THE ANALYSIS OF A PLATE OF IRREGULAR GEOMETRY USING POLAR COORDINATES AND POINT MATCHING

by 4589

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NOMENCLATURE

∇^4	biharmonic operator $\nabla^4 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2$
w	deflection
$\frac{\partial w}{\partial n}$	normal slope
q	transverse loading
D	flexural rigidity of the plate $D = \frac{Et^3}{12(1-\nu^2)}$
E	modulus of elasticity
ν	Poisson's ratio
R	radius of the large circle
w_c	maximum deflection of a clamped, uniformly-loaded, circular plate $w_c = \frac{qR^4}{64D}$
w_c'	maximum slope of a clamped, uniformly-loaded, circular plate $w_c' = -1.54 \frac{w_c}{R}$
r	radial coordinate
θ	angular coordinate
ϕ	normal direction
a	radius of the large circle $a = 4.016$ in.
b	radius of the small circle $b = 1.693$ in.
c	distance between the centers of the large and small circles $c = 4.685$ in.
n	positive integer
t	plate thickness

INTRODUCTION

Since exact solutions can not be obtained for many engineering problems, numerous approximate methods have been developed. One of the more recent techniques is the point-matching method. It is a form of collocation in which a solution is chosen which exactly satisfies the partial differential equation governing the problem while matching the boundary conditions at only a finite number of discrete points. Though the technique was employed by J.C. Slater (1934) and by J. Barta (1937), it was H. D. Conway (1960) who introduced the name "point matching."[1*] The point-matching method is accepted as a useful method to obtain the approximate solution of problems governed by harmonic and biharmonic differential equations, not only because it is a simple and direct method ideally suited for the computer, but also because it is a general method. A single computer program for a given type of problem can be written to give solutions for different boundary shapes and boundary conditions. Although the point-matching method is particularly suited to problems of irregular and unsymmetric boundaries, most of the published work has been for regular symmetric shapes.

In this report the point-matching method is illustrated by applying the technique to the analysis of a thin, clamped plate of irregular geometry under a uniform transverse load. Both single and double poles of expansion are used to determine the deflection of the plate. The advantages and disadvantages of both approaches are given and the values of deflection obtained in each case are compared with each other and to experimental values. Areas of difficulty are pinpointed and suggestions to improve accuracy are given. In addition,

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*Numeric superscripts refer to references presented at the end of the report symmetric and special cases are considered in order to compare the accuracy of the symmetric cases to unsymmetric cases. The use of radial slope along the boundary of the clamped plate instead of normal slope is also investigated.

THE POINT-MATCHING METHOD

For a thin plate of arbitrary shape under a uniform transverse load the governing differential equation is

$$\nabla^4 w = \frac{q}{D} \tag{1}$$

where ∇^4 is the biharmonic operator, w the deflection, q the transverse load, and D the flexural rigidity of the plate. The solution of the differential equation is of the form $w = w_c + w_p$ where w_c and w_p are the complementary and particular solutions respectively. The well-known solution in polar coordinates to the biharmonic differential equation is[2]

$$w = R_0 + \sum_{m=1}^{\infty} R_m \cos m\theta + \sum_{m=1}^{\infty} R'_m \sin m\theta + w_p \tag{2}$$

where w_p is the particular solution for the loading and

$$\begin{aligned} m = 0,\ R_0 &= A_0 + B_0 r^2 + C_0 \log r + D_0 r^2 \log r \\ m = 1,\ R_1 &= A_1 + B_1 r^3 + C_1 r^{-1} + D_1 \log r \\ m > 1,\ R_m &= A_m r^m + B_m r^{-m} + C_m r^{m+2} + D_m r^{-m+2} \end{aligned} \tag{3}$$

Omitting terms in order to obtain finite values at r = 0, one obtains the following algebraic-trigonometric polynomial solution suggested by Conway:[3]

$$w = \sum_{n=0}^{\infty} (A_n r^n \cos n\theta + B_n r^n \sin n\theta + C_n r^{n+2} \cos n\theta + D_n r^{n+2} \sin n\theta) + w_p \tag{4}$$

where for a uniformly loaded plate

$$w_p = \frac{qr^4}{64D} \tag{5}$$

For clamped plates, the boundary conditions are

$$w = \frac{\partial w}{\partial n} = 0 \tag{6}$$

where

$$\frac{\partial w}{\partial n} = \sum_{n=0}^{\infty}\Big[A_n n r^{n-1}\cos(n\theta + \phi) + B_n r^{n-1}\sin(n\theta + \phi) + C_n r^{n+1}\{2\cos n\theta \cos \phi + n\cos(n\theta + \phi)\} + D_n r^{n+1}\{2\sin n\theta \cos \phi + n\sin(n\theta + \phi)\}\Big] + \frac{\partial w}{\partial n}_p \tag{7}$$

and

$$\frac{\partial w}{\partial n}_p = \frac{4r^3}{64D} q\cos \phi \tag{8}$$

The plate coordinates and the sign convention used are shown in Fig. 1. The coordinates r and θ define points on the boundary and within the region. The φ coordinate measures the direction of the normal to the boundary and is positive when the normal direction leads the radial direction in the counterclockwise sense.

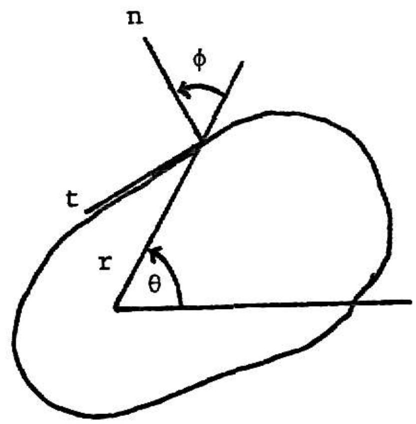


Fig. 1.

After having chosen a suitable series, each term of which satisfies the partial differential equation exactly, the procedure used in point matching is to determine the unknown coefficients so that the boundary conditions are satisfied at discrete points. As an example, to determine N terms in the series for the deflection of a clamped plate, N linear simultaneous equations are constructed which satisfy the boundary conditions of zero slope and zero deflection at N/2 points. This is done by substituting the values of r, θ, and φ for each collocation point into the appropriate equations expressing the boundary conditions at that point. The equations are then solved for the unknown coefficients. The deflection at interior points and at points on the boundary between matched points can then be calculated.

EXAMPLE PROBLEMS

The problem to be solved by the method of point matching was a thin, clamped, uniformly-loaded plate of constant thickness and of the shape shown in Fig. 2. The unsymmetric shape shown closely approximates the base of a sump pump, the dimensions of which were a = 4.016 in., b = 1.693 in., and c = 4.685 in. The method used to solve the simultaneous linear algebraic equations was Gauss-Elimination with complete pivoting. This subroutine was taken from the Scientific Subroutine Package and is included in Appendix A. The calculations in the subroutine were done in double precision as were all the calculations in the main programs. Two methods of expansion were used. In the first method a single pole was used for the expansion of the infinite series solution. In the second method two poles of expansion were used to obtain two independent series solutions which satisfied the continuity conditions on slope and deflection along a common boundary. The computer program for each method of expansion is given in Appendix A.

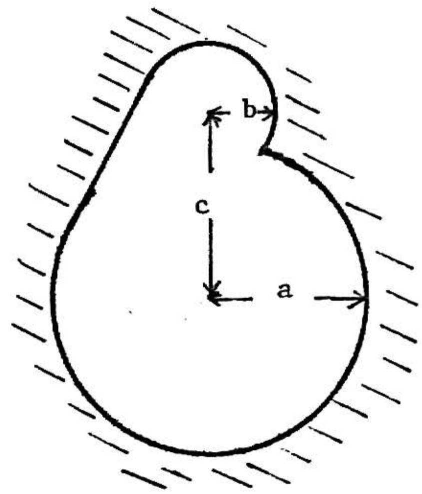


Fig. 2.

Single-Pole Expansion Method

Described below and shown in Fig. 3 are three cases where a single pole of expansion was applied to the unsymmetric shape in Fig. 2. The coordinates of the collocation points used are given in Appendix B.

Case I: The normal slope and the deflection were set equal to zero at thirty-five collocation points.

Case II: The radial slope and the deflection were set equal to zero at thirty-five collocation points.

Case III: The normal slope and the deflection were set equal to zero at thirty-five collocation points, and, in addition, corner point 32 was considered to be rounded.

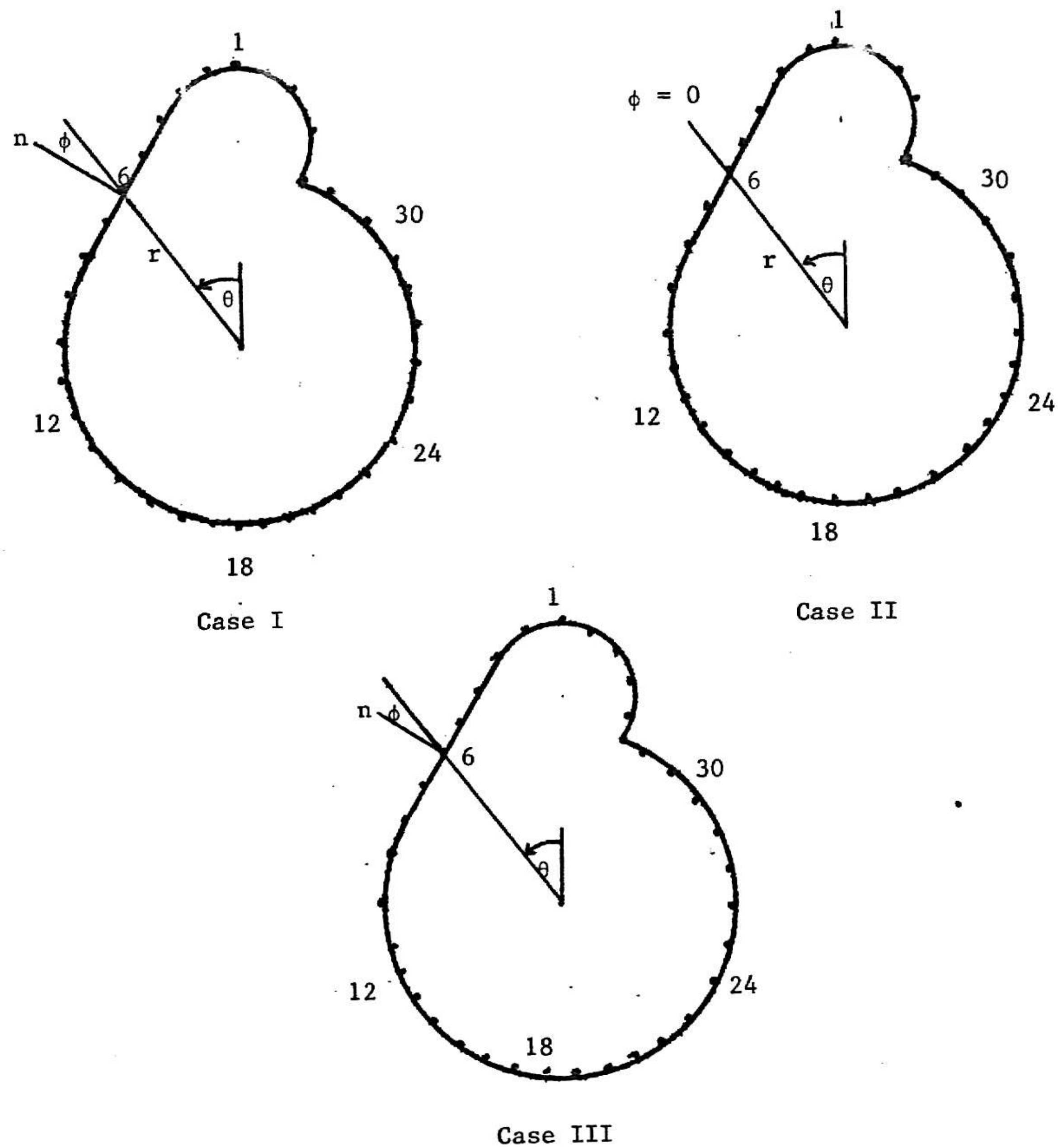


Fig. 3. Unsymmetric Cases

Since the maximum error occurs on the boundary, one can determine the accuracy of the solutions for deflection and slope by calculating the deviation of the deflection and slope between collocation points along the boundary. The maximum deflection of the unsymmetric plate should occur near the center of the large circle and should be approximately equal to the center deflection of a clamped, uniformly-loaded, circular plate, which is

$$w_c = \frac{qR^4}{64D} \tag{9}$$

Therefore, the deviation of the deflection along the boundary can be compared to this value. Likewise, the deviation of the slope along the boundary can be compared to the maximum slope of a clamped, uniformly-loaded, circular plate, which is

$$w_c' = -1.54 \frac{w_c}{R} \tag{10}$$

The deviations of the deflection along the boundary for the three unsymmetric cases of Fig. 3 are shown in Fig. 4.

