00 |
| 55 | 0.5078946E-01 | -0.2231984E 00 |
| 56 | 0.3670636E-01 | -0.2235833E 00 |
| 57 | 0.2262361E-01 | -0.2231981E 00 |
| 58 | 0.5361131E-01 | -0.2061793E 00 |
| 59 | 0.3670574E-01 | -0.2065646E 00 |
| 60 | 0.1980054E-01 | -0.2061790E 00 |
| 61 | 0.5643593E-01 | -0.1860581E 00 |
| 62 | 0.3670523E-01 | -0.1864438E 00 |
| 63 | 0.1697487E-01 | -0.1860579E 00 |
| 64 | 0.5926368E-01 | -0.1628316E 00 |
| 65 | 0.3670485E-01 | -0.1632178E 00 |
| 66 | 0.1414631E-01 | -0.1628315E 00 |
| 67 | 0.6209486E-01 | -0.1364964E 00 |
| 68 | 0.3670456E-01 | -0.1368830E 00 |
| 69 | 0.1131453E-01 | -0.1364962E 00 |
| 70 | 0.6492972E-01 | -0.1070484E 00 |
| 71 | 0.3670432E-01 | -0.1074358E 00 |
| 72 | 0.8479260E-02 | -0.1070483E 00 |
| 73 | 0.6776845E-01 | -0.7448375E-01 |
| 74 | 0.3670434E-01 | -0.7487124E-01 |
| 75 | 0.5640309E-02 | -0.7448429E-01 |
| 76 | 0.7061070E-01 | -0.3879423E-01 |
| 77 | 0.3670681E-01 | -0.3918636E-01 |
| 78 | 0.2796604E-02 | -0.3879792E-01 |
| 79 | 0.7345623E-01 | 0.0 |
| 80 | 0.3670652E-01 | -0.3706398E-03 |
| 81 | -0.5102914E-04 | 0.2269653E-04 |

COMPUTED STRESSES IN PLATE ELEMENTS...
QUAD NUM...LOCAL NODES...GLOBAL NODES...SIGX...SIGY...SIGXY....

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 0.78E 01 | 0.13E 01 | 0.12E-01 |
| 1 | 2 | 4 | 0.77E 01 | 0.12E 01 | 0.12E-01 |
| 1 | 3 | 5 | -0.37E 00 | -0.12E 01 | 0.12E-01 |
| 1 | 4 | 2 | -0.35E 00 | -0.12E 01 | 0.12E-01 |
| 2 | 1 | 2 | 0.37E 00 | 0.12E 01 | 0.11E-01 |
| 2 | 2 | 5 | 0.37E 00 | 0.12E 01 | 0.11E-01 |
| 2 | 3 | 6 | -0.78E 01 | -0.12E 01 | 0.11E-01 |
| 2 | 4 | 3 | -0.78E 01 | -0.12E 01 | 0.11E-01 |
| 3 | 1 | 4 | 0.77E 01 | 0.12E 01 | 0.83E-02 |
| 3 | 2 | 7 | 0.77E 01 | 0.12E 01 | 0.83E-02 |

| | | | | | |
|---|---|---|---|---|---|
| 3 | 3 | 8 | -0.36E 00 | -0.12E 01 | 0.83E-02 |
| 3 | 4 | 5 | -0.37E 00 | -0.12E 01 | 0.83E-02 |
| 4 | 1 | 5 | 0.37E 00 | 0.12E 01 | 0.15E-01 |
| 4 | 2 | 8 | 0.37E 00 | 0.12E 01 | 0.15E-01 |
| 4 | 3 | 9 | -0.78E 01 | -0.12E 01 | 0.15E-01 |
| 4 | 4 | 6 | -0.78E 01 | -0.12E 01 | 0.15E-01 |
| 5 | 1 | 7 | 0.77E 01 | 0.12E 01 | 0.11E-01 |
| 5 | 2 | 10 | 0.77E 01 | 0.12E 01 | 0.11E-01 |
| 5 | 3 | 11 | -0.37E 00 | -0.12E 01 | 0.11E-01 |
| 5 | 4 | 8 | -0.37E 00 | -0.12E 01 | 0.11E-01 |
| 6 | 1 | 8 | 0.36E 00 | 0.12E 01 | 0.11E-01 |
| 6 | 2 | 11 | 0.36E 00 | 0.12E 01 | 0.11E-01 |
| 6 | 3 | 12 | -0.77E 01 | -0.12E 01 | 0.11E-01 |
| 6 | 4 | 9 | -0.77E 01 | -0.12E 01 | 0.11E-01 |
| 7 | 1 | 10 | 0.77E 01 | 0.12E 01 | 0.10E-01 |
| 7 | 2 | 13 | 0.77E 01 | 0.12E 01 | 0.10E-01 |
| 7 | 3 | 14 | -0.37E 00 | -0.12E 01 | 0.10E-01 |
| 7 | 4 | 11 | -0.37E 00 | -0.12E 01 | 0.10E-01 |
| 8 | 1 | 11 | 0.36E 00 | 0.12E 01 | 0.11E-01 |
| 8 | 2 | 14 | 0.36E 00 | 0.12E 01 | 0.11E-01 |
| 8 | 3 | 15 | -0.77E 01 | -0.12E 01 | 0.11E-01 |
| 8 | 4 | 12 | -0.77E 01 | -0.12E 01 | 0.11E-01 |
| 9 | 1 | 13 | 0.77E 01 | 0.12E 01 | 0.97E-02 |
| 9 | 2 | 16 | 0.77E 01 | 0.12E 01 | 0.97E-02 |
| 9 | 3 | 17 | -0.37E 00 | -0.12E 01 | 0.97E-02 |
| 9 | 4 | 14 | -0.37E 00 | -0.12E 01 | 0.97E-02 |
| 10 | 1 | 14 | 0.36E 00 | 0.12E 01 | 0.99E-02 |
| 10 | 2 | 17 | 0.36E 00 | 0.12E 01 | 0.99E-02 |
| 10 | 3 | 18 | -0.77E 01 | -0.12E 01 | 0.99E-02 |
| 10 | 4 | 15 | -0.77E 01 | -0.12E 01 | 0.99E-02 |
| 11 | 1 | 16 | 0.77E 01 | 0.12E 01 | 0.89E-02 |
| 11 | 2 | 19 | 0.77E 01 | 0.12E 01 | 0.89E-02 |
| 11 | 3 | 20 | -0.37E 00 | -0.12E 01 | 0.89E-02 |
| 11 | 4 | 17 | -0.37E 00 | -0.12E 01 | 0.89E-02 |
| 12 | 1 | 17 | 0.36E 00 | 0.12E 01 | 0.90E-02 |
| 12 | 2 | 20 | 0.36E 00 | 0.12E 01 | 0.90E-02 |
| 12 | 3 | 21 | -0.77E 01 | -0.12E 01 | 0.90E-02 |
| 12 | 4 | 18 | -0.77E 01 | -0.12E 01 | 0.90E-02 |
| 13 | 1 | 19 | 0.77E 01 | 0.12E 01 | 0.79E-02 |
| 13 | 2 | 22 | 0.77E 01 | 0.12E 01 | 0.79E-02 |