From Fig. 4 it is apparent that the use of the boundary condition of zero radial slope (Case II) instead of zero normal slope (Case I) does not appreciably change the deviations of deflection along the boundary. However, the normal slope at the collocation points where zero radial slope was specified was as high as 3.2 w_c'. A comparison of the results of Case I and Case II shows that the values of the interior deflection at corresponding points are in relatively close agreement. At the center of the large circle, for example, the deflection for Case I is 1.09 w_c and the deflection for Case II is 1.12 w_c. Although the use of normal slope is a refinement over the use of radial slope, the accuracy obtained in each case is comparable. The main advantages of using radial slope over normal slope are that it eliminates the work of determining the normal direction and that it simplifies the computer program.

The accuracy of Case I is not very good, since the maximum deviation of

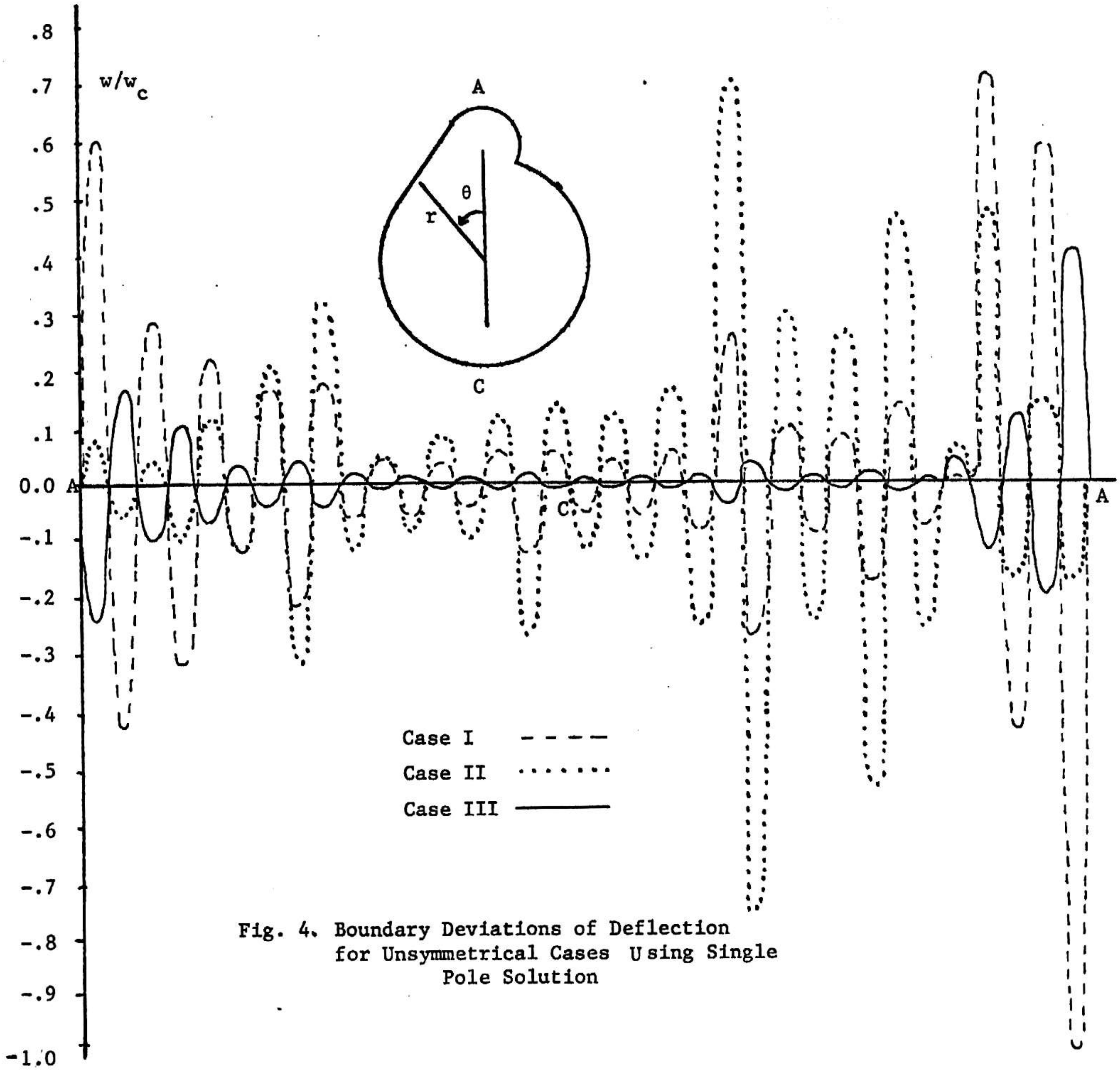


Fig. 4. Boundary Deviations of Deflection for Unsymmetrical Cases Using Single Pole Solution

the deflection along the boundary is of the same order as the maximum plate deflection w_c and the deviation of the deflection between collocation points is not very uniform. The solution does, however, represent considerable effort. The spacing of points and the number of points used proved to be very critical. Until the collocation points were more evenly distributed according to arc length, the deflection of interior points was often negative. In addition, until the number of collocation points was increased, very large deviations of slope and deflection along the boundary were obtained. Though there is no mathematical proof that convergence is obtained with an increase of collocation points,[1] in all cases the deviation between collocation points decreased as the number of collocation points increased. Although the maximum number of equations used for any case was 70, the number of equations could be increased since no difficulty with round-off error was encountered.

As seen in Fig. 4, the rounding of the corner (Case III) opposite the tangent line resulted in a more accurate solution with a maximum deflection deviation of .4 w_c. The reason the results of Case I are much worse than these results appears to be due in part to the fact that the values at this cusp are almost double valued and cause difficulty when several collocation points are used near the corner. To investigate this further and to compare the solutions of symmetric cases to unsymmetric cases, the four symmetric cases described below and shown in Fig. 5 were considered. A single pole of expansion was again used and in each of the following cases the boundary conditions were zero normal slope and deflection.

Case IV: Thirty-five symmetric collocation points were used with the points being the same as those used on the left-hand side of the cases shown in Fig. 3.

Case V: Thirty-five symmetric collocation points were used with the points being the same as those used on the right-hand side of Case I shown in Fig. 3.

Case VI: Thirty-five symmetric collocation points were used with the points being the same as those used on the right-hand side of Case III shown in Fig. 3. The corner point was rounded.

Case VII: Twenty-three collocation points were used with the points located every 15 degrees apart except for one deleted point in the upper right-hand side.

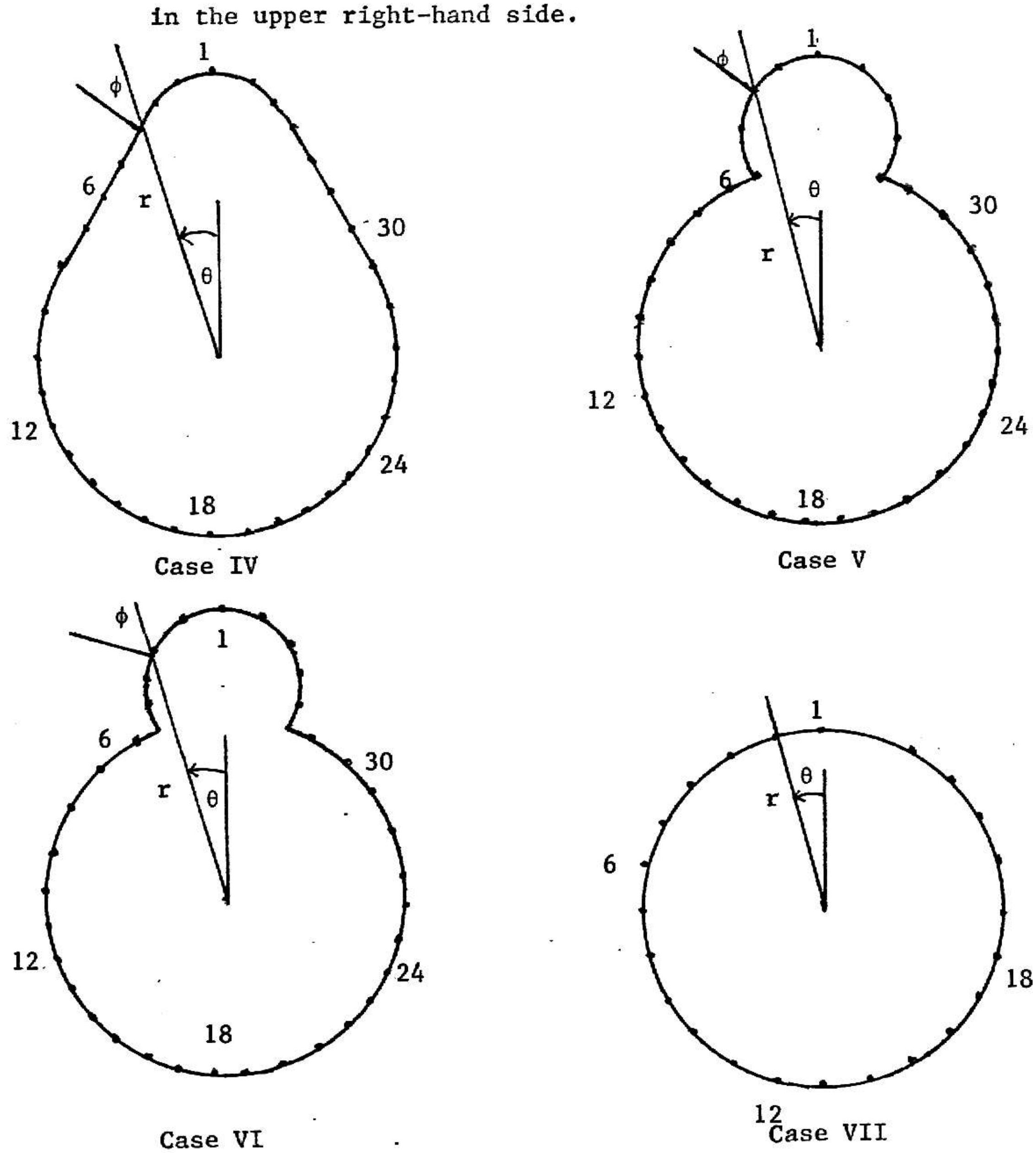


Fig. 5. Symmetric Cases

The graphs of Fig. 6 show the deviation of the deflection from the prescribed zero value for the symmetric cases presented in Fig. 5. A general comparison of these curves with the curves of the unsymmetric cases in Fig. 4 shows the overall increase in accuracy obtained when symmetrical cases are used. More specifically, the maximum deflection deviation for the symmetrical cases is .44 w_c, which is approximately one-half of the maximum deflection deviation for the unsymmetrical cases.

A comparison of the curves of Case V and Case VI in Fig. 6 shows that rounding the corner point reduced the maximum deflection deviation from .44 w_c to .07 w_c, which represents a considerable improvement. It is felt that the results obtained for Case VI are reasonable and that the solution would give good results for interior deflections. Since no loss of accuracy was detected in the solution of the 70 equations involved, it is felt that an even better solution could be obtained by increasing the number of collocation points.

The point-matching solution for Case IV resulted in a maximum boundary deflection deviation of .007 w_c and a maximum slope deviation of .0085 w'_c. The deviation curve for deflection was of such small order that it appears as a straight line in Fig. 6. Case IV illustrates the accuracy that can be obtained for a symmetric, well-behaved, irregular shape.

For the circular plate (Case VII) the point-matching solution gave the "exact" solution. This provided an excellent check for the computer program. The only non-zero coefficients of the series solution for this case were A_0 and C_0.

Double-Pole Expansion Method

In the two-pole method the plate is separated into two regions, each having its own series solution. These series are then truncated so that the