| | | | | | |
|---|---|---|---|---|---|
| 13 | 3 | 23 | -0.37E 00 | -0.12E 01 | 0.79E-02 |
| 13 | 4 | 20 | -0.37E 00 | -0.12E 01 | 0.79E-02 |
| 14 | 1 | 20 | 0.36E 00 | 0.12E 01 | 0.82E-02 |
| 14 | 2 | 23 | 0.36E 00 | 0.12E 01 | 0.82E-02 |
| 14 | 3 | 24 | -0.77E 01 | -0.12E 01 | 0.82E-02 |
| 14 | 4 | 21 | -0.77E 01 | -0.12E 01 | 0.82E-02 |
| 15 | 1 | 22 | 0.77E 01 | 0.12E 01 | 0.67E-02 |
| 15 | 2 | 25 | 0.77E 01 | 0.12E 01 | 0.67E-02 |
| 15 | 3 | 26 | -0.37E 00 | -0.12E 01 | 0.67E-02 |
| 15 | 4 | 23 | -0.37E 00 | -0.12E 01 | 0.67E-02 |
| 16 | 1 | 23 | 0.36E 00 | 0.12E 01 | 0.70E-02 |
| 16 | 2 | 26 | 0.36E 00 | 0.12E 01 | 0.70E-02 |
| 16 | 3 | 27 | -0.77E 01 | -0.12E 01 | 0.70E-02 |
| 16 | 4 | 24 | -0.77E 01 | -0.12E 01 | 0.70E-02 |
| 17 | 1 | 25 | 0.77E 01 | 0.12E 01 | 0.56E-02 |
| 17 | 2 | 28 | 0.77E 01 | 0.12E 01 | 0.56E-02 |
| 17 | 3 | 29 | -0.37E 00 | -0.12E 01 | 0.56E-02 |
| 17 | 4 | 26 | -0.37E 00 | -0.12E 01 | 0.56E-02 |
| 18 | 1 | 26 | 0.36E 00 | 0.12E 01 | 0.59E-02 |
| 18 | 2 | 29 | 0.36E 00 | 0.12E 01 | 0.59E-02 |
| 18 | 3 | 30 | -0.77E 01 | -0.12E 01 | 0.59E-02 |
| 18 | 4 | 27 | -0.77E 01 | -0.12E 01 | 0.59E-02 |
| 19 | 1 | 28 | 0.77E 01 | 0.12E 01 | 0.44E-02 |
| 19 | 2 | 31 | 0.77E 01 | 0.12E 01 | 0.44E-02 |
| 19 | 3 | 32 | -0.37E 00 | -0.12E 01 | 0.44E-02 |
| 19 | 4 | 29 | -0.37E 00 | -0.12E 01 | 0.44E-02 |
| 20 | 1 | 29 | 0.36E 00 | 0.12E 01 | 0.47E-02 |
| 20 | 2 | 32 | 0.36E 00 | 0.12E 01 | 0.47E-02 |
| 20 | 3 | 33 | -0.77E 01 | -0.12E 01 | 0.47E-02 |
| 20 | 4 | 30 | -0.77E 01 | -0.12E 01 | 0.47E-02 |
| 21 | 1 | 31 | 0.77E 01 | 0.12E 01 | 0.32E-02 |
| 21 | 2 | 34 | 0.77E 01 | 0.12E 01 | 0.32E-02 |
| 21 | 3 | 35 | -0.37E 00 | -0.12E 01 | 0.32E-02 |
| 21 | 4 | 32 | -0.37E 00 | -0.12E 01 | 0.32E-02 |
| 22 | 1 | 32 | 0.36E 00 | 0.12E 01 | 0.36E-02 |
| 22 | 2 | 35 | 0.36E 00 | 0.12E 01 | 0.36E-02 |
| 22 | 3 | 36 | -0.77E 01 | -0.12E 01 | 0.36E-02 |
| 22 | 4 | 33 | -0.77E 01 | -0.12E 01 | 0.36E-02 |
| 23 | 1 | 34 | 0.77E 01 | 0.12E 01 | 0.19E-02 |
| 23 | 2 | 37 | 0.77E 01 | 0.12E 01 | 0.19E-02 |