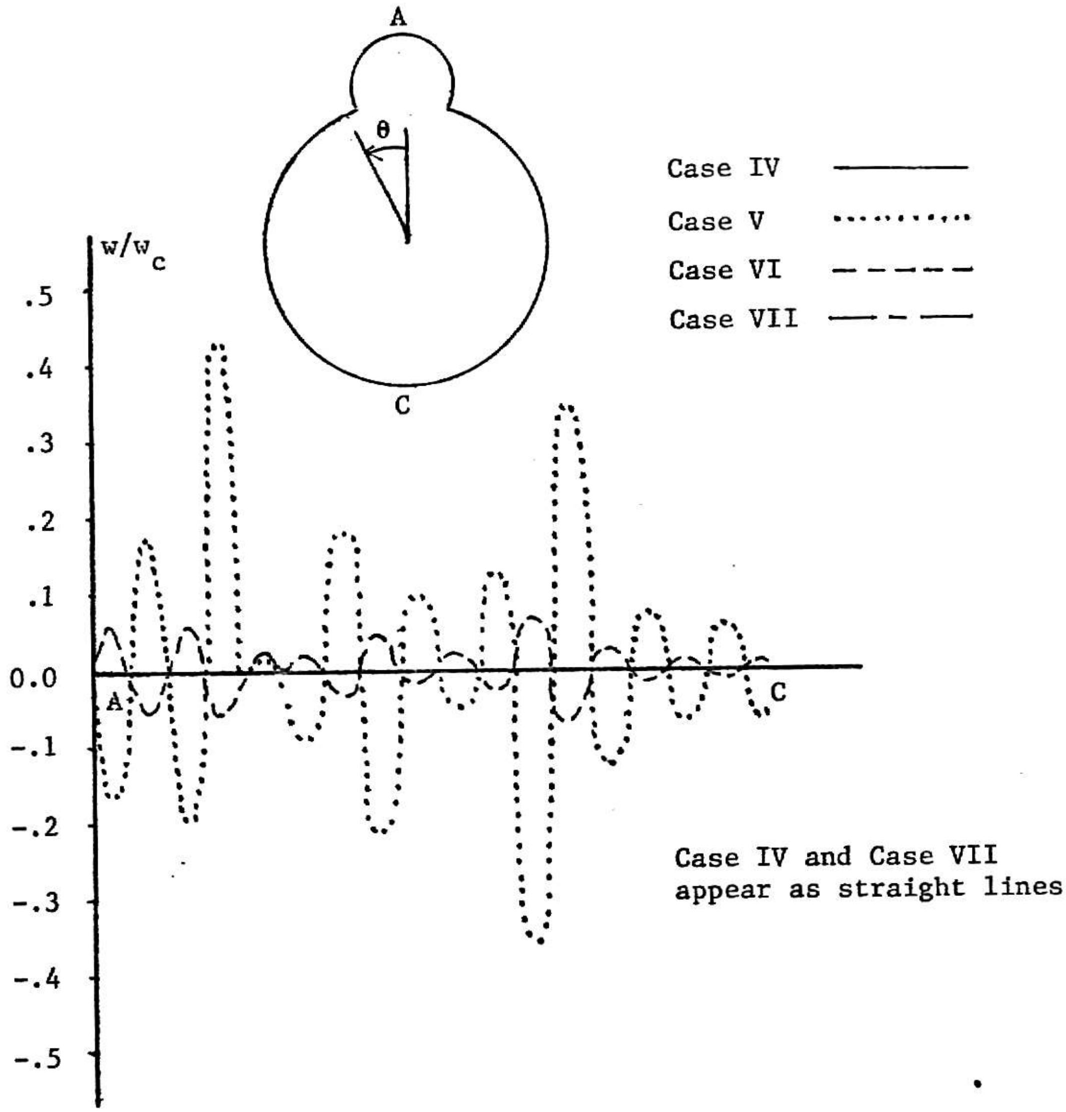


Fig. 6. Boundary Deviations of Deflection for Symmetrical Cases Using Single-Pole Solution

number of coefficients retained correspond to the number of matching conditions. For example, if there are 15 matching points on the fixed boundary of one region and 4 matching points on the common boundary, then that series would be truncated at 34 coefficients. The set of simultaneous equations to be solved to obtain the coefficients are then obtained by requiring the series to satisfy the boundary and continuity conditions at the matching points. Radial slopes, instead of normal slopes, were equated to zero at all boundary points. This was justified since the resulting normal slopes remained quite small. For the common boundary, however, the normal slopes were equated. Equating only deflections and normal slopes along the common boundary does not insure that the moments and the shearing forces will be equal along this section. An improved solution could perhaps be obtained by adding the continuity conditions on moments and shear at the common boundary.

The two-pole expansion method was used to obtain an approximate solution for the original unsymmetric problem which is shown in Fig. 2 and is repeated in Fig. 7 to show the coordinate system and the collocation points used. Several solutions were obtained, with the best overall results being found when 56 equations were used. This solution used 20 collocation points on the fixed boundary of the large circle, 6 points on the fixed boundary of the small circle, and 2 points on the common boundary. The maximum boundary deviation for the deflection was .03 w_c and the maximum boundary deviation for the radial slope was .13 w_c'. Thus the accuracy of this two-pole expansion solution was very much improved over the

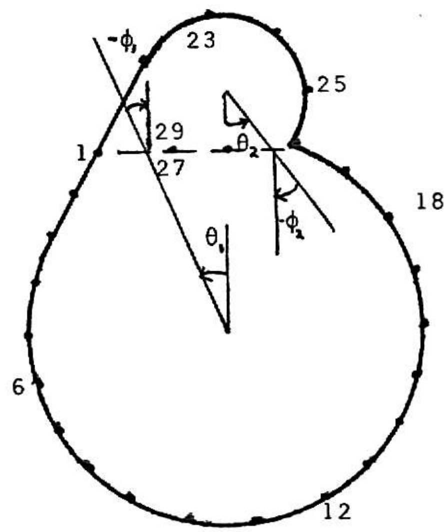


Fig. 7. Coordinate System and Identification of Collocation Points

accuracy of the single-pole expansion solution for the same problem.

Although the double pole of expansion handled the corner point more easily, it did have several disadvantages. Not only was the computer program more complicated, but the choice of collocation points proved to be even more critical than when a single pole of expansion was used. As an example, when two points were added to the clamped boundary of the small circle, negative values of deflection at interior points of the small circle were obtained. The proper spacing and number of collocation points proved to be critical in order to obtain reasonable results.

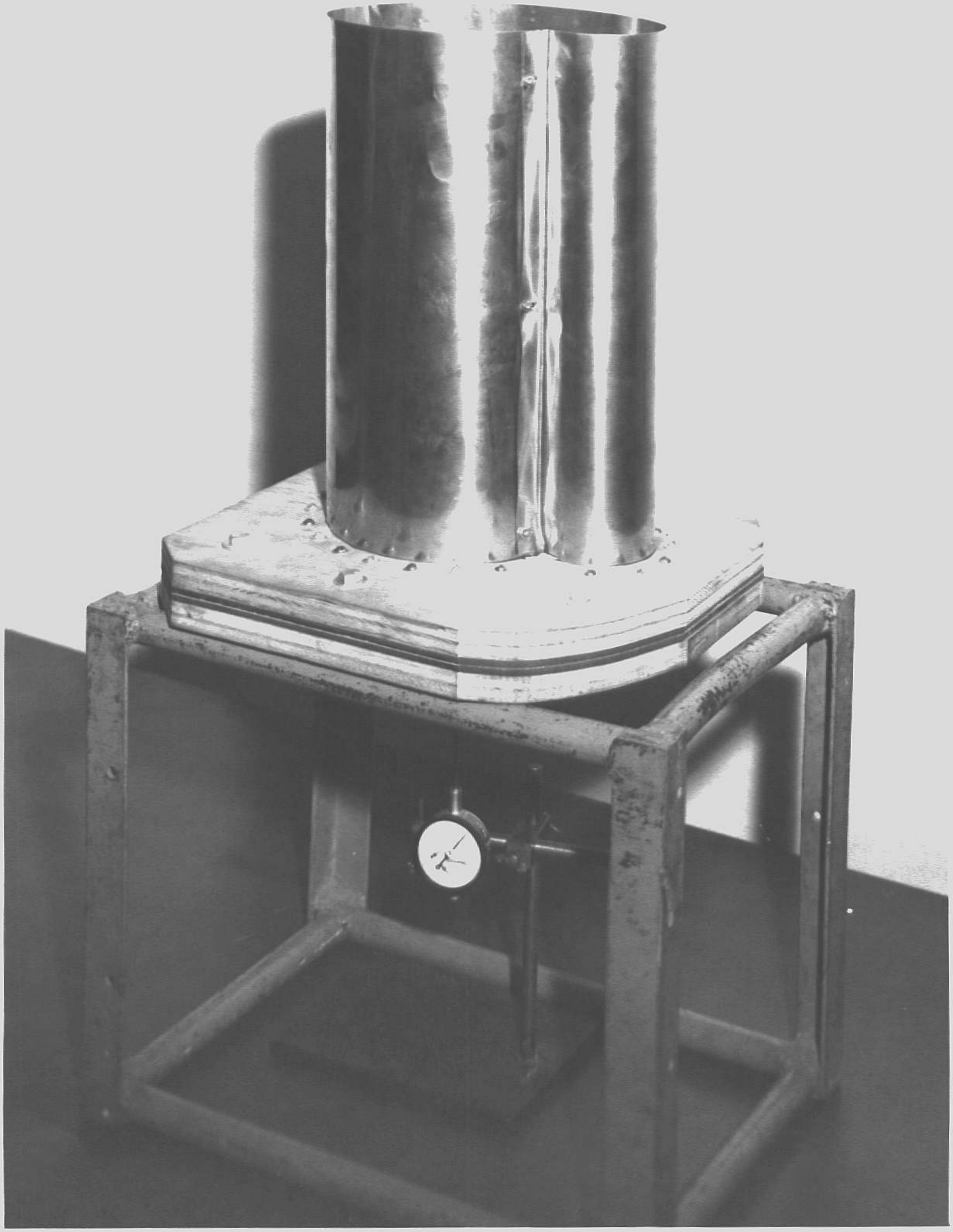
EXPERIMENTAL APPARATUS

The experiment described below and shown in Plate I was constructed to verify the results obtained by using both single and double poles of expansion. The shape shown in Fig. 2 was machined out of two bolted, 3/8-inch steel plates with 3/4-inch birch plywood backing in order to provide a suitable clamping device. A .032 inch thick sheet of aluminum (ALC-2024-T3) was then inserted and bolted between the two steel plates, and a stainless steel tank was constructed which conformed to the shape of the exposed aluminum plate. The plate was loaded by siphoning three inches of water into the tank. Three inches of water was used in order to obtain, according to small-deflection theory, a maximum deflection of approximately one-half of the plate's thickness. The deflection at selected points was determined by using a Starrett mechanical gage which was calibrated to read deflections as small as one ten-thousandths of an inch.

PLATE I

EXPERIMENTAL APPARATUS

PLATE I



VERIFICATION

The deflection at selected points along line AA' shown in Fig. 8 was determined experimentally. In Fig. 9 the experimental results are compared with the results obtained analytically. The assumed values for the modulus of elasticity and Poisson's ratio were 10.5×10^6 psi and .33 respectively.

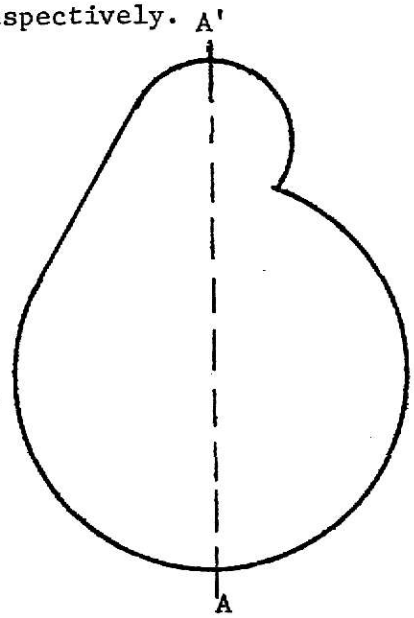


Fig. 8.

Figure 9 shows that the three different solutions for the deflection along line AA' are in close agreement. There are, however, some obvious irregularities in the upper region of the small circle. The single-pole solution with the rounded corner point gave negative deflections in this region, and the single-pole solution which contained the corner point produced an unexpected increase in the deflection over the small circle portion. These imperfections in the single-pole solutions indicate that more equations and a better spacing of collocation points is necessary in order to obtain an acceptable solution by the point-matching method. The double-pole solution appears to give the best results. The fact that this analytical solution produced smaller deflections, in the region of the small circle, than those obtained experimentally tends, however, to discredit the results for this case as well. The experimental results were expected to give smaller deflections than those predicted by the elementary plate theory in which the strain of the middle plane is neglected. Large-deflection plate theory predicts that the maximum deflection would be as much as 10 percent less than that given by the elementary theory. The difference between the experimental and theoretical

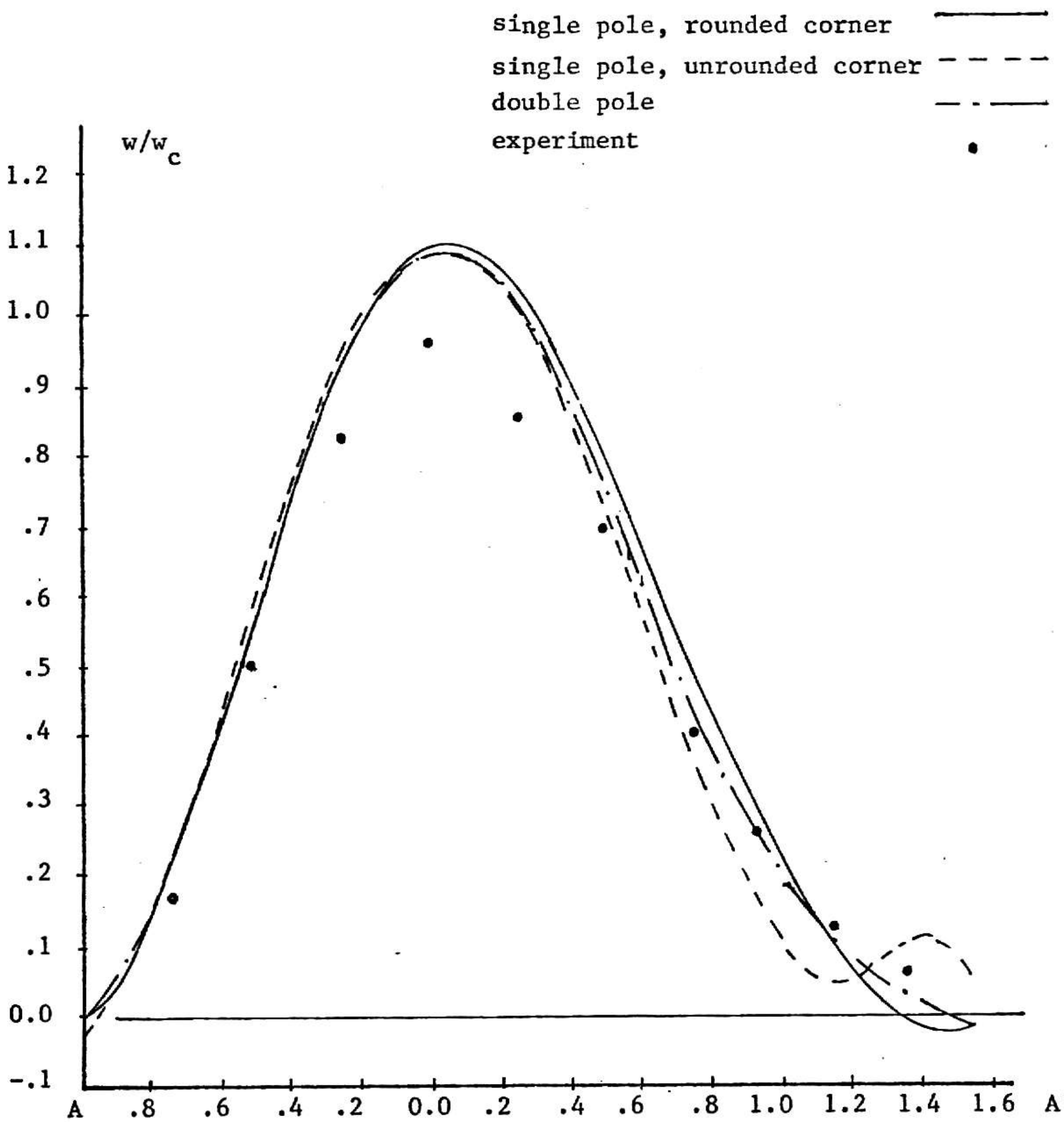


Fig. 9. Dimensionless Deflection Along AA'

results at the center of the large circle, as shown in Fig. 9, tends to verify these predictions.

It can also be seen from the curves of Fig. 9 that when the corner is rounded the values obtained for deflection by using a single pole of expansion are generally slightly larger than for the other cases. This was expected since with a rounded corner the plate is less constrained.

Table 1 was prepared to compare the experimental and theoretical deflection results at the interior points shown in Fig. 10. Good agreement is to be seen in this table for the points on the large circle.

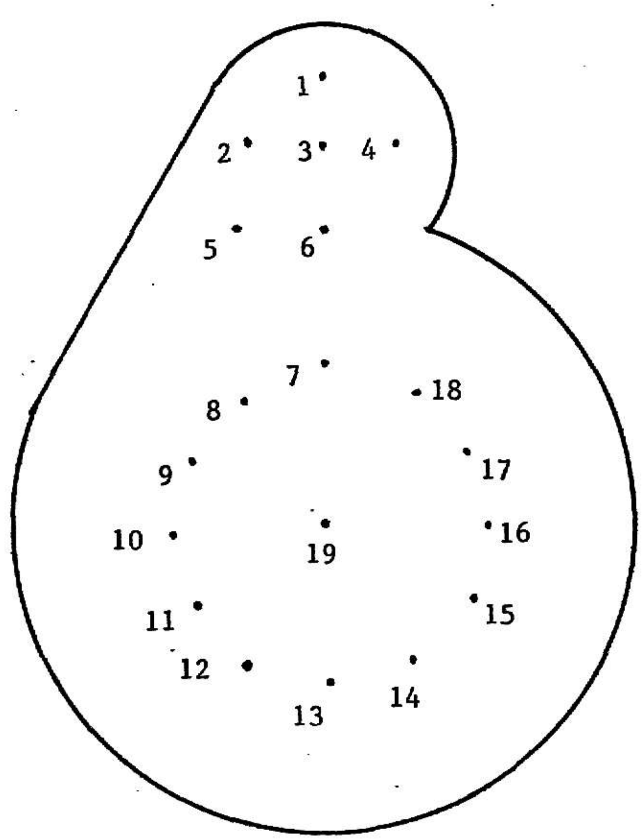


Fig. 10.

DEFLECTION W/W_c OBTAINED BY VARIOUS METHODS						
Pt. #	r	θ	Single Pole (70 eqns.) rounded corner	Single Pole (70 eqns.) unrounded corner	Double Pole (56 eqns.)	Exp.
1	1.3774	0.0	-.0167	.1023	.0277	.0647
2	1.1856	.1787	.0771	.0538	.1070	.1224
3	1.1856	0.0	.0909	.0506	.0991	.1368
4	1.1856	-.1787	.0442	-.0384	.0860	.0864
5	.9754	.2860	.2035	.1621	.2422	.2232
6	.9357	0.0	.2793	.1454	.2372	.2592
7	.5000	0.0	.8038	.7188	.7570	.6983
8	.5000	.5236	.7738	.7101	.7587	.6551
9	.5000	1.0472	.6772	.6438	.6692	.5327
10	.5000	1.5708	.6238	.5988	.6161	.5111
11	.5000	2.0944	.5987	.5966	.5951	.5255
12	.5000	2.6180	.5856	.6074	.5864	.5183
13	.5000	3.1416	.5836	.5991	.5831	.5039
14	.5000	-2.6180	.5875	.5844	.5827	.5183
15	.5000	-2.0944	.5881	.6033	.5854	.5255
16	.5000	-1.5708	.5945	.6323	.5926	.5255
17	.5000	-1.0472	.6264	.6381	.6108	.5615
18	.5000	- .5236	.7062	.6627	.6604	.5831
19	0.0	0.0	1.1026	1.0904	1.0866	.9647

Table 1

CONCLUSIONS

To obtain an approximate solution for a transversely loaded plate of irregular geometry, the point-matching method proved to be a relatively simple and direct method to employ. The main difficulty in applying the technique was in the proper choice and spacing of collocation points. It was only after more equally spacing the points according to arc length and increasing the number of collocation points that reasonable results were obtained. L. E. Hulbert and F. W. Niedenfuhr have suggested that some of these difficulties can be eliminated "by choosing a larger boundary point set than necessary, writing the boundary conditions at this set of points and solving the system of equations for the unknown constants by least squares."[4] The investigation of the addition of the least-squares technique to the point-matching method is recommended for further research.

The accuracy obtained using a single pole of expansion for the irregular plate was disappointing when compared to the accuracy that was obtained for the symmetric cases. This indicates that considerably less accuracy can be expected when working with completely unsymmetric shapes than when working with the relatively uniform symmetric shapes which have been considered in the literature.

The accuracy obtained by using double poles of expansion was good and produced the best solution. A multiple-pole scheme is therefore suggested whenever an irregular shape can be separated into more uniform regular shapes, even though, in so doing, more work is involved and the proper choice of collocation points is more difficult. Additional research could include moment and shear at the common boundary.

ACKNOWLEDGEMENT

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3. A. W. Leissa and F. W. Niedenfuhr, "A Study of the Cantilevered Square Plate Subjected to a Uniform Loading," Journal of the Aerospace Sciences, Feb. 1962, 162-169.

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Appendix A

COMPUTER PROGRAMS

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      THIS PROGRAM USES ONE ALBEGRAIC-TRIGONOMETRIC POLYNOMIAL TO DETERMINE
      RADIAL SLOPE, NORMAL SLOPE, MOMENTS, AND DEFLECTION.  THE BOUNDARY
      CONDITIONS ARE THAT THE DEFLECTION AND NORMAL SLOPE ARE ZERO
      IMPLICIT REAL*8(A-H,O-Z),INTEGER*4(I-N)
      DIMENSION RI(35),AN(35),BC(70),A(18),B(18),C(18),D(18),E(70,70),A1
     1(4900),CRI(222),CAN(222),CANR(222),ANR(35)
  101 FORMAT(4F20.8)
  113 FORMAT(1F80.8)
  102 FORMAT('0  SOLUTION CHECK IER= ',I4)
  103 FORMAT('1   COEFFICIENTS OBTAINED FOR THE ALGEBRAIC TRIGONOMETRIC
     1POLYNOMIAL')
  104 FORMAT('0   A(',I2,') =',D16.8,'  B(',I2,') =',D16.8,'  C(',I2,')
     1=',D16.8,'  D(',I2,') =',D16.8)
  106 FORMAT('1  THE DEFLECTION, RADIAL SLOPE, NORMAL SLOPE, AND MOMENTS
     1 ARE EVALUATED AT THE FOLLOWING SPECIFIED POINTS')
  107 FORMAT('1   SOLUTIONS AT ',I1,'/',I1,' THE RADIAL DISTANCE')
  108 FORMAT('-   AT THE POINT (',D16.8,',',D16.8,',',D16.8,')   DEFLECT
     1ION =',D16.8,5X,I3)
  118 FORMAT(68X,' NORMAL MOMENT =',D16.8)
  109 FORMAT(65X,'TANGENTIAL MOMENT =',D16.8)
  119 FORMAT(66X,' TWISTING MOMENT =',D16.8)
  112 FORMAT('0   AT THE POINT (',D16.8,',',D16.8,',',D16.8,')   RADIAL
     1SLOPE =',D16.8,5X,I3)
  122 FORMAT(71X,' NORMAL SLOPE =',D16.8)
  111 FORMAT('1    THE NORMAL AND RADIAL SLOPE AT SELECTED POINTS ALONG
     1THE BOUNDARY')
       WWWW IS THE DEFLECTION
       TNM IS THE NORMAL MOMENT
       TTM IS THE TANGENTIAL MOMENT
       TTWM IS THE TWISTING MOMENT
       RSLOPE IS THE RADIAL SLOPE
       PSLOPE IS THE NORMAL SLOPE
       K IS THE NUMBER OF POINTS MATCHED
       NE IS THE NUMBER OF EQUATIONS TO BE SOLVED
       M IS THE NUMBER OF EACH TYPE OF COEFFICIENT
       L IS THE NUMBER OF EACH TYPE OF COEFFICIENT MINUS ONE
       K=35
       NE=70
       M=18
       L=17
       EPS=1.0D-14
       READ(1,113)(RI(I),I=1,K)
       READ(1,113)(AN(I),I=1,K)
       READ(1,113)(ANR(I),I=1,K)
       KM=K+M
       KK=K+1
       DO 10 I=1,K
       R=RI(I)
       T=AN(I)
       TT=ANR(I)
       E(I,1)=1.0
       E(I,K+1)=R**2
```

```
      E(I+K,1)=0.0
      E(I+K,K+1)=(2*R**2)*DCOS(TT)
      DO 10 N=1,L
      E(I,N+1)=R**N*DCOS(N*T)
      E(I,N+M)=R**N*DSIN(N*T)
      E(I,N+KK)=R**(N+2)*DCOS(N*T)
      E(I,N+KM)=R**(N+2)*DSIN(N*T)
      E(I+K,N+1)=(N*R**N)*DCOS(N*T+TT)
      E(I+K,N+M)=(N*R**N)*DSIN(N*T+TT)
      E(I+K,N+KK)=R**(N+2)*(2*DCOS(N*T)*DCOS(TT)+N*DCOS(N*T+TT))
   10 E(I+K,N+KM)=R**(N+2)*(2*DSIN(N*T)*DCOS(TT)+N*DSIN(N*T+TT))
      DO 20 I=1,K
      R=RI(I)
      TT=ANR(I)
      BC(I)=-(R**4)
   20 BC(I+K)=-((4*R**4)*DCOS(TT))
      J=0
      DO 30 N=1,NE
      DO 30 I=1,NE
      J=J+1
   30 A1(J)=E(I,N)
      CALL DGELG(BC,A1,NE,1,EPS,IER)
      WRITE(3,102)IER
      WRITE(3,103)
      A(1)=BC(1)
      C(1)=BC(K+1)
      B(1)=0.0
      D(1)=0.0
      KL=K+L
      DO 40 I=2,M
      A(I)=BC(I)
      B(I)=BC(I+L)
      C(I)=BC(I+K)
   40 D(I)=BC(I+KL)
      DO 50 I=1,M
   50 WRITE(3,104)I,A(I),I,B(I),I,C(I),I,D(I)
      THE DEFLECTION, RADIAL SLOPE, NORMAL SLOPE, AND MOMENTS AT SELECTED
      POINTS ARE DETERMINED IN THE FOLLOWING
      J=222
      READ(1,101)(CRI(I),I=1,J)
      READ(1,101)(CAN(I),I=1,J)
      READ(1,101)(CANR(I),I=1,J)
      MM=1
      CHANGE THE ABOVE CARD IF MORE DIVISIONS IN THE RADIAL DISTANCES ARE
      DESIRED
      DO 60 K=1,MM
      WRITE(3,106)
      WRITE(3,107)K,MM
      DO 70 I=1,J
      R=(CRI(I)/MM)*K
      T=CAN(I)
      TT=CANR(I)
```