| | | | | | |
|---|---|---|---|---|---|
| 23 | 3 | 38 | -0.37E 00 | -0.12E 01 | 0.19E-02 |
| 23 | 4 | 35 | -0.37E 00 | -0.12E 01 | 0.19E-02 |
| 24 | 1 | 35 | 0.36E 00 | 0.12E 01 | 0.22E-02 |
| 24 | 2 | 38 | 0.36E 00 | 0.12E 01 | 0.22E-02 |
| 24 | 3 | 39 | -0.77E 01 | -0.12E 01 | 0.22E-02 |
| 24 | 4 | 36 | -0.77E 01 | -0.12E 01 | 0.22E-02 |
| 25 | 1 | 37 | 0.77E 01 | 0.12E 01 | 0.56E-03 |
| 25 | 2 | 40 | 0.77E 01 | 0.12E 01 | 0.56E-03 |
| 25 | 3 | 41 | -0.37E 00 | -0.12E 01 | 0.56E-03 |
| 25 | 4 | 38 | -0.37E 00 | -0.12E 01 | 0.56E-03 |
| 26 | 1 | 38 | 0.36E 00 | 0.12E 01 | 0.99E-03 |
| 26 | 2 | 41 | 0.36E 00 | 0.12E 01 | 0.99E-03 |
| 26 | 3 | 42 | -0.77E 01 | -0.12E 01 | 0.99E-03 |
| 26 | 4 | 39 | -0.77E 01 | -0.12E 01 | 0.99E-03 |
| 27 | 1 | 40 | 0.77E 01 | 0.12E 01 | -0.11E-02 |
| 27 | 2 | 43 | 0.77E 01 | 0.12E 01 | -0.11E-02 |
| 27 | 3 | 44 | -0.37E 00 | -0.12E 01 | -0.11E-02 |
| 27 | 4 | 41 | -0.37E 00 | -0.12E 01 | -0.11E-02 |
| 28 | 1 | 41 | 0.36E 00 | 0.12E 01 | -0.60E-03 |
| 28 | 2 | 44 | 0.36E 00 | 0.12E 01 | -0.60E-03 |
| 28 | 3 | 45 | -0.77E 01 | -0.12E 01 | -0.60E-03 |
| 28 | 4 | 42 | -0.77E 01 | -0.12E 01 | -0.60E-03 |
| 29 | 1 | 43 | 0.77E 01 | 0.12E 01 | -0.26E-02 |
| 29 | 2 | 46 | 0.77E 01 | 0.12E 01 | -0.26E-02 |
| 29 | 3 | 47 | -0.37E 00 | -0.12E 01 | -0.26E-02 |
| 29 | 4 | 44 | -0.37E 00 | -0.12E 01 | -0.26E-02 |
| 30 | 1 | 44 | 0.36E 00 | 0.12E 01 | -0.21E-02 |
| 30 | 2 | 47 | 0.36E 00 | 0.12E 01 | -0.21E-02 |
| 30 | 3 | 48 | -0.77E 01 | -0.12E 01 | -0.21E-02 |
| 30 | 4 | 45 | -0.77E 01 | -0.12E 01 | -0.21E-02 |
| 31 | 1 | 46 | 0.77E 01 | 0.12E 01 | -0.39E-02 |
| 31 | 2 | 49 | 0.77E 01 | 0.12E 01 | -0.39E-02 |
| 31 | 3 | 50 | -0.37E 00 | -0.12E 01 | -0.39E-02 |
| 31 | 4 | 47 | -0.36E 00 | -0.12E 01 | -0.39E-02 |
| 32 | 1 | 47 | 0.36E 00 | 0.12E 01 | -0.36E-02 |
| 32 | 2 | 50 | 0.36E 00 | 0.12E 01 | -0.36E-02 |
| 32 | 3 | 51 | -0.77E 01 | -0.12E 01 | -0.36E-02 |
| 32 | 4 | 48 | -0.77E 01 | -0.12E 01 | -0.36E-02 |
| 33 | 1 | 49 | 0.77E 01 | 0.12E 01 | -0.53E-02 |
| 33 | 2 | 52 | 0.77E 01 | 0.12E 01 | -0.53E-02 |

| | | | | | |
|---|---|---|---|---|---|
| 33 | 3 | 53 | -0.36E 00 | -0.12E 01 | -0.53E-02 |
| 33 | 4 | 50 | -0.36E 00 | -0.12E 01 | -0.53E-02 |
| 34 | 1 | 50 | 0.36E 00 | 0.12E 01 | -0.49E-02 |
| 34 | 2 | 53 | 0.36E 00 | 0.12E 01 | -0.49E-02 |
| 34 | 3 | 54 | -0.77E 01 | -0.12E 01 | -0.49E-02 |
| 34 | 4 | 51 | -0.77E 01 | -0.12E 01 | -0.49E-02 |
| 35 | 1 | 52 | 0.77E 01 | 0.12E 01 | -0.67E-02 |
| 35 | 2 | 55 | 0.77E 01 | 0.12E 01 | -0.67E-02 |
| 35 | 3 | 56 | -0.36E 00 | -0.12E 01 | -0.67E-02 |
| 35 | 4 | 53 | -0.36E 00 | -0.12E 01 | -0.67E-02 |
| 36 | 1 | 53 | 0.36E 00 | 0.12E 01 | -0.63E-02 |
| 36 | 2 | 56 | 0.36E 00 | 0.12E 01 | -0.63E-02 |
| 36 | 3 | 57 | -0.77E 01 | -0.12E 01 | -0.63E-02 |
| 36 | 4 | 54 | -0.77E 01 | -0.12E 01 | -0.63E-02 |
| 37 | 1 | 55 | 0.77E 01 | 0.12E 01 | -0.78E-02 |
| 37 | 2 | 58 | 0.77E 01 | 0.12E 01 | -0.78E-02 |
| 37 | 3 | 59 | -0.36E 00 | -0.12E 01 | -0.78E-02 |
| 37 | 4 | 56 | -0.36E 00 | -0.12E 01 | -0.78E-02 |
| 38 | 1 | 56 | 0.36E 00 | 0.12E 01 | -0.75E-02 |
| 38 | 2 | 59 | 0.36E 00 | 0.12E 01 | -0.75E-02 |
| 38 | 3 | 60 | -0.77E 01 | -0.12E 01 | -0.75E-02 |
| 38 | 4 | 57 | -0.77E 01 | -0.12E 01 | -0.75E-02 |
| 39 | 1 | 58 | 0.77E 01 | 0.12E 01 | -0.88E-02 |
| 39 | 2 | 61 | 0.77E 01 | 0.12E 01 | -0.88E-02 |
| 39 | 3 | 62 | -0.36E 00 | -0.12E 01 | -0.88E-02 |
| 39 | 4 | 59 | -0.36E 00 | -0.12E 01 | -0.88E-02 |
| 40 | 1 | 59 | 0.36E 00 | 0.12E 01 | -0.84E-02 |
| 40 | 2 | 62 | 0.36E 00 | 0.12E 01 | -0.84E-02 |
| 40 | 3 | 63 | -0.77E 01 | -0.12E 01 | -0.84E-02 |
| 40 | 4 | 60 | -0.77E 01 | -0.12E 01 | -0.84E-02 |
| 41 | 1 | 61 | 0.77E 01 | 0.12E 01 | -0.98E-02 |
| 41 | 2 | 64 | 0.77E 01 | 0.12E 01 | -0.98E-02 |
| 41 | 3 | 65 | -0.36E 00 | -0.12E 01 | -0.98E-02 |
| 41 | 4 | 62 | -0.36E 00 | -0.12E 01 | -0.98E-02 |
| 42 | 1 | 62 | 0.36E 00 | 0.12E 01 | -0.95E-02 |
| 42 | 2 | 65 | 0.36E 00 | 0.12E 01 | -0.95E-02 |
| 42 | 3 | 66 | -0.77E 01 | -0.12E 01 | -0.95E-02 |
| 42 | 4 | 63 | -0.77E 01 | -0.12E 01 | -0.95E-02 |
| 43 | 1 | 64 | 0.77E 01 | 0.12E 01 | -0.11E-01 |
| 43 | 2 | 67 | 0.77E 01 | 0.12E 01 | -0.11E-01 |