```
      WWWW=R**4+A(1)+C(1)*R**2
      DO 80 N=1,L
   80 WWWW=WWWW+(A(N+1)*R**N+C(N+1)*R**(N+2))*DCOS(N*T)+(B(N+1)*R**N+D(N
     1+1)*R**(N+2))*DSIN(N*T)
      CHANGE THE FOLLOWING CARD IF A DIFFERENT POISSON'S RATIO IS DESIRED
      VV=.33
      TNM=C(1)*2*(1+VV)+C(2)*2*R*(2*(1+VV)*DCOS(T)+(1-VV)*DCOS(T+2*TT))+
     1D(2)*2*R*(2*(1+VV)*DSIN(T)+(1-VV)*DSIN(T+2*TT))+A(3)*(1-VV)*2*DCOS
     1(2*T+2*TT)+B(3)*(1-VV)*2*DSIN(2*T+2*TT)+C(3)*3*R**2*(2*(1+VV)*DCOS
     1(2*T)+2*(1-VV)*DCOS(2*T+2*TT))+D(3)*3*R**2*(2*(1+VV)*DSIN(2*T)+2*(
     11-VV)*DSIN(2*T+2*TT))+4*R**2*(2*(1+VV)+(1-VV)*DCOS(2*TT))
      DO 90 N=3,L
   90 TNM=TNM+A(N+1)*(1-VV)*N*(N-1)*R**(N-2)*DCOS(N*T+2*TT)+B(N+1)*(1-VV
     1)*N*(N-1)*R**(N-2)*DSIN(N*T+2*TT)+C(N+1)*(N+1)*R**N*(2*(1+VV)*DCOS
     1(N*T)+N*(1-VV)*DCOS(N*T+2*TT))+D(N+1)*(N+1)*R**N*(2*(1+VV)*DSIN(N*
     1T)+N*(1-VV)*DSIN(N*T+2*TT))
      TNM=-(TNM/(64.0))
      THE NORMAL BENDING MOMENT SHOULD NOW BE MULTIPLIED BY Q*R**2
      TTM=-(C(1)*(2*(1+VV)))-(C(2)*2*R*(2*(1+VV)*DCOS(T)-(1-VV)*DCOS(T+2
     1*TT)))-(D(2)*2*R*(2*(1+VV)*DSIN(T)-(1-VV)*DSIN(T+2*TT)))+A(3)*(1-V
     1V)*2*DCOS(2*T+2*TT)+B(3)*(1-VV)*2*DSIN(2*T+2*TT)-(C(3)*3*R**2*(2*(
     11+VV)*DCOS(2*T)-(2*(1-VV)*DCOS(2*T+2*TT))))-(D(3)*3*R**2*(2*(1+VV)
     1*DSIN(2*T)-(2*(1-VV)*DSIN(2*T+2*TT))))-(4*R**2*(2*(1+VV)-(1-VV)*DC
     1OS(2*TT)))
      DO 100 N=3,L
  100 TTM=TTM+A(N+1)*(1-VV)*N*(N-1)*R**(N-2)*DCOS(N*T+2*TT)+B(N+1)*(1-VV
     1)*N*(N-1)*R**(N-2)*DSIN(N*T+2*TT)-(C(N+1)*(N+1)*R**N*(2*(1+VV)*DCO
     1S(N*T)-(N*(1-VV)*DCOS(N*T+2*TT))))-(D(N+1)*(N+1)*R**N*(2*(1+VV)*DS
     1IN(N*T)-(N*(1-VV)*DSIN(N*T+2*TT))))
      TTM=TTM/(64.0)
      TTWM=C(2)*2*R*DSIN(T+2*TT)-(D(2)*2*R*DCOS(T+2*TT))+A(3)*2*DSIN(2*T
     1+2*TT)-(B(3)*2*DCOS(2*T+2*TT))+C(3)*2*3*R**2*DSIN(2*T+2*TT)-(D(3)*
     12*3*R**2*DCOS(2*T+2*TT))+4*R**2*DSIN(2*TT)
      DO 110 N=3,L
  110 TTWM=TTWM+A(N+1)*N*(N-1)*R**(N-2)*DSIN(N*T+2*TT)-(B(N+1)*N*(N-1)*R
     1**(N-2)*DCOS(N*T+2*TT))+C(N+1)*N*(N+1)*R**N*DSIN(N*T+2*TT)-(D(N+1)
     1*N*(N+1)*R**N*DCOS(N*T+2*TT))
      TTWM=((1-VV)*TTWM)/(64.0)
      WRITE(3,108)R,T,TT,WWWW,I
      WRITE(3,118)TNM
      WRITE(3,109)TTM
   70 WRITE(3,119)TTWM
   60 CONTINUE
      WRITE(3,111)
      J=J-112
      DO 120 I=1,J
      R=CRI(I)
      T=CAN(I)
      TT=CANR(I)
      RSLOPE=2*C(1)*R+A(2)*DCOS(T)+B(2)*DSIN(T)+C(2)*3*R**2*DCOS(T)+D(2)
     1*3*R**2*DSIN(T)+4*R**3
      DO 130 N=2,L
```

```
130 PSLOPE=RSLOPE+A(N+1)*N*R**(N-1)*DCOS(N*T)+B(N+1)*N*R**(N-1)*DSIN(N
   1*T)+C(N+1)*(N+2)*R**(N+1)*DCOS(N*T)+D(N+1)*(N+2)*R**(N+1)*DSIN(N*T
   1)
    PSLOPE=4*R**3*DCOS(TT)+C(1)*R*2*DCOS(TT)+A(2)*DCOS(T+TT)+B(2)*DSIN
   1(T+TT)+C(2)*R**2*(2*DCOS(T)*DCOS(TT)+DCOS(T+TT))+D(2)*R**2*(2*DSIN
   1(T)*DCOS(TT)+DSIN(T+TT))
    DO 140 N=2,L
140 PSLOPE=PSLOPE+A(N+1)*N*R**(N-1)*DCOS(N*T+TT)+B(N+1)*N*R**(N-1)*DSI
   1N(N*T+TT)+C(N+1)*R**(N+1)*(2*DCOS(N*T)*DCOS(TT)+N*DCOS(N*T+TT))+D(
   1N+1)*R**(N+1)*(2*DSIN(N*T)*DCOS(TT)+N*DSIN(N*T+TT))
    WRITE(3,112)R,T,TT,RSLCPE,I
120 WRITE(3,122)PSLOPE
    STOP
    END
```

```
SUBROUTINE DGELG

PURPOSE
   TO SOLVE A GENERAL SYSTEM OF SIMULTANEOUS LINEAR EQUATIONS.

USAGE
   CALL DGELG(R,A,M,N,EPS,IER)

DESCRIPTION OF PARAMETERS
   R       - DOUBLE PRECISION M BY N RIGHT HAND SIDE MATRIX
             (DESTROYED). ON RETURN R CONTAINS THE SOLUTIONS
             OF THE EQUATIONS.
   A       - DOUBLE PRECISION M BY M COEFFICIENT MATRIX
             (DESTROYED).
   M       - THE NUMBER OF EQUATIONS IN THE SYSTEM.
   N       - THE NUMBER OF RIGHT HAND SIDE VECTORS.
   EPS     - SINGLE PRECISION INPUT CONSTANT WHICH IS USED AS
             RELATIVE TOLERANCE FOR TEST ON LOSS OF
             SIGNIFICANCE.
   IER     - RESULTING ERROR PARAMETER CODED AS FOLLOWS
             IER=0  - NO ERROR,
             IER=-1 - NO RESULT BECAUSE OF M LESS THAN 1 OR
                      PIVOT ELEMENT AT ANY ELIMINATION STEP
                      EQUAL TO 0,
             IER=K  - WARNING DUE TO POSSIBLE LOSS OF SIGNIFI-
                      CANCE INDICATED AT ELIMINATION STEP K+1,
                      WHERE PIVOT ELEMENT WAS LESS THAN OR
                      EQUAL TO THE INTERNAL TOLERANCE EPS TIMES
                      ABSOLUTELY GREATEST ELEMENT OF MATRIX A.

REMARKS
   INPUT MATRICES R AND A ARE ASSUMED TO BE STORED COLUMNWISE
   IN M*N RESP. M*M SUCCESSIVE STORAGE LOCATIONS. ON RETURN
   SOLUTION MATRIX R IS STORED COLUMNWISE TOO.
   THE PROCEDURE GIVES RESULTS IF THE NUMBER OF EQUATIONS M IS
   GREATER THAN 0 AND PIVOT ELEMENTS AT ALL ELIMINATION STEPS
   ARE DIFFERENT FROM 0. HOWEVER WARNING IER=K - IF GIVEN -
   INDICATES POSSIBLE LOSS OF SIGNIFICANCE. IN CASE OF A WELL
   SCALED MATRIX A AND APPROPRIATE TOLERANCE EPS, IER=K MAY BE
   INTERPRETED THAT MATRIX A HAS THE RANK K. NO WARNING IS
   GIVEN IN CASE M=1.

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
   NONE

METHOD
   SOLUTION IS DONE BY MEANS OF GAUSS-ELIMINATION WITH
   COMPLETE PIVOTING.

..................................................................

SUBROUTINE DGELG(R,A,M,N,EPS,IER)
```

```
      DIMENSION A(1),R(1)
      DOUBLE PRECISION R,A,PIV,TB,TOL,PIVI,DABS,EPS
      IF(M)23,23,1

      SEARCH FOR GREATEST ELEMENT IN MATRIX A
    1 IER=0
      PIV=0.D0
      MM=M*M
      NM=N*M
      DO 3 L=1,MM
      TB=DABS(A(L))
      IF(TB-PIV)3,3,2
    2 PIV=TB
      I=L
    3 CONTINUE
      TOL=EPS*PIV
      A(I) IS PIVOT ELEMENT. PIV CONTAINS THE ABSOLUTE VALUE OF A(I).


      START ELIMINATION LOOP
      LST=1
      DO 17 K=1,M

      TEST ON SINGULARITY
      IF(PIV)23,23,4
    4 IF(IER)7,5,7
    5 IF(PIV-TOL)6,6,7
    6 IER=K-1
    7 PIVI=1.D0/A(I)
      J=(I-1)/M
      I=I-J*M-K
      J=J+1-K
      I+K IS ROW-INDEX, J+K COLUMN-INDEX OF PIVOT ELEMENT

      PIVOT ROW REDUCTION AND ROW INTERCHANGE IN RIGHT HAND SIDE R
      DO 8 L=K,NM,M
      LL=L+I
      TB=PIVI*R(LL)
      R(LL)=R(L)
    8 R(L)=TB

      IS ELIMINATION TERMINATED
      IF(K-M)9,18,18

      COLUMN INTERCHANGE IN MATRIX A
    9 LEND=LST+M-K
      IF(J)12,12,10
   10 II=J*M
      DO 11 L=LST,LEND
      TB=A(L)
      LL=L+II
      A(L)=A(LL)
   11 A(LL)=TB
```

```
   ROW INTERCHANGE AND PIVOT ROW REDUCTION IN MATRIX A
12 DO 13 L=LST,MM,M
   LL=L+I
   TB=PIVI*A(LL)
   A(LL)=A(L)
13 A(L)=TB

   SAVE COLUMN INTERCHANGE INFORMATION
   A(LST)=J

   ELEMENT REDUCTION AND NEXT PIVOT SEARCH
   PIV=0.DO
   LST=LST+1
   J=0
   DO 16 II=LST,LEND
   PIVI=-A(II)
   IST=II+M
   J=J+1
   DO 15 L=IST,MM,M
   LL=L-J
   A(L)=A(L)+PIVI*A(LL)
   TB=DABS(A(L))
   IF(TB-PIV)15,15,14
14 PIV=TB
   I=L
15 CONTINUE
   DO 16 L=K,NM,M
   LL=L+J
16 R(LL)=R(LL)+PIVI*R(L)
17 LST=LST+M
   END OF ELIMINATION LOOP


   BACK SUBSTITUTION AND BACK INTERCHANGE
18 IF(M-1)23,22,19
19 IST=MM+M
   LST=M+1
   DO 21 I=2,M
   II=LST-I
   IST=IST-LST
   L=IST-M
   L=A(L)+.5DO
   DO 21 J=II,NM,M
   TB=R(J)
   LL=J
   DO 20 K=IST,MM,M
   LL=LL+1
20 TB=TB-A(K)*R(LL)
   K=J+L
   R(J)=R(K)
21 R(K)=TB
22 RETURN
```

```
C     ERROR RETURN
   23 IER=-1
      RETURN
      END
```

```
      IMPLICIT REAL*8(A-H,O-Z),INTEGER*4(I-N)
      DIMENSION RI(30),AN(30),BC(56),A(11),B(11),C(11),D(11),U(4),V(4),
     1(4),E(56,56),A1(3136),X(4),CRI(270),CAN(270),ANR(30),CANR(270)
  101 FORMAT(4F20.8)
  102 FORMAT('0    SOLUTION CHECK IER= ',I3)
  103 FORMAT('1    COEFFICIENTS OBTAINED FOR THE DEFLECTION EQUATION OF T
     1HE LARGER CIRCLE',/)
  104 FORMAT('0    A(',I2,') =',D16.8,'    C(',I2,') =',D16.8)
  105 FORMAT('0    B(',I2,') =',D16.8,'    D(',I2,') =',D16.8)
  106 FORMAT('1    COEFFICIENTS OBTAINED FOR THE DEFLECTION EQUATION OF T
     1HE SMALLER CIRCLE',/)
  107 FORMAT('0   AA(',I2,') =',D16.8,'   CC(',I2,') =',D16.8)
  108 FORMAT('0   BB(',I2,') =',D16.8,'   DD(',I2,') =',D16.8)
  109 FORMAT('1   THE DEFLECTION AND MOMENTS AT POINTS IN THE LARGE CIRCL
     1E')
  110 FORMAT('1   SOLUTIONS AT ',I1,'/',I1,' THE RADIAL DISTANCE')
  111 FORMAT('- AT (',D16.8,',',D16.8,',',D16.8,')  DEF =',D16.8,'     NO
     1RMAL MOMENT =',D16.8,5X,I3)
  112 FORMAT(1H0,44X,'TANGENTIAL MOMENT =',D16.8,'  TWISTING MOMENT =',D
     116.8)
  113 FORMAT('1      THE NORMAL AND RADIAL SLOPE AT SELECTED POINTS ALONG
     1THE BOUNDARY OF THE LARGE CIRCLE')
  114 FORMAT('0 AT (',D16.8,',',D16.8,',',D16.8,')  RSLOPE =',D16.8,'  N
     1SLOPE =',D16.8,5X,I3)
  116 FORMAT('   THE DEFLECTION AND MOMENTS AT POINTS IN THE SMALL CIRCL
     *E')
  117 FORMAT('1      THE NORMAL AND RADIAL SLOPE AT SELECTED POINTS ALONG
     1THE BOUNDARY OF THE SMALL CIRCLE')
  123 FORMAT(1F80.8)
      K IS THE NUMBER OF POINTS MATCHED ON THE LARGE AND SMALL CIRCLE
      PLUS TWICE THE NUMBER OF POINTS ON THE COMMON BOUNDARY
      NE IS THE NUMBER OF EQUATIONS TO BE SOLVED
      NCL IS THE NUMBER OF COEFFICIENTS OF EACH TYPE IN THE LARGE CIRCLE
      NCS IS THE NUMBER OF COEFFICIENTS OF EACH TYPE IN THE SMALL CIRCLE
      ML IS THE NUMBER OF POINTS MATCHED ON THE LARGE CIRCLE
      MS IS THE NUMBER OF POINTS MATCHED ON THE SMALL CIRCLE
      MB IS THE NUMBER OF POINTS MATCHED ALONG THE COMMON BOUNDARY
      WWWW IS THE DEFLECTION
      TNM IS THE NORMAL MOMENT
      TTM IS THE TANGENTIAL MOMENT
      TTWM IS THE TWISTING MOMENT
      RSLOPE IS THE RADIAL SLOPE
      PSLOPE IS THE NORMAL SLOPE
      K=30
      NE=56
      NCL=11
      NCS=4
      ML=20
      MS=6
      MB=2
      NL=NCL-1
      NS=NCS-1
```

```
      NCL2=NCL*2
      N2=(NCL*3)-1
      ML2=ML*2
      N3=(NCL*4)-1
      N4=(ML*2)+1
      N5=2*ML+2*MS
      N6=(4*NCL)-2
      ML1=ML+1
      N7=ML+MS
      N8=4*NCL+2*NS
      N9=N3+NS
      N11=N8+NS
      N12=N7+1
      N13=N7+MB
      N14=NCL+1
      N15=NCL2+1
      NCL3=NCL*3
      N16=N13+1
      N17=N7-MB
      N18=NCL*4
      N19=N18+NS
      N20=N8+1
      N21=N11+1
      N22=NCL2-1
      N23=NCL3-2
      N24=N9-1
      N25=N8-1
      N26=N11-1
      EPS=1.0D-14
      READ(1,123)(RI(I),I=1,K)
      READ(1,123)(AN(I),I=1,K)
      READ(1,123)(ANR(I),I=N12,K)
      LOWER CIRCLE - EQUATIONS IN MATRIX FOR DEFLECTION
      DO 10 I=1,ML
      R=RI(I)
      T=AN(I)
      E(I,1)=1.0
      E(I,NCL2)=R**2
      DO 10 N=1,NL
      E(I,N+1)=R**N*DCOS(N*T)
      E(I,N+NCL)=R**N*DSIN(N*T)
      E(I,N+NCL2)=R**(N+2)*DCOS(N*T)
   10 E(I,N+N2)=R**(N+2)*DSIN(N*T)
      LOWER CIRCLE - EQUATIONS IN MATRIX FOR RADIAL SLOPE
      DO 20 I=1,ML
      R=RI(I)
      T=AN(I)
      E(I+ML,1)=0.0
      E(I+ML,NCL2)=2*R**2
      DO 20 N=1,NL
      E(I+ML,N+1)=N*R**N*DCOS(N*T)
      E(I+ML,N+NCL)=N*R**N*DSIN(N*T)
```

```
      E(I+ML,N+NCL2)=(N+2)*R**(N+2)*DCOS(N*T)
   20 E(I+ML,N+N2)=(N+2)*R**(N+2)*DSIN(N*T)
      ZERO MATRIX ELEMENTS
      DO 30 I=1,ML2
      DO 30 N=N3,NE
   30 E(I,N)=0.0
      DO 40 I=N4,N5
      DO 40 N=1,N6
   40 E(I,N)=0.0
      UPPER CIRCLE - EQUATIONS IN MATRIX FOR DEFLECTION
      DO 50 I=ML1,N7
      R=RI(I)
      T=AN(I)
      E(I+ML,N3)=1.0
      E(I+ML,N8)=R**2
      DO 50 N=1,NS
      E(I+ML,N+N3)=R**N*DCOS(N*T)
      E(I+ML,N+N9)=R**N*DSIN(N*T)
      E(I+ML,N+N8)=R**(N+2)*DCOS(N*T)
   50 E(I+ML,N+N11)=R**(N+2)*DSIN(N*T)
      UPPER CIRCLE - EQUATIONS IN MATRIX FOR RADIAL SLOPE
      DO 60 I=ML1,N7
      R=RI(I)
      T=AN(I)
      E(I+N7,N3)=0.0
      E(I+N7,N8)=2*R**2
      DO 60 N=1,NS
      E(I+N7,N+N3)=N*R**N*DCOS(N*T)
      E(I+N7,N+N9)=N*R**N*DSIN(N*T)
      E(I+N7,N+N8)=(N+2)*R**(N+2)*DCOS(N*T)
   60 E(I+N7,N+N11)=(N+2)*R**(N+2)*DSIN(N*T)
      EQUATIONS IN MATRIX OF DEFLECTION FOR LOWER CIRCLE ALONG THE
      COMMON BOUNDARY
      DO 70 I=N12,N13
      R=RI(I)
      T=AN(I)
      E(I+N7,1)=1.0
      E(I+N7,NCL2)=R**2
      DO 70 N=1,NL
      E(I+N7,N+1)=R**N*DCOS(N*T)
      E(I+N7,N+NCL)=R**N*DSIN(N*T)
      E(I+N7,N+NCL2)=R**(N+2)*DCOS(N*T)
   70 E(I+N7,N+N2)=R**(N+2)*DSIN(N*T)
      EQUATIONS IN MATRIX OF NORMAL SLOPE FOR LOWER CIRCLE ALONG THE
      COMMON BOUNDARY
      DO 80 I=N12,N13
      R=RI(I)
      T=AN(I)
      TT=ANR(I)
      E(I+N13,1)=0.0
      E(I+N13,2)=DCOS(T+TT)
      E(I+N13,N14)=DSIN(T+TT)
```

```
      E(I+N13,NCL2)=2*R*DCOS(TT)
      E(I+N13,N15)=(R**2)*(2*DCOS(T)*DCOS(TT)+DCOS(T+TT))
      E(I+N13,NCL3)=(R**2)*(2*DSIN(T)*DCOS(TT)+DSIN(T+TT))
      DO 80 N=2,NL
      E(I+N13,N+1)=N*(R**(N-1))*DCOS(N*T+TT)
      E(I+N13,N+NCL)=N*(R**(N-1))*DSIN(N*T+TT)
      E(I+N13,N+NCL2)=(R**(N+1))*(2*DCOS(N*T)*DCOS(TT)+N*DCOS(N*T+TT))
   80 E(I+N13,N+N2)=(R**(N+1))*(2*DSIN(N*T)*DCOS(TT)+N*DSIN(N*T+TT))
      EQUATIONS IN MATRIX OF DEFLECTION FOR UPPER CIRCLE ALONG THE
      COMMON BOUNDARY
      DO 90 I=N16,K
      R=RI(I)
      T=AN(I)
      E(I+N17,N3)=-1.0
      E(I+N17,N8)=-(R**2)
      DO 90 N=1,NS
      E(I+N17,N+N3)=-(R**N*DCOS(N*T))
      E(I+N17,N+N9)=-(R**N*DSIN(N*T))
      E(I+N17,N+N8)=-(R**(N+2)*DCOS(N*T))
   90 E(I+N17,N+N11)=-(R**(N+2)*DSIN(N*T))
      EQUATIONS IN MATRIX OF NORMAL SLOPE FOR UPPER CIRCLE ALONG THE
      COMMON BOUNDARY
      DO 100 I=N16,K
      R=RI(I)
      T=AN(I)
      TT=ANR(I)
      E(I+N7,N3)=0.0
      E(I+N7,N18)=DCOS(T+TT)
      E(I+N7,N19)=DSIN(T+TT)
      E(I+N7,N8)=2*R*DCOS(TT)
      E(I+N7,N20)=(R**2)*(2*DCOS(T)*DCOS(TT)+DCOS(T+TT))
      E(I+N7,N21)=(R**2)*(2*DSIN(T)*DCOS(TT)+DSIN(T+TT))
      DO 100 N=2,NS
      E(I+N7,N+N3)=(N*(R**(N-1))*DCOS(N*T+TT))
      E(I+N7,N+N9)=(N*(R**(N-1))*DSIN(N*T+TT))
      E(I+N7,N+N8)=((R**(N+1))*(2*DCOS(N*T)*DCOS(TT)+N*DCOS(N*T+TT)))
  100 E(I+N7,N+N11)=((R**(N+1))*(2*DSIN(N*T)*DCOS(TT)+N*DSIN(N*T+TT)))
      BOUNDARY CONDITION - DEFLECTION AND SLOPE OF LOWER CIRCLE ZERO
      DO 115 I=1,ML
      R=RI(I)
      BC(I)=-(R**4)
  115 BC(I+ML)=-(4*R**4)
      BOUNDARY CONDITION - DEFLECTION AND SLOPE OF UPPER CIRCLE ZERO
      DO 120 I=ML1,N7
      R=RI(I)
      BC(I+ML)=-(R**4)
  120 BC(I+N7)=-(4*R**4)
      BOUNDARY CONDITION - NORMAL SLOPE ALONG THE COMMON BOUNDARY THE SAME
      DO 125 I=N12,N13
      R=RI(I)
      TT=ANR(I)
      RR=RI(I+MB)
```

```
      TTT=ANR(I+MB)
  125 BC(I+N13)=-(4*R**3*DCOS(TT))-(4*RR**3*DCOS(TTT))
      BOUNDARY CONDITION - DEFLECTION ALONG COMMON BOUNDARY THE SAME
      DO 126 I=N12,N13
      R=RI(I)
      RR=RI(I+MB)
  126 BC(I+N7)=(RR**4)-(R**4)
      J=0
      DO 130 N=1,NE
      DO 130 I=1,NE
      J=J+1
  130 A1(J)=E(I,N)
      CALL DGELG(BC,A1,NE,1,EPS,IER)
      WRITE(3,102)IER
      WRITE(3,103)
      A(1)=BC(1)
      B(1)=0.0
      C(1)=BC(NCL2)
      D(1)=0.0
      U(1)=BC(N3)
      V(1)=0.0
      W(1)=BC(N8)
      X(1)=0.0
      DO 140 I=2,NCL
      A(I)=BC(I)
      B(I)=BC(I+NL)
      C(I)=BC(I+N22)
  140 D(I)=BC(I+N23)
      DO 141 I=2,NCS
      U(I)=BC(I+N6)
      V(I)=BC(I+N24)
      W(I)=BC(I+N25)
  141 X(I)=BC(I+N26)
      DO 144 I=1,NCL
      WRITE(3,104)I,A(I),I,C(I)
  144 WRITE(3,105)I,B(I),I,D(I)
      WRITE(3,106)
      DO 150 I=1,NCS
      WRITE(3,107)I,U(I),I,W(I)
  150 WRITE(3,108)I,V(I),I,X(I)
      CHANGE THE FOLLOWING CARD WHEN DIFFERENT NUMBERS OF POINTS ARE USED
      J=270
      READ(1,101)(CRI(I),I=1,J)
      READ(1,101)(CAN(I),I=1,J)
      READ(1,101)(CANR(I),I=1,J)
      WRITE(3,109)
      MM=1
      CHANGE THE ABOVE CARD IF MORE DIVISIONS IN THE RADIAL DISTANCES OF
      THE LARGE CIRCLE ARE DESIRED.
      DO 61 K=1,MM
      WRITE(3,110)K,MM
      CHANGE THE FOLLOWING CARD WHEN DIFFERENT NUMBERS OF POINTS ARE USED
```