| | | | | | |
|---|---|---|---|---|---|
| 43 | 3 | 68 | -0.36E 00 | -0.12E 01 | -0.11E-01 |
| 43 | 4 | 65 | -0.36E 00 | -0.12E 01 | -0.11E-01 |
| 44 | 1 | 65 | 0.36E 00 | 0.12E 01 | -0.10E-01 |
| 44 | 2 | 68 | 0.36E 00 | 0.12E 01 | -0.10E-01 |
| 44 | 3 | 69 | -0.77E 01 | -0.12E 01 | -0.10E-01 |
| 44 | 4 | 66 | -0.77E 01 | -0.12E 01 | -0.10E-01 |
| 45 | 1 | 67 | 0.77E 01 | 0.12E 01 | -0.11E-01 |
| 45 | 2 | 70 | 0.77E 01 | 0.12E 01 | -0.11E-01 |
| 45 | 3 | 71 | -0.37E 00 | -0.12E 01 | -0.11E-01 |
| 45 | 4 | 68 | -0.36E 00 | -0.12E 01 | -0.11E-01 |
| 46 | 1 | 68 | 0.36E 00 | 0.12E 01 | -0.11E-01 |
| 46 | 2 | 71 | 0.36E 00 | 0.12E 01 | -0.11E-01 |
| 46 | 3 | 72 | -0.77E 01 | -0.12E 01 | -0.11E-01 |
| 46 | 4 | 69 | -0.77E 01 | -0.12E 01 | -0.11E-01 |
| 47 | 1 | 70 | 0.77E 01 | 0.12E 01 | -0.12E-01 |
| 47 | 2 | 73 | 0.77E 01 | 0.12E 01 | -0.12E-01 |
| 47 | 3 | 74 | -0.36E 00 | -0.12E 01 | -0.12E-01 |
| 47 | 4 | 71 | -0.36E 00 | -0.12E 01 | -0.12E-01 |
| 48 | 1 | 71 | 0.36E 00 | 0.12E 01 | -0.12E-01 |
| 48 | 2 | 74 | 0.36E 00 | 0.12E 01 | -0.12E-01 |
| 48 | 3 | 75 | -0.77E 01 | -0.12E 01 | -0.12E-01 |
| 48 | 4 | 72 | -0.77E 01 | -0.12E 01 | -0.12E-01 |
| 49 | 1 | 73 | 0.78E 01 | 0.12E 01 | -0.98E-02 |
| 49 | 2 | 76 | 0.78E 01 | 0.12E 01 | -0.98E-02 |
| 49 | 3 | 77 | -0.36E 00 | -0.12E 01 | -0.98E-02 |
| 49 | 4 | 74 | -0.36E 00 | -0.12E 01 | -0.98E-02 |
| 50 | 1 | 74 | 0.37E 00 | 0.12E 01 | -0.15E-01 |
| 50 | 2 | 77 | 0.37E 00 | 0.12E 01 | -0.15E-01 |
| 50 | 3 | 78 | -0.78E 01 | -0.12E 01 | -0.15E-01 |
| 50 | 4 | 75 | -0.78E 01 | -0.12E 01 | -0.15E-01 |
| 51 | 1 | 76 | 0.78E 01 | 0.12E 01 | -0.15E-01 |
| 51 | 2 | 79 | 0.78E 01 | 0.13E 01 | -0.15E-01 |
| 51 | 3 | 80 | -0.35E 00 | -0.12E 01 | -0.15E-01 |
| 51 | 4 | 77 | -0.37E 00 | -0.12E 01 | -0.15E-01 |
| 52 | 1 | 77 | 0.37E 00 | 0.12E 01 | -0.98E-02 |
| 52 | 2 | 80 | 0.37E 00 | 0.12E 01 | -0.98E-02 |
| 52 | 3 | 81 | -0.78E 01 | -0.12E 01 | -0.98E-02 |
| 52 | 4 | 78 | -0.78E 01 | -0.12E 01 | -0.98E-02 |

COMPUTED STRESSES IN BEAM ELEMENTS....

BEAM NUM...STRESS....

| | |
|---|---|
| 1 | 0.74E 01 |
| 2 | -0.74E 01 |
| 3 | 0.74E 01 |
| 4 | -0.74E 01 |
| 5 | 0.74E 01 |
| 6 | -0.74E 01 |
| 7 | 0.74E 01 |
| 8 | -0.74E 01 |
| 9 | 0.74E 01 |
| 10 | -0.74E 01 |
| 11 | 0.73E 01 |
| 12 | -0.74E 01 |
| 13 | 0.73E 01 |
| 14 | -0.73E 01 |
| 15 | 0.73E 01 |
| 16 | -0.73E 01 |
| 17 | 0.73E 01 |
| 18 | -0.73E 01 |
| 19 | 0.73E 01 |
| 20 | -0.73E 01 |
| 21 | 0.73E 01 |
| 22 | -0.73E 01 |
| 23 | 0.73E 01 |
| 24 | -0.73E 01 |
| 25 | 0.73E 01 |
| 26 | -0.73E 01 |
| 27 | 0.73E 01 |
| 28 | -0.73E 01 |
| 29 | 0.73E 01 |
| 30 | -0.73E 01 |
| 31 | 0.73E 01 |
| 32 | -0.73E 01 |
| 33 | 0.73E 01 |
| 34 | -0.73E 01 |
| 35 | 0.73E 01 |
| 36 | -0.73E 01 |
| 37 | 0.73E 01 |
| 38 | -0.73E 01 |
| 39 | 0.73E 01 |
| 40 | -0.73E 01 |
| 41 | 0.74E 01 |
| 42 | -0.74E 01 |
| 43 | 0.74E 01 |
| 44 | -0.74E 01 |
| 45 | 0.74E 01 |
| 46 | -0.74E 01 |
| 47 | 0.74E 01 |
| 48 | -0.74E 01 |
| 49 | 0.74E 01 |
| 50 | -0.74E 01 |
| 51 | 0.74E 01 |
| 52 | -0.74E 01 |