```
      DO 71 I=1,164
      R=(CRI(I)/MM)*K
      T=CAN(I)
      TT=CANR(I)
      WWWW=R**4+A(1)+C(1)*R**2
      DO 81 N=1,NL
   81 WWWW=WWWW+(A(N+1)*R**N+C(N+1)*R**(N+2))*DCOS(N*T)+(B(N+1)*R**N+D(N
     1+1)*R**(N+2))*DSIN(N*T)
C     CHANGE THE FOLLOWING CARD IF A DIFFERENT POISSON'S RATIO IS DESIRED
      VV=.33
      TNM=C(1)*2*(1+VV)+C(2)*2*R*(2*(1+VV)*DCOS(T)+(1-VV)*DCOS(T+2*TT))+
     1D(2)*2*R*(2*(1+VV)*DSIN(T)+(1-VV)*DSIN(T+2*TT))+A(3)*(1-VV)*2*DCOS
     1(2*T+2*TT)+B(3)*(1-VV)*2*DSIN(2*T+2*TT)+C(3)*3*R**2*(2*(1+VV)*DCOS
     1(2*T)+2*(1-VV)*DCOS(2*T+2*TT))+D(3)*3*R**2*(2*(1+VV)*DSIN(2*T)+2*(
     11-VV)*DSIN(2*T+2*TT))+4*R**2*(2*(1+VV)+(1-VV)*DCOS(2*TT))
      DO 91 N=3,NL
   91 TNM=TNM+A(N+1)*(1-VV)*N*(N-1)*R**(N-2)*DCOS(N*T+2*TT)+B(N+1)*(1-VV
     1)*N*(N-1)*R**(N-2)*DSIN(N*T+2*TT)+C(N+1)*(N+1)*R**N*(2*(1+VV)*DCOS
     1(N*T)+N*(1-VV)*DCOS(N*T+2*TT))+D(N+1)*(N+1)*R**N*(2*(1+VV)*DSIN(N*
     1T)+N*(1-VV)*DSIN(N*T+2*TT))
      TNM=-(TNM/(64.0))
C     THE NORMAL BENDING MOMENT SHOULD NOW BE MULTIPLIED BY Q*R**2
      TTM=-(C(1)*(2*(1+VV)))-(C(2)*2*R*(2*(1+VV)*DCOS(T)-(1-VV)*DCOS(T+2
     1*TT)))-(D(2)*2*R*(2*(1+VV)*DSIN(T)-(1-VV)*DSIN(T+2*TT)))+A(3)*(1-V
     1V)*2*DCOS(2*T+2*TT)+B(3)*(1-VV)*2*DSIN(2*T+2*TT)-(C(3)*3*R**2*(2*(
     11+VV)*DCOS(2*T)-(2*(1-VV)*DCOS(2*T+2*TT))))-(D(3)*3*R**2*(2*(1+VV)
     1*DSIN(2*T)-(2*(1-VV)*DSIN(2*T+2*TT))))-(4*R**2*(2*(1+VV)-(1-VV)*DC
     1OS(2*TT)))
      DO 151 N=3,NL
  151 TTM=TTM+A(N+1)*(1-VV)*N*(N-1)*R**(N-2)*DCOS(N*T+2*TT)+B(N+1)*(1-VV
     1)*N*(N-1)*R**(N-2)*DSIN(N*T+2*TT)-(C(N+1)*(N+1)*R**N*(2*(1+VV)*DCO
     1S(N*T)-(N*(1-VV)*DCOS(N*T+2*TT))))-(D(N+1)*(N+1)*R**N*(2*(1+VV)*DS
     1IN(N*T)-(N*(1-VV)*DSIN(N*T+2*TT))))
      TTM=TTM/(64.0)
      TTWM=C(2)*2*R*DSIN(T+2*TT)-(D(2)*2*R*DCOS(T+2*TT))+A(3)*2*DSIN(2*T
     1+2*TT)-(B(3)*2*DCOS(2*T+2*TT))+C(3)*2*3*R**2*DSIN(2*T+2*TT)-(D(3)*
     12*3*R**2*DCOS(2*T+2*TT))+4*R**2*DSIN(2*TT)
      DO 161 N=3,NL
  161 TTWM=TTWM+A(N+1)*N*(N-1)*R**(N-2)*DSIN(N*T+2*TT)-(B(N+1)*N*(N-1)*R
     1**(N-2)*DCOS(N*T+2*TT))+C(N+1)*N*(N+1)*R**N*DSIN(N*T+2*TT)-(D(N+1)
     1*N*(N+1)*R**N*DCOS(N*T+2*TT))
      TTWM=((1-VV)*TTWM)/(64.0)
      WRITE(3,111)R,T,TT,WWWW,TNM,I
   71 WRITE(3,112)TTM,TTWM
   61 CONTINUE
      WRITE(3,113)
C     CHANGE THE FOLLOWING CARD WHEN DIFFERENT NUMBERS OF POINTS ARE USED
      DO 121 I=1,88
      R=CRI(I)
      T=CAN(I)
      TT=CANR(I)
```

```
      RSLOPE=2*C(1)*R+A(2)*DCOS(T)+B(2)*DSIN(T)+C(2)*3*R**2*DCOS(T)+D(2)
     1*3*R**2*DSIN(T)+4*R**3
      DO 131 N=2,NL
  131 RSLOPE=RSLOPE+A(N+1)*N*R**(N-1)*DCOS(N*T)+B(N+1)*N*R**(N-1)*DSIN(N
     1*T)+C(N+1)*(N+2)*R**(N+1)*DCOS(N*T)+D(N+1)*(N+2)*R**(N+1)*DSIN(N*T
     1)
      PSLOPE=4*R**3*DCOS(TT)+C(1)*R*2*DCOS(TT)+A(2)*DCOS(T+TT)+B(2)*DSIN
     1(T+TT)+C(2)*R**2*(2*DCOS(T)*DCOS(TT)+DCOS(T+TT))+D(2)*R**2*(2*DSIN
     1(T)*DCOS(TT)+DSIN(T+TT))
      DO 143 N=2,NL
  143 PSLOPE=PSLOPE+A(N+1)*N*R**(N-1)*DCOS(N*T+TT)+B(N+1)*N*R**(N-1)*DSI
     1N(N*T+TT)+C(N+1)*R**(N+1)*(2*DCOS(N*T)*DCOS(TT)+N*DCOS(N*T+TT))+D(
     1N+1)*R**(N+1)*(2*DSIN(N*T)*DCOS(TT)+N*DSIN(N*T+TT))
  121 WRITE(3,114)R,T,TT,RSLCPE,PSLOPE,I
      THE DEFLECTION, RADIAL SLOPE, NORMAL SLOPE, AND MOMENTS AT SELECTED
      POINTS OF THE SMALL CIRCLE ARE DETERMINED HERE
      WRITE(3,116)
      LL=1
      CHANGE THE ABOVE CARD IF MORE DIVISIONS IN THE RADIAL DISTANCES OF
      THE SMALL CIRCLE ARE DESIRED.
      DO 62 K=1,LL
      WRITE(3,110)K,LL
      CHANGE THE FOLLOWING CARD WHEN DIFFERENT NUMBERS OF POINTS ARE USED
      DO 72 I=165,J
      R=(CRI(I)/LL)*K
      T=CAN(I)
      TT=CANR(I)
      WWWW=R**4+U(1)+W(1)*R**2
      DO 82 N=1,NS
   82 WWWW=WWWW+(U(N+1)*R**N+W(N+1)*R**(N+2))*DCOS(N*T)+(V(N+1)*R**N+X(N
     1+1)*R**(N+2))*DSIN(N*T)
      TNM=W(1)*2*(1+VV)+W(2)*2*R*(2*(1+VV)*DCOS(T)+(1-VV)*DCOS(T+2*TT))+
     1X(2)*2*R*(2*(1+VV)*DSIN(T)+(1-VV)*DSIN(T+2*TT))+U(3)*(1-VV)*2*DCOS
     1(2*T+2*TT)+V(3)*(1-VV)*2*DSIN(2*T+2*TT)+W(3)*3*R**2*(2*(1+VV)*DCOS
     1(2*T)+2*(1-VV)*DCOS(2*T+2*TT))+X(3)*3*R**2*(2*(1+VV)*DSIN(2*T)+2*(
     11-VV)*DSIN(2*T+2*TT))+4*R**2*(2*(1+VV)+(1-VV)*DCOS(2*TT))
      DO 92 N=3,NS
   92 TNM=TNM+J(N+1)*(1-VV)*N*(N-1)*R**(N-2)*DCOS(N*T+2*TT)+V(N+1)*(1-VV
     1)*N*(N-1)*R**(N-2)*DSIN(N*T+2*TT)+W(N+1)*(N+1)*R**N*(2*(1+VV)*DCOS
     1(N*T)+N*(1-VV)*DCOS(N*T+2*TT))+X(N+1)*(N+1)*R**N*(2*(1+VV)*DSIN(N*
     1T)+N*(1-VV)*DSIN(N*T+2*TT))
      TNM=-(TNM/(64.0))
      TTM=-(W(1)*(2*(1+VV)))-(W(2)*2*R*(2*(1+VV)*DCOS(T)-(1-VV)*DCOS(T+2
     1*TT)))-(X(2)*2*R*(2*(1+VV)*DSIN(T)-(1-VV)*DSIN(T+2*TT)))+U(3)*(1-V
     1V)*2*DCOS(2*T+2*TT)+V(3)*(1-VV)*2*DSIN(2*T+2*TT)-(W(3)*3*R**2*(2*(
     11+VV)*DCOS(2*T)-(2*(1-VV)*DCOS(2*T+2*TT))))-(X(3)*3*R**2*(2*(1+VV)
     1*DSIN(2*T)-(2*(1-VV)*DSIN(2*T+2*TT))))-(4*R**2*(2*(1+VV)-(1-VV)*DC
     1OS(2*TT)))
      DO 152 N=3,NS
  152 TTM=TTM+U(N+1)*(1-VV)*N*(N-1)*R**(N-2)*DCOS(N*T+2*TT)+V(N+1)*(1-VV
     1)*N*(N-1)*R**(N-2)*DSIN(N*T+2*TT)-(W(N+1)*(N+1)*R**N*(2*(1+VV)*DCO
     1S(N*T)-(N*(1-VV)*DCOS(N*T+2*TT))))-(X(N+1)*(N+1)*R**N*(2*(1+VV)*DS
```