INPUT...BENDING BOUNDARY CONDITIONS....
FROM NODE...TO NODE...INCR...DEFLZ...THETAX...THETAY...

```
  1    3    1         1    1    0
 79   81    1         1    1    0

....ELAPSED TIME BEFORE EIGENVALUE SOLN STARTS(SEC)=  10.56


 INPUT MODE SHAPE.....

   1   81    1    0.100E 01    0.100E 01    0.100E 01

 JMAX=  3      IMAX= 10      FAC=  0.800E 00     SHIFT=  0.0


 SHIFT=  0.0              TIME FOR THIS ITERATION(SEC)=   2.14
 FOR JJ=  1       II=  1      EIGENVALUE= -0.82565E 01

 SHIFT=  0.0              TIME FOR THIS ITERATION(SEC)=   0.35
 FOR JJ=  1       II=  2      EIGENVALUE= -0.25375E 01

 SHIFT=  0.0              TIME FOR THIS ITERATION(SEC)=   0.44
 FOR JJ=  1       II=  3      EIGENVALUE= -0.96346E 01

 SHIFT=  0.0              TIME FOR THIS ITERATION(SEC)=   0.42
 FOR JJ=  1       II=  4      EIGENVALUE= -0.27333E 01

 SHIFT=  0.0              TIME FOR THIS ITERATION(SEC)=   0.35
 FOR JJ=  1       II=  5      EIGENVALUE= -0.96824E 01

 SHIFT=  0.0              TIME FOR THIS ITERATION(SEC)=   0.46
 FOR JJ=  1       II=  6      EIGENVALUE= -0.27459E 01

 SHIFT=  0.0              TIME FOR THIS ITERATION(SEC)=   0.39
 FOR JJ=  1       II=  7      EIGENVALUE= -0.96665E 01
```

```
SHIFT=  0.0            TIME FOR THIS ITERATION(SEC)=   0.43
FOR JJ=  1       II=  8      EIGENVALUE= -0.27513E 01

SHIFT=  0.0            TIME FOR THIS ITERATION(SEC)=   0.43
FOR JJ=  1       II=  9      EIGENVALUE= -0.96485E 01

SHIFT=  0.0            TIME FOR THIS ITERATION(SEC)=   0.29
FOR JJ=  1       II= 10      EIGENVALUE= -0.27565E 01

SHIFT= -0.22052E 01    TIME FOR THIS ITERATION(SEC)=   2.20
FOR JJ=  2       II=  1      EIGENVALUE= -0.66117E 01

SHIFT= -0.22052E 01    TIME FOR THIS ITERATION(SEC)=   0.45
FOR JJ=  2       II=  2      EIGENVALUE= -0.46986E 01

SHIFT= -0.22052E 01    TIME FOR THIS ITERATION(SEC)=   0.41
FOR JJ=  2       II=  3      EIGENVALUE= -0.53881E 01

SHIFT= -0.22052E 01    TIME FOR THIS ITERATION(SEC)=   0.41
FOR JJ=  2       II=  4      EIGENVALUE= -0.50997E 01

SHIFT= -0.22052E 01    TIME FOR THIS ITERATION(SEC)=   0.42
FOR JJ=  2       II=  5      EIGENVALUE= -0.52137E 01

SHIFT= -0.22052E 01    TIME FOR THIS ITERATION(SEC)=   0.39
FOR JJ=  2       II=  6      EIGENVALUE= -0.51675E 01

SHIFT= -0.22052E 01    TIME FOR THIS ITERATION(SEC)=   0.33
FOR JJ=  2       II=  7      EIGENVALUE= -0.51860E 01
```

SHIFT= -0.22052E 01 TIME FOR THIS ITERATICN(SEC)= 0.51

FOR JJ= 2 II= 8 EIGENVALUE= -0.51786E 01

SHIFT= -0.22052E 01 TIME FOR THIS ITERATICN(SEC)= 0.37

FOR JJ= 2 II= 9 EIGENVALUE= -0.51816E 01

SHIFT= -0.22052E 01 TIME FOR THIS ITERATICN(SEC)= 0.46

FOR JJ= 2 II= 10 EIGENVALUE= -0.51804E 01

SHIFT= -0.41443E 01 TIME FOR THIS ITERATICN(SEC)= 2.11

FOR JJ= 3 II= 1 EIGENVALUE= -0.51858E 01

SHIFT= -0.41443E 01 TIME FOR THIS ITERATICN(SEC)= 0.3C

FOR JJ= 3 II= 2 EIGENVALUE= -0.51857E 01

NORMALIZED MODE SHAPE IS .......

| NODE | LATERAL DISPL | THETA-X | THETA-Y |
|---|---|---|---|
| 1 | 0.0 | 0.0 | -0.10483E-C1 |
| 2 | 0.0 | 0.0 | -0.68289E-02 |
| 3 | 0.0 | 0.0 | -0.32016E-C2 |
| 4 | 0.12066E 00 | -0.40054E-02 | -0.10406E-C1 |
| 5 | 0.78632E-01 | -0.40033E-02 | -0.67782E-02 |
| 6 | 0.36850E-01 | -0.39559E-C2 | -0.31779E-C2 |
| 7 | 0.23955E 00 | -0.79527E-02 | -0.10176E-01 |
| 8 | 0.15610E 00 | -0.79481E-02 | -0.66282E-02 |
| 9 | 0.73155E-01 | -0.78533E-02 | -0.31073E-C2 |
| 10 | 0.35492E 00 | -0.11784E-01 | -0.97976E-C2 |
| 11 | 0.23127E 0C | -0.11777E-C1 | -0.63804E-C2 |
| 12 | 0.10838E 00 | -0.11635E-C1 | -0.29911E-02 |
| 13 | 0.46509E 00 | -0.15444E-01 | -0.92757E-02 |
| 14 | 0.30305E 0C | -0.15433E-01 | -0.60406E-02 |
| 15 | 0.14201E 00 | -0.15246E-01 | -0.28311E-02 |
| 16 | 0.56845E 00 | -0.18878E-01 | -0.86183E-02 |
| 17 | 0.37038E 0C | -0.18863E-01 | -0.56115E-02 |
| 18 | 0.17356E 00 | -0.18635E-01 | -0.26299E-C2 |
| 19 | 0.66350E 00 | -0.22036E-C1 | -0.78355E-02 |
| 20 | 0.43229E 00 | -0.22018E-01 | -0.51013E-C2 |
| 21 | 0.20255E 0C | -0.21750E-01 | -0.23906E-02 |
| 22 | 0.74884E 00 | -0.24873E-C1 | -0.69391E-C2 |
| 23 | 0.48787E 00 | -0.24851E-01 | -0.45173E-02 |
| 24 | 0.22859E 00 | -0.24547E-01 | -0.21167E-02 |

| | | | |
|---|---|---|---|
| 25 | 0.82324E 00 | -0.27347E-01 | -0.59423E-02 |
| 26 | 0.53633E 00 | -0.27320E-01 | -0.38674E-02 |
| 27 | 0.25129E 00 | -0.26985E-01 | -0.18125E-02 |
| 28 | 0.88564E 00 | -0.29422E-01 | -0.48599E-02 |
| 29 | 0.57696E 00 | -0.29391E-01 | -0.31634E-02 |
| 30 | 0.27032E 00 | -0.29030E-01 | -0.14823E-02 |
| 31 | 0.93513E 00 | -0.31068E-01 | -0.37078E-02 |
| 32 | 0.60919E 00 | -0.31034E-01 | -0.24139E-02 |
| 33 | 0.28541E 00 | -0.30652E-01 | -0.11309E-02 |
| 34 | 0.97099E 00 | -0.32260E-01 | -0.25020E-02 |
| 35 | 0.63255E 00 | -0.32225E-01 | -0.16284E-02 |
| 36 | 0.29635E 00 | -0.31828E-01 | -0.76312E-03 |
| 37 | 0.99272E 00 | -0.32982E-01 | -0.12600E-02 |
| 38 | 0.64670E 00 | -0.32946E-01 | -0.82012E-03 |
| 39 | 0.30298E 00 | -0.32540E-01 | -0.38436E-03 |
| 40 | 0.10000E 01 | -0.33224E-01 | 0.29726E-06 |
| 41 | 0.65144E 00 | -0.33188E-01 | 0.87325E-06 |
| 42 | 0.30520E 00 | -0.32778E-01 | -0.20954E-07 |
| 43 | 0.99272E 00 | -0.32981E-01 | 0.12606E-02 |
| 44 | 0.64670E 00 | -0.32946E-01 | 0.81979E-03 |
| 45 | 0.30298E 00 | -0.32540E-01 | 0.38430E-03 |
| 46 | 0.97098E 00 | -0.32259E-01 | 0.25028E-02 |
| 47 | 0.63254E 00 | -0.32224E-01 | 0.16300E-02 |
| 48 | 0.29635E 00 | -0.31827E-01 | 0.76310E-03 |
| 49 | 0.93510E 00 | -0.31066E-01 | 0.37090E-02 |
| 50 | 0.60917E 00 | -0.31033E-01 | 0.24146E-02 |
| 51 | 0.28541E 00 | -0.30650E-01 | 0.11310E-02 |
| 52 | 0.88560E 00 | -0.29420E-01 | 0.48613E-02 |
| 53 | 0.57694E 00 | -0.29389E-01 | 0.31635E-02 |
| 54 | 0.27032E 00 | -0.29028E-01 | 0.14826E-02 |
| 55 | 0.82318E 00 | -0.27345E-01 | 0.59437E-02 |
| 56 | 0.53629E 00 | -0.27318E-01 | 0.38698E-02 |
| 57 | 0.25128E 00 | -0.26982E-01 | 0.18129E-02 |
| 58 | 0.74876E 00 | -0.24872E-01 | 0.69400E-02 |
| 59 | 0.48782E 00 | -0.24847E-01 | 0.45188E-02 |
| 60 | 0.22858E 00 | -0.24543E-01 | 0.21172E-02 |
| 61 | 0.66341E 00 | -0.22036E-01 | 0.78360E-02 |
| 62 | 0.43222E 00 | -0.22014E-01 | 0.51028E-02 |
| 63 | 0.20254E 00 | -0.21744E-01 | 0.23909E-02 |
| 64 | 0.56837E 00 | -0.18878E-01 | 0.86179E-02 |
| 65 | 0.37031E 00 | -0.18860E-01 | 0.56109E-02 |
| 66 | 0.17354E 00 | -0.18629E-01 | 0.26300E-02 |
| 67 | 0.46502E 00 | -0.15443E-01 | 0.92745E-02 |
| 68 | 0.30299E 00 | -0.15430E-01 | 0.60391E-02 |
| 69 | 0.14199E 00 | -0.15242E-01 | 0.28309E-02 |
| 70 | 0.35486E 00 | -0.11783E-01 | 0.97960E-02 |
| 71 | 0.23123E 00 | -0.11774E-01 | 0.63790E-02 |
| 72 | 0.10837E 00 | -0.11632E-01 | 0.29907E-02 |
| 73 | 0.23951E 00 | -0.79516E-02 | 0.10175E-01 |
| 74 | 0.15608E 00 | -0.79467E-02 | 0.66265E-02 |
| 75 | 0.73146E-01 | -0.78519E-02 | 0.31069E-02 |
| 76 | 0.12064E 00 | -0.40044E-02 | 0.10404E-01 |
| 77 | 0.78621E-01 | -0.40026E-02 | 0.67773E-02 |
| 78 | 0.36845E-01 | -0.39554E-02 | 0.31775E-02 |
| 79 | 0.0 | 0.0 | 0.10481E-01 |
| 80 | 0.0 | 0.0 | 0.68281E-02 |
| 81 | 0.0 | 0.0 | 0.32011E-02 |