```
     1IN(N*T)-(N*(1-VV)*DSIN(N*T+2*TT))))
      TTM=TTM/(64.0)
      TTWM=W(2)*2*R*DSIN(T+2*TT)-(X(2)*2*R*DCOS(T+2*TT))+U(3)*2*DSIN(2*T
     1+2*TT)-(V(3)*2*DCOS(2*T+2*TT))+W(3)*2*3*R**2*DSIN(2*T+2*TT)-(X(3)*
     12*3*R**2*DCOS(2*T+2*TT))+4*R**2*DSIN(2*TT)
      DO 162 N=3,NS
  162 TTWM=TTWM+U(N+1)*N*(N-1)*R**(N-2)*DSIN(N*T+2*TT)-(V(N+1)*N*(N-1)*R
     1**(N-2)*DCOS(N*T+2*TT))+W(N+1)*N*(N+1)*R**N*DSIN(N*T+2*TT)-(X(N+1)
     1*N*(N+1)*R**N*DCOS(N*T+2*TT))
      TTWM=((1-VV)*TTWM)/(64.0)
      WRITE(3,111)R,T,TT,WWWW,TNM,I
   72 WRITE(3,112)TTM,TTWM
   62 CONTINUE
      WRITE(3,117)
      CHANGE THE FOLLOWING CARD WHEN DIFFERENT NUMBERS OF POINTS ARE USED
      DO 122 I=165,238
      R=CRI(I)
      T=CAN(I)
      TT=CANR(I)
      RSLOPE=2*W(1)*R+U(2)*DCOS(T)+V(2)*DSIN(T)+W(2)*3*R**2*DCOS(T)+X(2)
     1*3*R**2*DSIN(T)+4*R**3
      DO 132 N=2,NS
  132 RSLOPE=RSLOPE+U(N+1)*N*R**(N-1)*DCOS(N*T)+V(N+1)*N*R**(N-1)*DSIN(N
     1*T)+W(N+1)*(N+2)*R**(N+1)*DCOS(N*T)+X(N+1)*(N+2)*R**(N+1)*DSIN(N*T
     1)
      PSLOPE=4*R**3*DCOS(TT)+W(1)*R*2*DCOS(TT)+U(2)*DCOS(T+TT)+V(2)*DSIN
     1(T+TT)+W(2)*R**2*(2*DCCS(T)*DCOS(TT)+DCOS(T+TT))+X(2)*R**2*(2*DSIN
     1(T)*DCOS(TT)+DSIN(T+TT))
      DO 142 N=2,NS
  142 PSLOPE=PSLOPE+U(N+1)*N*R**(N-1)*DCOS(N*T+TT)+V(N+1)*N*R**(N-1)*DSI
     1N(N*T+TT)+W(N+1)*R**(N+1)*(2*DCOS(N*T)*DCOS(TT)+N*DCOS(N*T+TT))+X(
     1N+1)*R**(N+1)*(2*DSIN(N*T)*DCOS(TT)+N*DSIN(N*T+TT))
  122 WRITE(3,114)R,T,TT,RSLCPE,PSLOPE,I
      STOP
      END
```

Appendix B

COLLOCATION POINTS

CASE I

#	r	θ	ϕ
1	1.58823490	0.0	0.0
2	1.56944847	0.09199870	0.25706393
3	1.51405334	0.17994565	0.51818699
4	1.42356110	0.26009887	0.79194361
5	1.26362896	0.39429474	0.65774769
6	1.12099552	0.58313787	0.46890461
7	1.03663349	0.78539997	0.26664245
8	1.00000000	1.05204201	0.0
9	1.00000000	1.30899906	0.0
10	1.00000000	1.57079983	0.0
11	1.00000000	1.74532986	0.0
12	1.00000000	1.91985989	0.0
13	1.00000000	2.09439945	0.0
14	1.00000000	2.26892948	0.0
15	1.00000000	2.44345951	0.0
16	1.00000000	2.61798954	0.0
17	1.00000000	2.87978935	0.0
18	1.00000000	3.05432987	0.0
19	1.00000000	3.22885513	0.0
20	1.00000000	3.40339565	0.0
21	1.00000000	3.57792568	0.0
22	1.00000000	3.75245571	0.0
23	1.00000000	3.92698574	0.0
24	1.00000000	4.18878937	0.0
25	1.00000000	4.45058537	0.0
26	1.00000000	4.62512589	0.0
27	1.00000000	4.79965591	0.0
28	1.00000000	4.97418594	0.0
29	1.00000000	5.23598957	0.0
30	1.00000000	5.49798584	0.0
31	1.00000000	5.75958920	0.0
32	1.00000000	-0.36044776	0.0
33	1.39804935	-0.27680981	-0.85765344
34	1.49478626	-0.20076752	-0.58462578
35	1.55895138	-0.11453408	-0.32179922

CASE II

#	r	θ	φ
1	1.58823490	0.0	0.0
2	1.56944847	0.09199870	0.0
3	1.51405334	0.17994565	0.0
4	1.42356110	0.26009887	0.0
5	1.26362896	0.39429474	0.0
6	1.12099552	0.58313787	0.0
7	1.03663349	0.78539997	0.0
8	1.00000000	1.05204201	0.0
9	1.00000000	1.30899906	0.0
10	1.00000000	1.57079983	0.0
11	1.00000000	1.74532986	0.0
12	1.00000000	1.91985989	0.0
13	1.00000000	2.09439945	0.0
14	1.00000000	2.26892948	0.0
15	1.00000000	2.44345951	0.0
16	1.00000000	2.61798954	0.0
17	1.00000000	2.87978935	0.0
18	1.00000000	3.05432987	0.0
19	1.00000000	3.22885513	0.0
20	1.00000000	3.40339565	0.0
21	1.00000000	3.57792568	0.0
22	1.00000000	3.75245571	0.0
23	1.00000000	3.92698574	0.0
24	1.00000000	4.18878937	0.0
25	1.00000000	4.45058537	0.0
26	1.00000000	4.62512589	0.0
27	1.00000000	4.79965591	0.0
28	1.00000000	4.97418594	0.0
29	1.00000000	5.23598957	0.0
30	1.00000000	5.49798584	0.0
31	1.00000000	5.75958920	0.0
32	1.00000000	−0.36044776	0.0
33	1.39804935	−0.27680981	0.0
34	1.49478626	−0.20076752	0.0
35	1.55895138	−0.11453408	0.0

CASE III

#	r	θ	φ
1	1.58823490	0.C	0.0
2	1.56944847	0.09199870	0.25706393
3	1.51405334	0.17994565	0.51818699
4	1.42356110	0.26009887	0.79194361
5	1.26362896	0.39429474	0.65774769
6	1.12099552	0.58313787	0.46890461
7	1.03663349	0.78539997	0.26664245
8	1.CCCCCCCC	1.C5204201	0.0
9	1.CCCCCCCC	1.30899906	0.0
10	1.00000000	1.57079983	0.0
11	1.00CCCCCC	1.74532986	0.0
12	1.000CCCCC	1.91985989	0.0
13	1.00000000	2.09439945	0.0
14	1.CCCCCCCC	2.26892948	0.0
15	1.CCCCCCCC	2.44345951	0.0
16	1.00C00000	2.61798954	0.0
17	1.00000000	2.87978935	0.0
18	1.00C0C000	3.05432987	0.0
19	1.CCCCCCCC	3.22885513	0.0
20	1.00000000	3.40339565	0.0
21	1.CCCCC000	3.57792568	0.0
22	1.COCOCCCC	3.75245571	0.0
23	1.00000000	3.92698574	0.0
24	1.00000000	4.18878937	0.0
25	1.CCCCCCCC	4.45058537	0.0
26	1.CCCCCCCC	4.62512589	0.0
27	1.00CCC000	4.79965591	0.0
28	1.00000000	4.97418594	0.0
29	1.CCCCC000	5.23598957	0.0
30	1.CCC00000	5.49798584	0.0
31	1.00000000	5.75958920	0.0
32	1.13324451	-0.36754555	-1.46504688
33	1.39804935	-0.27680981	-0.85765344
34	1.49478626	-0.20076752	-0.58462578
35	1.55895138	-0.11453408	-0.32179922

CASE IV

#	r	θ	φ
1	1.58823490	0.0	0.0
2	1.56944847	0.09199870	0.25706393
3	1.51405334	0.17994565	0.51818699
4	1.42356110	0.26009887	0.79194361
5	1.26362896	0.39429474	0.65774769
6	1.12099552	0.58313787	0.46890461
7	1.03663349	0.78539997	0.26664245
8	1.00000000	1.05204201	0.0
9	1.00000000	1.30899906	0.0
10	1.00000000	1.57079983	0.0
11	1.00000000	1.74532986	0.0
12	1.00000000	1.91985989	0.0
13	1.00000000	2.09439945	0.0
14	1.00000000	2.26892948	0.0
15	1.00000000	2.44345951	0.0
16	1.00000000	2.61798954	0.0
17	1.00000000	2.87978935	0.0
18	1.00000000	3.05432987	0.0
19	1.00000000	−3.05432987	0.0
20	1.00000000	−2.87978935	0.0
21	1.00000000	−2.61798954	0.0
22	1.00000000	−2.44345951	0.0
23	1.00000000	−2.26892948	0.0
24	1.00000000	−2.09439945	0.0
25	1.00000000	−1.91985989	0.0
26	1.00000000	−1.74532986	0.0
27	1.00000000	−1.57079983	0.0
28	1.00000000	−1.30899906	0.0
29	1.00000000	−1.05204201	0.0
30	1.03663349	−0.78539997	−0.26664245
31	1.12099552	−0.58313787	−0.46890461
32	1.26362896	−0.39429474	−0.65774769
33	1.42356110	−0.26009887	−0.79194361
34	1.51405334	−0.17994565	−0.51818699
35	1.56944847	−0.09199870	−0.25706393

CASE V

#	r	θ	φ
1	1.58823490	0.0	0.0
2	1.55895138	0.11453408	0.32179922
3	1.49478626	0.20076752	0.58462578
4	1.39804935	0.27680981	0.85765344
5	1.00000000	0.36044776	0.0
6	1.00000000	-5.75958920	0.0
7	1.00000000	-5.49778557	0.0
8	1.00000000	-5.23598957	0.0
9	1.00000000	-4.97418594	0.0
10	1.00000000	-4.79965591	0.0
11	1.00000000	-4.62512589	0.0
12	1.00000000	-4.45058537	0.0
13	1.00000000	-4.18878937	0.0
14	1.00000000	-3.92698574	0.0
15	1.00000000	-3.75245571	0.0
16	1.00000000	-3.57792568	0.0
17	1.00000000	-3.40339565	0.0
18	1.00000000	-3.22885513	0.0
19	1.00000000	3.22885513	0.0
20	1.00000000	3.40339565	0.0
21	1.00000000	3.57792568	0.0
22	1.00000000	3.75245571	0.0
23	1.00000000	3.92698574	0.0
24	1.00000000	4.18878937	0.0
25	1.00000000	4.45058537	0.0
26	1.00000000	4.62512589	0.0
27	1.00000000	4.79965591	0.0
28	1.00000000	4.97418594	0.0
29	1.00000000	5.23598957	0.0
30	1.00000000	5.49798584	0.0
31	1.00000000	5.75958920	0.0
32	1.00000000	-0.36044776	0.0
33	1.39804935	-0.27680981	-0.85765344
34	1.49478626	-0.20076752	-0.58462578
35	1.55895138	-0.11453408	-0.32179922

CASE VI

#	r	θ	φ
1	1.58823490	0.0	0.0
2	1.55895138	0.11453408	0.32179922
3	1.49478626	0.20076752	0.58462578
4	1.39804935	0.27680981	0.85765344
5	1.13324451	0.36754555	1.46504688
6	1.00000000	-5.75958920	0.0
7	1.00000000	-5.49778557	0.0
8	1.00000000	-5.23598957	0.0
9	1.00000000	-4.97418594	0.0
10	1.00000000	-4.79965591	0.0
11	1.00000000	-4.62512589	0.0
12	1.00000000	-4.45058537	0.0
13	1.00000000	-4.18878937	0.0
14	1.00000000	-3.92698574	0.0
15	1.00000000	-3.75245571	0.0
16	1.00000000	-3.57792568	0.0
17	1.00000000	-3.40339565	0.0
18	1.00000000	-3.22885513	0.0
19	1.00000000	3.22885513	0.0
20	1.00000000	3.40339565	0.0
21	1.00000000	3.57792568	0.0
22	1.00000000	3.75245571	0.0
23	1.00000000	3.92698574	0.0
24	1.00000000	4.18878937	0.0
25	1.00000000	4.45058537	0.0
26	1.00000000	4.62512589	0.0
27	1.00000000	4.79965591	0.0
28	1.00000000	4.97418594	0.0
29	1.00000000	5.23598957	0.0
30	1.00000000	5.49798584	0.0
31	1.00000000	5.75958920	0.0
32	1.13324451	-0.36754555	-1.46504688
33	1.39804935	-0.27680981	-0.85765344
34	1.49478626	-0.20076752	-0.58462578
35	1.55895138	-0.11453408	-0.32179922

CASE VII

#	r	θ	φ
1	1.COCCOOCC	C.C	0.0
2	1.00000000	0.26179934	C.0
3	1.C0000000	0.52359867	0.0
4	1.000CCOCO	0.78539783	0.0
5	1.0CCCC000	1.04719734	0.0
6	1.000CC000	1.30899620	C.0
7	1.00000000	1.57079506	0.0
8	1.0CCCCC00	1.83259487	0.0
9	1.000CCCCC	2.09439468	0.0
10	1.0000000C	2.35619354	0.0
11	1.00000000	2.61799240	0.0
12	1.0CC00CC0	2.87979221	0.0
13	1.000CC000	3.14159203	0.0
14	1.00000000	3.40339088	0.0
15	1.000CC000	3.66518974	0.0
16	1.CCCCCCC0	3.92698956	0.0
17	1.00000000	4.18878937	0.0
18	1.00000000	4.45058823	0.0
19	1.COCC0000	4.71238708	0.0
20	1.0000000C	4.97418690	0.0
21	1.00C00000	5.23598671	0.0
22	1.00000000	5.49778557	0.0
23	1.CCC0C0C0	5.75958443	0.0

CASE VIII

#	r	θ	φ
		Large Circle	
1	1.12099552	0.58313787	0.0
2	1.03663349	0.78539997	0.0
3	1.00000000	1.05204201	0.0
4	1.00000000	1.30899906	0.0
5	1.00000000	1.57079983	0.0
6	1.00000000	1.83259964	0.0
7	1.00000000	2.09439945	0.0
8	1.00000000	2.35618973	0.0
9	1.00000000	2.61798954	0.0
10	1.00000000	2.96705914	0.0
11	1.00000000	3.31612968	0.0
12	1.00000000	3.66518974	0.0
13	1.00000000	3.92698574	0.0
14	1.00000000	4.18878937	0.0
15	1.00000000	4.45058537	0.0
16	1.00000000	4.71238995	0.0
17	1.00000000	4.97418594	0.0
18	1.00000000	5.23598957	0.0
19	1.00000000	5.58504963	0.0
20	1.00000000	-0.36044776	0.0
		Small Circle	
21	0.65905398	-1.21280575	0.0
22	0.42300457	-2.00712967	0.0
23	0.42156863	-2.87978935	0.0
24	0.42156863	-3.66518974	0.0
25	0.42156863	-4.53785992	0.0
26	0.42156863	0.99108565	0.0
		Common Boundary of Large Circle	
27	0.97536963	0.28604054	-0.28604054
28	0.93662447	-0.04348692	0.04348692
		Common Boundary of Small Circle	
29	0.35925764	-0.87265998	0.87265998
30	0.23448992	0.17452997	-0.17452997

THE ANALYSIS OF A PLATE OF IRREGULAR
GEOMETRY USING POLAR COORDINATES AND POINT MATCHING

by

LOUIS DANIEL KOTTMANN, JR.

B. S., Kansas State University, 1967

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Manhattan, Kansas

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In this report the method of point matching is applied to the analysis of a plate of irregular geometry. Both single and double poles of expansion are used and the advantages and disadvantages of each method are investigated. Also, a comparison is made between the accuracy obtained for symmetric cases and unsymmetric cases. The results of this report show that in order to obtain reasonable results for irregular cases, a sufficiently large number of collocation points must be used and the points must be equally spaced according to arc length. The results also show that considerably less accuracy can be expected when working with completely unsymmetric shapes than when working with relatively uniform symmetric shapes. A multiple-pole scheme is suggested whenever an irregular shape can be separated into more uniform regular shapes. The investigation of the addition of the least squares technique to the point matching method is recommended for further research.