# APPENDIX V

## COMPUTER OUTPUT OF PROBLEM 6
(Short Output)

```
BASP ....
EXAMPLE PROBLEM....          CHECKED  BY  Y.H.HWANG.


PROB
  113      SS WF BEAM UNDER UNIFORM MOMENT (WHOLE BEAM) 2X26 DIVISIONS



NUM OF ELEMENTS                        52
NUM OF STIFF ELEMENTS                  52
NUM OF POINTS                          81
NUM OF LOADED POINTS                    4
NUM OF MEMBRANE BOUNDARY PTS            2
NUM OF ELAST SPRINGS(IN-PLANE)          0
NUM OF BENDING BOUNDARY PTS             6
NUM OF ELAST SPRINGS(BUCKLING)          0



NOTE... SHORT OUTPUT OPTION IS USED ....


NOTE... GEOMETRY PLOT OPTION       =      0
    ... STRESS PLOT OPTION         =     0
    ... BUCKLED SHAPE PLOT OPTION  =      2


MOD OF ELASTICITY            0.300E 05
POISSONS RATIO               0.300E 00


 INPUT... NODAL POINT COORDINATES ....
 FROM..TO..INCR...X1..Y1..X2..Y2..

    1   79    3    0.0          -0.105E 02    0.300E 03   -0.105E 02
    2   80    3    0.0           0.0          0.300E 03    0.0
    3   81    3    0.0           0.105E 02    0.300E 03    0.105E 02

 INPUT.. NODAL POINT NUMBER SYSTEM FOR QUADS ...
 FROM ELEM..TO ELEM..NODE INCR..ELEM INCR...I1,I2,I3,I4...

    1   51    3    2    1    4    5    2
    2   52    3    2    2    5    6    3

 INPUT... NODAL POINT NUMBER SYSTEM FOR BEAMS....
 FROM ELEM..TO ELEM..NODE INCR..ELEM INCR...I1,I2..
```

```
   1   51    3    2    1    4
   2   52    3    2    3    6

INPUT... PROPERTIES CF CUADS .....
FROM ELEM...TO ELEM...INCR....THICKNESS...

   1   52    1    0.100E 01

INPUT...PROPERTIES OF BEAM ELEM....
FROM ELEM...TO ELEM...INCR....THICKNESS...WIDTH...

   1   52    1    0.100E 01    0.100E 02

INPUT...APPLIED LOADS...
FROM NODE...TC NCDE...INCR...X-LCAD...Y-LCAD...

   1    1    1        -0.100E 03   0.0
   3    3    1         0.100E 03   0.0
  79   79    1         0.100E 03   0.0
  81   81    1        -0.100E 03   0.0

INPUT... IN-PLANE BCUNDARY CONDITIONS...
FROM NODE...TC NOCE...INCR...DEFLX...CEFLY...VALUEX...VALUEY...

   1    1    1    1    1        0.0          0.0
  79   79    1    0    1        0.0          0.0
```

....ELAPSED TIME BEFORE IN-PLANE SOLUTICN STARTS(SEC)= 1.49

....ELAPSED TIME FOR IN-PLANE SOLUTICN(SEC)= 0.71

INPUT...BENDING BCUNCARY CCNDITICNS....
FROM NODE...TO NODE...INCR...DEFLZ...THETAX...THETAY...

| | | | | | |
|---|---|---|---|---|---|
| 1 | 3 | 1 | 1 | 1 | 0 |
| 79 | 81 | 1 | 1 | 1 | 0 |

....ELAPSED TIME BEFORE EIGENVALUE SOLN STARTS(SEC)= 17.35

INPUT MODE SHAPE.....

1 81 1 0.100E 01 0.100E 01 0.100E 01

JMAX= 3 IMAX= 10 FAC= 0.800E 00 SHIFT= 0.519E 01

SHIFT= 0.51857E 01 TIME FOR THIS ITERATICN(SEC)= 0.45

FCR JJ= 1 II= 3 EIGENVALUE= 0.51982E 01

NORMALIZED MODE SHAPE IS .......

| NODE | LATERAL DISPL | THETA-X | THETA-Y |
|---|---|---|---|
| 1 | 0.0 | 0.0 | -0.32088E-02 |
| 2 | 0.0 | 0.0 | -0.68353E-02 |
| 3 | 0.0 | 0.0 | -0.10488E-01 |
| 4 | 0.36933E-01 | 0.39551E-02 | -0.31850E-02 |
| 5 | 0.78706E-01 | 0.40023E-02 | -0.67846E-02 |
| 6 | 0.12072E 00 | 0.40042E-02 | -0.10411E-01 |
| 7 | 0.73320E-01 | 0.78515E-02 | -0.31142E-02 |
| 8 | 0.15625E 00 | 0.79460E-02 | -0.66341E-02 |
| 9 | 0.23967E 00 | 0.79504E-02 | -0.10181E-01 |
| 10 | 0.10862E 00 | 0.11632E-01 | -0.29974E-02 |
| 11 | 0.23148E 00 | 0.11773E-01 | -0.63859E-02 |
| 12 | 0.35510E 00 | 0.11781E-01 | -0.98021E-02 |
| 13 | 0.14232E 00 | 0.15243E-01 | -0.28368E-02 |
| 14 | 0.30332E 00 | 0.15429E-01 | -0.60452E-02 |

| | | | |
|---|---|---|---|
| 15 | 0.46532E 00 | 0.15440E-01 | -0.92795E-02 |
| 16 | 0.17393E 00 | 0.18630E-01 | -0.26346E-02 |
| 17 | 0.37070E 00 | 0.18858E-01 | -0.56153E-02 |
| 18 | 0.56872E 00 | 0.18872E-01 | -0.86212E-02 |
| 19 | 0.20297E 00 | 0.21744E-01 | -0.23942E-02 |
| 20 | 0.43265E 00 | 0.22012E-01 | -0.51036E-02 |
| 21 | 0.66379E 00 | 0.22030E-01 | -0.78370E-02 |
| 22 | 0.22905E 00 | 0.24540E-01 | -0.21192E-02 |
| 23 | 0.48825E 00 | 0.24843E-01 | -0.45183E-02 |
| 24 | 0.74914E 00 | 0.24865E-01 | -0.69390E-02 |
| 25 | 0.25176E 00 | 0.26977E-01 | -0.18138E-02 |
| 26 | 0.53671E 00 | 0.27312E-01 | -0.38672E-02 |
| 27 | 0.82353E 00 | 0.27338E-01 | -0.59406E-02 |
| 28 | 0.27080E 00 | 0.29019E-01 | -0.14824E-02 |
| 29 | 0.57733E 00 | 0.29381E-01 | -0.31615E-02 |
| 30 | 0.88590E 00 | 0.29412E-01 | -0.48567E-02 |
| 31 | 0.28589E 00 | 0.30639E-01 | -0.11300E-02 |
| 32 | 0.60954E 00 | 0.31022E-01 | -0.24109E-02 |
| 33 | 0.93534E 00 | 0.31055E-01 | -0.37029E-02 |
| 34 | 0.29682E 00 | 0.31813E-01 | -0.76145E-03 |
| 35 | 0.63286E 00 | 0.32210E-01 | -0.16245E-02 |
| 36 | 0.97114E 00 | 0.32244E-01 | -0.24958E-02 |
| 37 | 0.30342E 00 | 0.32523E-01 | -0.38223E-03 |
| 38 | 0.64696E 00 | 0.32929E-01 | -0.81543E-03 |
| 39 | 0.99280E 00 | 0.32964E-01 | -0.12534E-02 |
| 40 | 0.30562E 00 | 0.32758E-01 | 0.24144E-05 |
| 41 | 0.65164E 00 | 0.33168E-01 | 0.56590E-05 |
| 42 | 0.10000E 01 | 0.33205E-01 | 0.68341E-05 |
| 43 | 0.30337E 00 | 0.32518E-01 | 0.38686E-03 |
| 44 | 0.64686E 00 | 0.32924E-01 | 0.82427E-03 |
| 45 | 0.99264E 00 | 0.32959E-01 | 0.12665E-02 |
| 46 | 0.29671E 00 | 0.31804E-01 | 0.76581E-03 |
| 47 | 0.63265E 00 | 0.32201E-01 | 0.16338E-02 |
| 48 | 0.97084E 00 | 0.32235E-01 | 0.25076E-02 |
| 49 | 0.28574E 00 | 0.30626E-01 | 0.11339E-02 |
| 50 | 0.60924E 00 | 0.31009E-01 | 0.24181E-02 |
| 51 | 0.93492E 00 | 0.31043E-01 | 0.37124E-02 |
| 52 | 0.27061E 00 | 0.29004E-01 | 0.14856E-02 |
| 53 | 0.57697E 00 | 0.29365E-01 | 0.31663E-02 |
| 54 | 0.88538E 00 | 0.29396E-01 | 0.48635E-02 |
| 55 | 0.25153E 00 | 0.26959E-01 | 0.18162E-02 |
| 56 | 0.53630E 00 | 0.27294E-01 | 0.38722E-02 |
| 57 | 0.82295E 00 | 0.27322E-01 | 0.59447E-02 |
| 58 | 0.22879E 00 | 0.24521E-01 | 0.21204E-02 |
| 59 | 0.48780E 00 | 0.24826E-01 | 0.45207E-02 |
| 60 | 0.74852E 00 | 0.24850E-01 | 0.69402E-02 |
| 61 | 0.20272E 00 | 0.21724E-01 | 0.23940E-02 |
| 62 | 0.43219E 00 | 0.21994E-01 | 0.51040E-02 |
| 63 | 0.66318E 00 | 0.22017E-01 | 0.78348E-02 |
| 64 | 0.17368E 00 | 0.18612E-01 | 0.26329E-02 |
| 65 | 0.37027E 00 | 0.18843E-01 | 0.56114E-02 |
| 66 | 0.56816E 00 | 0.18861E-01 | 0.86158E-02 |
| 67 | 0.14210E 00 | 0.15228E-01 | 0.28337E-02 |
| 68 | 0.30295E 00 | 0.15416E-01 | 0.60390E-02 |
| 69 | 0.46483E 00 | 0.15429E-01 | 0.92716E-02 |
| 70 | 0.10845E 00 | 0.11621E-01 | 0.29933E-02 |
| 71 | 0.23119E 00 | 0.11764E-01 | 0.63785E-02 |
| 72 | 0.35472E 00 | 0.11773E-01 | 0.97924E-02 |
| 73 | 0.73197E-01 | 0.78444E-02 | 0.31092E-02 |
| 74 | 0.15605E 00 | 0.79393E-02 | 0.66256E-02 |

| | | | |
|---|---|---|---|
| 75 | 0.23941E 00 | 0.79444E-02 | 0.10170E-01 |
| 76 | 0.36370E-01 | 0.39516E-02 | 0.31797E-02 |
| 77 | 0.78606E-01 | 0.39989E-02 | 0.67761E-02 |
| 78 | 0.12059E 00 | 0.40009E-02 | 0.10400E-01 |
| 79 | 0.0 | 0.0 | 0.32033E-02 |
| 80 | 0.0 | 0.0 | 0.68267E-02 |
| 81 | 0.0 | 0.0 | 0.10476E-01 |

WIDE-FLANGE BEAM BUCKLING ANALYSIS BY

FINITE ELEMENT METHOD

by

YUNG-HWEI HWANG

Diploma, Taipei Institute of Technology, 1974

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Civil Engineering

KANSAS STATE UNIVERSITY

Manhattan, Kansas

1978

ABSTRACT

A finite element procedure for determining the critical buckling load for wide-flange beams is described. The basic characteristics of three types of conventional element stiffnesses and geometric element stiffnesses (quadrilateral, rectangular and beam element) are reviewed.

A finite element computer program, which was developed in the University of Texas at Austin, was used to analyze the buckling of wide-flange beams having equal flanges. This computer program is applicable to structures whose instabilities involve small displacements and elastic behavior, i.e., linear elastic buckling. Local and overall structural instabilities may be treated together with complex loadings and support conditions.

The smallest eigenvalue corresponding to the smallest buckling load is determined by an inverse iteration procedure. The accuracy of this finite element computer program is evaluated by comparison with a number of problems for which classical solutions are available.